Yunus Tunçbilek tuncbilek@wisc.edu

Worksheet 15.1

1. Integrate $f(x, y) = x^2y$ over the domain $D = \{(x, y) : 0 \le x \le 2, -1 \le y \le 2\}$.

$f = x^2y$ is continuous.

$$\int_D x^2y\, dA \underset{\uparrow}{=} \int_0^2\int_{-1}^2 x^2y\, dy\, dx = \int_0^2 \left[\frac{x^2y^2}{2}\right]_{y=-1}^{y=2} dx$$

We can apply Fubini because f is continuous

$$= \int_0^2 \frac{3x^2}{2}\, dx = \left[\frac{x^3}{2}\right]_0^2 = \frac{8}{2} = \boxed{4}$$

2. Evaluate
$$\iint_D \cos(x + y)dA$$
where D is the rectangle with $0 \le x \le \pi$ and $-1 \le y \le 1$.

$\cos(x+y)$ is continuous.

$$\iint_D \cos(x+y)\, dA \underset{\uparrow}{=} \int_0^\pi\int_{-1}^1 \cos(x+y)\, dy\, dx = \int_0^\pi \left[\sin(x+y)\right]_{y=-1}^{y=1} dx$$

Fubini

$$= \int_0^\pi \sin(x+1) - \sin(x-1)\, dx = \left[-\cos(x+1) + \cos(x-1)\right]_0^\pi$$

$$= -\cos(\pi+1) + \cos(\pi-1) + \cos(1) - \cos(-1) = \boxed{0}$$

3. Calculate the following integrals.

(a)
$$\int_1^4\int_0^2 (6x^2y - 2x)dydx.$$

$$= \int_1^4 \left[3x^2y^2 - 2xy\right]_0^2 dx = \int_1^4 12x^2 - 4x\, dx$$

$$= \left[4x^3 - 2x^2\right]_1^4 = \boxed{256 - 32 - 4 + 2}$$

(b)

$$\int_0^1 \int_0^1 (x+y)^2 dxdy.$$

$$= \int_0^1 \left[\frac{(x+y)^3}{3}\right]_{x=0}^{x=1} dy = \int_0^1 \frac{(y+1)^3}{3} - \frac{y^3}{3}\, dy$$

$$= \left[\frac{(y+1)^4}{12} - \frac{y^4}{12}\right]_0^1 = \boxed{\frac{16}{12} - \frac{1}{12} - \frac{1}{12} + \frac{0}{12}}$$

(c)

$$\int_0^1 \int_1^2 (x + e^{-y}) dxdy.$$

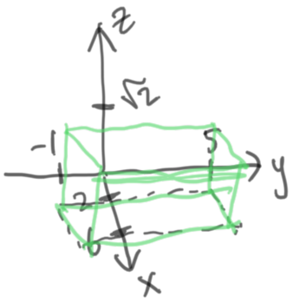
$$\int_0^1 \left[\frac{x^2}{2} + xe^{-y}\right]_{x=1}^{x=2} dy = \int_0^1 \frac{4}{2} + 2e^{-y} - \frac{1}{2} - e^{-y}\, dy$$

$$= \int_0^1 \frac{3}{2} + e^{-y}\, dy = \left(\frac{3y}{2} - e^{-y}\right)_0^1$$

$$= \frac{3}{2} - e^{-1} - \left(\frac{0}{2} - 1\right)$$

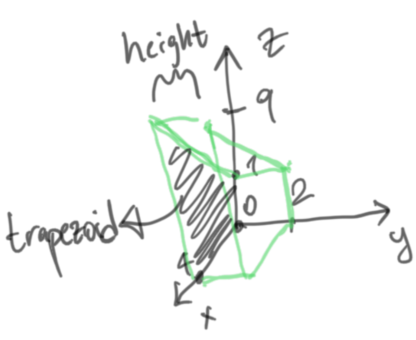
4. Evaluate the following integrals by first identifying them as the volume of a solid.

(a) $\iint_R \sqrt{2} dA$, $R = \{(x,y) : 2 \le x \le 6, -1 \le y \le 5\}$.



This integral is the volume of the green rectangular prism/cuboid. The sides are of length $4, 6, \sqrt{2} \Rightarrow$ Volume $= 24\sqrt{2}$

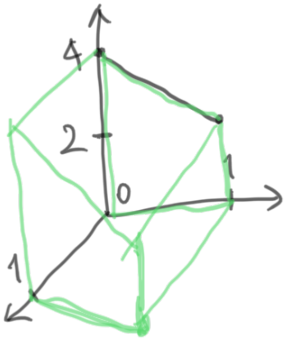
(b) $\iint_R 2x + 1dA$, $R = \{(x,y) : 0 \le x \le 4, 0 \le y \le 2\}$.



The integral is the volume of the green solid. Volume = Area of the red trapezoid times the "height"

$$= 4 \cdot \frac{1+9}{2} \cdot 2 = 40$$

(c) $\iint_R 4 - 2y dA$, $R = [0,1] \times [0,1]$.



Same as the last problem

$$\frac{2+4}{2}\cdot 1 \cdot 1 = 3$$

5. Evaluate the double integrals.

(a)

$$\iint_R (y + xy^2) dA, \quad R = \{(x,y) : 0 \le x \le 2, 1 \le y \le 2\}.$$

$y + xy^2$ is continuous.

$$\int_1^2\int_0^2 (y+xy^2)\,dx\,dy = \int_1^2 \left[xy + \frac{x^2y^2}{2}\right]_{x=0}^{x=2} dy = \int_1^2 2y + 2y^2\,dy$$

$$= \left(y^2 + \frac{2y^3}{3}\right)_1^2 = 4 + \frac{16}{3} - 1 - \frac{2}{3}$$

(b)

$$\iint_R \frac{xy^2}{x^2+1} dA, \quad R = \{(x,y) : 0 \le x \le 1, -3 \le y \le 3\}.$$

$\frac{xy^2}{1+x^2}$ is continuous. Use Fubini.

$$\int_{-3}^{3}\int_0^1 \frac{xy^2}{x^2+1}\,dx\,dy \underset{u=x^2}{=} \int_{-3}^{3} \frac{y^2}{2}\int_0^1 \frac{1}{u+1}\,du\,dy = \int_{-3}^{3} \frac{y^2}{2}\cdot \ln 2\,dy$$

$$= \left[\frac{\ln 2}{2}\cdot\frac{y^3}{3}\right]_{-3}^{3} = \ln 2 \cdot \frac{27}{3}$$

(c)

$$\iint_R \frac{x}{1+xy} dA, \quad R = [0,1] \times [0,1].$$

$1+xy \neq 0$ in the domain so $\frac{x}{1+xy}$ is continuous. Use Fubini.

$$\int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx \underset{u=xy}{=} \int_0^1\int_0^x \frac{1}{1+u}\,du\,dx = \int_0^1 \ln(1+x)\,dx = \Big[(x+1)\ln(x+1) - (x+1)\Big]_0^1$$

$$= 2\ln 2 - 2 - 0 + 1$$