In the previous sections, we have used bigroupoids, that is bicategories in which every 2-cell is an isomorphism and every 1-cell is an equivalence (*i.e.*, invertible up to isomorphism with respect to the (horizontal) "tensor product " composition). The link to simplicial topology now comes from the fact that every bicategory $\mathcal{G}$ has a simplicial set *nerve*, $\mathbf{Ner}(\mathcal{G})$, and *this simplicial set is is a Kan complex, precisely when the bicategory is a bigroupoid*, *i.e.*, has exactly the same invertibility requirements as those required for the bicategory to be a bigroupoid [10]. This nerve is minimal in dimensions ≥ 2 and if one chooses a base point $0 \in \mathbf{Ner}(\mathcal{G})_0$, which is the set of objects or 0-cells of $\mathcal{G}$, then $\mathbf{Ner}(\mathcal{G})$ has at that basepoint:

- $\pi_0(\mathbf{Ner}(\mathcal{G}))$ = the pointed set of categorical equivalence classes of the objects of $\mathcal{G}$, pointed by the class of 0.
- $\pi_1(\mathbf{Ner}(\mathcal{G}))$ = the group of homotopy classes of 1-cells of the form $f : 0 \longrightarrow 0$ under (horizontal) tensor composition of 1-cells
  = $\pi_0(\mathcal{G}(0,0))$, the set of connected components of the groupoid $\mathcal{G}(0,0) =_{\text{DEF}} \mathcal{G}(0)$, whose objects are 1-cells $f \colon 0 \longrightarrow 0$ and whose arrows are the 2-cell isomorphisms $\alpha \colon f \Longrightarrow g$ and whose nerve is $\Omega(\mathbf{Ner}(\mathcal{G}))$ at the basepoint 0.
  = $\pi_1(cl(\mathcal{G}))$, where $cl(\mathcal{G})$ denotes, as in Section 2, the groupoid which has the same objects as $\mathcal{G}$ but has 2-cell isomorphism classes of 1-cells for arrows and is the *fundamental groupoid* $\Pi_1(\mathbf{Ner}(\mathcal{G}))$ of the Kan complex $\mathbf{Ner}(\mathcal{G})$.
- $\pi_2(\mathbf{Ner}(\mathcal{G}))$ = the abelian group of 2-simplices all of whose 1-simplex faces are at the base point $s_0(0) \colon 0 \longrightarrow 0$[2]
  = $\text{Aut}(1_0)$ in the groupoid $\mathcal{G}(0,0))$, where $1_0 = s_0(0) \colon 0 \longrightarrow 0$ is the pseudo-identity 1-cell for 0 under tensor composition
  = $\pi_1(\mathcal{G}(0))$ in the notation of Section 2, and equivalently, $\pi_1(\Omega(\mathbf{Ner}(\mathcal{G}))$ in conventional simplicial notation.
- For $i \geq 3$, $\pi_i(\mathbf{Ner}(\mathcal{G})) = \mathbf{0}$, since, by definition, the canonical map
$$\mathbf{Ner}(\mathcal{G}) \longrightarrow \text{Cosk}^3(\mathbf{Ner}(\mathcal{G}))$$
  is an isomorphism and this forces all higher dimensional homotopy groups of the pointed Kan complex $\mathbf{Ner}(\mathcal{G})$ to be trivial.

Now it is easy to verify that simplicial maps between nerves of bigroupoids correspond exactly to strictly unitary homomorphisms $P \colon \mathcal{G} \longrightarrow \mathcal{H}$ of bigroupoids. With this in mind and choosing $P(0) = 0$ as the base point of $\mathcal{H}$, we obtain a pointed simplicial mapping of Kan complexes $\mathbf{Ner}(P) \colon \mathbf{Ner}(\mathcal{G}) \longrightarrow \mathbf{Ner}(\mathcal{H})$. Thus all one need note is that the nerve of the "homotopy fiber bigroupoid" $\mathcal{F}_0$ of Section 2 is just $\Gamma(\mathbf{Ner}(P))$, $\mathcal{F}_0(0) \simeq \Omega(\Gamma(\mathbf{Ner}(P)))$, and that the long exact sequence of the pointed simplicial mapping $\mathbf{Ner}(P)$ is precisely the nine term sequence of Corollary 2.3.

Similar remarks apply to Brown's original paper: every category $\mathcal{G}$ has canonically associated to a simplicial set, its Grothendieck nerve, whose n-simplices can be identified with "composable sequences of length n of arrows of the category". The resulting simplicial set is a Kan complex if, and only if, every arrow of $\mathcal{G}$ is invertible, *i.e.*, $\mathcal{G}$ is a groupoid. For any object 0 in $\mathbf{Ner}(\mathcal{G})$ as a base point, $\mathbf{Ner}(\mathcal{G})$ has only the pointed set of isomorphism classes of its objects as π_0 and $Aut(0)$ as π_1. The long exact sequence above then reduces to a six term one of exactly the same form.

[2]The complex is minimal in this dimension (and higher), so homotopic 2-simplices are equal.

Note that in both cases, the weakest tenable notion of $f\colon X \longrightarrow Y$ is a *fibration* is that which guarantees that for any choice of basepoint, the canonical simplicial mapping from the true fiber $Fib(f)$ of the mapping f to the "homotopy fiber" $\Gamma(f)$ of the same mapping be a weak equivalence.

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Applications of Categorical Galois Theory in Universal Algebra

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Abstract. We introduce factor permutable categories, which are a common generalization of Maltsev categories, of congruence modular varieties and of strongly unital varieties. We prove some basic centrality properties in these categories, which are useful to investigate several new aspects of the categorical theory of central extensions. We extend the equivalence between two natural and independent notions of central extension to exact factor permutable categories. These results are applied to semi-abelian categories, where several characterizations of central extensions are given, on the model of the category of groups. We finally show that the so-called Galois pregroupoid associated with an extension is an internal groupoid in any semi-abelian category.

Introduction

The categorical Galois theory developed by Janelidze [29] has contributed to clarify the deep relationship between several independent investigations in universal algebra and in homological algebra.

More specifically, the categorical theory of central extensions [31], which is a special case of it, provides a complete classification of the category $Centr(B)$ of central extensions of an object B in any exact category $\mathcal{C}$, where the property of centrality depends on the choice of an admissible subcategory $\mathcal{X}$ of $\mathcal{C}$. On the one hand, this theory includes several classical homological descriptions of $Centr(B)$ in the varieties of groups, rings, associative algebras and Lie algebras. More generally, it extends the theory developed by Frölich [22], Lue [39] and Furtado-Coelho [23] in the context of Ω-groups. On the other hand, the categorical notion of central extension includes the one naturally arising from the theory of commutators in

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universal algebra [32], [25]. More precisely, when $\mathcal{C}$ is a congruence modular variety and $\mathcal{X}$ is its admissible subvariety of abelian algebras, an extension $f\colon A \to B$ is *categorically central* (i.e. central in the sense of the categorical Galois theory) if and only if it is *algebraically central*, by which we mean that its kernel congruence $R[f]$ is contained in the centre of A, or, equivalently, $[R[f], \nabla_A] = \Delta_A$ (where ∇_A and Δ_A are the largest and the smallest congruences on A).

The notion of *factor permutable* category introduced in the present paper provides an appropriate framework to develop several new aspects of the categorical theory of central extensions. This notion is a categorical formulation of the axiom defining factor permutable varieties [26]: among the examples of factor permutable categories there are regular Maltsev categories, modular varieties and strongly unital varieties. We develop some basic aspects of a theory of centrality in these categories and establish some stability properties of the abelian objects. We prove that there is a direct product decomposition of central equivalence relations. This important property allows us to establish the equivalence between the notions of categorically central extension and of algebraically central extension in exact factor permutable categories. This theorem extends various previous results in this direction [32], [33], [25], and confirms the importance of the structural approach to commutator theory, which is at the same time simple and general.

We then consider some further aspects of the theory in the richer context of semi-abelian categories [34]. This part has been developed in collaboration with D. Bourn in [14]. We provide several characterizations of central extensions in semi-abelian categories on the model of the category of groups. We then show that the notion of central extension becomes intrinsic in this context: an extension $f\colon A \to B$ is central if and only if the diagonal $s\colon A \to R[f]$ is a kernel. We finally consider some applications of the categorical Galois theory, which provides in particular a complete classification of the category of central extensions. In the context of semi-abelian varieties, the category $Centr(B)$ can be decribed as a category of actions of the internal Galois groupoid of an appropriate extension.

The paper is structured as follows:

1. Internal structures
2. Connectors and commutators in algebra
3. Abelian objects in factor permutable categories
4. Algebraically central extensions
5. Categorically central extensions
6. Equivalence of the two notions
7. Central extensions in semi-abelian categories
8. Galois groupoids

In the first two sections we revise some basic categorical structures and we explain their relationship with the commutator of congruence relations as defined in universal algebra. In sections 3, 4, 5 and 6 we study the properties of abelian objects, of algebraically central extensions and of categorically central extensions in exact factor permutable categories. We include a brief and self-contained account of the categorical theory of central extensions in the form needed for our purposes. In the last two sections we work in the richer context of semi-abelian categories, where many stronger results can be established. For instance, the so-called Galois pregroupoid associated with an extension is proved to be always an internal groupoid.

1 Internal structures

In this section we briefly introduce some internal categorical structures, which will play a central role in the following. We refer to [2] or [40] for more details. In this article $\mathcal{C}$ will always denote a finitely complete category.

Internal categories and groupoids. We begin with some basic definitions:

1.1. Definition. An *internal category* X in $\mathcal{C}$ is a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_2"'] & X_0, \arrow[l, "s" description] \end{tikzcd}$$

where X_0 can be thought as the "object of objects", X_1 as the "object of arrows", $X_1 \times_{X_0} X_1$ as the "object of composable pairs of arrows" given by the following pullback

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_2"'] & X_0. \end{tikzcd}$$

The morphisms d_1, d_2 are called "domain" and "codomain" respectively, s is the "identity", p_1, p_2 are the projections, m is the "composition". These data must satisfy the usual axioms:

1. $d_1 \circ s = 1_{X_0} = d_2 \circ s$
2. $m \circ (1_{X_1}, s \circ d_2) = 1_{X_1} = m \circ (s \circ d_1, 1_{X_1})$
3. $d_1 \circ p_1 = d_1 \circ m, \quad d_2 \circ p_2 = d_2 \circ m$
4. $m \circ (1_{X_1} \times_{X_0} m) = m \circ (m \times_{X_0} 1_{X_1})$

(where the domain of the two composites in 4. is the "object of triples of composable arrows" $X_1 \times_{X_0} X_1 \times_{X_0} X_1$: it can be obtained as the pullback of $d_2 \circ p_2$ along d_1 or, equivalently, as the pullback of $d_1 \circ p_1$ along d_2).

1.2. Definition. An *internal groupoid* in $\mathcal{C}$ is an internal category in $\mathcal{C}$ equipped with a morphism $\sigma \colon X_1 \to X_1$, called the "inversion", such that

1. $d_1 \circ \sigma = d_2, \qquad d_2 \circ \sigma = d_1$
2. $m \circ (1_{X_1}, \sigma) = s \circ d_1, \qquad m \circ (\sigma, 1_{X_1}) = s \circ d_2$

Of course, when $\mathcal{C}$ is the category *Sets* of sets, an internal category (groupoid) is just an ordinary small category (groupoid). We denote by $Cat(\mathcal{C})$ and by $Grpd(\mathcal{C})$ the categories of internal categories and of internal groupoids in $\mathcal{C}$, respectively. The arrows in these categories are called *internal functors*: they are pairs (f_0, f_1) of arrows in $\mathcal{C}$ as in the diagram

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, "m_X"] \arrow[d, dotted, "f_2"] & X_1 \arrow[r, shift left=2, "d_2"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "f_1"] & X_0 \arrow[l, "s" description] \arrow[d, "f_0"] \\ Y_1 \times_{Y_0} Y_1 \arrow[r, "m_Y"] & Y_1 \arrow[r, shift left=2, "d_2"] \arrow[r, shift right=2, "d_1"'] & Y_0 \arrow[l, "s" description] \end{tikzcd}$$

such that

1. $d_1 \circ f_1 = f_0 \circ d_1, \qquad d_2 \circ f_1 = f_0 \circ d_2$

2. $f_1 \circ s = s \circ f_0$
3. $f_1 \circ m_X = m_Y \circ f_2$

(where f_2 is the arrow induced by the universal property of the pullback).
The following kind of internal functors will be useful:

1.3. Definition. If X and Y are internal categories in C, an internal functor $(f_0, f_1)\colon X \to Y$ is a *discrete fibration* if the commutative square

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "d_2"'] & Y_1 \arrow[d, "d_2"] \\ X_0 \arrow[r, "f_0"'] & Y_0 \end{tikzcd}$$

is a pullback.

When X and Y are groupoids, an internal functor is a discrete fibration if and only if the commutative square $f_0 \circ d_1 = d_1 \circ f_1$ is a pullback.

Internal equivalence relations. An *internal relation* R from X to Y in a category C is a pair of arrows in C

$$X \xleftarrow{d_1} R \xrightarrow{d_2} Y,$$

which is jointly monic, i.e. the factorization $(d_1, d_2)\colon R \to X \times Y$ is a monomorphism. Again, in the category *Sets* of sets, this is equivalent to saying that $(d_1, d_2)\colon R \to X \times Y$ determines a relation in the usual sense on the set X. When $X = Y$ a relation $(d_1, d_2)\colon R \to X \times X$ is called a relation on X, and is also denoted by (R, X), or by R.

A relation R on X in C is *reflexive* if there is an arrow $s\colon X \to R$ with $d_1 \circ s = 1_X = d_2 \circ s$.

A relation R on X in C is *symmetric* if there is an arrow $\sigma\colon R \to R$ with $d_1 \circ \sigma = d_2$ and $d_2 \circ \sigma = d_1$.

Let

$$\begin{tikzcd} R \times_X R \arrow[r, "p_2"] \arrow[d, "p_1"'] & R \arrow[d, "d_1"] \\ R \arrow[r, "d_2"'] & X \end{tikzcd}$$

denote the pullback of d_1 along d_2. Then the relation R in C is *transitive* if there is an arrow $m\colon R \times_X R \to R$ such that $d_1 \circ m = d_1 \circ p_1$ and $d_2 \circ m = d_2 \circ p_2$.

1.4. Definition. A relation R on X is an *equivalence relation* if it is reflexive, symmetric and transitive.

1.5. Remark. Again, an internal equivalence relation in the category of sets is an equivalence relation in the usual sense. When C is a variety of universal algebras, an internal equivalence relation R on an algebra X in C is simply a *congruence*, i.e. an equivalence relation (on the underlying set of X) which is at the same time a subalgebra of $X \times X$.

Any internal equivalence relation (R, X) as just defined is in particular an internal groupoid with m and σ defined as above. We shall write $Eq(C)$ for the full subcategory of the category $Grpd(C)$ whose objects are the internal equivalence relations in $\mathcal{C}$. The category whose objects are the equivalence relations on a fixed object X in $\mathcal{C}$, with obvious arrows, will be denoted by $Eq_X(\mathcal{C})$.

The *kernel pair* $(p_1, p_2)\colon R[f] \to X \times X$ of any arrow $f\colon X \to Y$ is an equivalence relation. When an equivalence relation R on X is a kernel pair, R is said to be *effective*. The largest and the smallest equivalence relations ∇_X and Δ_X on a given object X are always effective: indeed, if $\tau_X\colon X \to 1$ is the unique arrow from an object X to the terminal object 1, then $\nabla_X = R[\tau_X]$, while $\Delta_X = R[1_X]$.

If $f\colon X \to Y$ is an arrow in $\mathcal{C}$ and R is an equivalence relation on Y, the inverse image of R along f is the equivalence relation $f^{-1}(R)$ defined by the following pullback:

$$\begin{tikzcd} f^{-1}(R) \arrow[r, "\tilde{f}"] \arrow[d, "{(d_1,d_2)}"'] & R \arrow[d, "{(d_1,d_2)}"] \\ X \times X \arrow[r, "f\times f"'] & Y \times Y \end{tikzcd}$$

Internal connectors. The notion of internal connector, introduced in [13], slightly modifies the notion of internal pregroupoid, which was discovered by Pedicchio to play a very important role in commutator theory [42], [43]. Unlike pregroupoids, connectors are also suitable to deal with non-effective equivalence relations [15]. The notion of internal pregroupoid goes back to the work of Kock [38].

If R and S are two equivalence relations on X, we denote by $R \times_X S$ the pullback

$$\begin{tikzcd} R \times_X S \arrow[r, "p_2"] \arrow[d, "p_1"'] & S \arrow[d, "d_1"] \\ R \arrow[r, "d_2"'] & X. \end{tikzcd}$$

1.6. Definition. An (internal) *connector* on R and S is an arrow $p\colon R \times_X S \to X$ in $\mathcal{C}$ such that

1. $p(x,x,y) = y$
2. $xSp(x,y,z)$
3. $p(x,y,p(y,u,v)) = p(x,u,v)$

$1^\star$. $p(x,y,y) = x$
$2^\star$. $zRp(x,y,z)$
$3^\star$. $p(p(x,y,u),u,v) = p(x,y,v)$.

Let us point out that when an arrow $p\colon R \times_X S \to X$ satisfies the conditions 1, $1^\star$, 2 and $2^\star$, then it satisfies also 3 and $3^\star$ if and only if it satisfies the classical associativity $p(x,y,p(z,u,v)) = p(p(x,y,z),u,v)$.

The relationship between connectors and internal groupoids is clarified in the first of the following examples:

1.7. Example. Let X be a graph in a category $\mathcal{C}$

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_2"'] & X_0 \arrow[l, "s"'] \end{tikzcd}$$

with $d_1 \circ s = 1_{X_0} = d_2 \circ s$, namely a *reflexive graph*. The connectors on $R[d_1]$ and $R[d_2]$ are in bijection with the groupoid structures on this reflexive graph.

Indeed, on the one hand, if p is a connector on $R[d_1]$ and $R[d_2]$ then the arrow $m\colon X_1 \times_{X_0} X_1 \to X_1$ defined by $m(x,y) = p(y,(s \circ d_2)(x),x)$ (for any (x,y) in $X_1 \times_{X_0} X_1$) gives the composition of a groupoid structure. On the other hand, if X is equipped with a groupoid composition m and an inversion σ, a connector is obtained by setting $p(x,y,z) = m(m(z,\sigma(y)),x)$ (for any (x,y,z) in $R[d_1] \times_{X_1} R[d_2]$).

1.8. Example. A ternary operation $p\colon X \times X \times X \to X$ is called an *associative Maltsev operation* when it satisfies the properties $p(x,x,y) = y$, $p(x,y,y) = x$ and $p(x,y,p(z,u,v)) = p(p(x,y,z),u,v)$. An (internal) associative Maltsev operation is therefore the same thing as a connector between ∇_X and ∇_X.

1.9. Example. Given two objects X and Y, there is always a canonical connector arising from the product $X \times Y$. By using the suitable product projections let us form the pullback

$$\begin{tikzcd}
X \times X \times Y \times Y \arrow[r, "1_X \times 1_X \times p_2"] \arrow[d, "p_1 \times 1_Y \times 1_Y"'] & X \times X \times Y \arrow[d, "p_1 \times 1_Y"] \\
X \times Y \times Y \arrow[r, "1_X \times p_2"'] & X \times Y.
\end{tikzcd}$$

Then the canonical connector $p\colon X \times X \times Y \times Y \to X \times Y$ on $R[p_X]$ and $R[p_Y]$ (where $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ are the projections) is defined by

$$p(x,x',y,y') = (x',y).$$

Internal double equivalence relations. If $\mathcal{C}$ is a finitely complete category, the category $Eq(\mathcal{C})$ is itself finitely complete. Therefore, one can consider the category $Eq(Eq(\mathcal{C}))$ of internal equivalence relations in $Eq(\mathcal{C})$, which is called the category of internal double equivalence relations in $\mathcal{C}$.

A double equivalence relation C in $\mathcal{C}$ can be represented by a diagram of the form

$$\begin{tikzcd}
C \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & S \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] \\
R \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] & X,
\end{tikzcd}$$

where each pair of parallel arrows represents an internal equivalence relation in $\mathcal{C}$, and

$$d_1 \circ \pi_1 = d_1 \circ p_1, d_1 \circ \pi_2 = d_2 \circ p_1, d_2 \circ \pi_1 = d_1 \circ p_2, d_2 \circ \pi_2 = d_2 \circ p_2.$$

C is called a *double equivalence relation on R and S*. Given two equivalence relations R and S on X there is a canonical largest double equivalence relation on R and S, which is denoted by $R\Box S$ (see [20],[36]):

$$\begin{tikzcd} R\Box S \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & S \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] \\ R \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] & X. \end{tikzcd}$$

The double relation $R\Box S$ is defined by the following pullback:

$$\begin{tikzcd} R\Box S \arrow[r] \arrow[d] & R\times R \arrow[d, "(d_1\times d_1, d_2\times d_2)"] \\ S\times S \arrow[r, "(d_1,d_2)\times(d_1,d_2)"'] & X\times X\times X\times X \end{tikzcd}$$

$R\Box S$ is the subobject of X^4 consisting, in the set-theoretical context, of the quadruples (x,y,t,z) with xRy, tRz, xSt and ySz. Any double equivalence relation C on R and S is contained in $R\Box S$: an element in C will be denoted also by a matrix

$$\begin{pmatrix} x & t \\ y & z \end{pmatrix}$$

The following kind of double equivalence relations will be useful:

1.10. Definition. [20], [44] A double equivalence relation C on R and S in $\mathcal{C}$ as above is called a *centralizing relation* when the following square is a pullback:

$$\begin{tikzcd} C \arrow[r, "p_2"] \arrow[d, "\pi_1"'] & S \arrow[d, "d_1"] \\ R \arrow[r, "d_2"'] & X. \end{tikzcd}$$

Connectors and centralizing relations actually determine the same structure:

1.11. Lemma. [20], [15] *If R and S are two equivalence relations on the same object X, then the following conditions are equivalent:*

1. *there is a connector on R and S*
2. *there is a centralizing relation on R and S*

Proof 1. $\Rightarrow$ 2. If $p\colon R\times_X S \to X$ is a connector between R and S, then by defining $\pi_1(x,y,z) = (x,p(x,y,z))$ and $\pi_2(x,y,z) = (p(x,y,z),z)$ one gets a centralizing relation on R and S:

$$\begin{tikzcd} R \times_X S \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "\pi_2"] & S \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] \\ R \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] & X \end{tikzcd}$$

2. $\Rightarrow$ 1. If

$$\begin{tikzcd} C \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & S \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] \\ R \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] & X \end{tikzcd}$$

is a centralizing relation on R and S, then the arrow $d_2 \circ p_1 \colon C = R \times_X S \to S \to X$ defines a connector between R and S. $\square$

2 Connectors and commutators in algebra

In this section we recall some basic properties of regular, exact and Maltsev categories that will be useful in the following. We show how the notion of connector is deeply related to the notion of commutator of congruences in universal algebra. The discovery of this relationship is due to the pioneering work by Pedicchio [42], [43], and by Pedicchio and Janelidze [35], [36]. Recently, some new results in this direction have been obtained by Bourn and Gran [14], [15], [16], [12].

For the sake of simplicity we shall explain the relationship between connectors and commutators in the context of exact Maltsev categories. For more general varieties of universal algebras or categories the reader may consult [36], [15], [16].

Connectors in Maltsev categories. First of all, let us consider the case of the category *Grp* of groups. It is well known that in the category of groups there is a bijection between the congruences (=internal equivalence relations) and the normal subgroups. With a normal subgroup H of a group G one associates the congruence R_H defined by xR_Hy if and only if $x \cdot y^{-1}$ is in H. Conversely, with a congruence R on G one associates the normal subgroup $H = \{g \in G \mid gR1\}$ of G given by the elements in G which are in relation R with the unit element 1 of the group.

In the category of groups the existence of a connector between the congruences R_H and R_K on a group G is equivalent to the fact that the corresponding normal subgroups H and K commute in the usual sense, namely $[H, K] = \{1\}$. Let us recall the argument. If there is a connector $p \colon R_H \times_G R_K \to G$ on R_H and R_K, then we can construct an arrow $\alpha \colon H \times K \to G$ defined by $\alpha(h, k) = p(h, 1, k)$, which is a group homomorphism. Then, for all $h \in H$ and $k \in K$, we have: $h \cdot k = \alpha(h, 1) \cdot \alpha(1, k) = \alpha((h, 1) \cdot (1, k)) = \alpha(h, k) = \alpha((1, k) \cdot (h, 1)) = \alpha(1, k) \cdot \alpha(h, 1) = k \cdot h$. Conversely, suppose that $H \cdot K = K \cdot H$. Then the map $p \colon R_H \times_G R_K \to G$ defined by $p(h, k, l) = h \cdot k^{-1} \cdot l$ is a group homomorphism: indeed, for any $(a, b), (d, e) \in R_H$

and $(b,c),(e,f) \in R_K$ we have $(b^{-1}\cdot c)\cdot(d\cdot e^{-1}) = (d\cdot e^{-1})\cdot(b^{-1}\cdot c)$, since $b^{-1}\cdot c \in K$ and $d\cdot e^{-1} \in H$. This implies that $(a\cdot b^{-1}\cdot c)\cdot(d\cdot e^{-1}\cdot f) = (a\cdot d)\cdot(b\cdot e)^{-1}(c\cdot f)$, as desired. The axioms of connector are clearly satisfied by p, and it follows that p is a connector on R_H and R_K.

Similarly, in the category *Rng* of rings, two ideals I and J centralize each other in the usual sense, namely $IJ + JI = 0$, if and only if there is a connector on the corresponding congruences R_I and R_J.

It is useful to mention that a connector on two congruences in the category of groups or rings is necessarily unique, when it exists (a proof of this fact will be given in Theorem 2.6 in a more general context).

The fact that the existence of a connector on two congruences corresponds to the triviality of the commutator holds in very general categories. In order to explain this fact let us recall the following important notion due to Carboni, Lambek and Pedicchio:

2.1. Definition. [19] A finitely complete category $\mathcal{C}$ is a *Maltsev* category if any internal reflexive relation in $\mathcal{C}$ is an equivalence relation.

The terminology is motivated by the classical Maltsev Theorem:

2.2. Theorem. [41] *A variety $\mathcal{V}$ of universal algebras is Maltsev if and only if its theory has a ternary term $p(x,y,z)$ satisfying the axioms $p(x,x,y) = y$ and $p(x,y,y) = x$.*

Many varieties are Maltsev: groups, rings, Heyting algebras, Lie algebras, associative algebras, quasigroups and crossed modules. Among the examples of Maltsev categories there are also any abelian category, the dual of an elementary topos, the category of torsion-free abelian groups, the categories of Hausdorff groups and of topological groups [18].

As we shall see here below, the Maltsev property is equivalent to the fact that any internal relation is difunctional. Let us recall this latter notion:

2.3. Definition. A relation $R \rightarrowtail X \times Y$ from X to Y is *difunctional* if, whenever xRy, zRy and zRt, one then has that xRt.

Remark that the difunctionality property can be expressed in any finitely complete category (actually, pullbacks suffice). Indeed, let T be the object

$$T = \{(x,y,z,t) \in X \times Y \times X \times Y \mid xRy,\ zRy \,\text{and}\, zRt\},$$

which can be constructed as a limit. Then the relation R is difunctional if the canonical projections $p_1 \colon T \to X$ and $p_4 \colon T \to Y$ both factorize through R, i.e. there is an arrow $n \colon T \to R$ with $d_1 \circ n = p_1$ and $d_2 \circ n = p_4$. It is well known that a reflexive relation is an equivalence relation if and only if it is an internal groupoid: in the same way a relation $(d_1, d_2) \colon R \rightarrowtail X \times Y$ is difunctional if and only if it is a pregroupoid [36] (i.e. if and only if there is a connector on $R[d_1]$ and $R[d_2]$).

We are now ready to prove the following theorem:

2.4. Theorem. [20] *Let $\mathcal{C}$ be a finitely complete category. The following conditions are equivalent:*

1. *$\mathcal{C}$ is a Maltsev category*
2. *any reflexive relation in $\mathcal{C}$ is symmetric*
3. *any reflexive relation in $\mathcal{C}$ is transitive*
4. *for any $X, Y \in \mathcal{C}$ any relation $R \rightarrowtail X \times Y$ is difunctional*

5. *for any $X \in \mathcal{C}$ any relation $R \rightarrowtail X \times X$ is difunctional*

Proof For any relation $R \rightarrowtail X \times Y$ one defines a new relation $S \rightarrowtail R \times R$ as follows:

$$(a,b)S(c,d) \quad \Leftrightarrow \quad aRd.$$

This relation is reflexive by definition.
1. $\Rightarrow$ 2. Trivial.
2. $\Rightarrow$ 3. If aRb and bRc, then $(b,c)S(a,b)$ and by symmetry $(a,b)S(b,c)$. It follows that aRc.
3. $\Rightarrow$ 4. If aRb, cRb and cRd, then $(a,b)S(c,b)$ and $(c,b)S(c,d)$, so by transitivity $(a,b)S(c,d)$ and aRd.
4. $\Rightarrow$ 5. Trivial.
5. $\Rightarrow$ 1. Let R be a reflexive relation on X and let us prove that it is symmetric: if aRb, then from bRb aRb and aRa it follows that bRa. R is also transitive: if aRb and bRc, then aRb, bRb and bRc imply that aRc. □

In a Maltsev category the notion of internal connector becomes very simple:

2.5. Lemma. [42], [13] *Let $\mathcal{C}$ be a Maltsev category and let R and S be two equivalence relations on X. For an arrow $p\colon R \times_X S \to X$ the following conditions are equivalent:*

1. *p has the property that $p(x,y,y) = x$ and $p(x,x,y) = y$*
2. *p is a connector between R and S*

Proof 1. $\Rightarrow$ 2. Let us check the condition $zRp(x,y,z)$. For this, define a relation $T \rightarrowtail X^2 \times X$ as follows: $(x,y)Tz$ if and only if $(x,y,z) \in R \times_X S$ and $(z, p(x,y,z)) \in R$. Clearly, for any $(x,y,z) \in R \times_X S$, one has that $(x,y)Ty$, $(y,y)Ty$ and $(y,y)Tz$: by difunctionality it follows that $(x,y)Tz$. The condition $xSp(x,y,z)$ is proved similarly. In order to check that $p(p(x,y,z),u,v) = p(x,y,p(z,u,v))$, one defines a relation $W \rightarrowtail X^2 \times X^3$ by setting $(x,y)W(z,u,v)$ if and only if $(x,y,z) \in R \times_X S$, $(z,u,v) \in R \times_X S$ and $p(p(x,y,z),u,v) = p(x,y,p(z,u,v))$. From $(x,y)W(z,u,u)$,$(z,z)W(z,u,u)$ and $(z,z)W(z,u,v)$ it follows that $(x,y)W(z,u,v)$ for any x,y,z,u,v with the property that $(x,y,z) \in R \times_X S$ and $(z,u,v) \in R \times_X S$. □

Moreover, having a connector for two equivalence relations R and S on X becomes a property:

2.6. Theorem. [42], [13] *When there is a connector $p\colon R \times_X S \to X$ on R and S in a Maltsev category, it is necessarily unique.*

Proof Given two connectors p and p' on R and S, one defines a relation $D \rightarrowtail X^2 \times X$ as follows: $(x,y)Dz$ if and only if $(x,y,z) \in R \times_X S$ and $p(x,y,z) = p'(x,y,z)$. Then, for any $(x,y,z) \in R \times_X S$ one clearly has that $(x,y)Dy$, $(y,y)Dy$ and $(y,y)Dz$. By difunctionality of the relation D it follows that $(x,y)Dz$, as desired. □

Regular and exact Maltsev categories. We now revise some properties of the permutability of the composition of equivalence relations in regular and Barr-exact categories. We show that the Maltsev property is equivalent to the permutability of the composition of equivalence relations.

2.7. Definition. A finitely complete category $\mathcal{C}$ is *regular* when kernel pairs have coequalizers and regular epimorphisms are stable under pullbacks.

In a regular category any map $f\colon X \to Y$ factorizes through a unique smallest subobject $i\colon I \rightarrowtail Y$ of Y called its "regular image". Given two relations $(d_1, d_2)\colon R \rightarrowtail X \times Y$ and $(d_1, d_2)\colon S \rightarrowtail Y \times Z$ in a regular category, one can define the composite $S \circ R$ as the regular image of the arrow $(d_1 \circ p_1, d_2 \circ p_2)\colon R \times_Y S \to X \times Z$

$$\begin{tikzcd} & & R \times_Y S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\ & R \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & & S \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\ X & & Y & & Z, \end{tikzcd}$$

where $(R \times_Y S, p_1, p_2)$ is the pullback of $d_2\colon R \to Y$ along $d_1\colon S_2 \to Y$. In a regular category the composition of relations is associative. A map $f\colon X \to Y$ can be considered as a relation by identifying it with its graph $(1_X, f)\colon X \rightarrowtail X \times Y$. In particular the identity map $1_X\colon X \to X$ gives the equivalence relation Δ_X.

For any relation $(d_1, d_2)\colon R \rightarrowtail X \times Y$ one can consider the opposite relation R^o given by $(d_2, d_1)\colon R \rightarrowtail Y \times X$. The fact that a relation $(d_1, d_2)\colon R \rightarrowtail X \times X$ is reflexive can be expressed by saying that $\Delta_X \leq R$; similarly $R = R^o$ says that the relation R is symmetric, and $R \circ R \leq R$ that R is transitive.

Given two maps $f\colon X \to Y$ and $g\colon Z \to Y$ one can consider their composite as relations $g^o \circ f$ which is simply the pullback of f and g. In particular, when $f = g$, the relation $f^o \circ f\colon X \times_Y X \rightarrowtail X \times X$ is the kernel pair of f. One can check also the following useful facts: 1) for any relation $(d_1, d_2)\colon R \rightarrowtail X \times Y$ one has that $R = d_2 \circ d_1^o$, 2) for any regular epimorphism $f\colon X \to Y$ one has that $f \circ f^o = \Delta_Y$.

2.8. Remark. In a regular category a relation $R \rightarrowtail X \times Y$ is *difunctional* exactly when $R \circ R^o \circ R = R$. It is clear that any relation f that is a map is difunctional, so that $f \circ f^o \circ f = f$.

In a regular category the Maltsev property corresponds to the permutability of the equivalence relations:

2.9. Theorem. [18], [19] *Let $\mathcal{C}$ be a regular category. The following statements are equivalent:*

1. *$\mathcal{C}$ is a Maltsev category*
2. *for any $X \in \mathcal{C}$, $\forall R, S \in Eq_X(\mathcal{C})$ the composite $R \circ S$ of two equivalence relations on X is an equivalence relation*
3. *the composition of equivalence relations is permutable: for any $X \in \mathcal{C}$, $\forall R, S \in Eq_X(\mathcal{C})$, one has $R \circ S = S \circ R$*

Proof 1. $\Rightarrow$ 2. It follows from the fact that $\Delta_X \leq R \circ S$.

2. $\Rightarrow$ 3. By assumption $R \circ S$ and $S \circ R$ are equivalence relations; this implies that $(R \circ S) \circ (R \circ S) \leq R \circ S$ and $(S \circ R) \circ (S \circ R) \leq S \circ R$. Then

$$R \circ S = \Delta_X \circ R \circ S \circ \Delta_X \leq (S \circ R) \circ (S \circ R) \leq S \circ R$$

and similarly $S \circ R \leq R \circ S$.
3. $\Rightarrow$ 1. We are going to prove that any relation $R \rightarrowtail X \times Y$ is difunctional (the result will then follow by Theorem 2.4). If $R = d_2 \circ d_1^o$, one has

$$R \circ R^o \circ R = d_2 \circ d_1^o \circ d_1 \circ d_2^o \circ d_2 \circ d_1^o = d_2 \circ d_2^o \circ d_2 \circ d_1^o \circ d_1 \circ d_1^o = d_2 \circ d_1^o = R,$$

since by assumption the equivalence relations $d_1^o \circ d_1$ and $d_2^o \circ d_2$ permute, and the difunctionality always holds for relations that are maps. $\square$

Later on we shall be interested in regular categories which are also Barr-exact:

2.10. Definition. [1] A regular category $\mathcal{C}$ is *Barr-exact* if any equivalence relation in $\mathcal{C}$ is effective.

In an exact category, when two equivalence relations R and S permute, the following result holds:

2.11. Theorem. [18] *Let $\mathcal{C}$ be an exact category. Let r and s be two regular epimorphisms with the same domain X and kernel pairs R and S, respectively. Assume that $R \circ S = S \circ R$ and that the exterior part of the diagram*

$$\begin{tikzcd}
X \arrow[dr, dotted, "w"] \arrow[drr, bend left, "s"] \arrow[ddr, bend right, "r"'] & & \\
& Y \times_W Z \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & Z \arrow[d, "v"] \\
& Y \arrow[r, "u"] & W
\end{tikzcd}$$

is a pushout. Then the comparison map w to the pullback is a regular epimorphism.

Proof Let T be the kernel pair of $t = v \circ s = u \circ r$. Since the exterior of the diagram is a pushout, $t = r \wedge s$ as quotients of X, hence T is the join $R \vee S$ of R and S in $Eq_X(\mathcal{C})$. By assumption we have that

$$R \circ S = S \circ R = R \vee S = T.$$

Since $R = r^o \circ r$, $S = s^o \circ s$, $\Delta_Y = r \circ r^o$ and $\Delta_Z = s \circ s^o$, it follows that

$$s^o \circ s \circ r^o \circ r = S \circ R = T = t^o \circ t,$$

from which, by multiplying on the left by s and on the right by r^o,

$$s \circ s^o \circ s \circ r^o \circ r \circ r^o = s \circ (v \circ s)^o \circ (u \circ r) \circ r^o;$$

hence, $s \circ r^o = s \circ s^o \circ v^o \circ u \circ r \circ r^o = v^o \circ u$. Since $v^o \circ u$ is the pullback of v and u and $s \circ r^o$ is the image of $(r, s)\colon X \to Y \times Z$, one concludes that w is a regular epimorphism. $\square$

In particular, let us notice that any pushout of regular epimorphisms in an exact Maltsev category satisfies the assumptions of the previous Theorem.

Connectors and commutators. In any exact Maltsev category with coequalizers one can define the commutator of two equivalence relations R and S [42], [43], which extends the one originally defined by Smith in Maltsev varieties [44].

We assume that $\mathcal{C}$ is an exact Maltsev category with coequalizers. As in the case of varieties of universal algebra, one first defines an equivalence relation Δ_R^S on S: Δ_R^S is the kernel pair of the coequalizer $q\colon S \to Q$ of the morphisms $s \circ d_1 \colon R \to X \to S$ and $s \circ d_2 \colon R \to X \to S$. When $\mathcal{C}$ is a Maltsev variety, Δ_R^S is the smallest congruence on S containing all pairs of the form $((x,x),(y,y))$, in which (x,y) is in R.

By definition Δ_R^S is an equivalence relation on S. It is also an equivalence relation on R, because it is easily seen to be a reflexive relation, and $\mathcal{C}$ is a Maltsev category:

$$\begin{tikzcd}
\Delta_R^S \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, dotted, shift right, "d_1"'] \arrow[d, dotted, shift left, "d_2"] & S \arrow[r, "q"] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] & Q \arrow[d, dotted, shift right, "d_1"'] \arrow[d, dotted, shift left, "d_2"] \\
R \arrow[ur, shift left, "s\circ d_1"] \arrow[ur, shift right, "s\circ d_2"'] \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] & X \arrow[r, "p"'] & \frac{X}{R}.
\end{tikzcd}$$

Accordingly, Δ_R^S is a double relation on R and S. The commutator of R and S is then defined (see [42], [43]) as

$$[R,S] = \{(x,y) \in X \times X \mid \exists z \in X \quad \textit{with} \quad \begin{pmatrix} z & x \\ z & y \end{pmatrix} \in \Delta_R^S\}.$$

Categorically, the relation $[R,S]$ on X can be constructed as follows: let $R[d_1]$ denote the kernel pair of $d_1 \colon S \to X$ and let $R[d_1] \cap \Delta_R^S$ denote the intersection of $R[d_1]$ and Δ_R^S as relations on S. Then, if i is the inclusion of $R[d_1] \cap \Delta_R^S$ in Δ_R^S and $\nu = (d_2 \circ \pi_1, d_2 \circ \pi_2) \colon \Delta_R^S \to X \times X$, then the commutator $[R,S]$ is the regular image of the arrow $\nu \circ i$.

The fact that Δ_R^S is a double equivalence relation on R and S implies that $[R,S]$ is a reflexive relation on X, hence an equivalence relation, because $\mathcal{C}$ is Maltsev.

Now, the crucial observation, due to Pedicchio, relating commutators and connectors is the following

2.12. Theorem. [42] *Let R and S be two equivalence relations on X in an exact Maltsev category with coequalizers. Then the following conditions are equivalent:*

1. $[R,S] = \Delta_X$
2. *there is a connector on R and S*

Proof Let us first prove that $[R,S]=\Delta_X$ if and only if the canonical factorization $\alpha\colon \Delta_R^S \to R\times_X S$ through the usual pullback is a monomorphism:

$$\begin{tikzcd}
\Delta_R^S \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "\pi_2"] \arrow[ddr, bend right, "d_1"'] & & \\
& R\times_X S \arrow[r, "p_2"'] \arrow[d, "p_1"] & S \arrow[d, "d_1"'] \\
& R \arrow[r, "d_2"] & X.
\end{tikzcd}$$

For this, let us assume that $[R,S]=\Delta_X$ and that

$$\alpha\begin{pmatrix} x & t \\ y & z \end{pmatrix} = \alpha\begin{pmatrix} x & t' \\ y & z \end{pmatrix}.$$

By the symmetry and the transitivity of Δ_R^S on S, one knows that $\begin{pmatrix} x & t \\ x & t' \end{pmatrix}$ is in Δ_R^S, hence (t,t') is in $[R,S]$. It follows that $t=t'$. Conversely, let α be a mono and let (t,t') be in $[R,S]$. Then, on the one hand, $\begin{pmatrix} z & t \\ z & t' \end{pmatrix}$ is in Δ_R^S for some z in X; on the other hand $\begin{pmatrix} z & t' \\ z & t' \end{pmatrix}$ is in Δ_R^S because Δ_R^S is a double equivalence relation. Since α is a monomorphism, it follows that $t=t'$.

To complete the proof one needs to observe that this factorization α is always a regular epi thanks to Theorem 2.11. It follows that Δ_R^S is a centralizing relation on R and S if and only if $[R,S]=\Delta_X$. By Lemma 1.11 this fact is equivalent to the fact that there is a connector on R and S. $\square$

The discovery of the relationship between connectors and commutators opened the way to the categorical approach to centrality. Indeed, the main stability properties of the commutators correspond to the properties of some internal categorical structures [36], [15], [16], [12].

3 Abelian objects in factor permutable categories

In this section we define *factor permutable categories* and prove some basic centrality properties in these categories. Our definition is a categorical version of Gumm's definition of a factor permutable variety [26]. Abelian objects and central relations still behave well in these categories, and this fact makes them suitable to extend several results of the categorical theory of central extensions.

Let $p_A\colon A\times B\to A$ and $p_B\colon A\times B \to B$ denote the product projections:

3.1. Definition. A regular category $\mathcal{C}$ is *factor permutable* if any equivalence relation R on a direct product $A\times B$ permutes with $R[p_A]$ and with $R[p_B]$.

3.2. Examples. 1. By Theorem 2.9, any regular Maltsev category is factor permutable.
2. Any congruence modular variety is factor permutable. This important property was proved by Gumm (Corollary 4.5 in [26]). Hence in particular distributive varieties are factor permutable, as for instance the variety of lattices and the variety of implication algebras.

3. Any strongly unital variety is factor permutable. In particular, the variety of left closed magmas is strongly unital [10]. Strongly unital varieties have been characterized as follows [5]: a variety $\mathcal{V}$ is strongly unital if and only if its theory has exactly one constant 0 and a ternary operation $p(x,y,z)$ satisfying the axioms $p(x,x,y)=y$ and $p(x,0,0)=x$. To see that these varieties are factor permutable [26], consider a product $A\times B$ in a strongly unital variety $\mathcal{V}$. Let R be a congruence on $A\times B$, and let $(a,b)R[p_A](a,c)R(d,e)$. Then

$$p((a,c),(0,c),(0,b))\,R\,p((d,e),(0,c),(0,b))$$

gives $(a,b)R(d,p(e,c,b))$. This implies that $(a,b)R(d,p(e,c,b))R[p_A](d,e)$, as desired.

The following result will be very useful in the following:

3.3. Lemma. *If $\mathcal{C}$ is a factor permutable category, then the* weak shifting property *holds: for any equivalence relation R and S on $A\times B$ such that $R\cap R[p_A]\leq S$, given $(a,b),(a,c),(d,e)$ and (d,f) related as in the diagram*

$$\begin{tikzcd}
(a,b) \arrow[r, dash, "{R[p_A]}"] \arrow[d, dash, "R"'] \arrow[d, dash, bend right=70, "S"'] & (a,c) \arrow[d, dash, "R"] \\
(d,e) \arrow[r, dash, "{R[p_A]}"'] & (d,f),
\end{tikzcd}$$

one always has that $(a,c)S(d,f)$.

Proof By the factor permutability of $\mathcal{C}$, the assumption $(d,e)(R\cap S)\circ R[p_A](a,c)$ implies that there is an element (d,g) such that $(d,e)R[p_A](d,g)R\cap S(a,c)$, so that $(d,f)R(a,c)R\cap S(d,g)$ and then $(d,f)R\cap R[p_A](d,g)$. By the assumption that $R\cap R[p_A]\leq S$ it follows that $(d,f)S(d,g)$, and then $(d,f)S(a,c)$. □

Of course, the same kind of property holds when we consider equivalence relations R and S on $A\times B$ such that $R\cap R[p_B]\leq S$. The weak shifting property is equivalent to the following categorical property:

3.4. Lemma. *Given a product $A\times B$ in a factor permutable category, given an equivalence relation R on $A\times B$ the following property holds: for any $U\in Eq_{A\times B}(\mathcal{C})$ with $R\cap R[p_A]\leq U\leq R$ the canonical inclusion of equivalence relations*

$$\begin{tikzcd}
U\Box R[p_A] \arrow[r, tail, "j"] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] \arrow[dr, phantom, "(1)"] & R\Box R[p_A] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] \\
U \arrow[r, tail, "i"'] & R
\end{tikzcd}$$

is a discrete fibration.

Proof Observe that in the formulation of the weak shifting property we could assume that S is less or equal to R: indeed, if not, then we could replace S by its intersection $R\cap S=U$ with R. After making this assumption the discrete fibration condition becomes nothing but the direct translation of the weak shifting property into the language of pullbacks. □

3.5. Remark. From now on we shall adopt a simplification in the notations: in the diagrams we shall often write p_A also for the equivalence relation $R[p_A]$. Furthermore, if $f\colon A \times B \to C$ is an arrow, then the diagram on a product $A \times B$

$$\begin{tikzcd}
(x,y) \arrow[r, dash, "p_A"] \arrow[d, dash, "f"'] & (x,z) \arrow[d, dash, "f"] \\
(u,v) \arrow[r, dash, "p_A"'] & (u,t)
\end{tikzcd}$$

indicates that $(x,y)R[p_A](x,z)$, $(u,v)R[p_A](u,t)$, $(x,y)R[f](u,v)$ and $(x,z)R[f](u,t)$.

In the rest of this section $\mathcal{C}$ will denote a factor permutable category. First of all, we remark that the weak shifting property has some interesting consequences:

3.6. Lemma. *Given an equivalence relation R on an object X, there is at most one internal partial Maltsev operation $p\colon R \times X \to X$. In particular there is at most one connector on R and ∇_X.*

Proof Let $p\colon R \times X \to X$ be an internal partial Maltsev operation. We first prove that $R[p] \cap R[p_R] = \Delta_{R \times X}$. Consider two elements (x,y,z) and (x,y,u) in $R \times X$ with the property that $p(x,y,z) = p(x,y,u)$. Then the weak shifting property applied to

$$\begin{tikzcd}
(x,y,z) \arrow[r, dash, "p_X"] \arrow[d, dash, "p_R"'] \arrow[d, dash, bend right=60, "p"'] & (y,y,z) \arrow[d, dash, "p_R"] \\
(x,y,u) \arrow[r, dash, "p_X"'] & (y,y,u)
\end{tikzcd}$$

gives $z = p(y,y,z) = p(y,y,u) = u$. Let then $p\colon R \times X \to X$ and $p'\colon R \times X \to X$ be two internal Maltsev operations. The fact that $R[p] \cap R[p_R] = \Delta_{R\times X}$ allows one to apply the weak shifting property to

$$\begin{tikzcd}
(x,y,y) \arrow[r, dash, "p_R"] \arrow[d, dash, "p"'] \arrow[d, dash, bend right=60, "p'"'] & (x,y,z) \arrow[d, dash, "p"] \\
(x,x,x) \arrow[r, dash, "p_R"'] & (x,x,p(x,y,z))
\end{tikzcd}$$

for any (x,y,z) in $R \times X$. Accordingly, $p'(x,y,z) = p'(x,x,p(x,y,z)) = p(x,y,z)$. $\square$

If there is a connector on R and ∇_X, we say that R is *(algebraically) central.*

3.7. Corollary. *Given an object X in a factor permutable category $\mathcal{C}$, there is at most one internal Maltsev operation $p\colon X \times X \times X \to X$ on X. This operation is always associative.*

Proof Thanks to the previous lemma, only the associativity of p needs to be checked. Let us think of $X \times X \times X$ as being the product $(X \times (X \times X), p_1, p_{2,3})$.

Then the weak shifting property applied to the diagram

$$\begin{tikzcd}
(y,y,p(y,u,v)) \arrow[r, dash, "p_{2,3}"] \arrow[d, dash, "p_1"] & (x,y,p(y,u,v)) \arrow[d, dash, "p_1"] \\
(y,u,v) \arrow[r, dash, "p_{2,3}"'] \arrow[u, dash, bend left=60, "p"] & (x,u,v)
\end{tikzcd}$$

gives $p(x,y,p(y,u,v)) = p(x,u,v)$. Similarly one checks that $p(p(x,y,u),u,v) = p(x,y,v)$. Consequently,

$$p(x,y,p(z,u,v)) = p(p(x,y,z),z,p(z,u,v)) = p(p(x,y,z),u,v).$$

□

According to the previous corollary, being abelian for an object X in a factor permutable category is a property.

3.8. Definition. An object X in a factor permutable category is *abelian* if there is a (unique) Maltsev operation $p\colon X \times X \times X \to X$ on X.

Let $\mathcal{C}_{Ab}$ be the *category of abelian objects* in $\mathcal{C}$. Objects in $\mathcal{C}_{Ab}$ are the abelian objects in $\mathcal{C}$, and arrows in $\mathcal{C}_{Ab}$ are the arrows $f\colon X \to Y$ in $\mathcal{C}$ that respect the Maltsev operation, i.e. such that $p_Y \circ f^3 = f \circ p_X$.

3.9. Lemma. *$\mathcal{C}_{Ab}$ is full in $\mathcal{C}$.*

Proof Let $f\colon X \to Y$ be an arrow in $\mathcal{C}$, let $p_X\colon X^3 \to X$ and $p_Y\colon Y^3 \to Y$ be Maltsev operations on X and Y, respectively. Since $R[p_X] \cap R[p_{1,2}] = \Delta_{X^3}$ (as shown in the proof of Lemma 3.6) we can apply the weak shifting property to the diagram

$$\begin{tikzcd}
(x,y,y) \arrow[r, dash, "p_{1,2}"] \arrow[d, dash, "p_X"] & (x,y,z) \arrow[d, dash, "p_X"] \\
(x,x,x) \arrow[r, dash, "p_{1,2}"'] \arrow[u, dash, bend left=60, "p_Y \circ f^3"] & (x,x,p_X(x,y,z)).
\end{tikzcd}$$

This gives $(p_Y \circ f^3)(x,y,z) = (p_Y \circ f^3)(x,x,p_X(x,y,z)) = (f \circ p_X)(x,y,z)$, for any $(x,y,z) \in X^3$. □

Johnstone introduced in [37] the notion of a naturally Maltsev category: a finitely complete category is *naturally Maltsev* if there is a natural transformation p from $1_{\mathcal{C}} \times 1_{\mathcal{C}} \times 1_{\mathcal{C}}$ to $1_{\mathcal{C}}$ with the following property: for any A in $\mathcal{C}$, the A-component p_A of p is an internal Maltsev operation. It is easy to see that any naturally Maltsev category is Maltsev.

3.10. Corollary. *$\mathcal{C}_{Ab}$ is a naturally Maltsev category.*

Proof By definition of $\mathcal{C}_{Ab}$ any object in it is equipped with an internal Maltsev operation. Moreover, the fact that $\mathcal{C}_{Ab}$ is full in $\mathcal{C}$ means that the various Maltsev operations organize themselves in a natural transformation from $1_{\mathcal{C}} \times 1_{\mathcal{C}} \times 1_{\mathcal{C}}$ to $1_{\mathcal{C}}$. □

The following property of factor permutable categories will have some very strong consequences:

3.11. Proposition. *Any double equivalence relation D on ∇_X and R has the property that the canonical arrow $\alpha\colon D \to R \times X$ defined by*

$$\alpha\begin{pmatrix} x & t \\ y & z \end{pmatrix} = (x, y, z)$$

is a regular epi.

