Analogy and solutions for the harmonic oscillator

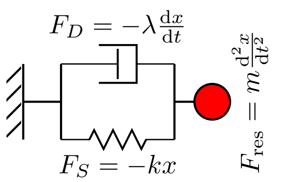
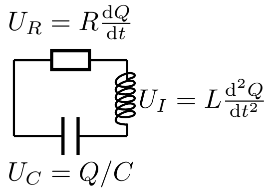
BK 2, Physics (Optional exercises)

January 2014

The aim of this note is to illustrate the analogy between the mechanical system studied by simulation and the electrical system studied by experiment. This is done by formulating the differential equation which controls the two systems. Furthermore, the analytic solution of this equation is discussed, and finally the case of two masses connected by a spring is discussed.

Analogy between systems

The following table summarize the components and physics (force and energy) for the two systems.

	Mechanical system	**Electric system**
Basic variables:		
	Displacement (x) and Force (F)	Charge (Q) and Voltage (U)
Components:		
	Spring: $F_S = -kx$	Capacitor: $U_C = Q/C$
	$E_{\text{elastic}} = \frac{1}{2}kx^2$	$E_{\text{electric}} = \frac{1}{2}\frac{1}{C}Q^2$
	Mass: $F_{\text{res}} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$	Inductor: $U_I = L\frac{\mathrm{d}I}{\mathrm{d}t} = L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2}$
	$E_{\text{kin}} = \frac{1}{2}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2$	$E_{\text{mag}} = \frac{1}{2}L\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)^2$
	Dashpot: $F_D = -\lambda\frac{\mathrm{d}x}{\mathrm{d}t}$	Resistor: $U_R = RI = R\frac{\mathrm{d}Q}{\mathrm{d}t}$
Topology:		
	Sum of forces equal to F_{res} $F_{\text{res}} = F_S + F_D$	Sum of voltage drops equal to zero $0 = U_C + U_I + U_R$
		
Differential equation:		
	$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - \lambda\frac{\mathrm{d}x}{\mathrm{d}t}$	$0 = L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + Q/C + R\frac{\mathrm{d}Q}{\mathrm{d}t}$ $L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} = -Q/C - R\frac{\mathrm{d}Q}{\mathrm{d}t}$

It can be seen that the two differential equations have the same structure, and therefore they will also have equivalent solutions (substituting the relevant quantities). It is common in physics, that different systems can be described by the same (differential) equations, and therefore share solutions and behavior.

It can be seen (as is hinted in the structure of the table on the previous page) that the variables and components are pairwise analogous.

Displacement	$\leftrightarrow$	Charge
Force	$\leftrightarrow$	Voltage
Spring	$\leftrightarrow$	Capacitor
Mass	$\leftrightarrow$	Inductor
Dashpot	$\leftrightarrow$	Resistor
Elastic energy	$\leftrightarrow$	Electrostatic energy
Kinetic energy	$\leftrightarrow$	Magnetic energy

Such analogies can be studied in great details and is the basis of a whole type of modeling in physics (this is studied in detail in the course “Physical Modeling”).

Analytic solution

The analytic solution to the formulated differential equations will be discussed in terms of the mechanical system.

Frictionless system

If the system is frictionless ($\lambda = 0$) the differential equation reduces to:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{k}{m} x \tag{1}$$

which has the solution:

$$x(t) = A\cos(\omega_0 t + \phi) \tag{2}$$

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{3}$$

Exercise 1 Check that equation 2 together with equation 3 is a solution to equation 1.

Exercise 2 Find A and ϕ so that $x(t)$ matches the *initial condition* where the system start in x_0 with $v = 0$ (that is $x(0) = x_0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0$).

Exercise 3 What is the relation between $1/\omega_0$ and the period, T, of the oscillation.

Exercise 4 (optional) Find A and ϕ so that $x(t)$ matches the initial condition where the system start with an arbitrary velocity (that is $x(0) = x_0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}(0) = v_0$).

Exercise 5 Compare the analytic solution to your simulations (are the results exactly equal).

