EXERCISES ON LIMITS & COLIMITS

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Exercise 1. Prove that pullbacks of epimorphisms in **Set** are epimorphisms and pushouts of monomorphisms in **Set** are monomorphisms. Note that these statements cannot be deduced from each other using duality. Now conclude that the same statements hold in **Top**.

Exercise 2. Let X be a set and $A, B \subset X$. Prove that the square

$$\begin{tikzcd} A \cap B \arrow[r, hook] \arrow[d, hook] & A \arrow[d, hook] \\ B \arrow[r, hook] & A \cup B \end{tikzcd}$$

is both a pullback and pushout in **Set**.

Exercise 3. Let R be a commutative ring. Prove that every R-module can be written as a filtered colimit of its finitely generated submodules.

Exercise 4. Let X be a set. Give a categorical definition of a topology on X as a subposet of the power set of X (regarded as a poset under inclusion) that is stable under certain categorical constructions.

Exercise 5. Let X be a space. Give a categorical description of what it means for a set of open subsets of X to form a basis for the topology on X.

Exercise 6. Let C be a category. Prove that if the identity functor $\mathrm{id}_C : C \to C$ has a limit, then $\lim_C \mathrm{id}_C$ is an initial object of C.

Definition. Let C be a category and $X \in C$. If the coproduct $X \sqcup X$ exists, the *codiagonal* or *fold morphism* is the morphism $\nabla_X : X \sqcup X \to X$ induced by the identities on X via the universal property of the coproduct.

If the product $X \times X$ exists, the *diagonal* morphism $\Delta_X : X \to X \times X$ is defined dually.

Exercise 7. In **Set**, show that the diagonal $\Delta_X : X \to X \times X$ is given by $\Delta_X(x) = (x, x)$ for all $x \in X$, so Δ_X embeds X as the diagonal in $X \times X$, hence the name.

The codiagonal $\nabla_X : X \sqcup X \to X$ is a bit more mysterious. Give a description of ∇_X (still in the category **Set**).

Exercise 8. Let C be a category and $X, Y \in C$. Suppose that the coproducts $X \sqcup X$ and $Y \sqcup Y$ exist, and let $\nabla_X : X \sqcup X \to X$ and $\nabla_Y : Y \sqcup Y \to Y$ denote the codiagonals. Show that for any morphism $f : X \to Y$ we have $f \circ \nabla_X = \nabla_Y \circ (f \sqcup f)$.

What is the dual statement?

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Exercise 9. Let C be a category with pullbacks and

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_0 \arrow[d] & X_2 \arrow[l] \arrow[d] \\ Z_1 \arrow[r] & Z_0 & Z_2 \arrow[l] \\ Y_1 \arrow[r] \arrow[u] & Y_0 \arrow[u] & Y_2 \arrow[l] \arrow[u] \end{tikzcd}$$

a commutative diagram in C. Prove that we have a natural isomorphism

$$(X_1 \times_{Z_1} Y_1) \times_{X_0 \times_{Z_0} Y_0} (X_2 \times_{Z_2} Y_2) \cong (X_1 \times_{X_0} X_2) \times_{Z_1 \times_{Z_0} Z_2} (Y_1 \times_{Y_0} Y_2)\,.$$

Definition. Let C be a category with pullbacks and $f\colon X \to Y$ a morphism in C. The *diagonal* of f is the morphism $\Delta_f\colon X \to X \times_Y X$ induced via the universal property of the pullback by the square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y\,. \end{tikzcd}$$

Remark. Note that $\Delta_{\mathrm{id}_X} = \Delta_X$.

Exercise 10 (magic square)**.** Let C be a category and let $f_1\colon X_1 \to Y$, $f_2\colon X_2 \to Y$, and $g\colon Y \to Z$ morphisms in C. Assuming that C has pullbacks, prove that the square

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r, "\Delta_g"'] & Y \times_Z Y \end{tikzcd}$$

is a pullback square in C (where the unlabeled morphisms are the morphisms naturally induced by the universal property of the pullback).

Definition. Let C be a category with pullbacks, $Z \in C$, and $f\colon X \to Y$ a morphism in the slice category $C_{/Z}$. The *graph morphism* of f is the morphism $f\colon X \to X \times_Z Y$ induced via the universal property of the pullback by the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d] \\ X \arrow[r] & Z\,. \end{tikzcd}$$

Exercise 11. Show that if $f\colon X \to Y$ is a map of sets, then the graph $\Gamma_f\colon X \to X \times Y$ is given by $x \mapsto (x, f(x))$.