Proof Thanks to the Barr embedding for regular categories, it is sufficient to prove this fact in the category of sets [1] (for an explanation of this fact see also Metatheorem 1.8 in [3]). Now, let D be any double equivalence relation on R and ∇_X, let (x, y, z) be an element in $R \times X$, and let $p_1\colon X \times X \to X$ be the first projection. Since the category is factor permutable, the equivalence relations $R[p_1]$ and D on $X \times X$ permute. Accordingly, from

$$(x, y)R[p_1](x, x)D(y, y)R[p_1](y, z)$$

it follows that there is an element (x, t) in $X \times X$ such that

$$(x, y)R[p_1](x, x)R[p_1](x, t)D(y, z).$$

Consequently, there is an element $\begin{pmatrix} x & t \\ y & z \end{pmatrix}$ in D, and this shows that the arrow α is a regular epi. □

3.12. Proposition. *Let R be an equivalence relation on Y. If R is central and $i\colon X \to Y$ is a monomorphism, then $i^{-1}(R)$ is central.*

Proof We denote by D the centralizing double relation on R and ∇_Y. Then consider the inverse image of the equivalence relation R along i:

$$\begin{tikzcd}
i^{-1}(R) \arrow[r, tail, "j"] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] & R \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] \\
X \arrow[r, tail, "i"'] & Y
\end{tikzcd}$$

The inverse image $j^{-1}(D)$ of D along j determines a double equivalence relation on $i^{-1}(R)$ and ∇_X:

$$\begin{tikzcd}
j^{-1}(D) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & X \times X \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] \\
i^{-1}(R) \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] & X.
\end{tikzcd}$$

There is an induced arrow α from $j^{-1}(D)$ to the pullback $i^{-1}(R) \times X$ of $d_2\colon i^{-1}(R) \to X$ along $d_1\colon X \times X \to X$. The arrow α is a regular epi thanks to the previous Proposition. On the other hand, the fact that $i \times i$ and j are monos implies that α is a mono, and then an iso. This means that $j^{-1}(D)$ is a centralizing relation on $i^{-1}(R)$ and ∇_X. □

3.13. Proposition. *Let R and S be two equivalence relations on X, with $R \leq S$. If S is central, then R is central.*

Proof Let $j\colon R \to S$ denote the inclusion of R in S. If $p\colon S \times X \to X$ is the connector on S and ∇_X, then the arrow $p \circ (j \times 1_X)\colon R \times X \to X$ is the connector between R and ∇_X. The only axiom in the definition of a connector that is not trivial is the one asserting that $p(x,y,z)Rz$ for any (x,y,z) in $R \times X$. In order to check this axiom, consider the relation D on $X \times X$ defined by

$$(x,t)D(y,z) \quad \Leftrightarrow \quad xRy \text{ and } t = p(x,y,z).$$

One can see that D is an equivalence relation on $X \times X$ with the property that $D \cap R[p_1] = \Delta_{X \times X}$. Consequently, for any $(x,y,z) \in R \times X$, we can apply the weak shifting property to the situation

$$\begin{tikzcd}
(x,x) \arrow[r, dash, "p_1"] \arrow[d, dash, "D"'] \arrow[d, dash, bend right=60, "R\square\nabla_X"'] & (x,p(x,y,z)) \arrow[d, dash, "D"] \\
(y,y) \arrow[r, dash, "p_1"'] & (y,z),
\end{tikzcd}$$

It follows that $(x,p(x,y,z))R\square\nabla_X(y,z)$, thus $p(x,y,z)Rz$, as desired. $\square$

3.14. Corollary. *If $i\colon R \to S$ is a monomorphism in the category of equivalence relations in C and S is central, then R is central.*

Proof It follows by the two previous propositions. $\square$

In particular we obtain a stability property of abelian objects:

3.15. Corollary. *C_{Ab} is closed under subobjects in C.*

Proof If Y is an abelian object and $i\colon X \to Y$ is a mono, then $(i, i \times i)\colon \nabla_X \to \nabla_Y$ is trivially an arrow in the category of equivalence relations, and the result follows by the previous corollary. $\square$

3.16. Lemma. *Let R be a central equivalence relation on X, and let S be an equivalence relation on Y. Then, if $\overline{f}$ in the commutative diagram*

$$\begin{tikzcd}
R \arrow[r, shift left, "d_2"] \arrow[r, shift right, "d_1"'] \arrow[d, "\overline{f}"'] & X \arrow[d, "f"] \\
S \arrow[r, shift left, "d_2"] \arrow[r, shift right, "d_1"'] & Y
\end{tikzcd}$$

is a regular epimorphism, then S is central.

Proof Let $p\colon R \times X \to X$ be the connector on R and ∇_X. Since f and $\overline{f}$ are regular epimorphisms, and the category is regular, both $f \times f\colon X \times X \to Y \times Y$ and $\overline{f} \times f\colon R \times X \to S \times Y$ are regular epimorphisms as well. Let $p_R\colon R \times X \to R$ and $p_X\colon R \times X \to X$ be the projections, let us first prove that $R[\overline{f} \times f] \cap R[p_R] \leq R[f \circ p]$.

Indeed, if $[(x,y,z),(x,y,u)]$ is in $R[\overline{f}\times f]\cap R[p_R]$, one has that $f(z)=f(u)$. By the weak shifting property applied to the situation

$$
\begin{tikzcd}
(y,y,z) \arrow[r, dash, "p_X"] \arrow[d, dash, "p_R"] \arrow[d, dash, bend right=60, "f\circ p"'] & (x,y,z) \arrow[d, dash, "p_R"] \\
(y,y,u) \arrow[r, dash, "p_X"'] & (x,y,u)
\end{tikzcd}
$$

it follows that $[(x,y,z),(x,y,u)]$ is in $R[f\circ p]$, as desired. Let us then show that the kernel pair $R[\overline{f}\times f]$ of $\overline{f}\times f$ is contained in $R[f\circ p]$. If $[(x,y,z),(u,v,w)]$ is an element in $R[\overline{f}\times f]$, then clearly

$$(f\circ p)(x,y,y)=f(x)=f(u)=(f\circ p)(u,v,v)$$

so that we can form the diagram

$$
\begin{tikzcd}
(x,y,y) \arrow[r, dash, "p_R"] \arrow[d, dash, "\overline{f}\times f"] \arrow[d, dash, bend right=60, "f\circ p"'] & (x,y,z) \arrow[d, dash, "\overline{f}\times f"] \\
(u,v,v) \arrow[r, dash, "p_R"'] & (u,v,w)
\end{tikzcd}
$$

We have just proved that $R[\overline{f}\times f]\cap R[p_R]\leq R[f\circ p]$, so that the weak shifting property implies that $[(x,y,z),(u,v,w)]$ is also in $R[f\circ p]$. Consequently, the universal property of the coequalizer $\overline{f}\times f$ yields a unique arrow $\pi\colon S\times Y\to Y$ such that $\pi\circ(\overline{f}\times f)=f\circ p$. This arrow π is the connector on S and ∇_Y (we leave the verification to the reader). □

3.17. Corollary. *$\mathcal{C}_{Ab}$ is closed in $\mathcal{C}$ under regular quotients.*

Proof Given a regular epi $f\colon X\to Y$ with X an abelian object, the commutative diagram

$$
\begin{tikzcd}
X\times X \arrow[r, shift left, "d_2"] \arrow[r, shift right, "d_1"'] \arrow[d, "f\times f"'] & X \arrow[d, "f"] \\
Y\times Y \arrow[r, shift left, "d_2"] \arrow[r, shift right, "d_1"'] & Y
\end{tikzcd}
$$

is clearly of the type considered in the previous lemma. Accordingly, Y is abelian. □

4 Algebraically central extensions

In this section we prove a useful direct product decomposition of central relations, and we introduce the so-called algebraically central extensions. $\mathcal{C}$ will always denote an exact factor permutable category.

4.1. Proposition. *An algebraically central equivalence relation R on A in $\mathcal{C}$ is canonically isomorphic to a product $A \times Q$, with Q an abelian object.*

Proof Let C be the centralizing relation on R and ∇_A:

$$\begin{tikzcd}
C \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & A \times A \arrow[d, shift right, "a_1"'] \arrow[d, shift left, "a_2"] \\
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A
\end{tikzcd}$$

Now, by taking the coequalizer q of π_1 and π_2 we obtain the commutative diagram

$$\begin{tikzcd}
C \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "p_1"'] & R \arrow[r, "q"] \arrow[d, "r_1"'] \arrow[dr, phantom, "(1)"] & Q \arrow[d, "d_1"] \\
A \times A \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] & A \arrow[r] & 1.
\end{tikzcd}$$

Since the category $\mathcal{C}$ is exact, the equivalence relation

$$\begin{tikzcd}
C \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & R
\end{tikzcd}$$

is the kernel pair of its coequalizer q, and this latter is a pullback stable regular epi. By assumption the arrow (p_1, r_1) from the equivalence relation (C, R) to (∇_A, A) is a discrete fibration of internal equivalence relations, so that the square (1) is a pullback (see Corollary 2 in [8]). Accordingly, the equivalence relation R is isomorphic to

$$\begin{tikzcd}
A \times Q \arrow[r, shift left, "\pi_A"] \arrow[r, shift right, "\delta_2"'] & A\ .
\end{tikzcd}$$

The object Q is abelian: indeed, the fact that there is a connector on the equivalence relations $R[r_1]$ and $R[r_2]$ (Example 1.7) implies that Q is abelian. This fact essentially follows from the fact that the induced arrows $\tilde{q_1}$ and $\tilde{q_2}$ in the diagram

$$\begin{tikzcd}
R[r_1] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\tilde{q_1}"'] & R \arrow[d, "q"] & R[r_2] \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "\tilde{q_2}"] \\
Q \times Q \arrow[r, shift left] \arrow[r, shift right] & Q & Q \times Q. \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

are regular epimorphisms, as is the arrow $q_3\colon R[r_1] \times_R R[r_2] \to Q \times Q \times Q$ induced by the universal property of pullbacks. $\square$

In the special case of exact Maltsev categories the previous proposition was proved in [15].
By an *extension of* B we mean a regular epi $f\colon A \to B$ whose codomain is B.

4.2. Definition. An extension $f\colon A \to B$ is *algebraically central* if its kernel pair $R[f]$ is algebraically central.

We remark that this definition has its plain meaning in factor permutable categories: indeed, for an extension being algebraically central becomes a property (thanks to Lemma 3.6). A consequence of the previous product decomposition is that any algebraically central extension $f\colon A \to B$ which is a split epi is isomorphic to the projection $\pi_B\colon B \times Q \to B$, where Q is the abelian object canonically associated with $R[f]$:

4.3. Corollary. *Let $f\colon A \to B$ be an algebraically central extension split by an arrow $i\colon B \to A$. Then $A \simeq B \times Q$, with Q an abelian object.*

Proof Let $A \times Q$ be the canonical product decomposition of $R[f]$, as obtained in the previous Proposition:

$$\begin{tikzcd} A \times Q \arrow[r, shift left, "\pi_A"] \arrow[r, shift right, "\delta_2"'] & A \arrow[r, "f"] & B. \end{tikzcd}$$

Then both squares in the following commutative diagram are pullbacks:

$$\begin{tikzcd} B \times Q \arrow[r, "i\times 1"] \arrow[d, "\pi_B"'] & A \times Q \arrow[r, "\delta_2"] \arrow[d, "\pi_A"'] & A \arrow[d, "f"] \\ B \arrow[r, "i"'] & A \arrow[r, "f"'] & B. \end{tikzcd}$$

Since $f \circ i = 1_B$, it follows that $\delta_2 \circ (i \times 1)$ is an isomorphism, and then f is isomorphic to the (split) extension $\pi_B\colon B \times Q \to B$. $\square$

The following result, asserting that algebraically central extensions are pullback stable, will be very useful:

4.4. Lemma. *If the extension $f\colon A \to B$ is algebraically central and the square*

$$\begin{tikzcd} C \arrow[r, "\pi_A"] \arrow[d, "\pi_E"'] & A \arrow[d, "f"] \\ E \arrow[r, "g"'] & B \end{tikzcd}$$

is a pullback, then π_E is algebraically central.

Proof Let R and $\overline{R}$ be the kernel pairs of f and π_E, respectively. Let D be the centralizing relation on R and ∇_A. Let $p\colon \overline{R} \to R$ denote the arrow induced by the universal property of kernel pairs. The relation $p^{-1}(D)$, defined by the pullback

$$\begin{tikzcd} p^{-1}(D) \arrow[r, "\overline{q}"] \arrow[d, "(\overline{\pi}_1{,}\overline{\pi}_2)"'] & D \arrow[d, "(\pi_1{,}\pi_2)"] \\ \overline{R} \times \overline{R} \arrow[r, "p\times p"'] & R \times R \end{tikzcd}$$

determines a double equivalence relation on $\overline{R}$ and ∇_C:

$$\begin{tikzcd}
& p^{-1}(D) \arrow[rr, shift left, dotted, "\overline{p}_1"] \arrow[rr, shift right, dotted, "\overline{p}_2"'] \arrow[dl, "\overline{q}"'] \arrow[dd, shift right, "\overline{\pi}_1"'] \arrow[dd, shift left, "\overline{\pi}_2"] & & C \times C \arrow[dl, "\pi_A \times \pi_A"'] \arrow[dd, shift right, "\overline{a}_1"'] \arrow[dd, shift left, "\overline{a}_2"] \\
D \arrow[rr, shift left, "p_1"] \arrow[rr, shift right, "p_2"'] \arrow[dd, shift right, "\pi_1"'] \arrow[dd, shift left, "\pi_2"] & & A \times A \arrow[dd, shift right, "a_1"'] \arrow[dd, shift left, "a_2"] & \\
& \overline{R} \arrow[rr, shift left, "\overline{r}_1"] \arrow[rr, shift right, "\overline{r}_2"'] \arrow[dl, "p"] & & C \arrow[dl, "\pi_A"] \\
R \arrow[rr, shift left, "r_1"] \arrow[rr, shift right, "r_2"'] & & A &
\end{tikzcd}$$

It is an easy exercise on pullbacks to check that $p^{-1}(D)$ is a centralizing relation on $\overline{R}$ and ∇_C. □

5 Categorically central extensions

We are now going to prove that whenever the subcategory $\mathcal{C}_{Ab}$ is reflective in a Barr-exact factor permutable category $\mathcal{C}$, it is necessarily "admissible" in the sense of the categorical theory of central extensions [31]. This will allow us to define the so-called categorically central extensions.

Let us first recall some terminology: for an object B in an exact category $\mathcal{C}$, the category $Ext(B)$ of "extensions" of B is the full subcategory of the comma category $\mathcal{C} \downarrow B$ whose objects are the regular epimorphisms with codomain B. We also write $\mathcal{C} \Downarrow B$ for $Ext(B)$.

When $\mathcal{X}$ is a full replete reflective subcategory of an exact category $\mathcal{C}$, closed in $\mathcal{C}$ under subobjects and regular quotients, one says that $\mathcal{X}$ is a *Birkhoff subcategory* of $\mathcal{C}$. This terminology is motivated by the classical Birkhoff's theorem which, in our terminology, says that a subcategory $\mathcal{X}$ of a variety $\mathcal{C}$ is a subvariety if and only if it is a Birkhoff subcategory.

We write $\eta_B \colon B \to HIB$ for the B-component of the unit of the adjunction, and we often drop H from the notations and write $\eta_B \colon B \to IB$. The left adjoint I to the inclusion functor induces, for all B in $\mathcal{C}$, a functor $I^B \colon \mathcal{C} \Downarrow B \to \mathcal{X} \Downarrow IB$ sending the extension $f \colon A \to B$ to the extension $If \colon IA \to IB$. This functor I^B has a right adjoint $H^B \colon \mathcal{X} \Downarrow IB \to \mathcal{C} \Downarrow B$ defined as follows: with an extension $\phi \colon X \to IB$ in $\mathcal{X}$ it associates the extension $s \colon C \to B$ given by the pullback

$$\begin{tikzcd}
C \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "(1)"] & HX \arrow[d, "H\phi"] \\
B \arrow[r, "\eta_B"'] & HIB.
\end{tikzcd}$$

The categorical theory of central extensions can be developed when $\mathcal{X}$ is a Birkhoff subcategory of the exact category $\mathcal{C}$ satisfying the following additional property:

5.1. Definition. A Birkhoff subcategory $\mathcal{X}$ of the exact category $\mathcal{C}$ is *admissible* when, for any B in $\mathcal{C}$, the functor $H^B \colon \mathcal{X} \Downarrow IB \to \mathcal{C} \Downarrow B$ is fully faithful.

Remark that for every extension ϕ lying in $\mathcal{X}$, the pullback (1) can be seen as the exterior rectangle of the following diagram:

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, "s"'] \arrow[dr, phantom, "(2)"] & IC \arrow[r, "\beta"] \arrow[d, "Is"] \arrow[dr, phantom, "(3)"] & X \arrow[d, "\phi"] \\
B \arrow[r, "\eta_B"'] & IB \arrow[r, "1_{IB}"'] & IB
\end{tikzcd}$$

where $\beta \colon IC \to X$ is the unique arrow such that $\beta \circ \eta_C = t$. The ϕ-component of the counit $\varepsilon^B \colon I^B H^B \to 1$ of the induced adjunction is this arrow $\beta \colon Is \to \phi$. The functor H^B is fully faithful, and $\mathcal{X}$ is admissible, exactly when the counit is an isomorphism [40] or, equivalently, when each β as above is an iso.

We are now going to show that, in our context, $\mathcal{C}_{Ab}$ is always admissible in $\mathcal{C}$, provided it is reflective in $\mathcal{C}$.

From now on we shall assume that $\mathcal{C}$ is an exact factor permutable category, and that the inclusion functor $H \colon \mathcal{C}_{Ab} \to \mathcal{C}$ (which is full by Lemma 3.9) has a left adjoint, that will be denoted by $I \colon \mathcal{C} \to \mathcal{C}_{Ab}$. Such a left adjoint does exist when $\mathcal{C}$ is a congruence modular variety [32], and also when $\mathcal{C}$ is an exact Maltsev category with coequalizers [24] (see also [12]). In both cases the functor $I \colon \mathcal{C} \to \mathcal{C}_{Ab}$ sends an object A to the quotient $\frac{A}{[\nabla_A, \nabla_A]}$ of A by the largest commutator $[\nabla_A, \nabla_A]$, this latter being constructed as explained at the end of section 2. Thanks to our previous results, we already know that the category $\mathcal{C}_{Ab}$ is then a Birkhoff subcategory in $\mathcal{C}$, being a full subcategory of $\mathcal{C}$ closed in $\mathcal{C}$ under subobjects and regular quotients (by Corollary 3.15 and 3.17). Let us now prove that $\mathcal{C}_{Ab}$ is also admissible:

5.2. Theorem. *Let $\mathcal{C}$ be an exact factor permutable category such that $\mathcal{C}_{Ab}$ is reflective in $\mathcal{C}$. Then $\mathcal{C}_{Ab}$ is admissible.*

Proof Let us consider the canonical decomposition (2) + (3) of any pullback (1) as indicated here above, and we are going to show that the arrow $\beta \colon IC \to X$ is always an isomorphism. By taking the kernel pairs of the arrows s and Is in the commutative square (2) we obtain the diagram

$$\begin{tikzcd}
R[s] \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] \arrow[d, "l"'] & C \arrow[r, "s"] \arrow[d, "\eta_C"'] \arrow[dr, phantom, "(2)"] & B \arrow[d, "\eta_B"] \\
R[I(s)] \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] & IC \arrow[r, "Is"'] & IB.
\end{tikzcd}$$

We shall first show that the induced arrow l is a regular epi: to check this, let us consider the regular epi-mono decomposition $i \circ p$ of l, which yields the following diagram

$$\begin{tikzcd}
R[s] \arrow[r, "p"] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] & R \arrow[r, "i"] \arrow[d, shift right, "\delta_1"'] \arrow[d, shift left, "\delta_2"] & R[Is] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] \\
C \arrow[r, "\eta_C"'] & IC \arrow[r, "1_{IC}"'] & IC
\end{tikzcd}$$

Now, the middle vertical arrows represent a reflexive relation R in the category $\mathcal{C}_{Ab}$, since $\mathcal{C}_{Ab}$ is closed in $\mathcal{C}$ under subobjects and products. Since $\mathcal{C}_{Ab}$ is naturally Maltsev (Corollary 3.10), R actually is an equivalence relation. It is easy to see that δ_1 and δ_2 have Is as coequalizer. The equivalence relations are effective in $\mathcal{C}$, then $R \simeq R[Is]$, i is an iso and l a regular epi.

Let us then recall that any extension f in $\mathcal{C}_{Ab}$ is algebraically central: in a naturally Maltsev category the connector $p\colon R[f] \times A \to A$ is the restriction of the A-component of the natural transformation from $1_{\mathcal{C}} \times 1_{\mathcal{C}} \times 1_{\mathcal{C}}$ to $1_{\mathcal{C}}$. Algebraically central extensions are stable under pulling back (by Lemma 4.4), hence the extension s is algebraically central because ϕ is so, and $R[s]$ is then isomorphic to $C \times Q$, where Q is an abelian object. Consequently in the following commutative diagram

$$\begin{tikzcd}
C \times Q \arrow[r, "l"] \arrow[d, shift right, "\pi_C"'] \arrow[d, shift left, "\delta_2"] & R[Is] \arrow[r, "\overline{\beta}"] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] & R[\phi] \arrow[d, shift right] \arrow[d, shift left] \\
C \arrow[r, "\eta_C"'] & IC \arrow[r, "\beta"'] & X
\end{tikzcd}$$

any of the left-hand commutative squares is a pushout of regular epimorphisms such that, by factor permutability, $R[\pi_C] \circ R[l] = R[l] \circ R[\pi_C]$. This implies that the canonical arrow $\alpha\colon C \times Q \to C \times_{IC} R[Is]$ to the corresponding pullback of η_C along $d_1\colon R[Is] \to IC$ is a regular epi (by Theorem 2.11). The arrow α is also a mono, since π_C and $\overline{\beta} \circ l$ are jointly monic (since the square (1) is a pullback), so that π_C and l are jointly monic as well. Accordingly, the arrow $(\eta_C, l)\colon (C \times Q, C) \to (R[Is], IC)$ is a discrete fibration, then (2) is a pullback, and this implies that (3) is a pullback and β is an iso, as desired. $\square$

As it was observed in [31], admissibility can be seen as an exactness condition on the left adjoint I. In our context, it can be expressed as follows:

5.3. Corollary. *Let $\mathcal{C}$ be an exact factor permutable category such that $\mathcal{C}_{Ab}$ is reflective in $\mathcal{C}$. Then the left adjoint $I\colon \mathcal{C} \to \mathcal{C}_{Ab}$ preserves all pullbacks of the form*

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, "s"'] & Z \arrow[d, "\psi"] \\
B \arrow[r, "v"'] & Y,
\end{tikzcd}$$

where Y, Z are in $\mathcal{C}_{Ab}$ and ψ is a regular epi.

Proof Let us write v as the composite $w \circ \eta_B$ for a unique $w \colon IB \to Y$, and let us consider the following composite of two pullbacks:

$$\begin{tikzcd} C \arrow[r, "t"] \arrow[d, "s"'] & X \arrow[r, "x"] \arrow[d, "\phi"] & Z \arrow[d, "\psi"] \\ B \arrow[r, "\eta_B"'] & IB \arrow[r, "w"'] & Y. \end{tikzcd}$$

The object X is in $\mathcal{C}_{Ab}$, and the arrow ϕ is a regular epi, being the pullback of ψ along w. Consequently the functor I preserves the left-hand pullback thanks to the previous theorem, and the right-hand pullback because it lies in $\mathcal{C}_{Ab}$. □

Let us now introduce some more terminology.

5.4. Definition. An extension $f \colon A \to B$ is *trivial* if it lies in the image of the functor H^B. This precisely means that the following square is a pullback:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & HIA \arrow[d, "HIf"] \\ B \arrow[r, "\eta_B"'] & HIB \end{tikzcd}$$

5.5. Definition. An extension $f \colon A \to B$ is (E,p)-split, where $p \colon E \to B$ is itself an extension of B, when $s \colon E \times_B A \to E$ in the following pullback is a trivial extension of E

$$\begin{tikzcd} E \times_B A \arrow[r, "t"] \arrow[d, "s"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B. \end{tikzcd}$$

We shall write $Spl(E,p)$ for the full subcategory of $Ext(B)$ whose objects are the extensions which are split by p.

5.6. Definition. An extension $f \colon A \to B$ is *categorically central* when there exists an extension $p \colon E \to B$ such that f is (E,p)-split.

If one denotes by $Triv(B)$ and by $Centr(B)$ the full subcategories of the category $Ext(B)$ whose objects are the trivial and the categorically central extensions of B, one has

$$Triv(B) \subseteq Centr(B) \subseteq Ext(B),$$

where the inclusions are proper in general. Trivial and categorically central extensions are pullback stable:

5.7. Proposition. [31] *For any extension $g \colon E \to B$ the pullback functor*

$$g^* \colon Ext(B) \to Ext(E)$$

takes trivial (categorically central) extensions of B to trivial (categorically central) extensions of E.

Proof Let the left-hand square below be the pullback along $g\colon E \to B$ of a trivial extension $f\colon A \to B$:

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "k"'] & A \arrow[r, "\eta_A"] \arrow[d, "f"] & HIA \arrow[d, "HIf"] \\ E \arrow[r, "g"'] & B \arrow[r, "\eta_B"'] & HIB \end{tikzcd} \tag{4}$$

Since both the squares are pullbacks, so is the exterior rectangle. By the naturality of η this last pullback is equal to the exterior rectangle in

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "k"'] & HIC \arrow[r, "HIh"] \arrow[d, "HIk"] & HIA \arrow[d, "HIf"] \\ E \arrow[r, "\eta_E"'] & HIE \arrow[r, "HIg"'] & HIB \end{tikzcd} \tag{5}$$

which is accordingly a pullback. Now applying HI to the exterior rectangle of (4) gives the right-hand square of (5), so this last too is a pullback by Corollary 5.3. This implies that the left-hand square of (5) is a pullback and $k\colon C \to E$ is a trivial extension of E.

Let us then prove that categorically central extensions are pullback stable. Let $f\colon A \to B$ be a categorically central extension, and let us assume that f belongs to $Spl(E,p)$ for an extension $p\colon E \to B$. For any arrow $g\colon D \to B$ we can then form the pullback

$$\begin{tikzcd} E \times_B D \arrow[r, "q"] \arrow[d, "h"'] & D \arrow[d, "g"] \\ E \arrow[r, "p"'] & B. \end{tikzcd}$$

Then the extension which is obtained by pulling back f along $g \circ q = p \circ h$ is trivial by the first part of the proof. Accordingly, the pullback of f along g belongs to $Spl(E \times_B D, q)$ and it is then (categorically) central. $\square$

5.8. Remark. From the previous proposition it follows that if $p\colon E \to B$ and $p'\colon E' \to B$ are two extensions of B, then $Spl(E',p') \subset Spl(E,p)$ whenever there is a map $g\colon E \to E'$ with $p' \circ g = p$.

6 Equivalence of the two notions

In this section we prove the equivalence between the two notions of categorically central and of algebraically central extension. A consequence of this result is that any central extension is normal, which means that it is split by itself.

In this section $\mathcal{C}$ will denote an exact factor permutable category such that $\mathcal{C}_{Ab}$ is a reflective (and hence admissible) subcategory of $\mathcal{C}$.

6.1. Theorem. *An extension $f\colon A \to B$ is algebraically central if and only if it is categorically central.*

Proof The category $\mathcal{C}_{Ab}$ is naturally Maltsev and consequently any extension in $\mathcal{C}_{Ab}$ is algebraically central. For this reason, in order to prove that any categorically central extension is algebraically central it is sufficient to prove that in any pullback

$$\begin{tikzcd} C \arrow[r, "\pi_A"] \arrow[d, "\pi_E"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, "f"] \\ E \arrow[r, "g"'] & B, \end{tikzcd}$$

where g is a regular epi, the extension f is algebraically central if and only if π_E is algebraically central. In Lemma 4.4 it was proved that algebraically central extensions are stable by pullbacks, so that we only need to prove that when π_E is an algebraically central extension then f is algebraically central. For this, let us consider the diagram

$$\begin{tikzcd} R[\pi_E] \arrow[r, shift left, "d_2"] \arrow[r, shift right, "d_1"'] \arrow[d, "p"'] & C \arrow[d, "\pi_A"] \\ R[f] \arrow[r, shift left, "d_2"] \arrow[r, shift right, "d_1"'] & A \end{tikzcd}$$

where p is the arrow induced by the universal property of the kernel pairs. This arrow p is a regular epi because the square (1) is a pullback, while π_A is a regular epi because g is a regular epi, so that also $\pi_A \times \pi_A$ is a regular epi. By Lemma 3.16 it follows that $R[f]$ is central, and f is algebraically central.

Conversely, let us assume that $f \colon A \to B$ is algebraically central. By Proposition 4.1 we know that its kernel pair is given (up to isomorphism) by

$$\begin{tikzcd} A \times Q \arrow[r, "\delta_2"'] \arrow[d, "\pi_A"] & A \arrow[d, "f"'] \\ A \arrow[r, "f"] & B, \end{tikzcd}$$

where Q is an object in $\mathcal{C}_{Ab}$. Let $t \circ m$ be the regular epi-mono factorization of the unique arrow from Q to 1, and let I be the regular image of this arrow. The arrow $m \colon Q \to I$ belongs to $\mathcal{C}_{Ab}$, and there is an induced arrow $i \colon A \to I$ with $i \circ \pi_A = m \circ \pi_Q$. The extension π_A appears then as the pullback of the trivial extension m along i, hence π_A is a trivial extension, and f is categorically central, as desired. □

Let us recall from [31] that an extension $f \colon A \to B$ is said to be *normal* when it is split by itself, i.e. when f is in $Spl(A, f)$. By definition any normal extension is central, and there are examples of central extensions which are not normal [31]. The proof of the previous theorem shows that, under our assumptions, we have:

6.2. Corollary. *Any central extension is normal.*

7 Central extensions in semi-abelian categories

In this section we are going to give several characterizations of central extensions in semi-abelian categories. The relationship with (the natural generalization of) the definition given by Frölich for varieties of Ω-groups is also clarified. The main results of this section were established in [14].

Let us first briefly recall the notions of protomodular category (see Bourn [6]) and of semi-abelian category (see Janelidze, Marki, Tholen [34]). We shall often refer to the survey by Borceux [3] in this volume, where many important properties of semi-abelian categories are proved in detail.

When $\mathcal{C}$ is a finitely complete category, let $Pt(\mathcal{C})$ denote the category whose objects are the split epimorphisms with a given splitting, and arrows pairs (f_0, f_1) of arrows f_0 and f_1 in $\mathcal{C}$ making commutative the two squares between these data:

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, shift left, "p"] & A' \arrow[d, shift left, "p'"] \\ B \arrow[r, "f_0"'] \arrow[u, shift left, "i"] & B' . \arrow[u, shift left, "i'"] \end{tikzcd}$$

Let $\pi\colon Pt(\mathcal{C}) \to \mathcal{C}$ be the functor sending any split epimorphism to its codomain. The functor π is a fibration, called the fibration of pointed objects. A *protomodular category* is a finitely complete category $\mathcal{C}$ such that every change of base functor with respect to the fibration π is conservative, i.e. it reflects isomorphisms [6]. When $\mathcal{C}$ has a zero object, this condition is equivalent to the split short five lemma.

7.1. Definition. [34] A *semi-abelian category* is an exact protomodular category with a zero object and finite coproducts.

An exact category is protomodular if and only if the following property holds: for any commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & B \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "(2)"] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

whenever (1) and (1) + (2) are pullbacks and f is a regular epi, then the square (2) is a pullback as well. This equivalent formulation of the protomodularity property will be useful in our study of central extensions.

Any semi-abelian category $\mathcal{C}$ is in particular an exact Maltsev category, (see, for instance, Theorem 3.7 in [3] for a proof of this fact, which is a special case of the more general results in [7]). In particular $\mathcal{C}$ is then an exact factor permutable category, so that all the results of the previous sections hold in any semi-abelian category.

7.2. Examples. Semi-abelian varieties of universal algebras have been recently characterized by Bourn and Janelidze [17] as follows: the theory has a unique constant 0, a $(n+1)$-ary operation β and n binary operations α_i satisfying the conditions $\alpha_i(x,x) = 0$ et $\beta(\alpha_1(x,y),, \alpha_n(x,y), y) = x$. The categories of groups, rings, Lie algebras, crossed modules and, more generally, any variety of Ω-groups [28] are semi-abelian. Further important examples of semi-abelian categories are given by the category of Heyting algebras, by any abelian category and by the dual category of the category of pointed sets.

If $f\colon A \to B$ is an arrow in a finitely complete pointed category $\mathcal{C}$, we denote by $Ker(f)\colon K[f] \to A$ the kernel of f, which is given by the pullback

$$\begin{tikzcd} K[f] \arrow[r, "Ker(f)"] \arrow[d] & A \arrow[d, "f"] \\ 0 \arrow[r] & B. \end{tikzcd}$$

7.3. Remark. The kernel of f can be built up from the kernel pair $R[f]$ in a very simple way. Consider the diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d] & R[f] \arrow[r, "d_2"] \arrow[d, "d_1"] & A \arrow[d, "f"] \\ 0 \arrow[r] & A \arrow[r, "f"'] & B \end{tikzcd}$$

where the left-hand square represents the kernel k of d_1: it is then clear that $d_2 \circ k = Ker(f)$.

In order to prove the main results of this section, a few preliminary lemmas will be needed. We begin with the following characterization of pushouts of regular epimorphisms, which holds more generally in any exact Maltsev category [18], [11]:

7.4. Lemma. *Consider a commutative diagram of regular epimorphisms*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & C \arrow[d, "g"] \\ B \arrow[r, "l"'] & D, \end{tikzcd}$$

in an exact Maltsev category $\mathcal{C}$. This square is a pushout if and only if the arrow $\tilde{h}\colon R[f] \to R[g]$ induced by the kernel pairs is a regular epimorphism.

Proof Let us assume that (1) is a pushout of regular epimorphisms in an exact Maltsev category. Consider the diagram

$$\begin{tikzcd} R[f] \arrow[r, "p"] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] & I \arrow[r, "m"] \arrow[d, shift right, "\delta_1"'] \arrow[d, shift left, "\delta_2"] & R[g] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] \\ A \arrow[r, "h"'] & C \arrow[r, "1_C"'] & C \end{tikzcd}$$

where $m \circ p = \tilde{h}$ is the regular epi-mono factorization of the arrow $\tilde{h}$ and the central part of the diagram represents the reflexive relation determined by this factorization. The relation I is then an equivalence relation on C because $\mathcal{C}$ is a Maltsev category, and the fact that the square (1) is a pushout implies that g is the coequalizer of δ_1 and δ_2. Since $\mathcal{C}$ is exact, I is then the kernel pair of g, m is an isomorphism and $\tilde{h}$ a regular epi, as desired. Conversely, if $\tilde{h}$ is a regular epi, then the square (1) is a pushout in any category. □

The following result due to Bourn [9] will be needed. Its proof can be also found in the survey [3].

7.5. Lemma. *Consider a commutative diagram in a semi-abelian category*

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & B \arrow[r, "p"] \arrow[d, "b"] \arrow[dr, phantom, "(2)"] & C \arrow[d, "c"] \\ A' \arrow[r, "k'"'] & B' \arrow[r, "p'"'] & C' \end{tikzcd}$$

where $k = Ker(p)$, $k' = Ker(p')$ and p, p' are regular epis. Then, when a and c are regular epis, b is a regular epi as well.

We are now in a position to prove the following

7.6. Proposition. [14] *Consider a commutative diagram of regular epimorphisms*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & C \arrow[d, "g"] \\ B \arrow[r, "l"'] & D, \end{tikzcd}$$

in a semi-abelian category. This square is a pushout if and only if the restriction $\overline{h}\colon K[f] \to K[g]$ of h to the kernels is a regular epimorphism.

Proof Thanks to Lemma 7.5 it is sufficient to prove that the restriction $\overline{h}\colon K[f] \to K[g]$ of h to the kernel is a regular epi if and only if the induced arrow $\tilde{h}\colon R[f] \to R[l]$ is a regular epi. Let us recall that, by Remark 7.3, the objects $K[f]$ and $K[d_1]$ are isomorphic (with $d_1\colon R[f] \to A$) and, for the same reason, $K[g] \simeq K[d_1]$ (where this time $d_1\colon R[g] \to C$). Consequently, whenever $\overline{h}$ is a regular epi, the previous lemma applied to the diagram

$$\begin{tikzcd} K[f] \arrow[r, "Ker(d_1)"] \arrow[d, "\overline{h}"'] & R[f] \arrow[r, "d_1"] \arrow[d, "\tilde{h}"] & A \arrow[d, "h"] \\ K[g] \arrow[r, "Ker(d_1)"'] & R[g] \arrow[r, "d_1"'] & C \end{tikzcd}$$

implies that $\tilde{h}$ is a regular epi.

Conversely, when $\tilde{h}$ is a regular epi, then the square

$$\begin{tikzcd} R[f] \arrow[r, "\tilde{h}"] \arrow[d, "d_1"'] & R[g] \arrow[d, "d_1"] \\ A \arrow[r, "h"'] & C \end{tikzcd}$$

is a pushout (because the vertical arrows are split epis). Since the category is exact Maltsev, it follows that the factorization $\gamma\colon R[f] \to A \times_C R[g]$ is a regular epi:

$$\begin{tikzcd}
R[f] \arrow[dr, dotted, "\gamma"] \arrow[drr, bend left, "\tilde{h}"] \arrow[ddr, bend right, "d_1"'] & & \\
& A \times_C R[g] \arrow[r, "\pi_{R[g]}"'] \arrow[d, "\pi_A"] & R[g] \arrow[d, "d_1"'] \\
& A \arrow[r, "h"] & C
\end{tikzcd}$$

There exists a unique arrow $\alpha\colon K[g] \to A \times_C R[g]$ which is the kernel of π_A and has the property that $\pi_{R[g]} \circ \alpha = Ker(d_1)$. If $k\colon K[f] \to R[f]$ is the kernel of $d_1\colon R[f] \to A$, then the square

$$\begin{tikzcd}
K[f] \arrow[r, "\bar{h}"] \arrow[d, "k"'] & K[g] \arrow[d, "\alpha"] \\
R[f] \arrow[r, "\gamma"'] & A \times_C R[g]
\end{tikzcd}$$

is a pullback. It follows that the arrow $\bar{h}$ is a regular epi because γ is a regular epi. □

When $\mathcal{C}$ is semi-abelian and $\mathcal{C}_{Ab}$ is its full subcategory of abelian objects in $\mathcal{C}$, there exists a reflection functor $I\colon \mathcal{C} \to \mathcal{C}_{Ab}$ which sends an object A to its quotient by the largest commutator $[\nabla_A, \nabla_A]$ on A. The reflective subcategory $\mathcal{C}_{Ab}$ is then always admissible in the sense of Definition 5.1. We denote by KA the kernel of the A-component of the unit η_A of the adjunction, by $K(g)\colon KA \to KB$ the restriction to KA of an arrow $g\colon A \to B$ in $\mathcal{C}$.

When $\mathcal{C}$ is semi-abelian there is a simple characterization of trivial extensions:

7.7. Lemma. *An extension $f\colon A \to B$ is trivial if and only if $K(f)\colon KA \to KB$ is a monomorphism or, equivalently, if and only if $K(f)$ is an isomorphism.*

Proof Let us consider the canonical commutative square

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & HIA \arrow[d, "HIf"] \\
B \arrow[r, "\eta_B"'] & HIB.
\end{tikzcd}$$

In any pointed category when (1) is a pullback, then the restriction $K(f)\colon KA \to KB$ to the kernel of η_A is an iso. Conversely, when $\mathcal{C}$ is semi-abelian and $K(f)$ is an iso, the exterior rectangle and the left-hand square in the diagram

$$\begin{tikzcd}
KA \arrow[r, "k"] \arrow[d] & A \arrow[r, "f"] \arrow[d, "\eta_A"] & B \arrow[d, "\eta_B"] \\
0 \arrow[r] & HIA \arrow[r, "HIf"'] & HIB
\end{tikzcd}$$

are pullbacks. By protomodularity it follows that the right-hand square is a pullback as well, because η_A is a regular epi. Thanks to the previous Proposition the proof is complete, since the square (1) is always a pushout of regular epimorphisms (this latter fact follows from the fact that $\mathcal{C}_{Ab}$ is closed under quotients in $\mathcal{C}$). □

If $f\colon A \to B$ is an extension, we denote by d_1, d_2 the projections of the kernel pair $R[f]$ and by $s\colon A \to R[f]$ the diagonal.

7.8. Theorem. [14] *For an extension $f\colon A \to B$ in a semi-abelian category $\mathcal{C}$ the following conditions are equivalent:*

1. *f is central*
2. *f is normal*
3. *$K(d_1)$ is a mono*
4. *$K(d_1)$ is an iso*
5. *$K(s)$ is a regular epi*
6. *$K(s)$ is an iso*
7. *$K(d_1) = K(d_2)$*
8. *For any $x, y\colon D \to A$ such that $f \circ x = f \circ y$, one has $K(x) = K(y)$.*

Proof 1. and 2. are equivalent by Corollary 6.2, 2. and 3. are equivalent by Lemma 7.7. The conditions 3., 4., 5. and 6. are trivially equivalent and 6. implies 7. To prove that 7. implies 3. consider two arrows γ and δ from any object D to $K(R[f])$ with the property that $K(d_1) \circ \gamma = K(d_1) \circ \delta$. One has $Ker(\eta_A) \circ K(d_1) \circ \gamma = Ker(\eta_A) \circ K(d_1) \circ \delta$, then $d_1 \circ Ker(\eta_{R[f]}) \circ \gamma = d_1 \circ Ker(\eta_{R[f]}) \circ \delta$. By assumption it follows that $Ker(\eta_A) \circ K(d_2) \circ \gamma = Ker(\eta_A) \circ K(d_2) \circ \delta$, so that $d_2 \circ Ker(\eta_{R[f]}) \circ \gamma = d_2 \circ Ker(\eta_{R[f]}) \circ \delta$ and then $\gamma = \delta$.

Since 8. clearly implies 7., the proof will be complete if we show that 7. implies 8. Let x and y be two arrows from an object D to A such that $f \circ x = f \circ y$. By the universal property of the kernel pair $R[f]$ there is an arrow $\sigma\colon D \to R[f]$ such that $d_1 \circ \sigma = x$ and $d_2 \circ \sigma = y$. It follows $K(x) = K(d_1) \circ K(\sigma) = K(d_2) \circ K(\sigma) = K(y)$. □

7.9. Remark. The condition 8. above literally extends the definition of central extension given by Frölich [22] and by Lue [39] for varieties of Ω-groups with respect to the subvariety of abelian algebras (see also [33]). We can then conclude that, when one considers a semi-abelian category $\mathcal{C}$ and its admissible subcategory $\mathcal{C}_{Ab}$, the three possible definitions of central extensions (categorically central, algebraically central and central in the sense of Frölich) are all equivalent.

It is also possible to give a characterization of central extensions which is intrinsic in $\mathcal{C}$, namely without referring to the subcategory $\mathcal{C}_{Ab}$:

7.10. Proposition. [14] *An extension $f\colon A \to B$ in a semi-abelian category is central if and only if the diagonal $s\colon A \to R[f]$ is a kernel.*

Proof Let us first assume that the extension f is central. Let D be the centralizing relation on $R[f]$ and ∇_A:

$$\begin{tikzcd}
D \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & A \times A \arrow[d, shift right, "a_1"'] \arrow[d, shift left, "a_2"] \\
R[f] \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A
\end{tikzcd}$$

If $\sigma\colon A \times A \to D$ is the arrow giving the reflexivity of the relation D on $A \times A$, then the internal functor

$$\begin{tikzcd}
A \times A \arrow[r, "\sigma"] \arrow[d, shift right, "a_1"'] \arrow[d, shift left, "a_2"] & D \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\
A \arrow[r, "s"'] & R[f]
\end{tikzcd}$$

is a discrete fibration. It follows that s is the kernel of the quotient $q\colon R[f] \to \frac{R[f]}{D}$ of $R[f]$ by the equivalence relation D.

Conversely, let us assume that s is a kernel of a map g, and let D be the kernel pair of g. There is a commutative diagram

$$\begin{tikzcd}
D \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] \arrow[d, dotted, shift right, "d_1"'] \arrow[d, dotted, shift left, "d_2"] & R[f] \arrow[r, "g"'] \arrow[d, shift right=2, "r_1"'] \arrow[d, "r_2"] & Q \arrow[d, dotted, shift right] \arrow[d, dotted, shift left] \\
A \times A \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] & A \arrow[u, shift right=2, "s"'] \arrow[r] & 0,
\end{tikzcd}$$

where the left vertical dotted arrows are induced by the universal property of kernel pairs, and determine a reflexive relation on $A \times A$, hence an equivalence relation, since $\mathcal{C}$ is a Maltsev category. By using the cancellation property for pullbacks in a semi-abelian category (recalled after Definition 7.1), one can check that D is a centralizing relation on $R[f]$ and ∇_A, and the extension f is then central. □

8 Galois groupoids

The categorical theory of central extensions is a special instance of the general categorical Galois theory developed by Janelidze (see [29],[30], [4]). In this section we show that the so-called Galois pregroupoid associated with an extension actually is an internal groupoid in the semi-abelian context. We conclude by briefly recalling how a description of the central extensions of an object B can be obtained by applying the results of the categorical Galois theory (the reader may find a thorough presentation on this subject in [4]).

Let us first recall that an internal *precategory* in a category $\mathcal{C}$ [30] is a diagram of the form

$$\begin{tikzcd}
P_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & P_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_2"'] & P_0 \arrow[l, "s" description]
\end{tikzcd}$$

with

1. $d_1 \circ s = 1_{P_0} = d_2 \circ s$

2. $d_2 \circ p_1 = d_1 \circ p_2$
3. $d_1 \circ p_1 = d_1 \circ m,\ d_2 \circ p_2 = d_2 \circ m$

Roughly speaking, a precategory is what remains of the definition of an internal category when one cancels all references to pullbacks. An arrow in the category $Precat(\mathcal{C})$ of internal precategories is simply a natural transformation between two such diagrams.

8.1. Definition. Let X be an internal groupoid in $\mathcal{C}$:

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, shift left=2, "p_1"] \arrow[r, "m"] \arrow[r, shift right=2, "p_2"'] & X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_2"'] & X_0 \arrow[l, "s"'] \end{tikzcd}.$$

An internal *covariant presheaf* P on the groupoid X (also called an *internal action* of X)

$$\begin{tikzcd} P_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m"] \arrow[r, shift right=2, "p_2"'] \arrow[d, "f_2"] & P_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_2"'] \arrow[d, "f_1"] & P_0 \arrow[l, "s"'] \arrow[d, "f_0"] \\ X_1 \times_{X_0} X_1 \arrow[r, shift left=2, "p_1"] \arrow[r, "m"] \arrow[r, shift right=2, "p_2"'] & X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_2"'] & X_0 \arrow[l, "s"'] \end{tikzcd}.$$

consists in a precategory in $\mathcal{C}$ (the upper line), a natural transformation (f_0, f_1, f_2) from the precategory to X with the property that the squares

$$f_0 \circ d_1 = d_1 \circ f_1 \quad \text{and} \quad p_1 \circ f_2 = f_1 \circ p_1$$

are pullbacks.

Given two covariant presheaves P and P' on X, an internal *natural transformation* α from P to P' is given by three arrows $(\alpha_0, \alpha_1, \alpha_2)$ such that $f'_i \circ \alpha_i = f_i$ for $i = 0, 1, 2$ and all the squares of "corresponding arrows" in the diagram

$$\begin{tikzcd} P_2 \arrow[r, shift left=3, "p_1"] \arrow[r, "m"] \arrow[r, shift right=3, "p_2"'] \arrow[d, "\alpha_2"] & P_1 \arrow[r, shift left=3, "d_1"] \arrow[r, shift right=3, "d_2"'] \arrow[d, "\alpha_1"] & P_0 \arrow[l, "s"'] \arrow[d, "\alpha_0"] \\ P_2' \arrow[r, shift left=3, "p_1'"] \arrow[r, "m'"] \arrow[r, shift right=3, "p_2'"'] & P_1' \arrow[r, shift left=3, "d_1'"] \arrow[r, shift right=3, "d_1'"'] & P_0' \arrow[l, "s'"'] \end{tikzcd}$$

are commutative.

Let now $\mathcal{C}$ be a semi-abelian category and let X be an internal groupoid in $\mathcal{C}_{Ab}$. We denote by $\{X, \mathcal{C}_{Ab}\}$ the category whose objects are the internal covariant presheaves in $\mathcal{C}_{Ab}$ on the internal groupoid X with the property that each f_i is a regular epi for $i = 0, 1, 2$ (or, equivalently, just for $i = 0$). The arrows in $\{X, \mathcal{C}_{Ab}\}$ are the internal natural transformations.

Now, each extension $f \colon A \to B$ in $\mathcal{C}$ determines the internal groupoid

$$\begin{tikzcd} R[f] \times_A R[f] \arrow[r, shift left=2, "p_1"] \arrow[r, "m"] \arrow[r, shift right=2, "p_2"'] & R[f] \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_2"'] & A \arrow[l, "s"'] \end{tikzcd}$$

in $\mathcal{C}$ which is its kernel pair. By applying the left adjoint $I\colon \mathcal{C} \to \mathcal{C}_{Ab}$ of the inclusion functor $H\colon \mathcal{C}_{Ab} \to \mathcal{C}$ to this internal groupoid one obtains an internal precategory

$$\begin{tikzcd} I(R[f] \times_A R[f]) \arrow[r, shift left=2, "I(p_1)"] \arrow[r, "I(m)" description] \arrow[r, shift right=2, "I(p_2)"'] & I(R[f]) \arrow[r, shift left=2, "I(d_1)"] \arrow[r, shift right=2, "I(d_2)"'] & I(A) \arrow[l, "I(s)" description] \end{tikzcd}$$

in $\mathcal{C}$. This special kind of precategory, which is called the *internal Galois pregroupoid of f*, turns out to be always an internal groupoid in our context. This fact will be a consequence of the following lemma:

8.2. Lemma. *Let $\mathcal{C}$ be a semi-abelian category. Then the functor $I\colon \mathcal{C} \to \mathcal{C}_{Ab}$ preserves pullbacks of split epimorphisms along regular epimorphisms.*

Proof Let

$$\begin{tikzcd} P \arrow[r, "p_B"] \arrow[d, shift left, "p_A"] & B \arrow[d, shift left, "g"] \\ A \arrow[u, shift left, "j"] \arrow[r, "f"'] & C \arrow[u, shift left, "i"] \end{tikzcd}$$

be a pullback of a split epi g along a regular epi f. By applying I to it one obtains a pushout of regular epis in $\mathcal{C}_{Ab}$

$$\begin{tikzcd} IP \arrow[r, "Ip_B"] \arrow[d, shift left, "Ip_A"] & IB \arrow[d, shift left, "Ig"] \\ IA \arrow[u, shift left, "Ij"] \arrow[r, "If"'] & IC. \arrow[u, shift left, "Ii"] \end{tikzcd}$$

By Theorem 2.11 we know that the factorization α from IP through the pullback $IA \times_{IC} IB$ of If and Ig is a regular epi. Let $\beta\colon P \to IA \times_{IC} IB$ be the unique arrow induced by the universal property of the pullback, so that $\beta = \alpha \circ \eta_P$. In order to prove that α is an isomorphism, it will be sufficient to check that $Ker(\eta_P) \simeq Ker(\beta)$. Indeed, in a semi-abelian category it is easy to see that two regular epis determine the same quotient if and only if their kernels are isomorphic. Now, by using the same notation as in the previous section, it is clear that the canonical factorization $\gamma\colon KP \to KA \times_{KC} KB \simeq Ker(\beta)$ is a monomorphism in any pointed category. It is also a regular epi, as one can see by applying Theorem 2.11 this time to the square

$$\begin{tikzcd} KP \arrow[r, "K(p_B)"] \arrow[d, shift left, "K(p_A)"] & KB \arrow[d, shift left, "K(g)"] \\ KA \arrow[u, shift left, "K(j)"] \arrow[r, "K(f)"'] & KC. \arrow[u, shift left, "K(i)"] \end{tikzcd}$$

Indeed, $K(p_B)$ and $K(f)$ are regular epis by Proposition 7.6, therefore the square above is a pushout of regular epimorphisms. □

8.3. Theorem. *If $\mathcal{C}$ is a semi-abelian category, any Galois pregroupoid in $\mathcal{C}_{Ab}$ is an internal groupoid.*

Proof It follows from the previous lemma, since all the pullbacks in the definition of an internal groupoid are then preserved by the functor I. □

The Galois groupoid associated with the extension $f\colon A \to B$ is denoted by $Gal(f)$. The results in [30] then give, in our terminology, an equivalence of categories between the category of extensions which are split by an extension $f\colon A \to B$ and the category $\{Gal(f), \mathcal{C}_{Ab}\}$:

8.4. Theorem. *For any extension $f\colon A \to B$ in $\mathcal{C}$, the categories $Spl(A, f)$ and $\{Gal(f), \mathcal{C}_{Ab}\}$ are equivalent.*

Thanks to Remark 5.8, if there exists a regular epi $\overline{p}\colon \overline{E} \to B$ with $\overline{E}$ projective with respect to regular epimorphisms, one has $Spl(E,p) \subset Spl(\overline{E},\overline{p})$ for all extensions $p\colon E \to B$. When $\mathcal{C}$ is a semi-abelian variety of universal algebras such a projective extension $\overline{p}\colon \overline{E} \to B$ always exists. Consequently, in this case, the category $Centr(B)$ is simply $Spl(\overline{E},\overline{p})$, since this last category is the union of all $Spl(E,p)$.

8.5. Theorem. [31] *Let $\mathcal{C}$ be a semi-abelian variety. If $\overline{p}\colon \overline{E} \to B$ is an extension of B projective with respect to surjective homomorphisms, then*

$$\{Gal(\overline{p}), \mathcal{C}_{Ab}\} \simeq Centr(B).$$

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Fibrations for Abstract Multicategories

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Abstract. Building upon the theory of 2-dimensional fibrations and that of (abstract) multicategories, we present the basics of a theory of *fibred multicategories*. We show their intrinsic role in the general theory: a multicategory is representable precisely when it is covariantly fibrant over the terminal one. Futhermore, such fibred structures allow for a treatment of *algebras for operads* in the internal category setting. We obtain thus a conceptual proof of the 'slices of categories of algebras are categories of algebras' property, which is instrumental in setting up Baez-Dolan's opetopes.

1 Introduction

We introduce the notion of *fibration for multicategories*, the latter understood in their most general sense of (normal) lax algebras on bimodules, as we recall below. Given the space constraints, we limit ourselves to a brief introduction of the attendant theory of fibred multicategories, taking it as an opportunity to review some aspects of our work on 2-fibrations [12] and the theory of representable multicategories [13, 14]. We omit most proofs, occasionally outlining interesting arguments.

In [13] we introduced the notion of **representable multicategory** as an alternative axiomatisation of the notion of **monoidal category**, *representability* being a *universal property* of a multicategory; it demands the existence of universal 'multilinear' morphisms, $\pi_{\vec{x}} : \vec{x} \to \otimes\, \vec{x}$ for every tuple of objects $\vec{x}$, whose codomain endows the underlying the underlying category of 'linear' morphisms with a 'tensor product'. The basics of this theory (axiomatics of universal morphisms, strictness, and coherence) were developed upon the heuristic

$$\textit{universal morphism} \sim \textit{cocartesian morphism} \tag{1}$$

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so that the theory of representable multicategories should parallel that of (co)fibred categories *cf.*[13, Table I]. Subsequently, in [14] we gave a general treatment of the above transformation

$$\boxed{\text{monoidal category} \mapsto \text{representable multicategory}}$$

in the setting of pseudo-algebras for a cartesian monad M on a '2-regular 2-category' $\mathcal{K}$, *i.e.* a 2-category admitting a 'calculus of bimodules' (the 2-dimensional analogue of the calculus of relations available in a regular category), so that M induces a pseudo-monad $\mathsf{Bimod}(\mathsf{M}) : \mathsf{Bimod}(\mathcal{K}) \to \mathsf{Bimod}(\mathcal{K})$ on bimodules. Given these data, we constructed a 2-category $\mathcal{K}_\dashv$ (consisting of normal lax algebras for $\mathsf{Bimod}(\mathsf{M})$) equipped with a 2-monad $\mathsf{T}_\dashv$ such that

1. $\mathsf{T}_\dashv$ has the adjoint-pseudo-algebra property, *i.e.* a psuedo-algebra structure $x : \mathsf{T}_\dashv X \to X$ on an object X is a left-adjoint to the unit $\eta_X : X \to \mathsf{T}_\dashv X$.
2. The 2-categories of pseudo-algebras, strong morphisms and transformations of M and $\mathsf{T}_\dashv$ are equivalent.

