Differential cohomology theories as sheaves on manifolds

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Contents

Overview

In these notes we set up differential cohomology theories as sheaves of spectra on the category of manifolds, and prove the basic structural results that we need to work with differential cohomology theories. Since we'll also want to work with sheaves of spaces or chain complexes, throughout we work with sheaves valued in an arbitrary presentable ∞-category.

Section 1 gives the definition of a sheaf on the category of manifolds and states a few immediate consequences from the definition. We also describe the embeddings of the categories of Banach and Fréchet manifolds into sheaves of sets on the category of manifolds. Section 2 gives a discussion of Dugger's Theorem that the **R**-invariant sheaves on the category of manifolds coincide with the constant sheaves. In Section 3 we use a construction of Morel–Suslin–Voevodsky to give a concrete description of the left adjoint to the inclusion of **R**-invariant sheaves into all sheaves. In Section 4 we explain how to deduce Quillen's Transfer Conjecture as a special case of work of Bachmann–Hoyois [4, Appendix C]. Section 5 is dedicated to the stable case. In this setting we show that there is a differential cohomology 'Fracture Theorem' that says that sheaves on manifolds can be reconstructed from the subcategories of **R**-invariant sheaves and sheaves with trivial global sections. We then explain how to construct the Simons–Sullivan 'differential cohomology diagram' in this setting. Throughout these notes there are a number of important results that have somewhat involved proofs. Since only the results are used, and their proofs are not immediately important to know, we have relegated them to Appendix A. Appendix A also breifly reviews a few subtle points from the theory of higher topoi that are relevant to these technical proofs.

1 Sheaves on the category of manifolds

1.0.1 Notation. We write **Man** for the (ordinary) category of smooth manifolds, including the empty manifold. The category **Man** has a Grothendieck topology where the covering families are families of open embeddings

$$\{j_\alpha \colon U_\alpha \hookrightarrow M\}_{\alpha \in A}$$

such that the family of open sets $\{j_\alpha(U_\alpha)\}_{\alpha\in A}$ is an open cover of M. Whenever we regard **Man** as a site, we use this topology. Note that representable functors are sheaves with respect to this topology on **Man**.

1.0.2 Remark. Since the category **Man** is equivalent to the category of manifolds with a fixed embedding into $\mathbf{R}^\infty$, the category **Man** is essentially small.

1.0.3 Definition. Let C be a presentable ∞-category. We write

$$\mathrm{PSh}(\mathbf{Man};C) \coloneqq \mathrm{Fun}(\mathbf{Man}^{\mathrm{op}}, C)$$

and write

$$\mathrm{Sh}(\mathbf{Man};C) \subset \mathrm{PSh}(\mathbf{Man};C)$$

for the full subcategory spanned by the C-valued sheaves on the site **Man** with respect to the Grothendieck topology given by open covers.

Explicitly, a C-valued presheaf $E\colon \mathbf{Man}^{\mathrm{op}} \to C$ is a sheaf if and only if for each manifold M, the restriction $E|_{\mathrm{Open}(M)}$ of E to the site $\mathrm{Open}(M)$ of open submanifolds of M is a sheaf on the topological space M.

1.0.4 Notation. We write $\mathrm{S}_{\mathbf{Man}}\colon \mathrm{PSh}(\mathbf{Man};C) \to \mathrm{Sh}(\mathbf{Man};C)$ for the left adjoint to the inclusion, that is, the *sheafification* functor.

1.0.5 Notation. We write **Spc** for the ∞-category of spaces and **Sp** for the ∞-category of spectra.

1.0.6 Definition. The ∞-category of ***differential cohomology theories*** is the ∞-category $\mathrm{Sh}(\mathbf{Man};\mathbf{Sp})$ of sheaves of spectra on **Man**.

For most of this text we work in the generality of sheaves with values in a general presentable ∞-category, or stable presentable ∞-category. The main reason for doing this is because we have reason to consider sheaves of spaces, sheaves of chain complexes, and sheaves of spectra, and want to treat them on the same footing.

1.0.7 Remark. We take the approach of Freed–Hopkins [18] and consider sheaves on the category of smooth manifolds. The general setup here is very robust, and one can take the basic objects to be manifolds with corners without essential change to how theory works; this the approach taken by Hopkins–Singer [22] and Bunke–Nikoulas–Völkl [14].

The first basic fact is that to check that a morphism of sheaves on **Man** is an equivalence, it suffices to check the claim on each Euclidean space.

1.0.8 Recollection. Let M be an n-manifold. An open cover $\mathcal{U}$ of M is ***good*** if for every finite set $U_1, \dots, U_m \in \mathcal{U}$ of opens in $\mathcal{U}$, the intersection $U_1 \cap \cdots \cap U_m$ is either empty or diffeomorphic to $\mathbf{R}^n$.

1.0.9 Notation. Let T be a topological space and $U \subset T$ be open. For every open cover $\mathcal{U}$ of U, write $\mathrm{I}(\mathcal{U}) \subset \mathrm{Open}(T)$ for the full subposet consisting of all nonempty finite intersections of elements in $\mathcal{U}$.

1.0.10 Lemma. *Let C be a presentable ∞-category. A morphism $f\colon E \to E'$ in $\mathrm{Sh}(\mathbf{Man};C)$ is an equivalence if and only if for all integers $n \geq 0$ the morphism $f(\mathbf{R}^n)\colon E(\mathbf{R}^n) \to E'(\mathbf{R}^n)$ is an equivalence in C.*

Proof. Let M be a manifold and $\mathcal{U}$ a good cover of M. The morphism f induces a commutative square

$$\begin{tikzcd}
E(M) \arrow[r, "\sim"] \arrow[d, "f(M)"'] & \lim_{U \in \mathrm{I}(\mathcal{U})^{\mathrm{op}}} E(U) \arrow[d] \\
E'(M) \arrow[r, "\sim"] & \lim_{U \in \mathrm{I}(\mathcal{U})^{\mathrm{op}}} E'(U) ,
\end{tikzcd}$$

where the horizontal morphisms are equivalences because E and E' are sheaves. Since the cover $\mathcal{U}$ is good and f is an equivalence on Euclidean spaces, we see that the induced morphism

$$f\colon E|_{\mathrm{I}(\mathcal{U})^{\mathrm{op}}} \to E'|_{\mathrm{I}(\mathcal{U})^{\mathrm{op}}}$$

of $\mathrm{I}(\mathcal{U})^{\mathrm{op}}$-indexed diagrams in C is an equivalence, which proves the claim. □

1.0.11 Remark. In fact, Lemma 1.0.10 can be refined. Let **Cart** ⊂ **Man** denote the full subcategory spanned by $\mathbf{R}^n$ for $n \geq 0$ and the empty manifold. We endow **Cart** ⊂ **Man** with the induced Grothendieck topology. Then right Kan extension along the inclusion $\mathbf{Cart}^{\mathrm{op}} \hookrightarrow \mathbf{Man}^{\mathrm{op}}$ defines an equivalence of ∞-categories

$$\mathrm{Sh}(\mathbf{Cart}; C) \xrightarrow{\sim} \mathrm{Sh}(\mathbf{Man}; C) . \tag{1.0.12}$$

Proving the equivalence (1.0.12) requires a bit of a technical digression about hypercompleteness and bases for hypercomplete ∞-topoi. Since we do not actually need to use the equivalence (1.0.12), we defer the proof to §A.8.

1.1 Checking equivalences on stalks

We now explain that equivalences of sheaves on **Man** with values in a *compactly generated* ∞-category can be checked on 'stalks' at the origins in $\mathbf{R}^n$ for $n \geq 0$. The proof of this requires a few technical detours which we defer to Section A.7.

1.1.1 Notation. Let M be a manifold and $x \in M$. We write $\mathrm{Open}_x(M) \subset \mathrm{Open}(X)$ for the full subposet spanned by the open neighborhoods of $x \in M$.

1.1.2 Definition. Let C be a presentable ∞-category, $E \in \mathrm{Sh}(\mathbf{Man}; C)$ a C-valued sheaf on **Man**, M a manifold, and $x \in M$. The ***stalk*** of E at $x \in M$ is the colimit

$$x^{\star}(E) \coloneqq \operatorname*{colim}_{U \in \mathrm{Open}_x(M)^{\mathrm{op}}} E(U)$$

in C.

1.1.3 Notation. For each integer $n \geq 0$, write $0_n \in \mathbf{R}^n$ for the origin, and write

$$\mathrm{B}_{\mathbf{R}^n}(1/k) \subset \mathbf{R}^n$$

for the open ball in $\mathbf{R}^n$ of radius $1/k$ centered at the 0_n.

1.1.4. Let $E\colon \mathbf{Man}^{\mathrm{op}} \to C$ be a sheaf on **Man**. Note that the stalk $0_n^{\star}(E)$ can be computed as the colimit

$$0_n^{\star}(E) \simeq \operatorname*{colim}_{k \to \infty} E(\mathrm{B}_{\mathbf{R}^n}(1/k)) .$$

The following result comes from the functoriality of a sheaf on **Man** in *all* manifolds, the fact that for ever n-manifold M and point $x \in M$, there exists an open embedding $j\colon \mathbf{R}^n \hookrightarrow M$ such that $j(0_n) = x$, and that equivalences in sheaves on M can be checked on stalks. In Section A.7 we provide a detailed proof.

1.1.5 Proposition. *Let C be a compactly generated ∞-category. A morphism f in $\mathrm{Sh}(\mathbf{Man}; C)$ is an equivalence if and only if $0_n^\star(f)$ is an equivalence in C for all integers $n \geq 0$.*

1.1.6 Remark. Proposition A.7.4 is important from our perspective. Freed and Hopkins work with differential cohomology theories using the language of simplicial sheaves and model categories [18]. Combining Proposition A.7.4 with [HTT, Remark 6.5.2.2 & Proposition 6.5.2.14] shows that the model structure on simplicial presheaves on **Man** considered in [18, §5] presents the ∞-category $\mathrm{Sh}(\mathbf{Man}; \mathbf{Spc})$.

1.1.7 Warning. Proposition 1.1.5 does now hold when C is replaced by an arbitrary presentable ∞-category.

1.2 Digression: Excision & the sheaf condition

The goal of this subsection is to prove a convenient reformulation of the sheaf condition in terms of an *excision* property. We do not make use of the reformation in this text, but present it here because it is the manifold analogue of *Nisnevich excision* from algebraic geometry [SAG, Proposition B.5.1.1; 2, §3.2; 32, §3.1, Proposition 1.4].

1.2.1 Theorem ([9, Theorem 5.1]). *Let C be a presentable ∞-category. A C-valued presheaf $F\colon \mathbf{Man}^{\mathrm{op}} \to C$ on **Man** is a sheaf if and only if F satisfies the following conditions:*

(1.2.1.1) *The object $F(\varnothing)$ is terminal in C.*

(1.2.1.2) *For every manifold M and pair of open subsets $U, V \subset M$ such that $U \cup V = M$, the induced square*

$$\begin{tikzcd} F(M) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \cap V) \end{tikzcd}$$

is a pullback square in C.

(1.2.1.3) *For every manifold M and $\mathbf{N}$-indexed sequence of open sets*

$$U_0 \subset U_1 \subset \cdots \subset M$$

such that $\bigcup_{n\geq 0} U_n = M$, the induced morphism

$$F(M) \to \lim_{n\geq 0} F(U_n)$$

is an equivalence in C.

The idea of Theorem 1.2.1 is as follows. Conditions (1.2.1.1) and (1.2.1.2) guarantee that F satisfies the sheaf condition with respect to finite open covers. Given descent with respect to finite open covers, by writing a countable cover as a union of a sequence of finite covers of smaller subspaces, (1.2.1.3) implies descent with respect to *countable* open covers. Note that implicit in Theorem 1.2.1 is the claim that descent with respect to countable open covers implies descent with respect to arbitrary open covers.

Since the sheaf condition on **Man** is defined after restriction to each manifold, Theorem 1.2.1 follows from an analogous rephrasing of the sheaf condition for a presheaf on an individual manifold (Proposition 1.2.5). The manifold structure isn't really used here; all that is necessary is that an open cover of an open subset of a manifold admits a countable subcover. Hence we work at this level of generality.

1.2.2 Observation. Let T be a topological space and C a presentable ∞-category. Since limits of finite cubes can be written as iterated pullbacks, the following are equivalent for a presheaf $F \in \mathrm{PSh}(T; C)$ on T:

(1.2.2.1) The presheaf F satisfies descent with respect to nonempty finite covers.

(1.2.2.2) For all opens $U, V \subset T$, the induced square

$$\begin{tikzcd} F(U \cup V) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \cap V) \end{tikzcd}$$

is a pullback square in C.

1.2.3 Recollection. A topological space T is ***Lindelöf*** if every open cover of T has a countable subcover.

The following conditions are equivalent for a topological space T:

(1.2.3.1) Every open subspace of T is Lindelöf.

(1.2.3.2) Every subspace of T is Lindelöf.

We say that T is ***hereditarily Lindelöf*** if T satisfies the equivalent conditions (1.2.3.1)–(1.2.3.2).

Note that every second-countable topological space (e.g., manifold) is hereditarily Lindelöf.

1.2.4 Lemma. *Let T be a hereditarily Lindelöf topological space and C a presentable ∞-category. The following are equivalent for a presheaf $F \in \mathrm{PSh}(T; C)$ on T:*

(1.2.4.1) *The presheaf F is a sheaf on T.*

(1.2.4.2) *The presheaf F satisfies descent with respect to countable open covers.*

Proof. Clearly (1.2.4.1)⇒(1.2.4.2). To see that (1.2.4.2)⇒(1.2.4.1), let $U \subset T$ be open and let $\mathcal{U}$ be an open cover of U Since T is hereditarily Lindelöf, there exists a countable subset $\mathcal{U}_0 \subset \mathcal{U}$ that also covers U. To conclude, note that have a commutative triangle

$$\begin{tikzcd} & F(U) \arrow[dl] \arrow[dr, "\sim"] & \\ \lim\limits_{V \in \mathrm{I}(\mathcal{U})^{\mathrm{op}}} F(V) \arrow[rr, "\sim"] & & \lim\limits_{V \in \mathrm{I}(\mathcal{U}_0)^{\mathrm{op}}} F(V) \end{tikzcd},$$

where the right-hand diagonal morphism is an equivalence by (1.2.4.2) and the horizontal morphism is an equivalence because the inclusion $\mathrm{I}(\mathcal{U}_0)^{\mathrm{op}} \subset \mathrm{I}(\mathcal{U}_0)^{\mathrm{op}}$ is limit-cofinal. □

Now we provide a characterization of sheaves on a hereditarily Lindelöf topological space in terms of an excision property. This characterization immediately implies Theorem 1.2.1.

1.2.5 Proposition. *Let T be a hereditarily Lindelöf topological space and C a presentable ∞-category. A C-valued presheaf $F \in \mathrm{PSh}(T;C)$ on T is a sheaf if and only if F satisfies the following conditions:*

(1.2.5.1) *The object $F(\varnothing)$ is terminal in C.*

(1.2.5.2) *For all opens $U, V \subset T$, the induced square*

$$\begin{tikzcd} F(U \cup V) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \cap V) \end{tikzcd}$$

is a pullback square in C.

(1.2.5.3) *For every $\mathbf{N}$-indexed sequence of open sets $U_0 \subset U_1 \subset \cdots \subset T$, the induced morphism*

$$F\left(\textstyle\bigcup_{n\geq 0} U_n\right) \to \lim_{n\geq 0} F(U_n)$$

is an equivalence in C.

Proof. First note that (1.2.5.1) and (1.2.5.2) are equivalent to saying that F satisfies descent with respect to finite covers. By Lemma 1.2.4, it suffices to show that F satisfies descent with respect to countable covers.

Let $V \subset T$ be open and $\mathcal{U} = \{V_i\}_{i\in\mathbf{N}}$ a countable open cover of V. For each $n \in \mathbf{N}$, define

$$U_n \coloneqq \bigcup_{i=0}^{n} V_i \qquad \text{and} \qquad \mathcal{U}_n \coloneqq \{V_0, \ldots, V_n\} .$$

Then $\mathcal{U}_n$ is a finite open cover of U_n and we have inclusions $U_n \subset U_{n+1}$ and $\mathcal{U}_n \subset \mathcal{U}_{n+1}$. Note that the poset $\mathrm{I}(\mathcal{U})$ is the filtered union

$$\mathrm{I}(\mathcal{U}) = \operatorname*{colim}_{n\geq 0} \mathrm{I}(\mathcal{U}_n) .$$

Since F satisfies descent with respect to finite covers, by (1.2.5.3) we see that we have natural equivalences

$$\begin{aligned} F(V) &\xrightarrow{\sim} \lim_{n\geq 0} F(U_n) \\ &\xrightarrow{\sim} \lim_{n\geq 0} \lim_{U\in \mathrm{I}(U_n)^{\mathrm{op}}} F(U) \\ &\simeq \lim_{U\in \mathrm{I}(\mathcal{U})^{\mathrm{op}}} F(U) . \end{aligned}$$

Hence F satisfies descent with respect to the countable cover $\mathcal{U}$, as desired. □

Proof of Theorem 1.2.1. Since manifolds are second-countable and open subsets of manifolds are manifolds, the claim is immediate from Proposition 1.2.5 and the definition of what it means to be a sheaf on **Man** (Definition 1.0.3). □

1.3 Digression: relation to infinite dimensional manifolds

We finish this section by describing 'Yoneda embeddings' of Banach and Fréchet manifolds into sheaves of *sets* on **Man**.