Exercise 12. Let C be a category with pullbacks, $Z \in C$, and $f\colon X \to Y$ a morphism in the slice category $C_{/Z}$. Write $g\colon Y \to Z$ for the structure morphism. Prove that the square

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_Z Y \arrow[d, "f \times_Z \mathrm{id}_Y"] \\ Y \arrow[r, "\Delta_g"'] & Y \times_Z Y \end{tikzcd}$$

is a pullback in C.

Definition. Let C be a category. We say that a collection of morphisms $P \subset \mathrm{Mor}(C)$ is *stable under pullback* if for any pullback square

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\bar{p}"] \arrow[d, "\bar{q}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "q"] \\ X \arrow[r, "p"'] & Z \end{tikzcd}$$

in C, if $p \in P$ then $\bar{p} \in P$.

Exercise 13. Let C be a category with pullbacks and $P \subset \mathrm{Mor}(C)$ a collection of morphisms in C stable under composition and pullback. Prove that given any commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

in C, if $p \in P$ and $\Delta_g \in P$, then $f \in P$.

Definition. A functor $F\colon C \to D$ is *left cofinal* if for all categories E and diagrams $G\colon D \to E$, the colimit $\mathrm{colim}_D\, G$ exists if and only if $\mathrm{colim}_C\, GF$ exists, in which case the natural morphism

$$\mathrm{colim}_C\, GF \to \mathrm{colim}_D\, G$$

is an isomorphism.

Dually, $F\colon C \to D$ is *right cofinal* if $F^{op}\colon C^{op} \to D^{op}$ is left cofinal.

Remark. The "co" in "cofinal" uses the non-mathematical English prefix meaning "jointly" — there's no duality involved here.

Exercise 14. Show that equivalences of categories are both left and right cofinal.

Exercise 15. Show that if a category C has a terminal object $*$, then the inclusion $\{*\} \hookrightarrow C$ of the full subcategory of C spanned by $*$ is left cofinal.

Exercise 16. Let $F\colon C \to D$ be a functor. Show that F is left cofinal if and only if for all diagrams $G\colon D \to \mathbf{Set}$ the natural morphism

$$\mathrm{colim}_C\, GF \to \mathrm{colim}_D\, G$$

is an isomorphism.

Notation. For a positive integer n, write $\mathbf{\Delta}_{\leq n}$ for the full subcategory of $\mathbf{\Delta}$ spanned by those sets of cardinality at most n.

Exercise 17. Show that the inclusion $\mathbf{\Delta}_{\leq 2} \hookrightarrow \mathbf{\Delta}$ is right cofinal.

Notation. Write $\mathbf{\Delta}^{inj} \subset \mathbf{\Delta}$ for the wide subcategory, i.e., subcategory containing all of the objects, of $\mathbf{\Delta}$ where the morphisms are injective maps of linearly ordered finite sets. For a positive integer n, write $\mathbf{\Delta}^{inj}_{\leq n}$ for the full subcategory of $\mathbf{\Delta}^{inj}$ spanned by those sets of cardinality at most n.

Exercise 18. Show that the inclusion $\mathbf{\Delta}^{inj}_{\leq 2} \hookrightarrow \mathbf{\Delta}^{inj}$ is right cofinal. Now deduce that the inclusion $\mathbf{\Delta}^{inj}_{\leq 2} \hookrightarrow \mathbf{\Delta}$ is right cofinal.

Exercise 19. Let C be a category with pullbacks and $f\colon X \to Y$ a morphism in C. Construct a functor $\check{C}(f)\colon \mathbf{\Delta}^{op} \to C$ whose value on $[n] = \{0 < \cdots < n\}$ is the $(n+1)$-fold iterated pullback $X \times_Y \cdots \times_Y X$ (so that the value on $[0]$ is simply X). The simplicial object $\check{C}(f)$ is called the *Čech nerve* of f.

Exercise 20. Let X be a topological space, $U \subset X$ an open set, $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ an open cover of U, and $\mathcal{F}$ a presheaf on X. Choose a well-ordering of A (this does not really matter, but is necessary to make the next step well-defined.) Extend the usual "sheaf condition diagram"

$$\prod_{\alpha_0 \in A} \mathcal{F}(U_{\alpha_0}) \rightrightarrows \prod_{\alpha_0, \alpha_1 \in A} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1})$$

to a diagram $\check{C}(U; \mathcal{F})\colon \mathbf{\Delta}^{inj} \to \mathbf{Set}$ of the form

$$\prod_{\alpha_0 \in A} \mathcal{F}(U_{\alpha_0}) \rightrightarrows \prod_{\alpha_0, \alpha_1 \in A} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1}) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \prod_{\alpha_0, \alpha_1, \alpha_2 \in A} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots .$$

Now reformulate the sheaf condition for a presheaf $\mathcal{F}$ in terms of the diagrams $\check{C}(\mathcal{U}; \mathcal{F})$.