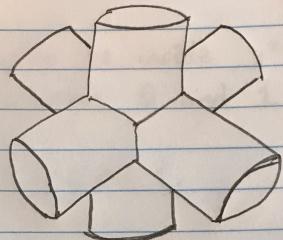


Volume of the Steinmetz 3-solid.

Question: Let E be the solid enclosed by the three circular cylinders, $x^2+y^2=1$, $x^2+z^2=1$, $y^2+z^2=1$.



What is the volume of E ?

(You can take their Q's here.)

Solution: First notice that the object E is symmetrical wrt the axes.
So it suffices to compute the volume of E ~~in the first octant~~ in the first octant, then multiply by 8.

$E' := E \cap \text{first octant.}$

Use cylindrical coordinates

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

Then to be inside $x^2 + y^2 = 1$, r must be between 0 and 1.

$$0 \leq r \leq 1.$$

To be in the first octant θ must be between 0 and $\frac{\pi}{2}$.

$$0 \leq \theta \leq \frac{\pi}{2}.$$

To be in the first octant, $z \geq 0$.

To be inside the two other cylinders,

$$z \leq \sqrt{1-x^2} \text{ and } z \leq \sqrt{1-y^2}.$$

Notice that if $\theta \leq \frac{\pi}{4}$, $\sqrt{1-x^2} \leq \sqrt{1-y^2}$ and $\sqrt{1-y^2} < \sqrt{1-x^2}$ otherwise.

So

$$\text{vol}(E) = 8 \text{vol}(E') = 8 \iiint_{E'} 1 \, dV$$

which is $\sqrt{1-r^2 \cos^2 \theta}$

$$= 8 \int_0^{\pi/4} \int_0^1 \int_0^1 1 + r \, dz \, dr \, d\theta$$

$$+ 8 \int_{\pi/4}^{\pi/2} \int_0^1 \int_0^1 1 + r \, dz \, dr \, d\theta.$$

The two integrals are the same after
a θ -sub like $\theta' = \pi/2 - \theta$.

$$\text{vol}(E) = 16 \int_0^{\pi/4} \int_0^1 \int_0^1 r \, dz \, dr \, d\theta,$$

$$= 16 \int_0^{\pi/4} \int_0^1 r \sqrt{1-r^2 \cos^2 \theta} \, dr \, d\theta.$$

(Let them guess the next step).

$$u = 1 - r^2 \cos^2 \theta \quad du = -2r \cos^2 \theta dr$$

$$= 16 \int_0^{\pi/4} \int_1^{1-\cos^2 \theta} u^{1/2} \cdot \frac{du}{-2\cos^2 \theta}$$

$$= 16 \int_0^{\pi/4} \left(\frac{2(1-\cos^2 \theta)^{3/2}}{3} - \frac{2}{3} \right) \cdot \frac{1}{-2\cos^2 \theta}$$

$$= \frac{-16}{3} \int_0^{\pi/4} \frac{(1-\cos^2 \theta)^{3/2}}{\cos^2 \theta} - \sec^2 \theta$$

$$= \frac{-16}{3} \int_0^{\pi/4} \frac{\sin^3 \theta}{\cos^2 \theta} + \frac{16}{3} \tan \frac{\pi}{4} - \tan 0$$

$$(\text{Exercise!}) = \frac{-16}{3} \left(\frac{3}{\sqrt{2}} - 2 \right) + \frac{16}{3} \cancel{-}$$

$$= 16 - \frac{16}{\sqrt{2}} = 8(2 - \sqrt{2})$$

We're done!