1.3.1 Notation. Write **Set** for the category of sets. We write **Ban** for the category of *Banach manifolds*, and **Fré** for the category of *Fréchet manifolds*.

1.3.2 Construction. Define 'Yoneda functors'

$$\begin{aligned} i_{\mathbf{Ban}} \colon \mathbf{Ban} &\to \mathrm{Sh}(\mathbf{Man}; \mathbf{Set}) \\ B &\mapsto [M \mapsto \mathbf{Ban}(M, B)] \end{aligned}$$

and

$$\begin{aligned} i_{\mathbf{Fré}} \colon \mathbf{Fré} &\to \mathrm{Sh}(\mathbf{Man}; \mathbf{Set}) \\ F &\mapsto [M \mapsto \mathbf{Fré}(M, F)] \ . \end{aligned}$$

1.3.3 Theorem (Hain [20], Losik [26; 27, Theorem 3.1.1; 39, Theorem A.1.5])**.** *The functors $i_{\mathbf{Ban}}$ and $i_{\mathbf{Fré}}$ from Construction 1.3.2 are fully faithful.*

The Fréchet manifold of smooth maps from a compact manifold to an arbitrary manifold is sent to the internal-Hom in $\mathrm{Sh}(\mathbf{Man}; \mathbf{Set})$ under the embedding $i_{\mathbf{Fré}}$. In particular, free loop spaces are correctly represented in $\mathrm{Sh}(\mathbf{Man}; \mathbf{Set})$. To state this result, let us first recall the internal-Hom in sheaves on **Man**.

1.3.4 Remark (cartesian closedness)**.** Like any topos, the category $\mathrm{Sh}(\mathbf{Man}; \mathbf{Set})$ of sheaves of sets on **Man** is *cartesian closed*. In particular, $\mathrm{Sh}(\mathbf{Man}; \mathbf{Set})$ has an internal-Hom defined by

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Set})}(E, E') \colon \mathbf{Man}^{\mathrm{op}} &\to \mathbf{Set} \\ M &\mapsto \mathrm{Map}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Set})}(E \times M, E') \ . \end{aligned}$$

1.3.5. If M and N are manifolds, and M is compact, then the topological space $\mathrm{C}^\infty(M, N)$ of smooth maps $M \to N$ has a natural Fréchet manifold structure. See [19, Chapter III, §1], in particular [19, Chapter III, Theorem 1.11], for details.

1.3.6 Theorem (Waldorf [39, Lemma A.1.7])**.** *Let M and N be manifolds. If M is compact, then there is a natural isomorphism*

$$i_{\mathbf{Fré}}(\mathrm{C}^\infty(M, N)) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Set})}(M, N) \ .$$

2 R-invariant sheaves

In this section we investigate **R**-*invariant* (or *homotopy invariant*) sheaves on **Man**. First, we give a proof of Dugger's observation that the global sections functor induces an equivalence from the subcategory $\mathrm{Sh}(\mathbf{Man}; C)$ of **R**-invariant sheaves to C (Proposition 2.0.9). Once we know that constant sheaves on **Man** are **R**-invariant, the claim is more-or-less formal, so the bulk of the work is in showing that constant sheaves are **R**-invariant. We then use this characterization of the constant sheaves to show that the constant sheaf functor $\Gamma^\star : C \to \mathrm{Sh}(\mathbf{Man}; C)$ is given by the assignment

$$X \mapsto [M \mapsto X^{\Pi_\infty(M)}] ,$$

where $X^{\Pi_\infty(M)}$ denotes the cotensor of $X \in C$ by the underlying homotopy type $\Pi_\infty(M)$ of the manifold M (Recollection 2.1.1).

2.0.1 Definition. Let C be a presentable ∞-category. We say that C-valued presheaf

$$F : \mathbf{Man}^{\mathrm{op}} \to C$$

is **R**-***invariant***, ***homotopy-invariant***, or ***concordance-invariant*** if for every manifold M, the morphism $F(M) \to F(M \times \mathbf{R})$ induced by the first projection $\mathrm{pr}_M : M \times \mathbf{R} \to M$ is an equivalence. Write

$$\mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C) \subset \mathrm{Sh}(\mathbf{Man}; C) \qquad \text{and} \qquad \mathrm{PSh}_{\mathbf{R}}(\mathbf{Man}; C) \subset \mathrm{PSh}(\mathbf{Man}; C)$$

for the full subcategories spanned by the **R**-invariant C-valued sheaves and presheaves, respectively.

2.0.2 Notation. Write $\Gamma_\star : \mathrm{Sh}(\mathbf{Man}; C) \to C$ for the ***global sections*** functor, defined by $\Gamma_\star(E) \coloneqq E(*)$. Write $\Gamma^\star : C \to \mathrm{Sh}(\mathbf{Man}; C)$ for the left adjoint to $\Gamma_\star$, i.e., the ***constant sheaf*** functor.

It turns out that the global sections functor also has a right adjoint.

2.0.3 Lemma. *Let C be a presentable ∞-category. Then the functor $\Gamma^! : C \to \mathrm{Sh}(\mathbf{Man}; C)$ defined by the formula*

$$\Gamma^!(X)(M) \coloneqq \prod_{m \in M} X$$

is fully faithful and right adjoint to the global sections functor $\Gamma_\star : \mathrm{PSh}(\mathbf{Man}; C) \to C$. (Here the product is over the underlying set of the manifold M.)

Proof. We define the unit and counit of the adjunction. The unit $\eta_F : F \to \Gamma^!\Gamma_\star(F)$ is defined by the natural map

$$F(M) \to \prod_{m \in M} F(\{m\}) \simeq \Gamma^!\Gamma_\star(F)(M)$$

induced by the inclusions $\{m\} \hookrightarrow M$ for all $m \in M$. The counit $\varepsilon_X : \Gamma_\star\Gamma^!(X) \to X$ is given by the natural identification $\prod_* X \simeq X$. The triangle identities are immediate from the definitions.

To conclude, note that since the counit ε is an equivalence, the functor $\Gamma^!$ is fully faithful. □

2.0.4 Corollary. *Let C be a presentable ∞-category. The global sections functor*

$$\Gamma_\star : \mathrm{Sh}(\mathbf{Man}; C) \to C$$

admits a fully faithful right adjoint $\Gamma^!$. Consequently, the constant sheaf functor $\Gamma^\star$ is fully faithful.

Proof. Note that it is immediate from Definition 1.0.3 that for each $X \in C$, the presheaf $\Gamma^!(X)$ is a sheaf on **Man**. Hence Lemma 2.0.3 shows that the fully faithful functor $\Gamma^! : C \hookrightarrow \mathrm{Sh}(\mathbf{Man}; C)$ is right adjoint to the global sections functor $\Gamma_\star$. Since $\Gamma^\star$ is left adjoint to $\Gamma_\star$, the fact that $\Gamma^!$ is fully faithful implies that $\Gamma^\star$ is fully faithful (see [30, Chapter VII, §4, Lemma 1]). ☐

2.0.5 Corollary. *Let C be a presentable ∞-category. For every $F \in \mathrm{PSh}(\mathbf{Man}; C)$, the unit $F \to \mathrm{S}_{\mathbf{Man}}(F)$ induces an equivalence on global sections.*

Proof. Combine Lemma 2.0.3 and the fact that $\Gamma^!(X)$ is a sheaf for all $X \in C$ (Corollary 2.0.4). ☐

2.0.6 Remark. The functor $\Gamma^!$ does not play a significant role in the approach to differential cohomology presented here. Rather, it serves as a convenient way to see that $\Gamma_\star$ preserves colimits and $\Gamma^\star$ is fully faithful.

2.0.7 Lemma. *Let C be a presentable ∞-category. A morphism $f : E \to E'$ in $\mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C)$ is an equivalence if and only if $\Gamma_\star(f)$ is an equivalence in C.*

Proof. This follows from Lemma 1.0.10 and the assumption that E and E' are **R**-invariant. ☐

It turns out that every constant sheaf on **Man** is **R**-invariant. While this is intuitively plausible, the proof requires some technical work. Since we only need the result and not the method of proof, we have relegated the proof to Appendix A.

2.0.8 Lemma (Lemma A.4.3)**.** *For any presentable ∞-category C, the constant sheaf functor*

$$\Gamma^\star : C \to \mathrm{Sh}(\mathbf{Man}; C)$$

factors through the full subcategory $\mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C) \subset \mathrm{Sh}(\mathbf{Man}; C)$

The following result of Dugger is now immediate from the fact that $\Gamma^\star$ is fully faithful, and $\Gamma_\star$ is conservative when restricted to the **R**-invariant sheaves (Lemma 2.0.7).

2.0.9 Proposition. *Let C be a presentable ∞-category. Then the global sections functor $\Gamma_\star : \mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C) \to C$ is an equivalence with inverse given by $\Gamma^\star$.*

Proof. Since $\Gamma^\star : C \hookrightarrow \mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C)$ is fully faithful, it suffices to show that for each **R**-invariant sheaf E, the counit $\varepsilon_E : \Gamma^\star\Gamma_\star(E) \to E$ is an equivalence. By Lemma 2.0.7, it suffices to show that

$$\Gamma_\star(\varepsilon_E) : \Gamma_\star\Gamma^\star\Gamma_\star(E) \to \Gamma_\star(E)$$

is an equivalence. The claim now follows from the triangle identity and the fact that the unit $\mathrm{id}_C \to \Gamma_\star\Gamma^\star$ is an equivalence (since $\Gamma^\star$ is fully faithful). ☐

2.0.10 Remark (history of Proposition 2.0.9)**.** Dugger [16, Theorem 3.4.3; 17, Proposition 8.3] sketches a proof of Proposition 2.0.9 in the special case that $C = \mathbf{Spc}$, using the language of model categories. See also [32, Proposition 3.3.3] for a related statement.

Bunke and collaborators [12; 13; 14] claim Proposition 2.0.9 in various degrees of generality. Bunke's notes claim a proof of Proposition 2.0.9 in the case $C = \mathbf{Spc}$ [12, Problem 4.21]. Work with Gepner proves Proposition 2.0.9 under restrictive hypotheses on C by checking equivalences on stalks [13, Lemma 6.60] in the sense of § 1.1. Work with Nikoulas and Völkl claims Proposition 2.0.9 in its full generality [14, Proposition 2.6], citing Dugger's work.

2.0.11 Remark. An analogue of Proposition 2.0.9 holds where the category of manifolds is replaced by the category of smooth complex analytic spaces, and $\mathbf{R}$ is replaced by the open unit disk in $\mathbf{C}$; see [3, Remarque 1.9].

2.0.12. By Proposition 2.0.9, the ∞-category $\mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C)$ is presentable and is closed under colimits in $\mathrm{Sh}(\mathbf{Man}; C)$. Moreover, since limits in $\mathrm{Sh}(\mathbf{Man}; C)$ are computed pointwise, the full subcategory $\mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C)$ is also closed under limits. The Adjoint Functor Theorem [HTT, Corollary 5.5.2.9] implies that the inclusion

$$\mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C) \hookrightarrow \mathrm{Sh}(\mathbf{Man}; C)$$

admits both a left and right adjoint, which we denote by L_{hi} and R_{hi}, respectively. We refer to the left adjoint L_{hi} as the ***homotopification*** functor.

Thus we have a chain of adjunctions

$$\begin{tikzcd} \mathrm{Sh}(\mathbf{Man}; C) \arrow[r, shift left=2, "\mathrm{L}_{\mathrm{hi}}"] \arrow[r, shift right=2, "\mathrm{R}_{\mathrm{hi}}"'] & \mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C) \arrow[l, hook'] \arrow[r, shift right=1, "\Gamma_\star"'] & C \arrow[l, shift right=1, "\Gamma^\star"'] \end{tikzcd} ,$$

where the right-hand adjunction is an adjoint equivalence. Consequently, the functor $\Gamma_\star \, \mathrm{L}_{\mathrm{hi}}$ is left adjoint to the constant sheaf functor $\Gamma^\star \colon C \to \mathrm{Sh}(\mathbf{Man}; C)$. We denote this left adjoint by

$$\Gamma_! \colon \mathrm{Sh}(\mathbf{Man}; C) \to C .$$

In this notation, we have equivalences $\mathrm{L}_{\mathrm{hi}} \simeq \Gamma^\star \Gamma_!$ and $\mathrm{R}_{\mathrm{hi}} \simeq \Gamma^\star \Gamma_\star$. Thus we have a chain of four adjoints

$$\begin{tikzcd} \mathrm{Sh}(\mathbf{Man}; C) \arrow[r, shift left=4.5, "\Gamma_!"] \arrow[r, shift right=1.5, "\Gamma_\star"'] & C \arrow[l, shift right=1.5, "\Gamma^\star"'] \arrow[l, shift left=4.5, "\Gamma^!"] \end{tikzcd} , \tag{2.0.13}$$

where functors lie above their right adjoints.

2.0.14 Remark (cohesion)**.** Much of the structure of sheaves on **Man** that we are interested in for studying differential cohomology (particularly Section 5) only depends on the existence of the chain of four adjoints (2.0.13). In the case where $C = \mathbf{Spc}$, the existence of these extra adjoints for the global sections geometric morphism (along with the condition that the extreme left adjoint $\Gamma_!$ preserve finite products) is what Schreiber calls a *cohesive* ∞*-topos* [36, Definition 3.4.1]. The primary examples of cohesive ∞-topoi are sheaves on **Man** and *global spaces* [35]. Cohesive ∞-topoi are a very general setting in which one can talk about a generalized form of 'differential cohomology'.

Many of the ideas about cohesive ∞-topoi go back to work of Simpson and Teleman [38].

2.1 Description of the constant sheaf functor

We now use Proposition 2.0.9 to give a concrete description of the constant sheaf functor (Lemma 2.1.5). To do this, we first recall the natural cotensoring of a presentable ∞-category over **Spc**.

2.1.1 Recollection (cotensoring over **Spc**). Every presentable ∞-category C is naturally *cotensored over* the ∞-category **Spc** of spaces [HTT, Remark 5.5.2.6]. That is, there is a functor

$$\begin{aligned} (-)^{(-)} \colon \mathbf{Spc}^{\mathrm{op}} \times C &\to C \\ (K, X) &\mapsto X^K \,, \end{aligned}$$

along with natural equivalences

$$\mathrm{Map}_C(X', X^K) \simeq \mathrm{Map}_{\mathbf{Spc}}(K, \mathrm{Map}_C(X', X)) \,.$$

2.1.2 Example. If $C = \mathbf{Sp}$ is the ∞-category of spectra, then the cotensoring is given by

$$X^K \coloneqq \mathrm{Hom}_{\mathbf{Sp}}(\Sigma^\infty_+ K, X) \,,$$

where $\mathrm{Hom}_{\mathbf{Sp}}$ denotes the mapping *spectrum* in **Sp**.

2.1.3 Notation. For a topological space T, we write $\Pi_\infty(T) \in \mathbf{Spc}$ for the underlying homotopy type of T.

2.1.4 Construction. Let C be a presentable ∞-category. Using the cotensoring of C over **Spc**, define a functor $\mathrm{sm} \colon C \to \mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C)$ by the assignment

$$X \mapsto \left[M \mapsto X^{\Pi_\infty(M)}\right] .$$

Given $X \in C$, the presheaf $\mathrm{sm}(X)$ is obviously $\mathbf{R}$-invariant. Moreover, the van Kampen Theorem [HA, Proposition A.3.2] implies that $\mathrm{sm}(X)$ is a sheaf on **Man**.

Construct a natural transformation $\alpha \colon \Gamma^\star \to \mathrm{sm}$ as follows. For each $X \in C$, note that we have a canonical identification $\Gamma_\star \mathrm{sm}(X) = X$. Then α is the natural transformation whose component at X corresponds to id_X under the equivalence

$$\mathrm{Map}_{\mathrm{Sh}(\mathbf{Man};C)}(\Gamma^\star(X), \mathrm{sm}(X)) \simeq \mathrm{Map}_C(X, \Gamma_\star \mathrm{sm}(X)) = \mathrm{Map}_C(X, X) \,.$$

2.1.5 Lemma. *For any presentable ∞-category C, the natural transformation*

$$\alpha \colon \Gamma^\star \to \mathrm{sm}$$

is an equivalence.

Proof. Since $\Gamma^\star$ and sm take values in $\mathbf{R}$-invariant sheaves, by Lemma 2.0.7 it suffices to check that $\Gamma_\star(\alpha) \colon \Gamma_\star\Gamma^\star \to \Gamma_\star \mathrm{sm}$ is an equivalence. This follows from the identification $\Gamma_\star \mathrm{sm} = \mathrm{id}_C$ and the fact that the unit $\mathrm{id}_C \to \Gamma_\star\Gamma^\star$ is an equivalence (Proposition 2.0.9). □

2.2 Criteria for R-invariance

In this subsection we collect two reformulations of **R**-invariance are due to Voevodsky [31, Lemma 2.16]. To state these reformulations, we first need some notation.

2.2.1 Notation. Let M be a manifold and $t \in \mathbf{R}$. We write $i_{M,t} \colon M \hookrightarrow M \times \mathbf{R}$ for the smooth embedding defined by $x \mapsto (x,t)$.