This construction achieves the transformation

$$\boxed{\text{coherent structure} \mapsto \text{universally characterised structure}}$$

which subsumes the case of monoidal categories above, as well as the classical Grothendieck transformation of psuedo-functors into (co)fibred categories (see Remark 1.1 below).

In [13, footnote p.169] we argued that in the analogy of representable multicategories with cofibred categories, the former lack a *base*. Here we rectify this statement, showing that representable multicategories are precisely those (covariantly) fibred over the *terminal* one (Theorem 4.1), thereby formalising the heuristic (1) above. Hence, when reasoning about certain categorical structures characterised by universal properties, we can soundly consider them as (covariantly) fibred structures. This correspondence provides yet another argument for the importance of fibred category theory in analysing categorical structure, complementary to the 'philosophical arguments' in [3].

Note on Terminology: Fibrations in categories involve the 'lifting' of morphisms from codomain to domain (or target to source) and thus give rise to contravariant pseudo-functors. For the dual notion, working in the locally dual 2-category $\mathcal{C}at^{co}$, there are two conflicting terminologies in the literature: Gray [11] advocates the use of the term *opfibration* while the Grothendieck school would use the term *cofibration*. Unfortunately, this latter clashes with the notion of cofibration of Quillen in the context of model structures on categories. Given that both notions would come to be used simultaneously in our subsequent work on coherence, it seems sensible to adopt a dissambiguating terminology right here: we shall refer to those fibred structures which give rise to covariant psuedo-functors as **covariant fibrations**, reserving the term **cofibration** for the algebraic topologists' stablished use.

In our foray into the theory of fibrations for multicategories, we would like the reader to keep in mind the following three levels of (increasing) generality and abstraction:

1. The ordinary *Set*-based notion of multicategory introduced by Lambek [17].

2. Multicategories as monads in a 'Kleisli bicategory of spans' $\mathbf{Spn}_{\mathsf{M}}(\mathbb{B})$, where $\mathsf{M} : \mathbb{B} \to \mathbb{B}$ is a cartesian monad on a category with pullbacks, as in [13](*cf.* [4, 18]).
3. Multicategories as normal lax algebras in $\mathsf{Bimod}(\mathcal{K})$, the bicategory of bimodules in a 2-regular 2-category $\mathcal{K}$ ([15]), with respect to the pseudo-monad $\mathsf{Bimod}(\mathsf{M})$ induced by a 2-monad $\mathsf{M} : \mathcal{K} \to \mathcal{K}$, compatible with the calculus of bimodules, as in [14].

1.1. Remark. Since the Grothendieck correspondence between covariant psuedo-functors and cofibrations (covariant fibrations)

$$\mathsf{Ps}\text{-}[\mathbb{C}, \mathcal{C}at] \simeq \mathcal{C}o\mathcal{F}ib/\mathbb{C}$$

on a category $\mathbb{C}$ is not exhibited explicitly in [14, §11] as a consequence of the general theory, we outline here how it can be achieved. This example shows that our 'abstract multicategories' may not look like multicategories at first sight.

Recall that a bimodule $\alpha \colon X \not\to Y$ is called *represpresentable* when it is of the form $x_{\#} = x \downarrow id$ (the span given by the two projections out of the comma-object) for a functor $x : X \to Y$. We observe that pseudo-functors out of $\mathbb{C}$ correspond to pseudo-algebras in the 2-category $[C_0, \mathcal{C}at]$, where C_0 is the object-of-objects of $\mathbb{C}$ (we are in an internal category setting $[C_0, \mathcal{C}at] \cong \mathcal{C}at[C_0, \mathcal{S}et]$). The appropriate 2-monad is $\mathsf{M} = \mathcal{C}at(\mathbb{C} \star \_)$, whose (1-dimensional) cartesian monad $\mathbb{C} \star \_ : [C_0, \mathcal{S}et] \to [C_0, \mathcal{S}et]$ expresses the free action of the (morphisms of) $\mathbb{C}$ on a C_0-family of sets. Remark 11.3 of *ibid.* shows that

$$\mathsf{Ps}\text{-}[\mathbb{C}, \mathcal{C}at] \simeq \mathit{Representable}{-}\mathsf{Lax}_{rep}[\mathbb{C}, \mathsf{Bimod}(\mathcal{C}at)]$$

where $\mathsf{Lax}_{rep}[\mathbb{C}, \mathsf{Bimod}(\mathcal{C}at)]$ is the 2-category of lax functors into bimodules and representable transformations between them (that is, induced by functors), obtained by our transformation process out of the 2-monad M and the 2-category $[C_0, \mathcal{S}et]$. The *Representable*– qualificative means the adjoint pseudo-algebras over such with respect to the 2-monad induced by the 'envelope' adjunction, which we recall in §2.1. See also Remark 2.2.(1) below for a reminder of the characterisation of such pseudo-algebras.

We want to show that the new basis of aximotisation is equivalent to the 2-category $\mathcal{C}at/\mathbb{C}$. Theorem 8.2 of *ibid.* entails :

$$\mathsf{Lax}_{rep}[\mathbb{C}, \mathsf{Bimod}(\mathcal{C}at)] \simeq \mathcal{M}ulticat_{\mathbb{C}\star\_}([C_0, \mathcal{S}et])$$

where C_0 is the object-of-objects of $\mathbb{C}$. This equivalence means that since we are working in an internal category setting we can simplify from bimodules to spans (and their easy composition via pullbacks). We now appeal to the well-known (and easy) equivalence $[C_0, \mathcal{S}et] \simeq Set/C_0$, which expresses the two canonical ways of viewing a family of sets. We obtain the following equivalences of 2-categories:

$$\mathcal{M}ulticat_{\mathbb{C}\star\_}([C_0, \mathcal{S}et]) \simeq \mathcal{M}ulticat_{\mathbb{C}\star\_}(\mathcal{S}et/C_0) \simeq \mathcal{C}at/\mathbb{C}$$

the last equivalence resulting from a mere inspection of the diagrams involved. Hence, 'multicategories' in this situation amount to functors into the category $\mathbb{C}$. Remark 11.2 of *ibid.* completes the argument, in the sense that the resulting adjoint pseudo-algebras over $\mathcal{C}at/\mathbb{C}$ obatined by our transformation are indeed cofibrations (covariant fibrations):

$$\mathit{Representable}{-}\mathcal{C}at/\mathbb{C} \simeq \mathcal{C}o\mathcal{F}ib/\mathbb{C} \quad \square$$

In [20], Street develops some basic aspects of the theory of fibrations internally in a 2-category $\mathcal{K}$ in a representable fashion, *i.e.* a morphism $p : E \to B$ is a fibration iff $\mathcal{K}(X,p) : \mathcal{K}(X,E) \to \mathcal{K}(X,B)$ is a fibration in $\mathcal{C}at$, for every object X. Starting with the concrete setting 1) above, we expect a fibration of multicategories to involve a lifting of (multi)morphisms. Here we ran into the problem that in the 2-category $\mathcal{M}ulticat$, the 2-cells refer only to *linear* morphisms, *i.e.* those morphisms whose domain lies in the image of the unit of the monad M (a 'singleton sequence'). Hence the representable definition is unsuitably weak for multicategories.

The expected lifting of (multi)morphisms is obtained, abstractly, by means of the 'fundamental' monadic adjunction

$$\begin{tikzcd} \mathsf{Lax\text{-}Bimod}_{\mathsf{T}}(\mathcal{K})\text{-}\mathsf{alg} \arrow[r, shift left, "\mathsf{Env}"] \arrow[r, phantom, "\perp"] & \mathsf{T\text{-}alg} \arrow[l, shift left, "U"] \end{tikzcd}$$

so that the left adjoint 'envelope' 2-functor *reflects (covariant) fibrations* (Theorem 2.4). This point of view allows us to reduce the situation to the ordinary case of (representable) fibrations in a 2-category mentioned above. We could draw an analogy with modules for a Lie algebra $\mathfrak{G}$, which correspond to ordinary modules for its universal envelope $U\mathfrak{G}$.

This incipient theory of fibred structures in a multicategory scenario has several foreseeable applications besides our motivational correspondence with representability above. We illustrate this by elucidating some aspects of the theory of *algebras for operads* (in the non-Σ case), giving a conceptual proof of the fact that the 'slices' of such a category of algebras are categories of algebras for a multicategory (Theorem 5.2). Among the topics we have left out for lack of space are the pseudo-functorial (or 'indexed') version of the fibred structures, the 'comprehensive' factorisation system associated to (discrete) covariant fibrations and a related Yoneda structure, which we will present elsewhere.

As for related work, we should mention that quite independently of our developments, Clementino, Hofmann and Tholen have used V-enriched multicategories (relative to a monad) as an abstract setting for categorical topology, with emphasis on the theory of descent [9, 5, 7]. In particular, their analysis of exponentiability involves liftings of factorisations of 'multimorphisms', analogous to Giraud's characterisation of exponentiability in $\mathcal{C}at$ [10] (so that covariant fibrations in our sense are exponentiable, just like in $\mathcal{C}at$). While their setting of V-enriched modules cannot deal with internal structures (which has been our emphasis), their developments and ours can be put into a common framework of 'abstract proarrows' in the sense of [21]. In fact, as we pointed out in [14, §2], the theory developed therein is essentially based on such an axiomatic of 'bicategories of bimodules'.

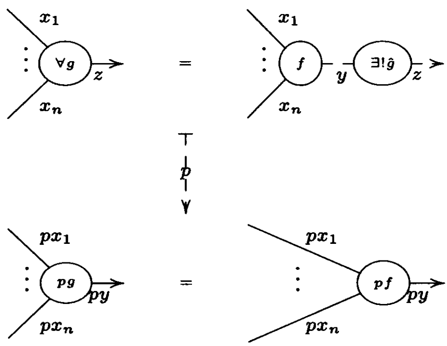
2 Fibrations for multicategories

We start concretely with $\mathcal{S}et$-based multicategories and introduce the elementary definitions of (covariant) fibrations for them. The covariant situation features more prominently in our applications than the contravariant one.

2.1. Definition. Let $p : \mathbb{T} \to \mathbb{B}$ be a morphism of multicategories.

- A morphism $f : \langle x_1 \ldots x_n \rangle \to y$ in $\mathbb{T}$ is (p-)**cocartesian** iff every morphism $g : \langle x_1 \ldots x_n \rangle \to z$, with the same image on the base $pg = pf$, admits a *unique* factorisation $g = \hat{g} \circ f$, with $\hat{g} : y \to z$ a *vertical* morphism ($p\hat{g} =$

$id_{py} = id_{pz}$). Diagrammatically,



- The morphism $p : \mathbb{T} \to \mathbb{B}$ is a **covariant fibration** if the following hold:
    1. for every list of objects $\vec{x} = \langle x_1, \ldots, x_n \rangle$ in $\mathbb{T}$ and every morphism $u : p\vec{x} \to j$ in $\mathbb{B}$, there is a cocartesian morphism $\underline{u} : \vec{x} \to u_! \vec{x}$ over u ($p\underline{u} = u$).
    2. Cocartesian morphisms are closed under composition.

2.2. Remarks.

1. Just like in the ordinary categorical situation, we could have phrased the definition of covariant fibration appealing to a stronger notion of cocartesian morphism (so that its universal property holds with respect to morphisms which factorise through its projection) and thereby dispense with the composition requirement above. But one of our basic results [14, Theorem 5.4] shows that the given formulation is more fundamental: a lax algebra $\alpha \colon MA \not\rightarrow A$ admits an adjoint pseudo-$\mathsf{T}_\dashv$-algebra structure iff
    (a) the bimodule α is representable (which in our context amounts to the existence of cocartesian morphisms), and
    (b) the structural 2-cell $\mu : \alpha \bullet \alpha \Rightarrow \alpha$ is an isomorphism (cocartesian morphisms are closed under multicategory composition).
2. We shall distinguish between fibrations of multicategories, as defined above, and *fibrations in $\mathcal{M}$ulticat* (in the representable sense), which have cartesian liftings of *linear* morphisms only.

Dually, we have a notion of (p-)cartesian morphism and of fibration (contravariant lifting).

2.3. Examples.

1. Given a functor $q : \mathbb{E} \to \mathbb{C}$ in *Cat*, consider the induced morphism of multicategories $q_\blacktriangleright : \mathbb{E}_\blacktriangleright \to \mathbb{C}_\blacktriangleright$ (the multicategories of discrete cocones [13, Example 2.2(2)]):
    - If q is a fibration in *Cat*, $q_\blacktriangleright$ is a fibration of multicategories (cartesian cocones consist of cartesian morphisms in $\mathbb{E}$). In particular, the multicategory $\mathbb{C}_\blacktriangleright$ is fibred over the terminal multicategory.
    - If q is a covariant fibration, and $\mathbb{E}$ has cofibred coproducts (coproducts in the fibres preserved by direct images), $q_\blacktriangleright$ is a covariant fibration of multicategories: given a list of objects $\langle x_1, \ldots, x_n \rangle$ of $\mathbb{E}_\blacktriangleright$ and a morphism $\langle u^1 : qx_1 \to j, \ldots, u^n : qx_n \to j \rangle$ we obtain a cocartesian

lifting by considering individual cocartesian liftings $\underline{u^i} : qx_i \to u^i_!(x_i)$ and forming the coproduct $\vec{u}_!(\vec{x}) = \coprod_i u^i_!(x_i)$ in $\mathbb{E}_j$ with coproduct injections $\kappa^i : u^i_!(x_i) \to \vec{u}_!(\vec{x})$. The composite cocone $\langle \kappa^i \circ \underline{u^i} \rangle_i$ is a cocartesian lifting of $\vec{u}$.

2. Let Rng be the category of commuatitive rings with unit and Rng_m the corresponding multicategory of multilinear maps: $\mathsf{Rng}_m(\langle R_1, \ldots, R_n \rangle, S) = \mathsf{Rng}(R_1 \otimes \ldots \otimes R_n, S)$. Let Mod_m be the multicategory whose objects are pairs (R, M), with R a ring and M an R-module. A morphism in $\mathsf{Mod}_m(\langle (R_1, M_1) \ldots (R_n, M_n) \rangle, (S, N))$ consists of a pair of morphisms (h, a), with $h : R_1 \otimes \ldots \otimes R_n \to S$ in Rng and a $R_1 \otimes \ldots \otimes R_n$-module morphism $a : M_1 \otimes \ldots \otimes M_n \to h^*(N)$ (the tensor product of the abelian groups M_is has componentwise action by the tensor product of the rings). The evident forgetful functor $U : \mathsf{Mod}_m \to \mathsf{Rng}_m$ is a covariant fibration of multicategories: a cocartesian lifting of $\langle (R_1, M_1) \ldots (R_n, M_n) \rangle$ at $h : R_1 \otimes \ldots \otimes R_n \to S$ is the direct image $(M_1 \otimes \ldots \otimes M_n) \otimes_{R_1 \otimes \ldots \otimes R_n} h^*(S)$, where $h^*(S)$ is S regarded as a $(R_1 \otimes \ldots \otimes R_n)$-module via h.

   A sophisticated variation of this example is explored in [19], where the total multicategory has 'multilinear maps with singularities' and the (implicit) base category consists of the full subcategory of Rng on the tensor powers of a Hopf algebra H. It provides a framework for *vertex algebras*.

2.1 Fibrations and the enveloping adjunction. We recall that the monadic adjunction $F \dashv U : \mathcal{M}on\mathcal{C}at \to \mathcal{M}ulticat$ (for our second view of multicategories (2) as monads in in the Kleisli bicategory of spans $\mathbf{Spn}_{\mathsf{M}}(\mathbb{B})$) acts as follows: given a monoidal category $\mathbb{C}$ with objects C_0 and arrows C_1, $U\mathbb{C}$ is

$$\begin{tikzcd}[column sep=small] & C_1 \arrow[dl, "d"'] \arrow[dr, "c"] & & & & & & \mathbf{M}(C_0) \circ C_1 \arrow[dl, "p"'] \arrow[dr, "q"] & & \\ C_0 & & C_0 & \longmapsto & & \mathbf{M}(C_0) \arrow[dl, "id"'] \arrow[dr, "m_0"] & & & C_1 \arrow[dl, "d"'] \arrow[dr, "c"] & \\ & & & & \mathbf{M}(C_0) & & C_0 & & & C_0 \end{tikzcd}$$

where $\mathbf{M}$ is the free-monoid monad in the ambient category (in $\mathcal{S}et$, $\mathbf{M}X = X^*$ the monoid of sequences under concatenation). Given a multicategory $\mathbb{D}$ with objects D_0 and arrows D_1, the free monoidal category $F\mathbb{D}$ is

$$\begin{tikzcd}[column sep=small] & D_1 \arrow[dl, "d"'] \arrow[dr, "c"] & & & & & \mathbf{M}(D_1) \arrow[dl, "\mathbf{M}d"'] \arrow[dr, "\mathbf{M}c"] & \\ \mathbf{M}(D_0) & & D_0 & \longmapsto & & \mathbf{M}^2(D_0) \arrow[dl, "\mu_{D_0}"'] & & \mathbf{M}(D_0) \\ & & & & \mathbf{M}(D_0) & & & \end{tikzcd}$$

For more details of how this construction works, see [13, §8.3]. A more involved construction (via a lax colimit for a monad in $\mathsf{Bimod}(\mathcal{K})$ [14, §2.2]) yields

$$F \dashv U : \mathbf{M}\text{-}\mathsf{alg} \to \mathsf{Lax\text{-}Bimod}_{\mathbf{M}}(\mathcal{K})\text{-}\mathsf{alg},$$

with the same intuitive content: a generalised 2-cell (or 'morphism') of Fx is a 'tuple' of 'morphims' of x (generalised 2-cells of the top object of the bimodule x) whose domain is the 'concatenation' of the domains of its components. The adjunction induces a cartesian 2-monad $\mathsf{T}_{\dashv} : \mathsf{Lax\text{-}Bimod}_{\mathbf{M}}(\mathcal{K})\text{-}\mathsf{alg} \to \mathsf{Lax\text{-}Bimod}_{\mathbf{M}}(\mathcal{K})\text{-}\mathsf{alg}$.

The basic relationship between the notions of fibrations for multicategories and categories is the following:

2.4. Theorem. *Let $p : \mathbb{T} \to \mathbb{B}$ be a morphism of multicategories. Tfae:*

1. *p is a (covariant) fibration of multicategories.*
2. *$Fp : F\mathbb{T} \to F\mathbb{B}$ is a (covariant) fibration of categories.*
3. *$Fp : F\mathbb{T} \to F\mathbb{B}$ is a (covariant) fibration of monoidal categories.*
4. *$\mathsf{T}_\dashv p : \mathsf{T}_\dashv \mathbb{T} \to \mathsf{T}_\dashv \mathbb{B}$ is a (covariant) fibration in $\mathcal{M}ulticat$ (in the sense of Remark 2.2.(2)).*

A fibration of monoidal categories in (3) above is the one-object case of a 2-fibration in the sense of [12]. We remind the reader that a 2-functor $P : \mathscr{E} \to \mathscr{B}$ is a *2-fibration* if it is a fibration at the 1-dimensional level and every 'local hom' functor $P_{X,Y} : \mathscr{E}(X, Y) \to \mathscr{B}(PX, PY)$ is a fibration, whose cartesian (2-)cells are preserved by precomposition with 1-cells (they have a pointwise nature). Hence, a fibration of strict monoidal categories amounts to a fibration of categories whose cartesian morphisms are closed under tensoring.

Notice that both (2) and (4) in the above characterisation make sense for our abstract multicategories as lax algebras in $\mathsf{Bimod}(\mathcal{K})$ and either of them can be adopted as a *definition* of (covariant) fibration in this setting.

From the above characterisation theorem, since both F and $\mathsf{T}_\dashv$ preserve pullbacks (because $\mathbf{M}$ is a cartesian monad), we deduce a change-of-base result for fibrations of multicategories, *i.e.* they are stable under pullback. Let $\mathcal{F}ib(\mathcal{M}ulticat)$ denote the 2-category whose objects are fibrations of multicategories, morphisms are commuting squares where the top morphism preserves cartesian morphisms of the total multicategories, and the evident 2-cells (*cf.* [12]).

2.5. Proposition. *The forgetful 2-functor base : $\mathcal{F}ib(\mathcal{M}ulticat) \to \mathcal{M}ulticat$ taking a fibration to its base multicategory, is a 2-fibration.*

Hence all the basic results of 2-fibrations of [12] (factorisation of adjunctions, construction of Kleisli objects, *etc.*) carry through to the setting of multicategories.

2.2 Adjoint characterisation. In [20] Street gave a characterisation of fibrations internal to a 2-category admitting comma-objects; the existence of cartesian liftings amounts to the existence of a right adjoint to the unit mapping a morphism (object over B) to its free fibration. We reformulate this characterisation using $\mathsf{Hom}\_$ (cotensors with the $\to$ category) and pullbacks, which we can fruitfully reinstantiate in the setting of lax algebras on bimodules.

2.6. Lemma. *A functor $p : \mathbb{E} \to \mathbb{B}$ in $\mathcal{C}at$ is fibration iff the functor η canonically induced into the pullback*

$$\begin{tikzcd}
E^{\to} \arrow[dr, "\eta"] \arrow[drr, bend left, "cod"] \arrow[ddr, bend right, "p^{\to}"'] & & \\
& B^{\to} \times_B E \arrow[r] \arrow[d, "\pi"] & E \arrow[d, "p"] \\
& B^{\to} \arrow[r, "cod"'] & B
\end{tikzcd}$$

admits a right adjoint in $\mathcal{C}at/(\mathbb{B}^{\to})$.

While in $\mathcal{C}at$ the situation for covariant fibrations is entirely dual (simply replacing *cod* by *dom* and *right* by *left* above), the asymmetry between the domain and codomain of (multi)morphisms means that we have to state the characterisations of covariant and contravariant fibrations of multicategories separately:

2.7. Proposition. (Adjoint characterisation of fibrations of multicategories)

Consider a morphism $p : E \to B$ in $\mathsf{Lax}\text{-}\mathsf{Bimod}_{\mathbf{M}}(\mathcal{K})\text{-}\mathsf{alg}$, with lax algebras on bimodules $E = \mathbf{M}E_0 \xleftarrow{d} E_1 \xrightarrow{c} E_0$ and $B = \mathbf{M}B_0 \xleftarrow{d} B_1 \xrightarrow{c} B_0$

1. *p is a fibration iff $p_0 : E_0 \to B_0$ is a fibration (in $\mathcal{K}$) and the canonical morphism η into the pullback*

$$\begin{tikzcd} E_1 \arrow[dr, dashed, "\eta"] \arrow[drr, bend left, "c"] \arrow[ddr, "p_1"'] & & \\ & B_1 \times_{B_0} E_0 \arrow[r] \arrow[d, "\pi"] & E_0 \arrow[d, "p_0"] \\ & B_1 \arrow[r, "c"'] & B_0 \end{tikzcd}$$

   admits a right adjoint in $\mathcal{K}/B_1$.

2. *p is a covariant fibration iff $p_0 : E_0 \to B_0$ is a covariant fibration (in $\mathcal{K}$) and the canonical morphism η into the pullback*

$$\begin{tikzcd} E_1 \arrow[dr, dashed, "\eta"] \arrow[drr, bend left, "d"] \arrow[ddr, "p_1"'] & & \\ & B_1 \times_{\mathbf{M}B_0} \mathbf{M}E_0 \arrow[r] \arrow[d, "\pi"] & \mathbf{M}E_0 \arrow[d, "\mathbf{M}p_0"] \\ & B_1 \arrow[r, "d"'] & \mathbf{M}B_0 \end{tikzcd}$$

   admits a left adjoint in $\mathcal{K}/B_1$.

2.8. Corollary. *Given algebras $x : \mathbf{M}X \to X$ and $y : \mathbf{M}Y \to Y$, and a morphism $f : x \to y$, such that the underlying morphism $f : X \to Y$ is a covariant fibration in $\mathcal{K}$, the induced morphism $Uf : x_{\#} \to y_{\#}$ between lax algebras is a covariant fibration.*

Proof Applying Proposition 2.7.(2), the corresponding left adjoint is obtained from the given one characterising f as a covariant fibration in $\mathcal{K}$, by pulling this latter back along the algebra structure $x : \mathbf{M}X \to X$.

$\square$

3 Coherence for fibrations of multicategories

Fixing a base multicategory $\mathbb{B}$, let $\mathcal{F}ib/\mathbb{B}$ denote the fibre over $\mathbb{B}$ of the 2-fibration *base* : $\mathcal{F}ib(\mathcal{M}ulticat) \to \mathcal{M}ulticat$ and similarly let $Split(\mathcal{F}ib/\mathbb{B})$ the corresponding sub-2-category of split fibrations (*i.e.* those with a choice of cartesian liftings closed under composition and identities) and morphisms between such preserving the splittings. Using the usual coherence theorem for fibrations of categories, the characterisation Theorem 2.4, and the fact that the unit of the monadic adjunction $F \dashv U$ is cartesian, we deduce the following coherence result:

3.1. Theorem. *The inclusion $Split(\mathcal{F}ib/\mathbb{B}) \hookrightarrow \mathcal{F}ib/\mathbb{B}$ has a left biadjoint whose unit is a pseudo-natural equivalence (with a section). Thus, every fibration is equivalent to a split one.*

The dual statement for covariant fibrations also holds.

3.2. Remark. An equivalence with a section is a split-epi at the 'object' level and thus both a covariant and (contravariant) fibration in any 2-category (in the representable sense). By Theorem 2.4, the same holds for equivalences with sections between abstract multicategories.

4 Covariant fibrations and representability

When the 2-category $\mathcal{K}$ has a terminal object $\mathbf{1}$, it bears a unique M-algebra structure, which makes it the terminal object in M-alg. Consequently, $U\mathbf{1}$ is the terminal object in Lax-$\mathsf{Bimod}_{\mathsf{M}}(\mathcal{K})$-$\mathsf{alg}$. For any M-algebra $x : \mathsf{M}X \to X$, the unique morphism $! : X \to \mathbf{1}$ is (rather trivially) a (covariant) fibration, since the terminal object is discrete. For multicategories, the situation is interestingly different (we work in the framework of a strong 2-regular 2-category in the sense of [15]):

4.1. Theorem. *A multicategory $\mathbb{B}$ (qua lax algebra) is representable iff the unique morphism $! : \mathbb{B} \to \mathbf{1}$ is a covariant fibration of multicategories.*

Proof

($\Leftarrow$) Since the unit $\eta : id \Rightarrow \mathsf{T}_\dashv$ of the adjunction

$$F \dashv U : \mathsf{M}\text{-}\mathsf{alg} \to \mathsf{Lax}\text{-}\mathsf{Bimod}_{\mathsf{M}}(\mathcal{K})\text{-}\mathsf{alg}$$

is cartesian with respect to (representable) covariant fibrations (this is where the axioms of 2-regularity come into play), the following square is a pullback

$$\begin{tikzcd} \mathbb{B} \arrow[r, "\perp", "\eta_{\mathbb{B}}"'] \arrow[d, "!"'] & \mathsf{T}_\dashv\mathbb{B} \arrow[l, bend right, dashed] \arrow[d, "\mathsf{T}_\dashv !"] \\ \mathbf{1} \arrow[r, "\perp", "\eta_{1}"'] & \mathsf{T}_\dashv\mathbf{1} \arrow[l, bend right] \end{tikzcd}$$

The existence of a left adjoint to the bottom morphism ($\mathbf{1}$ is clearly representable, since it comes from a M-algebra), the lifting of adjoints in a 2-fibration ([12, Lemma 4.1]) implies the existence of the 'dashed' left adjoint on top, which shows that $\mathbb{B}$ is representable.

($\Rightarrow$) For an M-algebra X, the unique $! : X \to \mathbf{1}$ is a covariant fibration, and so is $U! : UX \to U\mathbf{1}$ (Corollary 2.8). By coherence [14, Thm.7.4], any representable $\mathbb{B}$ is equivalent (via a covariant fibration, see Remark 2.8) to some UX (a strict M-algebra). The composite $\mathbb{B} \to UX \to \mathbf{1}$ is then a covariant fibration, as required. □

4.2. Remark. In the setting of $\mathcal{S}et$-based multicategories, the above theorem has the following concrete interpretation:

- The terminal multicategory has underlying multigraph

$$\begin{tikzcd} & \mathbf{N} \arrow[dl, "id"'] \arrow[dr, "!"] & \\ \mathbf{N} & & \mathbf{1} \end{tikzcd}$$

  where $\mathbf{N}$ is the set of natural numbers. Thus we have a unique arrow $n \succ \bullet$ for every n. Notice that this object is 'discrete' with respect to 2-cells into it, but it has non-trivial (multi)morphisms.
- Universal morphisms in a multicategory $\mathbb{C}$, $\pi_{\vec{x}} : \vec{x} \to \otimes\, \vec{x}$, are precisely the cocartesian morphisms for $! : \mathbb{C} \to \mathbf{1}$ over $|\vec{x}| \succ \bullet$ (see our heuristic (1) in §1).

Of course, in this simple setting, the above correspondence can be seen by mere inspection of the definitions involved. The general proof however requires vastly different methods. It is worth emphasizing that the ($\Rightarrow$) argument is quintessentially 2-fibrational. In the opposite direction, we have used coherence for adjoint pseudo-algebras. Although this use of coherence is not strictly necessary, the given argument does show several pieces of the theory at work.

4.3. Corollary. *The assignment* $\mathbb{M} \mapsto (! : \mathbb{M} \to \mathbf{1})$ *yields an equivalence of the 2-categories of representable multicategories and covariant fibrations over the terminal:*

$$\mathcal{R}ep\mathcal{M}ulticat \cong \mathcal{C}o\mathcal{F}ib/\mathbf{1}$$

Notice that the coherence theorem 3.1 and the above corollary do allow us to recover the coherence theorem for (abstract) representable multicategories.

The relationship between representability and covariant fibrations is completed by the following two results:

4.4. Proposition. *Let* $p : \mathbb{T} \to \mathbb{B}$ *be a morphism of multicategories.*

- *If both* $\mathbb{T}$ *and* $\mathbb{B}$ *are representable and* p *preserves universals, then* p *is a covariant fibration of multicategories iff* p *is a covariant fibration in* $\mathcal{M}ulticat$.
- *If* p *is a covariant fibration of multicategories and* $\mathbb{B}$ *is representable, then* $\mathbb{E}$ *is representable and* p *preserves universals.*

The first item above means that a covariant fibration between representable multicategories is the same thing as a *fibration of categories* (assuming that the induced functor is strong monoidal). The second result has the following logical interpretation: as we have argued in [12] and the references therein, the notion of *logical relation* between models of (various kinds of) type-theories can be fruitfully understood in terms of categorical structure in the total category of a fibration (over the base 'models'). The above result shows how to obtain a *logical tensor* in a *multicagory of predicates*, which is one covariantly fibred over a representable multicategory, that is, a base which admits a 'tensor'. For instance, in Example 2.3.(2), since the base multicategory Rng_m is representable, so is the multicategory Mod_m.

5 Operads and algebras

As an application of the theory of covariant fibrations, we show their role in the theory of *operads* and their *algebras*. From their origin in algebraic topology [16], these tools have found their way into various approaches to higher-dimensional category theory ([1, 2, 18]).

The basic setting of [16] is a (symmetric) monoidal category $\mathbb{C}$. In order to treat these notions with our multicategorical formulation, we would assume that $\mathbb{C}$ has finite limits and admits a free-monoid cartesian monad M, so that we consider multicategories as monads in $\mathbf{Spn}_{\mathsf{M}}(\mathbb{C})$. We have the following identification:

$$\boxed{\text{one-object multicategory} \equiv \text{(non-permutative) operad}}$$

Indeed, a one-object multicategory amounts to an $\mathbf{N}$-indexed family $O = \{O_n\}$ (elements of O_n should be thought of as n-ary operations) closed under composition and identities. Thus, an operad is a structure which groups together the *operations* of a (restricted) algebraic theory. On the other hand, a *monad* describes the result of applying such operations to some generators, thereby describing the *free algebras* of the theory. With these identifications in mind, it is easy to see that an operad O gives rise to a monad $(\_) \otimes_{\mathsf{M}} O$: the category $\mathbb{C}$ embeds in $\mathbf{Spn}_{\mathsf{M}}(\mathbb{C})(\mathbf{1},\mathbf{1})$ ($J : \mathbb{C} \to \mathbf{Spn}_{\mathsf{M}}(\mathbb{C})(\mathbf{1},\mathbf{1})$ regards an object X as a span $\mathsf{M}\mathbf{1} \xleftarrow{\eta} \mathbf{1} \leftarrow X \to \mathbf{1}$), while taking the top object of the span gives a functor $D : \mathbf{Spn}_{\mathsf{M}}(\mathbb{C})(\mathbf{1},\mathbf{1}) \to \mathbb{C}$. Given an operad O as an object in $\mathbf{Spn}_{\mathsf{M}}(\mathbb{C})(\mathbf{1},\mathbf{1})$, we set $(\_) \otimes_{\mathsf{M}} O = D \circ (\_) \bullet O \circ J$, which inherits the monoid structure from O, thereby yielding a monad $(\_) \otimes_{\mathsf{M}} O : \mathbb{C} \to \mathbb{C}$. Notice that the composite $(\_)\bullet O$ is the application of the operations to generators, the latter suitably reinterpreted as a family of operations. We can identify algebras for the operad O with those of its associated monad:

$$\boxed{O\text{-algebras} \equiv (\_ \otimes_{\mathsf{M}} O)\text{-}\mathsf{alg}}$$

In concrete terms, an O-algebra amounts to an object A of $\mathbb{C}$ endowed with actions $a(o) : A^n \to A$ for every n-ary operation o, associative and unitary (with respect to the monoid structure on O). By inspection of the resulting diagrams, we notice that such actions can be equivalently phrased in terms of discrete covariant fibrations:

$$\begin{tikzcd}
& A \otimes_{\mathbf{M}} O \arrow[dl, "p"'] \arrow[dr, "a"] \arrow[dd, "q"] & \\
\mathbf{M}A \arrow[dd, "\mathbf{M}!"'] \arrow[r, phantom, "p.b."] & & A \arrow[dd, "!"] \\
& O \arrow[dl, "ar"] \arrow[dr, "!"'] & \\
\mathbf{N} & & \mathbf{1}
\end{tikzcd}$$

where $\mathbf{N} = \mathbf{M1}$ and the left-hand square is a pullback. The top span is then a multicategory (because the actions are associative and unitary), which we write $(A, a)^+$, and $! : A \to \mathbf{1}$ is a covariant fibration. The pullback square means that the fibres are discrete, so that a (multi)morphism of $(A, a)^+$ is uniquely determined by its source and its image in O. We arrive to the following:

5.1. Proposition.

$$\boxed{O\text{-}algebras \equiv \textit{discrete covariant fibrations over the multicategory } O}$$

This identification indicates that we can consider more generally a notion of **algebra for a multicategory** as a discrete covariant fibration over it. From the algebraic-theory perspective above, operads correspond to single-sorted restricted theories (equations must involve the same variables in the same order on both sides [8], in the non-permutative case), while multicategories correspond to the many-sorted version.

We now consider slice categories of algebras. In order to see that such slice categories are themselves categories of algebras, we appeal to the following two 'fibrational facts':

- Given covariant fibrations of multicategories $p : \mathbb{A} \to \mathbb{B}$ and $q : \mathbb{C} \to \mathbb{B}$, a morphism of covariant fibrations $h : p \to q$ is a covariant fibration in $\mathcal{C}o\mathcal{F}ib/\mathbb{B}$ iff it is a covariant fibration of multicategories $h : \mathbb{A} \to \mathbb{C}$. More concisely

$$\{\mathcal{C}o\mathcal{F}ib - in(\mathcal{C}o\mathcal{F}ib/\mathbb{C})\}/q \equiv \mathcal{C}o\mathcal{F}ib/\mathbb{C}$$

  This property can be deduced from the corresponding one for ordinary fibrations in a 2-category, via the 2-fibrational argument in [12, §4.3] and the adjoint characterisation of covariant fibrations in Proposition 2.7.
- In the same situation, if the base covariant fibration q has discrete fibres, any morphism into it is a covariant fibration:

$$\{\mathcal{C}o\mathcal{F}ib - in(\mathcal{C}o\mathcal{F}ib/\mathbb{C})\}/q \cong (\mathcal{C}o\mathcal{F}ib/\mathbb{C})/q$$

Combining these two facts we obtain the following slicing result:

5.2. Theorem. *For a multicatgory O, any slice of the category of O-algebras is again a category of algebras:*

$$\boxed{O\text{–algebras}/(A,a) \equiv (A,a)^{+}\text{–algebras}}$$

5.3. Remark. For brevity we do not deal here with permutative operads (those whose operations have symmetric-group actions), which we will take us into the more involved setting of lax algebras on bimodules rather than spans (we would work with $\mathsf{Lax\text{-}Bimod}_{\mathsf{S}}(\mathcal{K})\text{-}\mathsf{alg}$, where S is the free-symmetric-monoidal-category monad). This is the set-up of [2]: an operad in their sense is a lax algebra. One technical subtlety of this extension is that S is not quite compatible with the calculus of bimodules and the resulting gadget $\mathsf{Bimod}_{\mathsf{S}}(\mathcal{K})$ is only a *lax* bicategory. Nevertheless, the notion of monad applies equally to this setting and the conceptual identifications above regarding algebras carry through. In particular Theorem 5.2 gives an alternative (fibrational) view of the slicing result claimed in *ibid.* See [6] for a more detailed account of the slicing process.

We conclude pointing out that the consideration of algebras for an operad O via the *endomorphism operad* $\mathsf{End}(A,A)$ of an object A (O-algebra structure on $A \equiv$ operad morphism $O \to \mathsf{End}(A,A)$) is available in our setting if the ambient category $\mathbb{C}$ is locally cartesian closed, but we forego the details for lack of space.

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Lie-Rinehart Algebras, Descent, and Quantization

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Abstract. A *Lie-Rinehart algebra* (A, L) consists of a commutative algebra A and a Lie algebra L with additional structure which generalizes the mutual structure of interaction between the algebra of smooth functions and the Lie algebra of smooth vector fields on a smooth manifold. Lie-Rinehart algebras provide the correct categorical language to solve the problem whether Kähler quantization commutes with reduction which, in turn, may be seen as a descent problem.

Introduction

The algebra of smooth functions $C^\infty(N)$ on a smooth manifold N and its Lie algebra of smooth vector fields $\mathrm{Vect}(N)$ have an interesting structure of interaction. For reasons which will become apparent below, we will refer to a pair (A, L) which consists of a commutative algebra A and a Lie algebra L with additional structure modeled on a pair of the kind $(C^\infty(N), \mathrm{Vect}(N))$ as a *Lie-Rinehart algebra*. In this article we will show that the notion of Lie-Rinehart algebra provides the correct categorical language to solve a problem which we will describe shortly. Lie-Rinehart algebras occur in other areas of mathematics as well; an overview will be given in Section 1 below.

According to a philosophy going back to DIRAC, the correspondence between a classical theory and its quantum counterpart should be based on an analogy between their mathematical structures. In one direction, this correspondence, albeit not well defined, is referred to as *quantization*. Given a classical system with constraints which, in turn, determine what is called the *reduced system*, the question arises whether quantization *descends* to the reduced system in such a way that, once the unconstrained system has been successfully quantized, imposing the symmetries on the quantized unconstrained system is equivalent to quantizing the reduced system. This question goes back to the early days of quantum mechanics and appears already in DIRAC'S work on the electron and positron [15], [16].

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In the framework of Kähler quantization, the problem may be phrased as a *descent problem* and, indeed, under favorable circumstances which, essentially come down to requiring that the unreduced and reduced spaces be both ordinary quantizable Kähler manifolds, the problem has been known for long to have a solution [22] which, among other things, involves a version of what is referred to as KEMPF'S *descent lemma* [46] in geometric invariant theory; see e. g. [17] and Remark 8.6 below. In the present article we will advertise the idea that the concept of *Lie-Rinehart algebra* provides the appropriate categorical language to solve the problem, spelled out as a *descent problem*, under suitable more general circumstances so that, in a sense, reduction after quantization is then equivalent to quantization after reduction; here the term "descent" should, perhaps, not be taken in too narrow a sense.

Given a classical system, its dynamical behaviour being encapsulated in a Poisson bracket among the classical observables, according to DIRAC's idea of correspondence between the classical and quantum system, the Poisson bracket should then be the classical analogue of the quantum mechanical commutator. Thus, on the physics side, the Poisson bracket is a crucial piece of structure. Mathematically, it is a crucial piece of structure as well; in particular, when the classical phase space involves singularities, these may be understood in terms of the Poisson structure. More precisely, when the classical phase space carries a stratified symplectic structure, the Poisson structure encapsulates the mutual positions of the symplectic structures on the strata. See e. g. [31], [32], [37], [39].

Up to now, the available methods have been insufficient to attack the problem of quantization of reduced observables, once the reduced phase space is no longer a smooth manifold; we will refer to this situation as the *singular case*. The singular case is the rule rather than the exception. For example, simple classical mechanical systems and the solution spaces of classical field theories involve singularities; see e. g. [3] and the references there. In the presence of singularities, restricting quantization to a smooth open dense part, the "top stratum", leads to a loss of information and in fact to inconsistent results, cf. Section 4 of [41]. To overcome these difficulties on the classical level, in [39], we isolated a certain class of "Kähler spaces with singularities", which we call *stratified Kähler spaces*. On such a space, the complex analytic structure alone is unsatisfactory for issues related with quantization because it overlooks the requisite Poisson structures. In [41] we developed the Kähler quantization scheme over (complex analytic) stratified Kähler spaces. A suitable notion of *prequantization*, phrased in terms of *prequantum modules* introduced in [30], yields the requisite representation of the Poisson algebra; in particular, this representation satisfies the Dirac condition. A suitably defined concept of stratified Kähler polarization then takes care of the irreducibility problem, as does an ordinary polarization in the smooth case. Over a stratified space, the appropriate quantum phase space is what we call a *costratified* Hilbert space; this is a system of Hilbert spaces, one for each stratum, which arises from quantization on the closure of that stratum, the stratification provides linear maps between these Hilbert spaces reversing the partial ordering among the strata, and these linear maps are compatible with the quantizations. The main result obtained in [41] says that, for a positive Kähler manifold with a hamiltonian action of a compact Lie group, when suitable additional conditions are imposed, reduction after quantization coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the *invariant* *unreduced and reduced quantum observables as well.* Examples abound; one such class of examples, involving holomorphic nilpotent orbits and in particular angular momentum zero spaces, has been treated in [41]. A particular case thereof will be reproduced in Section 6 below, for the sake of illustration.

A stratified polarization, see Section 5 below for details, is defined in terms of an appropriate Lie-Rinehart algebra which, for any Poisson algebra, serves as a replacement for the tangent bundle of a smooth symplectic manifold. The question whether quantization commutes with reduction includes the question whether what is behind the phrase "in terms of an appropriate Lie-Rinehart algebra" descends to the reduced level. This hints at interpreting this question as a descent problem.

To our knowledge, the idea of Lie-Rinehart algebra was first used by JACOBSON in [43] (without being explicitly identified as a structure in its own) to study certain field extensions. Thereafter this idea occurred in other areas including differential geometry and differential Galois theory. More details will be given below.

I am indebted to the organizers of the meeting for having given me the chance to illustrate an application of Lie-Rinehart algebras to a problem phrased in a language entirely different from that of Lie-Rinehart algebras. Perhaps one can build a general Galois theory including ordinary Galois theory, differential Galois theory, and ordinary principal bundles, in which Lie-Rinehart algebras appear as certain objects which capture infinitesimal symmetries.

1. Lie-Rinehart algebras

Let R be a commutative ring with 1 taken as ground ring which, for the moment, may be arbitrary. For a commutative R-algebra A, we denote by $\mathrm{Der}(A)$ the R-Lie algebra of derivations of A, with its standard Lie algebra structure. An *(R, A)-Lie algebra* [66] is a Lie algebra L over R which acts on (the left of) A (by derivations) and is also an A-module satisfying suitable compatibility conditions which generalize the usual properties of the Lie algebra of vector fields on a smooth manifold viewed as a module over its ring of functions; these conditions read

$$[\alpha, a\beta] = \alpha(a)\beta + a[\alpha, \beta],$$
$$(a\alpha)(b) = a(\alpha(b)),$$

where $a, b \in A$ and $\alpha, \beta \in L$. When the emphasis is on the pair (A, L) with the mutual structure of interaction between A and L, we refer to the pair (A, L) as a *Lie-Rinehart* algebra. Given an arbitrary commutative algebra A over R, an obvious example of a Lie-Rinehart algebra is the pair $(A, \mathrm{Der}(A))$, with the obvious action of $\mathrm{Der}(A)$ on A and obvious A-module structure on $\mathrm{Der}(A)$. There is an obvious notion of morphism of Lie-Rinehart algebras and, with this notion of morphism, Lie-Rinehart algebras constitute a category. More details may be found in RINEHART [66] and in our papers [29] and [30].

We will now briefly spell out some of the salient features of Lie-Rinehart algebras. Given an (R, A)-Lie algebra L, its *universal algebra* $(U(A, L), \iota_L, \iota_A)$ is an R-algebra $U(A, L)$ together with a morphism $\iota_A \colon A \longrightarrow U(A, L)$ of R-algebras and a morphism $\iota_L \colon L \longrightarrow U(A, L)$ of Lie algebras over R having the properties

$$\iota_A(a)\iota_L(\alpha) = \iota_L(a\,\alpha), \quad \iota_L(\alpha)\iota_A(a) - \iota_A(a)\iota_L(\alpha) = \iota_A(\alpha(a)),$$

and $(U(A, L), \iota_L, \iota_A)$ is *universal* among triples (B, ϕ_L, ϕ_A) having these properties. For example, when A is the algebra of smooth functions on a smooth manifold N

and L the Lie algebra of smooth vector fields on N, then $U(A, L)$ is the *algebra of (globally defined) differential operators on* N. An explicit construction for the R-algebra $U(A, L)$ is given in RINEHART [66]. See our paper [29] for an alternate construction which employs the MASSEY-PETERSON [59] algebra.

The universal algebra $U(A, L)$ admits an obvious filtered algebra structure $U_{-1} \subseteq U_0 \subseteq U_1 \subseteq \ldots$, cf. [66], where $U_{-1}(A, L) = 0$ and where, for $p \geq 0$, $U_p(A, L)$ is the left A-submodule of $U(A, L)$ generated by products of at most p elements of the image $\overline{L}$ of L in $U(A, L)$, and the associated graded object $E^0(U(A, L))$ inherits a commutative graded A-algebra structure. The Poincaré-Birkhoff-Witt Theorem for $U(A, L)$ then takes the following form where $S_A[L]$ denotes the symmetric A-algebra on L, cf. (3.1) of [66].

Theorem 1.1. (Rinehart) *For an (R, A)-Lie algebra L which is projective as an A-module, the canonical A-epimorphism $S_A[L] \longrightarrow E^0(U(A, L))$ is an isomorphism of A-algebras.*

Consequently, for an (R, A)-Lie algebra L which is projective as an A-module, the morphism $\iota_L : L \longrightarrow U(A, L)$ is injective.

The construction of the ordinary Koszul complex computing Lie algebra cohomology carries over as well: Let $\Lambda_A(sL)$ be the exterior Hopf algebra over A on the suspension sL of L, where "suspension" means that sL is L except that its elements are regraded up by 1. RINEHART [66] has proved that the ordinary Chevalley-Eilenberg operator induces an $U(A, L)$-linear operator d on $U(A, L) \otimes_A \Lambda_A(sL)$ (this is not obvious since L is not an ordinary A-Lie algebra unless L acts trivially on A) having square zero. We will refer to

$$K(A, L) = (U(A, L) \otimes_A \Lambda_A(sL), d) \tag{1.2}$$

as the *Rinehart complex* for (A, L). It is manifest that the Rinehart complex is functorial in (A, L). Moreover, as a graded A-module, the resulting chain complex $\mathrm{Hom}_{U(A,L)}(K(A, L), A)$ underlies the A-algebra $\mathrm{Alt}_A(L, A)$ of A-multilinear functions on L but, beware, the differential is linear only over the ground ring R and turns $\mathrm{Alt}_A(L, A)$ into a differential graded cocommutative algebra over R; we will refer to this differential graded R-algebra as the *Rinehart algebra* of (A, L). Rinehart also noticed that, when L is projective or free as a left A-module, $K(A, L)$ is a projective or free resolution of A in the category of left $U(A, L)$-modules according as L is a projective or free left A-module; details may be found in [66]. In particular, the Rinehart algebra $(\mathrm{Alt}_A(L, A), d)$ then computes the Ext-algebra $\mathrm{Ext}^*_{U(A,L)}(A, A)$.

Rinehart also noticed that, when A is the algebra of smooth functions on a smooth manifold N and L the Lie algebra of smooth vector fields on N, then $(\mathrm{Alt}_A(L, A), d)(= \mathrm{Hom}_{U(A,L)}(K(A, L), A))$ is the ordinary *de Rham complex of* N whence, as an algebra, the de Rham cohomology of N amounts to the Ext-algebra $\mathrm{Ext}^*_{U(A,L)}(A, A)$ over the algebra $U(A, L)$ of differential operators on N. Likewise, for a Lie algebra L over R acting trivially on R, $K(R, L)$ is the ordinary Koszul complex; in particular, when L is projective as an R-module, $K(R, L)$ is the ordinary Koszul resolution of the ground ring R. *Thus the cohomology of Lie-Rinehart algebras comprises de Rham- as well as Lie algebra cohomology.* In particular, this offers a possible explanation why CHEVALLEY and EILENBERG [12], when they first isolated Lie algebra cohomology, derived their formulas by abstracting from the de Rham operator of a smooth manifold. Suitable graded versions of the cohomology of Lie-Rinehart algebras comprise as well Hodge cohomology and coherent sheaf cohomology of complex manifolds [36, 38].

The classical differential geometry notions of connection, curvature, characteristic classes, etc. may be developed for arbitrary Lie-Rinehart algebras [29], [30], [34], and there are notions of duality for Lie-Rinehart algebras generalizing Poincaré duality [35]; the idea of duality has been shown in [33] to cast new light on Gerstenhaber- and Batalin-Vilkovisky algebras. In a sense these homological algebra interpretations of Batalin-Vilkovisky algebras push further Rinehart's observations related with the interpretation of de Rham cohomology as certain Ext-groups. Graded versions of duality for Lie-Rinehart algebras [36], [38] may be used to study e. g. complex manifolds, CR-structures, and the mirror conjecture.

Lie-Rinehart algebras were implicitly used already by JACOBSON [43] and later by HOCHSCHILD [27]. The idea of Lie-Rinehart algebra has been introduced by a very large number of authors, most of whom independently proposed their own terminology. I am indebted to K. Mackenzie for his help with compiling the following list in chronological order: Pseudo-algèbre de Lie: Herz, 1953 [24]—actually, Herz seems to be the first to describe the structure in a form which makes its generality clear—; Lie d-ring: Palais, 1961 [62]; (R,C)-Lie algebra: Rinehart, 1963 [66]; (R,C)-éspace d'Elie Cartan régulier et sans courbure: de Barros, 1964 [14]; (R,C)-algèbre de Lie: Bkouche, 1966 [7]; Lie algebra with an associated module structure: Hermann, 1967 [23]; Lie module: Nelson, 1967 [61]; Pseudo-algèbre de Lie: Pradines, 1967 [64]; (A, C) system: Kostant and Sternberg, 1971 [52]; Sheaf of twisted Lie algebras: Kamber and Tondeur, 1971 [44]; Algèbre de Lie sur C/R: Illusie, 1972 [42]; Lie algebra extension: N. Teleman, 1972 [76]; Lie-Cartan pair: Kastler and Stora, 1985 [45]; Atiyah algebra: Beilinson and Schechtmann, 1988 [6]; Lie-Rinehart algebra: Huebschmann, 1990 [29]; Differential Lie algebra: Kosmann–Schwarzbach and Magri, 1990 [50]. Hinich and Schechtman (1993) [26] have used the term Lie algebroid for the general algebraic concept. In differential Galois theory, Lie-Rinehart algebras occur under the name "algèbre différentielle" in a paper by FAHIM [18]. Lie-Rinehart algebras occur as well in CHASE [10] and STASHEFF [75]. We have chosen to use the terminology *Lie-Rinehart algebra* since, as already pointed out, Rinehart subsumed the cohomology of these objects under standard homological algebra and established a Poincaré-Birkhoff-Witt theorem for them. In differential geometry, (R, A)-Lie algebras arise as spaces of sections of Lie algebroids. These, in turn, were introduced in 1966 by PRADINES [63] and, in that paper, Pradines raised the issue whether Lie's third theorem holds for Lie algebroids in the sense that any Lie algebroid integrates to a Lie groupoid. CRAINIC and FERNANDES [13] have recently given a solution of this problem in terms of suitably defined obstructions. See MACKENZIE [56] for a complete account of Lie algebroids and Lie groupoids, as well as CANAS DA SILVA-WEINSTEIN [9] and MACKENZIE [57] for more recent surveys on particular aspects. The idea of Lie algebroid is lurking behind a construction in FUCHSSTEINER [19] (see Remark 2.6 below). A descent construction for Lie algebroids may be found in HIGGINS-MACKENZIE [25]. A general notion of morphism of Lie algebroids has been introduced by ALMEIDA-KUMPERA [2]. This notion has been used by S. CHEMLA [11] to study a version of duality for Lie algebroids in complex algebraic geometry generalizing Serre duality. Lie-Rinehart algebras are lurking as well behind the nowadays very active research area of D-modules.

Remark (out of context). At the end of his paper [66], RINEHART introduced an operator on the Hochschild complex of a commutative algebra which, some 20 years later, was reinvented by A. CONNES in order to define cyclic cohomology.

2. Poisson algebras

For intelligibility, we recall briefly how for an arbitrary Poisson algebra an appropriate Lie-Rinehart algebra serves as a replacement for the tangent bundle of a smooth symplectic manifold.

Let $(A, \{\cdot,\cdot\})$ be a Poisson algebra, and let D_A be the A-module of formal differentials of A the elements of which we write as du, for $u \in A$. For $u, v \in A$, the assignment to (du, dv) of $\pi(du, dv) = \{u, v\}$ yields an A-valued A-bilinear skew-symmetric 2-form $\pi = \pi_{\{\cdot,\cdot\}}$ on D_A, the *Poisson 2-form* for $(A, \{\cdot,\cdot\})$. Its adjoint

$$\pi^\sharp : D_A \to \mathrm{Der}(A) = \mathrm{Hom}_A(D_A, A) \tag{2.1}$$

is a morphism of A-modules, and the formula

$$[adu, bdv] = a\{u, b\}dv + b\{a, v\}du + abd\{u, v\} \tag{2.2}$$

yields a Lie bracket $[\cdot,\cdot]$ on D_A, viewed as an R-module. More details may be found in [29]. For the record we recall the following, established in [29] (3.8).

