Effects of overkill and regeneration on damage output rate in Oldschool Runescape

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1 Introduction

Efficient combat training in Oldschool Runescape is a problem of maximizing the damage output rate of the player (often measured by DPS). To optimize the DPS it is important to be able to determine its value also theoretically. Overkill and hitpoint regeneration are two major factors that have significant effects on DPS, yet the quantitative understanding of them is limited. While simulations provide quite good results, a more mathematical viewpoint can give further insight into the nature of these effects and result in some convenient approximations.

Overkill is an effect that occurs when the damage roll exceeds the remaining hitpoints of the enemy. It is important in combat training when the idle time between kills is short, and especially when the max hit is large relative to the enemy hitpoints.

The other mechanic that will be studied is health regeneration. Assuming no regeneration is a fair approximation when regeneration rate is much slower than the damage rate. But there are inevitably scenarios where this is not the case. Whereas overkill is most significant for high damage rates, regeneration affects mostly the slower fights. While there are many ways to regain health, we will be focusing on *natural regeneration* only.

2 Fight mechanics

A fight consists of two parties, attacker and the enemy. We only consider the damage dealt by the attacker on the enemy and not vice versa. In our model the enemy is attacked periodically until its remaining hitpoints are 0. The attack rate is characterized by the *attack period* T_A, which varies between weapons. Every time the enemy is hit (once per time period T_A) the damage dealt is calulated as follows.

1. The game determines if the hit will succeed by some random process which depends on various parameters, such as defensive bonuses of the enemy, offensive bonuses of the attacker and appropriate combat-related stats. We ignore the details of this process and just say the probability of the hit being successfull is a, which is called the *accuracy*.

2. If the hit was successful damage roll M, a uniformly distributed random number between 0 and m, is chosen, where m is the *maximum hit*. Like with accuracy we do not concern ourselves with the specifics of how the max hit is determined.

3. If $M > H$, where H is the remaining hitpoints of the enemy, the damage is capped to H and the final damage dealt is $\min(M, H)$. This is the overkill effect.

Unlike the damage calculation process, regeneration is not fundamentally random. It happens by periodically incrementing the remaining hitpoints by one until they are full. The period, T_R, of this healing cycle can vary between enemies giving rise to different regeneration rates. At the beginning of a fight the state of the healing cycle can be assumed to be unknown and thus treated as a random variable. This way only the timing of the first heal is random and the rest are perfectly periodic.

Throughout this study the parameters a, m, T_A and T_R will be treated as constants. The periods T_A and T_R can both be assumed integers if using gameticks as units.

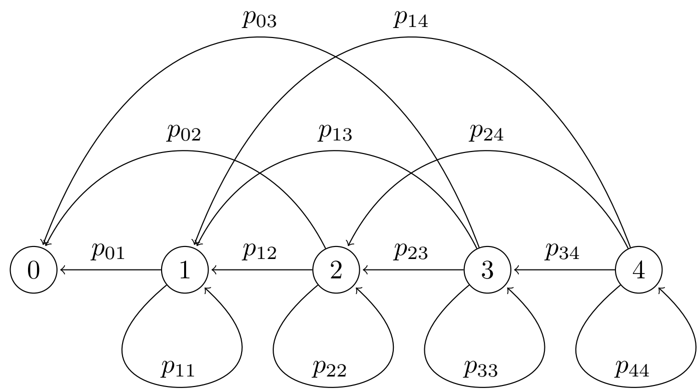

Figure 1: Random walk diagram for $m = 3$, $h = 4$ and no regeneration. The probability of walking to state i from state j is given with the transition probability p_{ij}. Only edges for which $p_{ij} > 0$ are shown.

3 Random walk description of a fight

We define the *state space* $\mathcal{H}$ as the set of all possible values remaining hitpoints can have during a fight. For a fight against an enemy with h maximum hitpoints the this is $\mathcal{H} = \{0, \ldots, h\}$. Fights can be thought of as random walks in $\mathcal{H}$ terminating in state 0. Each walk can be labeled with a sequence of hitpoint states $(H_k)_{k=0}^n$ visited during the fight such that $H_n = 0$ (fight ends in death) and $h \geq H_k > 0$ for $k < n$ (death does not happen before the last hit). The probability of transitioning to state i from state j is called the *transition probability* and is defined as

$$p_{ij} = \mathrm{P}(H_k{=}i \mid H_{k-1}{=}j). \tag{3.1}$$

The transition probabilities are said to have the *Markov property* as they only depend on the current state of the random walker.

