Chen Lie algebras and combinatorics of arrangements

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Lower Central Series & Chen Groups

G: a finitely-generated group. **LCS**:

$$G_1 = G, G_2 = G', \dots, G_{k+1} = [G_k, G], \dots$$

- **Associated graded Lie algebra**:

$$\boxed{\operatorname{gr}_* G = \bigoplus_{k \geq 1} G_k / G_{k+1}}$$

with $[\ ,\ ]\colon \operatorname{gr}_i \times \operatorname{gr}_j \to \operatorname{gr}_{i+j}$ from group commutator.

- **Derived series**:

$$G^{(0)} = G, G^{(1)} = G', \dots, G^{(i+1)} = [G^{(i)}, G^{(i)}], \dots$$

- **Chen Lie algebra**:

$$\boxed{\operatorname{gr}_*(G/G'')}$$

LCS ranks & Chen ranks:

$$\phi_k(G) = \operatorname{rank}(\operatorname{gr}_k G), \qquad \theta_k(G) = \operatorname{rank}(\operatorname{gr}_k(G/G''))$$

Clearly: $\phi_1 = \theta_1,\ \phi_2 = \theta_2,\ \phi_3 = \theta_3,\ \phi_k \geq \theta_k.$

Example: Free groups

F_n: free group of rank n

$\operatorname{gr} F_n = \mathbb{L}_n$ (free Lie algebra of rank n)

E. Witt [1937]:

$$\phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}}$$

K.-T. Chen [1951]:

$$\theta_k(F_n) = (k-1)\binom{n+k-2}{k} \quad \text{for } k \geq 2$$

Example: Pure braid groups

$P_n = F_{n-1} \rtimes \cdots \rtimes F_1$

T. Kohno [1985]

$$\phi_k(P_n) = \sum_{i=1}^{n-1} \phi_k(F_i)$$

D. Cohen–A. S. [1995]

$$\theta_k(P_n) = (k-1)\binom{n+1}{4} \quad \text{for } k \geq 3$$

Holonomy Lie Algebra & Derived Series

Assume $H_1(G)$ is torsion-free, and $b_2(G)$ finite.

Chen [1977] defined the **holonomy Lie algebra** of G:

$$\boxed{\mathfrak{H}_G = \mathrm{Lie}(H^1(G))/\mathrm{ideal}(\mathrm{im}(\partial_G))}$$

where: $\mathrm{Lie}(H^1(G))$ = free Lie algebra on $H^1(G)$
∂_G = dual of $\cup\colon H^1(G) \wedge H^1(G) \to H^2(G)$

The iso $H^1(G) \cong G/G'$ extends to Lie algebra map $\mathrm{Lie}(H^1(G)) \twoheadrightarrow \mathrm{gr}(G)$, which descends to natural map

$$\Psi_G \colon \mathfrak{H}_G \twoheadrightarrow \mathrm{gr}(G)$$

Theorem. *(D. Sullivan [1977]) If G is 1-formal, then:*

$$\mathfrak{H}_G \otimes \mathbb{Q} \cong \mathrm{gr}(G) \otimes \mathbb{Q}$$

Theorem 1. *If G is 1-formal, then, for all $i \geq 0$:*

$$\boxed{\Psi_G^{(i)} \otimes \mathbb{Q} \colon \mathfrak{H}_G/\mathfrak{H}_G^{(i)} \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{gr}(G/G^{(i)}) \otimes \mathbb{Q}}$$

Malcev Lie algebra L

$\mathbb{Q}$-Lie algebra, with complete, descending vector space filtration, $\{F_r L\}_{r\geq 1}$, s.t. $F_1 L = L$ and

- $[F_r L, F_s L] \subset F_{r+s} L$
- $\mathrm{gr}(L) = \bigoplus_{r\geq 1} F_r L / F_{r+1} L$ generated in degree 1

Exponential group $\exp(L)$

Set L, with group multiplication

$$x \cdot y = x + y + \tfrac{1}{2}[x,y] + \tfrac{1}{12}[x,[x,y]] + \tfrac{1}{12}[y,[y,x]] + \cdots$$

and filtration by normal subgroups $\{\exp(F_r L)\}_{r\geq 1}$.

