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Inter-universal Teichmüller theory. I: Construction of Hodge theaters. (English)

[Zbl 1465.14002](#)

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In this series of papers on Inter-Universal Teichmüller Theory [this paper; *ibid.* 57, No. 1–2, 209–401 (2021; [Zbl 1465.14003](#)); *ibid.* 57, No. 1–2, 403–626 (2021; [Zbl 1465.14004](#)); *ibid.* 57, No. 1–2, 627–723 (2021; [Zbl 1465.14005](#))], the author aims to prove the ABC conjecture of Masser and Oesterlé, in close to effective form.

Recall that in its simplest form, the ABC conjecture states that for all $\epsilon > 0$ there is some constant C_ϵ such that for all coprime positive integers a, b, c satisfying $a + b = c$, one has $c \leq C_\epsilon (\prod_{p|abc} p)^{1+\epsilon}$. Here the product runs over all primes p dividing one of a, b and c , but crucially not counting multiplicity. It is arguably the most central open Diophantine problem. For example, applied to $a = x^n$, $b = y^n$ and $c = z^n$ it formally implies that there are at most finitely many counterexamples to Fermat’s Last Theorem; if C_ϵ can be made explicit, it thus in principle reduces it to a finite computation. A proof of the ABC conjecture would also lead to a new proof of Mordell’s conjecture [*N. D. Elkies*, *Int. Math. Res. Not.* 1991, No. 7, 99–109 (1991; [Zbl 0763.11016](#))].

A closely related conjecture is the Szpiro conjecture, which relates conductor and discriminant of elliptic curves; the translation is given by passing to the Frey-Hellegouarch curve $y^2 = x(x - a)(x + b)$. The ABC conjecture is known to imply the Szpiro conjecture, but the converse fails, essentially because the discriminant of elliptic curves does not contain an “Archimedean factor”. The author observed in [*Math. J. Okayama Univ.* 52, 1–28 (2010; [Zbl 1221.14024](#))] that if one formulates suitably uniform versions of both conjectures for number fields, they become equivalent. (Additionally, one can arrange for auxiliary congruence conditions at finitely many primes, which is used 2-adically in the present paper.)

With this in mind, the author starts in these papers with the datum of an elliptic curve E over a number field F , satisfying a number of auxiliary conditions; in particular, E is semistable. Moreover, an auxiliary prime ℓ is chosen, of size roughly the square-root of the height of E . Two quantities attached to E at the primes v of bad reduction are of central importance. One is the q -invariant. Recall that by Tate uniformization, $E \times_F F_v$ can be uniformized as $\mathbb{G}_{m, F_v} / q_v^{\mathbb{Z}}$ (as a rigid-analytic variety), for a unique topologically nilpotent unit $q_v \in F_v^\times$. The other invariant are the values of the Θ -function attached to E at certain ℓ -torsion points. These can be described explicitly in terms of q_v , and are basically powers of q_v . The central claim of this series of papers is that through anabelian techniques it is possible to “identify” q -values and Θ -values up to “controlled indeterminacies”. This easily gives the desired Diophantine result.

More formally, the central claim in this series of papers is Corollary 3.12 in part III. In the fourth part, this somewhat abstract statement is shown to imply the ABC conjecture over general number fields. Unfortunately, the argument given for Corollary 3.12 is not a proof, and the theory built in these papers is clearly insufficient to prove the ABC conjecture.

A large part of the papers uses Mochizuki’s theory of the étale Θ -function [*S. Mochizuki*, *Publ. Res. Inst. Math. Sci.* 45, No. 1, 227–349 (2009; [Zbl 1170.14023](#))] whose goal is basically to recover the Θ -values of $E \times_F F_v$ from the fundamental group of the once-punctured elliptic curve $E \times_F F_v \setminus \{0\}$. It should however be noted that the author proved the much stronger result that one can recover $E \times_F F_v \setminus \{0\}$ from this fundamental group [*S. Mochizuki*, *J. Math. Sci., Tokyo* 22, No. 4, 939–1156 (2015; [Zbl 1358.14024](#))]. Many different kinds of hyperbolic curves are considered in these manuscripts, but to all of them this result of the author applies, showing that the passage from curves to fundamental groups is lossless. It is thus not clear which extra flexibility is gotten by passing from schemes to profinite groups.