2.2.2 Observation. For all manifolds M and $t \in \mathbf{R}$, the map $i_{M\times\mathbf{R},t}$ is given by the composite

$$M \times \mathbf{R} \xhookrightarrow{i_{M,t}\times \mathrm{id}_{\mathbf{R}}} M \times \mathbf{R} \times \mathbf{R} \xrightarrow[\sim]{\mathrm{id}_M \times \mathrm{swap}} M \times \mathbf{R}\,,$$

where $\mathrm{swap}\colon \mathbf{R}\times\mathbf{R} \xrightarrow{\sim} \mathbf{R}\times\mathbf{R}$ is the map that swaps the two factors.

2.2.3 Proposition. *Let C be a presentable ∞-category. The following are equivalent for a presheaf $F\colon \mathbf{Man}^{\mathrm{op}} \to C$:*

(2.2.3.1) *The presheaf F is **R**-invariant.*

(2.2.3.2) *For all manifolds M, the induced map*

$$i_{M,0}^{\star}\colon F(M\times\mathbf{R}) \to F(M)$$

is an equivalence.

(2.2.3.3) *For all manifolds M, the induced maps*

$$i_{M,0}^{\star}, i_{M,1}^{\star}\colon F(M\times\mathbf{R}) \to F(M)$$

are equivalent.

Proof. Since the embeddings

$$i_{M,0}, i_{M,1}\colon M \hookrightarrow M\times\mathbf{R}$$

are sections of the projection $\mathrm{pr}_M\colon M\times\mathbf{R}\to M$, it is clear that (2.2.3.1)⇔(2.2.3.2) and (2.2.3.1)⇒(2.2.3.3).

To complete the proof, we show that (2.2.3.3)⇒(2.2.3.1). Assuming (2.2.3.3), since $i_{M,0}$ is a section or the projection $\mathrm{pr}_M\colon M\times\mathbf{R}\to M$, it suffices to show that we have an equivalence $\mathrm{pr}_M^{\star}\, i_{M,0}^{\star} \simeq \mathrm{id}_{F(M\times\mathbf{R})}$. To see this, write $\mathrm{mult}\colon \mathbf{R}\times\mathbf{R}\to\mathbf{R}$ for the multiplication map, and notice that we have a commutative diagram in **Man**

(2.2.4)
$$\begin{tikzcd}
M\times\mathbf{R} \arrow[r, hook, "i_{M,0}\times\mathrm{id}_{\mathbf{R}}"] \arrow[dr, equal] & M\times\mathbf{R}\times\mathbf{R} \arrow[d, "\mathrm{id}_M\times\mathrm{mult}"] & M\times\mathbf{R} \arrow[l, hook', "i_{M,0}\times\mathrm{id}_{\mathbf{R}}"'] \arrow[d, "\mathrm{pr}"] \\
& M\times\mathbf{R} & M \arrow[l, hook', "i_{M,0}"]
\end{tikzcd}$$

Since $i_{M\times\mathbf{R},0}^{\star} \simeq i_{M\times\mathbf{R},1}^{\star}$ by assumption, Observation 2.2.2 shows that

(2.2.5)
$$(i_{M,0}\times\mathrm{id}_{\mathbf{R}})^{\star} \simeq (i_{M,1}\times\mathrm{id}_{\mathbf{R}})^{\star}\,.$$

Equation (2.2.5) and the commutativity of the diagram (2.2.4) now show that

$$
\begin{aligned}
\mathrm{pr}_M^\star \, i_{M,0}^\star &\simeq (i_{M,0} \times \mathrm{id}_{\mathbf{R}})^\star \circ (\mathrm{id}_M \times \mathrm{mult})^\star \\
&\simeq (i_{M,1} \times \mathrm{id}_{\mathbf{R}})^\star \circ (\mathrm{id}_M \times \mathrm{mult})^\star \\
&\simeq \mathrm{id}_{F(M \times \mathbf{R})} \ ,
\end{aligned}
$$

as desired. □

3 R-localization

In this section we show how to use the fact that the sheafification of an $\mathbf{R}$-invariant presheaf on $\mathbf{Man}$ is $\mathbf{R}$-invariant to provide a formula for the functors $\Gamma_!$ and L_{hi}. Since proving this first fact is a bit of a technical digression, we defer the proof to Appendix A.

3.0.1 Proposition (Proposition A.9.10)**.** *Let C be a presentable ∞-category. Then for every $\mathbf{R}$-invariant presheaf $F\colon \mathbf{Man}^{\mathrm{op}} \to C$, the counit $\Gamma^{\star}\Gamma_{\star}\,\mathrm{S}_{\mathbf{Man}}\,F \to \mathrm{S}_{\mathbf{Man}}\,F$ is an equivalence. In particular, $\mathrm{S}_{\mathbf{Man}}\,F$ is $\mathbf{R}$-invariant.*

Proposition 3.0.1 immediately gives a description of the homotopicification functor L_{hi} in terms of the $\mathbf{R}$-localization functor for presheaves.

3.0.2 Notation. Write $\mathrm{L}_{\mathbf{R}}\colon \mathrm{PSh}(\mathbf{Man}; C) \to \mathrm{PSh}_{\mathbf{R}}(\mathbf{Man}; C)$ for the left adjoint to the inclusion. We refer to the functor $\mathrm{L}_{\mathbf{R}}$ as the $\mathbf{R}$-***localization*** functor.

3.0.3 Corollary. *Let C be a presentable ∞-category. Then the composite*

$$\mathrm{S}_{\mathbf{Man}}\,\mathrm{L}_{\mathbf{R}}\colon \mathrm{Sh}(\mathbf{Man}; C) \to \mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C)$$

is left adjoint to the inclusion $\mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C) \hookrightarrow \mathrm{Sh}(\mathbf{Man}; C)$. That is, $\mathrm{L}_{\mathrm{hi}} \simeq \mathrm{S}_{\mathbf{Man}}\,\mathrm{L}_{\mathbf{R}}$.

Thus in order to describe the functors $\Gamma_!$ and L_{hi}, it suffices to describe the presheaf $\mathbf{R}$-localization functor $\mathrm{L}_{\mathbf{R}}$. In Subsection 3.1 we explain an explicit construction of $\mathrm{L}_{\mathbf{R}}$ due to Morel, Suslin, Voevodsky. In Subsection 3.2 we'll prove the Morel–Suslin–Voevodsky formula modulo one detail on simplicial homotopies in ∞-categories. In Subsection 3.3 we prove this minor detail after recalling the basics of simplicial homotopies in ∞-categories.

3.1 The Morel–Suslin–Voevodsky construction

By appealing to the 'Sing construction' of Morel–Suslin–Voevodsky [32, §2.3], we provide a simple description of the functor $\mathrm{L}_{\mathbf{R}}$. In light of Corollary 3.0.3, this also provides a more simple description of the homotopification functor L_{hi}.

3.1.1 Notation. Let $n \geq 0$ be an integer. Write Δ^n_{alg} for the hyperplane in $\mathbf{R}^{n+1}$ defined by

$$\Delta^n_{\mathrm{alg}} \coloneqq \{\,(t_0, \ldots, t_n) \in \mathbf{R}^{n+1} \mid t_0 + \cdots + t_n = 1\,\} \subset \mathbf{R}^{n+1}\,,$$

so that as a smooth manifold Δ^n_{alg} is diffeomorphic to $\mathbf{R}^n$. We call Δ^n_{alg} the ***algebraic** n**-simplex***.

In the usual way, the algebraic n-simplices for $n \geq 0$ assemble into a cosimplicial manifold

$$\Delta^{\bullet}_{\mathrm{alg}}\colon \mathbf{\Delta} \to \mathbf{Man}\,.$$

3.1.2 Proposition (Morel–Suslin–Voevodsky construction)**.** *Let C be a presentable ∞-category. The left adjoint*

$$\mathrm{L}_{\mathbf{R}}\colon \mathrm{PSh}(\mathbf{Man}; C) \to \mathrm{PSh}_{\mathbf{R}}(\mathbf{Man}; C)$$

is given by the geometric realization

$$\mathrm{L}_{\mathbf{R}}(F)(M) \coloneqq |F(M \times \Delta^{\bullet}_{\mathrm{alg}})|\,.$$

3.1.3 Remark. We call the construction

$$F \mapsto |F(- \times \Delta^\bullet_{\mathrm{alg}})|$$

the ***Morel–Suslin–Voevodsky*** construction. Morel and Voevodsky provide a very general version of the Morel–Suslin–Voevodsky construction for 'sites with an interval object' [32, §2.3], which covers the site **Man** with **R** as the interval object (see also [1, §4.3; 2, §4]). However, their arguments are model category-theoretic and apply to a more specific situation than what we're interested in, so we provide separate argument.

So as to not take us too far afield, we settle for working with the site of manifolds rather than a general site with an interval object. Our proof of Proposition 3.1.2 takes the approach used in Brazelton's notes on motivic homotopy theory [10, §3].

We defer the proof of Proposition 3.1.2 to §§3.2 and 3.3 and first derive some useful conseqences.

3.1.4 Corollary. *Let C be a presentable ∞-category. The left adjoint $\Gamma_! \colon \mathrm{Sh}(\mathbf{Man}; C) \to C$ to the constant sheaf functor is given by*

$$\Gamma_!(E) \simeq |E(\Delta^\bullet_{\mathrm{alg}})| \,.$$

Proof. By Corollary 3.1.4 and the identification $\Gamma_\star \mathrm{L}_{\mathrm{hi}} \simeq \Gamma_!$, it suffices to show that for every sheaf E on **Man**, the global sections of $\mathrm{S}_{\mathbf{Man}} \mathrm{L}_{\mathbf{R}} E$ are given by the geometric realization $|E(\Delta^\bullet_{\mathrm{alg}})|$. Since the unit

$$\mathrm{L}_{\mathbf{R}} E \to \mathrm{S}_{\mathbf{Man}} \mathrm{L}_{\mathbf{R}} E$$

of the sheafification adjunction induces an equivalence on global sections (Corollary 2.0.5), the claim now follows from Proposition 3.1.2. □

3.1.5 Corollary. *Let C be a presentable ∞-category. If geometric realizations commute with finite products in C (e.g., C is an ∞-topos), then the functor $\Gamma_! \colon \mathrm{Sh}(\mathbf{Man}; C) \to C$ preserves finite products.*

3.1.6. Since $\mathrm{L}_{\mathrm{hi}} \simeq \Gamma^\star \Gamma_!$, Lemma 2.1.5 and Corollary 3.1.4 show that L_{hi} is given by the formula

$$\mathrm{L}_{\mathrm{hi}}(E)(M) \simeq |E(\Delta^\bullet_{\mathrm{alg}})|^{\Pi_\infty(M)} \,.$$

3.2 Proof of the Morel–Suslin–Voevodsky formula

We prove Proposition 3.1.2 by applying the following recognition principle for localization functors.

3.2.1 Proposition ([HTT, Proposition 5.2.7.4])**.** *Let C be an ∞-category and $L \colon D \to D$ a functor with essential image $LD \subset D$. Then the following are equivalent:*

(3.2.1.1) *There exists a functor $F \colon D \to D'$ with fully faithful right adjoint $G \colon D' \hookrightarrow D$ such that $GF \simeq L$.*

(3.2.1.2) *The functor $L \colon D \to LD$ is left adjoint to the inclusion $LD \hookrightarrow D$.*

(3.2.1.3) *There is a natural transformation $\eta\colon \mathrm{id}_D \to L$ such that for all $d \in D$, the morphisms*

$$\eta_{L(d)}, L(\eta_d)\colon L(d) \to L(L(d))$$

are equivalences.

3.2.2 Notation. Let us temporarily write $\mathrm{H}\colon \mathrm{PSh}(\mathbf{Man}; C) \to \mathrm{PSh}(\mathbf{Man}; C)$ for the functor defined by

$$\mathrm{H}(F)(M) \coloneqq |F(M \times \Delta^{\bullet}_{\mathrm{alg}})| .$$

3.2.3 Construction. Let C be a presentable ∞-category. Define a natural transfoormation

$$\eta\colon \mathrm{id}_{\mathrm{PSh}(\mathbf{Man};C)} \to \mathrm{H}$$

as follows. Let M be a manifold, and also simply write M for the cosntant cosimplicial manifold at M. Projection onto the first factor defines a morphism of cosimplicial manifolds $\mathrm{pr}_M\colon M \times \Delta^{\bullet}_{\mathrm{alg}} \to M$ from the product cosimplicial manifold $M \times \Delta^{\bullet}_{\mathrm{alg}}$ to the constant cosimplicial manifold at M. For each C-valued presheaf $F \in \mathrm{PSh}(\mathbf{Man}; C)$, the morphism $\eta_F\colon F \to \mathrm{H}(F)$ is defined as the geometric realization

$$\eta_F(M) \coloneqq |\,\mathrm{pr}_M^{\star}\,|\colon F(M) \xrightarrow{\sim} |F(M)| \to |F(M \times \Delta^{\bullet}_{\mathrm{alg}})| = \mathrm{H}(F)(M) .$$

Equivalently, the morphism $\eta_F(M)$ is the composite

$$F(M) \simeq F(M \times \Delta^{0}_{\mathrm{alg}}) \to |F(M \times \Delta^{\bullet}_{\mathrm{alg}})|$$

of the equivalence $F(M) \xrightarrow{\sim} F(M \times \Delta^{0}_{\mathrm{alg}})$ induced by the projection $M \times \Delta^{0}_{\mathrm{alg}} \xrightarrow{\sim} M$ with the induced map $F(M \times \Delta^{0}_{\mathrm{alg}}) \to |F(M \times \Delta^{\bullet}_{\mathrm{alg}})|$ from the 0-simplices of the simplicial object $F(M \times \Delta^{\bullet}_{\mathrm{alg}})$ to its geometric realization.

In order to apply Proposition 3.2.1, the first thing to check is that $\mathrm{H}(F)$ is actually $\mathbf{R}$-invariant. This is not difficult, but for the proof we need to recall some background on simplicial homotopies in an arbitrary ∞-category, so we defer the proof of the following lemma to §3.3.

3.2.4 Lemma. *Let C be a presentable ∞-category. For any presheaf $F\colon \mathbf{Man}^{\mathrm{op}} \to C$, the presheaf $\mathrm{H}(F)$ is $\mathbf{R}$-invariant.*

The second thing to check is that $\eta_{H(G)}$ is an equivalence for every presheaf G. Combined with Lemma 3.2.4 this guarantees that the essential image of the functor

$$\mathrm{H}\colon \mathrm{PSh}(\mathbf{Man}; C) \to \mathrm{PSh}(\mathbf{Man}; C)$$

is $\mathrm{PSh}_{\mathbf{R}}(\mathbf{Man}; C)$.

3.2.5 Lemma. *Let C be a presentable ∞-category. If $F\colon \mathbf{Man}^{\mathrm{op}} \to C$ is $\mathbf{R}$-invariant, then the map $\eta_F\colon F \to \mathrm{H}(F)$ is an equivalence.*

Proof. Let M be a manifold. Since F is $\mathbf{R}$-invariant and $\Delta^n_{\mathrm{alg}} \cong \mathbf{R}^n$ for each $n \geq 0$, the projection $\mathrm{pr}_M \colon M \times \Delta^\bullet_{\mathrm{alg}} \to M$ from the cosimplicial manifold $M \times \Delta^\bullet_{\mathrm{alg}}$ to the constant cosimplicial manifold at M induces an equivalence

$$\mathrm{pr}_M^\star \colon F(M) \to F(M \times \Delta^\bullet_{\mathrm{alg}})$$

of simplicial objects in C. The claim now follows by passing to geometric realizations. □

3.2.6 Corollary. *Let C be a presentable ∞-category. The essential image of the functor* $\mathrm{H} \colon \mathrm{PSh}(\mathbf{Man}; C) \to \mathrm{PSh}(\mathbf{Man}; C)$ *is* $\mathrm{PSh}_{\mathbf{R}}(\mathbf{Man}; C)$.

Now we complete the proof by showing that see that $\mathrm{H}(\eta_F)$ is an equivalence.

3.2.7 Lemma. *Let C be a presentable ∞-category. For all $F \in \mathrm{PSh}(\mathbf{Man}; C)$, the maps*

$$\eta_{\mathrm{H}(F)}, \mathrm{H}(\eta_F) \colon \mathrm{H}(F) \to \mathrm{H}(\mathrm{H}(F))$$

are equivalences.

Proof. By Lemma 3.2.5 and Corollary 3.2.6, the morphism $\eta_{\mathrm{H}(F)}$ is an equivalence.