Proposition 2.3. *The A-module structure on D_A, the bracket $[\cdot,\cdot]$, and the morphism $\pi^\sharp$ of A-modules turn the pair (A, D_A) into a Lie-Rinehart algebra in such a way that $\pi^\sharp$ is a morphism of Lie-Rinehart algebras.*

We write $D_{\{\cdot,\cdot\}} = (D_A, [\cdot,\cdot], \pi^\sharp)$. The 2-form $\pi_{\{\cdot,\cdot\}}$, which is defined for *every* Poisson algebra, is plainly a 2-cocycle in the Rinehart algebra $(\mathrm{Alt}_A(D_{\{\cdot,\cdot\}}, A), d)$. In [29], we defined the *Poisson cohomology* $\mathrm{H}^*_{\mathrm{Poisson}}(A, A)$ of the Poisson algebra $(A, \{\cdot,\cdot\})$ to be the cohomology of this Rinehart algebra, that is,

$$\mathrm{H}^*_{\mathrm{Poisson}}(A, A) = \mathrm{H}^*\left(\mathrm{Alt}_A(D_{\{\cdot,\cdot\}}, A), d\right).$$

Henceforth we shall take as ground ring that of the reals $\mathbb{R}$ or that of the complex numbers $\mathbb{C}$. We shall consider spaces N with an algebra of continuous $\mathbb{R}$-valued or $\mathbb{C}$-valued functions, deliberately denoted by $C^\infty(N, \mathbb{R})$ or $C^\infty(N, \mathbb{C})$ as appropriate, or by just $C^\infty(N)$, for example ordinary smooth manifolds and ordinary smooth functions; such an algebra $C^\infty(N)$ will then be referred to as a *smooth* structure on N, and $C^\infty(N)$ will be viewed as part of the structure of N. A space may support *different* smooth structures, though. Given a space N with a smooth structure $C^\infty(N)$, we shall write $\Omega^1(N)$ for the space of formal differentials with those differentials divided out that are zero at each point, cf. [53]; for example, over the real line with its ordinary smooth structure, the formal differentials $d\sin x$ and $\cos x dx$ do not coincide but the formal differential $d\sin x - \cos x dx$ is zero at each point. At a point of N, the object $\Omega^1(N)$ comes down to the ordinary space of differentials for the smooth structure on N; see Section 1.3 of our paper [41] for details. When N is an ordinary smooth manifold, $\Omega^1(N)$ amounts to the space of smooth sections of the cotangent bundle. For a general smooth space N over the reals, when $A = C^\infty(N, \mathbb{R})$ is endowed with a Poisson structure, the formula (1.2) yields a Lie-bracket $[\cdot,\cdot]$ on the A-module $\Omega^1(N)$ and the 2-form $\pi_{\{\cdot,\cdot\}}$ is still defined on $\Omega^1(N)$; its adjoint then yields an A-linear map $\pi^\sharp$ from $\Omega^1(N)$ to $\mathrm{Der}(A)$ in such a way that $([\cdot,\cdot], \pi^\sharp)$ is an $(\mathbb{R}, A)$-Lie algebra structure on $\Omega^1(N)$

and that the adjoint $\pi^\sharp$ is a morphism of $(\mathbb{R}, A)$-Lie algebras. Here the notation $[\cdot,\cdot]$, $\pi^\sharp$, $\pi_{\{\cdot,\cdot\}}$ is abused somewhat. The obvious projection map from $D_{C^\infty(N)}$ to $\Omega^1(N)$ is plainly compatible with the Lie-Rinehart structures. The fact that $D_{C^\infty(N)}$ is "bigger" than $\Omega^1(N)$ in the sense that the surjection from the former to the latter has a non-trivial kernel causes no problem here since the A-dual of this surjection, that is, the induced map from $\mathrm{Hom}(\Omega^1(N), A)$ to $\mathrm{Hom}(D_{C^\infty(N)}, A)$, is an isomorphism. We shall write $\Omega^1(N)_{\{\cdot,\cdot\}} = (\Omega^1(N), [\cdot,\cdot], \pi^\sharp)$. When N is a smooth manifold in the usual sense, the range $\mathrm{Der}(A)$ of the adjoint map $\pi^\sharp$ from $\Omega^1(N)$ to $\mathrm{Der}(A)$ boils down to the space $\mathrm{Vect}(N)$ of smooth vector fields on N. In this case, the Poisson structure on N is symplectic, that is, arises from a (uniquely determined) symplectic structure on N, if and only if $\pi^\sharp$, which may now be written as a morphism of smooth vector bundles from T^*N to $\mathrm{T}N$, is an isomorphism.

The 2-form $\pi_{\{\cdot,\cdot\}}$ is a 2-cocycle in the Rinehart algebra $(\mathrm{Alt}_A(\Omega^1(N)_{\{\cdot,\cdot\}}, A), d)$ and the canonical map of differential graded algebras from $(\mathrm{Alt}_A(\Omega^1(N)_{\{\cdot,\cdot\}}, A), d)$ to $(\mathrm{Alt}_A(D_{\{\cdot,\cdot\}}, A), d)$ is an isomorphism. In particular, we may take the cohomology of the Rinehart algebra $(\mathrm{Alt}_A(\Omega^1(N)_{\{\cdot,\cdot\}}, A), d)$ as the definition of the Poisson cohomology $\mathrm{H}^*_{\mathrm{Poisson}}(A, A)$ of the Poisson algebra $(A, \{\cdot,\cdot\})$ as well. When N is an ordinary smooth manifold, its algebra of ordinary smooth functions being endowed with a Poisson structure, this notion of Poisson cohomology comes down to that introduced by LICHNEROWICZ [55]. For a general Poisson algebra A, the 2-form $\pi_{\{\cdot,\cdot\}}$, be it defined on $\Omega^1(N)_{\{\cdot,\cdot\}}$ for the case where A is the structure algebra $C^\infty(N)$ of a space N or on $D_{\{\cdot,\cdot\}}$ for an arbitrary Poisson algebra, generalizes the symplectic form of a symplectic manifold; see Section 3 of [29] for details. Suffice it to make the following observation, relevant for quantization: Consider a space N with a smooth structure $C^\infty(N)$ which, in turn, is endowed with a Poisson bracket $\{\cdot,\cdot\}$. The Poisson 2-form $\pi_{\{\cdot,\cdot\}}$ determines an *extension* of Lie-Rinehart algebras which is central as a Lie algebra extension. For technical reasons it is more convenient to take here the extension

$$0 \longrightarrow A \longrightarrow \overline{L}_{\{\cdot,\cdot\}} \longrightarrow \Omega^1(N)_{\{\cdot,\cdot\}} \longrightarrow 0 \tag{2.4}$$

which corresponds to the negative of the Poisson 2-form. Here, as A-modules, $\overline{L}_{\{\cdot,\cdot\}} = A \oplus \Omega^1(N)_{\{\cdot,\cdot\}}$, and the Lie bracket on $\overline{L}_{\{\cdot,\cdot\}}$ is given by

$$[(a, du), (b, dv)] = (\{u, b\} + \{a, v\} - \{u, v\}, d\{u, v\}), \quad a, b, u, v \in A. \tag{2.5}$$

Here we have written "$\overline{L}$" rather than simply L to indicate that the extension (2.4) represents the negative of the class of $\pi_{\{\cdot,\cdot\}}$ in the second cohomology group $\mathrm{H}^2(\mathrm{Alt}_A(D_{\{\cdot,\cdot\}}, A), d)$ of the corresponding Rinehart algebra, cf. [29], and the notation du, dv etc. is abused somewhat. Now, any principal circle bundle admits as its infinitesimal object an *Atiyah*-sequence [5] whose spaces of sections constitute a central extension of Lie-Rinehart algebras; see [56] for a complete account of Atiyah-sequences and [34] for a theory of characteristic classes for extensions of general Lie-Rinehart algebras. When the Poisson structure is an ordinary smooth symplectic Poisson structure whose symplectic form represents an integral cohomology class, the Lie-Rinehart algebra extension (2.4) comes down to the space of sections of the Atiyah-sequence of the principal circle bundle classified by that cohomology class.

Remark 2.6. For the special case where N is an ordinary smooth manifold, $C^\infty(N)$ its algebra of ordinary smooth functions, and where $\{\cdot,\cdot\}$ is a Poisson structure on $C^\infty(N)$, the Lie-Rinehart structure on the pair $(C^\infty(N), \Omega^1(N))$ (where $\Omega^1(N)$ amounts to the space of ordinary smooth 1-forms on N) was discovered by a number of authors during the 80's most of whom phrased the structure in terms of the corresponding Lie algebroid structure on the cotangent bundle of N; some historical comments may be found in [29]. The first reference I am aware of where versions of the Lie algebroid bracket and of the anchor map may be found is [19]; in that paper, the notion of "implectic operator" is introduced—this is the operator nowadays referred to as *Poisson tensor*—and the Lie bracket and anchor map are the formula (2) and morphism written as Ω_ϕ, respectively, in that paper. *The construction in terms of formal differentials carried out in* [29] (and reproduced above)—as opposed to the Lie algebroid construction—is *more general, though, since it refers to an arbitrary Poisson structure, not necessarily one which is defined on an algebra of smooth functions on an ordinary smooth manifold.* In fact, the aim of the present article is to demonstrate the significance of this more general construction which works as well for Poisson algebras of continuous functions defined on spaces with *singularities* where among other things it yields a tool to relate the Poisson structures on the strata of a stratified symplectic space; suitably translated into the language of sheaves, it also works over not necessarily non-singular varieties.

3. Quantization

According to DIRAC [15], [16], a *quantization* of a classical system described by a real Poisson algebra $(A, \{\cdot,\cdot\})$ is a representation $a \mapsto \hat a$ of a certain Lie subalgebra B of A, A and B being viewed merely as Lie algebras, by symmetric or, whenever possible, self-adjoint, operators $\hat a$ on a Hilbert space $\mathcal H$ such that (i) the Dirac condition

$$i\,[\hat a, \hat b] = \widehat{\{a,b\}}$$

holds; that (ii) for a constant c, the operator $\hat c$ is given by $\hat c = c\,\mathrm{Id}$; and that (iii) the representation is irreducible. Here the factor i in the Dirac condition is forced by the interpretation of quantum mechanics: Observables are to be represented by symmetric (or self-adjoint) operators but the ordinary commutator of two symmetric operators is not symmetric. The second requirement rules out the adjoint representation, and the irreducibility condition is forced by the requiremend that phase transitions be possible between two different states. See e. g. SNIATYCKI [71] or WOODHOUSE [79]. Also it is known that for $B = A$ the problem has no solution whence the requirement that only a sub Lie algebra of A be represented. The physical constant $\hbar$ is here absorbed in the Poisson structure. It has become common to refer to a procedure furnishing a representation that satisfies only (i) and (ii) above as *prequantization*.

Under suitable circumstances, over a smooth symplectic manifold, the *geometric quantization scheme* developed by KIRILLIOV [48], KOSTANT [51], SOURIAU [74], and I. SEGAL [67], furnishes a quantization; see [71] or [79] for complete accounts. We confine ourselves with the remark that geometric quantization proceeds in two steps. The first step, prequantization, yields a representation of the Lie algebra underlying the whole Poisson algebra which satisfies (i) and (ii) but such a representation is not irreducible; the second step involves a choice of *polarization* to force the irreducibility condition. In particular, a *Kähler polarization* leads to what is called *Kähler quantization*. The existence of a Kähler polarization entails that the underlying manifold carries an ordinary Kähler structure. In the singular case, the ordinary geometric quantization scheme is no longer available, though. In the rest of the paper we shall describe how, under certain favorable circumstances, the difficulties in the singular case can be overcome in the framework of Kähler quantization. An observation crucial in the singular case is that the notion of polarization can be given a meaning by means of appropriately defined Lie-Rinehart algebras. Before going into details, we will briefly explain one of the origins of singularities.

4. Symmetries

Recall that a *symplectic* manifold is a smooth manifold N together with a closed non-degenerate 2-form σ. Given a function f, the identity $\sigma(X_f, \cdot) = df$ then associates a uniquely determined vector field X_f to f, the *Hamiltonian vector field* of f and, given two functions f and h, their *Poisson bracket* $\{f,h\}$ is defined by $\{f,h\} = X_f h$. This yields a Poisson bracket $\{\cdot,\cdot\}$ on the algebra $C^\infty(N)$ of ordinary smooth functions on N, referred to as a *symplectic* Poisson bracket.

Given a Lie group G, a *hamiltonian* G-space is a smooth symplectic G-manifold (N,σ) together with a smooth G-equivariant map μ from N to the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G satisfying the formula

$$\sigma(X_N, \cdot) = X \circ d\mu \tag{4.1}$$

for every $X \in \mathfrak{g}$; here X_N denotes the vector field on N induced by $X \in \mathfrak{g}$ via the G-action, and X is viewed as a linear form on $\mathfrak{g}^*$. The map μ is called a *momentum mapping* (or *moment map*). We recall that (4.1) says that, given $X \in \mathfrak{g}$, the vector field X_N is the hamiltonian vector field for the smooth function $X \circ \mu$ on N. See e. g. [1] for details. Given a hamiltonian G-space (N,σ,μ), the space $N_{\mathrm{red}} = \mu^{-1}(0)/G$ is called the *reduced space*. When G is not compact, this space may have bad properties; for example, it is not even a Hausdorff space when there are non-closed G-orbits in N.

Let $C^\infty(N_{\mathrm{red}}) = (C^\infty(N))^G)/I^G$, where I^G refers to the ideal (in the algebra $(C^\infty(N))^G)$ of smooth G-invariant functions on N which vanish on the zero locus $\mu^{-1}(0)$; this is a smooth structure on the reduced space N_{red}. As observed by ARMS-CUSHMAN-GOTAY [3], the Noether Theorem implies that the symplectic Poisson structure on $C^\infty(N)$ descends to a Poisson structure $\{\ ,\ \}_{\mathrm{red}}$ on $C^\infty(N_{\mathrm{red}})$, and SJAMAAR-LERMAN [70] have shown that, when G is compact and when the momentum mapping is proper, the orbit type decomposition of N_{red} is a stratification in the sense of GORESKY-MACPHERSON [20]. The idea that the orbit type decomposition is a stratification (in a somewhat weaker sense) may be found already in [4]. For intelligibility, we recall some of the requisite technical details.

A decomposition of a space Y into pieces which are smooth manifolds such that these pieces fit together in a certain precise way is called a *stratification* [20]. More precisely: Let Y be a Hausdorff paracompact topological space and let $\mathcal{I}$ be a partially ordered set with order relation denoted by $\leq$. An *$\mathcal{I}$-decomposition* of Y is a locally finite collection of disjoint locally closed manifolds $S_i \subseteq Y$ called *pieces* (recall that a collection $\mathcal{A}$ of subsets of Y is said to be *locally finite* provided every $x \in Y$ has a neighborhood U_x in Y such that $U_x \cap A \neq \emptyset$ for at most finitely many A in $\mathcal{A}$) such that the following hold:

$Y = \cup S_i \ (i \in \mathcal{I})$,
$S_i \cap \overline{S}_j \neq \emptyset \Longleftrightarrow S_i \subseteq \overline{S}_j \Longleftrightarrow i \leq j \ (i, j \in \mathcal{I})$.
The space Y is then called a *decomposed* space. A decomposed space Y is said to be a *stratified space* if the pieces of Y, called *strata*, satisfy the following condition: Given a point x in a piece S there is an open neighborhood U of x in Y, an open ball B around x in S, a stratified space Λ, called the *link* of x, and a decomposition preserving homeomorphism from $B \times C^\circ(\Lambda)$ onto U. Here $C^\circ(\Lambda)$ refers to the open cone on Λ and, as a stratified space, Λ is less complicated than $C^\circ(\Lambda)$ whence the definition is not circular; the idea of complication is here made precise by means of the notion of *depth*.

A *stratified symplectic space* [70] is a stratified space Y together with a Poisson algebra $(C^\infty(Y), \{\ ,\ \})$ of continuous functions on Y which, on each piece of the decomposition, restricts to an ordinary smooth symplectic Poisson structure; in particular, $C^\infty(Y)$ is a smooth structure on Y.

Example 4.2. On the ordinary plane, with coordinates x_1, x_2, consider the algebra A of smooth functions in the coordinate functions x_1, x_2 together with, which is *crucial* here, an additional function r which is the radius function, subject to the relation $x_1^2 + x_2^2 = r^2$. Notice that r is *not* a smooth function in the usual sense whence the algebra A is strictly larger than that of ordinary smooth functions on the plane. The Poisson structure $\{\cdot,\cdot\}$ on A given by the formulas

$$\{x_1, x_2\} = 2r, \quad \{x_1, r\} = 2x_2, \quad \{x_2, r\} = -2x_1 \tag{4.2.1}$$

turns the plane into a stratified symplectic space. Geometrically, the plane is taken here as a half cone, the algebra A being that of *Whitney*-smooth functions on the half cone, with reference to the embedding into 3-space; there are two strata, the vertex of the half cone and the complement thereof. On the complement of the vertex, which is a punctured plane, the Poisson structure is symplectic. In physics, the Poisson algebra $(A, \{\cdot,\cdot\})$ arises, for $n \geq 2$, as the reduced Poisson algebra of a single particle in $\mathbb{R}^n$ with $\mathrm{O}(n, \mathbb{R})$-symmetry and angular momentum zero. For $n = 1$, the example still makes sense: the symmetry group is then just a copy of $\mathbb{Z}/2$, and the angular momentum is zero.

Given a hamiltonian G-space (N, σ, μ) with G compact, in view of an observation in [70], the Arms-Cushman-Gotay construction turns $(N_{\mathrm{red}}, C^\infty(N_{\mathrm{red}}), \{\ ,\ \}_{\mathrm{red}})$ (more precisely: each connected component of N_{red} in case the momentum mapping is not proper) into a stratified symplectic space. When N_{red} is smooth, i. e. has a single stratum, this space is just a smooth symplectic manifold, the ordinary MARSDEN-WEINSTEIN reduced space [58].

5. Stratified complex polarizations

Within the ordinary geometric quantization scheme, the irreducibility requirement is taken care of by means of a *polarization*. In particular, a *complex polarization* for an ordinary symplectic manifold N is an integrable Lagrangian distribution $F \subseteq \mathrm{T}^{\mathbb{C}}N$ of the complexified tangent bundle $\mathrm{T}^{\mathbb{C}}N$ [79]; under the identification of $\mathrm{T}^{\mathbb{C}}N$ with its (complex) dual coming from the symplectic structure, a complex polarization F then corresponds to a certain uniquely defined $(\mathbb{C}, C^\infty(N, \mathbb{C}))$-Lie subalgebra P of $\Omega^1(N, \mathbb{C})_{\{\cdot,\cdot\}}$.

Given a stratified symplectic space N, we refer to a $(\mathbb{C}, C^\infty(X, \mathbb{C}))$-Lie subalgebra P of $\Omega^1(X, \mathbb{C})_{\{\cdot,\cdot\}}$ as a *stratified complex polarization* for N if, for every stratum Y, under the identification of $\mathrm{T}^{\mathbb{C}}Y$ with its (complex) dual coming from the symplectic structure on that stratum, the $(\mathbb{C}, C^\infty(Y,\mathbb{C}))$-Lie subalgebra P_Y of $\Omega^1(Y,\mathbb{C})_{\{\cdot,\cdot\}}$ generated by the restriction of P to Y is identified with the space of sections of an ordinary complex polarization. A stratified complex polarization is, then, a *Kähler* polarization provided on any stratum it comes from an ordinary (not necessarily positive) Kähler polarization. We say that a stratified Kähler polarization is *complex analytic* provided it is induced from a complex analytic structure on N, and we define a complex analytic stratified Kähler structure to be a *normal Kähler structure* provided the complex analytic structure is normal. A normal Kähler structure is *positive* provided it is positive on each stratum. See Section 2 of [39] for more details.

Let G be a compact Lie group, denote its complex form by $G^{\mathbb{C}}$, and recall the following, cf. Proposition 4.2 of [39].

Proposition 5.1. *Given a positive Kähler manifold N with a holomorphic $G^{\mathbb{C}}$-action whose restriction to G preserves the Kähler structure and is hamiltonian, the Kähler polarization F induces a positive normal (complex analytic stratified) Kähler polarization P^{red} on the reduced space N^{red}, the latter being endowed with its stratified symplectic Poisson algebra $(C^\infty(N^{\mathrm{red}}), \{\cdot,\cdot\}^{\mathrm{red}})$.*

Under these circumstances, the underlying complex analytic structure of N^{red} is that of a geometric invariant theory quotient; the existence thereof may be found in [47] and [49]. The existence problem of this complex analytic structure may be seen as one of descent.

6. Examples

We will now illustrate the notions introduced so far by means of a number of examples. The interested reader will find more details in [39].

Example 6.1. For $\ell \geq 1$, consider the constrained system of ℓ particles in $\mathbb{R}^s$ with total angular momentum zero. Its unreduced phase space N is a product $(\mathrm{T}^*\mathbb{R}^s)^\ell$ of ℓ copies of $\mathrm{T}^*\mathbb{R}^s$, and we write the points of N in the form

$$(\mathbf{q},\mathbf{p}) = (\mathbf{q}_1,\mathbf{p}_1,\dots,\mathbf{q}_\ell,\mathbf{p}_\ell).$$

Let $H = \mathrm{O}(s,\mathbb{R})$. With reference to the obvious H-symmetry, the momentum mapping of this system has the form

$$\mu\colon N \longrightarrow \mathfrak{h}^*, \quad \mu(\mathbf{q}_1,\mathbf{p}_1,\dots,\mathbf{q}_\ell,\mathbf{p}_\ell) = \mathbf{q}_1 \wedge \mathbf{p}_1 + \dots + \mathbf{q}_\ell \wedge \mathbf{p}_\ell,$$

where the Lie algebra $\mathfrak{h} = \mathfrak{so}(s,\mathbb{R})$ is identified with its dual in the standard fashion. To elucidate the reduced space, observe that the assignment to $(\mathbf{q},\mathbf{p}) = (\mathbf{q}_1,\mathbf{p}_1,\dots,\mathbf{q}_\ell,\mathbf{p}_\ell)$ of the real symmetric $(2\ell \times 2\ell)$-matrix

$$\xi(\mathbf{q},\mathbf{p}) = \begin{bmatrix} \mathbf{q}_j\mathbf{q}_k & \mathbf{q}_j\mathbf{p}_k \\ \mathbf{p}_j\mathbf{q}_k & \mathbf{p}_j\mathbf{p}_k \end{bmatrix}_{1\leq j,k\leq \ell}$$

yields a real algebraic map $\xi\colon N \longrightarrow \mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^{2\ell}]$ from N to the real vector space $\mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^{2\ell}]$ of real symmetric $(2\ell \times 2\ell)$-matrices which passes to an embedding of the reduced space $N^{\mathrm{red}} = \mu^{-1}(0)/H$ into $\mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^{2\ell}]$, in fact, realizes N^{red} as a semi-algebraic set in $\mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^{2\ell}]$. Let J be the standard complex structure on $\mathbb{R}^{2\ell}$. Now, on the one hand, the association $S \mapsto JS$ identifies $\mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^{2\ell}]$ with $\mathfrak{sp}(\ell,\mathbb{R})$ in an $\mathrm{Sp}(\ell,\mathbb{R})$-equivariant fashion (with reference to the obvious actions) and hence identifies N^{red} with a subset of $\mathfrak{sp}(\ell,\mathbb{R})$ which, as observed in [54], is the closure of a certain nilpotent orbit which has been identified as a *holomorphic nilpotent orbit* in [39]. The Killing form transforms the Lie-Poisson structure on $\mathfrak{sp}(\ell,\mathbb{R})^*$ to a Poisson structure on $\mathfrak{sp}(\ell,\mathbb{R})$ which, restricted to N^{red}, yields a stratified symplectic structure. Another observation in [54] entails that this stratified symplectic structure coincides with the Sjamaar-Lerman stratified symplectic structure [70] mentioned earlier arising by symplectic reduction from $N = (\mathrm{T}^*\mathbb{R}^s)^\ell$. We mention in passing that, $\mathfrak{sp}(\ell,\mathbb{R})$ being identified with its dual by means of an appropriate positive multiple of the Killing form, as well as with $\mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^{2\ell}]$, the map ξ is essentially the momentum mapping for the obvious $\mathrm{Sp}(\ell,\mathbb{R})$-action on N.

On the other hand, the choice of J determines a maximal compact subalgebra of $\mathfrak{sp}(\ell,\mathbb{R})$ which is just a copy of $\mathfrak{u}(\ell)$ and, furthermore, a Cartan decomposition $\mathfrak{sp}(\ell,\mathbb{R}) = \mathfrak{u}(\ell)\oplus\mathfrak{p}$. Now matrix multiplication by J from the left induces a complex structure on $\mathfrak{p}$ and, with this structure, as a complex vector space, $\mathfrak{p}$ amounts to the complex symmetric square $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$ on $\mathbb{C}^\ell$. In particular, orthogonal projection to $\mathfrak{p}$ induces a linear surjection of real vector spaces from $\mathrm{S}^2_{\mathbb{R}}[\mathbb{R}^{2\ell}]$ to $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$, uniquely determined by J; it is given by the assignment to a real symmetric $(2\ell\times 2\ell)$-matrix of the corresponding complex symmetric $(\ell\times\ell)$-matrix with respect to the standard complex structure J on $\mathbb{R}^{2\ell}$. This projection, restricted to N^{red}, is injective and yields a complex analytic structure on N^{red}. The two structures are compatible and yield a normal (complex analytic stratified) Kähler structure on N^{red}; see [39] for details. We will describe the requisite (complex analytic) stratified Kähler polarization P shortly. For $\ell \geq s$, as a complex analytic space, N^{red} comes down to $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$ whereas, for $\ell < s$, as a complex analytic space, N^{red} may be described as a complex determinantal variety in $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$, that is, as an affine variety given by determinantal equations; see e. g. [8] for determinantal varieties. This may be deduced from standard geometric invariant theory results combined with the standard description of the invariants of the classical groups which, in turn, may be found e. g. in [78]. As a stratified symplectic space, the singularity structure of N^{red} is finer than that of the complex analytic structure, though: Once ℓ is fixed, for $s=\ell$, the smooth structure $C^\infty(N_\ell)$ and hence the Poisson structure on $N_\ell = N^{\mathrm{red}} \cong \mathbb{C}^d, d = \frac{\ell(\ell+1)}{2}$, is not standard and, as a stratified symplectic space, N_ℓ has $\ell+1$ strata. For $s<\ell$, the space $N^{\mathrm{red}} = N_s$ (say) may be described as the closure of a stratum in N_ℓ; moreover, a system of ℓ particles in $\mathbb{R}^{s-1}$ being viewed as a system of ℓ particles in $\mathbb{R}^s$ via the standard inclusion of $\mathbb{R}^{s-1}$ into $\mathbb{R}^s$ yields an injection of N_{s-1} into N_s. Thus we get a sequence

$$\{o\} = N_0 \subseteq N_1 \subseteq \ldots N_{s-1} \subseteq N_s \subseteq \cdots \subseteq N_\ell$$

of injections of normal (complex analytic stratified) Kähler spaces in such a way that, for $1 \leq s \leq \ell$, $N_{s-1} \subseteq N_s$ is the singular locus of N_s in the sense of stratified symplectic spaces, and the stratified Kähler structure on N_s, in particular the requisite Poisson structure, is then simply obtained by restriction from N_ℓ. For example, for $\ell = 1$, $(N_1, C^\infty(N_1), \{\cdot,\cdot\})$ is just the reduced space and reduced Poisson algebra of a system of a single particle in $\mathbb{R}^n$ $(n\geq 2)$ with angular momentum zero explained in the Example 4.2 above. For $\ell = s = 2$, the space $N_2 = N^{\mathrm{red}}$ is complex analytically a copy of $\mathbb{C}^3$ which, as a stratified symplectic space, sits inside $\mathfrak{sp}(2,\mathbb{R})$, and we need ten generators to describe the Poisson structure on N_2. The reduced space N_1 for $\ell = 2$ and $s = 1$ is here complex analytically realized inside $N_2 \cong \mathbb{C}^3$ as the quadric $Y^2 = XZ$.

To introduce coordinates, and to spell out a description of the complex analytic stratified Kähler polarizations, consider the complexification $\mathfrak{sp}(\ell,\mathbb{C})$ of $\mathfrak{sp}(\ell,\mathbb{R})$; this complexification sits inside the complex polynomial algebra

$$\mathbb{C}[z_1,\dots,z_\ell,\overline{z}_1,\dots,\overline{z}_\ell]$$

as its homogeneous quadratic constituent. The complexification $\mathfrak{k}^{\mathbb{C}} \cong \mathfrak{gl}(\ell,\mathbb{C})$ of the maximal compact subalgebra $\mathfrak{k} = \mathfrak{u}(\ell)$ of $\mathfrak{sp}(\ell,\mathbb{R})$ is the span of the $z_j\overline{z}_k$'s and, with reference to the Cartan decomposition $\mathfrak{sp}(\ell,\mathbb{R}) = \mathfrak{u}(\ell) \oplus \mathfrak{p}$ of $\mathfrak{sp}(\ell,\mathbb{R})$, the constituents $\mathfrak{p}^+$ and $\mathfrak{p}^-$ of the decomposition $\mathfrak{p}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ are the spans of the z_jz_k's and the $\overline{z}_j\overline{z}_k$'s, respectively; this gives an explicit description of $\mathfrak{p}^+$ and $\mathfrak{p}^-$ as $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$ and $\overline{\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]}$, respectively. Furthermore, $\mathfrak{k} = \mathfrak{u}(\ell)$ sits inside $\mathfrak{sp}(\ell,\mathbb{C})$ as the real span of the $z_j\overline{z}_k + \overline{z}_jz_k$'s and $i(z_j\overline{z}_k - \overline{z}_jz_k)$'s, and $\mathfrak{p}$ sits inside $\mathfrak{sp}(\ell,\mathbb{C})$ as the real span of the $z_jz_k + \overline{z}_j\overline{z}_k$'s and $i(z_jz_k - \overline{z}_j\overline{z}_k)$'s; the assignment to a real symmetric $(2\ell \times 2\ell)$-matrix of the corresponding complex symmetric $(\ell \times \ell)$-matrix is given by the association

$$z_jz_k + \overline{z}_j\overline{z}_k \longmapsto z_jz_k, \quad i(z_jz_k - \overline{z}_j\overline{z}_k) \longmapsto iz_jz_k.$$

The summands $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are the irreducible $\mathfrak{k}^{\mathbb{C}}$-representations in $\mathfrak{sp}(\ell,\mathbb{C})$ complementary to $\mathfrak{k}^{\mathbb{C}}$.

The homogeneous quadratic polynomials in the variables $z_1,\dots,z_\ell,\overline{z}_1,\dots,\overline{z}_\ell$ yield coordinates on $\mathfrak{sp}(\ell,\mathbb{R})$ and hence, via restriction, on N^{red}, that is, the smooth structure $C^\infty(N^{\mathrm{red}},\mathbb{C})$ may be described as the algebra of smooth functions in these variables, subject to the relations coming from the embedding of N^{red} into $\mathfrak{sp}(\ell,\mathbb{R})$. Now, the differentials $d(z_jz_k)$ of the coordinate functions z_jz_k $(1 \le j,k \le \ell)$ (that is, of those coordinate functions which do not involve any of the $\overline{z}_j$'s) generate the corresponding complex analytic stratified Kähler polarization $P \subseteq \Omega^1(N^{\mathrm{red}},\mathbb{C})$ as an $(\mathbb{C}, C^\infty(N^{\mathrm{red}},\mathbb{C}))$-Lie subalgebra of $\Omega^1(N^{\mathrm{red}},\mathbb{C})_{\{\cdot,\cdot\}}$.

In [39], we developed a theory of holomorphic nilpotent orbits of hermitian Lie algebras and established the fact that the (topological) closure of any holomorphic nilpotent orbit inherits a normal (complex analytic stratified) Kähler structure. The space N^{red}, realized as the closure of a holomorphic nilpotent orbit in $\mathfrak{sp}(\ell,\mathbb{R})$, is a special case thereof.

Example 6.2. A variant of the above example arises from the constrained system of ℓ harmonic oscillators in $\mathbb{R}^s$ with total angular momentum zero and constant energy. Its unreduced phase space Q is a copy of complex projective space $\mathbb{P}^{s\ell-1}\mathbb{C}$ of complex dimension $s\ell - 1$. For $\ell \ge s$, as a complex analytic space, Q^{red} coincides with the (complex) projectivization $\mathbb{P}\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$ of $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$ whereas for $\ell < s$, as a complex analytic space, Q^{red} may be described as a complex projective determinantal variety in $\mathbb{P}\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$. In fact, the determinantal equations mentioned in Example 6.1 above are homogeneous and yield the requisite homogeneous equations for the present case. In the same vein as before, we get a sequence

$$Q_1 \subseteq \dots Q_{s-1} \subseteq Q_s \subseteq \dots \subseteq Q_\ell \cong \mathbb{P}^d\mathbb{C},\ d = \frac{\ell(\ell+1)}{2} - 1,$$

of injections of compact normal (complex analytic stratified) Kähler spaces in such a way that, for $2 \le s \le \ell$, $Q_{s-1} \subseteq Q_s$ is the singular locus of Q_s in the sense of stratified symplectic spaces, each Q_s being the closure of a stratum in Q_ℓ, and the stratified Kähler structure on Q_s, in particular the requisite Poisson structure, is simply obtained by restriction from Q_ℓ. Complex analytically, each Q_s is a projective variety. Again, the smooth structure $C^\infty(Q_\ell)$ and hence the Poisson structure on $Q_\ell \cong \mathbb{P}^d\mathbb{C}$ $(s = \ell)$ is not the standard one (which arises from the Fubini-Study metric on complex projective space) and, as a stratified symplectic space, Q_ℓ has ℓ strata. For example, for $\ell = s = 2$, the space Q_2 is complex analytically a copy of $\mathbb{P}^2\mathbb{C}$, and the corresponding reduced space Q_1 (for $\ell = 2, s = 1$), which is abstractly just complex projective 1-space, sits complex analytically inside $Q_2 \cong \mathbb{P}^2\mathbb{C}$ as the projective conic $Y^2 = XZ$. These spaces are particular cases of a systematic class of examples of *exotic projective varieties*, introduced and explored in our paper [39].

Remark 6.3. Given a Lie group G, a smooth hamiltonian G-space, and a real G-invariant polarization, the question arises whether the statement of Proposition 5.1 still holds for this real polarization. When we try to identify, on the reduced level, a stratified version of such a polarization, we may run into the following difficulty, though: Under the circumstances of the Example 6.1, let $\ell = 1$, and consider the vertical polarization on $N = \mathrm{T}^*\mathbb{R}^n$. This polarization integrates to the foliation—even fibration—defined by the projection map from $\mathrm{T}^*\mathbb{R}^n$ to $\mathbb{R}^n$. This foliation is clearly $\mathrm{O}(n,\mathbb{R})$-invariant and, in terms of the standard coordinates $\mathbf{q} = (q^1, \dots, q^n)$ on $\mathbb{R}^n$, a leaf is given by the equation $\mathbf{q} = \mathbf{q}_0$ where $\mathbf{q}_0$ is a constant. We will now write the ordinary scalar product of two vectors $\mathbf{x}$ and $\mathbf{y}$ as $\mathbf{xy}$. With these preparations out of the way, under the present circumstances, the assignment to $(\mathbf{q},\mathbf{p}) \in \mathrm{T}^*\mathbb{R}^n$ of $x_1 = \mathbf{qq} - \mathbf{pp}$ and $x_2 = 2\mathbf{qp}$ yields a map from $\mathrm{T}^*\mathbb{R}^n$ to the plane $\mathbb{R}^2$ which induces an isomorphism of stratified symplectic spaces from the reduced space N^{red} onto the exotic plane described in the Example 4.2. In particular, the radius function r is given by $r = \mathbf{qq} + \mathbf{pp}$. Now $2\mathbf{qq} = x_1 + r$ whence, under reduction, the leaf $\mathbf{q} = \mathbf{q}_0$ passes to the subspace of the plane given by the equation

$$x_1 + r = 2\mathbf{q}_0\mathbf{q}_0 = c \text{ (say)}.$$

For $\mathbf{q}_0 \neq 0$, in the (x_1, x_2)-plane, this is just the parabola $x_2^2 + 2cx_1 = c^2$ since $r^2 = x_1^2 + x_2^2$ while, for $\mathbf{q}_0 = 0$, it is the non-positive half x_1-axis. The reason for this degeneracy is that the leaf $\mathbf{q} = 0$ is not transverse to the momentum mapping μ in the sense that, whatever $\mathbf{p} \in \mathbb{R}^n$, $\mu(0,\mathbf{p}) = 0$ while $\ker(d\mu(0,\mathbf{p}))$ and the tangent space of the leaf at $(0,\mathbf{p})$ do *not* together span the tangent space of N at $(0,\mathbf{p})$. Thus the reduced space is still foliated, but one leaf is singular; however even the restriction of this foliation to the top stratum still has a singular leaf, the negative half x_1-axis. A little thought reveals that this implies that this foliation cannot result from a stratified real polarization, the notion of stratified real polarization being defined in the same fashion as a stratified complex polarization, except that, on each stratum, the polarization should come down to a real polarization. As a side remark we mention that the assignment to a leaf of its intersection point with the non-negative x_1-axis identifies the space of leaves with the non-negative x_1-axis, and the latter in fact coincides with the orbit space $\mathbb{R}^n/\mathrm{O}(n,\mathbb{R})$. This description visualizes the exceptional role played by the non-positive x_1-axis. The distribution parallel to this foliation, though, is given by the hamiltonian vector field of the function $\mathbf{qq}$ in $C^\infty(N^{\mathrm{red}})$; it has the form

$$\{\mathbf{qq}, -\} = \frac{1}{2}\{x_1 + r, -\} = -x_2\frac{\partial}{\partial x_1} + (x_1 + r)\frac{\partial}{\partial x_2}$$

and in particular vanishes on the non-positive half x_1-axis. The function $\mathbf{qq}$ generates a maximal abelian Poisson subalgebra of $\left(C^\infty(N^{\mathrm{red}}), \{\cdot,\cdot\}^{\mathrm{red}}\right)$. This phenomenon is typical for cotangent bundles with a hamiltonian action of a Lie group arising from an action of that group on the base with more than a single orbit type. *Thus we see that the question whether a polarization other than a Kähler polarization descends to a stratified polarization on the reduced level leads to certain delicacies, and we do not know to what extent we can interpret it merely as a descent problem.*

The question whether, under suitable circumstances so that in particular the reduced space is still a smooth manifold, a real polarization descends has been studied in [21].

7. Prequantization on spaces with singularities

To develop prequantization over stratified symplectic spaces and to describe the behaviour of prequantization under reduction, in our paper [41], we introduced *stratified prequantum modules* over stratified symplectic spaces. A stratified prequantum module is defined in terms of the appropriate Lie-Rinehart algebra and determines what we call a *costratified prequantum space* but the two notions, though closely related, should not be confused.

Let N be a stratified symplectic space, and let $(A, \{\cdot,\cdot\})$ be its stratified symplectic Poisson algebra; a special case would be the ordinary symplectic Poisson algebra of a smooth symplectic manifold. Consider the extension (2.4) of Lie-Rinehart algebras. Given an $(A \otimes \mathbb{C})$-module M, we refer to an $(A, \overline{L}_{\{\cdot,\cdot\}})$-module structure $\chi\colon \overline{L}_{\{\cdot,\cdot\}} \longrightarrow \mathrm{End}_{\mathbb{R}}(M)$ on M as a *prequantum module structure for* $(A, \{\cdot,\cdot\})$ provided (i) the values of χ lie in $\mathrm{End}_{\mathbb{C}}(M)$, that is to say, the operators $\chi(a, \alpha)$ are complex linear transformations, and (ii) for every $a \in A$, $\chi(a, 0) = i\, a\, \mathrm{Id}_M$ [30, 41].

We recall from [29] that the assignment to $a \in A$ of $(a, da) \in \overline{L}_{\{\cdot,\cdot\}}$ yields a morphism ι of real Lie algebras from A to $\overline{L}_{\{\cdot,\cdot\}}$; this reduces the construction of Lie algebra representations of the Lie algebra which underlies the Poisson algebra A to the construction of representations of $\overline{L}_{\{\cdot,\cdot\}}$. Thus, for any prequantum module (M, χ), the composite of ι with $-i\chi$ is a representation $a \mapsto \widehat{a}$ of the A underlying real Lie algebra on M, viewed as a complex vector space, by $\mathbb{C}$-linear operators so that the constants in A act by multiplication and so that the Dirac condition holds, even though M does not necessarily carry a Hilbert space structure. These operators are given by the formula

$$\widehat{a}(x) = \frac{1}{i}\chi(0, da)(x) + ax, \quad a \in A,\ x \in M. \tag{7.1}$$

For illustration, consider an ordinary quantizable symplectic manifold (N, σ), with ordinary *prequantum bundle* $\zeta\colon \Lambda \to N$, that is, ζ is a complex line bundle with a connection ∇ whose curvature equals $-i\sigma$; the assignments $\chi_\nabla(a, 0) = i\, a\, \mathrm{Id}_M$ ($a \in A$) and $\chi_\nabla(0, \alpha) = \nabla_{\pi^\sharp(\alpha)}$ ($\alpha \in \Omega^1(N)_{\{\cdot,\cdot\}}$) then yield a prequantum module structure

$$\chi_\nabla\colon \overline{L}_{\{\cdot,\cdot\}} \to \mathrm{End}_{\mathbb{C}}(M) \subseteq \mathrm{End}_{\mathbb{R}}(M)$$

for $(A, \{\cdot,\cdot\})$. (Here $\pi^\sharp\colon \Omega^1(N) \to \mathrm{Vect}(N)$ refers to the adjoint of the 2-form π induced by the symplectic Poisson structure, cf. Section 2.) This is just the ordinary prequantization construction in another guise.

As before, consider a general stratified symplectic space N, with stratified symplectic Poisson algebra $(C^\infty(N), \{\cdot,\cdot\})$. For each stratum Y, let $(C^\infty(Y), \{\cdot,\cdot\}^Y)$ be its symplectic Poisson structure, and let

$$0 \to C^\infty(Y) \to \overline{L}_{\{\cdot,\cdot\}^Y} \to \Omega^1(Y)_{\{\cdot,\cdot\}^Y} \to 0$$

be the corresponding extension (2.4) of Lie-Rinehart algebras. As in (1.5) of [39], we define a *stratified prequantum module* for N to consist of
— a prequantum module (M, χ) for $(C^\infty(N), \{\cdot,\cdot\})$, together with,
— for each stratum Y, a prequantum module structure χ_Y for $(C^\infty(Y), \{\cdot,\cdot\}^Y)$ on $M_Y = C^\infty(Y) \otimes_{C^\infty(N)} M$ in such a way that the canonical linear map of complex vector spaces from M to M_Y is a morphism of prequantum modules from (M, χ) to (M_Y, χ_Y).

Given a stratified prequantum module (M, χ) for N, when Y runs through the strata of N, we refer to the system

$$\left(M_{\overline{Y}}, \chi_{\overline{Y}}\colon \overline{L}_{\{\cdot,\cdot\}^{\overline{Y}}} \to \mathrm{End}_{\mathbb{R}}(M_{\overline{Y}})\right)$$

of prequantum modules, together with, for every pair of strata Y, Y' such that $Y' \subseteq \overline{Y}$, the induced morphism

$$(M_{\overline{Y}}, \chi_{\overline{Y}}) \to (M_{\overline{Y}'}, \chi_{\overline{Y}'})$$

of prequantum modules, as a *costratified prequantum space*. More formally: Consider the category $\mathcal{C}_N$ whose objects are the strata of N and whose morphisms are the inclusions $Y' \subseteq \overline{Y}$. We define a *costratified complex vector space* on N to be a contravariant functor from $\mathcal{C}_N$ to the category of complex vector spaces, and a *costratified prequantum space* on N to be a costratified complex vector space together with a compatible system of prequantum module structures. Thus a stratified prequantum module (M, χ) for $(N, C^\infty(N), \{\cdot,\cdot\})$ determines a costratified prequantum space on N; see the (1.4) and (1.5) of [41] for details.

Theorem 7.2. *Given a symplectic manifold N with a hamiltonian action of a compact Lie group G, a G-equivariant prequantum bundle ζ descends to a stratified prequantum module $(\chi^{\mathrm{red}}, M^{\mathrm{red}})$ for the stratified symplectic space*

$$(N^{\mathrm{red}}, C^\infty(N^{\mathrm{red}}), \{\cdot,\cdot\}^{\mathrm{red}}).$$

Proof. See Theorem 2.1 of [41]. □

Thus, phrased in the language of prequantum modules, the relationship between the unreduced and reduced prequantum object may be interpreted as one of descent.

In particular, consider a complex analytic stratified Kähler space $(N, C^\infty(N), \{\cdot,\cdot\}, P)$ (cf. Section 5 above or Section 2 of [39]), and let (M, χ) be a stratified prequantum module for $(C^\infty(N), \{\cdot,\cdot\})$. We refer to (M, χ) as a *complex analytic* stratified prequantum module provided M is the space of ($C^\infty(N)$-) sections of a complex V-line bundle ζ on N in such a way that P endows ζ via χ with a complex analytic structure. If this happens to be the case, M^P necessarily amounts to the space of global sections of the sheaf of germs of holomorphic sections of ζ. See Section 3 of [39].

8. Kähler quantization and reduction

Let G be a compact Lie group, let (N, σ, μ) be a hamiltonian G-space of the kind as that in the circumstances of Proposition 5.1, and suppose that N is quantizable. Thus N is, in particular, a positive Kähler manifold with a holomorphic $G^{\mathbb{C}}$-action whose restriction to G preserves the Kähler structure and is hamiltonian. Write P for the corresponding Kähler polarization, necessarily G-invariant, viewed as a $(\mathbb{C}, C^\infty(N, \mathbb{C}))$-Lie subalgebra of the $(\mathbb{C}, C^\infty(N, \mathbb{C}))$-Lie algebra $\Omega^1(N, \mathbb{C})_{\{\cdot,\cdot\}}$, and let ζ be a prequantum bundle. Via its connection, it acquires a holomorphic structure, and the connection is the unique hermitian connection for a corresponding hermitian structure. The momentum mapping induces, in particular, an infinitesimal action of the Lie algebra $\mathfrak{g}$ of G on ζ preserving the connection and hermitian structure. Suppose that this action lifts to a G-action on ζ preserving the connection and lifting the G-action on N. For connected G, the assumption that the G-action lift to one on ζ is (well known to be) redundant (since the infinitesimal action is essentially given by the momentum mapping) and it will suffice to replace G by an appropriate covering group if need be. Prequantization turns the space of smooth sections of ζ into a prequantum module for the ordinary smooth symplectic Poisson algebra of N. We write this prequantum module as M; it inherits a G-action preserving the polarization P. Hence the quantum module M^P, that is, the space $\Gamma(\zeta)$ of global holomorphic sections of ζ, is a complex representation space for G. This quantum module is the corresponding *unreduced* quantum state space, except that there is no Hilbert space structure present yet, and *reduction after quantization*, for the *quantum state spaces*, amounts to taking the space $(M^P)^G$ of G-invariant holomorphic sections.

The projection map from the space of smooth G-invariant sections of ζ to M^{red} restricts to a linear map

$$\rho\colon \Gamma(\zeta)^G \longrightarrow (M^{\mathrm{red}})^{P^{\mathrm{red}}} \tag{8.1}$$

of complex vector spaces, defined on the space $(M^P)^G = \Gamma(\zeta)^G$ of G-invariant holomorphic sections of ζ. Here and below P^{red} refers to the stratified Kähler polarization the existence of which is asserted in Proposition 5.1 above, M^{red} to the prequantum module for the stratified Kähler space mentioned in Theorem 7.2 (without having been made explicit there), and $(M^{\mathrm{red}})^{P^{\mathrm{red}}}$ to the P^{red}-invariants; notice that P^{red} is, in particular, a Lie algebra whence it makes sense to talk about P^{red}-invariants. A module of the kind $(M^{\mathrm{red}})^{P^{\mathrm{red}}}$ is referred to as a *reduced quantum module* in [41]. It acquires a costratified Hilbert space structure, the requisite scalar products being induced from appropriate hermitian structures via integration.

As far as the comparison of G-invariant unreduced quantum observables and reduced quantum observables is concerned, the statement that *Kähler quantization commutes with reduction* amounts to the following, cf. Theorem 3.6 in [41].

Theorem 8.2. *The data (N, σ, μ, M, P) being fixed so that, in particular, (N, σ, μ) is a smooth hamiltonian G-space structure on a quantizable positive Kähler manifold N with a holomorphic $G^{\mathbb{C}}$-action, the Kähler polarization being written as P, let f be a smooth G-invariant function on N which is quantizable in the sense that it preserves P. Then its class*

$$[f] \in C^\infty(N^{\mathrm{red}}) (= (C^\infty(N))^G / I^G)$$

is quantizable, i. e. preserves P^{red} and, for every $h \in (M^P)^G$,

$$\rho(\widehat{f}(h)) = \widehat{[f]}(\rho(h)). \tag{8.2.1}$$

So far, we did not make any claim to the effect that the reduced quantum module $(M^{\mathrm{red}})^{P^{\mathrm{red}}}$ amounts to a space of global holomorphic sections. We now recall that, under the circumstances of Theorem 8.2, the momentum mapping is said to be *admissible* provided, for every $m \in N$, the path of steepest descent through m is contained in a compact set [68], [49] (§9). For example, when the momentum mapping is proper it is admissible. Likewise, the momentum mapping for a unitary representation of a compact Lie group is admissible in this sense, see Example 2.1 in [68].

The statement "*Kähler quantization commutes with reduction*" is then completed by the following two observations, cf. [41] ((3.7) and (3.8)).

Proposition 8.3 *Under the circumstances of Theorem* 8.2, *when μ is admissible and when N^{red} has a top stratum (i. e. an open dense stratum), for example when μ is proper, the reduced stratified prequantum module $(M^{\mathrm{red}}, \chi^{\mathrm{red}})$ is complex analytic, that is, as a complex vector space, M^{red} amounts to the space of global holomorphic sections of a suitable holomorphic V-line bundle on N^{red}.*

The relevant V-line bundle on N^{red} may be found in [68] (Proposition 2.11).

Theorem 8.4 [68] (Theorem 2.15) *Under the circumstances of Theorem* 8.2, *when the momentum mapping μ is proper, in particular, when N is compact, the map ρ is an isomorphism of complex vector spaces.*

In this result, the properness condition, while sufficient, is not necessary, that is, the map ρ may be an isomorphism without the momentum mapping being proper.

A version of Theorem 8.4 has been established in (4.15) of [60]; cf. also [69] and the literature there, as well as [65] and [77] for generalizations to higher dimensional sheaf cohomology.

Remark 8.5. The statements of Theorems 8.2 and 8.4 are logically independent; in particular the statement of Theorem 8.2 makes sense whether or not ρ is an isomorphism, and its proof does not rely on ρ being an isomorphism.

Thus we have consistent Kähler quantizations on the unreduced and reduced spaces, including a satisfactory treatment of observables, as indicated by the formula (8.2.1). We have already pointed out in the introduction that examples in finite dimensions abound. We hope that this kind of approach, suitably adapted, will eventually yield the quantization of certain infinite dimensional systems arising from field theory.

Remark 8.6. KEMPF'S descent lemma [46] mentioned earlier characterizes, among the holomorphic V-line bundles which arise on a geometric invariant theory quotient by the standard geometric invariant theory construction, those which are ordinary (holomorphic) line bundles. In the circumstances of Theorem 8.4, complex analytically, the space N^{red} is a geometric invariant theory quotient, and the V-line bundle which underlies the reduced quantum module arises by the standard geometric invariant theory construction. Here the term "descent" is used in its strict sense.

Illustration 8.7. Under the circumstances of the Example 6.2, let $\mathcal{O}(1)$ be the ordinary hyperplane bundle on $Q = \mathbb{P}^{s\ell-1}\mathbb{C}$ and, as usual, for $k \geq 1$, write its k'th power as $\mathcal{O}(k)$. The unitary group $\mathrm{U}(s\ell)$ acts on $\mathbb{P}^{s\ell-1}\mathbb{C}$ in a hamiltonian fashion having as momentum mapping the familiar embedding of $\mathbb{P}^{s\ell-1}\mathbb{C}$ into $\mathfrak{u}(s\ell)^*$, and the adjoint thereof yields a morphism of Lie algebras from $\mathfrak{u}(s\ell)$ to $C^\infty(\mathbb{P}^{s\ell-1}\mathbb{C})$, the latter being endowed with its symplectic Poisson structure coming from the Fubini-Study metric. It is a standard fact that, for $k \geq 1$, Kähler quantization, with reference to $k\omega$ and $\mathcal{O}(k)$ (where ω is the Fubini-Study form), yields the k'th symmetric power of the standard representation defining the Lie algebra $\mathfrak{u}(s\ell)$, and this representation integrates to the k'th symmetric power E_s^k of the standard representation E_s defining the group $\mathrm{U}(s\ell)$. (We use the subscript $-_s$ since here and below ℓ is fixed while s varies.) The symmetry group $H = \mathrm{O}(s, \mathbb{R})$ of the constrained system in (6.1) above appears as a subgroup of $\mathrm{U}(s\ell)$ in an obvious fashion and, viewed as this subgroup, H centralizes the subgroup $\mathrm{U}(\ell) = \mathrm{Sp}(\ell, \mathbb{R}) \cap \mathrm{U}(s\ell)$ (the maximal compact subgroup $\mathrm{U}(\ell)$ of $\mathrm{Sp}(\ell, \mathbb{R})$); hence, for $k \geq 1$, the subspace $(E_s^k)^H$ of H-invariants is a $\mathrm{U}(\ell)$-representation. On the other hand, with an abuse of notation, let $\mathcal{O}(1)$ be the hyperplane bundle on the reduced space $Q_\ell = \mathbb{P}^d\mathbb{C}$, $d = \frac{\ell(\ell+1)}{2} - 1$ and, for $k \geq 1$, let $\mathcal{O}(k)$ be its k'th power. The space of holomorphic sections thereof, $\Gamma(\mathcal{O}(k))$, amounts to the k'th symmetric power $\mathrm{S}^k_{\mathbb{C}}[\mathfrak{p}^*]$ of the dual of $\mathfrak{p} = \mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$ (the space of homogeneous degree k polynomial functions on $\mathfrak{p}$). For $1 \leq s \leq \ell$ and $k \geq 1$, maintain the notation $\mathcal{O}(k)$ for the restriction of the k'th power of the hyperplane bundle to $Q_s \subseteq Q_\ell$; for $s < \ell$, the space of holomorphic sections $\widetilde{E}_s^k$ of $\mathcal{O}(k)$ is now a certain quotient of $\widetilde{E}_\ell^k = \mathrm{S}^k_{\mathbb{C}}[\mathfrak{p}^*]$ which will be made precise below.

