Chapter 12: Vectors and the Geometry of Space: Discovery Project The Geometry of a Tetrahedron
Book Title: Calculus
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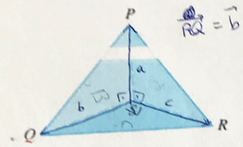
Discovery Project The Geometry of a Tetrahedron

A tetrahedron is a solid with four vertices, P, Q, R, and S, and four triangular faces, as shown in the figure.

$\vec{PR} = \vec{c} - \vec{a}$

$\vec{RQ} = \vec{b} - \vec{a}$ $\quad c \times a + a \times b + b \times c + c - a \times b - a = 0$



1. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be vectors with lengths equal to the areas of the faces opposite the vertices P, Q, R, and S, respectively, and directions perpendicular to the respective faces and pointing outward. Show that

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$$

2. The volume V of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face.

   a. Find a formula for the volume of a tetrahedron in terms of the coordinates of its vertices P, Q, R, and S.

   b. Find the volume of the tetrahedron whose vertices are $P(1, 1, 1)$, $Q(1, 2, 3)$, $R(1, 1, 2)$, and $S(3, -1, 2)$.

3. Suppose the tetrahedron in the figure has a trirectangular vertex S. (This means that the three angles at S are all right angles.) Let A, B, and C be the areas of the three faces that meet at S, and let D be the area of the opposite face PQR. Using the result of Problem 1, or otherwise, show that

$$D^2 = A^2 + B^2 + C^2$$

(This is a three-dimensional version of the Pythagorean Theorem.)