To see that $\mathrm{H}(\eta_F) \colon \mathrm{H}(F) \to \mathrm{H}(\mathrm{H}(F))$ is an equivalence, note that for all manifolds M,

$$\mathrm{H}(F)(M) = \operatorname*{colim}_{[m] \in \mathbf{\Delta}^{\mathrm{op}}} F(M \times \Delta^m_{\mathrm{alg}})$$

and

$$\begin{aligned} \mathrm{H}(\mathrm{H}(F))(M) &= \operatorname*{colim}_{[m] \in \mathbf{\Delta}^{\mathrm{op}}} \operatorname*{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} F(M \times \Delta^m_{\mathrm{alg}} \times \Delta^n_{\mathrm{alg}}) \\ &\simeq \operatorname*{colim}_{([m],[n]) \in \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}} F(M \times \Delta^m_{\mathrm{alg}} \times \Delta^n_{\mathrm{alg}}) \, . \end{aligned}$$

Moreover, the map $\mathrm{H}(\eta_F) \colon \mathrm{H}(F) \to \mathrm{H}(\mathrm{H}(F))$ is induced on colimits by the fully functor $\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$ given by the assignment $[m] \mapsto ([m], [0])$. First taking the colimit over the variable $[m] \in \mathbf{\Delta}^{\mathrm{op}}$, we see that the map $\mathrm{H}(\eta_F)(M)$ is induced by the map from the 0-simplices $\mathrm{H}(F)(M)$ of the simplicial object $\mathrm{H}(F)(M \times \Delta^\bullet_{\mathrm{alg}})$ to its geometric realization. Since $\mathrm{H}(F)$ is $\mathbf{R}$-invariant (Lemma 3.2.4), the simplicial object $\mathrm{H}(F)(M \times \Delta^\bullet_{\mathrm{alg}})$ is equivalent to the constant simplicial object at $\mathrm{H}(F)(M)$, hence the induced map

$$\mathrm{H}(F)(M) \to \operatorname*{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{H}(F)(M \times \Delta^n_{\mathrm{alg}})$$

from the 0-simplices is an equivalence. □

Proof of Proposition 3.1.2. Combine Corollary 3.2.6, Lemma 3.2.7, and Proposition 3.2.1. □

3.3 Simplicial homotopies in ∞-categories

In order to prove Lemma 3.2.4, we need to use a little bit on simplicial homotopies. Since we're working natively to ∞-categories and not in simplicial sets or simplicial presheaves, we need to reformulate the definition if a simplicial homotopy to make sense in our context.

3.3.1 Notation. We write $u\colon \mathbf{\Delta}_{/[1]} \to \mathbf{\Delta}$ for the forgetful functor. For $i \in [1]$, we write $j_i\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_{/[1]}$ for the fully faithful functor given on objects by the assignment

$$[n] \mapsto \big[[n] \to \{i\} \hookrightarrow [1]\big]\,,$$

with the obvious assignment on morphisms.

3.3.2 Observation. For each $i \in [1]$, the fully faithful functor $j_i\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_{/[1]}$ is left adjoint to the functor $\mathbf{\Delta}_{/[1]} \hookrightarrow \mathbf{\Delta}$ that sends an object $s\colon [m] \to [1]$ to the fiber $s^{-1}(i)$ of s over i (with the induced ordering), and the obvious assignment on morphisms.

3.3.3 Definition ([HA, Definition 7.2.1.6])**.** Let D be an ∞-category and let

$$f_0, f_1\colon X_\bullet \to Y_\bullet$$

be morphisms in the ∞-category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, D)$ of simplicial objects in D. A ***simplicial homotopy*** from f_0 to f_1 consists of the following data:

(3.3.3.1) A morphism $h\colon u^\star(X_\bullet) \to u^\star(Y_\bullet)$ in $\mathrm{Fun}((\mathbf{\Delta}_{/[1]})^{\mathrm{op}}, D)$.

(3.3.3.2) Equivalences $j_0^\star(h) \simeq f_0$ and $j_1^\star(h) \simeq f_1$ of morphisms $X_\bullet \to Y_\bullet$ in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, D)$.

We often simply write $h\colon u^\star(X_\bullet) \to u^\star(Y_\bullet)$ for the entire data of a simplicial homotopy from f_0 to f_1.

3.3.4 Remark (relation to classical simplicial homotopies)**.** It is not immediately clear that Definition 3.3.3 recovers the classical notion of a simplicial homotopy in the category $s\mathbf{Set} \coloneqq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set})$ of simplicial sets. Let us explain why, given simplicial sets $X_\bullet$ and $Y_\bullet$, the set of simplicial homotopies $X_\bullet \times \Delta^1 \to Y_\bullet$ is in natural bijection with simplicial homotopies $u^\star(X_\bullet) \to u^\star(Y_\bullet)$ in the sense of Definition 3.3.3.

First we recall the relationship between presheaf categories and slice categories. Let S be a small category and $s \in S$. Then the colimit-preservng extension of the 'sliced Yoneda embedding'

$$\begin{aligned} S_{/s} &\hookrightarrow \mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Set})_{/よ(s)} \\ [s' \to s] &\mapsto [よ(s') \to よ(s)] \end{aligned}$$

defines an equivalence of categories $\mathrm{Fun}((S_{/s})^{\mathrm{op}}, \mathbf{Set}) \xrightarrow{\sim} \mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Set})_{/よ(s)}$. Under this identification, the functor $\mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Set}) \to \mathrm{Fun}((S_{/s})^{\mathrm{op}}, \mathbf{Set})$ given by precomposition with the forgetful functor $S_{/s} \to S$ is identified with the functor

$$よ(s) \times (-)\colon \mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Set}) \to \mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Set})_{/よ(s)}\,.$$

Moreover, functor $よ(s) \times (-)$ is right adjoint to the forgetful functor

$$\mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Set})_{/よ(s)} \to \mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Set}) .$$

Now we specialize to the case $S = \mathbf{\Delta}$ and $s = [1]$. Write

$$u_! \colon s\mathbf{Set}_{/\Delta^1} \to s\mathbf{Set}$$

for the forgetful functor. For all simplicial sets $X_\bullet$ and $Y_\bullet$, we have have natural bijections

$$\begin{aligned} s\mathbf{Set}_{/\Delta^1}(u^\star(X_\bullet), u^\star(Y_\bullet)) &\cong s\mathbf{Set}(u_! u^\star(X_\bullet), Y_\bullet) \\ &\cong s\mathbf{Set}(X_\bullet \times \Delta^1, Y_\bullet) . \end{aligned}$$

Hence simplicial homotopies in the sense of Definition 3.3.3 are in natural bijection with simplicial homotopies in the classical sense.

The main useful fact about simplicial homotopies is that if $h \colon u^\star(X_\bullet) \to u^\star(Y_\bullet)$ is a simplicial homotopy from f_0 to f_1, then f_0 and f_1 induce the same map $|X_\bullet| \to |Y_\bullet|$ on geometric realizations.

3.3.5 Lemma. *Let D be an ∞-category that admits geometric realizations of simplicial objects. Let $f_0, f_1 \colon X_\bullet \to Y_\bullet$ be morphisms of simplicial objects in D and let h be a simplicial homotopy from f_0 to f_1. Then the simplicial homotopy h induces an equivalence $|f_0| \simeq |f_1|$ between the induced morphisms*

$$|f_0|, |f_1| \colon |X_\bullet| \to |Y_\bullet|$$

on geometric realizations.

Proof of Lemma 3.3.5. Since the functors $j_0, j_1 \colon \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow (\mathbf{\Delta}_{/[1]})^{\mathrm{op}}$ are right adjoints (Observation 3.3.2), both j_0 and j_1 are colimit-cofinal. Since $u j_0 = \mathrm{id}_{\mathbf{\Delta}^{\mathrm{op}}}$ and $u j_1 = \mathrm{id}_{\mathbf{\Delta}^{\mathrm{op}}}$, we see that the functors $j_0, j_1 \colon \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow (\mathbf{\Delta}_{/[1]})^{\mathrm{op}}$ are also colimit-cofinal. Hence the simplicial homotopy h provides equivalences

$$|f_0| \simeq |j_0^\star(h)| \simeq \operatorname*{colim}_{(\mathbf{\Delta}_{/[1]})^{\mathrm{op}}} h \colon |X_\bullet| \simeq \operatorname*{colim}_{(\mathbf{\Delta}_{/[1]})^{\mathrm{op}}} u^\star(X_\bullet) \to \operatorname*{colim}_{(\mathbf{\Delta}_{/[1]})^{\mathrm{op}}} u^\star(Y_\bullet) \simeq |Y_\bullet|$$

and

$$|f_1| \simeq |j_1^\star(h)| \simeq \operatorname*{colim}_{(\mathbf{\Delta}_{/[1]})^{\mathrm{op}}} h \colon |X_\bullet| \simeq \operatorname*{colim}_{(\mathbf{\Delta}_{/[1]})^{\mathrm{op}}} u^\star(X_\bullet) \to \operatorname*{colim}_{(\mathbf{\Delta}_{/[1]})^{\mathrm{op}}} u^\star(Y_\bullet) \simeq |Y_\bullet| .$$

Hence $|f_0| \simeq |f_1|$, as desired. $\square$

Now we are ready to apply the technology of simplicial homotopies to prove that $\mathrm{H}(F)$ is $\mathbf{R}$-invariant (Lemma 3.2.4).

3.3.6 Lemma. *For all manifolds M, there is a natural simplicial homotopy in $\mathbf{Man}^{\mathrm{op}}$ from the map*

$$i_{M \times \Delta^\bullet_{\mathrm{alg}}, 0}\, \mathrm{pr}_{M \times \Delta^\bullet_{\mathrm{alg}}} \colon M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbf{R} \to M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbf{R}$$

to the identity.

Proof. Define a simplicial homotopy

$$h\colon u^\star(M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbf{R}) \to u^\star(M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbf{R})$$

as follows. For each map $\sigma\colon [n] \to [1]$ in $\mathbf{\Delta}$, write $h'_\sigma\colon \Delta^n_{\mathrm{alg}} \times \mathbf{R} \to \Delta^n_{\mathrm{alg}} \times \mathbf{R}$ for the smooth map defined by the formula

$$h'_\sigma(t_0, \dots, t_n, x) \coloneqq \left(t_0, \dots, t_n, x \textstyle\sum_{k \in \sigma^{-1}(1)} t_k\right) .$$

Define $h_\sigma\colon M \times \Delta^n_{\mathrm{alg}} \times \mathbf{R} \to M \times \Delta^n_{\mathrm{alg}} \times \mathbf{R}$ by setting $h_\sigma \coloneqq \mathrm{id}_M \times h'_\sigma$. It is immediate from the definitions that h defines a simplicial homotopy

$$u^\star(M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbf{R}) \to u^\star(M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbf{R}) ,$$

and, moreover,

$$j_0^\star(h) = i_{M \times \Delta^\bullet_{\mathrm{alg}},0}\, \mathrm{pr}_{M \times \Delta^\bullet_{\mathrm{alg}}} \qquad \text{and} \qquad j_1^\star(h) = \mathrm{id}_{M \times \Delta^\bullet_{\mathrm{alg}} \times \mathbf{R}} . \qquad \square$$

Proof of Lemma 3.2.4. Let M be a manifold. Since $\mathrm{pr}_M\, i_{M,0} = \mathrm{id}_M$, to see that

$$\mathrm{pr}_M^\star\colon \mathrm{H}(F)(M) \to \mathrm{H}(F)(M \times \mathbf{R})$$

is an equivalence, it suffices to show that

$$\mathrm{pr}_M^\star\, i_{M,0}^\star \simeq \mathrm{id}_{\mathrm{H}(F)(M \times \mathbf{R})} .$$

This follows from combining Lemmas 3.3.5 and 3.3.6. □

4 The Transfer Conjecture

Let X be a space. We have seen that the constant sheaf $\Gamma^\star(X)$ on $\mathbf{Man}$ is given by the formula

$$\Gamma^\star(X) = \mathrm{Map}_{\mathbf{Spc}}(\Pi_\infty(M), X)$$

(Lemma 2.1.5). If $X = \Omega^\infty E$ is the infinite loop space of a spectrum E, then the sheaf $\Gamma^\star(X)$ acquires additional functoriality: for any finite covering map between manifolds $f \colon N \to M$, the Becker–Gottleib transfer $\Sigma^\infty_+ \Pi_\infty(M) \to \Sigma^\infty_+ \Pi_\infty(N)$ induces a *transfer* map

$$f_\star \colon \Gamma^\star(X)(N) \to \Gamma^\star(X)(M) .$$

This enhanced functoriality can be used to make $\Gamma^\star(X)$ into a copresheaf on a 2-category $\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man})$ with objects smooth manifolds and morphisms *correspondences*

$$\begin{tikzcd} & N \arrow[dl] \arrow[dr, "f"] & \\ M_0 & & M_1 , \end{tikzcd}$$

where f is a finite covering map.

For a sheaf F on $\mathbf{Man}$, Quillen conjectured that an extension of F to $\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man})$ is just another way of encoding an E_∞-structure on F. However, when Quillen originally formulated this *Transfer Conjecture*, the language to express the higher coherences necessary for the validity of the result was not available. Moreover, Quillen's original formulation was *dis*proven by Kraines and Lada [25; 34].

The goal of this section is explain how to deduce the following corrected version of the Transfer Conjecture from very general results of Bachmann–Hoyois on commutative algebgras and ∞-categories of spans [4, Appendix C].

4.0.1 Theorem (Transfer Conjecture; Corollaries 4.3.5 and 4.3.6)**.** *Let C be a presentable ∞-category. There is an equivalence of ∞-categories*

$$\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}), C) \xrightarrow{\sim} \mathrm{Sh}(\mathbf{Man}; \mathrm{CMon}(C))$$

between functors $\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}) \to C$ *whose restriction to* $\mathbf{Man}^{\mathrm{op}}$ *is a sheaf and sheaves of commutative monoids C. This further restricts to an equivalence*

$$\mathrm{Fun}_{\mathrm{loc},\mathbf{R}}(\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}), C) \xrightarrow{\sim} \mathrm{CMon}(C)$$

between functors $\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}) \to C$ *whose restriction to* $\mathbf{Man}^{\mathrm{op}}$ *is an* $\mathbf{R}$*-invariant sheaf and commutative monoids in C.*

4.0.2 Example. Setting $C = \mathbf{Spc}$ in Theorem 4.0.1 gives an equivalence between functors

$$\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}) \to \mathbf{Spc}$$

whose restriction to $\mathbf{Man}^{\mathrm{op}}$ is an $\mathbf{R}$-invariant sheaf and E_∞-spaces. Restricting to grouplike objects on both sides and applying the recognition principle [HA, Remark 5.2.6.26] provides an equivalence between grouplike objects of $\mathrm{Fun}_{\mathrm{loc},\mathbf{R}}(\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}), \mathbf{Spc})$ and the ∞-category $\mathbf{Sp}_{\geq 0}$ of connective spectra.

In order to give a more precise formulation of Theorem 4.0.1, we'll first review constructing 2-categories of *correspondences* or *spans* from 1-categories (§ 4.1). We then briefly recall the role that ∞-categories of spans play in encoding E_∞-structures (§4.2). Finally, we walk through [4, Appendix C] in the case of interest and explain how to deduce the Transfer Conjecture from their results (§4.3).

4.1 Categories of spans

In this subsection we explain how to construct the 2-category $\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man})$ of correspondences of manifolds appearing in the Transfer Conjecture. This is a special case of a general construction for ∞-categories due to Barwick [6, §§3–5]. If D is an n-category, then Barwick's ∞-category of spans in D is an $(n+1)$-category. In order to avoid explaining how to deal with the homotopy coherence problems that arise, we only present the 1-categorical case since we can give a simple definition as a 2-category.

4.1.1 Construction (2-category of spans)**.** Let D be a 1-category, and let $L, R \subset \mathrm{Mor}(D)$ be two classes of morphisms in D satisfying the following properties:

(4.1.1.1) The classes L and R contain all isomorphisms.

(4.1.1.2) The classes L and R are each stable under composition.

(4.1.1.3) Given a morphism $\ell \colon X \to Z$ in L and morphism $r \colon Y \to Z$ in R, there exists a pullback diagram

$$\begin{tikzcd} W \arrow[r, "\bar{r}"] \arrow[d, "\bar{\ell}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "r"] \\ X \arrow[r, "\ell"'] & Z \end{tikzcd}$$

in D where $\bar{\ell} \in L$ and $\bar{r} \in R$.

Define a 2-category $\mathrm{Span}(D; L, R)$ as follows. The objects of $\mathrm{Span}(D; L, R)$ are the objects of D. Given objects $X_0, X_1 \in D$, the groupoid $\mathrm{Map}_{\mathrm{Span}(D;L,R)}(X_0, X_1)$ has objects diagrams

$$\begin{tikzcd} & Y \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\ X_0 & & X_1 , \end{tikzcd}$$

in D where $\ell \in L$ and $r \in R$, and morphisms isomorphisms of diagrams. Composition is given by pullback of spans: given morphisms $X_0 \to X_1$ and $X_1 \to X_2$ corresponding to spans

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X_0 & & X_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 , \end{tikzcd}$$

the composite morphism $X_0 \to X_2$ in $\mathrm{Span}(D; L, R)$ is defined as the large pullback span

$$\begin{tikzcd}[column sep=small] & & Y \times_{X_1} Z \arrow[dl] \arrow[dr] & & \\ & Y \arrow[dl] \arrow[dr] & & Z \arrow[dl] \arrow[dr] & \\ X_0 & & X_1 & & X_2 . \end{tikzcd}$$

4.1.2 Notation. Let D be a 1-category. We write $\mathrm{all} \coloneqq \mathrm{Mor}(D)$ for the class of all morphisms in D. If D has pullbacks, we write

$$\mathrm{Span}(D) \coloneqq \mathrm{Span}(D; \mathrm{all}, \mathrm{all})$$

for the 2-category of spans of arbitrary morphisms in D.