For $1 \leq s \leq \ell$, the composite of the embedding of N_s into $\mathfrak{sp}(\ell, \mathbb{R})^*$ with the surjection from $\mathfrak{sp}(\ell, \mathbb{R})^*$ to $\mathfrak{u}(\ell)^*$ induced from the injection of $\mathfrak{u}(\ell)$ into $\mathfrak{sp}(\ell, \mathbb{R})$ yields a map from N_s to $\mathfrak{u}(\ell)^*$ which descends to a map from Q_s to $\mathfrak{u}(\ell)^*$, the adjoint of which induces a morphism of Lie algebras from $\mathfrak{u}(\ell)$ to $C^\infty(Q_s)$, the latter being endowed with its *stratified symplectic Poisson structure* explained earlier. For $k \geq 1$, the space of sections M^{red} (cf. Theorem 7.2) of $\mathcal{O}(k)$, with reference to a $C^\infty(Q_s)$-module structure constructed in [41] and not made precise here, inherits a stratified prequantum module structure; and stratified Kähler quantization yields a $\mathrm{U}(\ell)$-representation on the space $\widetilde{E}_s^k$, which amounts to that written earlier as $(M^{\mathrm{red}})^{P^{\mathrm{red}}}$, cf. (8.1), in such a way that the map ρ given as (8.1) above identifies the representation written above as $(E_s^{2k})^H$ with $\widetilde{E}_s^k$; moreover, the spaces $(E_s^{2k-1})^H$ are zero.

We conclude with an explicit description of the spaces $(E_s^{2k})^H$ or, equivalently, of the spaces $\widetilde{E}_s^k$: Introduce coordinates $x_1, \ldots, x_\ell$ on $\mathbb{C}^\ell$. These give rise to coordinates $\{x_{i,j} = x_{j,i};\ 1 \leq i, j \leq \ell\}$ on $\mathfrak{p} = S^2_{\mathbb{C}}[\mathbb{C}^\ell]$, and the determinants

$$\delta_1 = x_{1,1},\ \delta_2 = \begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{1,2} & x_{2,2} \end{vmatrix},\ \delta_3 = \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{1,2} & x_{2,2} & x_{2,3} \\ x_{1,3} & x_{2,3} & x_{3,3} \end{vmatrix},\ \text{etc.}$$

are highest weight vectors for certain $\mathrm{U}(\ell)$-representations. For $1 \leq s \leq r$ and $k \geq 1$, the $\mathrm{U}(\ell)$-representation $\widetilde{E}_s^k$ is the sum of the irreducible representations having as highest weight vectors the monomials

$$\delta_1^\alpha \delta_2^\beta \ldots \delta_s^\gamma, \quad \alpha + 2\beta + \cdots + s\gamma = k,$$

and the morphism from $\widetilde{E}^k_s$ to $\widetilde{E}^k_{s-1}$ coming from restriction from Q_s to Q_{s-1} is an isomorphism on the span of those irreducible representations which do not involve δ_s and has the span of the remaining ones as its kernel. In particular, this explains how $\widetilde{E}^k_s$ arises from $\widetilde{E}^k_\ell = \mathrm{S}^k_{\mathbb{C}}[\mathfrak{p}^*]$. For $1 \leq s \leq \ell$, the system $(\widetilde{E}^k_1, \widetilde{E}^k_2, \ldots, \widetilde{E}^k_s)$ is an example of a costratified quantum space.

The alerted reader is invited to consult [41] for more details.

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A Note on the Semiabelian Variety of Heyting Semilattices

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A *Heyting semilattice* is a partially ordered set which is cartesian closed when regarded as a category: equivalently, the variety **HSLat** of Heyting semilattices corresponds to the algebraic theory generated by a constant $\top$ and two binary operations $\wedge, \Rightarrow$ satisfying the equations

$$\begin{gathered}\top \wedge x = x\ ,\quad x \wedge x = x\ ,\quad x \wedge y = y \wedge x\ ,\\ x \wedge (y \wedge z) = (x \wedge y) \wedge z\ ,\quad (x \Rightarrow x) = \top\ ,\\ x \wedge (x \Rightarrow y) = x \wedge y\ ,\quad y \wedge (x \Rightarrow y) = y\ ,\\ \text{and } x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)\ .\end{gathered}$$

In this note we shall also wish to consider the variety **HAlg** of *Heyting algebras*, obtained by adding a further constant $\perp$ and a binary operation $\vee$ satisfying

$$\begin{gathered}\perp \vee x = x\ ,\quad x \vee x = x\ ,\quad x \vee y = y \vee x\ ,\\ x \vee (y \vee z) = (x \vee y) \vee z\ ,\quad x \wedge (x \vee y) = x\ ,\\ \text{and } x \vee (x \wedge y) = x\ ,\end{gathered}$$

and the variety **BAlg** of Boolean algebras, obtained from **HAlg** by adding the further equation $x \vee (x \Rightarrow \perp) = \top$. (As usual, we shall normally abbreviate $(x \Rightarrow \perp)$ to $\neg x$, and we shall write $(x \Leftrightarrow y)$ for $((x \Rightarrow y) \wedge (y \Rightarrow x))$.)

The category of Heyting algebras is protomodular, by an argument due to the present author (but first published in [1]); the argument also works for Heyting semilattices, and since the latter variety is pointed (equivalently, the theory of Heyting semilattices has a unique constant) it is a semiabelian category. In [2], Bourn and Janelidze showed that a pointed variety is semiabelian iff its theory contains, for some positive integer n, a family of n binary operations α_i $(1 \leq i \leq n)$ and an $(n+1)$-ary operation β satisfying the equations $\alpha_i(x,x) = \top$ for all i (where $\top$ denotes the unique constant of the theory) and

$$\beta(\alpha_i(x,y), \alpha_2(x,y), \ldots, \alpha_n(x,y), y) = x\ .$$

(We shall refer to these equations as condition (BJ), although in fact they were first considered by Ursini [5].)

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In his talk at the Fields Institute workshop, Francis Borceux noted that most familiar examples of semiabelian varieties satisfy condition (BJ) with $n = 1$, and asked whether there was a 'naturally occurring' variety where $n > 1$ was required. In this note we shall show that **HSLat** is such a variety: we shall exhibit a set of Heyting semilattice operations satisfying condition (BJ) for $n = 2$, but show that the condition cannot be satisfied for $n = 1$ even in the variety of Heyting algebras.

For the positive result, we set $\alpha_1(x, y) = (x \Rightarrow y)$, $\alpha_2(x, y) = (((x \Rightarrow y) \Rightarrow y) \Rightarrow x)$ and $\beta(u, v, w) = (u \Rightarrow w) \wedge v$. It is easily seen that we have $\alpha_1(x, x) = \top$ and

$$\alpha_2(x, x) = ((\top \Rightarrow x) \Rightarrow x) = (x \Rightarrow x) = \top \ ;$$

and

$$\beta(\alpha_1(x, y), \alpha_2(x, y), y) = ((x \Rightarrow y) \Rightarrow y) \wedge (((x \Rightarrow y) \Rightarrow y) \Rightarrow x) \leq x \ ,$$

but we also have $x \leq ((x \Rightarrow y) \Rightarrow y)$ since $x \wedge (x \Rightarrow y) \leq y$, and $x \leq (((x \Rightarrow y) \Rightarrow y) \Rightarrow x)$ since $x \wedge ((x \Rightarrow y) \Rightarrow y) \leq x$, so $\beta(\alpha_1(x, y), \alpha_2(x, y), y) = x$, as required.

For the negative result, note first that if we have binary operations α and β in any algebraic theory satisfying condition (BJ) for $n = 1$, then the implication '$x = y$ implies $\alpha(x, y) = \top$' must be reversible; for we must have $\beta(\top, y) = y$ for all y. Hence, if we have such operations in the theory of Heyting algebras, then $\alpha(x, y)$ must coincide in any Boolean algebra with the operation $(x \Leftrightarrow y)$, since bi-implication is the only Boolean binary operation with this property. In particular, the unary operation $\alpha(x, \bot)$ must coincide in any Boolean algebra with $(x \Leftrightarrow \bot) = \neg x$. However, we have an explicit description of the unary operations in the theory of Heyting algebras (equivalently, of the elements of the free Heyting algebra on one generator); they are pictured on p. 35 of [3], for example. From this description, it is easy to see that $\neg x$ is the unique unary operation in Heyting algebras which reduces to $\neg x$ in Boolean algebras; so we must actually have $\alpha(x, \bot) = \neg x$ in all Heyting algebras. But then, for any x, we have

$$x = \beta(\alpha(x, \bot), \bot) = \beta(\alpha(\neg\neg x, \bot), \bot) = \neg\neg x \ ;$$

so we have derived a contradiction.

In conclusion, we remark that if an algebraic theory contains a constant $\top$ and a Mal'cev operation μ which is (left) weakly associative in the sense of [4], then it satisfies condition (BJ) with $n = 1$: we simply set $\alpha(x, y) = \mu(x, y, \top)$ and $\beta(x, y) = \mu(x, \top, y)$. In [4], we showed that the theory of Heyting algebras does not contain a weakly associative Mal'cev operation by exhibiting a topological Heyting algebra whose underlying space is not homogeneous. The foregoing argument provides another proof of this fact.

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Monoidal Functors Generated by Adjunctions, with Applications to Transport of Structure

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Abstract. Bénabou pointed out in 1963 that a pair $f \dashv u : A \to B$ of adjoint functors induces a monoidal functor $[f,u] : [A,A] \to [B,B]$ between the (strict) monoidal categories of endofunctors. We show that this result about adjunctions in the monoidal 2-category **Cat** extends to adjunctions in any right-closed monoidal 2-category $\mathcal{V}$, or more generally in any 2-category $\mathcal{A}$ with an action $*$ of a monoidal 2-category $\mathcal{V}$ admitting an adjunction $\mathcal{A}(T * A, B) \cong \mathcal{V}(T, \langle A, B\rangle)$; certainly such an adjunction exists when $*$ is the canonical action of $[\mathcal{A},\mathcal{A}]$ on $\mathcal{A}$, provided that $\mathcal{A}$ is complete and locally small. This result allows a concise and general treatment of the transport of algebraic structure along an equivalence.

1 Introduction

We suppose given a *monoidal 2-category* $\mathcal{V}$: that is, a 2-category $\mathcal{V}$ along with a monoidal structure $(\otimes, I, a, l, r)$ for which $\otimes$ is a 2-functor and a, l, r are 2-natural. We further suppose given a 2-category $\mathcal{A}$ and an *action* of $\mathcal{V}$ on $\mathcal{A}$: that is, a 2-functor $* : \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ together with 2-natural isomorphisms $\alpha : (X \otimes Y) * A \cong X * (Y * A)$ and $\lambda : I * A \cong A$ satisfying the usual two coherence axioms. Finally

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we suppose each 2-functor $- * A : \mathcal{V} \to \mathcal{A}$ to have a right adjoint $\langle\!\langle A, - \rangle\!\rangle$, so that we have a 2-natural isomorphism

$$\Phi : \mathcal{A}(X * A, B) \cong \mathcal{V}(X, \langle\!\langle A, B \rangle\!\rangle). \tag{1.1}$$

A first example is that where $\mathcal{A}$ is $\mathcal{V}$ itself, with $\otimes$ for $*$ and a, l for α, λ; then $\langle\!\langle A, B \rangle\!\rangle$ is an "internal hom", more commonly denoted by $[A, B]$, whose existence makes of $\mathcal{V}$ a *right-closed* monoidal 2-category. A second example is that where $\mathcal{A}$ is any 2-category which is locally small and complete, while $\mathcal{V}$ is the monoidal 2-category $[\mathcal{A}, \mathcal{A}]$ of endofunctors of $\mathcal{A}$ (meaning of course *endo-2-functors*, since $\mathcal{A}$ is a 2-category), with composition for its tensor product and the identity functor $1 = 1_{\mathcal{A}}$ for its unit object. The action $[\mathcal{A}, \mathcal{A}] \times \mathcal{A} \to \mathcal{A}$ we intend here is that given by evaluation, sending (T, A) to TA and similarly defined on morphisms. Now (1.1) takes the form

$$\Phi : \mathcal{A}(XA, B) \cong [\mathcal{A}, \mathcal{A}](X, \langle A, B \rangle), \tag{1.2}$$

where $\langle A, B \rangle$ is the right Kan extension of $B : 1 \to \mathcal{A}$ along $A : 1 \to \mathcal{A}$ given by $\langle A, B \rangle C = B^{\mathcal{A}(C,A)}$.

There is a sense in which the second example is "extremal". For in the context of a general example as in (1.1), we can still apply (1.2) (provided $\mathcal{A}$ is locally small and complete) to get

$$\mathcal{A}(X * A, B) \cong [\mathcal{A}, \mathcal{A}](X * -, \langle A, B \rangle),$$

so that we have a natural isomorphism

$$\mathcal{V}(X, \langle\!\langle A, B \rangle\!\rangle) \cong [\mathcal{A}, \mathcal{A}](X * -, \langle A, B \rangle). \tag{1.3}$$

Moreover, as was discussed in [6], it is common in examples of such actions for the 2-functor $\mathcal{V} \to [\mathcal{A}, \mathcal{A}]$ sending X to $X * -$ to have a right adjoint $\theta : [\mathcal{A}, \mathcal{A}] \to \mathcal{V}$; and when this is so, (1.3) gives a natural isomorphism $\langle\!\langle A, B \rangle\!\rangle \cong \theta \langle A, B \rangle$. In these circumstances, our main results below for the general case (1.1) are consequences of those for the extremal case (1.2).

However, it in fact costs nothing to consider the general case throughout, especially if we use the coherence to simplify the notation as follows. Forget for the moment that $\mathcal{V}$ and $\mathcal{A}$ are 2-categories. To give a monoidal category $\mathcal{V}$ and an action $* : \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ of $\mathcal{V}$ on $\mathcal{A}$ is equally to give a bicategory $\mathbb{B}$ with just two object 0 and 1, having

$$\mathbb{B}(0,0) = \mathcal{V},\ \mathbb{B}(1,0) = \mathcal{A}\ , \mathbb{B}(1,1) = 1,\ \mathbb{B}(0,1) = 0,$$

where the last 0 denotes the empty category. As shown by Mac Lane and Paré [15] — for a more elegant alternative proof attributed to Gordon and Power see also [7] — we can replace $\mathbb{B}$ by an equivalent bicategory $\mathbb{C}$, with the same objects 0 and 1, in which composition is strictly associative. (Recall that this is indeed an equivalence, and not merely a biequivalence: there are homomorphisms $\mathbb{B} \to \mathbb{C}$ and $\mathbb{C} \to \mathbb{B}$, each of whose composites is *isomorphic* to the identity via (invertible) strong transformations.) When $\mathcal{V}$ and $\mathcal{A}$ are in fact 2-categories as above, the 2-cells of $\mathcal{V}$ and of $\mathcal{A}$, which are 3-cells in $\mathbb{B}$, just go along for the ride in the equivalence. Accordingly, so long as we deal with properties stable under such an equivalence, we may simplify by supposing henceforth that both $\otimes$ and $*$ are strictly associative — which allows us to write XY for $X \otimes Y$ in $\mathcal{V}$ and XA for $X * A$ in $\mathcal{A}$, with 1 for I.

Moreover, because of the importance of the extremal case, we shall henceforth write $\langle A, B\rangle$ rather than $\langle\!\langle A, B\rangle\!\rangle$ in the general case, so that (1.1) becomes

$$\Phi : \mathcal{A}(XA, B) \cong \mathcal{V}(X, \langle A, B\rangle); \tag{1.4}$$

and we shall henceforth use Φ without further explanation to denote this isomorphism.

Of course $\langle -, -\rangle$ admits a unique structure of 2-functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{V}$ for which Φ is 2-natural in each variable. Let us write

$$e = e_{A,B} : \langle A, B\rangle A \to B \tag{1.5}$$

for the 2-natural counit of the adjunction, and recall that we have a multiplication

$$M = M^{B}_{A,C} : \langle B, C\rangle\langle A, B\rangle \to \langle A, C\rangle \tag{1.6}$$

determined, using the adjunction, by the commutativity of

$$\begin{tikzcd} \langle B,C\rangle\langle A,B\rangle A \arrow[r, "MA"] \arrow[d, "\langle B,C\rangle e"'] & \langle A,C\rangle A \arrow[d, "e"] \\ \langle B,C\rangle B \arrow[r, "e"'] & C, \end{tikzcd} \tag{1.7}$$

as well as a "unit map" $J = J_A : 1 \to \langle A, A\rangle$ which is the mate under Φ of $\lambda : 1A \to A$ (here given by the identity), so that we have

$$1A \xrightarrow{JA} \langle A, A\rangle A \xrightarrow{e} A$$

equal to the identity. As is well known — see for example [6] — M and J provide the composition and the unit for a $\mathcal{V}$-category $\mathbb{A}$, whose underlying ordinary category is $\mathcal{A}$ and whose $\mathcal{V}$-valued hom $\mathbb{A}(A, B)$ is $\langle A, B\rangle$.

For each $A \in \mathcal{A}$ we have on $\langle A, A\rangle$ the structure of a monoid $(\langle A, A\rangle, i, m)$, where $m : \langle A, A\rangle\langle A, A\rangle \to \langle A, A\rangle$ is M^{A}_{AA} and $i : 1 \to \langle A, A\rangle$ is J_A. For a second object B of $\mathcal{A}$, let us write $(\langle B, B\rangle, j, n)$ for the monoid structure; in the extremal case where $\mathcal{V} = [\mathcal{A}, \mathcal{A}]$, these monoids are of course *monads on* $\mathcal{A}$ (meaning 2-monads, since $\mathcal{A}$ is a 2-category).

Our central result concerns an adjunction

$$\eta, \varepsilon : f \dashv f^* : A \to B \tag{1.8}$$

in the 2-category $\mathcal{A}$. Write w for $\langle f, f^*\rangle : \langle A, A\rangle \to \langle B, B\rangle$, noting that it is the image under Φ of the composite

$$\langle A, A\rangle B \xrightarrow{\langle A,A\rangle f} \langle A, A\rangle A \xrightarrow{e} A \xrightarrow{f^*} B,$$

which we shall denote more briefly by $t : \langle A, A\rangle B \to B$.

In the very simple case where $\mathcal{A} = \mathcal{V} = \mathbf{Cat}$ with its cartesian monoidal structure, A is a category and $\langle A, A\rangle = [A, A]$ is the strict monoidal category of endofunctors of A. Now an adjunction $f \dashv f^* : A \to B$ in $\mathcal{A}$ is just an adjunction in the original sense of the word; and Bénabou [1] observed that here $w = \langle f, f^*\rangle$ is part of a *monoidal functor* $(w, w^\circ, \widetilde{w})$. Indeed w sends $u : A \to A$ to f^*uf, and we have only to take $w^\circ : 1 \to f^*1f$ to be $\eta : 1 \to f^*f$, and to take $\widetilde{w}_{u,v} : f^*uff^*vf \to f^*uvf$ to be $f^*u\varepsilon vf$. Our central aim is to prove a similar result in the general case, providing for w the structure of a *lax map of monoids in* $\mathcal{V}$. Doing so is equivalent to providing for $t : \langle A, A\rangle B \to B$ the structure of a *lax action on* B

of the monoid $\langle A, A\rangle$; and this observation allows us to enrich the central result as follows. The evaluation $e : \langle A, A\rangle A \to A$ is itself a strict action of $\langle A, A\rangle$ on A, and we show $f^* : A \to B$ to admit the structure of a lax map of lax $\langle A, A\rangle$-algebras, while $f : B \to A$ becomes a colax map of such algebras. Under further hypotheses on the adjunction $f \dashv f^*$, which are certainly satisfied when it is an equivalence (that is, when η and ε are invertible), the whole adjunction enriches to one in the 2-category Ps-$\langle A, A\rangle$-Alg of pseudo $\langle A, A\rangle$-algebras. When A has the structure of a T-algebra, the corresponding map $T \to \langle A, A\rangle$ of monoids provides a 2-functor from Ps-$\langle A, A\rangle$-Alg to Ps-T-Alg carrying the adjunction to one in Ps-T-Alg, which can be seen as a rule for transporting pseudo T-algebra structures along an equivalence. Finally, the 2-functor from Ps-T-Alg to T-Alg, which we have in the case of a *flexible* monoid T, carries the adjunction in Ps-T-Alg to one in T-Alg, giving a rule for transporting (strict) T-algebra structures along an equivalence when T is flexible.

We provide below the detailed statements of these and related results, along with their proofs. First, we recall in the next section the definitions of lax maps of monoids, of lax algebras, and of lax morphisms of lax algebras.

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2 The definitions

The 2-category Colax$[\mathbf{2}, \mathcal{V}]$ has for objects the arrows $f : X \to Y$ of $\mathcal{V}$, for arrows $f \to f'$ the triples (u, ρ, v) of the form

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\rho}{\Rightarrow}"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "v"'] & Y', \end{tikzcd}$$

and for 2-cells $(u, \rho, v) \to (\bar{u}, \bar{\rho}, \bar{v})$ the pairs (α, β) where $\alpha : u \to \bar{u}$ and $\beta : v \to \bar{v}$ satisfy the obvious coherence condition [10, p.221]. This 2-category has an evident monoidal structure in which the tensor product of $f : X \to Y$ and $g : W \to Z$ is $fg : XW \to YZ$. For monoids $T = (T, i, m)$ and $S = (S, j, n)$ in $\mathcal{V}$ (recall that these are 2-monads in the case $\mathcal{V} = [\mathcal{A}, \mathcal{A}]$), a *lax map of monoids* or *lax monoid map* $w = (w, w^\circ, \widetilde{w}) : T \to S$ consists of a map $w : T \to S$ in $\mathcal{V}$ along with 2-cells

$$\begin{tikzcd} & T \arrow[d, "w"] \\ 1 \arrow[ur, "i"] \arrow[dr, "j"'] \arrow[r, phantom, "\overset{w^\circ}{\Rightarrow}"] & S, \end{tikzcd} \qquad \begin{tikzcd} TT \arrow[r, "m"] \arrow[d, "ww"'] \arrow[dr, phantom, "\overset{\widetilde{w}}{\Rightarrow}"] & T \arrow[d, "w"] \\ SS \arrow[r, "n"'] & S, \end{tikzcd}$$

satisfying the three equations [9, (4.2-4.4)] which make of $w : T \to S$ a monoid in Colax$[\mathbf{2}, \mathcal{V}]$. If now $z = (z, z^\circ, \widetilde{z})$ is another lax monoid map, a 2-cell $\theta : w \to z$ is said to be a *monoid 2-cell* if it makes commutative the diagrams

$$\begin{tikzcd} & wi \arrow[dd, "\theta i"] \\ j \arrow[ur, "w^\circ"] \arrow[dr, "z^\circ"'] & \\ & zi, \end{tikzcd} \qquad \begin{tikzcd} n.ww \arrow[r, "\widetilde{w}"] \arrow[d, "n.\theta\theta"'] & wm \arrow[d, "\theta m"] \\ n.zz \arrow[r, "\widetilde{z}"'] & zm, \end{tikzcd}$$

wherein $\theta\theta$ denotes the common value of

$$S\theta.\theta T : Sw.wT \to Sz.zT \quad \text{and} \quad \theta S.T\theta : wS.Tw \to zS.Tz.$$

Thus we have a 2-category $\mathrm{Mon}_l\mathcal{V}$ of monoids in $\mathcal{V}$, lax maps of these, and monoid 2-cells. A *pseudo map of monoids* is a lax one for which w° and $\widetilde{w}$ are invertible; with the same notion of 2-cell, these form a 2-category $\mathrm{Mon}_p\mathcal{V}$. And of course a *strict map of monoids*, or simply a *monoid map*, is just a lax one for which w° and $\widetilde{w}$ are identities; with the same notion of 2-cell once again, these form a 2-category $\mathrm{Mon}\mathcal{V}$, which may also be called $\mathrm{Mon}_s\mathcal{V}$ if we wish to emphasize the strictness of the maps.

When, in the definition above of lax monoid map, (S, j, n) is the monoid $(\langle B, B\rangle, j, n)$ described in Section 1, to give the arrow $w : T \to \langle B, B\rangle$ is equally, by (1.4), to give an arrow $t : TB \to B$ in $\mathcal{A}$; similarly to give the 2-cells w° and $\widetilde{w}$ is equally to give 2-cells

$$\begin{tikzcd} & TB \arrow[dd, "t"] \\ B \arrow[ur, "iB"] \arrow[dr, "1"'] \arrow[r, phantom, "\overset{\widehat{t}}{\Rightarrow}"] & {} \\ & B, \end{tikzcd} \qquad \begin{tikzcd} T^2B \arrow[r, "mB"] \arrow[d, "Tt"'] \arrow[dr, phantom, "\overset{\overline{t}}{\Rightarrow}"] & TB \arrow[d, "t"] \\ TB \arrow[r, "t"'] & B, \end{tikzcd}$$

and the three axioms on $(w, w^\circ, \widetilde{w})$ are easily converted to the three axioms [9, (4.6-4.8)] for $(t, \widehat{t}, \overline{t})$ to be a *lax action* of the monoid T on the object B of $\mathcal{A}$. It is a *pseudo action* when $\widehat{t}$ and $\overline{t}$ are invertible, and a *strict action* — or merely an *action* — when $\widehat{t}$ and $\overline{t}$ are identities. Reversing the sense of $\widehat{t}$ and $\overline{t}$ produces the notion of a *colax action* of T on B. When $(t, \widehat{t}, \overline{t})$ is a lax action of T on B, we call the quadruple $(B, t, \widehat{t}, \overline{t})$ a *lax T-algebra*; similarly for the notions of *pseudo T-algebra*, of *strict T-algebra* (or merely *T-algebra*), and of *colax T-algebra*.

If $(B, b, \widehat{b}, \overline{b})$ and $(A, a, \widehat{a}, \overline{a})$ are lax T-algebras, a *lax morphism* (or *lax map*) from $(B, b, \widehat{b}, \overline{b})$ to $(A, a, \widehat{a}, \overline{a})$ is a pair $(f, \overline{f})$ where $f : B \to A$ is a morphism in $\mathcal{A}$ while $\overline{f}$ is a 2-cell $a.Tf \to fb$ satisfying the two axioms [9, (4.10) and (4.11)]. (Note that we have explained lax monoid maps as monoids in a suitable monoidal 2-category, and explained lax actions as lax monoid maps $T \to \langle B, B\rangle$; the corresponding rationale for the definition of lax morphisms of lax algebras will be given a little later.)

The lax morphism is said to be a *pseudo morphism*, or just a *morphism*, of the lax T-algebras when $\overline{f}$ is invertible, and to be a *strict morphism* when $\overline{f}$ is the identity; while reversing the sense of $\overline{f}$ gives the notion of a *colax morphism*. In the case where $\widehat{b}$, $\overline{b}$, $\widehat{a}$, and $\overline{a}$ are identities, we recover the usual notions of lax, pseudo, strict, or colax morphisms of (strict) T-algebras; even for these, following the lead of [4], we use "morphism" without a modifier to mean "pseudo morphism". In the same way, when $\widehat{b}$, $\overline{b}$, $\widehat{a}$, and $\overline{a}$ are invertible, we call $(f, \overline{f})$ a lax morphism of pseudo T-algebras, and so on.

The notion of *algebra 2-cell* $\varphi : f \to g : (B, b, \widehat{b}, \overline{b}) \to (A, a, \widehat{a}, \overline{a})$ is the same for lax algebras, pseudo ones, or strict ones: namely a 2-cell $\varphi : f \to g$ in $\mathcal{A}$ satisfying the single obvious equation. So we have 2-categories and inclusions

$$\text{Lax-}T\text{-Alg}_s \to \text{Lax-}T\text{-Alg}_p = \text{Lax-}T\text{-Alg} \to \text{Lax-}T\text{-Alg}_l$$

of lax T-algebras with, respectively, strict morphisms, pseudo morphisms (often just called morphisms), and lax morphisms; as well as the 2-category Lax-T-Alg$_c$ whose morphisms are the colax ones. Similarly, there are strings of inclusions with Lax-T-Alg$_l$ replaced by Ps-T-Alg$_l$ (when we restrict to the pseudo algebras) or by T-Alg$_l$ (when we restrict to the strict ones).

We promised to give a "rationale" for the definition of lax morphism of lax T-algebras; in fact our needs below in proving the central result make it more convenient to work with colax morphisms. We therefore define a 2-category Lax$[\mathbf{2}, \mathcal{A}]$, analogous to Colax$[\mathbf{2}, \mathcal{A}]$, in which an object is once again a morphism $f : B \to A$ in $\mathcal{A}$, while an arrow $(b, r, a) : f \to g$ consists of morphisms b and a together with a 2-cell r as in

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow r"] & D \arrow[d, "g"] \\ A \arrow[r, "a"'] & C, \end{tikzcd}$$

and we have the obvious definition of 2-cell. There is an evident action of the monoidal 2-category $\mathcal{V}$ on Lax$[\mathbf{2}, \mathcal{A}]$, sending $(T, f : B \to A)$ to $Tf : TB \to TA$, and defined in the obvious way on morphisms and 2-cells. To give a map $(b, r, a) : Tf \to g$ in Lax$[\mathbf{2}, \mathcal{A}]$ is equally to give β, ρ, and α as in

$$\begin{tikzcd} & \langle B, D\rangle \arrow[dr, "{\langle B,g\rangle}"] & \\ T \arrow[ur, "\beta"] \arrow[dr, "\alpha"'] \arrow[rr, phantom, "\Downarrow\rho"] & & \langle B, C\rangle, \\ & \langle A, C\rangle \arrow[ur, "{\langle f,C\rangle}"'] & \end{tikzcd}$$

where $\beta = \Phi b$, $\alpha = \Phi a$, and $\rho = \Phi r$; so that b, r, and a are recovered, using the evaluation e, as $b = e_{B,D} . \beta B$, $r = e_{B,C} . \rho B$, and $a = e_{A,C} . \alpha A$. If we now write

$$\begin{tikzcd} & \langle B, D\rangle \arrow[dr, "{\langle B,g\rangle}"] & \\ \{f, g\} \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[rr, phantom, "\Downarrow\lambda"] & & \langle B, C\rangle \\ & \langle A, C\rangle \arrow[ur, "{\langle f,C\rangle}"'] & \end{tikzcd}$$

for the comma object, to give (β, ρ, α) is equally to give a map $\gamma : T \to \{f, g\}$ in $\mathcal{V}$: namely the unique map for which $u\gamma = \beta$, $\lambda\gamma = \rho$, and $v\gamma = \alpha$. These bijections sending γ to $(u, \lambda, v)\gamma = (\beta, \rho, \alpha)$ and sending (β, ρ, α) to $(e.\beta B, e.\rho B, e.\alpha A) = (b, r, a)$ clearly extend to 2-cells and become isomorphisms of categories, their composite being a natural isomorphism

$$\mathcal{V}(T, \{f, g\}) \cong \text{Lax}[\mathbf{2}, \mathcal{A}](Tf, g) \tag{2.1}$$

exhibiting $\{f,-\}$ as the right adjoint of the 2-functor $\mathcal{V} \to \mathrm{Lax}[\mathbf{2},\mathcal{A}]$ sending T to Tf. The counit $E_{f,g} : \{f,g\}f \to g$ of the adjunction, which we may again call the *evaluation*, has the form

$$\begin{tikzcd} \{f,g\}B \arrow[r, "E^0_{f,g}"] \arrow[d, "\{f,g\}f"'] \arrow[dr, phantom, "\Downarrow E^{\to}_{f,g}"] & D \arrow[d, "g"] \\ \{f,g\}A \arrow[r, "E^1_{f,g}"'] & C, \end{tikzcd}$$

and is obtained by setting $T = \{f,g\}$ and $\gamma = 1$, so that $E^0_{f,g} = e.uB$, $E^{\to}_{f,g} = e.\lambda B$, and $E^1_{f,g} = e.vA$. When $g = f$, the comma object $\{f,g\}$ becomes

$$\begin{tikzcd} & \langle B,B\rangle \arrow[dr, "\langle B,f\rangle"] & \\ \{f,f\} \arrow[ur, "u"] \arrow[dr, "v"'] \arrow[rr, phantom, "\Downarrow\lambda"] & & \langle B,A\rangle. \\ & \langle A,A\rangle \arrow[ur, "\langle f,A\rangle"'] & \end{tikzcd}$$

By the general results on actions in Section 1, for any $f : B \to A$ the object $\{f,f\}$ admits a canonical structure $(\{f,f\},k,l)$ of monoid in $\mathcal{V}$, where $l : \{f,f\}\{f,f\} \to \{f,f\}$ is determined by the commutativity in $\mathrm{Lax}[\mathbf{2},\mathcal{A}]$ of

$$\begin{tikzcd} \{f,f\}\{f,f\}f \arrow[r, "lf"] \arrow[d, "\{f,f\}E_{f,f}"'] & \{f,f\}f \arrow[d, "E_{f,f}"] \\ \{f,f\}f \arrow[r, "E_{f,f}"'] & f, \end{tikzcd} \tag{2.2}$$

and similarly $k : 1 \to \{f,f\}$ is determined by the equation $E_{f,f}.kf = 1_f$ in $\mathrm{Lax}[\mathbf{2},\mathcal{A}]$. Moreover, for a monoid $T = (T,i,m)$ in $\mathcal{V}$, to give a lax monoid map $\gamma : T \to \{f,f\}$ is equivalently, by the earlier part of this section, to give (as its image under (2.1)) a lax action $(c,\widehat{c},\overline{c}) : Tf \to f$. Here c consists of maps $b : TB \to B$ and $a : TA \to A$ in $\mathcal{A}$ and a 2-cell $\overline{f} : fb \to a.Tf$, while $\widehat{c}$ is a pair $(\widehat{b},\widehat{a})$ of 2-cells $\widehat{b} : 1 \to b.iB$ and $\widehat{a} : 1 \to a.iA$, and $\overline{c}$ is a pair $(\overline{b},\overline{a})$ of 2-cells $\overline{b} : b.Tb \to b.mB$ and $\overline{a} : a.Ta \to a.mA$; all these data satisfying equations which assert precisely that $(b,\widehat{b},\overline{b})$ and $(a,\widehat{a},\overline{a})$ are lax actions of T on B and A and that $(f,\overline{f})$ is a colax morphism $(B,b,\widehat{b},\overline{b}) \to (A,a,\widehat{a},\overline{a})$ of lax T-algebras.

In particular, a strict monoid map $\gamma : T \to \{f,f\}$ corresponds to strict actions $b : TB \to B$ and $a : TA \to A$, along with an $\overline{f}$ making $(f,\overline{f}) : (B,b) \to (A,a)$ a colax morphism of T-algebras. The strict monoid maps $\beta : T \to \langle B,B\rangle$ and $\alpha : T \to \langle A,A\rangle$ corresponding to the strict actions b and a are the composites of γ with $u : \{f,f\} \to \langle B,B\rangle$ and $v : \{f,f\} \to \langle A,A\rangle$; since γ here may be the identity map of $\{f,f\}$, we conclude that *u and v are themselves strict monoid maps.*

We can be more explicit about the value of $k : 1 \to \{f,f\}$: it corresponds of course under (2.1) to the identity $1f \to f$, and hence is the unique k for which uk and vk are the units $j : 1 \to \langle B,B\rangle$ and $i : 1 \to \langle A,A\rangle$ of the monoids $\langle B,B\rangle$ and $\langle A,A\rangle$ while $\lambda k = id$.

The explicit description of the multiplication $l : \{f,f\}\{f,f\} \to \{f,f\}$ is slightly more complicated. First we unravel (2.2) to obtain

$$
\begin{tikzcd}[column sep=small]
\{f,f\}\{f,f\}B \arrow[dr, "\{f,f\}\{f,f\}f"] \arrow[dd, "\{f,f\}(e.uB)"'] & \\
& \{f,f\}\{f,f\}A \arrow[dd, "\{f,f\}(e.vA)"] \\
\{f,f\}B \arrow[dr, "\{f,f\}f"] \arrow[dd, "e.uB"'] \arrow[ur, phantom, "\Uparrow\{f,f\}(e.\lambda B)"] & \\
& \{f,f\}A \arrow[dd, "e.vA"] \\
B \arrow[dr, "f"'] \arrow[ur, phantom, "e.\lambda B \Uparrow"] & \\
& A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=small]
\{f,f\}\{f,f\}B \arrow[dr, "\{f,f\}\{f,f\}f"] \arrow[dd, "lB"'] & \\
& \{f,f\}\{f,f\}A \arrow[dd, "lA"] \\
\{f,f\}B \arrow[dr, "\{f,f\}f"] \arrow[dd, "e.uB"'] & \\
& \{f,f\}A \arrow[dd, "e.vA"] \\
B \arrow[dr, "f"'] \arrow[ur, phantom, "e.\lambda B \Uparrow"] & \\
& A,
\end{tikzcd}
\tag{2.3}
$$

and then apply the isomorphism Φ to this equality. The resulting equality, at the level of 1-cells, asserts that ul and vl are the composites

$$\{f,f\}\{f,f\} \xrightarrow{uu} \langle B,B\rangle\langle B,B\rangle \xrightarrow{n} \langle B,B\rangle,$$

$$\{f,f\}\{f,f\} \xrightarrow{vv} \langle A,A\rangle\langle A,A\rangle \xrightarrow{m} \langle A,A\rangle,$$

(repeating our observation above that u and v are strict monoid maps); at the level of 2-cells, it reduces, as we indicate below, to the assertion that λl is given by

$$
\begin{tikzcd}[column sep=small]
& & \langle B,B\rangle\langle B,B\rangle \arrow[rr, "n"] \arrow[dr, "\langle B,f\rangle\langle B,B\rangle"] & & \langle B,B\rangle \arrow[dd, "\langle B,f\rangle"] \\
& \{f,f\}\langle B,B\rangle \arrow[ur, "u\langle B,B\rangle"] \arrow[dr, "v\langle B,B\rangle"'] \arrow[rr, phantom, "\Downarrow\lambda\langle B,B\rangle"] & & \langle B,A\rangle\langle B,B\rangle \arrow[dr, "M"] & \\
\{f,f\}\{f,f\} \arrow[ur, "\{f,f\}u"] \arrow[dr, "v\{f,f\}"'] & & \langle A,A\rangle\langle B,B\rangle \arrow[ur, "\langle f,A\rangle\langle B,B\rangle"] \arrow[dr, "\langle A,A\rangle\langle B,f\rangle"'] & & \langle B,A\rangle \\
& \langle A,A\rangle\{f,f\} \arrow[ur, "\langle A,A\rangle u"] \arrow[dr, "\langle A,A\rangle v"'] \arrow[rr, phantom, "\Downarrow\langle A,A\rangle\lambda"] & & \langle A,A\rangle\langle B,A\rangle \arrow[ur, "M"'] & \\
& & \langle A,A\rangle\langle A,A\rangle \arrow[ur, "\langle A,A\rangle\langle f,A\rangle"] \arrow[rr, "m"'] & & \langle A,A\rangle. \arrow[uu, "\langle f,A\rangle"']
\end{tikzcd}
\tag{2.4}
$$

Since the image under Φ of the right side of (2.3) is the composite λl, we must exhibit (2.4) as the image under Φ of the left side of (2.3); and this left side is the "vertical" composite of $e.\lambda B.\{f,f\}e.\{f,f\}uB$ with $e.vA.\{f,f\}e.\{f,f\}\lambda B$. Because the action $* : \mathcal{V} \times \mathcal{A} \to \mathcal{A}$, denoted by juxtaposition, is a 2-functor, we have $\lambda B.\{f,f\}e = \langle B,A\rangle e.\lambda\langle B,B\rangle B$, so that

$$\begin{aligned} e.\lambda B.\{f,f\}e.\{f,f\}uB &= e.\langle B,A\rangle e.\lambda\langle B,B\rangle B.\{f,f\}uB \\ &= e.MB.\lambda\langle B,B\rangle B.\{f,f\}uB, \end{aligned}$$

where the second step uses (1.7) to replace $e.\langle B,A\rangle e$ by $e.MB$; and the image $M.\lambda\langle B,B\rangle.\{f,f\}u$ of $e.MB.\lambda\langle B,B\rangle B.\{f,f\}uB$ under Φ is the top half of (2.4). Similar arguments justify the steps in

$$\begin{aligned} e.vA.\{f,f\}e.\{f,f\}\lambda B &= e.\langle A,A\rangle e.v\langle B,A\rangle B.\{f,f\}\lambda B \\ &= e.MB.\langle A,A\rangle\lambda B.v\{f,f\}B; \end{aligned}$$

and the image $M.\langle A,A\rangle\lambda.v\{f,f\}$ of this last under Φ is the bottom half of (2.4).

3 The central result

We begin with the following, due to Street [16, Proposition 5]:

Lemma 3.1 *Let*

$$\begin{tikzcd} & B \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow\lambda"] & \\ D \arrow[ur, "u"] \arrow[dr, "v"'] & & A \\ & C \arrow[ur, "g"'] & \end{tikzcd}$$

be a comma object in the 2-category $\mathcal{A}$. If f has a right adjoint given by $\eta,\varepsilon : f \dashv f^* : A \to B$, then v has a right adjoint given by $\zeta, id : v \dashv v^* : C \to D$, where v^* is the unique map satisfying $uv^* = f^*g$, $\lambda v^* = \varepsilon g$, and $vv^* = 1$; while $\zeta : 1 \to v^*v$ is the unique 2-cell for which $v\zeta : v \to vv^*v$ is the identity on $v(= vv^*v)$ and $u\zeta : u \to uv^*v$ is the composite

$$u \xrightarrow{\eta u} f^*fu \xrightarrow{f^*\lambda} f^*gv = uv^*v.$$

Remark 3.2 There is of course a similar result where we replace the comma object by an iso-comma object, and the adjunction $f \dashv f^*$ by an equivalence; but we shall not need to refer to this below.

Returning now to the general situation of a monoidal 2-category $\mathcal{V}$ acting on a 2-category $\mathcal{A}$ with a right adjoint expressed by the 2-natural isomorphism Φ of (1.4), consider an arbitrary adjunction $\eta,\varepsilon : f \dashv f^* : A \to B$ in $\mathcal{A}$, and note that the 2-functor $\langle B,-\rangle : \mathcal{A} \to \mathcal{V}$ takes this adjunction into an adjunction

$$\langle B,\eta\rangle, \langle B,\varepsilon\rangle : \langle B,f\rangle \dashv \langle B,f^*\rangle : \langle B,A\rangle \to \langle B,B\rangle$$

in $\mathcal{V}$. Supposing henceforth $\mathcal{V}$ to admit comma objects, we can apply Lemma 3.1 to this adjunction and to the comma object

$$\begin{tikzcd} & \langle B,B\rangle \arrow[dr, "\langle B,f\rangle"] \arrow[dd, phantom, "\Downarrow\lambda"] & \\ \{f,f\} \arrow[ur, "u"] \arrow[dr, "v"'] & & \langle B,A\rangle \\ & \langle A,A\rangle \arrow[ur, "\langle f,A\rangle"'] & \end{tikzcd}$$

to get:

Proposition 3.3 *In the presence of the adjunction $\eta, \varepsilon : f \dashv f^* : A \to B$, the map $v : \{f, f\} \to \langle A, A\rangle$ has a right adjoint given by $\zeta, id : v \dashv z : \langle A, A\rangle \to \{f, f\}$, where z is the unique map satisfying $uz = \langle f, f^*\rangle$, $\lambda z = \langle f, \varepsilon\rangle : \langle f, ff^*\rangle \to \langle f, A\rangle$, and $vz = 1$; while $\zeta : 1 \to zv$ is the unique 2-cell for which $v\zeta = id$ and $u\zeta$ is the composite*

$$u \xrightarrow{\langle B,\eta\rangle u} \langle B, f^*f\rangle u \xrightarrow{\langle B,f^*\rangle\lambda} \langle f, f^*\rangle v\,.$$

The result of the following lemma is very like that of [8, Theorem 1.2], of which it is not, however, a consequence; the situation is rather that the proof-techniques of that paper adapt so readily to the present lemma that we can safely leave the details to the reader.

Lemma 3.4 *Let $\alpha, \beta : \rho \dashv \sigma : S \to T$ be an adjunction in the monoidal 2-category $\mathcal{V}$, where T and S have monoid structures (T, i, m) and (S, j, n). Then there is a bijection between enrichments of ρ to a colax monoid map $(\rho, \rho', \rho^{\#})$ and enrichments of σ to a lax monoid map $(\sigma, \sigma^\circ, \widetilde{\sigma})$, where σ° and $\widetilde{\sigma}$ are given respectively by the pasting composites*

$$\begin{tikzcd}
 & T \arrow[rr, "1"] \arrow[dr, "\rho"] \arrow[d, phantom, "\Downarrow\rho'"] & \arrow[d, phantom, "\Downarrow\alpha"] & T \\
1 \arrow[ur, "i"] \arrow[rr, "j"'] & & S \arrow[ur, "\sigma"'] &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
 & TT \arrow[rr, "m"] \arrow[dr, "\rho\rho"] \arrow[d, phantom, "\Downarrow\beta\beta"] & \arrow[d, phantom, "\Downarrow\rho^{\#}"] & T \arrow[rr, "1"] \arrow[dr, "\rho"] & \arrow[d, phantom, "\Downarrow\alpha"] & T \\
SS \arrow[ur, "\sigma\sigma"] \arrow[rr, "1"'] & & SS \arrow[rr, "n"'] & & S \arrow[ur, "\sigma"'] &
\end{tikzcd}\quad .$$

We now apply the lemma to the adjunction $\zeta, id : v \dashv z : \langle A, A\rangle \to \{f, f\}$. We saw in Section 2 that v is a strict monoid map, which we can see as a colax monoid map (v, id, id); it follows from the lemma that z admits an enrichment to a lax monoid map $(z, z^\circ, \widetilde{z})$ where z° is given by

$$\begin{tikzcd}
 & \{f,f\} \arrow[rr, "1"] \arrow[dr, "v"] & \arrow[d, phantom, "\Downarrow\zeta"] & \{f,f\} \\
1 \arrow[ur, "k"] \arrow[rr, "i"'] & & \langle A,A\rangle \arrow[ur, "z"'] &
\end{tikzcd}$$

or simply ζk, while $\widetilde{z}$ is given by

$$\begin{tikzcd}
 & \{f,f\}\{f,f\} \arrow[rr, "l"] \arrow[dr, "vv"] & & \{f,f\} \arrow[rr, "1"] \arrow[dr, "v"] & \arrow[d, phantom, "\Downarrow\zeta"] & \{f,f\} \\
\langle A,A\rangle\langle A,A\rangle \arrow[ur, "zz"] \arrow[rr, "1"'] & & \langle A,A\rangle\langle A,A\rangle \arrow[rr, "m"'] & & \langle A,A\rangle \arrow[ur, "z"'] &
\end{tikzcd}$$

or simply $\zeta l.zz$. Note that, since $v\zeta = id$ by Proposition 3.3, we have $vz^\circ = id$ and $v\widetilde{z} = id$; thus the composite $(vz, vz^\circ, v\widetilde{z})$ of the strict monoid map v and the lax monoid map $z = (z, z^\circ, \widetilde{z})$ is the identity monoid map $1 = (1, id, id) : \langle A, A\rangle \to \langle A, A\rangle$.

Let us set $w = \langle f, f^*\rangle : \langle A, A\rangle \to \langle B, B\rangle$, as indicated in Section 1. Then $w = uz$ by the definition of z; and since u is a strict monoid map while $z = (z, z^\circ, \widetilde{z})$ is a lax monoid map, we have a lax monoid map

$$(w, w^\circ, \widetilde{w}) = (uz, uz^\circ, u\widetilde{z}) : \langle A, A\rangle \to \langle B, B\rangle.$$

To complete our central result, therefore, it remains to describe more explicitly w° and $\widetilde{w}$; or equivalently to describe the $\widehat{t}$ and the $\overline{t}$ which enrich the $t : \langle A, A\rangle B \to B$ of Section 1, given by

$$\langle A, A\rangle B \xrightarrow{\langle A,A\rangle f} \langle A, A\rangle A \xrightarrow{e} A \xrightarrow{f^*} B,$$

to the lax action $(t, \widehat{t}, \overline{t})$ of $\langle A, A\rangle$ on B corresponding to the lax monoid map $(w, w^\circ, \widetilde{w})$.

Now $w^\circ = uz^\circ = u\zeta k$, which, by Proposition 3.3 and the observation in Section 2 that $\lambda k = id$, is just the 2-cell $\langle B, \eta\rangle uk$ in

$$\begin{tikzcd}
 & & & \langle B,A\rangle \arrow[dr, "\langle B,f^*\rangle"] \arrow[d, phantom, "\Downarrow\langle B,\eta\rangle"] & \\
1 \arrow[r, "k"'] & \{f,f\} \arrow[r, "u"'] & \langle B,B\rangle \arrow[ur, "\langle B,f\rangle"] \arrow[rr, "1"'] & & \langle B,B\rangle;
\end{tikzcd}$$

and since $uk = j$, this is just $\langle B, \eta\rangle j$. Moreover, applying Φ^{-1} to each side of the equality $w^\circ = \langle B, \eta\rangle j$ shows that $\widehat{t} : 1 \to t.iB$ is given by $\eta : 1 \to f^*f$; observe here, using the naturality of i, that $f^*f = f^*e.iA.f = f^*e.\langle A, A\rangle f.iB = t.iB$.