Let $\mathcal{S}_j^n$ be the set of all possible fights of length n against an enemy with j hitpoints remaining. The simplest possible case is the set of all 1-hit fights $\mathcal{S}_j^1 = \{(j, 0)\}$. The sets of all longer fights can be defined recursively by noticing that if the first hit lowers the hitpoints to i, the remaining sequence of states is equivalent to a fight of length $n - 1$ against an enemy with i hitpoints remaining. In other words

$$\mathcal{S}_j^n = \{j\} \times \bigcup_{i=1}^{h} \mathcal{S}_i^{n-1} \quad \text{for } n > 1. \tag{3.2}$$

For the set of all 2-hit fights this gives $\mathcal{S}_j^2 = \{(j, h, 0), (j, h-1, 0), \ldots, (j, 2, 0), (j, 1, 0)\}$.

Using the transition probabilities (eq3.1) the probability that an n-hit fight $f \in \mathcal{S}_j^n$ occurs is

$$\mathrm{P}(f) = \prod_{k=1}^{n} p_{f_k f_{k-1}} \tag{3.3}$$

where f_k is the number of hitpoints the enemy has remaining after k hits.

Let L_j be the length of a fight against an enemy with j hitpoints remaining. The probability of a 1-hit fight is now

$$\mathrm{P}(L_j{=}1) = \sum_{f \in \mathcal{S}_j^1} \mathrm{P}(f) = \mathrm{P}((j{,}0)) = p_{0j} \tag{3.4}$$

For longer fights the probability is obtained by summing over all fights of length n and invoking eq3.2

$$\mathrm{P}(L_j{=}n) = \sum_{f \in \mathcal{S}_j^n} \mathrm{P}(f) = \sum_{i=1}^{h} p_{ij} \sum_{f \in \mathcal{S}_i^{n-1}} \mathrm{P}(f) = \sum_{i=1}^{h} p_{ij} \mathrm{P}(L_i{=}n{-}1) \tag{3.5}$$

The expected length of a fight can now be expressed as

$$\begin{aligned}
\langle L_j \rangle &= \sum_{n=1}^{\infty} n\mathrm{P}(L_j{=}n) \\
&= \mathrm{P}(L_j{=}1) + \sum_{n=2}^{\infty} n \sum_{i=1}^{h} p_{ij} \mathrm{P}(L_i{=}n{-}1) \\
&= p_{0j} + \sum_{i=1}^{h} p_{ij} \sum_{n=2}^{\infty} n\mathrm{P}(L_i{=}n{-}1). \qquad (3.6)
\end{aligned}$$

The inner sum can be worked out to be

$$\begin{aligned}
\sum_{n=2}^{\infty} n\mathrm{P}(L_i{=}n{-}1) &= \sum_{n=1}^{\infty} (n+1)\mathrm{P}(L_i{=}n) \\
&= \sum_{n=1}^{\infty} n\mathrm{P}(L_i{=}n) + \sum_{n=1}^{\infty} \mathrm{P}(L_i{=}n) \\
&= \langle L_i \rangle + 1. \qquad (3.7)
\end{aligned}$$

In the last equality we used the definition of expected value as well as the fact that an enemy will be guaranteed to die if hit infinitely many times. Inserting this back into eq3.6 gives

$$\langle L_j \rangle = p_{0j} + \sum_{i=1}^{h} p_{ij}(\langle L_i \rangle + 1) = \sum_{i=0}^{h} p_{ij} + \sum_{i=1}^{h} p_{ij} \langle L_i \rangle \qquad (3.8)$$

Since the transition probabilities must add up to 1 when summed over the entire state space, we have $\sum_{i=0}^{h} p_{ij} = 1$ and the recurrence relation for the expected length of a fight becomes

$$\boxed{\langle L_j \rangle = 1 + \sum_{i=1}^{h} p_{ij} \langle L_i \rangle}. \qquad (3.9)$$

Since eq3.9 holds for all $j \in \{1, \ldots, h\}$ it defines a linear system of h equations. In the next chapters we will study them further by taking a deeper look at the transition probabilities.

4 No regeneration

4.1 Transition probabilities

To compute the length of a fight from the recurrence (eq3.9) we need to know the transition probabilities. If regeneration is ignored all the state transitions are caused by hitting the enemy. From the fight mechanics as stated in Chapter 2 we can determine the probability distribution of the accuracy-corrected damage roll X. This is the amount of damage dealt before capping it by overkill.

$$\mathrm{P}(X{=}k) = \begin{cases} 1 - \frac{am}{m+1}, & \text{if } k = 0 \\ \frac{a}{m+1}, & \text{if } 1 \le k \le m \end{cases} \tag{4.1}$$

where a is the accuracy. As expected, the distribution is uniform everywhere except at 0, where the chance of missing skews it. In terms of eq4.1 the transition probabilities are given by

$$p_{ij} = \begin{cases} \mathrm{P}(X{=}j{-}i) & \text{if } i > 0 \\ \mathrm{P}(X{\ge}j{-}i) & \text{if } i = 0 \end{cases}. \tag{4.2}$$