Malcev completion $\widehat{G} = \exp(L)$

$$\begin{tikzcd}
G \arrow[r, "\rho"] \arrow[d, twoheadrightarrow] & \exp(L) \arrow[d, twoheadrightarrow] \\
G/G_r \arrow[r, "\rho_r"] \arrow[dr, "f_r"'] & \exp(L/F_r) \arrow[d, dotted, "\tilde{f}_r"] \\
& \exp(\widetilde{L})
\end{tikzcd}$$

Construction: $\widehat{G} = \exp(\mathrm{Prim}(\widehat{\mathbb{Q}G})) = \varprojlim_n (G/G_n \otimes \mathbb{Q})$

Lemma.

$$\rho\big(G^{(i)}\big) \subset \exp\left(L^{(i)}\right) = \exp(L)^{(i)}$$

Proposition.

$$\widehat{G/G^{(i)}} = \exp\left(L/L^{(i)}\right)$$

Proof. Chase diagram:

$$\begin{tikzcd}
& G \arrow[rrr, "\rho"] \arrow[dl] \arrow[dd] & & & \exp(L) \arrow[dl] \arrow[dd] \\
G/G^{(i)} \arrow[rrr, "\rho^{(i)}"] \arrow[ddd] & & & \exp(L/L^{(i)}) \arrow[ddd] & \\
& G/G_r \arrow[rrr, "\rho_r"] \arrow[ddl] \arrow[dr, "f_r"] & & & \exp(L/F_r) \arrow[dll, dashed, "\tilde{f}_r"'] \arrow[ddl] \\
& & \exp(\widetilde{L}) & & \\
G/G_r \cdot G^{(i)} \arrow[rrr, "\rho_r^{(i)}"] \arrow[urr, "f"] & & & \exp(L/(F_r + L^{(i)})) \arrow[ul, dotted, "\tilde{f}"'] &
\end{tikzcd}$$

A group G is called **1-formal** if

$$\boxed{\widehat{G} = \exp(\widehat{\mathfrak{H}_G \otimes \mathbb{Q}})}$$

Proof of Theorem 1.

Set $\mathfrak{H} = \mathfrak{H}_G$ and $L = \widehat{\mathfrak{H} \otimes \mathbb{Q}}$.

By 1-formality: $\quad \widehat{G} = \exp(L)$.

Clearly: $\quad L/L^{(i)} = \widehat{\mathfrak{H}} \otimes \mathbb{Q}/\widehat{\mathfrak{H}^{(i)}} \otimes \mathbb{Q} = \widehat{\mathfrak{H}/\mathfrak{H}^{(i)}} \otimes \mathbb{Q}$.

By the Proposition:

$$\widehat{G/G^{(i)}} = \exp\left(\widehat{\mathfrak{H}_G/\mathfrak{H}_G^{(i)}} \otimes \mathbb{Q}\right).$$

Passing to associated graded Lie algebras:

$$\mathrm{gr}(G/G^{(i)}) \otimes \mathbb{Q} \cong \left(\mathfrak{H}_G/\mathfrak{H}_G^{(i)}\right) \otimes \mathbb{Q}$$

Thus

$$\Psi_G^{(i)} \otimes \mathbb{Q} \colon \left(\mathfrak{H}_G/\mathfrak{H}_G^{(i)}\right) \otimes \mathbb{Q} \twoheadrightarrow \mathrm{gr}\big(G/G^{(i)}\big) \otimes \mathbb{Q}$$

is a surjection between $\mathbb{Q}$-vector spaces of the same (finite) dimension, hence an isomorphism. $\square$

Let X be a path-connected space with $b_1(X)$ and $b_2(X)$ finite, and $H_1(X)$ torsion-free. Set:

$$\mathfrak{H}_X = \operatorname{Lie}(H^1(X))/\operatorname{ideal}(\operatorname{im}(\partial_X))$$

where $\partial_X = (\cup_X)^\sharp$. If $G = \pi_1(X)$, then:

$$\Pi_X \colon \mathfrak{H}_X \twoheadrightarrow \mathfrak{H}_G$$

Always: $\Pi_X \otimes \mathbb{Q} \colon \mathfrak{H}_X \otimes \mathbb{Q} \xrightarrow{\cong} \mathfrak{H}_G \otimes \mathbb{Q}$.