It remains to describe the 2-cell

$$\begin{tikzcd}
\langle A,A\rangle\langle A,A\rangle \arrow[r, "m"] \arrow[d, "ww"'] \arrow[dr, phantom, "\overset{\widetilde{w}}{\Rightarrow}"] & \langle A,A\rangle \arrow[d, "w"] \\
\langle B,B\rangle\langle B,B\rangle \arrow[r, "n"'] & \langle B,B\rangle,
\end{tikzcd}$$

or equivalently the component

$$\begin{tikzcd}
\langle A,A\rangle\langle A,A\rangle B \arrow[r, "mB"] \arrow[d, "\langle A,A\rangle t"'] \arrow[dr, phantom, "\overset{\overline{t}}{\Rightarrow}"] & \langle A,A\rangle B \arrow[d, "t"] \\
\langle A,A\rangle B \arrow[r, "t"'] & B
\end{tikzcd}$$

of the lax action $(t, \widehat{t}, \overline{t})$ of $\langle A, A\rangle$ on B. Now $\widetilde{w} = u\widetilde{z} = u\zeta l.zz$, so from the definition of $u\zeta$ in Proposition 3.3 it follows that $\widetilde{w}$ is given by the pasting composite

$$\begin{tikzcd}
 & & & \langle B,B\rangle \arrow[rr, "1"] \arrow[dr, "\langle B,f\rangle"'] & \arrow[d, phantom, "\Downarrow\langle B,\eta\rangle"] & \langle B,B\rangle \\
\langle A,A\rangle\langle A,A\rangle \arrow[r, "zz"] & \{f,f\}\{f,f\} \arrow[r, "l"] & \{f,f\} \arrow[ur, "u"] \arrow[dr, "v"'] & \Downarrow\lambda & \langle B,A\rangle \arrow[ur, "\langle B,f^*\rangle"'] & \\
 & & & \langle A,A\rangle\ . \arrow[ur, "\langle f,A\rangle"'] & &
\end{tikzcd} \tag{3.1}$$

Using the description (2.4) of λl and the equations $vz = 1$ and $uz = \langle f, f^*\rangle$, we see that $\lambda l.zz$ may be written as

$$\begin{tikzcd}
 & & \langle B,B\rangle\langle B,B\rangle \arrow[rr, "n"] \arrow[dr, "\langle B,f\rangle\langle B,B\rangle"] \arrow[dd, phantom, "\Downarrow\lambda\langle B,B\rangle"] & & \langle B,B\rangle \arrow[dd, "\langle B,f\rangle"] \\
 & \{f,f\}\langle B,B\rangle \arrow[ur, "u\langle B,B\rangle"] \arrow[dr, "v\langle B,B\rangle"] & & \langle B,A\rangle\langle B,B\rangle \arrow[dr, "M"] & \\
\langle A,A\rangle\langle A,A\rangle \arrow[ur, "z\langle f,f^*\rangle"] \arrow[dr, "\langle A,A\rangle z"'] & & \langle A,A\rangle\langle B,B\rangle \arrow[ur, "\langle f,A\rangle\langle B,B\rangle"] \arrow[dr, "\langle A,A\rangle\langle B,f\rangle"] \arrow[dd, phantom, "\Downarrow\langle A,A\rangle\lambda"] & & \langle B,A\rangle \\
 & \langle A,A\rangle\{f,f\} \arrow[ur, "\langle A,A\rangle u"] \arrow[dr, "\langle A,A\rangle v"'] & & \langle A,A\rangle\langle B,A\rangle \arrow[ur, "M"'] & \\
 & & \langle A,A\rangle\langle A,A\rangle \arrow[rr, "m"'] \arrow[ur, "\langle A,A\rangle\langle f,A\rangle"] & & \langle A,A\rangle . \arrow[uu, "\langle f,A\rangle"']
\end{tikzcd} \tag{3.2}$$

Using the equation $\lambda z = \langle f, \varepsilon\rangle$ to simplify (3.2), and substituting the result into (3.1), we conclude that $\widetilde{w}$ is the composite:

$$\begin{tikzcd}
 & & \langle B,B\rangle\langle B,B\rangle \arrow[r, "n"] \arrow[d, "\langle B,f\rangle\langle B,B\rangle"] & \langle B,B\rangle \arrow[dl, "\langle B,f\rangle"'] \arrow[dd, "\langle B,B\rangle"] \\
 & \langle A,A\rangle\langle B,B\rangle \arrow[ur, "\langle f,f^*\rangle\langle B,B\rangle"] \arrow[r, "\langle f,A\rangle\langle B,B\rangle"'] \arrow[d, "\langle A,A\rangle\langle B,f\rangle"] \arrow[ur, phantom, "\Downarrow\langle f,\varepsilon\rangle\langle B,B\rangle"] & \langle B,A\rangle\langle B,B\rangle \arrow[d, "M"] & \\
\langle A,A\rangle\langle A,A\rangle \arrow[ur, "\langle A,A\rangle\langle f,f^*\rangle"] \arrow[r, "\langle A,A\rangle\langle f,A\rangle"'] \arrow[ur, phantom, "\Downarrow\langle A,A\rangle\langle f,\varepsilon\rangle"] & \langle A,A\rangle\langle B,A\rangle \arrow[r, "M"'] & \langle B,A\rangle \arrow[r, "\langle B,f^*\rangle"'] \arrow[ur, phantom, "\Leftarrow\langle B,\eta\rangle"] & \langle B,B\rangle .
\end{tikzcd}$$

By the "extraordinary" naturality of the M's and one of the triangular equations, this reduces to

$$\begin{tikzcd}
 & \langle A,A\rangle\langle B,B\rangle \arrow[dr, "\langle A,A\rangle\langle B,f\rangle"] \arrow[d, phantom, "\Downarrow\langle A,A\rangle\langle f,\varepsilon\rangle"] & & & \\
\langle A,A\rangle\langle A,A\rangle \arrow[ur, "\langle A,A\rangle\langle f,f^*\rangle"] \arrow[rr, "\langle A,A\rangle\langle f,A\rangle"'] & {} & \langle A,A\rangle\langle B,A\rangle \arrow[r, "M"'] & \langle B,A\rangle \arrow[r, "\langle B,f^*\rangle"'] & \langle B,B\rangle ,
\end{tikzcd}$$

and so, using extraordinary naturality once again, to

$$\begin{tikzcd}
 & & & \langle B,B\rangle \arrow[dr, "\langle B,f\rangle"] \arrow[d, phantom, "\Downarrow\langle B,\varepsilon\rangle"] & & \\
\langle A,A\rangle\langle A,A\rangle \arrow[r, "m"'] & \langle A,A\rangle \arrow[r, "\langle f,A\rangle"'] & \langle B,A\rangle \arrow[ur, "\langle B,f^*\rangle"] \arrow[rr, "1"'] & {} & \langle B,A\rangle \arrow[r, "\langle B,f^*\rangle"'] & \langle B,B\rangle .
\end{tikzcd} \tag{3.3}$$

Finally $\overline{t}$ is obtained from $\widetilde{w}$ by applying $(\ )B$ and composing with the evaluation $e : \langle B, B\rangle B \to B$; by the ordinary and the extraordinary 2-naturality of e this gives

$$\begin{tikzcd}
 & & & & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & & \\
\langle A,A\rangle\langle A,A\rangle B \arrow[r, "mB"'] & \langle A,A\rangle B \arrow[r, "\langle A,A\rangle f"'] & \langle A,A\rangle A \arrow[r, "e"'] & A \arrow[ur, "f^*"] \arrow[rr, "1"'] & {} & A \arrow[r, "f^*"'] & B,
\end{tikzcd}$$

which is perhaps more readily seen as a 2-cell $t.\langle A, A\rangle t \to t.mB$ by using the 2-naturality of e once more, to display it in the form

$$\begin{tikzcd}
\langle A,A\rangle\langle A,A\rangle B \arrow[rrr, "mB"] \arrow[d, "\langle A,A\rangle\langle A,A\rangle f"'] & & & \langle A,A\rangle B \arrow[d, "\langle A,A\rangle f"] \\
\langle A,A\rangle\langle A,A\rangle A \arrow[rrr, "mA"] \arrow[d, "\langle A,A\rangle e"'] & & & \langle A,A\rangle A \arrow[d, "e"] \\
\langle A,A\rangle A \arrow[rrr, "e"] \arrow[d, "\langle A,A\rangle f^*"'] \arrow[dr, "1"] \arrow[dr, phantom, "\Uparrow\langle A,A\rangle\varepsilon"', shift right=2ex] & & & A \arrow[d, "f^*"] \\
\langle A,A\rangle B \arrow[r, "\langle A,A\rangle f"'] & \langle A,A\rangle A \arrow[r, "e"'] & A \arrow[r, "f^*"'] & B.
\end{tikzcd} \tag{3.4}$$

Summing up, we have as our central result:

Theorem 3.5 *Let the monoidal 2-category $\mathcal{V}$ admit comma objects, and let it so act on the 2-category $\mathcal{A}$ that we have the adjunction $\Phi : \mathcal{A}(XA, B) \cong \mathcal{V}(X, \langle A, B\rangle)$. Then each adjunction $\eta, \varepsilon : f \dashv f^* : A \to B$ in $\mathcal{A}$ gives rise to a lax map of monoids $(w, w^\circ, \widetilde{w}) : \langle A, A\rangle \to \langle B, B\rangle$ in $\mathcal{V}$, where $w = \langle f, f^*\rangle$ and w° is given by $\langle B, \eta\rangle j$, while $\widetilde{w}$ is given by* (3.3). *In fact to give a lax map $(w, w^\circ, \widetilde{w}) : \langle A, A\rangle \to \langle B, B\rangle$ of monoids is equally to give a lax action $(t, \widehat{t}, \overline{t})$ of the monoid $\langle A, A\rangle$ on B; and here $t : \langle A, A\rangle B \to B$ is the composite*

$$\langle A, A\rangle B \xrightarrow{\langle A,A\rangle f} \langle A, A\rangle A \xrightarrow{e} A \xrightarrow{f^*} B,$$

while $\widehat{t} = \Phi^{-1}(w^\circ)$ is given by η and $\overline{t} = \Phi^{-1}(\widetilde{w})$ is given by (3.4). *When η and ε are invertible, so that the adjunction $f \dashv f^*$ is an equivalence, the 2-cells w°, $\widetilde{w}$, $\widehat{t}$, and $\overline{t}$ are invertible, so that $(w, w^\circ, \widetilde{w})$ is a pseudo map of monoids, while $(t, \widehat{t}, \overline{t})$ is a pseudo action of $\langle A, A\rangle$ on B.*

4 The enrichments of f and f^*

We continue to suppose satisfied the hypotheses of Theorem 3.5. As we saw in Section 3, the lax monoid map $(z, z^\circ, \widetilde{z}) : \langle A, A\rangle \to \{f, f\}$ satisfies $u(z, z^\circ, \widetilde{z}) = (w, w^\circ, \widetilde{w})$ and $v(z, z^\circ, \widetilde{z}) = 1$ in $\mathrm{Mon}_l\, \mathcal{V}$. Since the lax monoid map $(w, w^\circ, \widetilde{w}) : \langle A, A\rangle \to \langle B, B\rangle$ corresponds to the lax action $(t, \widehat{t}, \overline{t}) : \langle A, A\rangle B \to B$ and the strict monoid map $1 : \langle A, A\rangle \to \langle A, A\rangle$ corresponds to the strict action $e : \langle A, A\rangle A \to A$, it follows from our observations in Section 2 that we have a colax map $(f, \bar{\bar{f}}) : (B, t, \widehat{t}, \bar{t}) \to (A, e)$ of lax $\langle A, A\rangle$-algebras, where the diagram

$$\begin{tikzcd}
\langle A,A\rangle B \arrow[r, "t"] \arrow[d, "\langle A,A\rangle f"'] \arrow[dr, phantom, "\Downarrow\bar{\bar{f}}"] & B \arrow[d, "f"] \\
\langle A,A\rangle A \arrow[r, "e"'] & A
\end{tikzcd}$$

is the image under Φ^{-1} of λz. Since $\lambda z = \langle B, \varepsilon\rangle\langle f, A\rangle$ by Proposition 3.3, an easy calculation exhibits $\bar{\bar{f}}$ as the 2-cell

$$\begin{tikzcd}
\langle A,A\rangle B \arrow[r, "\langle A,A\rangle f"] \arrow[d, "\langle A,A\rangle f"'] & \langle A,A\rangle A \arrow[r, "e"] & A \arrow[r, "f^*"] \arrow[dr, "1"'] & B \arrow[d, "f"] \arrow[dl, phantom, "\Leftarrow\varepsilon"] \\
\langle A,A\rangle A \arrow[rrr, "e"'] & & & A.
\end{tikzcd}$$

It now follows from [8, Theorem 1.2] that we have a lax map $(f^*, \overline{f^*}) : (A, e) \to (B, t, \widehat{t}, \bar{t})$ of lax $\langle A, A\rangle$-algebras, where the 2-cell $\overline{f^*}$ is the composite

$$\begin{tikzcd}
\langle A,A\rangle A \arrow[rr, "1"] \arrow[dr, "\langle A,A\rangle f^*"'] & \arrow[d, phantom, "\langle A,A\rangle\varepsilon\Uparrow"] & \langle A,A\rangle A \arrow[rr, "e"] & \arrow[d, phantom, "\bar{\bar{f}}\Uparrow"] & A \arrow[dr, "f^*"] \arrow[d, phantom, "\eta\Uparrow"] & \\
& \langle A,A\rangle B \arrow[ur, "\langle A,A\rangle f"'] \arrow[rr, "t"'] & & B \arrow[ur, "f"] \arrow[rr, "1"'] & & B;
\end{tikzcd}$$

which, on substituting for $\bar{\bar{f}}$ its explicit value above and using one of the triangular equations, gives

$$\begin{tikzcd}
\langle A,A\rangle A \arrow[rrr, "e"] \arrow[d, "\langle A,A\rangle f^*"'] \arrow[dr, "1"] & & & A \arrow[d, "f^*"] \\
\langle A,A\rangle B \arrow[r, "\langle A,A\rangle f"'] \arrow[ur, phantom, "\Uparrow\langle A,A\rangle\varepsilon"] & \langle A,A\rangle A \arrow[r, "e"'] & A \arrow[r, "f^*"'] & B
\end{tikzcd}$$

as the value of $\overline{f^*}$.

When $\bar{\bar{f}} = \varepsilon e.\langle A, A\rangle f$ is invertible — and so in particular when ε itself is invertible — we have a lax map $(f, \bar{f})$ of lax $\langle A, A\rangle$-algebras, where $\bar{f} = \bar{\bar{f}}^{-1}$, and by [8, Proposition 1.3] we have an adjunction $(f, \bar{f}) \dashv (f^*, \overline{f^*})$ in Lax-$\langle A, A\rangle$-Alg$_l$. We state this formally only in the most important case where ε is itself invertible; then both $\bar{f}$ and $\overline{f^*}$ are invertible so that $(f, \bar{f})$ and $(f^*, \overline{f^*})$ become pseudomorphisms:

Theorem 4.1 *Let the counit $\varepsilon : ff^* \to 1$ of the adjunction $\eta, \varepsilon : f \dashv f^* : A \to B$ be invertible. Then we have an adjunction $\eta, \varepsilon : (f, \bar{f}) \dashv (f^*, \overline{f^*}) : (A, e) \to (B, t, \widehat{t}, \bar{t})$ in* Lax-$\langle A, A\rangle$-Alg, *where $\bar{f}$ is given by $\varepsilon^{-1}e.\langle A, A\rangle f$ and $\overline{f^*}$ is given by $f^*e.\langle A, A\rangle\varepsilon$. When η too is invertible, so that the original adjunction is an equivalence in $\mathcal{A}$, both $\widehat{t}$ and $\bar{t}$ are invertible, so that the adjunction $(f, \bar{f}) \dashv (f^*, \overline{f^*})$ becomes an equivalence in* Ps-$\langle A, A\rangle$-Alg.

A somewhat different case that has been useful historically, as the motivation for introducing the concept of *flexibility* for 2-monads, is that where we suppose the invertibility only of $\overline{f^*} = f^*e.\langle A, A\rangle\varepsilon$, which gives us an adjunction

$$\eta, \varepsilon : (f, \bar{\bar{f}}) \dashv (f^*, (\overline{f^*})^{-1}) : (A, e) \to (B, t, \hat{t}, \bar{t})$$

(with $\bar{t}$ too invertible) in the 2-category Lax-$\langle A, A\rangle$-Alg_c of lax $\langle A, A\rangle$-algebras and *colax* maps. The historical example supposed η too to be invertible — indeed, to be an identity — so that also $\hat{t}$ was invertible, and we were dealing with pseudo $\langle A, A\rangle$-algebras. To regain lax maps instead of colax ones, we need only to pass to the dual case by supposing the original adjunction $\eta, \varepsilon : f \dashv f^* : A \to B$ to lie in $\mathcal{A}^{co}$ rather than $\mathcal{A}$. Leaving the reader to work through the simple dualizing process, we merely state the result (which essentially repeats [8, Theorem 3.2], itself a generalization of [5].)

Theorem 4.2 *With $\mathcal{V}$ and $\mathcal{A}$ as before, let $\eta, \varepsilon : f \dashv f^* : A \to B$ have ε invertible, and let $fe.\langle A, A\rangle\eta$ be invertible. Then we can enrich the adjunction $f \dashv f^*$ to an adjunction $\eta, \varepsilon : (f, \bar{f}) \dashv (f^*, \overline{f^*}) : (A, s, \hat{s}, \bar{s}) \to (B, e)$ in* Ps-$\langle B, B\rangle$-Alg_l.

5 The monoid $\langle f, f^*\rangle : \langle A, A\rangle \to \langle B, B\rangle$ as an endo-object

Before turning to the applications of the above to transport of structure, we revisit our central results in Theorem 3.5 and in the prologue to Theorem 4.1, to cast a new light on them. The first of these asserts that, under the conditions of the theorem, the map $w = \langle f, f^*\rangle : \langle A, A\rangle \to \langle B, B\rangle$ underlies a lax monoid map $(w, w^\circ, \widetilde{w})$. By Section 2, such a lax monoid map is the same thing as a monoid in the monoidal 2-category Colax$[\mathbf{2}, \mathcal{V}]$. Now probably the most direct way of showing an object of a monoidal 2-category to admit a monoid structure is to exhibit it as an "object $\langle\langle A, A\rangle\rangle$ of endomorphisms" in the context of an action admitting the adjunction Φ of (1.1); in this way we saw $\langle A, A\rangle$ to be a monoid in Section 1, and $\{f, f\}$ in Section 2: the latter involving the action of $\mathcal{V}$ on Lax$[\mathbf{2}, \mathcal{V}]$ and the adjunction (2.1). Of course $\mathcal{V}$ also acts on Colax$[\mathbf{2}, \mathcal{A}]$ in the dual fashion, with a right adjoint say $\{f, g\}'$, so that a strict monoid map $T \to \{f, f\}'$ corresponds to a lax map $(f, \bar{f})$ of T-algebras. These three are all examples of monoids in $\mathcal{V}$; and the question suggests itself whether the monoid $(w, w^\circ, \widetilde{w})$ in Colax$[\mathbf{2}, \mathcal{V}]$, enriching $w = \langle f, f^*\rangle : \langle A, A\rangle \to \langle B, B\rangle$, is an object of endomorphisms for some suitable action.

To this end, we consider an action of Colax$[\mathbf{2}, \mathcal{V}]$ on Colax$[\mathbf{2}, \mathcal{A}]$ which extends the action above of $\mathcal{V}$ on Colax$[\mathbf{2}, \mathcal{A}]$. The 2-functor $* :$ Colax$[\mathbf{2}, \mathcal{V}] \times$ Colax$[\mathbf{2}, \mathcal{A}] \to$ Colax$[\mathbf{2}, \mathcal{A}]$ on objects sends $(\rho : T \to S, g : A \to B)$ to $\rho g : TA \to SB$; on morphisms it sends $((\alpha, \lambda, \beta), (a, \theta, b))$ to $(\alpha a, \lambda\theta, \beta b)$; and on 2-cells it sends $((\gamma, \delta), (\xi, \eta))$ to $(\gamma\xi, \delta\eta)$. That this is indeed an action is immediate. Consider now what it is to give a morphism $(a, \theta, b) : \rho g \to h$, as in

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "\rho g"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & C \arrow[d, "h"] \\ SB \arrow[r, "b"'] & D. \end{tikzcd} \tag{5.1}$$

It comes to giving the images (α, φ, β) of (a, θ, b) under the isomorphism Φ, as in

$$\begin{tikzcd}
T \arrow[r, "\alpha"] \arrow[dd, "\rho"'] & \langle A,C\rangle \arrow[dr, "{\langle A,h\rangle}"] & \\
 & \varphi\Uparrow & \langle A,D\rangle. \\
S \arrow[r, "\beta"'] & \langle B,D\rangle \arrow[ur, "{\langle g,D\rangle}"'] &
\end{tikzcd} \tag{5.2}$$

In general, this is not of the form $\rho \to \sigma$ for some object σ of Colax$[\mathbf{2}, \mathcal{V}]$: the present action does not admit a right adjoint like that in (1.1). Suppose however that the morphism g is a right adjoint — say $g = f^*$ where, as before, we have $\eta, \varepsilon : f \dashv f^* = g : A \to B$ in $\mathcal{A}$. These same data constitute, in $\mathcal{A}^{\mathrm{op}}$, an adjunction $\eta, \varepsilon : g \dashv f : A \to B$, which is sent by the 2-functor $\langle -, D\rangle : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ to the adjunction

$$\langle \eta, D\rangle, \langle \varepsilon, D\rangle : \langle g, D\rangle \dashv \langle f, D\rangle : \langle A, D\rangle \to \langle B, D\rangle$$

in $\mathcal{V}$. Accordingly, to give the $\varphi : \langle g, D\rangle\beta\rho \to \langle A, h\rangle\alpha$ of (5.2) is equally (see [13]) to give a 2-cell $\psi : \beta\rho \to \langle f, D\rangle\langle A, h\rangle\alpha = \langle f, h\rangle\alpha$, where ψ is given in terms of φ as the pasting composite

$$\begin{tikzcd}
T \arrow[r, "\alpha"] \arrow[dd, "\rho"'] & \langle A,C\rangle \arrow[dr, "{\langle A,h\rangle}"] & & \\
 & \varphi\Uparrow & \langle A,D\rangle \arrow[dr, "{\langle f,D\rangle}"] \arrow[d, phantom, "\Uparrow\langle\eta,D\rangle"] & \\
S \arrow[r, "\beta"'] & \langle B,D\rangle \arrow[ur, "{\langle g,D\rangle}"] \arrow[rr, "1"'] & {} & \langle B,D\rangle,
\end{tikzcd} \tag{5.3}$$

with a similar formula giving φ in terms of ψ. The passage from φ to ψ is clearly 2-natural in the ρ and in the h of (5.1), so that we have a 2-natural isomorphism

$$\mathrm{Colax}[\mathbf{2}, \mathcal{A}](\rho g, h) \cong \mathrm{Colax}[\mathbf{2}, \mathcal{V}](\rho, \langle f, h\rangle).$$

Thus, although we have for a general g no adjunction of the form

$$\mathrm{Colax}[\mathbf{2}, \mathcal{A}](\rho g, h) \cong \mathrm{Colax}[\mathbf{2}, \mathcal{V}](\rho, [g, h]),$$

yet we do have such a $[g, h]$ when g is of the form f^*, it being given by $[g, h] = \langle f, h\rangle$; more succinctly, we have $[f^*, h] = \langle f, h\rangle$. In particular, $\langle f, f^*\rangle : \langle A, A\rangle \to \langle B, B\rangle$ is the value of $[f^*, f^*]$, which is a monoid in Colax$[\mathbf{2}, \mathcal{V}]$ because it has the form of an object of endomorphisms.

We now turn to our second main result, namely the observation in Section 4 that $f : B \to A$ underlies a colax map of lax $\langle A, A\rangle$-algebras, or equivalently that $f^* : A \to B$ underlies a lax map of such algebras. We can approach the latter, too, in terms of the present action of Colax$[\mathbf{2}, \mathcal{V}]$ on Colax$[\mathbf{2}, \mathcal{A}]$.

We may identify an object T of $\mathcal{V}$ with the object $1_T : T \to T$ of Colax$[\mathbf{2}, \mathcal{V}]$; and a monad structure on T gives rise to one on 1_T, with the same notation. To give a [lax] action of such a monad on an object $g : A \to B$ of Colax$[\mathbf{2}, \mathcal{A}]$ is clearly to give [lax] actions of T on A and on B, along with a lax map $(g, \overline{g}) : A \to B$ of such [lax] T-algebras; and the same is true when we omit each "[lax]". Accordingly to enrich $f^* : A \to B$ to a lax map of lax $\langle A, A\rangle$-algebras, we have only to provide in Colax$[\mathbf{2}, \mathcal{A}]$ a lax action $\langle A, A\rangle f^* \to f^*$, or equivalently to provide in Colax$[\mathbf{2}, \mathcal{V}]$ a lax monoid map $\langle A, A\rangle \to [f^*, f^*]$. Recall that $\langle A, A\rangle$ here stands for $1 : \langle A, A\rangle \to \langle A, A\rangle$, while $[f^*, f^*] = \langle f, f^*\rangle : \langle A, A\rangle \to \langle B, B\rangle$. We simplify now by writing C for $\langle A, A\rangle$ and D for $\langle B, B\rangle$, with $k : C \to D$ for $\langle f, f^*\rangle$; recall that $C = \langle A, A\rangle$ is a monoid (C, i, m) in $\mathcal{V}$, while $D = \langle B, B\rangle$ is a monoid (D, j, n), and $k : C \to D$ is a monoid in Colax$[\mathbf{2}, \mathcal{V}]$, or equally a lax map $(k, k^\circ, \widetilde{k})$ of monoids in $\mathcal{V}$, where k° and $\widetilde{k}$ have the forms

$$\begin{tikzcd} & C \arrow[d, "k"] \\ 1 \arrow[ur, "i"] \arrow[dr, "j"'] \arrow[r, phantom, "\overset{k^\circ}{\Rightarrow}"] & {} \\ & D, \end{tikzcd} \qquad \begin{tikzcd} CC \arrow[r, "m"] \arrow[d, "kk"'] \arrow[dr, phantom, "\overset{\widetilde{k}}{\Rightarrow}"] & C \arrow[d, "k"] \\ DD \arrow[r, "n"'] & D. \end{tikzcd}$$

What we seek is a lax monoid map $(h, h^\circ, \widetilde{h}) : 1_C \to k$ in Colax$[\mathbf{2}, \mathcal{V}]$. For h we take the map $(1, id, k)$ as in

$$\begin{tikzcd} C \arrow[r, "1"] \arrow[d, "1"'] & C \arrow[d, "k"] \\ C \arrow[r, "k"'] & D. \end{tikzcd}$$

Next, h° has to be a 2-cell

from $\begin{tikzcd} 1 \arrow[r, "i"] \arrow[d, "1"'] \arrow[dr, phantom, "\overset{k^\circ}{\Rightarrow}"] & C \arrow[d, "k"] \\ 1 \arrow[r, "j"'] & D \end{tikzcd}$ to $\begin{tikzcd} 1 \arrow[r, "i"] \arrow[d, "1"'] & C \arrow[r, "1"] \arrow[d, "1"] & C \arrow[d, "k"] \\ 1 \arrow[r, "i"'] & C \arrow[r, "k"'] & D, \end{tikzcd}$

for which we take the pair (id, k°). Similarly the 2-cell $\widetilde{h}$

from $\begin{tikzcd} CC \arrow[r, "11"] \arrow[d, "11"'] & CC \arrow[r, "m"] \arrow[d, "kk"] \arrow[dr, phantom, "\overset{\widetilde{k}}{\Rightarrow}"] & C \arrow[d, "k"] \\ CC \arrow[r, "kk"'] & DD \arrow[r, "n"'] & D \end{tikzcd}$ to $\begin{tikzcd} CC \arrow[r, "m"] \arrow[d, "11"'] & C \arrow[r, "1"] \arrow[d, "1"] & C \arrow[d, "k"] \\ CC \arrow[r, "m"'] & C \arrow[r, "k"'] & D \end{tikzcd}$

is provided by the pair $(id, \widetilde{k})$. The easy verification that $(h, h^\circ, \widetilde{h})$ is indeed a lax map of monoids provides us with the desired enrichment $(f^*, \overline{f^*})$ of f^* to a lax map of lax $\langle A, A\rangle$-algebras.

We carry this analysis no further, since the calculations which give the explicit values of $\langle f, f^*\rangle : \langle A, A\rangle \to \langle B, B\rangle$ and of $(f^*, \overline{f^*}) : A \to B$ are no shorter if we begin with these present observations than were our calculations above based on the observations of Sections 3 and 4.

6 Transport of structure along an equivalence

We restrict ourselves here to the important case where the adjunction $\eta, \varepsilon : f \dashv f^* : A \to B$ is an *equivalence* in $\mathcal{A}$; the reader interested in the more general situations of the previous section will easily make the necessary extensions. We are used in universal algebra to the transport of structure along an isomorphism: if T is a monad on the mere category $\mathcal{A}$, if $a : TA \to A$ is an action of T on $A \in \mathcal{A}$, and if $f : B \to A$ is an isomorphism in $\mathcal{A}$ with inverse $f^* : A \to B$, there is a unique action $b : TB \to B$ of T on B for which f becomes an isomorphism of T-algebras — namely that given by $b = f^*a.Tf$. What replaces this result when $\mathcal{A}$ is a 2-category, the monad $T = (T, i, m)$ is a 2-monad, and the isomorphism $f : B \to A$ is replaced by the adjoint equivalence $f \dashv f^*$? We make use of the results above, taking for $\mathcal{V}$ the monoidal 2-category $[\mathcal{A}, \mathcal{A}]$, and supposing $\mathcal{A}$ complete and locally small, so that we have the adjunction $\Phi : \mathcal{A}(XA, B) \cong [\mathcal{A}, \mathcal{A}](X, \langle A, B\rangle)$. It is well known — see for instance [13] — that a strict map $\alpha : T \to S$ of monads on $\mathcal{A}$ (2-monads, of course, since $\mathcal{A}$ is a 2-category) induces a 2-functor $\alpha^* : S\text{-Alg} \to T\text{-Alg}$, commuting with the forgetful 2-functors to $\mathcal{A}$, and restricting to a 2-functor $S\text{-Alg}_s \to T\text{-Alg}_s$. We need a less strict analogue of this: we show that a pseudo map $\alpha = (\alpha, \alpha^\circ, \widetilde{\alpha}) : T \to S$ of monads induces a 2-functor $\alpha^* : \text{Ps-}S\text{-Alg} \to \text{Ps-}T\text{-Alg}$, again commuting with the forgetful 2-functors to $\mathcal{A}$, and again admitting a restriction $\alpha^*_s : \text{Ps-}S\text{-Alg}_s \to \text{Ps-}T\text{-Alg}_s$ to the *strict* maps of pseudo algebras. First, a pseudo action $s = (s, \widehat{s}, \overline{s})$ of S on B corresponds as in Section 2 to a pseudo map $\sigma = (\sigma, \sigma^\circ, \widetilde{\sigma}) : S \to \langle B, B\rangle$ of monads, which composes with $\alpha = (\alpha, \alpha^\circ, \widetilde{\alpha})$ to give a pseudo map $\rho = (\rho, \rho^\circ, \widetilde{\rho}) : T \to \langle B, B\rangle$, corresponding to a pseudo action $r = (r, \widehat{r}, \overline{r})$ of T on B; and $(B, r, \widehat{r}, \overline{r})$ is the image under α^* of $(B, s, \widehat{s}, \overline{s})$. Next, given a morphism $f : B \to B'$ where $(B, s, \widehat{s}, \overline{s})$ and $(B', s', \widehat{s}', \overline{s}')$ are pseudo S-algebras, to give f the structure of a morphism (that is, a pseudo morphism) of pseudo S-algebras is to give a pseudo monad map $\gamma : S \to \langle\langle f, f\rangle\rangle$ with $\chi\gamma = \sigma$ and $\chi'\gamma = \sigma'$, where

$$\begin{tikzcd} & \langle B, B\rangle \arrow[dr, "{\langle B,f\rangle}"] & \\ \langle\langle f, f\rangle\rangle \arrow[ur, "\chi"] \arrow[dr, "\chi'"'] \arrow[rr, phantom, "\mu\Uparrow"] & & \langle B, B'\rangle \\ & \langle B', B'\rangle \arrow[ur, "{\langle f,B'\rangle}"'] & \end{tikzcd} \tag{6.1}$$

is the iso-comma object in $[\mathcal{A}, \mathcal{A}]$; and then the composite pseudo monad-map $\gamma\alpha : T \to \langle\langle f, f\rangle\rangle$ corresponds to a morphism $(f, \overline{\overline{f}}) : \alpha^*(B, s, \widehat{s}, \overline{s}) \to \alpha^*(B', s', \widehat{s}', \overline{s}')$ which is the desired $\alpha^*(f, \overline{f})$. The isomorphism $\overline{f}$ is an identity — that is, the morphism $(f, \overline{f})$ is strict — when $\gamma : S \to \langle\langle f, f\rangle\rangle$ factorizes through the canonical $\delta : (f, f) \to \langle\langle f, f\rangle\rangle$, where (f, f) is the pullback

$$\begin{tikzcd} & \langle B, B\rangle \arrow[dr, "{\langle B,f\rangle}"] & \\ (f, f) \arrow[ur] \arrow[dr] & & \langle B, B'\rangle\ ; \\ & \langle B', B'\rangle \arrow[ur, "{\langle f,B'\rangle}"'] & \end{tikzcd}$$

in which case $\gamma\alpha$ factorizes through δ. Thus α^* does indeed send strict morphisms to strict morphisms, and we have established the 2-functors $\alpha^* : \text{Ps-}S\text{-Alg} \to \text{Ps-}T\text{-Alg}$ and $\alpha^*_s : \text{Ps-}S\text{-Alg}_s \to \text{Ps-}T\text{-Alg}_s$.

With these tools at hand, we return to the question of transporting structure: let us have the adjoint equivalence $\eta, \varepsilon : f \dashv f^* : A \to B$ in $\mathcal{A}$, and a pseudo action $(a, \widehat{a}, \overline{a})$ of T on A. This corresponds to a pseudo monad-map $(\alpha, \alpha^\circ, \widetilde{\alpha}) : T \to \langle A, A\rangle$, where $\alpha : T \to \langle A, A\rangle$ is the image under Φ of $a : TA \to A$. This $\alpha = (\alpha, \alpha^\circ, \widetilde{\alpha})$ induces a 2-functor $\alpha^* : \text{Ps-}\langle A, A\rangle\text{-Alg} \to \text{Ps-}T\text{-Alg}$ which carries the adjoint equivalence $(f, \overline{f}) \dashv (f^*, \overline{f^*})$ in $\text{Ps-}\langle A, A\rangle\text{-Alg}$ of Theorem 4.1 to an adjoint equivalence

$$\eta, \varepsilon : \alpha^*(f, \overline{f}) \dashv \alpha^*(f^*, \overline{f^*}) : \alpha^*(A, e) \to \alpha^*(B, t, \widehat{t}, \overline{t})$$

in Ps-T-Alg. Since the strict action $e : \langle A, A\rangle A \to A$ corresponds to the identity morphism $\langle A, A\rangle \to \langle A, A\rangle$, the pseudo T-algebra $\alpha^*(A, e)$ is the $(A, a, \widehat{a}, \overline{a})$ we started with. Calculating $\alpha^*(B, t, \widehat{t}, \overline{t})$ is also straightforward, but we do it explicitly below only for the important case where A is a strict T-algebra, with $\widehat{a}$ and $\overline{a}$ identities. In fact there is a theoretical sense in which it suffices to study this case: it is shown in [3] that, under modest conditions on T, a pseudo T-algebra is just a T'-algebra for another monad T'; we shall return to this observation below, in connection with *flexible* monads.

We take (A, a), then, to be a strict T-algebra, observing that $\alpha : T \to \langle A, A\rangle$, as the image under Φ of $a : TA \to A$, satisfies $e.\alpha A = a$. (Note that we have earlier used i for the unit and m for the multiplication not only of $\langle A, A\rangle$ but also of T; but continuing to do so will lead to no confusion.) Let us write $(B, b, \widehat{b}, \overline{b})$ for the T-algebra $\alpha^*(B, t, \widehat{t}, \overline{t})$. Since $b : TB \to B$ is the composite $t.\alpha B$, while t is given by the composite

$$\langle A, A\rangle B \xrightarrow{\langle A, A\rangle f} \langle A, A\rangle A \xrightarrow{e} A \xrightarrow{f^*} B\ ,$$

the naturality of α along with the equation $e.\alpha A = a$ gives b as the composite

$$TB \xrightarrow{Tf} TA \xrightarrow{a} A \xrightarrow{f^*} B.$$

Similarly $\widehat{b}$ is simply $\eta : 1 \to f^*f = f^*a.Tf.iB$, while by the description (3.4) of $\overline{t}$ we see that $\overline{b}$ is given by

$$\begin{tikzcd}
TTB \arrow[rrr, "mB"] \arrow[d, "TTf"'] & & & TB \arrow[d, "Tf"] \\
TTA \arrow[rrr, "mA"] \arrow[d, "Ta"'] & & & TA \arrow[d, "a"] \\
TA \arrow[rrr, "a"] \arrow[d, "Tf^*"'] \arrow[dr, "1"] & & & A \arrow[d, "f^*"] \\
TB \arrow[r, "Tf"'] \arrow[ur, phantom, "\Uparrow T\varepsilon"] & TA \arrow[r, "a"'] & A \arrow[r, "f^*"'] & B.
\end{tikzcd} \tag{6.2}$$

Similarly, $\alpha^*(f, \overline{f}) = (f, \overline{\overline{f}})$ and $\alpha^*(f^*, \overline{f^*}) = (f^*, \overline{\overline{f^*}})$, where $\overline{\overline{f}}$ is given by $\varepsilon^{-1}a.Tf : a.Tf \to ff^*a.Tf (= fb)$, and $\overline{\overline{f^*}}$ by $f^*a.T\varepsilon : (b.Tf^* =) f^*a.Tf.Tf^* \to f^*a$. Summing up, we have:

Theorem 6.1 *Given the equivalence $\eta, \varepsilon : f \dashv f^* : A \to B$ in the complete and locally small 2-category $\mathcal{A}$, and an algebra (A, a) for the monad $T = (T, i, m)$ on $\mathcal{A}$, the equivalence enriches to an equivalence*

$$\eta, \varepsilon : (f, \overline{\overline{f}}) \dashv (f^*, \overline{\overline{f^*}}) : (A, a) \to (B, b, \widehat{b}, \overline{b})$$

in Ps-T-Alg, *where $\widehat{b} = \eta$ and $\overline{b}$ is given by $f^*a.T\varepsilon.Ta.T^2f$ as in* (6.2), *and where $\overline{\overline{f}} = \varepsilon^{-1}a.Tf$ and $\overline{\overline{f^*}} = f^*a.T\varepsilon$.*

Consider the case where $\mathcal{A} = \mathbf{Cat}$ and $T = (T, i, m)$ is the 2-monad whose algebras are the strict monoidal categories. A consequence of the coherence theorem for monoidal categories is that for any monoidal category B there is a strict monoidal category A — that is, a strict T-algebra (A, a) — and an equivalence $\eta, \varepsilon : f \dashv f^* : A \to B$ with f and f^* strong monoidal functors. Now suppose that B is a skeleton of the category of countable sets, equipped with the cartesian monoidal structure; then the monoidal structure on A is again the cartesian one, for some choice of binary products and terminal object. If the equivalence $\eta, \varepsilon : f \dashv f^*$ underlay an equivalence in T-Alg, then the monoidal structure on B would be both cartesian and strict. By an argument due to Isbell [14, p.160], however, this is impossible.

In general, then, it is not possible to enrich an adjoint equivalence to one in T-Alg. However such an enrichment does exist when the monad T is *flexible* — a notion, originally introduced in [8], which we now recall. First note that, in the present case where $\mathcal{V} = [\mathcal{A}, \mathcal{A}]$, so that a monoid in $\mathcal{V}$ is a monad on $\mathcal{A}$, the 2-categories $\mathrm{Mon}_l\,\mathcal{V}$, $\mathrm{Mon}_p\,\mathcal{V}$, and $\mathrm{Mon}\,\mathcal{V}$ of Section 2 are conveniently renamed $\mathrm{Mnd}_l\,\mathcal{A}$, $\mathrm{Mnd}_p\,\mathcal{A}$, and $\mathrm{Mnd}\,\mathcal{A}$. In particular we have the inclusion 2-functor $J : \mathrm{Mnd}\,\mathcal{A} \to \mathrm{Mnd}_p\,\mathcal{A}$; and it was shown in Blackwell's thesis [3] that a partial left adjoint to J is defined at the monad T if $\mathcal{A}$ is cocomplete and T has some rank. (An endofunctor T of $\mathcal{A}$ is said to have rank κ, where κ is a regular cardinal, if T preserves κ-filtered colimits.) To say that the partial adjoint is defined at T means, of course, that there is a pseudo map $p : T \to T'$ of monads on $\mathcal{A}$ such that, for any monad S on $\mathcal{A}$, the 2-functor $\mathrm{Mnd}\,\mathcal{A}(T', S) \to \mathrm{Mnd}_p\,\mathcal{A}(T, S)$ given by composition with p is an isomorphism of 2-categories. In more elementary terms, every pseudo map $g : T \to S$ is of the form hp for a unique strict map $h : T' \to S$, and every monad 2-cell $\alpha : hp \to h'p$, where the monad maps h and h' are strict, is βp for a unique monad 2-cell $\beta : h \to h'$.

In particular, there is a unique strict monad map $q : T' \to T$ for which $qp = 1_T$. Even before Blackwell's result, Kelly had shown in [9] that, whenever the partial left adjoint is defined at T, there is an invertible 2-cell $\rho : 1_{T'} \cong pq$ with $\rho p = id$ and $q\rho = id$, so that we have in $\mathrm{Mnd}_p\,\mathcal{A}$ the equivalence

$$\rho, id : q \dashv p : T \to T'.$$

Taking $S = \langle B, B\rangle$ in the universal property of $p : T \to T'$ shows that to give a pseudo action of T on B is just to give a strict action of T' on B. And taking for S the $\langle\!\langle f, f\rangle\!\rangle$ of (6.1) shows that enriching $f : B \to B'$ to a morphism $(f, \overline{f})$ of pseudo T-algebras is the same as enriching it to a morphism of T'-algebras. Accordingly we have an isomorphism of 2-categories

$$\text{Ps-}T\text{-Alg} \cong T'\text{-Alg}$$

which commutes with the underlying 2-functors to $\mathcal{A}$, and which restricts to an isomorphism of 2-categories

$$\text{Ps-}T\text{-Alg}_s \cong T'\text{-Alg}_s;$$

moreover, by a similar argument, it extends to an isomorphism

$$\text{Ps-}T\text{-Alg}_l \cong T'\text{-Alg}_l.$$

This is the intent of our earlier remark that a pseudo T-algebra is just a T'-algebra for a certain monad T'. The strict monad map $q : T' \to T$ induces a 2-functor $q^* : T\text{-Alg} \to T'\text{-Alg}$, restricting to $T\text{-Alg}_s \to T'\text{-Alg}_s$ and extending to

T-Alg$_l \to T'$-Alg$_l$. If we identify T'-Alg with Ps-T-Alg via the isomorphism above, $q^* : T$-Alg $\to T'$-Alg is of course nothing but the inclusion T-Alg $\to$ Ps-T-Alg.

The notion of flexibility was introduced by Kelly in [8] as a property of 2-monads, which is the case of interest here; later it was generalized to be a property of algebras for a 2-monad, of which a 2-monad itself is a special case — see [4]; in another special case introduced there, flexibility is a property of a *weight* for **Cat**-enriched limits, the corresponding *flexible limits* being studied in [2].

Supposing the monad T on $\mathcal{A}$ to be such that $p : T \to T'$ exists as above, with $q : T' \to T$ the unique strict map for which $qp = 1$, we say that T is *flexible* if there is some *strict* map $r : T \to T'$ for which $qr = 1$. Since we have $\rho : 1 \cong pq$ as above, we have $\rho r : r \cong pqr = p$; so that besides the equivalence $\rho, id : q \dashv p : T \to T'$ in $\mathrm{Mnd}_p \mathcal{A}$, we now have an equivalence

$$\sigma, id : q \dashv r : T \to T' \tag{6.3}$$

in $\mathrm{Mnd}\,\mathcal{A}$ itself. One easily sees that (supposing the left adjoints to exist) the monad T' is always flexible, and in fact a monad S is flexible precisely when it is a retract in $\mathrm{Mnd}\,\mathcal{A}$ of some T'; the details can be found in [4]. For a flexible T, the equivalence of 2-categories

$$q^* \dashv r^* : T'\text{-Alg} \cong \text{Ps-}T\text{-Alg} \to T\text{-Alg}$$

induced by the equivalence (6.3) restricts of course to an equivalence

$$q^* \dashv r^* : T'\text{-Alg}_s \cong \text{Ps-}T\text{-Alg}_s \to T\text{-Alg}_s$$

between the Eilenberg-Moore 2-categories for the monads.

We can now give our main result on flexible monads:

Theorem 6.2 *Let $\eta, \varepsilon : f \dashv f^* : A \to B$ be an equivalence in the complete, cocomplete, and locally-small 2-category $\mathcal{A}$, let $T = (T, i, m)$ be a flexible monad on $\mathcal{A}$ having some rank, let $a : TA \to A$ be an action (meaning a strict one) of T on A, and let $qr = 1_T$, where $q : T' \to T$ is as above and r is a strict monad map. Then the given equivalence has an enrichment to an equivalence*

$$\eta, \varepsilon : (f, \check{f}) \dashv (f^*, \check{f}^*) : (A, a) \to (B, \check{b})$$

in T-Alg.

Proof Identifying Ps-T-Alg with the isomorphic T'-Alg, we find the desired equivalence as the image under $r^* :$ Ps-T-Alg $\to T$-Alg of the equivalence of Theorem 6.1. Note here that the (A, a) in the equivalence of Theorem 6.1 really denotes $q^*(A, a)$ — the T-algebra (A, a) seen as a pseudo T-algebra — and that $r^*q^*(A, a)$ is (A, a) itself, since $qr = 1$. □

Remark 6.3 That the scope of the theorem is extremely broad will be clear from the forthcoming article [11], where it is shown that a monad T on **Cat** is flexible if the structure of a T-algebra can be presented by operations and equations, in the sense of [12], in such a way that there are no equations between objects, only between maps; with similar results for many other 2-categories in place of **Cat**.

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On the Cyclic Homology of Hopf Crossed Products

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Abstract. We consider Hopf crossed products of the the type $A\#_\sigma\mathcal{H}$, where $\mathcal{H}$ is a cocommutative Hopf algebra, A is an $\mathcal{H}$-module algebra and σ is a "numerical" convolution invertible 2-cocycle on $\mathcal{H}$. we give an spectral sequence that converges to the cyclic homology of $A\#_\sigma\mathcal{H}$ and identify the E^1 and E^2 terms of the spectral sequence.

1 Introduction

A celebrated problem in noncommutative geometry, more precisely in cyclic homology theory, is to compute the cyclic homology of a crossed product algebra. The interest in this problem stems from the fact that, according to a guiding principle in noncommutative geometry [3], crossed products play the role of "noncommutative quotients" in situations where the usual set theoretic quotients are ill behaved. For example, when a (locally compact) group G acts on a locally compact Hausdorff space X, the quotient space X/G may not be well behaved, e.g. may not be even a Hausdorff space. The crossed product (C^*-) algebra $C_0(X) \ltimes G$, however, is a good replacement for X/G [3]. In fact, if the action of G is free and proper, then by a theorem of Rieffel [16] the C^*-algebra of continuous functions vanishing at infinity on X/G, denoted by $C_0(X/G)$ is in a suitable C^*-algebraic sense, Morita equivalent to the crossed product algebra $C_0(X) \ltimes G$. Since K-Theory, Hochschild homology and cyclic homology are Morita invariant functors, replacing the commutative algebra $C_0(X/G)$ by the noncommutative algebra $C_0(X) \ltimes G$ results in no loss of information.

It is therefore natural and desirable to develop tools to compute the cyclic homology of crossed product algebras. Most of the results obtained so far are

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concerned with the action of groups on algebras [7, 8, 15]. For Hopf algebra crossed products, [9] gives a complete answer for Hochschild homology but it is not clear how to extend its method to cyclic homology. If one is only interested in smash products as opposed to crossed products, one can find a complete answer in [1] in terms of a spectral sequence converging to the cyclic homology of the smash product algebra $A\#\mathcal{H}$.

The goal of this article is to extend the results of [1] to Hopf algebra crossed products. Due to the fact that very complicated formulas appear in our constructions, however, we had to assume that the Hopf algebra is cocommutative and the 2-cocycle takes values in the ground field. Under these conditions, we give a spectral sequence for the cyclic homology of a crossed product algebra $A\underset{\sigma}{\#}\mathcal{H}$, when the cocycle σ is convolution invertible and takes values in the ground field k. The method of proof is similar to the one used in [1] which is based on the generalized cyclic Eilenberg-Zilber theorem of [8]. Though we think the same method should apply to arbitrary crossed products $A\underset{\sigma}{\#}\mathcal{H}$ (with convolution invertible cocycles), due to technical difficulties we are not able to verify this.

One of our main motivations to consider cocycle crossed products is to find simple methods to compute the cyclic cohomology of " noncommutative toroidal orbifolds" considered in [12, Sec. 9] and [6]. These examples are suggested by applications of noncommutative geometry to string theory and M(atrix) theory. In these examples one considers algebras of the type $B^d_{\theta,\sigma} = A^d_\theta\underset{\sigma}{\#}\mathbb{C}G$, where G is a finite group acting by automorphisms on the noncommutative d-dimensional torus A^d_θ and $\sigma \in H^2(G, U(1))$ is a group 2-cocycle on G. Corollary 3.8 shows that the cyclic cohomology of $B^d_{\theta,\sigma}$ can always be computed from a cyclic complex much simpler than the original cyclic complex of the algebra $B^d_{\theta,\sigma}$.

2 Preliminaries

In this paper we work over a fixed ground field k. All algebras are unital associative algebras over k and all modules are unitary. The unadorned tensor product $\otimes$ means tensor product over k. We denote the coproduct of a Hopf algebra by Δ, the counit by ϵ and the antipode by S. We use Sweedler's notation and write $\Delta(h) = h^{(1)} \otimes h^{(2)}$ to denote the coproduct, where summation is understood. Similarly, we write $\Delta^{(n)}(h) = h^{(1)} \otimes h^{(2)} \otimes \ldots \otimes h^{(n+1)}$ to denote the iterated coproducts defined by $\Delta^{(1)} = \Delta$ and $\Delta^{(n+1)} = (\Delta \otimes 1) \circ \Delta^{(n)}$.

We recall the concept of Hopf crossed product, introduced for the first time in [5], and independently [2]. A good reference for this notion is Chapter 7 of [14]. Let $\mathcal{H}$ be a Hopf algebra and A an algebra. Recall from [2] and [5] that a *weak action* of $\mathcal{H}$ on A is a linear map $\mathcal{H} \otimes A \longrightarrow A$, $h \otimes a \to h(a)$ such that, for all $h \in \mathcal{H}$, and $a, b \in A$

1) $h(ab) = h^{(1)}(a)h^{(2)}(b)$,
2) $h(1) = \epsilon(h)1$,
3) $1(a) = a$.

By an *action* of $\mathcal{H}$ on A, we mean a weak action such that A is an $\mathcal{H}$-module, i.e. for all $h, l \in \mathcal{H}$ and $a \in A$ we have $h(l(a)) = hl(a)$. In the latter case we say A is an *$\mathcal{H}-$module algebra.*

Let A be an $\mathcal{H}$-module algebra. The *smash product* $A\#\mathcal{H}$ of A and $\mathcal{H}$ is an associative algebra whose underlying vector space is $A\otimes\mathcal{H}$ and whose multiplication is defined by

$$(a \otimes h)(b \otimes l) = ah^{(1)}(b) \otimes h^{(2)}l.$$

If, on the other hand, we have only a weak action of $\mathcal{H}$ on A the above formula does not define an associative multiplication, and a modification is needed. Given a linear map $\sigma : \mathcal{H} \otimes \mathcal{H} \to A$ one defines a (not necessarily unital or associative) multiplication on $A \otimes \mathcal{H}$ by [2, 5]

$$(a \otimes h)(b \otimes l) = ah^{(1)}(b)\sigma(h^{(2)}, l^{(1)}) \otimes h^{(3)}l^{(2)}.$$

It can be shown that the above formula defines an associative product with $1 \otimes 1$ as its unit, if and only if σ and the weak action enjoy the following properties:

1) (Normality) For all $h \in \mathcal{H}$, $\sigma(h, 1) = \sigma(1, h) = \epsilon(h)1$.
2) (Cocycle property) For all $h, l, m \in \mathcal{H}$,
$\sum h^{(1)}(\sigma(l^{(1)}, m^{(1)}))\sigma(h^{(2)}, l^{(2)}m^{(2)}) = \sum \sigma(h^{(1)}, l^{(1)})\sigma(h^{(2)}l^{(2)}, m)$,
3) (Twisted module property) For all $h, l \in \mathcal{H}$ and $a \in A$,
$\sum h^{(1)}(l^{(1)}(a))\sigma(h^{(2)}, l^{(2)}) = \sum \sigma(h^{(1)}, l^{(1)})l^{(2)}h^{(2)}(a))$.

The cocycle σ is said to be *convolution invertible* if it is an invertible element of the convolution algebra $Hom_k(\mathcal{H} \otimes \mathcal{H}, A)$. Now assume the Hopf algebra $\mathcal{H}$ is cocommutative, $\sigma : \mathcal{H} \otimes \mathcal{H} \longrightarrow k1_A$ takes values in the ground field k, and σ is invertible. Then it follows that A is an $\mathcal{H}$-module algebra, i.e. the weak action in the above situation is in fact an action. To prove this let $a \in A$ be fixed. Define two functions in $Hom_k(\mathcal{H} \otimes \mathcal{H}, A)$ by $F(h, l) = \sum h^{(1)}(l^{(1)}(a))\sigma(h^{(2)}, l^{(2)})$ and $G(h, l) = \sum \sigma(h^{(1)}, l^{(1)})l^{(2)}h^{(2)}(a)$. Then $F = G$ by the twisted module property of σ, so $F * \sigma^{-1} = G * \sigma^{-1}$. In other word

$$h(l(a)) = F * \sigma^{-1}(h, l) = G * \sigma^{-1}(h, l) = hl(a).$$

One notes that the above proof remains valid when $\mathcal{H}$ is cocommutative and σ takes its values in the center of A, instead of k.

One of the main technical tools used in [1] to derive a spectral sequence for the cyclic homology of smash products is the generalized cyclic Eilenberg-Zilber theorem. This result was first stated in [8] but its first algebraic proof appeared in [11]. The idea of using an Eilenberg-Zilber type theorem to derive a spectral sequence for cyclic homology of smash products (for the action of groups) is due to Getzler and Jones [8]. We find it remarkable that the same idea works in the case of Hopf algebra crossed product (with convolution invertible cocycle). In the following we recall the definitions of (para)cyclic modules, cylindrical modules and state the Eilenberg-Zilber theorem for cylindrical modules.

Recall that a *paracyclic module* is a simplicial k-module $\{M_n\}_{n \geq 0}$, such that the following extra relations are satisfied [7, 8]:

$$\delta_i\tau = \tau\delta_{i-1}, \qquad \delta_0\tau = \delta_n, \qquad 1 \leq i \leq n,$$
$$\sigma_i\tau = \tau\sigma_{i+1}, \qquad \sigma_0\tau = \tau^2\sigma_n, \qquad 1 \leq i \leq n,$$

where $\delta_i : M_n \to M_{n-1}$, $\sigma_i : M_n \to M_{n+1}$, $0 \leq i \leq n$, are faces and degeneracies of the simplicial module $\{M_n\}_{n \geq 0}$ and $\tau : M_n \to M_n$, $n \geq 0$ are k-linear maps. If furthermore we have $\tau^{n+1} = id_{M_n}$, for all $n \geq 0$, then we say that we have a *cyclic module*.

We denote the cyclic module of an associative unital k-algebra A, by $A^\natural$. It is defined by $A^\natural_n = A^{\otimes(n+1)}$, $n \geq 0$, and simplicial and cyclic operations defined by

$$\begin{aligned}
\delta_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \cdots \otimes a_n, \quad 0 \leq i \leq n-1, \\
\delta_n(a_0 \otimes \cdots \otimes a_n) &= a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, \\
\sigma_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \cdots \otimes a_n, \quad 0 \leq i \leq n, \\
\tau_n(a_0 \otimes \cdots \otimes a_n) &= a_n \otimes a_0 \cdots \otimes a_{n-1}.
\end{aligned}$$

To any cyclic module one associates its cyclic homology groups [13, 4]. In particular, the cyclic homology groups of $A^\natural$ are denoted by $HC_n(A)$, $n \geq 0$, and are called cyclic homology of A.

By a *biparacyclic* module we mean a doubly graded sequence of k-modules $\{M_{p,q}\}_{p,q\geq 0}$ such that each row and each column is a paracyclic module and all vertical operators commute with all horizontal operators. In particular a *bicyclic* module is a biparacyclic module such that each row and each column is a cyclic module. We denote the horizontal and vertical operators of a biparacyclic module by $(\delta_i, \sigma_i, \tau)$ and (d_i, s_i, t) respectively. By a *cylindrical module* we mean a biparacyclic module such that for all $p, q \geq 0$,

$$\tau^{p+1} t^{q+1} = id_{M_{p,q}}. \tag{2.1}$$

Given a cylindrical module M, its *diagonal*, denoted by dM, is a cyclic module defined by $(dM)_n = M_{n,n}$ and with simplicial and cyclic operators given by $\delta_i d_i$, $\sigma_i s_i$, and τt. In view of (2.1), it is a cyclic module. The total complex of a cylindrical module, denoted by $Tot(M)$, is a mixed complex with operators given by $b + \bar{b}$ and $B + T\bar{B}$, where $T = 1 - (bB + Bb)$. Here b (resp. $\bar{b}$) and B (resp. $\bar{B}$) are the vertical (resp. horizontal) Hochschild and Connes boundary operators of cyclic modules. Note that it differs from the usual notion of total complex in that we use $B + T\bar{B}$ instead of $B + \bar{B}$. In fact the latter choice won't give us a mixed complex [8]. It can be checked that $Tot(M)$ is a mixed complex. Given a cylindrical or cyclic module M, we denote its normalization by $N(M)$.

The following theorem is the main technical result that enables us to derive spectral sequences for the cyclic homology of crossed product algebras.