A nice feature of the transition probabilities for the regenerationless case is that the hitpoints can never increase because $p_{ij} = 0$ for $i > j$. In particular, this means that it makes no difference if the enemy being fought has full hitpoints or if some of them are lost. In the absence of regeneration a fight against an enemy with j hitpoints remaining is identical to a fight against an enemy whose maximum hitpoints *are* j. Therefore, we can without loss of generality assume $j = h$ studying only fights that start at maximum hitpoints.

4.2 Recurrence relations

Inserting the transition probabilities into eq3.9 gives

$$\begin{aligned} \langle L_h \rangle &= 1 + \sum_{i=1}^{h} \mathrm{P}(X{=}h{-}i)\langle L_i \rangle \\ &= 1 + \mathrm{P}(X{=}0)\langle L_h \rangle + \sum_{i=1}^{h-1} \mathrm{P}(X{=}h{-}i)\langle L_i \rangle \\ \implies (1 - \mathrm{P}(X{=}0))\langle L_h \rangle &= 1 + \sum_{i=1}^{h-1} \mathrm{P}(X{=}h{-}i)\langle L_i \rangle \end{aligned} \tag{4.3}$$

Now there are three cases to be considered.
Case: $h = 1$

$$\langle L_1 \rangle = \frac{1}{1 - \mathrm{P}(X{=}0)} = \frac{m+1}{am} \tag{4.4}$$

Case: $1 < h \le m + 1$

$$\begin{aligned} \langle L_h \rangle &= \frac{m+1}{am} + \frac{m+1}{am} \sum_{i=1}^{h-1} \frac{a}{m+1} \langle L_i \rangle \\ &= \langle L_1 \rangle + \frac{1}{m} \sum_{i=1}^{h-1} \langle L_i \rangle \\ &= \underbrace{\langle L_1 \rangle + \frac{1}{m} \sum_{i=1}^{h-2} \langle L_i \rangle}_{\langle L_{h-1} \rangle} + \frac{1}{m} \langle L_{h-1} \rangle \\ &= \frac{m+1}{m} \langle L_{h-1} \rangle \end{aligned} \tag{4.5}$$

Case: $h > m+1$

$$\begin{aligned}\langle L_h\rangle &= \langle L_1\rangle + \frac{1}{m}\sum_{i=h-1-m}^{h-1}\langle L_i\rangle \\ &= \underbrace{\langle L_1\rangle + \frac{1}{m}\sum_{i=h-2-m}^{h-2}\langle L_i\rangle}_{\langle L_{h-1}\rangle} + \frac{1}{m}\langle L_{h-1}\rangle - \frac{1}{m}\langle L_{h-m-1}\rangle \\ &= \frac{m+1}{m}\langle L_{h-1}\rangle - \frac{1}{m}\langle L_{h-m-1}\rangle \\ &= \frac{m+1}{m}\left(\langle L_{h-1}\rangle - \frac{1}{m+1}\langle L_{h-m-1}\rangle\right) \end{aligned} \tag{4.6}$$

4.3 Solving the recurrences

We begin by defining $r \equiv \frac{m+1}{m}$ and $p \equiv \frac{-1}{m+1}$. This allows rewriting the recurrence relations more compactly as

$$\boxed{\langle L_h\rangle = \begin{cases} r\left(\langle L_{h-1}\rangle - p\langle L_{h-m-1}\rangle\right) & \text{if } h \leq m+1 \\ r\langle L_{h-1}\rangle & \text{if } h > m+1 \end{cases}} \tag{4.7}$$

with the initial condition $\langle L_1\rangle = \frac{r}{a}$. The case $h \leq m+1$ is simply the recurrence relation for a geometric sequence. Therefore,

$$\langle L_h\rangle = \frac{r^h}{a} \quad \text{if } h \leq m+1. \tag{4.8}$$

The recurrence relation for the case $h > m+1$ is not as easy to bring in a non-recursive form. It is a linear recurrence of order m, the initial m elements of which are given by eq4.8. This recurrence is perhaps most effectively approached with generating functions. The ordinary generating function of the number sequence $\langle L_h\rangle$ is the function $g(z)$ whose power series expansion has $\langle L_h\rangle$ as coefficients.

$$g(z) = \sum_{h=1}^{\infty}\langle L_h\rangle z^h \tag{4.9}$$

Using the generating function formulation the recurrence (eq4.7) translates into an algebraic equation for $g(z)$.

