On the homotopy theory of Stratified Spaces

Outline.

(1) Background / motivation

(2) Exit-path ∞-categories

(3) Main result

(4) Comments on the proof

1. Background / motivation

Longstanding Problem. Prove a 'homotopy hypothesis' for stratified spaces

$$\left(\begin{array}{c}\text{A homotopy theory}\\ \text{of stratified}\\ \text{topological spaces}\end{array}\right) \xrightarrow{\sim} \left(\begin{array}{c}\text{Certain}\\ \infty\text{-categories}\end{array}\right)$$

Where:

(1) The RHS has excellent formal properties: is a presentable ∞-cat, has good functoriality in the poset, ...

(2) The 'topological' side captures all differential-topological examples → topologically stratified spaces in the sense of Goresky-MacPherson (e.g., Whitney Strat. Spaces)

(3) The equivalence is given by MacPherson's $\underbrace{\text{exit-path}}_{\text{will explain}}$ construction.

Previous work. Henriques, D. Miller, Ayala-Francis-Tanaka, Ayala-Francis-Rozenblyum, Chataur (+ collaborators), Douteau, Andrade, Woolf, Nand-Lal, ...

Stratified topological spaces

Definition. Let P be a poset. The Alexandroff topology on (the set) P is the topology where

$$[U \subset P \text{ open}] \iff [\underbrace{p \in U \text{ and } p' \geq p \Rightarrow p' \in U}_{P \text{ `upwards closed'}}]$$

> $\mathrm{Top}_{/P}$: Category of P-stratified topological spaces.

Given $s: T \longrightarrow P$ and $p \in P$, we call

$$T_p := s^{-1}(p) \text{ the } p\text{-th stratum of } T.$$

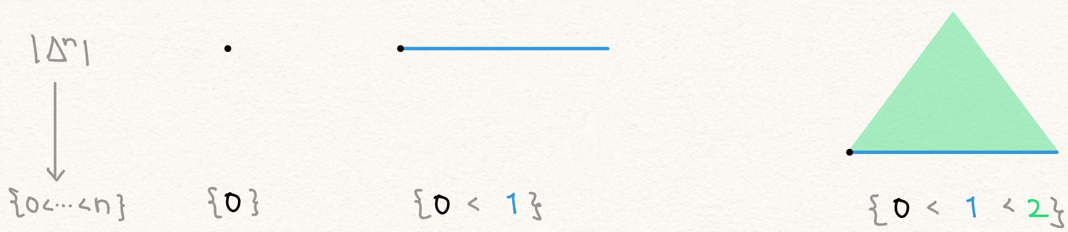
Example. Consider the poset $\{0 < \cdots < n\}$. There is a bijection

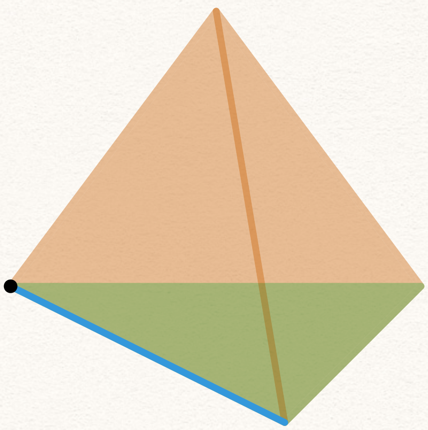
$$\left\{ \begin{array}{c} \text{stratifications} \\ T \longrightarrow \{0 < \cdots < n\} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{filtrations by} \\ \text{closed subspaces} \\ F_0(T) \subset \cdots \subset F_n(T) = T \end{array} \right\}$$

$$[s : T \longrightarrow \{0 < \cdots < n\}] \longmapsto [F_i(T) := s^{-1}(\{0 < \cdots < i\})]$$

$$[\text{strat w/ strata } T_i := F_i(T) \setminus F_{i-1}(T)] \longleftarrow [F_0(T) \subset \cdots \subset F_n(T)].$$