Theorem 2.1 (Generalized cyclic Eilenberg-Zilber theorem ([11, 8])) *For any cylindrical module M there is a natural quasi-isomorphism of mixed complexes $f_0 + uf_1 : Tot(N(M)) \longrightarrow N(dM)$, where f_0 is the shuffle map.*

3 A Spectral sequence for Hopf crossed products

Let $\mathcal{H}$ be a cocommutative Hopf algebra, A a left $\mathcal{H}$- module algebra and $\sigma : \mathcal{H} \otimes \mathcal{H} \longrightarrow k1_A$ a two cocycle satisfying the cocycle conditions 1), 2), and 3) in Section 2. We further assume that σ is convolution invertible. We introduce a cylindrical module

$$A\natural_\sigma \mathcal{H} = \{\mathcal{H}^{\otimes(p+1)} \otimes A^{\otimes(q+1)}\}_{p,q\geq 0}$$

with vertical and horizontal simplicial and cyclic operators (δ, σ, τ) and (d, s, t), defined as follows

$$\begin{aligned}
\tau(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_0^{(2)}, \ldots, g_p^{(2)} \mid \\
&\qquad S(g_0^{(1)} \ldots g_p^{(1)})(a_q), a_0, a_1, \ldots, a_{q-1}) \\
\delta_i(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_0, \ldots, g_p \mid a_0, \ldots, a_i a_{i+1}, \ldots, a_q) \quad 0 \le i < q \\
\delta_q^{p,q}(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_0^{(2)}, \ldots, g_p^{(2)} \mid \\
&\qquad S(g_0^{(1)} \ldots g_p^{(2)})(a_q) a_0, a_1 \ldots, a_{q-1}) \\
\sigma_i(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_0, \ldots, g_p \mid a_0, \ldots, a_i, 1, a_{i+1}, \ldots, a_q) \quad 0 \le i \le q \\
t(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_p^{(q+2)}, g_0, \ldots, g_{p-1} \mid g_p^{(1)}(a_0), \ldots, g_p^{(q+1)}(a_q)) \\
d_i(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_0, \ldots, g_i^{(1)} g_{i+1}^{(1)}, \ldots, g_p \mid \\
&\qquad \sigma(g_i^{(2)}, g_{i+1}^{(2)}) a_0, \ldots, a_q) \quad 0 \le i < p \\
d_p(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_p^{(q+2)} g_0^{(1)}, g_1, \ldots, g_{p-1} \mid \\
&\qquad \sigma(g_p^{(q+3)}, g_0^{(2)}) g_p^{(1)}(a_0), \ldots, g_p^{(q+1)}(a_q)) \\
s_i(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_0, \ldots, g_i, 1, g_{i+1}, \ldots, g_p \mid a_0, \ldots, a_q) \quad 0 \le i \le p.
\end{aligned}$$

Theorem 3.1 *Endowed with the above operators, $A\natural_\sigma \mathcal{H}$ is a cylindrical module.*

Proof We should check that every row and every column is a paracyclic module, and vertical operator commutes with each horizontal operator, and in addition the identity (2.1) holds. Since the weak action in our situation is actually an action and the vertical operators are the same as the vertical operators in ([1] Theorem 3.1), we refer the reader to [1] for the proof that the columns form paracyclic modules. To check that the rows are paracyclic modules we need to verify the following identities

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i \qquad i < j \\
s_i s_j &= s_{j+1} s_i \qquad i \le j \\
d_i s_j &= \begin{cases} s_{j-1} d_i & i < j \\ \text{identity} & i = j \text{ or } i = j+1 \\ s_j d_{i-1} & i > j+1. \end{cases}
\end{aligned}$$

$$\begin{aligned}
d_i t_n &= t_{n-1} d_{i-1} \qquad 1 \le i \le n\,, \quad d_0 t_n = d_n \\
s_i t_n &= t_{n+1} s_{i-1} \qquad 1 \le i \le n\,, \quad s_0 t_n = t_{n+1}^2 s_n
\end{aligned}$$

We just check $d_i d_{i+1} = d_i d_i$ and the cylindrical module condition (2.1). The rest can be proved by the same techniques. We have:

$$\begin{aligned}
&d_i d_{i+1}(g_0, \ldots, g_p \mid a_0, \ldots, a_q) \\
&= \; d_i(g_0, \ldots, g_{i+1}^{(1)} g_{i+2}^{(1)}, \ldots g_p \mid \sigma(g_{i+1}^{(2)}, g_{i+2}^{(2)}) a_0, \ldots, a_q) \\
&= \; (g_0, \ldots, g_i^{(1)} g_{i+1}^{(1)} g_{i+2}^{(1)}, \ldots g_p \mid \sigma(g_i^{(2)}, g_{i+1}^{(2)} g_{i+2}^{(2)}) \sigma(g_{i+1}^{(3)}, g_{i+2}^{(3)}) a_0 \ldots, a_q),
\end{aligned}$$

that by using the cocycle property 2) in Section 2 is equal to

$$\begin{aligned}(g_0,\dots,g_i^{(1)}g_{i+1}^{(1)}g_{i+2}^{(1)},\dots g_p \mid \sigma(g_i^{(2)},g_{i+1}^{(2)})\sigma(g_i^{(3)}g_{i+1}^{(3)},g_{i+2}^{(3)})a_0\dots,a_q)\\ = d_id_i(g_0,\dots,g_p \mid a_0,\dots,a_q).\end{aligned}$$

Next we check the cylindrical module condition (2.1). We have:

$$\begin{aligned}
&t^{p+1}\tau^{q+1}(g_0,\dots,g_p \mid a_0,\dots,a_q)\\
&= t^{p+1}\tau^{q}(g_0^{(2)},\dots,g_p^{(2)} \mid S(g_0^{(1)}g_1^{(1)}\dots g_p^{(1)})\cdot a_q, a_0,\dots,a_{q-1})\\
&= t^{p+1}(g_0^{(q+1)},\dots,g_p^{(q+1)} \mid S(g_0^{(q)}\dots g_p^{(q)})\cdot a_0, S(g_0^{(q-1)}\dots g_p^{(q-1)})\cdot a_1,\\
&\qquad\qquad \dots, S(g_0^{(0)}\dots g_p^{(0)})\cdot a_q)\\
&= t^{p}(g_p^{(2q+2)},g_0^{(q+1)},\dots,g_{p-1}^{(q+1)} \mid (g_p^{(q+1)}S(g_0^{(q)}\dots g_p^{(q)}))\cdot a_0,\\
&\qquad (g_p^{(q+2)}S(g_0^{(q-1)}\dots g_p^{(q-1)})\cdot a_1,\dots,(g^{(2q+1)}S(g_0^{(0)}\dots g_p^{(0)}))\cdot a_q)\\
&= t^{p}(g_p^{(2q)},g_0^{(q+1)},\dots,g_{p-1}^{(q+1)} \mid (S(g_0^{(q)}\dots g_{p-1}^{(q)}))\cdot a_0,\\
&\qquad (g_p^{(q)}S(g_0^{(q-1)}\dots g_p^{(q-1)})\cdot a_1,\dots,(g_p^{(2q-1)}S(g_0^{(0)}\dots g_p^{(0)}))\cdot a_q)\\
&= t^{p}(g_p,g_0^{(q+1)},\dots,g_{p-1}^{(q+1)} \mid (S(g_0^{(q)}\dots g_{p-1}^{(q)}))\cdot a_0,\\
&\qquad (S(g_0^{(q-1)}\dots g_{p-1}^{(q-1)})\cdot a_1,\dots,(S(g_0^{(0)}\dots g_{p-1}^{(0)}))\cdot a_q)\\
&= (g_0,\dots,g_p \mid a_0,\dots,a_q).
\end{aligned}$$

The theorem is proved. □

Next we show that the diagonal of the above cylindrical module, $d(A\natural_\sigma\mathcal{H})$, is isomorphic with the cyclic module $(A\#_\sigma\mathcal{H})^\natural$ associated with the crossed product algebra. To this end we define maps $\Phi : (A\#_\sigma\mathcal{H})^\natural \to d(A\natural_\sigma\mathcal{H})$ and $\Psi : d(A\natural_\sigma\mathcal{H}) \to (A\#_\sigma\mathcal{H})^\natural$ by the following formulas

$$\begin{aligned}
&\Phi(a_0\otimes g_0,\dots,a_n\otimes g_n)\\
&= (g_0^{(2)},g_1^{(3)},\dots,g_n^{(n+2)} \mid S(g_0^{(1)}g_1^{(2)}\dots g_n^{(n+1)})\cdot a_0, S(g_1^{(1)}g_2^{(2)}\dots g_n^{(n)})\cdot a_1,\dots,\\
&\qquad S(g_{n-1}^{(1)}g_n^{(2)})\cdot a_{n-1}, S(g_n^{(1)})\cdot a_n),
\end{aligned}$$

$$\begin{aligned}
&\Psi(g_0,\dots,g_n \mid a_0,\dots,a_n)\\
&= ((g_0^{(1)}g_1^{(1)}\dots g_n^{(1)})\cdot a_0\otimes g_0^{(2)}, (g_1^{(2)}\dots g_n^{(2)})\cdot a_1\otimes g_1^{(3)},\dots,g_n^{(n+1)}\cdot a_n\otimes g_n^{(n+2)}).
\end{aligned}$$

Theorem 3.2 *The above maps, Φ, Ψ, are morphisms of cyclic modules and are inverse to one another.*

Proof It is not hard to see that Φ and Ψ are inverse of each other. We just prove that Φ is a cyclic map. We first verify the commutativity of Φ and the cyclic operators, i.e., the relation $t\tau\Phi = \Phi\tau_{A\#_\sigma\mathcal{H}}$. We have:

$$
\begin{aligned}
&(t\tau)\Phi(a_0\otimes g_0,\ldots,a_n\otimes g_n)\\
&= t\tau(g_0^{(2)},g_1^{(3)},\ldots,g_n^{(n+2)} \mid S(g_0^{(1)}g_1^{(2)}\ldots g_n^{(n+1)})\cdot a_0, S(g_1^{(1)}g_2^{(2)}\ldots g_n^{(n)})\cdot a_1,\\
&\qquad \ldots, S(g_{n-1}^{(1)}g_n^{(2)})\cdot a_{n-1}, S(g_n^{(1)})\cdot a_n)\\
&= t(g_0^{(3)},g_1^{(4)},\ldots,g_n^{(n+3)} \mid S(g_0^{(2)}g_1^{(3)}\ldots g_n^{(n+2)})S(g_n^{(1)})\cdot a_n,\\
&\qquad S(g_0^{(1)}g_1^{(2)}\ldots g_n^{(n+1)})\cdot a_0\ldots, S(g_{n-1}^{(1)}g_n^{(2)})\cdot a_{n-1})\\
&= (g_n^{((2n+4)},g_0^{(3)},g_1^{(4)},\ldots,g_{n-1}^{(n+2)} \mid g_n^{(n+3)}S(g_0^{(2)}g_1^{(3)}\ldots g_n^{(n+1)})S(g_n^{(1)})\cdot a_n,\\
&\qquad g_n^{(n+4)}S(g_0^{(1)}g_1^{(2)}\ldots g_n^{(n+1)})\cdot a_0\ldots, g_n^{(2n+3)}S(g_{n-1}^{(1)}g_n^{(2)})\cdot a_{n-1})\\
&= (g_n^{((2n+3)},g_0^{(3)},g_1^{(4)},\ldots,g_{n-1}^{(n+2)} \mid \epsilon(g_n^{(n+2)})S(g_n^{(1)}g_0^{(2)}\ldots g_{n-1}^{(n+1)})\cdot a_n,\\
&\qquad g_n^{(n+3)}S(g_0^{(1)}g_1^{(2)}\ldots g_n^{(n+1)})\cdot a_0,\ldots, g_n^{(2n+2)}S(g_{n-1}^{(1)}g_n^{(2)})\cdot a_{n-1})\\
&= (g_n^{((2n+2)},g_0^{(3)},g_1^{(4)},\ldots,g_{n-1}^{(n+2)} \mid S(g_n^{(1)}g_0^{(2)}\ldots g_{n-1}^{(n+1)})\cdot a_n,\\
&\qquad \epsilon(g_n^{(n+1)})S(g_0^{(1)}g_1^{(2)}\ldots g_{n-1}^{(n)})\cdot a_0,\ldots, g_n^{(2n)}S(g_{n-1}^{(1)}g_n^{(2)})\cdot a_{n-1})\\
&= (g_n^{(2)},g_0^{(3)},g_1^{(4)},\ldots,g_{n-1}^{(n+2)} \mid S(g_n^{(1)}g_0^{(2)}\ldots g_{n-1}^{(n+1)})\cdot a_n,\\
&\qquad S(g_0^{(1)}g_1^{(2)}\ldots g_{n-1}^{(n)})\cdot a_0,\ldots,\ldots, \epsilon(g_n^{(2)})S(g_{n-1}^{(1)})\cdot a_{n-1})\\
&= (g_n^{(2)},g_0^{(3)},\ldots,g_{n-1}^{(n+2)} \mid S(g_n^{(1)}g_0^{(2)}\ldots g_{n-1}^{(n+1)})\cdot a_n,\\
&\qquad S(g_0^{(1)}g_1^{(2)}\ldots g_{n-1}^{(n)})\cdot a_0,\ldots,\ldots, S(g_{n-1}^{(1)})\cdot a_{n-1})\\
&= \Phi(a_n\otimes g_n, a_0\otimes g_0,\ldots,a_{n-1}\otimes g_{n-1}) = \Phi(\tau_{A\#_\sigma\mathcal{H}}(a_0\otimes g_0,\ldots,a_n\otimes g_n)).
\end{aligned}
$$

Next we check the commutativity of Φ and the face operators, i.e. the relation $d_i\delta_i\Phi = \Phi d_i^{A\#_\sigma\mathcal{H}}$. For $0\le i<n$, we have:

$$
\begin{aligned}
&d_i\delta_i\Phi(a_0\otimes g_0,\ldots,a_n\otimes g_n)\\
&= d_i\delta_i(g_0^{(2)},g_1^{(3)},\ldots,g_n^{(n+2)} \mid S(g_0^{(1)}g_1^{(2)}\ldots g_n^{(n+1)})\cdot a_0, S(g_1^{(1)}g_2^{(2)}\ldots g_n^{(n)})\cdot a_1,\\
&\qquad \ldots, S(g_{n-1}^{(1)}g_n^{(2)})\cdot a_{n-1}, S(g_n^{(1)})\cdot a_n)\\
&= d_i((g_0^{(2)},g_1^{(3)},\ldots,g_n^{(n+2)} \mid S(g_0^{(1)}g_1^{(2)}\ldots g_n^{(n+1)})\cdot a_0, S(g_1^{(1)}g_2^{(2)}\ldots g_n^{(n)})\cdot a_1,\\
&\qquad \ldots, S(g_{i+1}^{(1)},\ldots,g_n^{(n+1-i)})\cdot(S(g_i^{(1)})(a_i)a_{i+1}),\\
&\qquad \ldots, S(g_{n-1}^{(1)}g_n^{(2)})\cdot a_{n-1}, S(g_n^{(1)})\cdot a_n))\\
&= ((g_0^{(2)},g_1^{(3)},\ldots,g_i^{(i+2)}g_{i+1}^{(i+3)},\ldots g_n^{(n+2)} \mid \sigma(g_i^{(i+3)},g_{i+1}^{(i+4)})\\
&\quad S(g_0^{(1)}g_1^{(2)}\ldots g_n^{(n+1)})\cdot a_0, S(g_1^{(1)}g_2^{(2)}\ldots g_n^{(n)})\cdot a_1,\ldots, S(g_{i+1}^{(1)}\cdots g_n^{(n+1-i)})\\
&\quad (S(g_i^{(1)})(a_i)a_{i+1}),\ldots, S(g_{n-1}^{(1)}g_n^{(2)})\cdot a_{n-1}, S(g_n^{(1)})\cdot a_n))\\
&= \Phi d_i^{A\#_\sigma\mathcal{H}}(a_0\otimes g_0,\ldots,a_n\otimes g_n).
\end{aligned}
$$

For $i = n$, we have:

$$\begin{aligned}
&d_n\delta_n\Phi(a_0 \otimes g_0, \dots, a_n \otimes g_n)\\
&= d_n\delta_n(g_0^{(2)}, g_1^{(3)}, \dots, g_n^{(n+2)} \mid S(g_0^{(1)}g_1^{(2)}\dots g_n^{(n+1)}) \cdot a_0, S(g_1^{(1)}g_2^{(2)}\dots g_n^{(n)}) \cdot a_1,\\
&\qquad \dots, S(g_{n-1}^{(1)}g_n^{(2)}) \cdot a_{n-1}, S(g_n^{(1)}) \cdot a_n)\\
&= d_n(g_0^{(3)}, g_1^{(4)}, \dots, g_n^{(n+3)} \mid\\
&\qquad (S(g_0^{(2)}g_1^{(3)}\dots g_n^{(n+2)})S(g_n^{(1)}) \cdot a_n)(S(g_0^{(1)}g_1^{(2)}\dots g_n^{(n+1)}) \cdot a_0),\\
&\qquad S(g_1^{(1)}g_2^{(2)}\dots g_n^{(n)}) \cdot a_1, \dots, S(g_{n-1}^{(1)}g_n^{(2)}) \cdot a_{n-1})\\
&= (g_n^{(2n+4)}g_0^{(3)}, g_1^{(4)}, \dots, g_{n-1}^{(n+2)} \mid\\
&\qquad \sigma(g_n^{(2n+5)}, g_0^{(4)})g_n^{(n+3)} \cdot ((S(g_0^{(2)}g_1^{(3)}\dots g_n^{(n+2)})S(g_n^{(1)}) \cdot a_n)\\
&\qquad (S(g_0^{(1)}g_1^{(2)}\dots g_n^{(n+1)}) \cdot a_0)), S(g_1^{(1)}g_2^{(2)}\dots g_n^{(n)}) \cdot a_1, \dots,\\
&\qquad S(g_{n-1}^{(1)}g_n^{(2)}) \cdot a_{n-1})\\
&= \Phi d_n^{A\#_\sigma\mathcal{H}}(a_0 \otimes g_0, \dots, a_n \otimes g_n).
\end{aligned}$$

The commutativity of Φ and the degeneracies are easier to check and is left to the reader. The theorem is proved. □

Let $H_\sigma = k\underset{\sigma}{\#}\mathcal{H}$ denote the crossed product of $\mathcal{H}$ and k where $\mathcal{H}$ acts on k via the counit ϵ. One can check that the q-th row of the cylindrical module $A\underset{\sigma}{\natural}\mathcal{H}$ is the standard Hochschild complex of the algebra $\mathcal{H}_\sigma$ with coefficients in the bimodule $M_q = \mathcal{H} \otimes A^{\otimes(q+1)}$. Here H_σ acts on M_q on the left and right by

$$h \cdot (g \otimes a_0 \otimes \dots \otimes a_q) = \sigma(h^{(q+3)}, g^{(2)})h^{(q+2)}g^{(1)} \otimes h^{(1)}a_0 \otimes \dots \otimes h^{(q+1)}a_q$$

$$(g \otimes a_0 \otimes \dots \otimes a_q) \cdot h = \sigma(g^{(2)}, h^{(2)})g^{(1)}h^{(1)} \otimes a_0 \otimes \dots \otimes a_q.$$

For the proof of Theorem 3.4 we need an extension of Mac Lane's isomorphism, which relates group homology to Hochschild homology, to Hopf algebras.

We recall that the Hopf homology of a Hopf algebra $\mathcal{H}$ with coefficients in a left $\mathcal{H}$-module M is the homology of the following complex

$$M \xleftarrow{d_0} \mathcal{H} \otimes M \xleftarrow{d_1} \mathcal{H} \otimes \mathcal{H} \otimes M \xleftarrow{d_2} \dots \mathcal{H}^{\otimes n} \otimes M \xleftarrow{d_n} \mathcal{H}^{\otimes(n+1)} \otimes M \leftarrow \dots,$$

where the differential d_n is given by

$$d_n(h_0 \otimes h_1 \otimes \dots \otimes h_n \otimes m) = \epsilon(h_0)h_1 \otimes \dots \otimes h_n \otimes m +$$
$$\sum_{1\le i\le n-1} (-1)^i h_0 \otimes \dots \otimes h_ih_{i+1} \otimes \dots \otimes h_n \otimes m + (-1)^n h_0 \otimes h_1 \otimes \dots \otimes h_{n-1} \otimes hm.$$

We denote the nth Hopf homology group of $\mathcal{H}$ with coefficients in M by $H_n(\mathcal{H}; M)$.

Let M be an $\mathcal{H}_\sigma$-bimodule. We can convert M to a new left $\mathcal{H}$-module, $\widetilde{M} = M$, where the action of $\mathcal{H}$ on $\widetilde{M}$ is defined by

$$h \blacktriangleright m = \sigma^{-1}(S(h^{(2)}), h^{(3)})\overline{h^{(4)}}m\overline{S(h^{(1)})}, \tag{3.1}$$

where $\bar{h}$ denotes the image of h in $\mathcal{H}_\sigma$ under the map $h \to 1\#h$. Note that in the proof of the following lemma the cocommutativity of $\mathcal{H}$ is used.

Lemma 3.3 *Let M be an $\mathcal{H}_\sigma$-bimodule. Then by the above definition M is a left $\mathcal{H}$-module, i.e., $g \blacktriangleright (h \blacktriangleright m) = (gh) \blacktriangleright m$, for all $g, h \in \mathcal{H}$ and $m \in M$.*

Proof We have

$$
\begin{aligned}
&g \blacktriangleright (h \blacktriangleright m)\\
&= g \blacktriangleright (\sigma^{-1}(S(h^{(2)}), h^{(3)})\overline{h^{(4)}}m\overline{S(h^{(1)})})\\
&= \sigma^{-1}(S(g^{(2)}), g^{(3)})\sigma^{-1}(S(h^{(2)}), h^{(3)})\overline{g^{(4)}}(\overline{h^{(4)}}m\overline{S(h^{(1)})})\overline{S(g^{(1)})}\\
&= \sigma^{-1}(S(g^{(3)}), g^{(4)})\sigma^{-1}(S(h^{(3)}), h^{(4)})\sigma(g^{(5)}, h^{(5)})\\
&\qquad \sigma(S(h^{(2)}), S(g^{(2)}))\overline{g^{(5)}h^{(5)}}m\overline{S(g^{(1)}h^{(1)})}\\
&= \sigma^{-1}(S(g^{(3)}), g^{(4)})\sigma^{-1}(S(h^{(3)}), h^{(4)})\sigma(g^{(5)}, h^{(5)})\sigma(S(h^{(2)}), S(g^{(2)}))\\
&\qquad \sigma(S(h^{(9)})S(g^{(9)}), g^{(6)}h^{(6)})\sigma^{-1}(S(h^{(8)}S(g^{(8)}), g^{(7)}h^{(7)})\\
&\qquad \overline{g^{(5)}h^{(5)}}m\overline{S(g^{(1)}h^{(1)})}\\
&= \sigma^{-1}(S(g^{(3)}), g^{(4)})\sigma^{-1}(S(h^{(3)}), h^{(4)})\sigma(g^{(5)}, h^{(5)})\sigma(S(h^{(2)}), S(g^{(2)}))\\
&\qquad \sigma(S(h^{(8)})S(g^{(8)}), g^{(6)}h^{(6)})\sigma^{-1}(S(h^{(9)}S(g^{(9)}), g^{(7)}h^{(7)})\\
&\qquad \overline{g^{(5)}h^{(5)}}m\overline{S(g^{(1)}h^{(1)})}\\
&= \sigma(S(h^{(9)})S(g^{(9)}), g^{(5)})\sigma(S(h^{(8)}), h^{(5)})\sigma^{-1}(S(g^{(3)}), g^{(4)})\sigma^{-1}(S(h^{(3)}), h^{(4)})\\
&\qquad \sigma(S(h^{(2)}), S(g^{(2)}))\sigma^{-1}(S(h^{(9)}S(g^{(9)}), g^{(7)}h^{(7)})\overline{g^{(5)}h^{(5)}}m\overline{S(g^{(1)}h^{(1)})}\\
&= \sigma(S(h^{(5)})S(g^{(5)}), g^{(8)})\sigma^{-1}(S(g^{(5)}), g^{(6)})\sigma(S(h^{(6)}), S(g^{(6)}))\\
&\qquad \sigma^{-1}(S(h^{(4)}S(g^{(4)}), g^{(3)}h^{(3)})\overline{g^{(2)}h^{(2)}}m\overline{S(g^{(1)}h^{(1)})}\\
&= \sigma^{-1}(S(g^{(8)}), g^{(7)})\sigma(S(g^{(6)}), g^{(5)})\sigma^{-1}(S(h^{(4)}S(g^{(4)}), g^{(3)}h^{(3)})\\
&\qquad \overline{g^{(2)}h^{(2)}}m\overline{S(g^{(1)}h^{(1)})}\\
&= \sigma^{-1}(S(h^{(2)})S(g^{(2)}), g^{(3)}h^{(3)})\overline{g^{(4)}h^{(4)}}m\overline{S(g^{(1)}h^{(1)})} = (gh) \blacktriangleright m.
\end{aligned}
$$

□

The following result was first proved in [10] for σ a trivial cocycle.

Theorem 3.4 (Mac Lane Isomorphism for Hopf crossed products) *Let M be an $\mathcal{H}_\sigma$-bimodule and $\widetilde{M}$ be defined as above. Then the following map defines an isomorphism between Hochschild and Hopf homology complexes:*

$$
\begin{gathered}
\Theta : C_n(\mathcal{H}_\sigma, M) \longrightarrow C_n(\mathcal{H}; \widetilde{M})\\
\Theta(\bar{h}_1 \otimes \ldots \otimes \bar{h}_n \otimes m) = h_1^{(2)} \otimes h_2^{(2)} \otimes \ldots \otimes h_n^{(2)} \otimes m\overline{h_1^{(1)}} \ldots \overline{h_n^{(1)}}.
\end{gathered}
$$

Proof We show more than what we need for the proof, namely we show that Θ is an isomorphisms of of simplicial modules. We have:

$$
\begin{aligned}
&\Theta\delta_0(\bar{h}_1 \otimes \ldots \otimes \bar{h}_n \otimes m) = \Theta(\bar{h}_2 \otimes \ldots \otimes \bar{h}_n \otimes m\bar{h}_1)\\
&\qquad = h_2^{(2)} \otimes \ldots \otimes h_n^{(2)} \otimes m\bar{h}_1\overline{h_2^{(1)}} \ldots \overline{h_n^{(1)}} = \delta_0\Theta(\bar{h}_1 \otimes \ldots \otimes \bar{h}_n \otimes m).
\end{aligned}
$$

For $0 \leq i \leq n$, we have

$$
\begin{aligned}
&\Theta\delta_i(\bar{h}_1 \otimes \ldots \otimes \bar{h}_n \otimes m)\\
&= \Theta(\bar{h}_1 \otimes \ldots \otimes (\overline{h_i})(\overline{h_{i+1}}) \otimes \ldots \otimes \bar{h}_n \otimes m)\\
&= h_1^{(2)} \otimes h_2^{(2)} \otimes \ldots \otimes h_i^{(2)}h_{i+1}^{(2)} \otimes \ldots \otimes h_n^{(2)} \otimes m\overline{h_1^{(1)}} \ldots \overline{h_n^{(1)}}\\
&= \delta_i\Theta(\bar{h}_1 \otimes \ldots \otimes \bar{h}_n \otimes m).
\end{aligned}
$$

We leave it to the reader to check the commutativity of Θ with the last face and the degeneracies.

To finish the proof one can check that the following map is the inverse of Θ

$$\mathfrak{I} : C_n(\mathcal{H}; \widetilde{M}) \longrightarrow C_n(\mathcal{H}_\sigma, M)$$

$$\begin{aligned} &\mathfrak{I}(h_1 \otimes \ldots \otimes h_n \otimes m) \\ &= \sigma^{-1}(S(h_1^{(2)}), h_1^{(3)}) \ldots \sigma^{-1}(S(h_n^{(2)}), h_n^{(3)}) \overline{h_1^{(4)}} \otimes \ldots \otimes \overline{h_n^{(4)}} \\ &\quad \otimes m \overline{S(h_n^{(1)})} \ldots \overline{S(h_1^{(1)})}. \end{aligned}$$

□

We apply the generalized cyclic Eilenberg-Zilber theorem (Theorem 2.1) to the cylindrical module $A\natural_\sigma\mathcal{H}$ to derive a spectral sequence for the cyclic homology of $A\#_\sigma\mathcal{H}$. We have

$$Tot(A\natural_\sigma\mathcal{H}) \rightarrow d(A\natural_\sigma\mathcal{H}) \cong (A\#_\sigma\mathcal{H})^\natural,$$

where the first map is a quasi-isomorphism of mixed complexes given in Theorem 2.1 and the second map is the isomorphism given in Theorem 3.2. We filter the mixed complex $Tot(A\natural_\sigma\mathcal{H})$ by sub mixed complexes

$$F^i(Tot(A\natural_\sigma\mathcal{H}))_n = \bigoplus_{\substack{p+q=n \\ q\leq i}} A^{\otimes(p+1)} \otimes \mathcal{H}^{\otimes(q+1)}.$$

This gives us an spectral sequence that converges to $HC_\bullet(A\#_\sigma\mathcal{H})$. We can then apply Theorem 3.3 to identify the E^1-term of this spectral sequence, i.e. the homology of rows, as Hopf homologies of $\mathcal{H}$, with coefficients in $M_q = \mathcal{H} \otimes A^{\otimes(q+1)}$, where $\mathcal{H}$ acts on M_q by

$$\begin{aligned} h \blacktriangleright (g \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_q) &= \sigma^{-1}(S(h^{(3)}), h^{(4)})\sigma(h^{(q+6)}, g^{(1)})\sigma(h^{(q+7)}g^{(2)}, S(h^2)) \\ & h^{(q+8)}g^{(3)}S(h^{(1)}) \otimes h^{(5)}(a_0) \otimes \ldots \otimes h^{(q+5)}(a_q). \end{aligned} \tag{3.2}$$

This proves the following theorem.

Theorem 3.5 *There is a spectral sequence that converges to $HC_{p+q}(A\#_\sigma\mathcal{H})$. The E^1-term of this spectral sequence is given by*

$$E^1_{p,q} = H_p(\mathcal{H}; M_q).$$

Given any cylindrical module $X = \{X_{p,q}\}_{p,q\geq 0}$, if we compute the Hochschild homologies of rows of X we obtain a new bigraded k-module $X' = \{X'_{p,q}\}_{p,q\geq 0}$. We claim that the columns of X', i.e. $\{X'_{p,q}\}_{q\geq 0}$ form a cyclic module for each $p \geq 0$. For some special cases one can find the proof in [8, 1]. The same proof, however, works in the general case. This observation proves the following proposition.

Proposition 3.6 *The p^{th} column of E^1, i.e. $\{H_p(\mathcal{H}; M_q)\}_{q\geq 0}$ is a cyclic module for each $p \geq 0$.*

We denote the p^{th} column of E^1 by N_p. One can observe that the induced differential d^1 on E^1 is simply the differential $b+B$ associated to the cyclic modules N_p. This finishes the proof of the following theorem.

Theorem 3.7 *The E^2 term of the spectral sequence in Theorem 3.5 is*

$$E^2_{p,q} = HC_q(N_p).$$

Recall that if $\mathcal{H}$ is a semisimple Hopf algebra, then for any $\mathcal{H}$-module M, $H_i(\mathcal{H}, M) = 0$ for $i \geq 1$. From this and Theorem 3.5, we obtain the following corollary.

Corollary 3.8 *Let $\mathcal{H}$ be semisimple. Then the above spectral sequence collapses and we obtain*

$$HC_q(A \underset{\sigma}{\#} \mathcal{H}) \cong HC_q(N_0).$$

Since $N_0 = H_0(\mathcal{H}, M_\bullet)$, we obtain

$$N_{0,q} = M_q^{\mathcal{H}} = (\mathcal{H} \otimes A^{\otimes(q+1)})^{\mathcal{H}},$$

where the action of $\mathcal{H}$ is defined by 3.2.

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On Sequentially h-complete Groups

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Abstract. A topological group G is *sequentially h-complete* if all the continuous homomorphic images of G are sequentially complete. In this paper we give necessary and sufficient conditions on a complete group for being compact, using the language of sequential h-completeness. In the process of obtaining such conditions, we establish a structure theorem for ω-precompact sequentially h-complete groups. As a consequence we obtain a reduction theorem for the problem of c-compactness.

All topological groups in this paper are assumed to be Hausdorff.

A topological group G is *sequentially h-complete* if all the continuous homomorphic images of G are sequentially complete (i.e., every Cauchy-sequence converges). G is called *precompact* if for any neighborhood U of the identity element there exists a finite subset F of G such that $G = UF$.

In [6, Theorem 3.6] Dikranjan and Tkačenko proved that nilpotent sequentially h-complete groups are precompact (also see [4]). Thus, if a group is nilpotent, sequentially h-complete and complete, then it is compact.

Inspired by this result, the aim of this paper is to give necessary and sufficient conditions on a complete group for being compact, using the language of sequential h-completeness. This aim is carried out in Theorem 6.

For an infinite cardinal τ, a topological group G is *τ-precompact* if for any neighborhood U of the identity element there exists $F \subset G$ such that $G = UF$ and $|F| \leq \tau$. In order to prove Theorem 6, we will first establish a strengthened version of the Guran's Embedding Theorem for ω-precompact sequentially h-completely groups (Theorem 5).

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A topological group G is *c-compact* if for any topological group H the projection $\pi_H : G \times H \to H$ maps closed subgroups of $G \times H$ onto closed subgroups of H (see [12], [5] and [2], as well as [3]). The problem of whether every c-compact topological group is compact has been an open question for more than ten years. As a consequence of Theorem 6, we obtain that the problem of c-compactness can be reduced to the second-countable case (Theorem 9).

The following Theorem is a slight generalization of Theorem 3.2 from [8]:

Theorem 1 *Let G be an ω-precompact sequentially h-complete topological group. Then every continuous homomorphism $f : G \to H$ onto a group H of countable pseudocharacter is open.*

In order to prove Theorem 1, we need the following three facts, two of which are due to Guran:

Fact A (Guran's Embedding Theorem) *A topological group is τ-precompact if and only if it is topologically isomorphic to a subgroup of a direct product of topological groups of weight $\leq \tau$.* (Theorem 4.1.3 in [14].)

Fact B (Banach's Open Map Theorem) *Any continuous homomorphism from a separable complete metrizable group onto a Baire group is open.* (Corollary V.4 in [10].)

Fact C *Let G be an ω-precompact topological group of countable pseudocharacter. Then G admits a coarser second countable group topology.* (Corollary 4 in [9].)

The proof below is just a slight modification of the proof of Theorem 3.2 from [8] mentioned above:

Proof First, suppose that H is metrizable. Let U be a neighborhood of e in G. Since, by Fact A, G embeds into a product of separable metrizable group, we may assume that $U = g^{-1}(V)$ for some continuous homomorphism $g : G \to M$ onto a separable metrizable group M and some neighborhood V of e in M. Let $h = (f, g) : G \to H \times M$, and put $L = h(G)$. Let $p : L \to H$ and $q : L \to M$ be the restrictions of the canonical projections $H \times M \to H$ and $H \times M \to M$. Clearly, one has $f = ph$ and $g = qh$. Since q is continuous, $W = q^{-1}(V)$ is open. We have $h(U) = h(g^{-1}(V)) = h(h^{-1}q^{-1}(V)) = W$ and $f(U) = ph(U) = p(W)$. Since G is sequentially h-complete, the groups L and H, being homomorphic images of G, are sequentially complete. Since H and L are metrizable, this means that they are simply complete. They are also separable, because they are metrizable and ω-precompact. Thus, by Fact B, $p : L \to H$ is open, and hence $f(U) = p(W)$ is open in H.

To show the general case, suppose that the topology $\mathcal{T}$ of the group H is of countable pseudocharacter; then H admits a coarser second countable topology $\mathcal{T}'$ (by Fact C); put $\iota : (H, \mathcal{T}) \to (H, \mathcal{T}')$ to be the identity map. Since $\mathcal{T}'$ is metrizable, the continuous homomorphism $\iota \circ \varphi : G \to (H, \mathcal{T}')$ is open by what we have already proved, and thus φ is also open. □

Corollary 2 *Let G be an ω-precompact sequentially h-complete group. The following statements are equivalent:*

(i) *G is second countable;*

(ii) *G contains a countable network;*

(iii) *G has a countable pseudocharacter;*
(iv) *G is metrizable.*

Proof The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious, and so is the equivalence of (i) and (iv).

(iii) $\Rightarrow$ (iv): If $(G, \mathcal{T})$ is of countable pseudocharacter, it admits a coarser second countable topology $\mathcal{T}'$(by Fact C); put $\iota : G \to G_1$ to be the identity map. By Theorem 1, ι is open, and thus $\mathcal{T}$ is second countable, as desired. □

A topological group G is *minimal* if every continuous isomorphism $\varphi : G \to H$ is a homeomorphism, or equivalently, if the topology of G is a coarsest (Hausdorff) group topology on G. The group G is *totally minimal* if every continuous surjective homomorphism $f : G \to H$ is open; in other words, G is totally minimal if every quotient of G is minimal.

Corollary 3 *Every ω-precompact sequentially h-complete topological group of countable pseudocharacter is totally minimal and metrizable.*

Proof Let G be an ω-precompact sequentially h-complete group of countable pseudocharacter. By Corollary 2, G is metrizable and contains a countable network. Sequential h-completeness and the property of having a countable network are preserved under continuous homomorphic images, so for every continuous homomorphism $\varphi : G \to H$ onto a topological group H, the group H is sequentially h-complete and contains a countable network; in particular, H is ω-precompact. Thus, by Corollary 2, H is of countable pseudocharacter. Therefore, by Theorem 1, φ is open. □

Corollary 4 generalizes [8, 3.3] to the sequentially h-complete topological groups.

Corollary 4 *Every sequentially h-complete topological group with a countable network is totally minimal and metrizable.* □

A topological group is *h-complete* if all its continuous homomorphic images are complete (see [7]).

Using Theorem 1, we obtain the following strengthening of the Guran's Embedding Theorem for sequentially h-complete groups:

Theorem 5 *Every ω-precompact sequentially h-complete group G densely embeds into the (projective) limit of its metrizable quotients. In particular, if G is h-complete, then it is equal to the limit of its metrizable quotients.*

Proof By Fact A, since G is ω-precompact, it embeds into $\Sigma = \prod\limits_{\alpha \in I} \Sigma_\alpha$, a product of second countable groups. Denote by $\pi_\alpha : G \to \Sigma_\alpha$ the restriction of the canonical projections to G; without loss of generality, we may assume that the π_α are onto. Since G is also sequentially h-complete, the π_α are open (by Theorem 1). Thus, one has $G/N_\alpha \cong \Sigma_\alpha$, where $N_\alpha = \ker \pi_\alpha$. We may also assume that *all* metrizable quotients of G appear in the product constituting Σ, because by adding factors we do not ruin the embedding.

Let ι be the embedding of G into the product of its metrizable quotients, and put $L = \varprojlim\limits_{\alpha \in I} G/N_\alpha$. The image of ι is obviously contained in L. (We note that if G/N_α and G/N_β are metrizable quotients, then $G/N_\alpha N_\beta$ and $G/N_\alpha \cap N_\beta$ are also metrizable quotients; the latter one is metrizable, because the continuous homomorphism $G \to G/N_\alpha \times G/N_\beta$ is open onto its image, as the codomain is metrizable.) In order to show density, let $x = (x_\alpha N_\alpha)_{\alpha\in I} \in L$ and let

$$U = U_{\alpha_1} \times \cdots \times U_{\alpha_k} \times \prod_{\alpha\neq\alpha_i} G/N_\alpha$$

be a neighborhood of x. By the consideration above, the quotient $G/\bigcap_{i=1}^{k} N_{\alpha_i}$ is metrizable, so $\bigcap_{i=1}^{k} N_{\alpha_i} = N_\gamma$ for some $\gamma \in I$. Thus, $\pi_{\alpha_i}(x_\gamma) = x_{\alpha_i} N_{\alpha_i}$, therefore $\iota(G)$ intersects U, and hence $\iota(G)$ is dense in L. $\square$

A topological group G is *maximally almost-periodic* (or briefly, *MAP*) if it admits a continuous monomorphism $m : G \to K$ into a compact group K, or equivalently, if the finite-dimensional unitary representations of G separate points.

The Theorem below is a far-reaching generalization of the main result of [11]:

Theorem 6 *Let G be a complete topological group. Then the following assertions are equivalent:*

(i) *the closed normal subgroups of closed separable subgroups of G are h-complete and MAP;*
(ii) *every closed separable subgroup H of G is sequentially h-complete, and its metrizable quotients H/N are MAP;*
(iii) *G is compact.*

The following easy consequence of a result by Dikranjan and Tkačenko plays a very important role in proving Theorem 6:

Fact D *G is precompact if and only if every closed separable subgroup of G is precompact.* (Theorem 3.5 in [6].)

Proof (i) $\Rightarrow$ (ii): If H/N is a quotient described in (ii), then it is clearly h-complete, as the homomorphic image of H, which is assumed to be h-complete in (i). Let $m : H \to K$ be a continuous injective homomorphism into a compact group K. Since H is h-complete, $m(H)$ is closed in K, so we may assume that m is onto. The subgroup $m(N)$ is normal in K, because m is bijective, and it is closed because N is h-complete. Therefore, $\bar{m} : H/N \to K/m(N)$ is a continuous injective homomorphism, showing that H/N is MAP.

(ii) $\Rightarrow$ (iii): Since G is complete, in order to show that G is compact, we show that it is also precompact. By Fact D, it suffices to show that every closed separable subgroup of G is precompact.

Let H be a closed separable subgroup of G. The group H is ω-precompact (because it is separable), and by (ii) H is sequentially h-complete. Applying Theorem 5, H densely embeds into the the product P of its metrizable quotient. In order to show that H is precompact, we show that each factor of the product P is precompact.

Let $Q = H/N$ be a metrizable quotient of H; since H is sequentially h-complete, so is Q. The group Q is second countable, because (being the continuous image of H) it is separable. Thus, by Corollary 4, Q is totally minimal, because it is sequentially h-complete. According to (ii), Q is MAP, and together with minimality this implies that Q is precompact. $\square$

Remark The implication (i) $\Rightarrow$ (iii) can also be proved directly, without applying Theorem 5, by using Fact D and [8, 3.4].

Since subgroups and continuous homomorphic images of c-compact groups are c-compact again, and are in particular h-complete, we obtain:

Corollary 7 *If G is c-compact and MAP, then G is compact.*

Proof If G is MAP, then in particular all its subgroups are so, and all its closed subgroups are c-compact, thus h-complete. Hence Theorem 6 applies. □

A group is *minimally almost periodic* (or briefly, *m.a.p.*) if it has no non-trivial finite-dimensional unitary representations.

Corollary 8 *Every c-compact group G has a maximal compact quotient G/M, where M is a closed characteristic m.a.p. subgroup of G.*

Proof Let G be a c-compact group, and let $M = n(G)$, the von Neumann radical of G (the intersection of the kernels of all the finite-dimensional unitary representations of G). By its definition, G/M is the maximal MAP quotient of G, and according to Corollary 7 this is the same as the maximal compact quotient of G, because each quotient of G is c-compact.

The subgroup M is also c-compact, so by what we have proved so far, it has a maximal compact quotient M/L, where L is characteristic in M, thus L is normal in G. By the Third Isomorphism Theorem, $G/M \cong (G/L)/(M/L)$, and since both M/L and G/M are compact, by the three space property of compactness in topological groups, the quotient G/L is compact too. Thus, $M \subseteq L$, and therefore $M = L$. Hence, M is minimally almost periodic. □

We conclude with a reduction theorem. If G is c-compact, then so is every closed subgroup of G. Thus, each closed subgroup H and separable metrizable quotient H/N mentioned in (ii) of Theorem 6 is also c-compact. Therefore, the equivalence of (ii) and (iii) in Theorem 6 yields:

Theorem 9 *The following statements are equivalent:*

(i) *every c-compact group is compact;*
(ii) *every second countable c-compact group is MAP (and thus compact);*
(iii) *every second countable c-compact m.a.p. group is trivial.* □

It is not known whether (ii) is true, but by Corollary 4 every second-countable c-compact group is totally minimal, a fact that may help in proving or giving a counterexample to (ii).

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Embeddings of Algebras

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This paper is dedicated to Saunders Mac Lane.

Abstract. Conditions are given for determining when the unit of an adjunction is monic when the domain of the left adjoint is a category **B** of algebras. Key necessary and sufficient conditions are given in terms of a graph assigned to object G of **B**. The general method is then applied to present a proof of a Clifford algebra embedding theorem which exactly parallels that of the Birkhoff-Witt theorem for Lie algebras. A contrasting application related to work of Schreier and Serre on amalgams of groups further illustrates the method.

1 Introduction

We consider here the general problem of determining when the unit morphisms $\eta_G : G \to UFG$ of an adjunction $(F, U, \eta, \varepsilon) : \mathbf{B} \rightharpoonup \mathbf{A}$ are monic when the domain **B** of F is a category of algebras.

When $\mathbf{B} = \mathbf{Sets}$ it is clear from the commutativity of the diagram

$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "f"'] & UFG \arrow[d] \\ & UA \end{tikzcd} \tag{1.1}$$

that η_G is always monic provided that there exists A in **A** such that UA has more than one element. There are, in fact, two monads on **Sets** which have only trivial algebras. On the empty set ϕ one of them takes value ϕ and the other 1. On $X \neq \phi$ both monads take value 1(cf. Manes [17]). Pareigis [18] makes similar remarks in his discussion of consistent algebraic theories.

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In the case when **B** is a more general algebraic category the problem becomes more intricate and has been a subject of research in such diverse areas as Lie algebras, Clifford algebras, amalgams of groups and categorical coherence (cf. [6], [8], [9], [12], [17] and [21]).

In earlier papers ([11], [12] and [13]) the author developed a general method for solving this problem and applied it to generalizations of the Birkhoff-Witt theorem, to coherence problems and to embeddings of partially defined algebras.

In this paper the general method has been reformulated in an isomorphism invariant way and a new application to the embedding of Clifford algebras has been made which demonstrates as well a strong parallel between the proof techniques used for Clifford and Lie algebra embeddings. It is shown explicitly here how embeddings involving group amalgams follow the same pattern as the Clifford algebra embedding. The Diamond, Embedding and Connectedness Principles used in support of the general method in earlier work have been reformulated in terms of costrict objects to ensure isomorphism invariance of results.

The method employed when looking at a unit morphism $\eta_G : G \to UFG$ is to associate to G a category $\mathbf{C}(S_V)$ called a *V picture of the adjoint F to U at G* and then to embed G as a costrict subcategory of $\mathbf{C}(S_V)$. The principles developed in Section 2 then will provide a set of necessary and sufficient test conditions to determine whether η_G is monic or not. $\mathbf{C}(S_V)$ is a certain quotient category of a free category generated by a graph S_V obtained by forgetting part of the structure of G using V and then taking a left adjoint to VU.

Section 2 considers various principles that ensure the embeddability of distinct isomorphism classes of objects of a subcategory **G** of **C** in the components of **C**. For the purposes of applications in the later sections it is important to require that the objects of **G** are costrict in **C**. Then diamond completeness for a pair of morphisms with common reducible domain is equivalent to the Diamond Principle, namely, that objects $X \to A$ of $X/\mathbf{C}$ with A in **G** are weakly terminal. There is another equivalent Embedding Principle and finally a Principle of Connectedness which is equivalent to the first two in the presence of a rank functor. Roughly speaking an Embedding Principle is one that ensures a monic η and the other equivalent principles are the ones tested on examples. If the objects of **G** satisfy the further property of being t-costrict (Section 2), then there is a corresponding set of equivalent principles.

Sections 3 and 4 translate these conditions into the context of adjoints with left adjoint domain a category of algebras. Instead of looking at the unit directly sufficiently many operations and equations are dropped until there is a monic unit. Then we take an adjoint and create a graph by reintroducing the operations and do a couple of other constructions to create an appropriate category for applying the results of Section 2. The starting object may then be viewed as a costrict subcategory of this newly created category and then the principles developed in Section 2 are applied.

The next two sections bring in the embedding of Clifford algebras as a new application of the results of the preceding sections. This is formulated in such a way that the close parallel with earlier development of a Lie algebra embedding result is apparent. This explicit comparison and introduction appears in Sections 5 and 6. Finally in Section 7 an application to group amalgams is given based on earlier work. This serves to give an application of the main theorem of a type contrasting with the Clifford algebra example. It should be noted that the method used here is based on structures with partially defined operations, thus providing a view of amalgamations of groups with common subgroup, distinct from the one appearing in Serre([21]).

2 Embedding principles for costrict subcategories

Let **G** be a subcategory of **C**. In this section various equivalent principles are formulated which ensure that distinct isomorphism classes of objects of **G** appear in distinct components of **C**. These details are developed for use in proving the embedding results of later sections.

Let **G** be a subcategory of a category **C** and let $\mathcal{P}(\mathbf{G})$ be the *power category* of **G**. The objects of $\mathcal{P}(\mathbf{G})$ are the subclasses of objects of **G** and the morphisms are the inclusions.

The *reduction functor* $\mathcal{R}_{\mathbf{G}} : \mathbf{C}^{op} \to \mathcal{P}(\mathbf{G})$ is defined by

$$\mathcal{R}_{\mathbf{G}}X = \{A|A \text{ is an object of } \mathbf{G} \text{ and there exists a } \mathbf{C}\text{-morphism } X \to A\}$$

with the obvious definition on morphisms.

An object X of **C** is **G**-*reducible* if $\mathcal{R}_{\mathbf{G}}X$ is nonempty.

An object A is *costrict* in **C** if each **C**-morphism with domain A is an isomorphism. If the only such isomorphism is the identity, then we call A *t-costrict.*

We say that W is *weakly terminal* in a category **W** if for each object S of **W** there is a morphism $S \to W$ and any two such morphisms differ by an automorphism of the codomain.

Diamond Principle for G. Let **G** be a subcategory of **C**, then the objects $\alpha : X \to A$ of $X/\mathbf{C}$ with A an object of **G** are weakly terminal in $X/\mathbf{C}$ for each object X of **C**.

If we change the principle by requiring the objects α to be terminal and not just weakly terminal, then the result is called the Strong Diamond Principle.

Lemma 2.1 *Let* **G** *be a costrict subcategory of* **C**. *Then the following statements are equivalent:*

(a) *The Diamond Principle for* **G**.
(b) *Each pair* $Y \leftarrow X \rightarrow Z$ *of* **C**-*morphisms with* X *a* **G**-*reducible object can be completed to a commutative diamond in* **C**.

Proof Suppose that (b) holds and let $\alpha : X \to A$ be an object of $X/\mathbf{C}$ with A in **G** and $\beta : X \to Y$ be any other object. Then, by hypothesis there exists a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ A \arrow[dr, "\delta"'] & & Y \arrow[dl, "\gamma"] \\ & C & \end{tikzcd}$$

in **C**. But A in **G** and **G** costrict implies that δ is an isomorphism. Thus $\delta^{-1}\gamma : \beta \to \alpha$. If $\gamma' : \beta \to \alpha$ then, by hypothesis, $A \overset{\delta^{-1}\gamma}{\longleftarrow} Y \overset{\gamma'}{\to} A$ can be completed to a commutative square since Y is G-reducible. Thus

$$\begin{tikzcd} & Y \arrow[dl, "\delta^{-1}\gamma"'] \arrow[dr, "\gamma'"] & \\ A \arrow[dr, "\rho"'] & & A \arrow[dl, "\rho'"] \\ & D & \end{tikzcd}$$

where ρ and ρ' are isomorphisms. Thus $\gamma' = ({\rho'}^{-1}\rho)(\delta^{-1}\gamma)$ and γ' differs from $\delta^{-1}\gamma$ by an automorphism. Thus α is weakly terminal in $X/\mathbf{C}$.

It is trivial to show that (a) implies (b). □

The following Lemma illustrates how the Strong Diamond Principle corresponds to t-costrict objects.

Lemma 2.2 *Let* $\mathbf{G}$ *be an t-costrict subcategory of* $\mathbf{C}$*, that is, the objects of* $\mathbf{G}$ *are t-costrict in* $\mathbf{C}$*. Then the following statements are equivalent:*

(a) *The Strong Diamond Principle for* $\mathbf{G}$.
(b) *Each pair* $Y \leftarrow X \rightarrow Z$ *of* $\mathbf{C}$*-morphisms with* X *a* $\mathbf{G}$*-reducible object can be completed to a commutative diamond in* $\mathbf{C}$.

The *component class* $[X]$ of an object X of a category $\mathbf{C}$ (or a graph $\mathbf{C}$) is the class of all objects Y which can be connected to X by a finite sequence of morphisms (e.g. $X \rightarrow X_1 \leftarrow X_2 \rightarrow Y$). We let *Comp*$\mathbf{C}$ denote the collection of component classes.

Embedding Principle for G. If $[X] = [Y]$ in *Comp*$\mathbf{C}$, then $\mathcal{R}_{\mathbf{G}}X = \mathcal{R}_{\mathbf{G}}Y$. Furthermore there is at most one morphism $X \rightarrow A$ up to an automorphism of A, for each pair (X, A) consisting of an object X of $\mathbf{C}$ and an object A of $\mathbf{G}$.

If we require there to be at most one morphism and not just at most one morphism up to automorphism, then we have the Strong Embedding Principle.

Theorem 2.3 *Let* $\mathbf{G}$ *be a costrict subcategory of* $\mathbf{C}$*. Then the following statements are equivalent:*

(a) *The Embedding Principle for* $\mathbf{G}$.
(b) *The Diamond Principle for* $\mathbf{G}$.

Proof Suppose (a) holds. Let $A \xleftarrow{\alpha} X \xrightarrow{\beta} Y$ be a diagram in $\mathbf{C}$ with A in $\mathbf{G}$. Thus $[A] = [Y] = [X]$ in *Comp*$\mathbf{C}$ and $\mathcal{R}_{\mathbf{G}}X = \mathcal{R}_{\mathbf{G}}Y$, by hypothesis. Hence A is in $\mathcal{R}_{\mathbf{G}}Y$ and there exists $\gamma : Y \rightarrow A$. But then $\gamma\beta$ and α are morphisms $X \rightarrow A$ and α and $\gamma\beta$ differ by an automorphism ρ of A, by hypothesis. Thus $\alpha = \rho\gamma\beta$ and $\rho\gamma : \beta \rightarrow \alpha$ in $X/\mathbf{C}$ and $\rho\gamma$ is unique up to automorphism of α since, by hypothesis, as a $\mathbf{C}$ morphism $Y \rightarrow A$ it is unique up to automorphism of A.

Conversely, suppose (b) holds and $[X] = [Y]$ in *Comp*$\mathbf{C}$. Then X and Y are connected by a finite sequence of morphisms. Thus to see that $\mathcal{R}_{\mathbf{G}}X = \mathcal{R}_{\mathbf{G}}Y$ it is sufficient to show that $\mathcal{R}_{\mathbf{G}}Z = \mathcal{R}_{\mathbf{G}}W$ for each morphism $\beta : Z \rightarrow W$. Clearly $\mathcal{R}_{\mathbf{G}}W \subseteq \mathcal{R}_{\mathbf{G}}Z$ since $\mathcal{R}_{\mathbf{G}}$ is a functor $\mathbf{C}^{op} \rightarrow \mathcal{P}(\mathbf{G})$. If $\mathcal{R}_{\mathbf{G}}Z$ is empty, then $\mathcal{R}_{\mathbf{G}}Z = \mathcal{R}_{\mathbf{G}}W$. Otherwise there is an object A of $\mathcal{R}_{\mathbf{G}}Z$ and a diagram $A \xleftarrow{\alpha} Z \xrightarrow{\beta} W$ for some $\alpha : Z \rightarrow A$. By (b) $\alpha : Z \rightarrow A$ is weakly terminal in $Z/\mathbf{C}$. Thus there exists a $\gamma : W \rightarrow A$ with $\alpha = \gamma\beta$. Thus A is in $\mathcal{R}_{\mathbf{G}}W$ and $\mathcal{R}_{\mathbf{G}}Z = \mathcal{R}_{\mathbf{G}}W$. Suppose $\alpha, \alpha' : X \rightarrow A$ with A in $\mathbf{G}$. We need to show that these are the same up to an automorphism of A. By (b) $\alpha : X \to A$ is weakly terminal in $X/\mathbf{C}$, thus there is a $\beta : A \to A$ with $\alpha = \beta\alpha'$. But A is costrict, thus β is an isomorphism, hence α and α' are the same up to isomorphism of A. $\square$

We remark that for **G** a t-costrict subcategory of **C** the Strong Embedding Principle is equivalent to the Strong Diamond Principle.