4.1.3 Example. Let D be a category and R a class of morphisms in D such that the pullback of a morphism in R along an arbitrary morphism of D exists, and the class R is stable under pullback. Then there is a natural faithful functor

$$D^{\mathrm{op}} \to \mathrm{Span}(D; \mathrm{all}, R)$$

given by the identity on objects, and on morphisms by sending a morphism $f \colon X \to Y$ to the span

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, equal] & \\ Y & & X . \end{tikzcd}$$

4.1.4 Example. Write fcov for the class of finite covering maps of manifolds. Note that the pullback of a finite covering map of manifolds along any morphism exists, and the class of finite covering maps is stable under pullback. We write

$$\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}) \coloneqq \mathrm{Span}(\mathbf{Man}; \mathrm{all}, \mathrm{fcov})$$

for the 2-category with objects manifolds and morphisms *correspondences*[1] of manifolds

$$\begin{tikzcd} & N \arrow[dl] \arrow[dr, "f"] & \\ M_0 & & M_1 , \end{tikzcd}$$

where f is a finite covering map.

4.1.5 Example. Write fold for the class of maps that are finite coproducts of fold maps of manifolds, i.e., finite coproducts of fold maps $\nabla \colon M^{\sqcup i} \to M$ from a finite disjoint union of copies of M to M. Note that coproduct decompositions are stable under all pullbacks

[1] The term 'correspondence' is just another name for a span; 'correspondence' seems to be the more common term in geometry.

that exist in the category of manifolds, hence the class fold is stable under pullback. We write

$$\mathrm{Corr}_{\mathrm{fold}}(\mathbf{Man}) \coloneqq \mathrm{Span}(\mathbf{Man}; \mathrm{all}, \mathrm{fold})$$

for the 2-category with objects manifolds and morphisms correspondences of manifolds

$$\begin{tikzcd} & N \arrow[dl] \arrow[dr, "f"] & \\ M_0 & & M_1 \end{tikzcd},$$

where f is a finite coproduct of fold maps.

Note that fold $\subset$ fcov, so that $\mathrm{Corr}_{\mathrm{fold}}(\mathbf{Man})$ defines a subcategory of $\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man})$ that contains all objects.

4.2 Spans and commutative monoids

In this subsection we briefly recall the role that ∞-categories of spans play in encoding E_∞-structures. We begin by introducing the relevant 2-category of spans.

4.2.1 Notation. Write **Fin** for the category of finite sets.

4.2.2 Recollection. Let C be an ∞-category with finite products. A *commutative monoid* or E_∞*-monoid* in C is a functor $M \colon \mathbf{Fin}_* \to C$ such that for every integer $n \geq 0$, the collapse maps $\{1, \dots, n\}_+ \to \{i\}_+$ induce an equivalence

$$M(\{1, \dots, n\}_+) \xrightarrow{\sim} \prod_{i=1}^{n} M(\{i\}_+) .$$

We write $\mathrm{CMon}(C) \subset \mathrm{Fun}(\mathbf{Fin}_*, C)$ for the full subcategory spanned by the commutative monoids.

4.2.3 Observation. The 2-category $\mathrm{Span}(\mathbf{Fin})$ is semiadditive: the direct sum in $\mathrm{Span}(\mathbf{Fin})$ is given by disjoint union of finite sets. See [4, Lemma C.3; 6, Proposition 4.3] for more general results on the semiadditivity of ∞-categories of spans.

4.2.4 Observation. Write inj for the class of injective maps in **Fin** There is an equivalence of categories $\mathbf{Fin}_* \xrightarrow{\sim} \mathrm{Span}(\mathbf{Fin}; \mathrm{inj}, \mathrm{all})$ given by sending $X_+ \mapsto X$ and a morphism $f \colon X_+ \to Y_+$ to the span

$$\begin{tikzcd} & f^{-1}(Y) \arrow[dl, hook'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}.$$

The importance of transfers in E_∞-structures is explained by the following universal property of the 2-category $\mathrm{Span}(\mathbf{Fin})$ of spans of finite sets.

4.2.5 Proposition (Cranch [4, Proposition C.1; 15, §5])**.** *Let C be an ∞-category with finite products. Then the restriction*

$$\mathrm{Fun}(\mathrm{Span}(\mathbf{Fin}), C) \to \mathrm{Fun}(\mathbf{Fin}_*, C)$$

along the inclusion $\mathbf{Fin}_* \to \mathrm{Span}(\mathbf{Fin})$ *restricts to an equivalence between:*

(4.2.5.1) *Functors $M\colon \mathrm{Span}(\mathbf{Fin}) \to C$ that preserve finite products (equivalently, $M|_{\mathbf{Fin}^{\mathrm{op}}}$ preserves finite products).*

(4.2.5.2) *Commutative monoids in C.*

The inverse is given by right Kan extension.

The 2-category $\mathrm{Span}(\mathbf{Fin})$ has a second (related) universal property: $\mathrm{Span}(\mathbf{Fin})$ is the free semiadditive ∞-category generated by a single object.

4.2.6 Proposition (Harpaz [21, Theorem 1.1])**.** *Let C be a semiadditive ∞-category. Then evaluation at $* \in \mathrm{Span}(\mathbf{Fin})$ defines an equivalence*

$$\mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathbf{Fin}), C) \xrightarrow{\sim} C\,.$$

4.3 The Transfer Conjecture after Bachmann–Hoyois

In this subsection we outline work of Bachmann–Hoyois that implies the Transfer Conjecture [4, Appendix C]. The perspective on commutative monoids in D as finite product-preserving functors $\mathrm{Span}(\mathbf{Fin}) \to D$ (Proposition 4.2.5) is fundamental to proving the transfer conjecture.

The first step is to relate finite product-preserving functors $\mathrm{Corr}_{\mathrm{fold}}(\mathbf{Man}) \to D$ to presheaves of commutative monoids on $\mathbf{Man}$. Then we impose the sheaf condition to pass from $\mathrm{Corr}_{\mathrm{fold}}(\mathbf{Man})$ to $\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man})$.

4.3.1 Notation. Write $\Theta\colon \mathbf{Man}^{\mathrm{op}} \times \mathrm{Span}(\mathbf{Fin}) \to \mathrm{Corr}_{\mathrm{fold}}(\mathbf{Man})$ for the functor given on objects by the assignment

$$(M, I) \mapsto M^{\sqcup I}$$

and on morphisms by the assignment

$$(M \to N, I_0 \leftarrow J \to I_1) \quad \mapsto \quad \begin{tikzcd} & M^{\sqcup J} \arrow[dl] \arrow[dr] & \\ N^{\sqcup I_0} & & M^{\sqcup I_1}\,. \end{tikzcd}$$

The functor Θ is the universal functor that preserves finite products in each variable:

4.3.2 Proposition ([4, Proposition C.5])**.** *Let C be an ∞-category with finite products. Then the restriction functor*

$$\Theta^{\star}\colon\ \mathrm{Fun}(\mathrm{Corr}_{\mathrm{fold}}(\mathbf{Man}), C) \to \mathrm{Fun}(\mathbf{Man}^{\mathrm{op}} \times \mathrm{Span}(\mathbf{Fin}), C)$$

restricts to an equivalence

$$\mathrm{Fun}^{\times}(\mathrm{Corr}_{\mathrm{fold}}(\mathbf{Man}), C) \xrightarrow{\sim} \mathrm{Fun}^{\times}(\mathbf{Man}^{\mathrm{op}}, \mathrm{CMon}(C))\,.$$

The inverse is given by right Kan extension along Θ.

Since every finite covering map is locally a fold map, we see:

4.3.3 Proposition ([4, Proposition C.11]). *Let C be an ∞-category with finite products. Then the restriction functor*

$$\mathrm{Fun}(\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}), C) \to \mathrm{Fun}(\mathrm{Corr}_{\mathrm{fold}}(\mathbf{Man}), C)$$

induces an equivalence between the full subcategories of those functors whose restrictions to $\mathbf{Man}^{\mathrm{op}}$ are sheaves. The inverse is given by right Kan extension.

4.3.4 Notation. Write

$$\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}), C) \subset \mathrm{Fun}(\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}), C)$$

for the full subcategory spanned by those functors F whose restrictions to $\mathbf{Man}^{\mathrm{op}}$ are sheaves.

We now arrive at Quillen's Transfer Conjecture:

4.3.5 Corollary (Transfer Conjecture). *Let C be an ∞-category with all limits. Restriction along the inclusion $\mathbf{Man}^{\mathrm{op}} \hookrightarrow \mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man})$ defines an equivalence of ∞-categories*

$$\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}), C) \xrightarrow{\sim} \mathrm{Sh}(\mathbf{Man}; \mathrm{CMon}(C)) .$$

4.3.6 Corollary. *Let C be a presentable ∞-category. Restriction along the inclusion $\mathbf{Man}^{\mathrm{op}} \hookrightarrow \mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man})$ defines an equivalence of ∞-categories*

$$\mathrm{Fun}_{\mathrm{loc},\mathbf{R}}(\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}), C) \xrightarrow{\sim} \mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; \mathrm{CMon}(C)) .$$

Post-composing with the global sections functor $\Gamma_\star$ defines an equivalence

$$\mathrm{Fun}_{\mathrm{loc},\mathbf{R}}(\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}), C) \xrightarrow{\sim} \mathrm{CMon}(C) .$$

4.3.7. Unwinding the definitions we see that restriction along the inclusion

$$\mathrm{Span}(\mathbf{Fin}) \subset \mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man})$$

defines an equivalence

$$\mathrm{Fun}_{\mathrm{loc},\mathbf{R}}(\mathrm{Corr}_{\mathrm{fcov}}(\mathbf{Man}), C) \xrightarrow{\sim} \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbf{Fin}), C) \simeq \mathrm{CMon}(C) .$$

5 Structures in the stable case

In ordinary differential cohomology, we had the Simons–Sullivan 'differential cohomology diagram'

$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& \mathrm{H}^{*-1}(M;\mathbf{R}/\mathbf{Z}) \arrow[rr, "-\beta"] \arrow[dr] & & \mathrm{H}^{*}(M;\mathbf{Z}) \arrow[ur] \arrow[dr] & \\
\mathrm{H}^{*-1}_{\mathrm{dR}}(M) \arrow[ur] \arrow[dr] & & \hat{\mathrm{H}}^{*}(M;\mathbf{Z}) \arrow[ur] \arrow[dr] & & \mathrm{H}^{*}_{\mathrm{dR}}(M) \\
& \frac{\Omega^{*-1}(M)}{\Omega^{*-1}_{c\ell}(M)_{\mathbf{Z}}} \arrow[rr, "d"'] \arrow[ur] & & \Omega^{*}_{c\ell}(M)_{\mathbf{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0 \, ,
\end{tikzcd}
\tag{5.0.1}
$$

which actually characterized ordinary differential cohomology [37, Theorem 1.1]. We want to be able to reproduce an analogue of the differential cohomology diagram for *any* sheaf of spectra on **Man**. To do this, we need to identify how cohomology with coefficients in $\mathbf{R}/\mathbf{Z}$, $\mathbf{Z}$, and $\mathbf{R}$ as well as $\Omega^{*-1}(M)/\Omega^{*-1}_{c\ell}(M)_{\mathbf{Z}}$ and $\Omega^{*}_{c\ell}(M)_{\mathbf{Z}}$ fit into the story.

First we define the sheaf of spectra on **Man** that plays the role of closed forms with integer periods. We use this to prove that the category of sheaves of spectra on **Man** is the *recollement* of the subcategory of $\mathbf{R}$-invariant sheaves and the subcageory of 'purely geometric' sheaves; i.e., those sheaves with trivial global sections (Theorem 5.1.7). We conclude by producing the remainder of the differential cohomology diagram (Proposition 5.2.4 and (5.2.7)).

5.1 The fracture square

The first thing to notice is that for any sheaf E on **Man**, the sheaf $\mathrm{R}_{\mathrm{hi}}(E)$ is $\mathbf{R}$-invariant, and we have a counit morphism $\mathrm{R}_{\mathrm{hi}}(E) \to E$. This naively suggests that $\mathrm{R}_{\mathrm{hi}}(E)$ should play the role of $\mathbf{R}/\mathbf{Z}$.

5.1.1 Definition. Let C be a stable presentable ∞-category. Define a functor

$$\mathrm{Z}\colon \mathrm{Sh}(\mathbf{Man};C) \to \mathrm{Sh}(\mathbf{Man};C)$$

and a ***curvature*** natural transformation $\mathrm{curv}\colon \mathrm{id} \to \mathrm{Z}$ by the cofiber sequence

$$\mathrm{R}_{\mathrm{hi}} \xrightarrow{\varepsilon} \mathrm{id} \xrightarrow{\mathrm{curv}} \mathrm{Z}\,,$$

where $\varepsilon\colon \mathrm{R}_{\mathrm{hi}} \to \mathrm{id}$ is the counit. For a C-valued sheaf E on **Man**, we call $\mathrm{Z}(E)$ the sheaf of ***differential cycles*** associated to E.

5.1.2 Observations.

(5.1.2.1) Since the global sections functor $\Gamma_\star : \mathrm{Sh}(\mathbf{Man}; C) \to C$ is exact and the counit $\mathrm{R}_{\mathrm{hi}}(E) \to E$ induces an equivalence on global sections, we have $\Gamma_\star \circ \mathrm{Z} \simeq 0$. Hence $\mathrm{R}_{\mathrm{hi}} \circ Z \simeq 0$.

(5.1.2.2) Since $\mathrm{R}_{\mathrm{hi}} \mathrm{L}_{\mathrm{hi}} \simeq \mathrm{L}_{\mathrm{hi}}$, for any C-valued sheaf E on $\mathbf{Man}$, the cofiber sequence defining Z gives a cofiber sequence

$$\mathrm{R}_{\mathrm{hi}} \mathrm{L}_{\mathrm{hi}}(E) \xrightarrow{\sim} \mathrm{L}_{\mathrm{hi}}(E) \longrightarrow \mathrm{Z}\,\mathrm{L}_{\mathrm{hi}}(E) .$$

Hence $\mathrm{Z} \circ \mathrm{L}_{\mathrm{hi}} \simeq 0$.

5.1.3 Definition. Let C be a stable presentable ∞-category. A sheaf $E \colon \mathbf{Man}^{\mathrm{op}} \to C$ is ***pure*** if

$$\Gamma_\star(E) = E(*) \simeq 0 .$$

Write

$$\mathrm{Sh}_{\mathrm{pu}}(\mathbf{Man}; C) \subset \mathrm{Sh}(\mathbf{Man}; C)$$

for the full subcategory spanned by the pure sheaves.

5.1.4 Observation. Since the global sections functor $\Gamma_\star$ preserves all limits and colimits, the subcategory of pure sheaves is stable under limits and colimits. Hence $\mathrm{Sh}_{\mathrm{pu}}(\mathbf{Man}; C)$ is presentable and the inclusion $\mathrm{Sh}_{\mathrm{pu}}(\mathbf{Man}; C) \hookrightarrow \mathrm{Sh}(\mathbf{Man}; C)$ admits both a left and a right adjoint. The curvature map $\mathrm{curv} \colon \mathrm{id} \to \mathrm{Z}$ is a unit morphism that exhibits $\mathrm{Z} \colon \mathrm{Sh}(\mathbf{Man}; C) \to \mathrm{Sh}_{\mathrm{pu}}(\mathbf{Man}; C)$ as the left adjoint to the inclusion. Thus we have a chain of adjunctions

$$\mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C) \underset{\mathrm{R}_{\mathrm{hi}}}{\overset{\mathrm{L}_{\mathrm{hi}}}{\leftrightarrows}} \mathrm{Sh}(\mathbf{Man}; C) \overset{\mathrm{Z}}{\rightleftarrows} \mathrm{Sh}_{\mathrm{pu}}(\mathbf{Man}; C) .$$

5.1.5 Remark. Note that the subcategory $\mathrm{Sh}_{\mathrm{pu}}(\mathbf{Man}; C)$ of pure sheaves is the right-orthogonal complement of $\mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C)$ in $\mathrm{Sh}(\mathbf{Man}; C)$; i.e., the full subcategory spanned by those objects E such that for every $\mathbf{R}$-invariant sheaf H we have

$$\mathrm{Map}_{\mathrm{Sh}(\mathbf{Man};C)}(H, E) \simeq * . \tag{5.1.6}$$

To see this, note that since the $\mathbf{R}$-invariant sheaves are exactly the constant sheaves, the condition that the mapping space $\mathrm{Map}_{\mathrm{Sh}(\mathbf{Man};C)}(H, E)$ is contractible for each $\mathbf{R}$-invariant sheaf H is equivalent to the condition that for every object $X \in C$,

$$\mathrm{Map}_{\mathrm{Sh}(\mathbf{Man};C)}(\Gamma^\star(X), E) \simeq \mathrm{Map}_C(X, \Gamma_\star(E))$$

is contractible. That is, $\Gamma_\star(E)$ is a terminal object of C.

The following is now a general fact about stable recollements [HA, §A.8; 7, Lemma 5].

