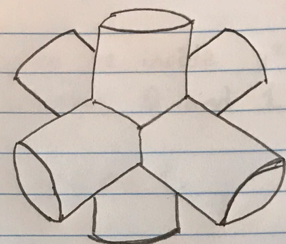
Volume of the Steinmenz 3-solid.

Question: Let E be the solid enclosed by the three circular cylinders, $x^2+y^2=1$, $x^2+z^2=1$, $y^2+z^2=1$.



What is the volume of E?

(You can take their Q's here.)

Solution: First notice that the object E is symmetrical wrt the axes.
So it suffices to compute the volume of E ~~the first~~ in the first octant, then multiply by 8.

$E' := E \cap$ first octant.

$u = 1 - r^2\cos^2\theta \qquad du = -2r\cos^2\theta\, dr$

$$= 16 \int_0^{\pi/4} \int_1^{1-\cos^2\theta} u^{1/2} \cdot \frac{du}{-2\cos^2\theta}$$

$$= 16 \int_0^{\pi/4} \left(\frac{\cancel{2}(1-\cos^2\theta)^{3/2}}{3} - \frac{\cancel{2}}{3} \right) \cdot \frac{1}{-\cancel{2}\cos^2\theta}$$

$$= \frac{-16}{3} \int_0^{\pi/4} \frac{(1-\cos^2\theta)^{3/2}}{\cos^2\theta} - \sec^2\theta$$

$$= \frac{-16}{3} \int_0^{\pi/4} \frac{\sin^3\theta}{\cos^2\theta} + \frac{16}{3} \tan\frac{\pi}{4} - \tan 0$$

(Exercise!) $= \frac{-16}{3}\left(\frac{3}{\sqrt{2}} - 2\right) + \frac{16}{3}$

$$= 16 - \frac{16}{\sqrt{2}} = 8(2-\sqrt{2})$$

We're done!

So

$$\mathrm{vol}(E) = 8\,\mathrm{vol}(E') = 8\iiint_{E'} 1\, dV$$

which is

$$= 8\int_0^{\pi/4}\int_0^1\int_0^{\sqrt{1-r^2\cos^2\theta}} 1\cdot r\, dz\, dr\, d\theta$$

$$+\, 8\int_{\pi/4}^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2\sin^2\theta}} 1\cdot r\, dz\, dr\, d\theta.$$

The two integrals are the same after a θ-sub like $\theta' = \pi/2 - \theta$.

$$\mathrm{vol}(E) = 16\int_0^{\pi/4}\int_0^1\int_0^{\sqrt{1-r^2\cos^2\theta}} r\, dz\, dr\, d\theta.$$

$$= 16\int_0^{\pi/4}\int_0^1 r\sqrt{1-r^2\cos^2\theta}\, dr\, d\theta.$$

(Let them guess the next step).

Use cylindrical coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Then to be inside $x^2+y^2=1$, r must be between 0 and 1.

$$0 \leq r \leq 1.$$

To be in the first octant θ must be between 0 and $\frac{\pi}{2}$.

$$0 \leq \theta \leq \frac{\pi}{2}.$$

To be in the first octant, $z \geq 0$.

To be inside the two other cylinders,

$$z \leq \sqrt{1-x^2} \quad \text{and} \quad z \leq \sqrt{1-y^2}.$$

Notice that if $\theta \leq \frac{\pi}{4}$, $\sqrt{1-x^2} \leq \sqrt{1-y^2}$ and $\sqrt{1-y^2} < \sqrt{1-x^2}$ otherwise.