Underdamped system

If the system is damped (that is $\lambda > 0$) the structure of the solution depends on the ratio between the parameters (m, k, and λ).

If the damping is weak enough, the solution is given by:

$$\begin{aligned} x(t) &= Ae^{-\frac{t}{\tau_{\mathrm{damp}}}}\cos(\omega t + \phi) && (4)\\ \omega &= \omega_0\sqrt{1 - \left(\frac{1}{\omega_0\tau_{\mathrm{damp}}}\right)^2} && (5)\\ \tau_{\mathrm{damp}} &= \frac{2m}{\lambda}. && (6)\end{aligned}$$

The solution holds as long as the time constant of the damping (τ_{damp}) is larger than the characteristic time of the oscillation ($1/\omega_0$), that is:

$$1 < \omega_0\tau_{\mathrm{damp}}. \tag{7}$$

In this case the system is said to be "underdamped".

Exercise 6 What is the physical interpretation of τ_{damp}, and the product $\omega_0\tau_{\mathrm{damp}}$.

In the case where the system start in x_0 with $v = 0$, A and ϕ can be found as:

$$\begin{aligned} A &= x_0\sqrt{\frac{1}{1 - \left(\frac{1}{\omega_0\tau_{\mathrm{damp}}}\right)^2}} && (8)\\ \phi &= -\tan^{-1}\left(\frac{1}{\omega\tau_{\mathrm{damp}}}\right) && (9)\end{aligned}$$

Exercise 7 Compare the formulas for A, and ϕ given in equation 8 and 9 to the results from exercise 2. Under which conditions is it important to use there more complicated formulas?

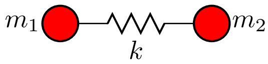
Exercise 8 Compare the analytic solutions with simulation results.

- Vary λ all the way to the limit given in equation 7
  Hint: Matlab might have problems if you set λ to exact the limit where the solution breaks down.
- Show the "effective amplitude" ($Ae^{-\frac{t}{\tau_{\mathrm{damp}}}}$) on the plot.
- Indicate $t = \tau_{\mathrm{damp}}$ on the plot. Compare to your results from exercise 6.

Two masses and a spring

In many applications of the harmonic oscillator, the system consists of a number of masses connected by springs. This is in contrast to the system which you have been working with until now, where one end of the spring is attached to a rigid unmovable wall.

In the following the simplest case, of two masses and one spring, will be analyzed.



Two differential equations are needed to describe the system:

$$m_1 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = k\Delta l \tag{10}$$

$$m_2 \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = -k\Delta l, \tag{11}$$

where x_1 and x_2 are the positions of the two masses respectively, and Δl is the change of the length of the spring from its "rest length" (l_0).

Exercise 9 Consider why the sign is different between the two equations.

The system can be greatly simplified, by setting up a differential equation for the change in length, Δl, instead of the positions of the individual masses.

The connection between the positions (x_1 and x_2) and Δl is described as:

$$\Delta l = l - l_0 \tag{12}$$

$$= (x_2 - x_1) - l_0 \tag{13}$$

By differentiation equation 13 yields

$$\frac{\mathrm{d}^2 \Delta l}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} - \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2}. \tag{14}$$

Equation 10 and 11 can now be substituted into equation 14, giving:

$$\frac{\mathrm{d}^2 \Delta l}{\mathrm{d}t^2} = -k \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \Delta l, \tag{15}$$

which is a differential equation of the same form as equation 1, but in this case describing the time evolution of the distance between the two masses, Δl.

The quantity $\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$ plays the role of the inverse mass, and normally one defines the so called "reduced mass" μ as:

$$\mu = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}} = \frac{m_1 m_2}{m_1 + m_2}. \tag{16}$$

With this definition equation 15 can be reformulate as:

$$\frac{\mathrm{d}^2 \Delta l}{\mathrm{d}t^2} = -\frac{k}{\mu} \Delta l, \tag{17}$$

completely equivalent to equation 1.

Exercise 10 How is the characteristic vibration frequency of a diatomic molecule related to the "spring constant" and masses of the atoms.

Exercise 11 Verify the steps leading to equation 17.