5.1.7 Theorem (fracture square)**.** *Let C be a stable presentable ∞-category. For every sheaf $E \colon \mathbf{Man}^{\mathrm{op}} \to C$, the* ***fracture square***

$$\begin{tikzcd} E \arrow[r, "\mathrm{curv}"] \arrow[d] & \mathrm{Z}(E) \arrow[d] \\ \mathrm{L_{hi}}(E) \arrow[r, "\mathrm{L_{hi}}(\mathrm{curv})"'] & \mathrm{L_{hi}}\,\mathrm{Z}(E) \end{tikzcd} \tag{5.1.8}$$

is a pullback square, where the vertical morphisms are induced by the unit $\mathrm{id} \to \mathrm{L_{hi}}$. *Equivalently,* $\mathrm{Sh}(\mathbf{Man}; C)$ *is the recollement of* $\mathrm{Sh}_{\mathbf{R}}(\mathbf{Man}; C)$ *with* $\mathrm{Sh}_{\mathrm{pu}}(\mathbf{Man}; C)$*. That is,* $\mathrm{Sh}(\mathbf{Man}; C)$ *is the ∞-category of triples*

$$(E_{\mathbf{R}}, E_{\mathrm{pu}}, \phi \colon E_{\mathbf{R}} \to \mathrm{L_{hi}}\, E_{\mathrm{pu}})\,,$$

where $E_{\mathbf{R}}$ is a $\mathbf{R}$*-invariant sheaf, E_{pu} is a pure sheaf, and ϕ is any morphism.*

5.1.9 Recollection. Let C be a pointed ∞-category and

$$\begin{tikzcd} W \arrow[r, "\bar{f}"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \tag{5.1.10}$$

a commutative square in C. Then we have a natural equivalence

$$\mathrm{fib}(W \to X \times_Z Y) \simeq \mathrm{fib}(\mathrm{fib}(\bar{f}) \to \mathrm{fib}(f))\,.$$

In particular, if C is stable and $\mathrm{fib}(\bar{f}) \xrightarrow{\sim} \mathrm{fib}(f)$, then the square (5.1.10) is a pullback square. See [5, §2; 33] for more details.

Proof of Theorem 5.1.7. We give a proof that the fracture square is a pullback. For the fact that $\mathrm{Sh}(\mathbf{Man}; C)$ is a recollement of the $\mathbf{R}$-invariant and pure sheaves, see [7, Lemma 5].

By Recollection 5.1.9, to show that the square (5.1.8) is a pullback, it suffices to show that the fibers of the top and bottom rows of (5.1.8) are equivalent. By definition we have $\mathrm{fib}(\mathrm{curv}) \simeq \mathrm{R_{hi}}(E)$. On the other hand, since $\mathrm{L_{hi}}$ is left exact, we have equivalences

$$\begin{aligned} \mathrm{fib}(\mathrm{L_{hi}}(\mathrm{curv})) &\simeq \mathrm{L_{hi}}(\mathrm{fib}(\mathrm{curv})) \\ &\simeq \mathrm{L_{hi}}(\mathrm{R_{hi}}(E)) \\ &= \mathrm{R_{hi}}(E)\,, \end{aligned}$$

where the last identification follows from the fact that $\mathrm{R_{hi}}(E)$ is $\mathbf{R}$-invariant. □

5.2 The differential cohomology diagram

We now finish constructing the differential cohomology diagram for a general differential cohomology theory.

5.2.1 Definition. Let C be a stable presentable ∞-category. Define a functor

$$\mathrm{A}\colon \mathrm{Sh}(\mathbf{Man};C) \to \mathrm{Sh}(\mathbf{Man};C)$$

by the fiber sequence

$$\mathrm{A} \longrightarrow \mathrm{id} \xrightarrow{\eta} \mathrm{L}_{\mathrm{hi}} ,$$

where $\eta\colon \mathrm{id} \to \mathrm{L}_{\mathrm{hi}}$ is the unit. For a C-valued sheaf E on **Man**, we call $\mathrm{A}(E)$ the sheaf of ***differential deformations*** associated to E.

5.2.2 Observations.

(5.2.2.1) Since L_{hi} is idempotent and exact, we see that $\mathrm{L}_{\mathrm{hi}} \circ \mathrm{A} \simeq 0$.

(5.2.2.2) Since $\mathrm{L}_{\mathrm{hi}}\,\mathrm{R}_{\mathrm{hi}} \simeq \mathrm{R}_{\mathrm{hi}}$, for any C-valued sheaf E on **Man**, the fiber sequence defining A gives a fiber sequence

$$\mathrm{A}\,\mathrm{R}_{\mathrm{hi}}(E) \longrightarrow \mathrm{R}_{\mathrm{hi}}(E) \xrightarrow{\sim} \mathrm{L}_{\mathrm{hi}}\,\mathrm{R}_{\mathrm{hi}}(E) ,$$

hence $\mathrm{A} \circ \mathrm{R}_{\mathrm{hi}} \simeq 0$.

5.2.3 Notation. We write $\mathrm{d}\colon \mathrm{A} \to \mathrm{Z}$ for the composite

$$\mathrm{d}\colon \mathrm{A} \longrightarrow \mathrm{id} \xrightarrow{\mathrm{curv}} \mathrm{Z} .$$

5.2.4 Proposition. *Let C be a stable presentable ∞-category. Then there is a commutative diagram*

(5.2.5)
$$\begin{tikzcd}
\Sigma^{-1}\mathrm{L}_{\mathrm{hi}}\,\mathrm{Z} \arrow[r] \arrow[d] & \mathrm{A} \arrow[r, "\mathrm{d}"] \arrow[d] & \mathrm{Z} \arrow[d, equal] \\
\mathrm{R}_{\mathrm{hi}} \arrow[r, "\varepsilon"] \arrow[d] & \mathrm{id} \arrow[r, "\mathrm{curv}"'] \arrow[d, "\eta"] & \mathrm{Z} \\
\mathrm{L}_{\mathrm{hi}} \arrow[r, equal] & \mathrm{L}_{\mathrm{hi}} &
\end{tikzcd} ,$$

where the rows and columns are fiber sequences and the upper left square is a pullback.

Proof. The right vertical and bottom horizontal sequences are fiber sequences by definition. To see that the left vertical sequence is a fiber sequence, first note that L_{hi} is exact. Because R_{hi} already lands in $\mathbf{R}$-invariant sheaves, we see that $\mathrm{L}_{\mathrm{hi}}\,\mathrm{R}_{\mathrm{hi}} = \mathrm{R}_{\mathrm{hi}}$. Thus applying L_{hi} to the fiber sequence

$$\mathrm{R}_{\mathrm{hi}} \xrightarrow{\varepsilon} \mathrm{id} \xrightarrow{\mathrm{curv}} \mathrm{Z}$$

gives a fiber sequence

(5.2.6)
$$\mathrm{R}_{\mathrm{hi}} \xrightarrow{\varepsilon} \mathrm{L}_{\mathrm{hi}} \xrightarrow{\mathrm{L}_{\mathrm{hi}}(\mathrm{curv})} \mathrm{L}_{\mathrm{hi}}\,\mathrm{Z} .$$

We conclude by rotating the fiber sequence.

Now we show that the upper left square of (5.2.5) is a pullback, and use this to deduce that the top horizontal squence is a fiber sequence. By Recollection 5.1.9, to see that the upper left square is a pullback it suffices to show that the vertical fibers are equivalent. This follows from the fact that the vertical sequences in (5.2.5) are fiber sequences (so the vertical fibers are given by $\Sigma^{-1}\,\mathrm{L}_{\mathrm{hi}}$). To see that the top sequence is a fiber sequence, note that $\mathrm{fib}(\mathrm{d})$ can be computed as the iterated pullback

$$\begin{tikzcd}
\Sigma^{-1}\,\mathrm{L}_{\mathrm{hi}}\,\mathrm{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{A} \arrow[d] \\
\mathrm{R}_{\mathrm{hi}} \arrow[r, "\varepsilon"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{id} \arrow[d, "\mathrm{curv}"] \\
0 \arrow[r] & \mathrm{Z}\,.
\end{tikzcd}$$

$\square$

5.2.7. Rearranging the diagram (5.2.5), for each $E \in \mathrm{Sh}(\mathbf{Man}; C)$ we get the following 'differential cohomology diagram'

$$\begin{tikzcd}
 & \mathrm{R}_{\mathrm{hi}}(E) \arrow[rr] \arrow[dr] & & \mathrm{L}_{\mathrm{hi}}(E) \arrow[dr] & \\
\Sigma^{-1}\,\mathrm{L}_{\mathrm{hi}}\,\mathrm{Z}(E) \arrow[ur] \arrow[dr] & & E \arrow[ur] \arrow[dr, "\mathrm{curv}"] & & \mathrm{L}_{\mathrm{hi}}\,\mathrm{Z}(E) \\
 & \mathrm{A}(E) \arrow[rr, "\mathrm{d}"'] \arrow[ur] & & \mathrm{Z}(E) \arrow[ur] &
\end{tikzcd} ,$$

where the diagonals are fiber sequences and the top and bottom rows are extensions of fiber sequences by one term.

5.3 Differential refinements

5.3.1 Definition. Let C be a presentable stable ∞-category. A ***differential refinement*** of a an object $E \in C$ is pair (E, ϕ) of a sheaf $E \in \mathrm{Sh}(\mathbf{Man}; C)$ together with an equivalence $\phi\colon \Gamma_!(E) \simeq E$ in C.

5.3.2. From the fracture square (Theorem 5.1.7), a differential refinement of $E \in C$ is equivalently the data of a pure sheaf $\hat{P} \in \mathrm{Sh}_{\mathrm{pu}}(\mathbf{Man}; C)$ along with a morphism $E \to \Gamma_!(\hat{P})$ in C. Given this data, we can construct a differential refinement E in the sense of

Definition 5.3.1 as the pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{P} \arrow[d] \\ \Gamma^{\star}(E) \arrow[r] & \Gamma^{\star}\Gamma_{!}(\hat{P}) . \end{tikzcd}$$

In this case, we have:

(5.3.2.1) $\mathrm{A}(E) \xrightarrow{\sim} \mathrm{A}(\hat{P})$.

(5.3.2.2) $\mathrm{Z}(E) \xrightarrow{\sim} \hat{P}$.

(5.3.2.3) $\Gamma_{\star}(E)$ fits into a fiber sequence

$$\Gamma_{\star}(E) \to E \to \Gamma_{!}(\hat{P}) .$$

5.3.3 Construction (pullback of a differential refinement)**.** Let C be a presentable stable ∞-category, $f : E \to E'$ a morphism in C, and (E', ϕ') a differential refinement of E'. Form the pullback

(5.3.4)
$$\begin{tikzcd} E \arrow[r, "\hat{f}"] \arrow[d] \arrow[dr, phantom, "\square"] & E' \arrow[d] \\ \Gamma^{\star}(E) \arrow[r, "\Gamma^{\star}(f)"'] & \Gamma^{\star}(E') , \end{tikzcd}$$

where the morphism $E' \to \Gamma^{\star}(E')$ is adjoint to the given equivalence $\phi' : \Gamma_{!}(E') \xrightarrow{\sim} E'$. Since $\Gamma_{!}$ is exact, applying $\Gamma_{!}$ to the square (5.3.5) gives a pullback square

(5.3.5)
$$\begin{tikzcd} \Gamma_{!}(E) \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\square"] & \Gamma_{!}(E') \arrow[d, "\phi'"', "\wr"] \\ E \arrow[r, "f"'] & E' , \end{tikzcd}$$

which provides an equivalence $\phi \colon \Gamma_{!}(E) \xrightarrow{\sim} E$. The ***pullback differential refinement*** of (E', ϕ') along f is the differential refinement (E, ϕ) of E.

5.3.6 Lemma. *In the notation of Construction 5.3.3, the following*

(5.3.6.1) *The morphism* $\mathrm{A}(\hat{f}) \colon \mathrm{A}(E) \to \mathrm{A}(E')$ *is an equivalence.*

(5.3.6.2) *The morphism* $\mathrm{Z}(\hat{f}) \colon \mathrm{Z}(E) \to \mathrm{Z}(E')$ *is an equivalence.*

(5.3.6.3) *The global sections of E is given by the pullback*

$$\begin{tikzcd}
\Gamma_\star(E) \arrow[r, "\Gamma_\star(\hat{f})"] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma_\star(E') \arrow[d] \\
E \arrow[r, "f"'] & E' .
\end{tikzcd}$$

A Technical details from topos theory

The purpose of this appendix is to prove a number of technical results used throughout the text. We have relegated these proofs to this appendix either because they are lengthy and, while the result is important, the proof is not important to know, or because they require some knowlege from the theory of higher topoi. Since we are mostly interested in sheaves of spaces in this appendix, we adopt the following notational convention.

A.0.1 Notation. We write $\mathrm{Sh}(\mathbf{Man}) \coloneqq \mathrm{Sh}(\mathbf{Man}; \mathbf{Spc})$ for the ∞-topos of sheaves of spaces on $\mathbf{Man}$.

A.0.2 Remark. For this appendix, it is sufficient to know that the ∞-category of sheaves of spaces on a site is an ∞-topos, and that a ***geometric morphism*** of ∞-topoi is a right adjoint functor $f_\star \colon \boldsymbol{X} \to \boldsymbol{Y}$ whose left adjoint $f^\star$ is left exact.

A.1 From sheaves of spaces to C-valued sheaves

Let C be a presentable ∞-category. In this section we explain a formal procedure that allows us to pass from the ∞-category $\mathrm{Sh}(\mathbf{Man}; \mathbf{Spc})$ of sheaves of spaces on $\mathbf{Man}$ to the ∞-category $\mathrm{Sh}(\mathbf{Man}; C)$ of C-valued sheaves on $\mathbf{Man}$. We'll also recall the basics of tensor products of presentable ∞-categories and explain how to describe $\mathrm{Sh}(\mathbf{Man}; C)$ as the tensor product $\mathrm{Sh}(\mathbf{Man}; \mathbf{Spc}) \otimes C$.

The first thing to observe is that if $G \colon \mathrm{Sh}(\mathbf{Man}; \mathbf{Spc})^{\mathrm{op}} \to C$ is a functor that preserves limits, then the restriction $G \colon \mathbf{Man}^{\mathrm{op}} \to C$ is a sheaf. It turns out that all C-valued sheaves arise in this way.

A.1.1 Proposition ([SAG, Proposition 1.3.1.7])**.** *Let (S, τ) be an ∞-site and C an ∞-category with all limits. Write $よ \colon S \to \mathrm{Sh}_\tau(S; \mathbf{Spc})$ for the sheafification of the Yoneda embedding. Then pre-composition with $よ$ defines an equivalence*

$$\mathrm{Fun}^{\lim}(\mathrm{Sh}_\tau(S; \mathbf{Spc})^{\mathrm{op}}, C) \xrightarrow{\sim} \mathrm{Sh}_\tau(S; C) .$$

Now we give the ∞-category $\mathrm{Fun}^{\lim}(\mathrm{Sh}(\mathbf{Man})^{\mathrm{op}}, C)$ a description in terms of a universal property of presentable ∞-categories.

A.1.2 Recollection ([HA, Proposition 4.8.1.17])**.** Let C and D be presentable ∞-categories. The *tensor product* of presentable ∞-categories $C \otimes D$ along with the functor $\otimes \colon C \times D \to C \otimes D$ are characterized by the following universal property: for any presentable ∞-category E, restriction along $\otimes$ defines an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(C \otimes D, E) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L,L}}(C \times D, E) ,$$

where $\mathrm{Fun}^{\mathrm{L,L}}(C \times D, E) \subset \mathrm{Fun}(C \times D, E)$ is the full subcategory spanned by those functors $C \times D \to E$ that preserve colimits separately in each variable. The tensor product of presentable ∞-categories defines a functor

$$\otimes \colon \mathbf{Pr}^{\mathrm{L}} \times \mathbf{Pr}^{\mathrm{L}} \to \mathbf{Pr}^{\mathrm{L}}$$

and can be used to equip $\mathbf{Pr}^{\mathrm{L}}$ with the structure of a symmetric monoidal ∞-category.

The tensor product $C \otimes D$ admits the following useful asymmetric description:

$$C \otimes D \simeq \mathrm{Fun}^{\lim}(C^{\mathrm{op}}, D) .$$

If $F \colon D \to D'$ is a right adjoint functor of presentable ∞-categories, then the induced right adjoint

$$\mathrm{id}_C \otimes F \colon C \otimes D \simeq \mathrm{Fun}^{\lim}(C^{\mathrm{op}}, D) \to \mathrm{Fun}^{\lim}(C^{\mathrm{op}}, D') \simeq C \otimes D'$$

is given by post-composition with F. Unfortunately, the left adjoint to $\mathrm{id}_C \otimes F$ does not generally admit a simple description. However, if C is compactly generated and the left adjoint to F is left exact, then the left adjoint to $\mathrm{id}_C \otimes F$ admits a simple description (see Observation A.6.3).

A.1.3 Example. For any presentable ∞-category C, we have a natural equivalence

$$\mathrm{Sh}(\mathbf{Man}) \otimes C \xrightarrow{\sim} \mathrm{Sh}(\mathbf{Man}; C) .$$

A.2 Restriction to a manifold

We now give an alternative description of the functor $\mathrm{Sh}(\mathbf{Man}; C) \to C$ that sends a sheaf to its value on a manifold M.

A.2.1 Notation. Let T be a topological space and C a presentable ∞-category. Write

$$\mathrm{PSh}(T; C) \coloneqq \mathrm{Fun}(\mathrm{Open}(T)^{\mathrm{op}}, C)$$

and write $\mathrm{Sh}(T; C) \subset \mathrm{PSh}(T; C)$ for the ∞-category of C-valued sheaves on T. Write

$$\Gamma_{T,\star} \colon \mathrm{Sh}(T; C) \to C$$

for the *global sections* functor, defined by $\Gamma_{T,\star}(F) \coloneqq F(T)$, and write $\Gamma_T^\star \colon C \to \mathrm{Sh}(T; C)$ for the left adjoint to $\Gamma_{T,\star}$, i.e., the *constant sheaf* functor.