If $\cup_X$ is onto: $\Pi_X \colon \mathfrak{H}_X \xrightarrow{\cong} \mathfrak{H}_G$.

Now suppose X is *formal* (its $\mathbb{Q}$-homotopy type is determined by $H^*(X,\mathbb{Q})$). Then $\pi_1(X)$ is 1-formal [Sullivan 1977], and so:

$$\boxed{\operatorname{gr}\big(G/G^{(i)}\big) \otimes \mathbb{Q} \cong \Big(\mathfrak{H}_X/\mathfrak{H}_X^{(i)}\Big) \otimes \mathbb{Q}}$$

Examples of formal spaces:

- Compact Kähler manifolds [DGMS 1975].
- Certain (but not all) complements of normal-crossing divisors in smooth projective varieties [Morgan 1978]. All are 1-formal [Kohno 1983].

Hyperplane arrangements

$\mathcal{A}$: central arrangement of n hyperplanes in $\mathbb{C}^\ell$

$\mathcal{L}(\mathcal{A})$: intersection lattice (the "combinatorics" of $\mathcal{A}$)

$X = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H, \quad G = \pi_1(X)$

$H^*(X;\mathbb{Z})$ computed by Brieskorn [1973]; combinatorial presentation $A = E/I$ by Orlik-Solomon [1980].

Consequences:

- X is formal;
  thus, G is 1-formal.

- $\cup_X : \bigwedge^k H^1(X) \to H^k(X)$ surjective;
  hence, $\mathfrak{H}_X = \mathfrak{H}_G$.

- $\mathfrak{H} = \mathfrak{H}_X$ determined by $\mathcal{L}_2(\mathcal{A})$:

$$\boxed{\mathfrak{H} = \mathbb{L}(x_1,\dots,x_n)\Big/\Big([x_j, \sum_{i\in Y} x_i] = 0 \mid V \in \mathcal{L}_2, j \in V\Big)}$$

Theorem 2. *Let $\mathcal{A}$ be a complex hyperplane arrangement, with complement X, fundamental group G, and holonomy Lie algebra $\mathfrak{H}$. Then:*

$$\boxed{\operatorname{gr}(G/G'') \otimes \mathbb{Q} = (\mathfrak{H}/\mathfrak{H}'') \otimes \mathbb{Q}}$$

Consequently, the rational Chen Lie algebra of $\mathcal{A}$ is combinatorially determined (as a graded Lie algebra) by $\mathcal{L}_2(\mathcal{A})$.

In particular, the Chen ranks, $\theta_k(G) = \operatorname{rank} \operatorname{gr}_k(G/G'')$, are combinatorially determined.

Comments/Questions

- Can $\operatorname{gr}(G/G'')$ have torsion? (There are now indications that it may.)
- There are examples where $\operatorname{gr}(G)$ does have torsion.
- Is such torsion combinatorially determined?

Alexander invariants

G f.g. group

$I = \ker(\epsilon\colon \mathbb{Z}G \to \mathbb{Z})$ augmentation ideal

- **Alexander module**

$$A_G = \mathbb{Z}(G/G') \otimes_{\mathbb{Z}G} I$$

with G/G' acting on the left

- **Alexander invariant**

$$B_G = G'/G''$$

with G/G' acting by conjugation: $gG' \cdot hG'' = ghg^{-1}G''$

Fit into exact sequence

$$0 \to B_G \to A_G \to I \to 0$$

Asociated graded module (w.r.t. I-adic filtration):

$$\operatorname{gr}(B_G) = \bigoplus_{k\geq 0} I^k B_G / I^{k+1} B_G$$

W.S. Massey [1980]:

$$\boxed{\operatorname{gr}_k(G/G'') = \operatorname{gr}_{k-2}(B_G)} \quad \text{for } k \geq 2$$

Now suppose G is a *commutator-relators* group:

$$G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle, \quad r_j \in [F_n, F_n]$$

Then $G/G' = \mathbb{Z}^n$, generated by $t_i = \mathrm{ab}(x_i)$.