$$\begin{aligned} g(z) &= \sum_{h=1}^{m+1}\langle L_h\rangle z^h + \sum_{h=m+2}^{\infty}\langle L_h\rangle z^h \\ &= \sum_{h=1}^{m+1}\langle L_h\rangle z^h + \sum_{h=m+2}^{\infty} r\left(\langle L_{h-1}\rangle + p\langle L_{h-m-1}\rangle\right) z^h \\ &= \sum_{h=1}^{m+1}\langle L_h\rangle z^h + rz\sum_{h=m+2}^{\infty}\langle L_{h-1}\rangle z^{h-1} + rpz^{m+1}\sum_{h=m+2}^{\infty}\langle L_{h-m-1}\rangle z^{h-m-1} \\ &= \sum_{h=1}^{m+1}\langle L_h\rangle z^h + rz\sum_{h=m+1}^{\infty}\langle L_h\rangle z^h + rpz^{m+1}\sum_{h=1}^{\infty}\langle L_h\rangle z^h \\ &= \sum_{h=1}^{m}\langle L_h\rangle z^h + \langle L_{m+1}\rangle z^{m+1} + rz\left(\sum_{h=1}^{\infty}\langle L_h\rangle z^h - \sum_{h=1}^{m}\langle L_h\rangle z^h\right) + rpz^{m+1}g(z) \\ &= (1-rz)\sum_{h=1}^{m}\langle L_h\rangle z^h + \langle L_{m+1}\rangle z^{m+1} + \left(rz + rpz^{m+1}\right)g(z) \end{aligned} \tag{4.10}$$

Now, moving all terms containing $g(z)$ to the left and substituting the values for $\langle L_h \rangle$ from eq4.8 gives

$$(1 - rz - rpz^{m+1})g(z) = \frac{1}{a}\left((rz)^{m+1} + (1 - rz)\sum_{h=1}^{m}(rz)^h\right) \tag{4.11}$$

The partial sum on the right is the m first terms of a geometric series which has the following closed form expression.

$$\sum_{h=1}^{m}(rz)^h = rz\frac{1-(rz)^m}{1-rz}$$

Plugging it back to eq4.11 to gives

$$(1 - rz - rpz^{m+1})g(z) = \frac{1}{a}\left((rz)^{m+1} + rz\left(1-(rz)^m\right)\right) = \frac{1}{a}rz$$
$$\implies g(z) = \frac{rz/a}{1 - rz - rpz^{m+1}}. \tag{4.12}$$

To read off the explicit formula for $\langle L_h \rangle$ the generating function must be expanded back into a power series. This is possible by noticing that $g(z)$ is an infinite geometric sum with $rz(1 - pz^m)$ as its coefficient.

$$g(z) = \frac{rz/a}{1 - rz\left(1 + pz^m\right)} = \frac{rz}{a}\sum_{n=0}^{\infty}(rz(1+pz^m))^n = \frac{1}{a}\sum_{n=0}^{\infty}(rz)^{n+1}(1+pz^m)^n$$

Then, we expand the nth order binomial term using the binomial theorem.

$$\begin{aligned} g(z) &= \frac{1}{a}\sum_{n=0}^{\infty}(rz)^{n+1}\sum_{i=0}^{n}(pz^m)^i\binom{n}{i} \\ &= \sum_{n=0}^{\infty}\frac{1}{a}\sum_{i=0}^{n}r^{n+1}p^i\binom{n}{i}z^{mi+n+1}. \end{aligned} \tag{4.13}$$

$\langle L_h \rangle$ is now the coefficient of the term z^h. Looking at the exponent of z in eq4.13 we see that for the coefficient of interest the indices i and n must satisfy $h = mi + n + 1$. For each i there is exactly one valid n, namely $n = h - mi - 1$. Therefore, the coefficient of the hth order term in the series expansion of $g(z)$ is

$$\langle L_h \rangle = \frac{1}{a}\sum_{i \in \mathcal{I}} r^{h-mi}p^i\binom{h-mi-1}{i}$$

where $\mathcal{I}$ is the set of indices i such that

$$0 \leq i \leq n = h - mi - 1 \iff (m+1)i \leq h - 1 \iff i \leq \frac{h-1}{m+1}.$$

Since i must also be an integer the index set becomes $\mathcal{I} = \left\{0, 1, \ldots, \left\lfloor \frac{h-1}{m+1} \right\rfloor\right\}$. Expressing the constants r and p explicitly, the solution to the recursion problem (eq4.7) can be written as

$$\langle L_h \rangle = \frac{1}{a}\sum_{i=0}^{\left\lfloor \frac{h-1}{m+1} \right\rfloor}\left(\frac{m+1}{m}\right)^{h-mi}\left(\frac{-1}{m+1}\right)^i\binom{h-mi-1}{i}. \tag{4.14}$$

Naturally this can also be proven to satisfy the recursion using induction (see Appendix A). With this result the damage dealt per hit is simply $h/\langle L_h \rangle$ and the damage per second can be obtained from it by dividing with the attack interval T_A.

