MATHEMATICS

Is height $\mathfrak{p} + \dim A/\mathfrak{p} = \dim A$ true?

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Let A be an integral domain of finite Krull dimension. Let $\mathfrak{p}$ be a prime ideal. Is it true that

$$\text{height } \mathfrak{p} + \dim A/\mathfrak{p} = \dim A$$

where dim refers to the Krull dimension of a ring?

Hartshorne states it as Theorem 1.8A in Chapter I (for the case A a finitely-generated k-algebra which is an integral domain) and cites Matsumura and Atiyah–Macdonald, but I haven't been able to find anything which looks relevant in either. (Disclaimer: I know nothing about dimension theory, and very little commutative algebra.) If it is true (under additional assumptions, if need be), where can I find a complete proof?

It is obvious that

$$\text{height } \mathfrak{p} + \dim A/\mathfrak{p} \leq \dim A$$

by a lifting argument, but the reverse inequality is eluding me. Localisation doesn't seem to be the answer, since localisation can change the dimension...

reference-request commutative-algebra krull-dimension dimension-theory-algebra

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edited May 10, 2019 at 11:16
Martin Sleziak
53.2k 20 189 365

asked Jul 3, 2011 at 4:13
Zhen Lin
89.5k 11 186 331

6 This is not true in general. The keyword is the catenary ring: en.wikipedia.org/wiki/Catenary_ring.
– user325 Jul 3, 2011 at 6:36

3 As George's answer shows, "catenary" is not enough. – Manos Oct 1, 2020 at 5:58

This is explained in Matsumura's Commutative algebra book though. See (14.H) COROLLARY 3.
– Zero Apr 29, 2021 at 16:39

4 Answers

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61

Yours is a very interesting and subtle question, which often generates confusion. First let us give a name to the property you are interested in: a ring A will be said to satisfy **(DIM)** if for all $\mathfrak{p} \in \operatorname{Spec}(A)$ we have

$$\operatorname{height}(\mathfrak{p}) + \dim A/\mathfrak{p} = \dim(A) \qquad \text{(DIM)}$$

The main misconception is to believe that this follows from catenarity:
Fact 1: A catenary ring, or even a universally catenary ring, does not satisfy (DIM) in general.
Counterexample: Let $(R, \mathfrak{m})$ be a discrete valuation ring whose maximal ideal has uniformizing parameter π, i.e. $\mathfrak{m} = (\pi)$. Let $A = R[T]$, the polynomial ring over R. The ring A has dimension 2. Then for the maximal ideal $\mathfrak{p} = (\pi T - 1)$, the relation (DIM) is false: $\operatorname{height}(\mathfrak{p}) + \dim A/\mathfrak{p} = 1 + 0 = 1 \neq 2 = \dim(A)$.
And this even though A is as nice as can be: an integral domain, noetherian, regular, *universally catenary*,...

Happily here are two positive results:

Fact 2: A finitely generated integral algebra over a field satisfies (DIM) (and is universally catenary).
So, by the algebro-geometric dictionary, an affine variety X has the pleasant property that for each integral subvariety $Y \subset X$ we have, as hoped, $\operatorname{dimension}(Y) + \operatorname{codimension}(Y) = \operatorname{dimension}(X)$.

Fact 3: A Cohen-Macaulay local ring satisfies (DIM) (and is universally catenary).
For example a regular ring is Cohen-Macaulay. This "explains" why my counter-example above was not local.