Identify $\mathbb{Z}(G/G')$ with $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

A_G and B_G are finitely-generated modules over the Noetherian ring Λ, thus admit finite presentations:

$$\Lambda^m \xrightarrow{D_G = \left(\frac{\partial r_j}{\partial x_i}\right)^{\mathrm{ab}}} \Lambda^n \to A_G \to 0,$$

$$\Lambda^{\binom{n}{3}+m} \xrightarrow{\Delta_G = \delta_3 + \vartheta_G} \Lambda^{\binom{n}{2}} \to B_G \to 0,$$

where δ_k are Koszul differentials, and $\delta_2 \circ \vartheta_G = D_G$.

Magnus embedding:

$$\begin{aligned} \mu \colon \Lambda &\hookrightarrow P = \mathbb{Z}[[s_1, \dots, s_n]] \\ t_i &\mapsto 1 + s_i \end{aligned}$$

Passing to associated graded rings w.r.t. the filtrations by powers of $\mathfrak{I} = (t_1 - 1, \dots, t_n - 1)$ and $\mathfrak{m} = (s_1, \dots, s_n)$:

$$\mathrm{gr}(\mu) \colon \ \mathrm{gr}(\Lambda) \xrightarrow{\cong} S = \mathbb{Z}[s_1, \dots, s_n]$$

Let $\mu^{(q)} \colon \Lambda \to P/\mathfrak{m}^{q+1}$ be the q-th truncation.

Since G commutator-relators, $\mu^{(0)}(D_G) = 0$. Set:

$$D_G^{(1)} = \mu^{(1)}(D_G)$$

Entries belong to $\mathfrak{m}/\mathfrak{m}^2 = \mathrm{gr}_1(P) \subset S$:

$$\left(D_G^{(1)}\right)_{k,j} = \sum_{i=1}^{n} \epsilon\left(\frac{\partial^2 r_k}{\partial x_i \partial x_j}\right) s_i$$

- **Linearized Alexander module** of G: the S-module

$$\boxed{\mathfrak{A}_G = \operatorname{coker} D_G^{(1)}}$$

Let $\left(S \otimes \bigwedge^k H^1 G, d_k\right)$ be the Koszul complex of S.

Since $d_1 \circ D_G^{(1)} = 0$, d_1 factors through $\mathfrak{A}_G \twoheadrightarrow \mathfrak{m}$.

- **Linearized Alexander invariant**: the S-module

$$\boxed{\mathfrak{B}_G = \ker\left(\mathfrak{A}_G \twoheadrightarrow \mathfrak{m}\right)}$$

Proposition. *$\mathfrak{B}_G$ has presentation*

$$S \otimes (\wedge^3 H^1 G \oplus H^2 G) \xrightarrow{d_3 + \mathrm{id} \otimes \partial_G} S \otimes \wedge^2 H^1 G \to \mathfrak{B}_G \to 0$$

where $\partial_G = (\cup_G)^\sharp$.

Let $\mathfrak{H} = \mathfrak{H}_G$. Consider exact sequence:

$$0 \to \mathfrak{H}'/\mathfrak{H}'' \to \mathfrak{H}/\mathfrak{H}'' \to \mathfrak{H}/\mathfrak{H}' \to 0$$

- **Infinitesimal Alexander invariant** of $\mathfrak{H}$:

$$\boxed{B(\mathfrak{H}) = \mathfrak{H}'/\mathfrak{H}''}$$

Module over $S = U(\mathfrak{H}/\mathfrak{H}')$ via adjoint rep.