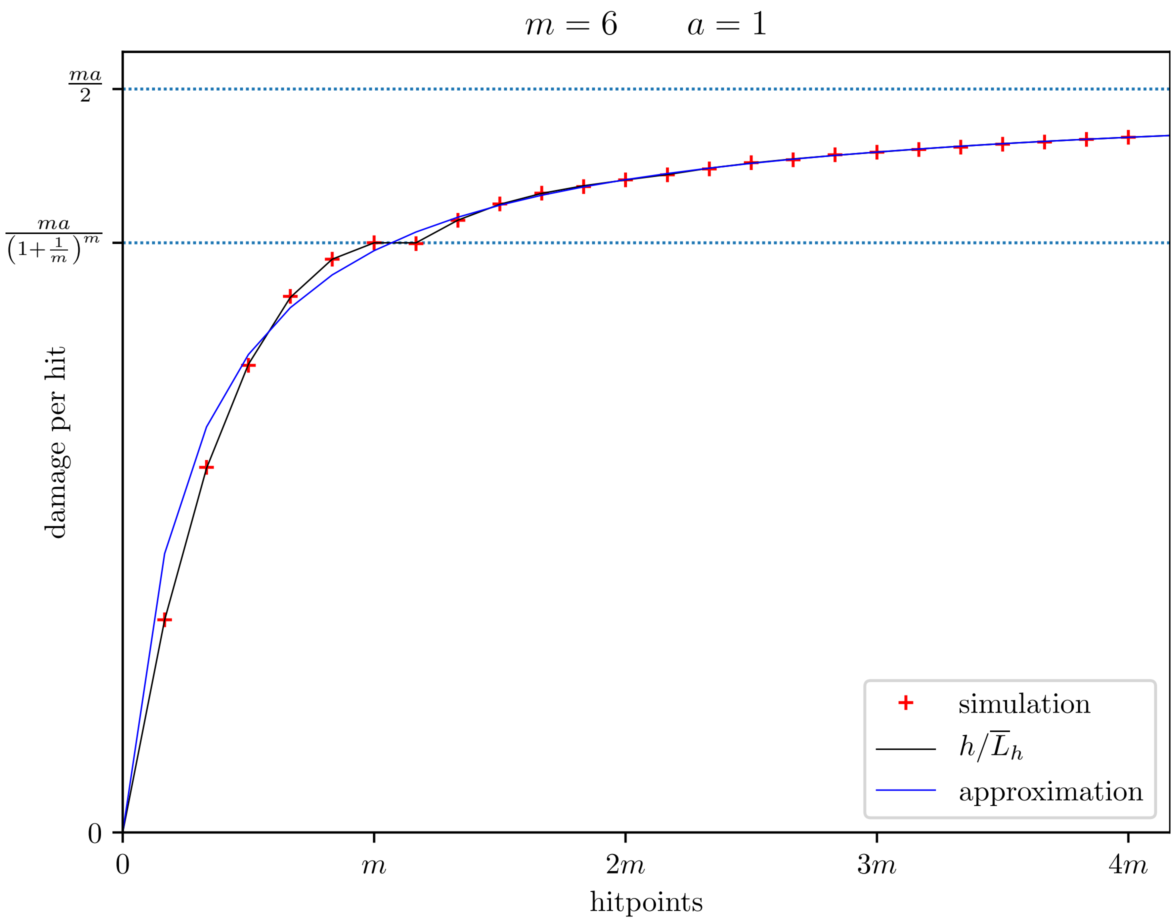

Figure 2: Comparison of damage per hit (DPH) calculated using the explicit formula (eq4.14), asymptotic approximation (eq4.15) and a simulation. The simulation was written using the damage calculation mechanics described in Chapter 3. For each datapoint 10^5 fights were simulated and their average lengths calculated.

4.4 Asymptotic approximation

Eq4.14 can be annoying to work with, especially for large h, so an approximation for the case $h \gg m$ is needed. Intuitively the length of a fight should have linear dependency on hitpoints as $h \to \infty$. This is because on average one can expect each hit to do $am/2$ damage and therefore it should take approximately $2h/am$ hits to kill an enemy with h hitpoints. Due to overkill however, the last hit will do slightly less than $am/2$ damage on average introducing a constant correction term to this linear approximation. The average length of a fight with no regeneration has the following asymptotic behaviour (derivation not shown here).

$$\overline{L}_h \sim \frac{2}{ma}\left(h + \frac{m-1}{3}\right) \tag{4.15}$$

The correctness of both the results was validated by comparing them to a computer simulation. The comparison is illustrated in figure 2.