A.2.2 Observation. Let C be a presentable ∞-category and M a manifold. The forgetful functor $\mathrm{Open}(M) \to \mathbf{Man}$ induces a restriction functor

$$(-)|_M \colon \mathrm{PSh}(\mathbf{Man}; C) \to \mathrm{PSh}(M; C) .$$

By the definition of the Grothendieck topology on $\mathbf{Man}$, the functor $(-)|_M$ sends sheaves on $\mathbf{Man}$ to sheaves on M. Note that the functor given by sending a sheaf E on $\mathbf{Man}$ to its value on M is given by the composite

$$\mathrm{Sh}(\mathbf{Man}; C) \xrightarrow{(-)|_M} \mathrm{Sh}(M; C) \xrightarrow{\Gamma_{M,\star}} C .$$

Moreover, if $p \colon N \to M$ is a morphism in $\mathbf{Man}$, then there is a canonical natural transformation fitting into the triangle

$$\begin{tikzcd}
\mathrm{Sh}(\mathbf{Man}; C) \arrow[rr, "(-)|_M"] \arrow[dr, "(-)|_N"'] & \arrow[d, phantom, "\Uparrow \mathrm{can}_p"] & \mathrm{Sh}(M; C) \\
& \mathrm{Sh}(N; C) \arrow[ur, "p_\star"'] &
\end{tikzcd}$$

defined as follows: given a sheaf E on **Man** and an open subset $U \subset M$, the morphism $E(U) \to E(p^{-1}(U))$ is induced by the projection $p^{-1}(U) \to U$ by the functoriality of E. In particular, upon taking global sections, the morphism

$$\mathrm{can}_p \colon E(M) = \Gamma_{M,\star}(E|_M) \to \Gamma_{M,\star}(p_\star(E|_N)) = E(N)$$

is the morphism $E(M) \to E(N)$ induced by p by the functoriality of E.

A.3 Sheafification

Next we show that restriction from $\mathrm{Sh}(\mathbf{Man}; C)$ to $\mathrm{Sh}(M; C)$ commutes with sheafification. While this is intuitively clear, a bit of work is required. We first recall a construction of the sheafification functor in the higher-categorical setting.

A.3.1 Recollection (sheafification in higher category theory)**.** Let (S, τ) be an ∞-site (e.g., $S = \mathbf{Man}$), C a presentable ∞-category, and $F \colon S^{\mathrm{op}} \to C$ be a C-valued presheaf on S. Write $F^\dagger \colon S^{\mathrm{op}} \to C$ for the C-valued presheaf given by the assignment

$$F^\dagger(V) \coloneqq \operatorname*{colim}_{I \in \mathrm{Cov}_\tau(V)} \lim_{U \in I^{\mathrm{op}}} F(U) ,$$

where $\mathrm{Cov}_\tau(V)$ denotes the category of τ-covering sieves of V (see [HTT, Construction 6.2.2.9]). The proof of [HTT, Proposition 6.2.2.7] shows that the sheafification of F can be obtained as a transfinite iteration of the construction $F \mapsto F^\dagger$ (see the discussion at the beginning of [HTT, §6.5.3]).

A.3.2 Remark. If C is a 1-category, then for any presheaf $F \colon S^{\mathrm{op}} \to C$, the presheaf $F^\dagger$ is separated, and if F is separated, then $F^\dagger$ is a sheaf. Moreover, the sheafification of F is equivalent to the sheaf $F^{\dagger\dagger}$ obtained by applying the $\dagger$-construction twice. This is the 'usual' construction of sheafification in ordinary category theory.

A.3.3 Lemma. *For any presentable ∞-category C and manifold M, the square*

$$\begin{tikzcd} \mathrm{PSh}(\mathbf{Man}; C) \arrow[r, "(-)|_M"] \arrow[d, "\mathrm{S}_{\mathbf{Man}}"'] & \mathrm{PSh}(M; C) \arrow[d, "\mathrm{S}_M"] \\ \mathrm{Sh}(\mathbf{Man}; C) \arrow[r, "(-)|_M"'] & \mathrm{Sh}(M; C) , \end{tikzcd}$$

commutes, where the vertical functors are given by sheafification.

Proof. Since the square

$$\begin{tikzcd} \mathrm{PSh}(\mathbf{Man}; C) \arrow[r, "(-)|_M"] & \mathrm{PSh}(M; C) \\ \mathrm{Sh}(\mathbf{Man}; C) \arrow[r, "(-)|_M"'] \arrow[u, hook] & \mathrm{Sh}(M; C) \arrow[u, hook] \end{tikzcd}$$

commutes, we have a *basechange transformation*

$$\mathrm{S}_M \circ (-)|_M \to (-)|_M \circ \mathrm{S}_{\mathbf{Man}}$$

which we wish to show is an equivalence (see [HA, Definition 4.7.4.13]). In light of Recollection A.3.1, it suffices to show that if $E\colon \mathbf{Man}^{\mathrm{op}} \to C$ is a presheaf, then we have a natural identification

$$(E^\dagger)|_M = (E|_M)^\dagger ,$$

where on the left-hand side, the † denotes the †-construction for **Man**, and on the right-hand side, the † denotes the †-construction for $\mathrm{Open}(M)$. Now note that by the definition of the Grothendieck topology on **Man**, both $(E^\dagger)|_M$ and $(E|_M)^\dagger$ are given by the assignment

$$V \mapsto \operatorname*{colim}_{I \in \mathrm{Cov}(V)} \lim_{U \in I^{\mathrm{op}}} E(U) ,$$

where $\mathrm{Cov}(V)$ denotes the category of covering sieves of the topological space V. □

A.3.4 Corollary. *Let C be a presentable ∞-category, $X \in C$, and M a manifold. Then we have a natural identification $\Gamma^\star(X)|_M = \Gamma^\star_M(X)$ of the restriction of $\Gamma^\star(X)$ to M with the constant sheaf on M at X.*

A.4 Constant sheaves are R-invariant

In this subsection, we prove that constant sheaves on **Man** are **R**-invariant. In light of Corollary A.3.4, this easily follows from the following lemma due to Lurie.

A.4.1 Notation. Let T be a topological space. We write $\mathrm{pr}\colon T \times \mathbf{R} \to T$ for the projection onto the first factor.

A.4.2 Lemma ([HA, Lemma A.2.9])**.** *Let C be a presentable ∞-category and T a topological space. Then the pullback functor*

$$\mathrm{pr}^\star\colon \mathrm{Sh}(T;C) \to \mathrm{Sh}(T \times \mathbf{R};C)$$

is fully faithful and admits a left adjoint $\mathrm{pr}_!\colon \mathrm{Sh}(T \times \mathbf{R};C) \to \mathrm{Sh}(T;C)$.

Since sheafification commutes with restriction to a particular manifold, Lemma A.4.2 easily implies that constant sheaves are **R**-invariant:

A.4.3 Lemma (Lemma 2.0.8)**.** *For any presentable ∞-category C, the constant sheaf functor*

$$\Gamma^\star\colon C \to \mathrm{Sh}(\mathbf{Man};C)$$

factors through the full subcategory $\mathrm{Sh}_{\mathbf{R}}(\mathbf{Man};C) \subset \mathrm{Sh}(\mathbf{Man};C)$.

Proof. In light of Observation A.2.2, it suffices to show that for any manifold M and object $X \in C$, the morphism

$$\mathrm{can}_{\mathrm{pr}}\colon \Gamma_{M,\star}(\Gamma^\star(X)|_M) \to \Gamma_{M\times\mathbf{R},\star}(\Gamma^\star(X)|_{M\times\mathbf{R}})$$

is an equivalence in C. By Corollary A.3.4 we have natural identifications

$$\Gamma^\star(X)|_M = \Gamma^\star_M(X) \qquad \text{and} \qquad \Gamma^\star(X)|_{M\times\mathbf{R}} = \Gamma^\star_{M\times\mathbf{R}}(X) \simeq \mathrm{pr}^\star\,\Gamma^\star_M(X) .$$

Under these identifications, the morphism

$$\mathrm{can}_{\mathrm{pr}}\colon \Gamma_{M,\star}(\Gamma^\star_M(X)) \to \Gamma_{M\times\mathbf{R},\star}(\mathrm{pr}^\star\,\Gamma^\star_M(X)) \simeq \Gamma_{M,\star}(\mathrm{pr}_\star\,\mathrm{pr}^\star\,\Gamma^\star_M(X))$$

is induced by the unit $\mathrm{id}_{\mathrm{Sh}(M;C)} \to \mathrm{pr}_\star\,\mathrm{pr}^\star$. Applying Lemma A.4.2 concludes the proof. □

A.5 Background on notions of completeness for higher topoi

There are three notions of 'completeness' for an ∞-topos $\boldsymbol{X}$:

(1) *Hypercompleteness:* Whitehead's Theorem holds in $\boldsymbol{X}$.

(2) *Convergence of Postnikov towers:* Every object of $\boldsymbol{X}$ is the limit of its Postnikov tower.

(3) *Postnikov completeness:* $\boldsymbol{X}$ can be recovered as the limit $\lim_n \boldsymbol{X}_{\leq n}$ of its subcategories $\boldsymbol{X}_{\leq n} \subset \boldsymbol{X}$ of n-truncated objects along the truncation functors $\tau_{\leq n} \colon \boldsymbol{X}_{\leq n+1} \to \boldsymbol{X}_{\leq n}$.

While all of these properties hold for the ∞-topos **Spc** of spaces, they need not hold for a general ∞-topos. We have implications (3)⇒(2)⇒(1), and none of the implications are reversible in general. In this subsection we give a brief overview of hypercompletness as it plays a role in relating the Freed–Hopkins approach to differential cohomology from [18] to the ∞-categorical approach we have taken here. Detailed accounts of hypercompleteness and Postnikov completeness can be found in [HTT, §6.5] and [SAG, §A.7], respectively.

A.5.1 Definition. Let $\boldsymbol{X}$ be an ∞-topos. A morphism f in $\boldsymbol{X}$ is ∞-***connective*** if for every integer $n \geq -2$ the n-truncation $\tau_{\leq n}(f)$ of f is an equivalence.

A.5.2 Definition. Let $\boldsymbol{X}$ be an ∞-topos. An object $U \in \boldsymbol{X}$ is ***hypercomplete*** if U is local with respect to the class of ∞-connective morphisms in $\boldsymbol{X}$. We write $\boldsymbol{X}^{\mathrm{hyp}} \subset \boldsymbol{X}$ for the full subcategory spanned by the hypercomplete objects of $\boldsymbol{X}$. An ∞-topos is ***hypercomplete*** if $\boldsymbol{X}^{\mathrm{hyp}} = \boldsymbol{X}$.

A.5.3. The ∞-category $\boldsymbol{X}^{\mathrm{hyp}} \subset \boldsymbol{X}$ is a left exact localization of $\boldsymbol{X}$, hence an ∞-topos [HTT, p. 699]. Moreover, the ∞-topos $\boldsymbol{X}^{\mathrm{hyp}}$ is hypercomplete [HTT, Lemma 6.5.2.12].

A.5.4. The ∞-topos $\boldsymbol{X}^{\mathrm{hyp}}$ is the universal hypercomplete ∞-topos equipped with a geometric morphism to $\boldsymbol{X}$ [HTT, Proposition 6.5.2.13]. For this reason we call $\boldsymbol{X}^{\mathrm{hyp}}$ the ***hypercompletion*** of $\boldsymbol{X}$.

A.5.5 Observation. Let $\boldsymbol{X}$ be an ∞-topos. Then $\boldsymbol{X}$ is hypercomplete if and only if the pullback functor $p^* \colon \boldsymbol{X} \to \boldsymbol{X}^{\mathrm{post}}$ is conservative.

The standard way of working with sheaves of spaces on a site (S, τ) in the language of model-categories is to use the Brown–Joyal–Jardine model structure on simplicial presheaves [11; 24]. However, this model structure only presents the hypercompletion of the ∞-topos of sheaves of spaces on (S, τ).

A.5.6 Proposition ([HTT, Proposition 6.5.2.14])**.** *Let (S, τ) be a site. Then the underlying ∞-category of the category of simplicial presheaves on S in the Brown–Joyal–Jardine model structure is equivalent to the ∞-topos $\mathrm{Sh}_\tau(S; \mathbf{Spc})^{\mathrm{hyp}}$ of hypercomplete sheaves of spaces on S.*

A.5.7 Definition. Let $\boldsymbol{X}$ be an ∞-topos. A ***point*** of $\boldsymbol{X}$ is a left exact left adjoint functor $x^* \colon \boldsymbol{X} \to \mathbf{Spc}$. Given an object $U \in \boldsymbol{X}$ and point x^* of $\boldsymbol{X}$, we call $x^*(U)$ the ***stalk*** of U at x^*.

A.5.8 Example. Let T be a topological space and $t \in T$. Then the stalk functor

$$(-)_t \colon \mathrm{Sh}(T) \to \mathbf{Spc}$$

defines a point of $\mathrm{Sh}(T)$.

A.5.9 Definition. An ∞-topos $\boldsymbol{X}$ ***has enough points*** if a morphism f in $\boldsymbol{X}$ is an equivalence if and only if for every point $x^\star$ of $\boldsymbol{X}$, the stalk $x^\star(f)$ is an equivalence in $\mathbf{Spc}$.

A.5.10 Example. An ∞-topos with enough points is hypercomplete.

A.5.11 Remark. The existence of enough points is incomparable with the convergence of Postnikov towers and is also incomparable with Postnikov completeness (both of which imply hypercompleteness).

A.5.12 Example. Let M be a manifold. Then the ∞-topos $\mathrm{Sh}(M)$ is Postnikov complete [HTT, Proposition 7.2.1.10 & Theorem 7.2.3.6].

A.6 Tensor products with compactly generated ∞-categories

Our next goal is to provide a conservative family of points for the ∞-topos $\mathrm{Sh}(\mathbf{Man})$. Since we're also interested in working with sheaves of spectra on $\mathbf{Man}$ (i.e., differential cohomology theories) and sheaves on $\mathbf{Man}$ with values in the derived ∞-category of a ring, we'd like to know that we can check equivalences in these ∞-categories on stalks. These ∞-categories are obtained with taking a tensor product of presentable ∞-categories with the ∞-category $\mathrm{Sh}(\mathbf{Man}; \mathbf{Spc})$; one might try to deduce the desired conservativity result formally from the claim for $\mathrm{Sh}(\mathbf{Man}; \mathbf{Spc})$. However, if C is a presentable ∞-category and $f^\star \colon D' \to D$ is a conservative left adjoint between presentable ∞-categories, the functor $\mathrm{id}_C \otimes f^\star \colon C \otimes D' \to C \otimes D$ is not generally conservative. In this section we show that $\mathrm{id}_C \otimes f^\star$ is conservative provided that C is *compactly generated* and $f^\star$ is left exact (Lemma A.6.4). In Section A.7 we provide a conservative family of points for the ∞-topos $\mathrm{Sh}(\mathbf{Man})$.

A.6.1 Notation. Let C be an ∞-category. We write $C^\omega \subset C$ for the full subcategory spanned by the compact objects.

Recall that if C is compactly generated, then $C^\omega \subset C$ is closed under finite colimits and $C \simeq \mathrm{Ind}(C^\omega)$.

A.6.2 Observation. Let C be a *compactly generated* ∞-category and D a presentable ∞-category. Then restriction along the inclusion $C^{\omega,\mathrm{op}} \hookrightarrow C^{\mathrm{op}}$ defines an equivalence of ∞-categories

$$\mathrm{Fun}^{\lim}(C^{\mathrm{op}}, D) \simeq \mathrm{Fun}^{\lim}(\mathrm{Ind}(C^\omega)^{\mathrm{op}}, D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}(C^{\omega,\mathrm{op}}, D) .$$

A.6.3 Observation. Let C be a presentable ∞-category and $f^\star \colon D' \to D$ be a *left exact* left adjoint between presentable ∞-categories with right adjoint $f_\star$. The right adjoint to the $\mathrm{id}_C \otimes f^\star$ is identified with the functor

$$\mathrm{Fun}^{\lim}(C^{\mathrm{op}}, D) \to \mathrm{Fun}^{\lim}(C^{\mathrm{op}}, D')$$

given by post-composition with $f_\star$. Note that we have a commutative square of ∞-categories

$$\begin{tikzcd} \mathrm{Fun}^{\lim}(C^{\mathrm{op}}, D) \arrow[r, "\sim"] \arrow[d, "f_\star\circ-"'] & \mathrm{Fun}^{\mathrm{lex}}(C^{\omega,\mathrm{op}}, D) \arrow[d, "f_\star\circ-"] \\ \mathrm{Fun}^{\lim}(C^{\mathrm{op}}, D') \arrow[r, "\sim"] & \mathrm{Fun}^{\mathrm{lex}}(C^{\omega,\mathrm{op}}, D') . \end{tikzcd}$$

Moreover, since $f^\star$ is left exact, the functor

$$f^\star \circ - \colon \mathrm{Fun}^{\mathrm{lex}}(C^{\omega,\mathrm{op}}, D') \to \mathrm{Fun}^{\mathrm{lex}}(C^{\omega,\mathrm{op}}, D)$$

given by post-composition with $f^\star$ is left adjoint to the functor given by post-composition with $f_\star$. Hence we have a commutative square of ∞-categories

$$\begin{tikzcd} C \otimes D' \arrow[r, "\sim"] \arrow[d, "\mathrm{id}_C \otimes f^\star"'] & \mathrm{Fun}^{\mathrm{lex}}(C^{\omega,\mathrm{op}}, D') \arrow[d, "f^\star\circ-"] \\ C \otimes D \arrow[r, "\sim"] & \mathrm{Fun}^{\mathrm{lex}}(C^{\omega,\mathrm{op}}, D) . \end{tikzcd}$$

A.6.4 Lemma. *Let $\{f_i^\star \colon X \to D_i\}_{i\in I}$ be a collection of left exact left adjoint functors between presentable ∞-categories. Assume that the functors $\{f_i^\star\}_{i\in I}$ are jointly conservative. Then for any compactly generated ∞-category C, the family*

$$\{\mathrm{id}_C \otimes f_i^\star \colon C \otimes X \to C \otimes D_i\}_{i\in I}$$

is jointly conservative.