Proposition. *If G is a comm-rels group, then:*

$$\boxed{B(\mathfrak{H}_G) \cong \mathfrak{B}_G}$$

as modules over $S = \mathrm{Sym}(H^1(G))$

Theorem 3. *Let G be a commutator-relators group. Assume G is 1-formal. Then:*

$$\boxed{\mathrm{Hilb}(\mathrm{gr}(B_G) \otimes \mathbb{Q}, t) = \mathrm{Hilb}(B(\mathfrak{H}_G) \otimes \mathbb{Q}, t)}$$

Hence:

$$\boxed{\sum_{k\geq 0} \theta_{k+2}\, t^k = \mathrm{Hilb}(\mathfrak{B}_G \otimes \mathbb{Q}, t)}$$

The 1-formality hypothesis is crucial:

Example. Let

$$G = \langle x_1, x_2 \mid ((x_1, x_2), x_2) = 1 \rangle$$

G comm-rels group, $H_1(G) = \mathbb{Z}^2$, $H_2(G) = \mathbb{Z}$, $\cup_G = 0$

G not 1-formal (Malcev algebra is not quadratic)

$\mathfrak{H}_G = \mathbb{L}_2$, $B(\mathfrak{H}_G) = \mathbb{Z}[s_1, s_2]$,

$$\mathrm{Hilb}(B(\mathfrak{H}_G), t) = 1/(1-t)^2$$

$B_G = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]/(t_2 - 1)$, $\mathrm{gr}(B_G) = \mathbb{Z}[s_1, s_2]/(s_2)$,

$$\mathrm{Hilb}(\mathrm{gr}(B_G), t) = 1/(1-t)$$

Hence:

$$\mathrm{gr}(B_G) \otimes \mathbb{Q} \not\cong \mathfrak{B}_G \otimes \mathbb{Q}$$

$$\mathfrak{H}_G/\mathfrak{H}_G'' \otimes \mathbb{Q} \not\cong \mathrm{gr}(G/G'') \otimes \mathbb{Q}$$

Towards a Resonance Formula for θ_k

(Work in progress with Hal Schenck)

$\mathcal{A} = \{H_1, \dots, H_n\}, \quad A = H^*(X, \mathbb{C}) = E/I, \;\; G = \pi_1(X)$

Theorem A. *For all $k \geq 2$,*

$$\boxed{\theta_k(G) = \dim \operatorname{Tor}^E_{k-1}(A, \mathbb{C})_k}$$

Proof 1: Use Theorem 1 and result of Fröberg–Löfwall.

Proof 2: Use Theorem 3 and results of Einsenbud et al.

Recall from D. Cohen–A.S. [1999]:

$$\boxed{V(\operatorname{ann} \mathfrak{B}_G) = \mathcal{R}_1(\mathcal{A})}$$

where $\mathcal{R}_1(\mathcal{A}) = \{\lambda \in \mathbb{C}^n \mid \dim H^1(A, \cdot \sum_{i=1}^n \lambda_i e_i) > 0\}$

Write $\mathcal{R}_1(\mathcal{A}) = \bigcup_{i=1}^q L_i, \quad \dim L_i = l_i.$

Conjecture. For all $k \gg 4$:

$$\boxed{\theta_k(G) = \sum_{i=1}^{q} (k-1) \binom{k + l_i - 2}{k}}$$

■ Graphic arrangements

$\mathsf{G} = (\mathcal{V}, \mathcal{E})$ graph. Let

$$\mathcal{A}_{\mathsf{G}} = \{\ker(z_i - z_j) \mid \{i, j\} \in \mathcal{E}\}$$

be the **graphic arrangement** associated to G, and

$$f_s(\mathsf{G}) = \sharp\{\text{complete subgraphs on } s+1 \text{ vertices}\}$$

Theorem B. *For groups of graphic arrangements:*

$$\boxed{\theta_k(G) = (k-1)(f_2 + f_3)} \quad \textit{for } k \geq 2$$

Follows from H. Schenck–A.S. [2002] and Theorem A.

Example. $\mathsf{G} = K_n$, complete graph on n vertices

$\mathcal{A}_{K_n}$ = braid arrangement in $\mathbb{C}^n$

$G = P_n$, pure braid group on n strings

Since $f_s(K_n) = \binom{n}{s+1}$, get:

$$\boxed{\theta_k(P_n) = (k-1)\binom{n+1}{4}} \quad \text{for } k \geq 2$$

This recovers computation by D. Cohen–A.S. [1995].