5 Regeneration

5.1 Regenerating random walker

When regeneration is considered, the random walk that was used to describe a fight in Chapter 3 gets new edges. Now it will be possible to walk backwards as well as forwards in the state space. Unfortunately, since regeneration attempts happen with a fixed period T_R, the transition probabilities will vary with time. Furthermore, when the regeneration attempts do occur they might not coincide with the damaging events.

To incorporate the two events in the same random walker formalism, consider the time interval $\Delta t_k = [t_k, t_{k+1} - 1]$, where t_k is the (game)tick on which the kth hit occurs. A single transition of the random walker is now determined by the overall state change that takes place during time interval Δt_k. Since we have assumed $T_R \geq T_A$ the number of regeneration attempts during Δt_k is at most 1. The assumption $T_A \leq T_B$ should hold for nearly all cases in practice as the the typical regeneration period is 60 seconds and even the slowest weapons have attack periods of just 4.2 seconds. For some rarer cases such as flinching an enemy with unusually high regeneration rate where this could become a problem one could simply allow the regeneration of more than 1 hitpoint per attack interval.

The only source of randomness in the regeneration model is the tick τ on which the first regeneration attempt occurs. We treat it as a random variable distributed uniformly in the interval $[1, T_R]$. Alternatively τ can be thought of as the time until the next regeneration attempt at the beginning of a fight. Now the expected length of a fight is

$$
\begin{aligned}
\langle L_j \rangle &= \sum_{n=1}^{\infty} n \mathrm{P}(L_j{=}n) \\
&= \sum_{n=1}^{\infty} n \sum_{\tau=1}^{T_R} \mathrm{P}(\tau)\mathrm{P}(L_j{=}n \mid \tau) \\
&= \frac{1}{T_R} \sum_{\tau=1}^{T_R} \sum_{n=1}^{\infty} n \mathrm{P}(L_j{=}n \mid \tau) \\
&= \frac{1}{T_R} \sum_{\tau=1}^{T_R} \langle L_j^\tau \rangle
\end{aligned}
\tag{5.1}
$$

where L_j^τ is the length of a fight at the beginning of which the next regeneration attempt is τ ticks away. While not shown here, by almost identical reasoning to that in Chapter 3 one can derive the recurrence relation

$$
\langle L_j^\tau \rangle = 1 + \sum_{i=1}^{h} p_{ij}^\tau \langle L_i^{\tau - T_A} \rangle. \tag{5.2}
$$

The only critical difference besides the transition probabilities is the recursion argument. Now in addition to modifying the remaining hitpoints, hitting the enemy also shifts τ backwards by T_A ticks since this is the amount of time that passes between hits. The time-dependent transition probabilities are given by

$$
p_{ij}^\tau = \begin{cases} p_{ij} & \text{if } \tau > T_A \pmod{T_R} \\ p_{i,\min(j+1,h)} & \text{if } \tau \leq T_A \pmod{T_R} \end{cases}. \tag{5.3}
$$

where p_{ij} is the transition probability of the non-regenerating case (eq4.2). The minimum is taken to prevent hitpoints from exceeding h and modular arithmetic used to handle the periodicity of the regeneration cycle.

Because of the backwards transition that regeneration has made possible, all states are now dependent on one another. Therefore, a recursive solution similar to that in Chapter 4 is no longer possible and eq5.2 should instead be treated as a linear system of h equations. Furthermore, the time shift in the τ-dependency splits them further making the system actually $hT_R/\gcd(T_R, T_A)$-dimensional. For example in case of fighting ankous with a scimitar ($h = 60$, $T_R = 100$, $T_A = 4$) we would have to solve a system of 1500 equations.

5.2 Random healing cycle approximation

Instead of regenerating deterministically at realistic time intervals we could assume that the healing happens right after each hit with such a probability that the correct regeneration rate is achieved. This reduces the complexity of equation 5.2 significantly as it removes the τ-dependence entirely. We define the *regeneration rate* as

$$\rho = \frac{T_A}{T_R} \tag{5.4}$$

and interpret it as the probability that a regeneration event occurs during an attack cycle.