A central notion is the notion of the Θ -link. This links two so-called Hodge theaters. The relevant part here is the datum of such a fundamental group Π acting on a monoid. Two relevant monoids are the monoid of powers of q_v , and the monoid of powers of the Θ -value(s). These are both abstractly isomorphic to \mathbb{N} , with a distinguished generator. The Θ -link is given by (what the author calls) the full poly-isomorphism between Π ’s, together with an isomorphism of these abstract monoids. Here, the full poly-isomorphism is the collection of all isomorphisms. The argument given by the author for taking not the obvious

isomorphism between Π 's but the full poly-isomorphism is that this is necessary because the Θ -link does not respect the interpretation of the elements of these monoids as actual numbers. Certainly the Θ -link does not respect this interpretation, but this cannot be rescued by taking any other isomorphism between Π 's, let alone an indetermined one!

In any case, at some point in the proof of Corollary 3.12, things are so obfuscated that it is completely unclear whether some object refers to the q -values or the Θ -values, as it is somehow claimed to be definitionally equal to both of them, up to some blurring of course, and hence you get the desired result.

Finally, let me briefly summarize the content of the individual papers. In parts II and III, with the exception of the critical Corollary 3.12, the reader will not find any proof that is longer than a few lines; the typical proof reads "The various assertions of Corollary 2.3 follow immediately from the definitions and the references quoted in the statements of these assertions.", which is in line with the amount of mathematical content. In part I, the first two sections deal with certain group-theoretic results, typical in anabelian geometry, for example about how profinite groups can sit in tempered fundamental groups; these may be of interest to specialists. The rest of part I is largely about the definition of the so-called Hodge-theaters, and some proofs are a bit longer. The category of Hodge theaters has an extremely complicated definition, but the content of these nontrivial proofs is that their category is equivalent to the category with one object and automorphism $\mathbb{Z}/2\mathbb{Z}$, and in fact is canonically equivalent to the category of elliptic curves over F isomorphic to the given E (we note that the functors in both directions are even constructive). In other words, any Hodge theater comes in a unique way from an elliptic curve isomorphic to E . Thus, when the author later chooses an infinite collection of such Hodge theaters, he might as well choose an infinite collection of elliptic curves isomorphic to E . Finally, part IV contains certain technical computations standard in number theory to translate Corollary 3.12 of part III into the ABC conjecture.

Together with *J. Stix*, the reviewer has spent a week in Kyoto to discuss these issues with the author, and has detailed the findings in a manuscript entitled "Why ABC is still a conjecture" [<https://www.math.uni-bonn.de/people/scholze/WhyABCisStillaConjecture.pdf>] that discusses the issues in slightly more detail. The concerns expressed in this manuscript have not been addressed in the published version.

Reviewer: [Peter Scholze \(Bonn\)](#)

MSC:

- [14-02](#) Research exposition (monographs, survey articles) pertaining to algebraic geometry
- [14H25](#) Arithmetic ground fields for curves
- [14H30](#) Coverings of curves, fundamental group
- [14G32](#) Universal profinite groups (relationship to moduli spaces, projective and moduli towers, Galois theory)

<p>Cited in 4 Reviews</p> <p>Cited in 7 Documents</p>

Keywords:

[inter-universal Teichmüller theory](#); [Hodge theater](#); [global multiplicative subspace](#); [canonical generator](#); [punctured elliptic curve](#); [theta-link](#); [étale theta function](#); [absolute anabelian geometry](#)

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