The paradox resolved. How is it possible for a catenary ring A not to satisfy (DIM)? Here is how. If you have an inclusion of two primes $\mathfrak{p} \subsetneq \mathfrak{q}$ catenarity says that you can complete it to a saturated chain of primes $\mathfrak{p} \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_{r-1} \subsetneq \mathfrak{q}$ and that all such completions will have length the same length r. Fine. But what can you say if you have just one prime $\mathfrak{p}$? Not much! The catenary ring A may have dimension $\dim(A) > \operatorname{height}(\mathfrak{p}) + \dim(A/\mathfrak{p})$ because it possesses a long chain of primes *avoiding* the prime $\mathfrak{p}$ altogether. In my counterexample above the only saturated chain of primes containing $\mathfrak{p} = (\pi T - 1)$ is $0 \subsetneq \mathfrak{p}$. However the ring A has dimension 2 because of the saturated chain of primes $0 \subsetneq (\pi) \subsetneq (\pi, T)$, which *avoids* $\mathfrak{p}$.

Addendum. Here is why the ideal $\mathfrak{p}$ in the counter-example is maximal. We have $A/\mathfrak{p} = R[T]/(\pi T - 1) = R[1/\pi] = \operatorname{Frac}(R)$, since the fraction field of a discrete valuation ring can be obtained just by inverting a uniformizing parameter. So $A/\mathfrak{p}$ is a field and $\mathfrak{p}$ is maximal.

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edited Mar 26, 2013 at 9:56
user26857

answered Jul 3, 2011 at 20:37
Georges Elencwajg
150k 12 289 475

6 Dear Soarer, don't worry: that catenary implies (DIM) is one of the most widespread misconceptions I have ever met (and I spent quite some time trying to clear these issues for myself). – Georges Elencwajg Jul 3, 2011 at 23:01

1 Equidimensional is defined in EGA 0-IV.14 as all irreducible components having the same dimension; equicodimensional is that all minimal irreducible closed sets have the same codimension. A noetherian space of finite dimension is biequidimensional if and only if the equality (DIM) is verified, if I am not mistaken. – Akhil Mathew Jul 4, 2011 at 2:54

1 Ah, ok. Then I agree with your claim. Also, I agree that 0-IV.14 is mostly content-free. The real content, that (DIM) holds for integral domains finitely generated over a field, is in IV-5.2.1 (though catenary-ness is deduced from the fact that a regular local ring is universally catenary; the way I always thought of the result was via the transcendence degree additivity. (I certainly haven't read all of EGA.) – Akhil Mathew Jul 4, 2011 at 13:23

1 I think it is also worth noticing that when a ring satisfies (DIM) for prime ideals, then the dimensional equality holds for any ideal: If I is an arbitrary ideal and $\mathfrak{p}$ a prime ideal over I with $ht(I) = ht(p)$, then the quotient map $R/I \to R/\mathfrak{p}$ is surjective, so the induced spectra map is injective, yielding $\dim R/I \geq \dim R/\mathfrak{p}$. So we have $\dim R/\mathfrak{p} \leq \dim R/I \leq \dim R - ht(I) = \dim R - ht(\mathfrak{p})$, and since DIM holds for prime ideals, the equality follows. – Sebastian Jul 31, 2014 at 13:24

1 @Cyclicduck The Stacks Project Lemma 10.104.4 – Georges Elencwajg Jun 25, 2021 at 17:06

7

Although this is an old question, I thought it was worth mentioning a recent paper by Heinrich that corrects the statements in $\text{EGA}0_{\text{IV}}$ mentioned in the comments.

Let us start with some definitions (following [Heinrich, Def. 1.2, Prop. 4.1]):

Definition. Let X be a topological space which is T_0, noetherian, and finite dimensional.

1. The space X is *biequidimensional* if all maximal chains of irreducible closed subsets of X have the same length.
2. The space X is *weakly biequidimensional* if it is equidimensional, equicodimensional, and catenary.