In this model, there are two ways of lowering the hitpoints by k: hit k and heal 0 or hit $k+1$ and heal 1. In terms of the regeneration probability ρ and the damage roll X the transition probabilities are

$$p_{ij} = \begin{cases} \mathrm{P}(X{=}j{-}i) + \rho\mathrm{P}(X{=}j{-}i{+}1) & \text{if } i = h \\ (1-\rho)\mathrm{P}(X{=}j{-}i) + \rho\mathrm{P}(X{=}j{-}i{+}1) & \text{if } 1 < i < h \\ (1-\rho)\mathrm{P}(X{=}j{-}i) & \text{if } i = 1 \\ \mathrm{P}(X{\geq}j{-}i) & \text{if } i = 0 \end{cases} . \tag{5.5}$$

Notice that transitioning to state 0 does not depend on regeneration because dead enemies cannot regenerate. For the same reason it is impossible to land in state 1 by first reaching 0 hitpoints and then healing. Transition probability for going to state h on the other hand, is different because it is not possible to heal past maximum hitpoints.

Naturally it is still possible for the remaining hitpoints to climb up the state space making a recursive solution impossible. However, because of the eliminated dependence on the pahse of the regeneration cycle the walker is now fully described by equation 3.9 just like in the regenerationless case. This reduces the size of the linear system down to h equations, which is a much more manageable number for practical calculations.

5.3 Effective hitpoints approach

Consider a fight of length L. Since the fight lasts $T_A L$ ticks the number of hitpoints regenerated can be estimated by $\frac{T_A}{T_R}L$. Assuming that the enemy has h maximum hitpoints, the total damage dealt during the fight is $y \equiv h + \frac{T_A}{T_R}L$. This quantity is called the *effective hitpoints* because the fight is nearly equivalent to one against a non-regenerating enemy with y hitpoints. If regeneration rate is small compared to the damage rate the expected length of a fight should be approximately $\langle L_y \rangle$. This gives us the equation

$$y = h + \rho\langle L_y \rangle \tag{5.6}$$

where we have defined the *regeneration rate* $\rho \equiv \frac{T_A}{T_R}$.

To solve y from this equation we use the asymptotic approximation (eq4.15).

$$\begin{aligned} y &= h + \frac{2\rho}{ma}\left(y + \frac{m-1}{3}\right) \\ \implies \left(1 - \frac{2\rho}{ma}\right) y &= h + \frac{2\rho}{ma}\frac{m-1}{3} \\ \implies y &= \frac{1}{ma - 2\rho}\left(mah + 2\rho\frac{m-1}{3}\right) \\ &= h + \rho\left(\frac{h + \frac{m-1}{3}}{\frac{ma}{2} - \rho}\right) \end{aligned} \tag{5.7}$$

Writing the effective hitpoints in this form reveals some nice properties. Comparing the result to eq5.6 allows reading off the expected length of a fight as

$$\langle L_y \rangle = \frac{h + \frac{m-1}{3}}{\frac{ma}{2} - \rho} \tag{5.8}$$

and in turn the damage per hit as

$$\frac{y}{\langle L_y \rangle} = \frac{\frac{ma}{2} - \rho}{1 + \frac{m-1}{3h}} + \rho \tag{5.9}$$

If the regeneration rate ρ is larger than the damage rate $\frac{ma}{2}$ the equation breaks down as the term $\frac{ma}{2} - \rho$ becomes negative. This seems to suggest that in this case the fight would go on forever. Of course this is just a limitation of the model since in reality the fight would eventually terminate as long as $T_R > T_A$ and $m > 0$, which is a very resonable assumption.

A Inductive proof of equation 4.14

We will use the binomial coefficient identities $\binom{k}{n} = \binom{k-1}{n} + \binom{k-1}{n-1}$ and $\binom{k}{0} = 1$ to prove that

$$\overline{L}_h = \frac{1}{a}\sum_{i=0}^{N} r^{h-im}p^i\binom{h-im-1}{i}, \quad \text{where } N = \left\lfloor \frac{h-1}{m+1} \right\rfloor$$

is a solution to the recursion problem stated by eq4.7.

Proof. If $0 < h \leq m+1$, then $N = 0$ and

$$\overline{L}_h = \frac{1}{a}\sum_{i=0}^{0} r^{h-im}p^i\binom{h-im-1}{i} = a^{-1}r^h\binom{h-1}{0} = a^{-1}r^h$$