Example (Simplices). There is a natural strat $|\Delta^n| \longrightarrow \{0 < \cdots < n\}$





$\{0 < 1 < 2 < 3\}$

In general:

$$|\Delta^{\{0\}}| \subset |\Delta^{\{0<1\}}| \subset \cdots \subset |\Delta^{\{0<\cdots<n-1\}}| \subset |\Delta^n|$$

$$(t_0, \dots, t_n) \longmapsto \max\{i \in [n] \mid t_i \neq 0\}$$

Example (geometric realizations of posets). For any poset P,

$$P \cong \operatorname*{colim}_{\{p_0 < \cdots < p_n\} \subset P} \{p_0 < \cdots < p_n\}.$$

Hence the natural stratifications $|\Delta^n| \to \{0 < \cdots < n\}$ give rise to a natural stratification

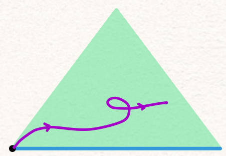
$$|N(P)| \longrightarrow P.$$

2. Exit-paths

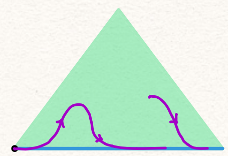
Idea (MacPherson...). The 'stratified homotopy type' of $T \in \mathrm{Top}_{/P}$ should be determined by its 'exit-path ∞-category' $\mathrm{Exit}_P(T)$ with:

(0) Objects: points of T

(1) 1-morphisms: exit-paths from lower to higher strata



Exit-path



not exit-paths

(2) 2-morphisms: homotopies respecting stratifications

⋮

Takeaway. $\mathrm{Exit}_P(T)$ should be an ∞-category with a functor

$$\begin{aligned} \mathrm{Exit}_P(T) &\longrightarrow N(P) \\ T_p \ni t &\longmapsto t \end{aligned}$$

$\underbrace{\text{with fibers } \infty\text{-groupoids.}}$

$\Leftrightarrow$ $\mathrm{Exit}_P(T) \longrightarrow N(P)$ conservative

Definition. The ∞-category of abstract P-stratified homotopy types is

$$\boxed{\mathrm{Str}_P := \mathrm{Cat}^{\mathrm{cons}}_{\infty,/N(P)}} \subset \mathrm{Cat}_{\infty,/N(P)}$$

> This is the RHS of the desired stratified homotopy hypothesis.

Key Observation. A morphism $f: C \to D$ in Str_P is an equivalence iff:

(1) For all $p \in P$, $\quad C_p \xrightarrow{\sim} D_p$. 'equivalence on strata'

(2) For all $p < q$, $\quad \underbrace{\mathrm{Fun}_{/P}(\{p<q\}, C)}_{\text{an } \infty\text{-groupoid}} \xrightarrow{\sim} \mathrm{Fun}_{/P}(\{p<q\}, D)$

'equivalence on links'

sSet model for exit-paths

Construction (Henriques–Lurie). The functor

$$\mathrm{sSet}_{/N(P)} \xrightarrow{|-|_P} \mathrm{Top}_{/P}$$

$$\begin{array}{ccc} X & & |X| \\ \downarrow & \longmapsto & \downarrow \\ N(P) & & |N(P)| \longrightarrow P \end{array}$$

Natural strat.

is a left adjoint. The right adjoint $\mathrm{Sing}_P : \mathrm{Top}_{/P} \to \mathrm{sSet}_{/N(P)}$ is given by

$$\begin{tikzcd} \boxed{\mathrm{Sing}_P(T)} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sing}(T) \arrow[d] \\ N(P) \arrow[r, hook] & \mathrm{Sing}(P). \end{tikzcd}$$

adjoint to $|N(P)| \to P$

Observe. For $p \in P$,

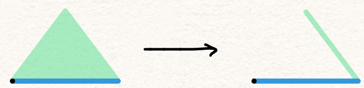
$$\mathrm{Sing}_P(T) \underset{N(P)}{\times} \{p\} \simeq \mathrm{Sing}(T_p).$$

> So $\mathrm{Sing}_P(T)$ is a sSet with a map to $N(P)$ & fibers Kan-complexes.