The often cited result from $\text{EGA}0_{\text{IV}}$ is the following:

Claim [$\text{EGA}0_{\text{IV}}$, Prop. 14.3.3]**.** *Let X be a topological space which is T_0, noetherian, and finite dimensional. The following are equivalent:*

1. *The space X is biequidimensional.*
2. *The space X is weakly biequidimensional.*
3. *The space X is equicodimensional and for every inclusion of irreducible closed subsets $Y \subseteq Z$ in X, we have*

$$\dim(Z) = \dim(Y) + \operatorname{codim}(Y, Z).$$

4. *The space X is equicodimensional and for every inclusion of irreducible closed subsets $Y \subseteq Z$ in X, we have*

$$\operatorname{codim}(Y, X) = \operatorname{codim}(Y, Z) + \operatorname{codim}(Z, X).$$

This is not quite correct, as was found independently by Gabber and by Chen (see [ILO, Exp. XV, §2.4, footnote (i) on p. 196]), and also by Heinrich [Heinrich].

Gabber and Heinrich both noted that (1), (3), and (4) are equivalent (see [Heinrich, Lem. 2.3] for a proof), and Heinrich showed that these conditions imply (2) [Heinrich, Lem. 2.1]. Gabber and Heinrich both gave examples where (2) does not imply (3); we reproduce Heinrich's here:

Example [Heinrich, Ex. 3.7]**.** The ring A obtained by localizing the ring

$$\frac{k[v, w, x, y]}{(vy, wy)}$$

away from the union $(v, w, x, y - 1) \cup (v, w, y)$ is weakly biequidimensional but does not satisfy (3): setting $Y = V(v, w, x, y - 1) \subsetneq V(v, w) = Z$, we have

$$\dim(Z) = 2 > 0 + 1 = \dim(Y) + \operatorname{codim}(Y, Z).$$

See [Heinrich, Ex. 3.7] for details.

A preprint by Emerton and Gee gives a correct variant of the Claim above; see [Emerton and Gee, Lem. 2.32]. The basic difference is that the Claim is true if X is assumed to be irreducible. Gabber also gives a variant where (2) is replaced by "X is catenary and equidimensional and its irreducible components are equicodimensional" [ILO, Exp. XV, §2.4, footnote (i) on p. 196].

The statement with the hypotheses given in Hartshorne is true.

For a reference, see COR 13.4 on pg. 290 of Eisenbud's *Commutative Algebra*.

The general idea of proof is this: Consider a maximal chain of prime ideals in A which includes the given prime $\mathfrak{p}$, the length of which is $\dim A$ (see Thm A, pg. 290 of Eisenbud). It follows that $\dim A = \operatorname{height} \mathfrak{p} + \dim A/\mathfrak{p}$.

You are specifying a prime p here, while OP was asking about arbitrary prime. – user325 Jul 3, 2011 at 6:37

@Soarer: I'm not sure what you mean. Given an arbitrary prime $\mathfrak{p}$, construct a maximal chain of prime ideals which includes $\mathfrak{p}$. Every maximal chain of primes in A has the same length, i.e. the Krull dimension of A, although this assertion itself is not trivial, but the full proof is in Eisenbud. – John M Jul 3, 2011 at 6:59

1

The following is a useful situation where A is not necessarily an integral domain. It generalizes Fact 2 from Georges's answer and specializes (1) $\to$ (3) from Takumi's answer.

Definition 1. Let A be a finitely generated algebra over a field k. A is called *equidimensional* if all minimal primes of A have the same height.

Theorem 1. Suppose A is an equidimensional k-algebra. Then height $P + \dim A/P = \dim A$ for any $P \in \operatorname{Spec} A$.

Here is the scheme-theoretic picture.

Definition 2. Let X be a k-scheme. X is called *equidimensional* if its irreducible components have all the same dimension.

Theorem 2. Let X be an equidimensional k-scheme locally of finite type. Let Y be an irreducible closed subscheme with generic point η. Then $\dim \mathcal{O}_{X,\eta} + \dim Y = \dim X$.

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user26857
51.9k 13 72 145

answered Jan 26, 2020 at 13:14
Manos
25.7k 5 66 172

Theorem 1 is a mild generalization of the affine domain case. – user26857 Jan 26, 2020 at 21:14