Proof. In light of Observation A.6.3, it suffices to show that the collection of functors

$$\{f_i^\star \circ - \colon \mathrm{Fun}^{\mathrm{lex}}(C^{\omega,\mathrm{op}}, X) \to \mathrm{Fun}^{\mathrm{lex}}(C^{\omega,\mathrm{op}}, D_i)\}_{i\in I}$$

is jointly conservative. This is immediate from the assumption that the functors $\{f_i^\star\}_{i\in I}$ are jointly conservative. □

A.7 A conservative family of points

In this section we show the stalks at the origins in $\mathbf{R}^n$ for $n \geq 0$ form a conservative family of points for the ∞-topos $\mathrm{Sh}(\mathbf{Man})$ (Proposition A.7.4). This implies that the model structure on simplicial presheaves on **Man** considered by Freed–Hopkins in [18, §5] presents the ∞-topos $\mathrm{Sh}(\mathbf{Man})$. We also present an observation of Hoyois that shows that the ∞-topos $\mathrm{Sh}(\mathbf{Man})$ is Postnikov complete (Proposition A.7.5).

We begin by discussing the stalk of a sheaf on **Man** at a point of a manifold.

A.7.1 Observation. Let M be a manifold. Note that the presheaf restriction

$$(-)|_M \colon \mathrm{PSh}(\mathbf{Man}; C) \to \mathrm{PSh}(M; C)$$

is both a left and right adjoint. Since limits of sheaves are computed pointwise and the presheaf restriction preserves sheaves, the restriction $(-)|_M \colon \mathrm{Sh}(\mathbf{Man}; C) \to \mathrm{Sh}(M; C)$

preserves all limits. Since restriction to M commutes with sheafification (Lemma A.3.3) and colimits of sheaves are obtained by sheafifying colimits of presheaves, we see that $(-)|_M \colon \mathrm{Sh}(\mathbf{Man}; C) \to \mathrm{Sh}(M; C)$ commutes with colimits. The Adjoint Functor Theorem implies that restriction of sheaves on **Man** to M admits both a left and a right adjoint.

A.7.2 Construction. Let M be a manifold and $x \in M$. In light of Observation A.7.1, the composition of the restriction to M with the stalk at x defines a left exact left adjoint

$$\mathrm{Sh}(\mathbf{Man}; C) \xrightarrow{(-)|_M} \mathrm{Sh}(M; C) \xrightarrow{(-)_x} C \,,$$

which we denote by $x^\star$. Given a sheaf E on **Man**, we call $x^\star(E)$ the ***stalk*** of E at $x \in M$.

A.7.3 Observation. Let M be a manifold and $j \colon U \hookrightarrow M$ an open embedding. Then, by definition, the triangle

$$\begin{tikzcd}
\mathrm{Sh}(\mathbf{Man}; C) \arrow[r, "(-)|_M"] \arrow[dr, "(-)|_U"'] & \mathrm{Sh}(M; C) \arrow[d, "j^\star"] \\
 & \mathrm{Sh}(U; C)
\end{tikzcd}$$

commutes. Thus for any $x \in U$, then there is a canonical identification of the stalk functor $\mathrm{Sh}(\mathbf{Man}; C) \to C$ at $x \in U$ with the stalk functor at $j(x) \in M$.

Recall that for each integer $n \geq 0$, write $0_n \in \mathbf{R}^n$ for the origin (Notation 1.1.3).

A.7.4 Proposition. *Let C be a compactly generated ∞-category. Then the set of stalk functors $\{0_n^\star \colon \mathrm{Sh}(\mathbf{Man}; C) \to C\}_{n\geq 0}$ is jointly conservative. In particular, the ∞-topos* $\mathrm{Sh}(\mathbf{Man})$ *is hypercomplete.*

Proof. In light if Lemma A.6.4, it suffices to treat the case $C = \mathbf{Spc}$. In this case, first note that the family of restriction functors

$$(-)|_M \colon \mathrm{Sh}(\mathbf{Man}) \to \mathbf{Spc}$$

for $M \in \mathbf{Man}$ is conservative (Observation A.2.2). For each manifold M, the ∞-topos $\mathrm{Sh}(M)$ is a hypercomplete ∞-topos and the points of M provide conservative family of points for $\mathrm{Sh}(M)$ [HTT, Corollary 7.2.1.17]. Thus the stalk functors

$$x^\star \colon \mathrm{Sh}(\mathbf{Man}) \to \mathbf{Spc}$$

for all $M \in \mathbf{Man}$ and $x \in M$ form a conservative family of points for $\mathrm{Sh}(\mathbf{Man})$. To conclude, note that for every manifold M and point $x \in M$, there exists an open embedding $j \colon \mathbf{R}^n \hookrightarrow M$ such that $j(0_n) = x$ and apply Observation A.7.3. □

We now give a quick argument showing that the ∞-topos $\mathrm{Sh}(\mathbf{Man})$ is Postnikov complete. We learned the following argument from Hoyois; it is a slight refinement of the argument for the convergence of Postnikov towers that Hoyois gave in [23].

A.7.5 Proposition. *The ∞-topos* $\mathrm{Sh}(\mathbf{Man})$ *is Postnikov complete.*

Proof. Since $\mathrm{Sh}(\mathbf{Man})$ is hypercomplete, by Observation A.5.5 it suffices to show that the right adjoint $p_\star \colon \mathrm{Sh}(\mathbf{Man})^{\mathrm{post}} \to \mathrm{Sh}(\mathbf{Man})$ is fully faithful. That is, we need to show that for every collection of objects $\{F_n\}_{n \geq -2}$ of $\mathrm{Sh}(\mathbf{Man})$ equipped with compatible equivalences $\tau_{\leq n}(F_{n+1}) \simeq F_n$, and integer $k \geq -2$, the natural morphism

$$\tau_{\leq k}\left(\lim_{n \geq -2} F_n\right) \to F_k \tag{A.7.6}$$

is an equivalence. To see this, note that since the restriction functors

$$\{(-)|_M \colon \mathrm{Sh}(\mathbf{Man}) \to \mathrm{Sh}(M)\}_{M \in \mathbf{Man}}$$

are jointly conservative and commute with limits and truncations, it suffices to show that the morphism (A.7.6) becomes an equivalence after trestriction to each manifold M. This last claim follows from the fact that the ∞-topos $\mathrm{Sh}(M)$ is Postnikov complete (Example A.5.12). □

A.8 Sheaves on the cartesian site

Since every manifold admits an open cover by Euclidean spaces, the category of sheaves of *sets* on **Man** is equivalent to sheaves of sets on the full subcategory spanned by the Euclidean spaces. We prove an analogous result for sheaves of *spaces*; this follows from the hypercompleteness of $\mathrm{Sh}(\mathbf{Man})$ along with the general fact that the inclusion of a basis for a site induces an equivalence on *hyper*sheaves. We include a discussion of this here as some authors work with sheaves on this smaller site.

A.8.1 Definition. The ***cartesian site*** is the full subcategory $\mathbf{Cart} \subset \mathbf{Man}$ spanned by the empty manifold and Euclidean spaces $\mathbf{R}^n$ for $n \geq 0$, with the induced Grothendieck topology.

Since every manifold admits a cover by Euclidean spaces, the cartesian site is a *basis* for the Grothendieck topology on **Man** (see [28, §B.6] for more about bases for Grothendieck topologies).

A.8.2 Lemma. *Let C be a presentable ∞-category. Then right Kan extension along the inclusion $\mathbf{Cart}^{\mathrm{op}} \hookrightarrow \mathbf{Man}^{\mathrm{op}}$ defines an equivalence of ∞-categories*

$$\mathrm{Sh}(\mathbf{Cart}; C) \simeq \mathrm{Sh}(\mathbf{Man}; C)$$

with inverse given by restriction of presheaves.

Proof. Since $\mathrm{Sh}(\mathbf{Cart}; C)$ and $\mathrm{Sh}(\mathbf{Man}; C)$ are the tensor products of presentable ∞-categories

$$\mathrm{Sh}(\mathbf{Cart}; C) \simeq \mathrm{Sh}(\mathbf{Cart}) \otimes C \qquad \text{and} \qquad \mathrm{Sh}(\mathbf{Man}; C) \simeq \mathrm{Sh}(\mathbf{Man}) \otimes C\,,$$

it suffices to treat the case where $C = \mathbf{Spc}$ is the ∞-category of spaces. In this case, since the ∞-topos $\mathrm{Sh}(\mathbf{Man})$ is hypercomplete (Proposition A.7.4), the claim follows from the fact that $\mathbf{Cart} \hookrightarrow \mathbf{Man}$ is a basis for the topology on **Man** [8, Corollary 3.12.13]. □

A.9 The sheafification of an R-invariant presheaf

In this subsection we show that if F is an $\mathbf{R}$-invariant presheaf on **Man**, then the sheafification of F is $\mathbf{R}$-invariant (Proposition A.9.10). This provides a description of the homotopification functor L_{hi}. First we provide a plausibility argument for why the sheafification of an $\mathbf{R}$-invariant presheaf should be $\mathbf{R}$-invariant.

A.9.1. Recall that the sheafification of a C-valued presheaf F on **Man** can computed as a transfinite composite of the $\dagger$-construction of Recollection A.3.1. If F is $\mathbf{R}$-invariant, then for any n-manifold M we have equivalences

$$
\begin{aligned}
&\text{(A.9.2)} \qquad F^{\dagger}(M) = \operatorname*{colim}_{I \in \mathrm{Cov}(M)} \lim_{U \in I^{\mathrm{op}}} F(U) \\
&\text{(A.9.3)} \qquad \qquad \xleftarrow{\sim} \operatorname*{colim}_{I \in \mathrm{Cov}^{\mathrm{good}}(M)} \lim_{U \in I^{\mathrm{op}}} F(U) \\
&\text{(A.9.4)} \qquad \qquad \simeq \operatorname*{colim}_{I \in \mathrm{Cov}^{\mathrm{good}}(M)} \lim_{U \in I^{\mathrm{op}}} F(U)^{\Pi_\infty(U)} \\
&\text{(A.9.5)} \qquad \qquad \xleftarrow{\sim} \operatorname*{colim}_{I \in \mathrm{Cov}^{\mathrm{good}}(M)} \lim_{U \in I^{\mathrm{op}}} F(*)^{\Pi_\infty(U)} \\
&\text{(A.9.6)} \qquad \qquad \xleftarrow{\sim} \operatorname*{colim}_{I \in \mathrm{Cov}^{\mathrm{good}}(M)} F(*)^{\operatorname{colim}_{U \in I^{\mathrm{op}}} \Pi_\infty(U)} \\
&\text{(A.9.7)} \qquad \qquad \xleftarrow{\sim} \operatorname*{colim}_{I \in \mathrm{Cov}^{\mathrm{good}}(M)} F(*)^{\Pi_\infty(M)} = F(*)^{\Pi_\infty(M)} \, .
\end{aligned}
$$

Here $\mathrm{Cov}^{\mathrm{good}}(M) \subset \mathrm{Cov}(M)$ denotes the full subcategory spanned by the *good* covering sieves, i.e., those covering sieves of M where every open is diffeomorphic to $\mathbf{R}^n$. The equivalence (A.9.3) is because good covering sieves are cofinal in all covering sieves, (A.9.4) is because U is contractible, (A.9.5) is due to the assumption that F is $\mathbf{R}$-invariant, (A.9.6) is by the definition of the cotensor, and the (A.9.7) is by the van Kampen Theorem [HA, Proposition A.3.2].

One might hope that this argument provides an equivalence $\Gamma^{\star} F(*) \xrightarrow{\sim} F^{\dagger}$ (recall the description of $\Gamma^{\star}$ given by Lemma 2.1.5), so that $F^{\dagger}$ is $\mathbf{R}$-invariant, from which one concludes that the sheafification of F is $\mathbf{R}$-invariant. However, the equivalence (A.9.3) is not natural in M as good covers are not functorial. What we have shown is that abstractly $F^{\dagger}(M)$ is given by $F(*)^{\Pi_\infty(M)}$, but have not provided a morphism

$$\Gamma^{\star} F(*) \to F^{\dagger} \, .$$

Moreover, it is not so easy to provide a such a morphism as we do not yet know that $F^{\dagger}$ is a sheaf, and it is easy to map *into* cotensors, but not out of them. Even though the sheafification of F is less concrete than $F^{\dagger}$, using the fact that $\Gamma^{\star}$ is a left adjoint, we can construct a morphism $\Gamma^{\star} F(*) \to \mathrm{S}_{\mathbf{Man}} F$ without much difficulty.

A.9.8 Recollection. Let C be a compactly generated ∞-category, T be a topological space, $t \in T$, and F a C-valued presheaf on T. Then the morphism $F_t \to \mathrm{S}_T(F)_t$ on stalks at t induced by the unit $F \to \mathrm{S}_T F$ is an equivalence. See [29, Proposition 4.1.4].

A.9.9 Lemma. *Let C be a compactly generated ∞-category, $F \in \mathrm{PSh}(\mathbf{Man}; C)$, M a manifold, and $x \in M$. Then the morphism*

$$x^{\star} F \to x^{\star} \, \mathrm{S}_{\mathbf{Man}} F$$

induced by the unit is an equivalence.

Proof. By definition, if F' is a presheaf on **Man**, then $x^\star F' \coloneqq (F'|_M)_x$. By Lemma A.3.3 we have a canonical identification $\mathrm{S}_{\mathbf{Man}}(F)|_M = \mathrm{S}_M(F|_M)$. The claim now follows from Recollection A.9.8. □

A.9.10 Proposition. *Let C be a presentable ∞-category and $F\colon \mathbf{Man}^{\mathrm{op}} \to C$ an $\mathbf{R}$-invariant presheaf on* **Man**. *Then the counit $\Gamma^\star\Gamma_\star \mathrm{S}_{\mathbf{Man}} F \to \mathrm{S}_{\mathbf{Man}} F$ is an equivalence. In particular, $\mathrm{S}_{\mathbf{Man}} F$ is $\mathbf{R}$-invariant.*

Proof. Since the left adjoints

$$\Gamma^\star\Gamma_\star \mathrm{S}_{\mathbf{Man}}, \mathrm{S}_{\mathbf{Man}} \colon \mathrm{PSh}_{\mathbf{R}}(\mathbf{Man}; C) \to \mathrm{Sh}(\mathbf{Man}; C)$$

are obtained by applying the tensor product of presentable ∞-categories $- \otimes C$ to the left adjoints

$$\Gamma^\star\Gamma_\star \mathrm{S}_{\mathbf{Man}}, \mathrm{S}_{\mathbf{Man}} \colon \mathrm{PSh}_{\mathbf{R}}(\mathbf{Man}; \mathbf{Spc}) \to \mathrm{Sh}(\mathbf{Man}; \mathbf{Spc}) ,$$

it suffices to prove the claim in the case that $C = \mathbf{Spc}$. In this case, we show that the counit

$$\varepsilon_F \colon \Gamma^\star\Gamma_\star \mathrm{S}_{\mathbf{Man}} F \to \mathrm{S}_{\mathbf{Man}} F$$

is an equivalence by checking that ε_F is an equivalence on stalks (Proposition A.7.4). Let M be a manifold and $x \in M$, and write $\Gamma^\star_{\mathrm{pre}} \colon C \to \mathrm{Fun}(\mathbf{Man}^{\mathrm{op}}, C)$ for the constant presheaf functor. By Lemma A.9.9 it suffices to show that the counit

$$x^\star\Gamma^\star_{\mathrm{pre}} F(*) \to x^\star F . \tag{A.9.11}$$

By definition, $x^\star\Gamma^\star_{\mathrm{pre}} F(*) = F(*)$, and

$$x^\star F = \operatorname*{colim}_{U \in \mathrm{Open}_x(M)^{\mathrm{op}}} F(U) ,$$

where $\mathrm{Open}_x(M) \subset \mathrm{Open}(M)$ is the full subposet spanned by those opens containing $x \in M$. Let $\mathrm{Open}'_x(M) \subset \mathrm{Open}_x(M)$ denote the full subposet with elements those opens diffeomorphic to $\mathbf{R}^{\dim(M)}$. Note that the inclusion

$$\mathrm{Open}'_x(M)^{\mathrm{op}} \subset \mathrm{Open}_x(M)^{\mathrm{op}}$$

is colimit-cofinal. Since F is $\mathbf{R}$-invariant, we see that

$$\begin{aligned} x^\star F &\simeq \operatorname*{colim}_{U \in \mathrm{Open}'_x(M)^{\mathrm{op}}} F(U) \\ &\simeq \operatorname*{colim}_{U \in \mathrm{Open}'_x(M)^{\mathrm{op}}} F(*) \\ &\simeq F(*) . \end{aligned}$$

Unwinding the definitions we see that the counit morphism (A.9.11) is an equivalence. □

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