> If $\mathrm{Sing}_P(T)$ is a quasicategory, it deserves to be called the exit-path ∞-category of T!

⚠ $\mathrm{Sing}_P(T)$ is not generally a quasicategory

$\mathrm{Sing}_{[2]}\Big($ $\Big)$ is not a qcat because there isn't a stratified retraction



> Often $\mathrm{Sing}_P(T)$ is a quasicategory:

[HA, Theorem A.6.4]

Theorem (Lurie). If the stratification $T \to P$ is 'conical', then $\mathrm{Sing}_P(T)$ is a quasicategory.

locally looks like $\mathrm{Cone}(T') \times V$ where

- T' is $P_{>p}$-stratified
- V is any top. space
- The strat is via

$$\mathrm{Cone}(T') \times V \xrightarrow{pr} \mathrm{Cone}(T) \to (P_{>p})^{\triangleleft} \cong P_{\geq p} \subset P$$

3. Main results

Notation. Write $\mathrm{Top}^{ex}_{/P} \subset \mathrm{Top}_{/P}$ for the full subcategory of those $T \to P$ such that $\mathrm{Sing}_P(T)$ is a quasicategory.

Stratified homotopy hypothesis (H.). The functor

$$\begin{aligned} \mathrm{Exit}_P : \mathrm{Top}^{ex}_{/P}[(\text{equiv. on strata \& links})^{-1}] &\longrightarrow \mathrm{Str}_P \\ T &\longmapsto \mathrm{Sing}_P(T) \end{aligned}$$

(1) $T_p \xrightarrow{\sim} S_p$ whe

(2) $\mathrm{Map}_{/P}(|\{p<q\}|, T) \xrightarrow{\sim} \mathrm{Map}_{/P}(|\{p<q\}|, S)$ whe

is an equivalence of ∞-categories.

Remark. This comes from an equivalence involving all of $\mathrm{Top}_{/P}$, but the functor is not so simple on the bigger category.

Corollary. Write

$\mathrm{StrTop}^{ex} \subset \mathrm{Fun}([1], \mathrm{Top})$ $\qquad$ $\mathrm{Str} \subset \mathrm{Fun}([1], \mathrm{Cat}_\infty)$

T $\quad$ $\mathrm{Sing}_P(T)$ qcat $\qquad$ C

$\downarrow$ $\qquad\qquad\qquad\qquad\qquad$ $\downarrow$ — conservative

P $\quad$ poset $\qquad\qquad\qquad\qquad$ P $\quad$ poset

The functor

$$\mathrm{Exit} : \mathrm{StrTop}^{ex}[\underbrace{W^{-1}}_{\text{iso on posets \& equiv. on strata \& links}}] \longrightarrow \mathrm{Str} \quad \text{is an equivalence.}$$

Remark. Joint work with Barwick & Glasman on exit-path categories in algebraic geometry has demonstrated how useful this perspective on strat. spaces is!

Relation to work of Ayala-Francis-Tanaka-Rozenblyum

AFTR introduce a category Con of conically smooth stratified spaces:

$$\begin{array}{c} T \\ \downarrow \\ P \end{array} \quad + \text{properties on } P \quad + \quad \begin{array}{l} \text{atlas of conical} \\ \text{nbhds on } T \end{array}$$

↳ strata are manifolds

- There is a forgetful functor $\mathrm{Con} \longrightarrow \mathrm{StrTop}^{ex}$.
- AFR prove

$$\begin{tikzcd} \mathrm{Con}[(\text{strat homotopy equivs})^{-1}] \arrow[r, hook, "\mathrm{Exit}"] \arrow[d, "\text{forget}"] & \mathrm{Str} \\ \mathrm{StrTop}^{ex}[W^{-1}] \arrow[ur, "\mathrm{Exit}", "\sim"'] & \end{tikzcd}$$