Therefore the statement holds for the initial condition.
Assume that the statement is true for the $m+1$ consecutive elements $\overline{L}_{h-1-m}, \ldots, \overline{L}_{h-1}$. Then

$$\begin{aligned}
\overline{L}_h &= r\left(\overline{L}_{h-1} + p\overline{L}_{h-m-1}\right) \\
&= r\left(\frac{1}{a}\sum_{i=0}^{\left\lfloor \frac{(h-1)-1}{m+1} \right\rfloor} r^{(h-1)-im}p^i\binom{(h-1)-im-1}{i} + p\frac{1}{a}\sum_{i=0}^{\left\lfloor \frac{(h-1-m)-1}{m+1} \right\rfloor} r^{(h-1-m)-im}p^i\binom{(h-1-m)-im-1}{i}\right) \\
\implies a\overline{L}_h &= r\sum_{i=0}^{\left\lfloor \frac{h-2}{m+1} \right\rfloor} r^{h-1-im}p^i\binom{h-2-im}{i} + rp\sum_{i=0}^{\left\lfloor \frac{h-1-(m+1)}{m+1} \right\rfloor} r^{h-1-(i+1)m}p^i\binom{h-2-(i+1)m}{i} \\
&= \sum_{i=0}^{\left\lfloor \frac{h-2}{m+1} \right\rfloor} r^{h-im}p^i\binom{h-2-im}{i} + \sum_{i=0}^{\left\lfloor \frac{h-1-(m+1)}{m+1} \right\rfloor} r^{h-(i+1)m}p^{i+1}\binom{h-2-(i+1)m}{i} \\
&= \sum_{i=0}^{\left\lfloor \frac{h-2}{m+1} \right\rfloor} r^{h-im}p^i\binom{h-2-im}{i} + \sum_{i=0}^{\left\lfloor \frac{h-1}{m+1} \right\rfloor - 1} r^{h-(i+1)m}p^{i+1}\binom{h-2-(i+1)m}{i} \qquad \text{(A.1)}
\end{aligned}$$

If $h-1 = N(m+1)$, then $\left\lfloor \frac{h-2}{m+1} \right\rfloor = \left\lfloor \frac{h-1}{m+1} \right\rfloor - 1 = N-1$. Now eqA.1 reduces to

$$\begin{aligned}
a\overline{L}_h &= \sum_{i=0}^{N-1} r^{h-im}p^i\binom{h-2-im}{i} + \sum_{i=0}^{N-1} r^{h-(i+1)m}p^{i+1}\binom{h-2-(i+1)m}{i} \\
&= \sum_{i=0}^{N-1} r^{h-im}p^i\binom{h-2-im}{i} + \sum_{i=1}^{N} r^{h-im}p^i\binom{h-2-im}{i-1} \\
&= \sum_{i=1}^{N} r^{h-im}p^i\left(\binom{h-2-im}{i} + \binom{h-2-im}{i-1}\right) + r^h\binom{h-2}{0} - r^{h-Nm}p^N\binom{h-2-Nm}{N} \\
&= \sum_{i=1}^{N} r^{h-im}p^i\binom{h-1-im}{i} + r^h - r^{N(m+1)+1-Nm}p^N\binom{N(m+1)-1-Nm}{N} \\
&= \sum_{i=1}^{N} r^{h-im}p^i\binom{h-1-im}{i} + r^{h-0m}p^0\binom{h-1-0m}{0} - r^{N+1}p^N\cancelto{0}{\binom{N-1}{N}} \\
&= \sum_{i=0}^{N} r^{h-im}p^i\binom{h-1-im}{i}
\end{aligned}$$

which is the required form.

The other case to consider is $h - 1 \neq N(m+1) \implies \left\lfloor \frac{h-2}{m+1} \right\rfloor = \left\lfloor \frac{h-1}{m+1} \right\rfloor = N$. In this case eqA.1 gives

$$\begin{aligned}
a\overline{L}_h &= \sum_{i=0}^{N} r^{h-im} p^i \binom{h-2-im}{i} + \sum_{i=0}^{N-1} r^{h-(i+1)m} p^{i+1} \binom{h-2-(i+1)m}{i} \\
&= \sum_{i=0}^{N} r^{h-im} p^i \binom{h-2-im}{i} + \sum_{i=1}^{N} r^{h-im} p^i \binom{h-2-im}{i-1} \\
&= \sum_{i=1}^{N} r^{h-im} p^i \left(\binom{h-2-im}{i} + \binom{h-2-im}{i-1} \right) + r^{h-0m} p^0 \binom{h-2-0m}{0} \\
&= \sum_{i=1}^{N} r^{h-im} p^i \binom{h-1-im}{i} + r^{h-0m} p^0 \binom{h-1-0m}{0} \\
&= \sum_{i=0}^{N} r^{h-im} p^i \binom{h-1-im}{i}
\end{aligned}$$

This covers all the cases. If the statement is true for $h \in \{k-m-1, \dots, k-1\}$, it will also be true for $h = k$. Since we have shown that the statement holds for the initial condition ($0 < h \leq m+1$) it follows by induction that it holds for all $h > 0$. $\square$