Our results
⇓
fully faithful

- Good: it is not known if Whitney strat. spaces admit conically smooth structures! [AFR, Conjecture 0.0.7]

Note. The Ayala-Francis-Tanaka-Rozenblyum theory of conically smooth stratified spaces imposes regularity conditions in order to be able to do factorization homology before inverting stratified homotopies.

- Our theory is not designed to be used for factorization homology.

4. Comments on the proof

Notation. P poset. Let $\mathrm{sd}(P)$ denote the poset of nonempty linearly ordered finite subsets of P ordered by inclusion.

Key Observation (Barwick - Glasman - H.). The functor

$$\begin{aligned} N_P : \mathrm{Str}_P &\longrightarrow \mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathrm{Spc}) \\ C &\longmapsto [\Sigma \longmapsto \mathrm{Fun}_{/P}(\Sigma, C)] \end{aligned}$$

is a fully faithful right adjoint.

The ess. image consists of those $F : \mathrm{sd}(P)^{op} \longrightarrow \mathrm{Spc}$ such that for all $\{p_0 < \cdots < p_n\} \subset P$,

$$F\{p_0 < \cdots < p_n\} \xrightarrow{\sim} F\{p_0 < p_1\} \underset{F\{p_1\}}{\times} F\{p_1 < p_2\} \underset{F\{p_2\}}{\times} \cdots \underset{F\{p_{n-1}\}}{\times} F\{p_{n-1} < p_n\}.$$

'Segal condition' satisfied for $\mathrm{Fun}_{/P}(-, C)$ since

$$\{p_0 < \cdots < p_n\} \cong \{p_0 < p_1\} \underset{\{p_1\}}{\cup} \cdots \underset{\{p_{n-1}\}}{\cup} \{p_{n-1} < p_n\}$$

Outline of proof of main theorem.

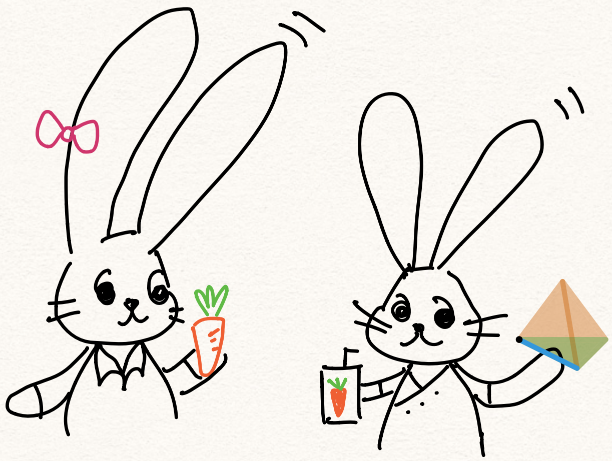
(1) Prove that the projective model structure $\underbrace{\text{right-transfers}}_{\substack{\text{weak equivs \& fibs} \\ \text{checked after applying rt. adjoint}}}$

$$\begin{aligned} \mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathrm{sSet})^{\mathrm{proj}} &\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Top}_{/P} \\ [\Sigma \mapsto \mathrm{Sing}\,\mathrm{Map}_{/P}(|\Sigma|_P, T)] &\longleftarrow\!\shortmid\ T \end{aligned}$$

(2) Prove the above Quillen adjunction is an equivalence (Douteau)

(3) Localize & check that $\mathrm{Top}^{ex}_{/P}[W^{-1}] \longrightarrow \mathrm{Str}_P$ is essentially surj.

↳ Makes use of our Joyal-Kan model structure on $sSet_{/N(P)}$.



Thanks for listening!