Given an object X of **C** let $(X/\mathbf{C})_{\mathcal{P}}$ be the full subcategory of the slice category $X/\mathbf{C}$ obtained by omitting those objects which are isomorphisms in **C**.

Principle of Connectedness for T. The categories $(X/\mathbf{C})_{\mathcal{P}}$ are connected, or empty, for each object X of **T**, where **T** is a subcategory of **C**.

Lemma 2.4 *If the Diamond Principle holds for an costrict subcategory* **G** *of* **C**, *then the Principle of Connectedness holds for the full subcategory* $\mathbf{T_G}$ *of* **C** *consisting of all* **G**-*reducible objects of* **C**.

Proof Let X be in $\mathbf{T_G}$. Then there is a morphism $X \to A$ in **C** with A in **G**. By the Diamond Principle for **G** the morphism $X \to A$ is weakly terminal in $X/\mathbf{C}$. Thus $(X/\mathbf{C})_{\mathcal{P}}$ is connected or empty. $\square$

Let $\mathbb{P}$ be a preorder, considered as a category, and write $x \geq y$ when the hom set $\mathbb{P}(x, y)$ has exactly one member. We will use the following definition.

Definition 2.5 If $\mathbb{P}$ is a preorder, then a subset $\mathbb{S}$ is *inductive* iff for all $x \in \mathbb{P}$, if for all $y, (y < x) \Rightarrow y \in \mathbb{S}$, then $x \in \mathbb{S}$.
A preorder $\mathbb{P}$ is *inductive* iff for all $S \subset \mathbb{P}$, if S is inductive, then $S = \mathbb{P}$.

Definition 2.6 Let **C** be a category and $\mathbb{P}$ be an inductive preorder. A *rank functor* for **C** is a functor $R : \mathbf{C} \to \mathbb{P}$ with $R\alpha \neq 1$ whenever α is not an isomorphism, that is, R is strictly rank reducing on non isomorphisms.

Example 2.7 Let $\mathbb{N}$ be the preorder of nonnegative integers with $n \to m$ iff $n \geq m$. A functor $R : \mathbf{C} \to \mathbb{N}$ with $R\alpha \neq 1$ whenever α is not an isomorphism is then a rank functor.

The following theorem uses an inductive argument to show that (a) implies (b). The other implications follow from Theorem 2.3 and Lemma 2.4.

Theorem 2.8 *Let* **C** *be a category with a given rank functor and* **G** *a costrict subcategory. Then the following are equivalent.*

(a) *The Principle of Connectedness for the full subcategory* $\mathbf{T_G}$ *of* **C** *of all* **G**-*reducible objects of* **C**.
(b) *The Diamond Principle for* **G**.
(c) *The Embedding Principle for* **G**.

Proof Suppose (a) holds. Let $\alpha : X \to A$ with $A \in \mathbf{G}$ and let $\beta : X \to Y$. We show, inductively that α is weakly terminal. If α is an isomorphism, then A costrict implies that $\beta\alpha^{-1}$ is an isomorphism. Let $\varphi : Y \to A$ be its inverse. Then $\varphi\beta = \alpha$. If $\varphi'\beta = \alpha$, then $\varphi'\beta\alpha^{-1} = 1$ Applying the inverse φ of $\beta\alpha^{-1}$ on the right we get $\varphi' = \varphi$. We now assume inductively that $Z \to A$ is weakly terminal in $Z/\mathbf{C}$ for all Z of lower rank than X. The case when β is an isomorphism is trivial since then $\gamma = \alpha\beta^{-1} : \beta \to \alpha$ and $\gamma'\beta = \gamma\beta$ implies that $\gamma' = \gamma$. It is sufficient to show that there is a morphism $\gamma : \beta \to \alpha$, unique up to automorphism of α whenever α and

β are not isomorphisms, that is, when they are objects of $(X/\mathbf{C})_{\mathcal{P}}$. By hypothesis the category $(X/\mathbf{C})_{\mathcal{P}}$ is connected, hence there is a diagram

$$\begin{tikzcd}
 & & X \arrow[dll, "\alpha=\alpha_0"'] \arrow[d, "\alpha_i"'] \arrow[dr, "\alpha_{i+1}"] \arrow[drr, "\beta=\alpha_n"] & & \\
A & \cdots \arrow[l, "\gamma_0"] & W_i \arrow[r, "\gamma_i"'] & W_{i+1}\cdots & Y
\end{tikzcd} \tag{2.1}$$

in $\mathbf{C}$ for which the α_is are not isomorphisms and $\gamma_i : W_i \to W_{i+1}$ for i odd and $\gamma_i : W_{i+1} \to W_i$ for i even. Note that γ_0 can always be regarded as having codomain A since the only morphisms with domain A are isomorphisms. Let $A = W_0$ and $Y = W_n$. If n = 1, then $\gamma_0 : \beta \to \alpha$. Assume inductively that $n > 1$ and that if the objects α and β are connected by a sequence of length $< n$, then there exists $\gamma : \beta \to \alpha$. Consider

$$\begin{tikzcd}
 & X \arrow[dl, "\alpha_0"'] \arrow[d, "\alpha_1"] \arrow[dr, "\alpha_2"] & \\
A & W_1 \arrow[l, "\gamma_0"] \arrow[r, "\gamma_1"'] & W_2
\end{tikzcd} \tag{2.2}$$

Rank W_1 is less than rank X since α_1 is not an isomorphism. Thus, by induction on rank, γ_0 is weakly terminal in $W_1/\mathbf{C}$. Thus there is a morphism $\delta : \gamma_1 \to \gamma_0$ unique up to automorphism of γ_0 and $\delta\gamma_1 = \gamma_0$ and $\delta\alpha_2 = \delta\gamma_1\alpha_1 = \gamma_0\alpha_1 = \alpha_0$. This connects α and β by a sequence of length $< n$ and thus there exists $\gamma : \beta \to \alpha$. Suppose γ, $\gamma^{'} : \beta \to \alpha$. As $\mathbf{C}$ morphisms both are morphisms $Y \to A$. But β is not an isomorphism and thus rank of Y is less than the rank of X. Thus by the induction hypothesis $\gamma, \gamma^{'} : Y \to A$ are weakly terminal. Thus there is a $\mathbf{C}$ morphism ξ such that $\gamma^{'} = \xi\gamma$. Furthermore ξ is an isomorphism since A is costrict. Thus α is weakly terminal since any two morphisms $\gamma, \gamma^{'} : \beta \to \alpha$ differ by ξ and ξ is an automorphism of α since $\xi\alpha = \xi\gamma\beta = \gamma^{'}\beta = \alpha$.

□

3 Necessary and sufficient conditions

Let $\mathbf{Alg}(\Omega, E)$ denote the category of algebras defined by a set of operators Ω and identities E as described in [16]. In this section we describe a set of necessary and sufficient conditions for an adjunction of a certain type defined on a category $\mathbf{B}$ of algebras to have its unit a monomorphism. In particular, we see that this condition involves the Embedding Principle, which is actually the same principle used in case of classical coherence(cf. [11], [14], [15]).

Let

$$\mathbf{A} \xrightarrow{U} \mathbf{B} \xrightarrow{V} \mathbf{D} \tag{3.1}$$

be a diagram such that

(a) $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ are the categories of (Ω, E), (Ω', E') and (Ω'', E'') algebras, respectively, with $\Omega'' \subseteq \Omega'$ and $E'' \subseteq E'$, and

(b) V is the forgetful functor on operators $\Omega' - \Omega''$ and identities $E' - E''$ and U is a functor commuting with the underlying set functors on $\mathbf{A}$ and $\mathbf{B}$.

Note that U is *not* necessarily a functor forgetting part of Ω and E.

We next describe a functor $C_V : \mathbf{B} \to \mathbf{Grph}$ associated to each pair consisting of a diagram (3.1) of algebras and an adjunction $(L, VU, \eta', \varepsilon') : \mathbf{D} \rightharpoonup \mathbf{A}$, where **Grph** is the category of directed graphs in the sense of [16].

Given an object G of $\mathbf{B}$ let the objects of the graph $C_V(G)$ be the elements of the underlying set $|LVG|$ of LVG.

Recursive definition of the arrows of $C_V(G)$:

$$\omega_{ULVG}(|\eta'_{VG}|x_1, \cdots, |\eta'_{VG}|x_n) \to |\eta'_{VG}|\omega_G(x_1, \cdots, x_n)$$

is an arrow if ω is in the set $\Omega' - \Omega''$ of operators forgotten by V and $(x_1, \cdots, x_n)$ is an n-tuple of elements of $|G|$ for which $\omega_G(x_1, \cdots, x_n)$ is defined.

If $d \to e$ is an arrow of $C_V(G)$, then so is

$$\rho_{LVG}(d_1, \cdots, d, \cdots, d_q) \to \rho_{LVG}(d_1, \cdots, e, \cdots, d_q)$$

for ρ an operator of arity q in Ω and $d_1, \cdots, d_{i-1}, d_{i+1}, \cdots, d_q$ arbitrary elements of $|LVG|$.

If $\beta : G \to G' \in \mathbf{B}$, then $C_V(\beta) : C_V(G) \to C_V(G')$ is the graph morphism which is just the function $|LV\beta| : |LVG| \to |LVG'|$ on objects and defined recursively on arrows in the obvious way.

In the following proposition note that if, in the diagram (3.1), $\mathbf{D}$ is the category of sets, then an adjunction $(L, VU, \eta', \varepsilon')$ is given by letting LX be the free algebra on the set X.

Proposition 3.1 *Suppose*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "U"] & \mathbf{B} \arrow[r, "V"] & \mathbf{D} \arrow[ll, bend left, "L"] \end{tikzcd}$$

is a diagram of algebra categories as in (3.1), with adjunction $(L, VU, \eta', \varepsilon') : \mathbf{D} \rightharpoonup \mathbf{A}$ given. Then there is an adjunction $(F, U, \eta, \varepsilon) : \mathbf{B} \rightharpoonup \mathbf{A}$ with the following specific properties:

(a) *The underlying set of FG is $CompC_V(G)$.*

(b) *If ρ is an operator of arity n in Ω, then ρ_{FG} is defined by*

$$\rho_{FG}([c_1], \cdots, [c_n]) = [\rho_{LVG}(c_1, \cdots, c_n)]$$

where $c_1, \cdots, c_n$ are members of the set $|LVG|$ of objects of the graph $C_V(G)$.

(c) *The unit morphism $\eta_G : G \to UFG$ of $(F, U, \eta, \varepsilon)$ has an underlying set map which is the composition $[\ ]\cdot \mid \eta'_{VG}|$, where $|\eta'_{VG}| : |G| \to |LVG| = ObjC_V(G)$ is the set map underlying the unit $\eta'_{VG} : VG \to VULVG$ of the adjunction $(L, VU, \eta', \varepsilon')$ and $[\ ] : ObjC_V(G) \to CompC_V(G)$ is the function which sends each object to its component.*

Proof Suppose the hypotheses of the proposition hold. Let S_V be a subgraph of $C_V(G)$ having the same objects $|LVG|$ and the same components as $C_V(G)$. Then the proposition remains valid under substitution of S_V for $C_V(G)$ throughout. This allows us to "picture" the adjoint using a possibly smaller set of arrows than those present in $C_V(G)$. Accordingly, we define a *V picture of the adjoint F to U at* $G \in |\mathbf{B}|$ to be any quotient category $\mathbf{C}(= \mathbf{C}(S_V))$ of the free category generated by such a subgraph S_V of $C_V(G)$. This proposition is then valid upon substitution of the underlying graph of a V picture $\mathbf{C}$ for $C_V(G)$ throughout. $\square$

Theorem 3.2 *Let*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "U"] & \mathbf{B} \arrow[r, "V"] \arrow[l, bend left, "F"] & \mathbf{D} \arrow[ll, bend left=50, "L"] \end{tikzcd}$$

be given with adjunctions $(L, VU, \eta', \varepsilon') : \mathbf{D} \rightharpoonup \mathbf{A}$ *and* $(F, U, \eta, \varepsilon) : \mathbf{B} \rightharpoonup A$ *as described in Proposition* 3.1.

Given $G \in |\mathbf{B}|$ *let* $C(S_V)$ *be any* V *picture of the adjoint* F *to* U *at* G.

Then the unit morphism $\eta_G : G \to UFG$ *of the adjunction* $(F, U, \eta, \varepsilon)$ *is monic if and only if the following hold:*

(a) *The discrete subcategory* $\mathbf{G} = \eta'_{VG}(|G|)$ *is costrict in* $\mathbf{C}(S_V)$ *for* η'_{VG} *the unit of* $(L, VU, \eta', \varepsilon')$.
(b) *If* $[A] = [B]$ *in* $CompC(S_V)$, *then* $\mathcal{R}_{\mathbf{G}}A = \mathcal{R}_{\mathbf{G}}B$ *for all* $A, B \in |\mathbf{G}|$, *where* $\mathcal{R}_{\mathbf{G}} : C(S_V)^{op} \to \mathcal{P}(\mathbf{G})$ *is the reduction functor.*
(c) *The unit morphism* η'_{VG} *is monic.*

Strict Embedding Principle for G. If $[X] = [Y]$ in $Comp\mathbf{C}$, then $\mathcal{R}_{\mathbf{G}}X = \mathcal{R}_{\mathbf{G}}Y$.

Note that in Theorem 3.2 this principle is a restatement of condition (b).

4 Sufficient conditions

In the presence of a rank functor we have seen in Theorem 2.8 that the three principles of section 2 are equivalent. Applying this equivalence to Theorem 3.2 we obtain the following:

Theorem 4.1 *Let*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "U"] & \mathbf{B} \arrow[r, "V"] \arrow[l, bend left, "F"] & \mathbf{D} \arrow[ll, bend left=50, "L"] \end{tikzcd}$$

be given with hypotheses as in Theorem 3.2.

Let G *be an object of* $\mathbf{B}$. *Then the unit morphism* $\eta_G : G \to UFG$ *of* $(F, U, \eta, \varepsilon)$ *is monic provided there exists a* V *picture* $\mathbf{C}$ *of the adjoint* F *to* U *at* G *for which the following conditions hold:*

(a) $\mathbf{C}$ *has a rank functor.*
(b) *The discrete subcategory* $\mathbf{G} = \eta'_{VG}(|G|)$ *is costrict in* $\mathbf{C}$ *for* η'_{VG} *the unit of* $(L, VU, \eta', \varepsilon')$.
(c) *The categories* $(X/\mathbf{C})_{\mathcal{P}}$ *are connected, or empty, for each* $\mathbf{G}$*-reducible object* $X \in |\mathbf{C}|$.
(d) *The unit morphism* η'_{VG} *of* $(L, VU, \eta', \varepsilon')$ *is monic.*

Proof Condition (c) is the Principle of Connectedness which by Theorem 2.8 is equivalent to the Embedding Principle. This, in turn, implies the Strict Embedding Principle, which is condition (b) of Theorem 3.2. □

Remark 4.2 Assuming (a) in 4.1, then conditions (b), (c) and (d) are not in general equivalent to the monicity of η. See [12] for an example where η is monic and (a), (b) and (d) hold but not (c).

Remark 4.3 The example referred to in 4.2 shows that the Embedding Principle is stronger than the Strict Embedding Principle used in the necessary and sufficient conditions of Theorem 3.2 since, in the presence of a rank functor, condition (c) of 4.1 is equivalent to the Embedding Principle.

5 Associative embedding of Lie algebras

Given a commutative ring K, let $U : \mathbf{A} \to \mathbf{L}$ be the usual algebraic functor from associative algebras to Lie algebras which replaces the associative multiplication ab of A by the multiplication $[a, b] = ab - ba$ of UA.

The left adjoint $F : \mathbf{L} \to \mathbf{A}$ assigns to each Lie algebra L its universal enveloping algebra $\mathbf{F}L$ but the question as to whether the unit $\eta_L : L \to UFL$ is an embedding is a subtle one since not all Lie algebras can be so embedded(cf. Higgins [9]).

The Birkhoff-Witt theorem gives a positive answer to the embedding question for any K when L is projective as a K-module and for any L when K is a Dedekind domain as well as in various other special cases(cf. [6], [9], [20]). Generalizing from Lie algebras to colour Lie algebras leads to new positive answers in some cases as well as to "Non Birkhoff-Witt" results in other cases such as that of colour Lie algebras and their envelopes over the cyclic group with three elements. For a discussion of colour Lie algebras and special cases, including Lie superalgebras, see [2], [4], [5], [19] and [20].

In the following theorems 5.1 and 5.2 we apply general results of the preceding sections to the case of Lie algebras over a commutative ring K. The results are applied in the case when the underlying module is free.

We do not strive for maximum generality in the case of Lie algebras since our purpose here is rather to demonstrate the commonality of approach in different types of examples, which in this paper have been chosen to be group amalgams, Clifford algebras and Lie algebras.

We now consider specifics. Considering a Lie algebra as a K-module we have a diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, "U"] & \mathbf{L} \arrow[r, "V"] & K\text{-}\mathbf{Mod} \arrow[ll, bend left, "L"] \end{tikzcd}$$

where the conditions (a) and (b) of (3.1) hold. The adjunction $(L, VU, \eta', \varepsilon')$: K-**Mod** $\rightharpoonup$ **A** can be described explicitly as follows.

Given $G \in |\mathbf{L}|$ it is known that LVG is the tensor algebra of VG. Thus

$$LVG = \oplus_{n \geq 0}(\otimes_{i=1}^{n}(VG)).$$

Furthermore $\eta'_{VG} : VG \to K \oplus VG \oplus (VG \otimes VG) \oplus \cdots$ is monic. In this section let $\mathbf{G}$ be the discrete subgraph $\eta'_{VG}(|G|)$ of $C_V(G)$. Thus $\mathbf{G}$ is a discrete subcategory of any V picture of F at G. Applying Theorem 3.2 the following embedding result holds.

Theorem 5.1 *Let G be a Lie-algebra and $\mathbf{C}$ any V picture of F at G. Then a Lie-algebra G can be embedded in its universal associative algebra FG if and only if the following hold:*

(a) *$[A] = [B]$ in CompC implies that $\mathcal{R}_{\mathbf{G}}A = \mathcal{R}_{\mathbf{G}}B$ for all $A, B \in |\mathbf{G}|$ where $\mathcal{R}_{\mathbf{G}} : \mathbf{C}^{op} \to \mathcal{P}(\mathbf{G})$ is the reduction functor.*
(b) *$\mathbf{G}$ is an costrict subcategory of $\mathbf{C}$.*

Similarly, by applying Theorem 4.1 and again noting that η'_{VG} is monic we have the following sufficient conditions:

Theorem 5.2 *A Lie-algebra G can be embedded in its universal associative algebra FG if there exists any V picture $\mathbf{C}$ of the adjoint F to U at G with the following properties.*

(a) *The categories $(X/\mathbf{C})_{\mathcal{P}}$ are connected for each $X \in |\mathbf{C}|$ which is $\mathbf{G}$-reducible.*
(b) *$\mathbf{C}$ has a rank functor.*
(c) *$\mathbf{G} = \eta'_{VG}(|G|)$ is a costrict subcategory of of $\mathbf{C}$.*

Theorem 5.3 (Birkhoff-Witt). *A Lie-algebra G whose underlying module VG is free can be embedded in its universal associative algebra FG.*

Proof The conditions of the previous theorem are to be verified for the following V picture of the adjoint F to U at G. Let $\mathbf{C}$ be the preorder which is a quotient of the free category on the following subgraph S_V of $C_V(G)$. The objects of S_V are the elements of the free K module LVG on all finite strings $x_{i_1}\cdots x_{i_n}$ of elements from a basis $(x_i)_{i\in I}$ of the free K module VG. Given a well ordering of I we let the arrows of S_V be those of the form

$$k_ix_{i_1}\cdots x_{i_n}+\alpha \to k_ix_{i_1}\cdots x_{i_{j+1}}x_{i_j}\cdots x_{i_n}+k_ix_{i_1}\cdots [x_{i_j},x_{i_{j+1}}]\cdots x_{i_n}+\alpha$$

for $i_{j+1}<i_j$, $k_i{\in}K$, and α any element of LVG (not involving $x_{i_1}\cdots x_{i_n}$).

To show that the categories $(X/\mathbf{C})_{\mathcal{P}}$ are connected for each $\mathbf{G}$-reducible object in $\mathbf{C}$, it turns out that the key idea is to show that for $c<b<a$ in I the objects

$$\beta : x_ax_bx_c{\to}x_bx_ax_c+[x_a,x_b]x_c$$

and

$$\gamma : x_ax_bx_c{\to}x_ax_cx_b+x_a[x_b,x_c]$$

can be connected in $((x_ax_bx_c)/\mathbf{C})_{\mathcal{P}}$.

The next step in the process is to further reduce the ranges of β and γ.

From β we have

$$x_bx_ax_c+[x_a,x_b]x_c{\to}x_bx_cx_a+x_b[x_a,x_c]+[x_a,x_b]x_c$$
$${\to}x_cx_bx_a+[x_b,x_c]x_a+x_b[x_a,x_c]+[x_a,x_b]x_c$$

and from γ

$$x_ax_cx_b+x_a[x_b,x_c]{\to}x_cx_ax_b+[x_a,x_c]x_b+x_a[x_b,x_c]$$
$${\to}x_cx_bx_a+x_c[x_a,x_b]+[x_a,x_c]x_b+x_a[x_b,x_c]$$

Finally we use the Jacobi identity and the identity $[x,y]=-[y,x]$ to connect the arrows as required(cf. [12]).

The rank functor for $\mathbf{C}$ is given as follows. Given $X=kx_{a_1}\cdots x_{a_n}$ let $R(X)=(R_n(X))$ be a sequence of nonnegative integers defined by $R_n(X)=\sum_{i=1}^{n}p_{a_i}$ where p_{a_i} is the number of x_{a_j} to the right of x_{a_i} with $a_j<a_i$ and $R_s(X)=0$ for $s\neq n$. We extend by linearity to all elements of $LVG=|\mathbf{C}|$. If $X\to Y$ is an arrow, then $R(Y)<R(X)$ where the latter inequality means that $R_n(Y)<R_n(X)$ for n the largest integer with $R_n(Y)\neq R_n(X)$. Thus R extends to a rank functor.

Finally, we verify condition (c) by observing that any element of $|G|$ may be expressed in the form $\sum_{i\in I}k_ix_i$ in terms of the basis $(x_i)_{i\in I}$ of G, which is regarded as a subset $\mathbf{G}$ of LVG via the embedding ${\eta'}_{VG}$. From the preceding description of arrows of S_V there is no arrow with domain an element of $\mathbf{G}={\eta'}_{VG}(|G|)$. Thus $\mathbf{G}$ is costrict in $\mathbf{C}$. □

6 Embedding into Clifford algebras

In this section we show how the embedding of a vector space with symmetric bilinear form into its universal associative Clifford algebra follows the pattern of previous sections.

Clifford algebras are associative algebras generated using symmetric bilinear forms. They are important in mathematics because of their relationship to orthogonal Lie groups and in physics because they arise as the type of algebras generated by the Dirac matrices (cf. Hermann [8]).

Let X be a vector space over K with a symmetric bilinear form $\beta : X \times X \to K$. Then the Clifford algebra of the bilinear form β is the associative algebra $Cl(X, \beta)$ $= TX/I(\beta)$ where TX is the tensor algebra of X and $I(\beta)$ is the two-sided ideal generated by all elements

$$x_1 x_2 + x_2 x_1 - 2\beta(x_1, x_2)$$

for $x_1, x_2 \in X$ (cf. [8]).

The diagram used here is a generalization of the type used in section 3, namely,

$$\begin{tikzcd} \mathbf{A} & \mathbf{Cl} \arrow[l, "I"'] \arrow[r, "W"] & \mathbf{B} \arrow[r, "R"] \arrow[l, bend left, "F"] & \mathbf{D} \arrow[lll, bend left, "L"] \end{tikzcd}$$

where the functor $U : \mathbf{A} \to \mathbf{B}$ of section 3 is replaced by a relation $(I, W) : \mathbf{A} \leftarrow \mathbf{Cl} \to \mathbf{B}$ in **Cat** for I the inclusion.

For the rest of this section let **A** be the category of associative K-algebras, **Cl** the Clifford algebras $Cl(X, \beta)$, **B** the K-vector spaces X with bilinear form β and **D** the K-vector spaces. Furthermore I is the inclusion, W and R are forgetful, $F(X, \beta)$ $= Cl(X, \beta)$ and LX is the tensor algebra on X.

We show how results of earlier sections yield the following embedding.

Theorem 6.1 *A vector space X over K with symmetric bilinear form β : $X \times X \to K$ can be embedded in its universal associative Clifford algebra $Cl(X, \beta)$*

Proof We follow the pattern of Theorems 5.2 and 5.3. That is, we verify the same three conditions (a), (b) and (c) as in 5.2, only now for a V picture **C** of the adjoint F to W. In fact, for condition (c), we use the same tensor algebra monad $(L, VU, \eta', \varepsilon')$ and the same discrete subcategory **G**, only now arising from the vector space X $= R(X, \beta)$, instead of VG as in 5.2.

In this case the preorder **C** arises as a quotient of the free category on the following subgraph S_V of $C_V(G)$. The objects of S_V are exactly the same as in Theorem 5.3, namely, they are the elements of the vector space $LR(X, \beta)$ on all finite strings $x_{i_1} \cdots x_{i_n}$ of elements from a basis $(x_i)_{i \in I}$ of the space $R(X, \beta)$. The arrows of S_V are different from those of 5.3. Given a well ordering of I we let the arrows of S_V be those of the form

$$k_i x_{i_1} \cdots x_{i_n} + \alpha \to -k_i x_{i_1} \cdots x_{i_{j+1}} x_{i_j} \cdots x_{i_n} + k_i x_{i_1} \cdots 2\beta(x_{i_j}, x_{i_{j+1}}) \cdots x_{i_n} + \alpha$$

for $i_{j+1} < i_j$, $k_i \in K$, and α any element of $LR(X, \beta)$ (not involving $x_{i_1} \cdots x_{i_n}$).

To show that the categories $(X/\mathbf{C})_{\mathcal{P}}$ are connected for each **G**-reducible object in **C**, it turns out that the key idea is to show that for $c < b < a$ in I the objects

$$\delta : x_a x_b x_c \to -x_b x_a x_c + 2\beta(x_a, x_b) x_c$$

and

$$\gamma : x_a x_b x_c \rightarrow -x_a x_c x_b + x_a 2\beta(x_b, x_c)$$

can be connected in $((x_a x_b x_c)/\mathbf{C})_{\mathcal{P}}$. This is done by further reduction of the ranges of δ and γ until they reach a common endpoint.

Following δ we have

$$-x_b x_a x_c + 2\beta(x_a, x_b)x_c \rightarrow x_b x_c x_a - x_b 2\beta(x_a, x_c) + 2\beta(x_a, x_b)x_c$$

$$\rightarrow -x_c x_b x_a + 2\beta(x_b, x_c)x_a - x_b 2\beta(x_a, x_c) + 2\beta(x_a, x_b)x_c$$

and following γ

$$-x_a x_c x_b + x_a 2\beta(x_b, x_c] \rightarrow x_c x_a x_b - 2\beta(x_a, x_c)x_b + x_a 2\beta(x_b, x_c)$$

$$\rightarrow -x_c x_b x_a + x_c 2\beta(x_a, x_b) - 2\beta(x_a, x_c)x_b + x_a 2\beta(x_b, x_c)$$

The category $\mathbf{G}$ is costrict for the same reasons as in 5.3 and rank on objects is defined the same way as well. Note the close parallel between the proof of this theorem and Theorem 5.2.

□

7 Amalgams of groups

We show how the following classical theorem follows from sections 3 and 4.

Theorem 7.1 (Schreier). *If S is a common subgroup of the groups X and Y and if*

$$\begin{tikzcd} S \arrow[r, "\subseteq"] \arrow[d, "\subseteq"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & P \end{tikzcd} \tag{7.1}$$

is the pushout in the category of groups, then α and β are monomorphisms. The group P is referred to as the free product of X and Y with amalgamated subgroup S.

Let $\mathbf{B}$ be the category of sets with a single partially defined binary operation. The diagram $X \longleftarrow S \longrightarrow Y$ of groups can be regarded as a diagram in $\mathbf{B}$ and can be completed to a diagram

$$\begin{tikzcd} S \arrow[r, "\subseteq"] \arrow[d, "\subseteq"'] & X \arrow[d, "\gamma"] \\ Y \arrow[r, "\delta"'] & Z \end{tikzcd} \tag{7.2}$$

commuting in $\mathbf{B}$ where Z is the disjoint union of X and Y with common subset S identified and ab is defined if both $a, b \in X$ or if both $a, b \in Y$, otherwise it is undefined. Clearly (7.2) is a pushout in **Set** and in $\mathbf{B}$. The morphisms γ and δ are the obvious monomorphisms. The next Lemma and Proposition describe how this approach yields the Schreier Theorem. (cf. Baer [1], Serre [21]).

Lemma 7.2 *The Schreier Theorem holds if in (7.2) the pushout codomain Z in $\mathbf{B}$ is embeddable in a group.*

Proof Let $\iota : Z \to P'$ be a monomorphism in $\mathbf{B}$ with P' a group. Then

$$\begin{tikzcd} S \arrow[r, "\subseteq"] \arrow[d, "\subseteq"'] & X \arrow[d, "\iota\gamma"] \\ Y \arrow[r, "\iota\delta"'] & P' \end{tikzcd}$$

commutes in groups. Thus for some group homomorphism ϕ we have $\iota\gamma = \phi\alpha$ and $\iota\delta = \phi\beta$ since (7.1) is a pushout in groups. Thus α, β are monic since ι, γ and δ are. □

Proposition 7.3 *Let Z be as in (7.2). Then Z is embeddable in a group.*

Proof We embed Z in a particular semigroup which turns out to be a group. Begin with the diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, "U"] & \mathbf{B} \arrow[r, "V"] & \mathbf{Set} \arrow[ll, bend left, "L"] \end{tikzcd}$$

where $\mathbf{A}$ is the category of semigroups (not necessarily with 1), U forgetful, V forgetful and $(L, VU, \eta', \varepsilon')$ an adjunction. We then have a preorder $\mathbf{C}_Z$ which is a quotient of the free category $\mathbf{F}_Z$ generated by $C_V(Z)$. Proposition 3.1 (which also holds for the category $\mathbf{B}$ of partial algebras, see [12]) shows that an adjunction $(F, U, \eta, \varepsilon) : \mathbf{B} \rightharpoonup \mathbf{A}$ exists and describes it. It is sufficient to show that the unit $\eta_Z : Z \to UFZ$ of the adjunction is a monomorphism. By Theorem 4.1 it is sufficient to verify conditions (a) through (d). These conditions are trivial except for (c), which requires that the categories $(X/\mathbf{C})_{\mathcal{P}}$ be connected for each $\mathbf{G}$-reducible object X of $\mathbf{C}$. The objects of $\mathbf{C}_Z$ are elements of the free semigroup LVZ on VZ. An object X may be written as a string $(a_1, \cdots, a_n)$ of length $n \geq 1$ where $a_i \in VZ$ for $i = 1, \cdots, n$. It is sufficient to show that C_V arrows

$$\begin{tikzcd} & (a_1, \cdots, a_n) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ (\cdots, a_i a_{i+1}, \cdots) & & (\cdots, a_j a_{j+1}, \cdots) \end{tikzcd}$$

regarded as $(X/\mathbf{C}_Z)_{\mathcal{P}}$ objects can be connected by a finite sequence of morphisms in the same category. This requires a detailed argument when $i = j - 1$ or $i = j + 1$, otherwise it is trivial (cf. Baer [1], MacDonald [12]). □

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Universal Covers and Category Theory in Polynomial and Differential Galois Theory

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Abstract. The category of finite dimensional modules for the proalgebraic differential Galois group of the differential Galois theoretic closure of a differential field F is equivalent to the category of finite dimensional F spaces with an endomorphism extending the derivation of F. This paper presents an expository proof of this fact modeled on a similar equivalence from polynomial Galois theory, whose proof is also presented as motivation.

1 Introduction

We begin by recalling some notation, definitions, and standard results:

k denotes a field.

$K \supset k$ is a *splitting field*, or *polynomial Galois* extension, for the degree n monic separable polynomial

$$p = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0, \quad a_i \in k$$

if:

1. K is a field extension of k generated over k by $W = \{y \in K \mid p(y) = 0\}$("generated by solutions"); and

2. p is a product of linear factors in $K[X]$ ("full set of solutions").

For polynomial Galois extensions, let $G(K/k) = \mathrm{Aut}_k(K)$; note that $G(K/k) \to S_n(W)$ is an injection.

Then we have the familiar Fundamental Theorem for polynomial Galois extensions:

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Theorem (***Fundamental Theorem for Polynomial Galois Extensions***) *Let $K \supset k$ be a polynomial Galois extension. Then $G = G(K/k)$ is a finite group and there is a one-one lattice inverting correspondence between subfields M, $K \supset M \supset k$, and subgroups H of G given by $M \mapsto G(K/M)$ and $H \mapsto K^H$. If M is itself a polynomial Galois extension, then the restriction map $G \to G(M/k)$ is a surjection with kernel $G(K/M)$. If H is normal in G, then K^H is a polynomial Galois extension.*

There is a completely analogous theory for differential fields:

F denotes a differential field of characteristic zero with derivation $D = D_F$ and algebraically closed field of constants C.

$E \supset F$ is a *Picard–Vessiot*, or *Differential Galois*, extension for an order n monic linear homogeneous differential operator

$$L = Y^{(n)} + a_{n-1}Y^{(n-1)} + \cdots + a_1 Y^{(1)} + a_0 Y, \quad a_i \in F$$

if:

1. E is a differential field extension of F generated over F by $V = \{y \in E \mid L(y) = 0\}$ ("generated by solutions").

2. The constants of E are those of F ("no new constants").

3. $\dim_C(V) = n$ ("full set of solutions").

For Picard–Vessiot extensions, let $G(E/F) = \mathrm{Aut}_F^{\mathrm{diff}}(E)$; then $G(E/F) \to GL(V)$ is an injection with Zariski closed image.

There is a "Fundamental Theorem" for differential Galois extensions:

Theorem (***Fundamental Theorem for Picard–Vessiot Extensions***) *Let $E \supset F$ be a Picard–Vessiot extension. Then $G = G(E/F)$ has a canonical structure of affine algebraic group and there is a one-one lattice inverting correspondence between differential subfields K, $E \supset K \supset F$, and Zariski closed subgroups H of G given by $K \mapsto G(E/K)$ and $H \mapsto E^H$. If K is itself a Picard–Vessiot extension, then the restriction map $G \to G(K/F)$ is a surjection with kernel $G(E/K)$. If H is normal in G, then E^H is a Picard–Vessiot extension.*

There are Fundamental Theorems for infinite extensions as well:

Theorem (***Fundamental Theorem for Infinite Polynomial Galois Extensions***) *Let k be a field and let $K \supseteq k$ be a directed union of polynomial Galois field extensions of k. Then the group of automorphisms $G = G(K/k)$ has a canonical structure of topological (in fact profinite) group and there is a bijection between the set of closed subgroups of G, and the set of subfields of K containing k, under which a subgroup H corresponds to the subfield K^H of K fixed element–wise by H and the subfield M corresponds to the subgroup $Aut_M(K)$ of G which fixes each element of M. If M is itself a union of polynomial Galois extensions, then the restriction map $G \to G(M/k)$ is a surjection with kernel $G(K/M)$. If H is (closed and) normal in G, then M^H is a union of polynomial Galois extensions.*

Theorem (***Fundamental Theorem for Infinite Picard–Vessiot Extensions***) *Let $E \supset F$ be a directed union of Picard–Vessiot extensions. Then the group of differential automorphisms $G = G(E/F)$ has a canonical structure of proaffine group and there is a one-one lattice inverting correspondence between differential subfields K, $E \supset K \supset F$, and Zariski closed subgroups H of G given by $K \mapsto G(E/K)$ and $H \mapsto E^H$. If K is itself an infinite Picard–Vessiot extension, then the restriction map $G \to G(K/F)$ is a surjection with kernel $G(E/K)$. If H is (Zariski closed and) normal in G, then K^H is an infinite Picard–Vessiot extension.* [5]

We shall call these theorems (and their finite dimensional versions stated previously) "Correspondence Theorem Galois Theory". These theorems are about the pair consisting of the base field and the extension . There is a another aspect of Galois theory, which we will call "Universal Cover Galois Theory", which focuses on the base field and hopes to understand all possible (polynomial or differential) Galois extensions of the base by constructing a closure (or universal cover) and looking at its group of automorphisms.

The field k has a *separable closure*, which can be defined as a union of polynomial Galois extensions of k such that every polynomial Galois extension of k has an isomorphic copy in it. (More generally, every algebraic separable extension of k embeds over k in a separable closure of k.)

For various reasons, the direct analogues of "algebraic closure" and its properties for differential Galois extensions do not hold. However, the following notion is of interest, and can be shown to exist [7]:

A *Picard–Vessiot closure* $E \supset F$ of F is a differential field extension which is a union of Picard–Vessiot extensions of F and such that every such Picard–Vessiot extension of F has an isomorphic copy in E.

For a differential field extension of F to embed in a Picard–Vessiot closure, it is necessary and sufficient that it have the same constants as F and be generated over F as a differential field by elements that satisfy monic linear homogeneous differential equations over F [8, Prop. 13].

As noted, the goal of what we are calling Universal Cover Galois Theory is produce the (profinite or proalgebraic) Galois group of the (polynomial or differential) universal cover of the base field. In the next section, we will see how this is done in the polynomial case, and then in the following how it is done in the differential case.

1.1 History and literature. This is an expository article. Section 2 is basically an account of the special (one point) case of A. Grothendieck's theory of Galois categories and the fundamental group. I learned this material from J. P. Murre's account of it [9] which is still an excellent exposition. Differential Galois theory is the work of Ellis Kolchin [4]. For a comprehensive modern introduction, see M. van der Put and M. Singer [10]. The survey article by Singer [11] is also a good introduction. Less advanced are the author's introductory expository lectures on the subject [6], which is a reference for much of the terminology used here. Section 3 is an account of a version of the Tannakian Categories methods in differential Galois Theory. This originated in work of P. Deligne [3]; there are explanations of this in both [11] and [10]. A compact explanation of the theory as well as how to do the Fundamental Theorem in this context is also found in D. Bertrand's article

[1]. Information about the Picard–Vessiot closure is found in [7] and [8]. And of course the standard reference for Galois theory is categories is F. Borceaux and G. Janelidze [2].

2 The Galois group of the separable closure

The Fundamental Theorems recalled above were called "Correspondence Theorem Galois Theory". Even in their infinite forms, they are just special cases of Categorical Galois Theory [2]. The point of view of Correspondence Theory is from the extension down to the base: somehow the extension has been constructed and the group of automorphisms obtained (if only in principle) and then the lattice of intermediate fields is equivalent to the lattice of subgroups.

Now the naive view of Galois theory, say the one often adopted by students, is often the opposite: the point of view is from the base up. Despite the fact that, as Aurelio Carboni for example has noted, Galois Theory is about *not* solving equations, not about solving equations, the base up point of view begins with the base field, the (polynomial or differential) equation and asks for the solutions. Even though this doesn't work, let us imagine how to conduct such a project. We will deal first with the polynomial situation.

Let $p = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ be a separable polynomial over k which we will assume has no repeated irreducible factors (and hence no multiple roots). The set W of roots of p in a separable closure K of k has, by itself, only the structure of a finite set. The elements of W, however, are not completely interchangeable (remember we are taking the point of view of the base k): the elements of W are grouped according to the irreducible factor of p of which they are a root. Within these groupings they are interchangeable, but there are still limitations, namely any multivariable polynomial relations (with k coefficients) should be preserved as well. Of course the "interchanging" we are talking about is the action of the group $\pi_1(k) = G(K/k)$ on W.

(The reader will note the obvious circularity here: we are trying to describe the set of roots of p from the point of view of k, and to do so we introduce the separable closure and its group of k automorphisms. But this means that be have in principle found the roots not only of p but of every (separable) polynomial over of k!).

It is easy to check that the action of $\pi_1(k)$ on W is topological (which means simply that the stabilizers of points are open). And some natural questions arise, for example, do all finite sets with continuous $\pi_1(k)$ action come from polynomials over k, and if so, how? To take up the first, one should consider all finite sets with $\pi_1(k)$ action, and therefore the category $\mathcal{M}(\pi_1(k))$ of all of them, morphisms in the category being $\pi_1(k)$ equivariant maps. (It is a theorem of Grothendieck [9] that $\pi_1(k)$ can be recovered from $\mathcal{M}(\pi_1(k))$, as we will recall later.)

Now let us ask about how finite sets with continuous $\pi_1(k)$ action might come from polynomials. For the set $W = p^{-1}(0) \subset K$ we considered above, we could find p as $\prod_{\alpha \in W}(X - \alpha)$. But suppose we start with an arbitrary finite set X with continuous $\pi_1(k)$ action. If we want to repeat the above construction, then the first thing that should be considered is how to embed X in K, $\pi_1(k)$ equivariantly of course . While we do not know if such a map even exists, it is clear that no one such should be privileged. Thus the natural thing is to consider the set of all $\pi_1(k)$

equivariant maps $X \to K$. This set is a ring, under pointwise operations on K, and is even a k algebra (the latter sitting in it as constant functions).

We use $C(X,K)$ to denote all the functions from X to K. Then the set $C(X,K)$ is a commutative k algebra under pointwise operations, and $\pi_1(k)$ acts on it via $\sigma \cdot \phi(x) = \sigma(\phi(\sigma^{-1}x))$. We consider the ring of invariants $C(X,K)^{\pi_1(k)}$, which is the ring of $\pi_1(k)$ equivariant functions from X to K. Suppose that ϕ is such a function and x is an element of X. Let $\{x = x_1, \ldots, x_n\}$ be the orbit of x and let H be the intersection of the stabilizers of the x_i. Note that H is closed and normal and of finite index in $\pi_1(k)$. Then all the $\phi(x_i)$ lie in $M = K^H$, which is a polynomial Galois extension of k. Since X is a finite union of orbits, it follows, by taking the compositum of such extensions for each orbit, that there is a finite, normal separable extension $N \supset k$ such that $C(X,K)^{\pi_1(k)} = C(X,N)^{\pi_1(k)}$. Now $C(X,N)$ is a finite product of finite, separable extensions of k, and it follows that its subalgebra $C(X,N)^{\pi_1(k)}$ is as well.

In other words, our search for a polynomial related to X led to a commutative k algebra which is a finite product of finite separable extensions of k. We consider the category of all such:

Let $\mathcal{A}(k)$ be the category whose objects are finite products of finite separable field extensions of k and whose morphisms are k algebra homomorphisms. From the discussion above, we have a contrafunctor

$$\mathcal{U} = C(\cdot, K)^{\pi_1(k)} : \mathcal{M}(\pi_1(k)) \to \mathcal{A}(k).$$

On the other hand, for any object $A = K_1 \times \cdots \times K_n$ in $\mathcal{A}(k)$, we can consider the set $\mathcal{V}(A) = \mathrm{Alg}_k(A, K)$. We have that $\mathcal{V}(A)$ is a finite set (its cardinality is the dimension of A as an k vector space) and there is a left $\pi_1(k)$ action on $\mathcal{F}(A)$, given by following an embeding by an automorphism of K. All the embedings of A into K lie in a fixed finite, separable, normal subextension $K_0 \supseteq k$ of K, and this implies that the action of $\pi_1(k)$ on $\mathcal{V}(A)$ is continuous. Thus we also have a contrafunctor

$$\mathcal{V} = \mathrm{Alg}_k(\cdot, K) : \mathcal{A}(k) \to \mathcal{M}(\pi_1(k))$$

to the category $\mathcal{M}(\pi_1(k))$ of finite sets with continuous $\pi_1(k)$ action.

We will see that $\mathcal{U}$ and $\mathcal{V}$ are equivalences of categories.

Here are some properties of the functor $\mathcal{V}$: if $K_0 \supseteq k$ is a finite separable extension, then $\mathcal{V}(K_0) = \mathrm{Alg}_k(K_0, K)$ has cardinality the dimension of K_0 over k. If $A = K_1 \times \cdots \times K_n$ is a finite product of finite separable extensions of k, then every homomorphism from A to a field must factor through a projection onto a K_i, so it follows that $\mathcal{V}(A)$ is the (disjoint) union $\mathcal{V}(K_1) \amalg \cdots \amalg \mathcal{V}(K_n)$ and hence has cardinality $\sum |\mathcal{V}(K_i)| = \sum \dim_k(K_i) = \dim_k(A)$. We also note that since K is the separable closure of k, $\mathcal{V}(A) = \mathrm{Alg}_k(A, K)$ is always non–empty.

And some properties of the functor $\mathcal{U}$: if X is a finite set with continuous $\pi_1(k)$ action and $X = X_1 \amalg X_2$ is a disjoint union of two proper $\pi_1(k)$ subsets, then the inclusions $X_i \to X$ induce maps $C(X,K) \to C(X_1,K)$ which in turn give an isomorphism $C(X,K) \to C(X_1,K) \times C(X_2,K)$. All these are $\pi_1(k)$ equivariant, and hence give an isomorphism $\mathcal{U}(X) \to \mathcal{U}(X_1) \times \mathcal{U}(X_2)$. Now suppose that X does not so decompose, which means that X is a single orbit, say of the element x with stabilizer H (which is, of course, closed and of finite index in $\pi_1(k)$). Then

$\pi_1(k)$ equivariant maps from X to K are determined by the image of x, which may be any element with stabilizer H. Thus $C(X,K)^{\pi_1(k)} \to K^H$ by $\phi \mapsto \phi(x)$ is a bijection. Combining this observation with the previous then yields the following formula for $\mathcal{U}$: If $X = X_1 \amalg \cdots \amalg X_n$ is a disjoint union of orbits with orbit representatives x_i with stabilizers H_i, then $\mathcal{U}(X) = \prod K^{H_i}$. The index of H_i in $\pi_1(k)$ is both the cardinality of X_i and the dimension of K^{H_i} over k, and it follows that $\dim_k(\mathcal{U}(X)) = |X|$. We also note that $\mathcal{U}(X) = \prod K^{H_i}$ is always non–zero.

Combining, we have cardinality/dimension equalities, $|\mathcal{U}(\mathcal{V}(X))| = |X|$ and $\dim_k(\mathcal{V}(\mathcal{U}(A)) = \dim_k(A)$

We also have "double dual" maps

$$A \to \mathcal{U}(\mathcal{V}(A)) = C(\mathrm{Alg}_k(A,K),K)^{\pi_1(k)} \text{ by } a \mapsto \hat{a}, \text{ where } \hat{a}(\tau) = \tau(a).$$

and

$$X \to \mathcal{V}(\mathcal{U}(X)) = \mathrm{Alg}_k(C(X,K)^{\pi_1(k)},K) \text{ by } x \mapsto \hat{x}, \text{ where } \hat{x}(\phi) = \phi(x).$$

It follows from the cardinality/dimension equalities that in the case that A is a field or X is a single orbit that the double dual maps are bijections, and then that they are bijections in general from the product formulae above.

The above remarks imply that the functors $\mathcal{U}$ and $\mathcal{V}$ give an (anti)equivalence of categories, a result which we now state as a theorem:

Theorem (*Categorical Classification Theorem*) *Let k be a field, let K be a separable closure of k and let $\pi_1(k) = Aut_k(K)$. Then $\pi_1(k)$ has a natural topological structure as a profinite group. Let $\mathcal{A}(k)$ denote the category of commutative k algebras which are finite products of finite separable field extensions of k and k algebra homomorphisms. Let $\mathcal{M}(\pi_1(k))$ denote the category of finite sets with continuous $\pi_1(k)$ action, and $\pi_1(k)$ equivariant functions. Consider the contravariant functors*

$$\mathcal{U} = C(\cdot,K)^{\pi_1(k)} : \mathcal{M}(\pi_1(k)) \to \mathcal{A}(k)$$

and

$$\mathcal{V} = Alg_k(\cdot,K) : \mathcal{A}(k) \to \mathcal{M}(\pi_1(k)).$$

Then both compositions $\mathcal{U}\circ\mathcal{V}$ and $\mathcal{V}\circ\mathcal{U}$ are naturally isomorphic to the identity using the double dual maps, and hence the categories $\mathcal{A}(k)$ and $\mathcal{M}(\pi_1(k))$ are equivalent.

The proofs of the assertions summarized as the Categorical Classification Theorem depended on the Fundamental Theorem of Galois Theory. Conversely, the Categorical Classification Theorem can be used to prove the Fundamental Theorem:

Suppose $K_0 \supset k$ is a finite, normal, separable extension, and that $\tau : K_0 \to K$ is an embeding over k. By normality, $\mathcal{V}(K_0) = \mathrm{Alg}_k(K_0,K)$ is a single orbit, and the stabilizer H of τ is a closed normal subgroup of $\pi_1(k)$ with $\pi_1(k)/H$ isomorphic to $G = \mathrm{Aut}_k(K_0)$. Also, $\mathcal{V}(k) = \{\mathrm{id}_k\}$ is a final object. The transitive $\pi_1(k)$ sets X which fit into a diagram $\mathcal{V}(K_0) \to X \to \mathcal{V}(k)$ are the $\pi_1(k)$ sets between $\pi_1(k)/H$ and $\pi_1(k)/\pi_1(k)$, namely those of the form $\pi_1(k)/K$ where K is a closed subgroup of $\pi_1(k)$ containing H, and thus correspond one–one to subgroups of $\pi_1(k)/H$. The quotients of $\mathcal{V}(K_0)$ correspond, by $\mathcal{U}$, to the subobjects of E. Thus once the Categorical Classification Theorem is available, the Fundamental Theorem of Galois Theory (for finite field extensions) translates into the simple correspondence between (homogeneous) quotients of a finite homogeneous space and the subgroups of the transformation group. We state this result, noting that it implies the Fundamental Theorem of Galois Theory:

Theorem (Fundamental Theorem for Faithful Transitive G Sets) *Let G be a finite group and regard G as a finite set on which G acts transitively and with trivial stabilizers, and let e be the identity of G. Let Z be a one point set and $p : X \to Z$ a map. Then transitive G sets Y and classes of G equivariant surjective maps $q : G \to Y$ which factor through p are in one–one correspondence with subgroups H of G as follows: to the subgroup H, make correspond the G set G/H and the map $G \to G/H$ by $g \mapsto gH$; to the surjective G map $q : G \to X$, make correspond the stabilizer H of $q(e)$.*

3 The Galois group of the Picard–Vessiot closure

In the preceding section, we saw how the category $\mathcal{M}(\pi_1(k))$ of finite sets on which the profinite Galois group of the separable closure of k acts continuously was (anti)equivalent to a category of k algebras. And we recalled that for any profinite group, the category of finite sets on which it acts continuously determines it. A similar statement is true about proalgebraic groups: such a group is determined by the category of vector spaces (or modules) on which it acts algebraically (this is the general Tannaka Duality Theorem [3]). The group of differential automorphisms $\Pi(F) = G(E/F)$ of the Picard–Vessiot closure E of the differential field F is a proalgebraic group, and it is therefore natural to consider the analogue of the functors of the preceding section in the differential case.

Thus we consider the category $\mathcal{M}(\Pi(F))$ of finite dimensional, algebraic, $\Pi(F)$ modules, and the functor $\mathrm{Hom}_{\Pi(F)}(\cdot, E)$ defined on it. The proalgebraic group $\Pi(F)$ is over the field C of constants of F, and the vector spaces in $\mathcal{M}(\Pi(F))$ are over C. The field E is not a $\Pi(F)$ module, although of course $\Pi(F)$ acts on it, since not every element in E has a $\Pi(F)$ orbit that spans a finite dimensional C vector space. Those elements that do form an F subalgebra S of E, which is additionally characterized by the fact that it consists of the elements of E which satisfy a linear homogeneous differential equation over F (see [6, Prop. 5.1, p.61], and [8]). Any $\Pi(F)$ equivariant homomorphism from an algebraic $\Pi(F)$ module to E must have image in S, so the functor to be considered is actually $\mathcal{V}(U) = \mathrm{Hom}_{\Pi(F)}(U, S)$.

It is clear that $\mathcal{V}(U)$, for U an object of $\mathcal{M}(\Pi(F))$, is an abelian group under pointwise addition of of functions. It is also true that $\mathcal{V}(U)$ is an F vector space via multiplication in the range of functions. We will see below that $\mathcal{V}(U)$ is finite dimensional over F. The derivation D of E preserves S, and this derivation of S defines an operator, which we also call D, on $\mathcal{V}(U)$ as follows: let $f \in \mathcal{V}(U)$ and let $u \in U$. Then $D(f)(u)$ is defined to be $D(f(u))$. It is easy to check that D on $\mathcal{V}(U)$ is additive and in fact is C linear. It is not F linear, but we do have the following formula: for $\alpha \in F$ and $f \in \mathcal{V}(U)$, $D(\alpha f) = D(\alpha)f + \alpha D(f)$.

This suggests we consider the category $\mathcal{M}(F \cdot D)$ of finite dimensional F vector spaces V equipped with C linear endomorphisms D_V (usually abbreviated D) such that for $\alpha \in F$ and $v \in V$, $D(\alpha v) = D(\alpha)v + \alpha D(v)$, morphisms being F linear maps which commute with D action. We call objects of $\mathcal{M}(F \cdot D)$ *$F \cdot D$ modules*, and morphisms of $\mathcal{M}(F \cdot D)$ *$F \cdot D$ morphisms*. (Sometimes $F \cdot D$ modules are known as connections [11, 2.4.1 p.536].) The contrafunctor $\mathcal{V}$ sends all objects and morphisms $\mathcal{M}(\Pi(F))$ to $\mathcal{M}(F \cdot D)$ (we still have to establish that $\mathcal{V}(U)$ is always finite dimensional over F). We note that, except for the finite dimensionality, S is like an object in $\mathcal{M}(F \cdot D)$ in that it has an operator D satisfying the appropriate relation, and for an object in $\mathcal{M}(F \cdot D)$ we will use $\mathrm{Hom}_{F \cdot D}(V, S)$ to denote the F linear D preserving homomorphisms from V to S.

It is clear that $\mathrm{Hom}_{F \cdot D}(V, S)$ is an abelian group under pointwise addition of functions, and a C vector space under the usual scalar multiplication operation.

The group $\Pi(F)$ acts on $\mathrm{Hom}_{F \cdot D}(V, S)$: for $\sigma \in \Pi(F), T \in \mathrm{Hom}_{F \cdot D}(V, S)$, and $v \in V$, define $\sigma(T)(v) = \sigma(T(v))$. We will see later that $\mathrm{Hom}_{F \cdot D}(V, S)$ is a finite dimensional C vector space, and that the action of $\Pi(F)$ on it is algebraic. Thus we will have a contrafunctor $\mathcal{U}(V) = \mathrm{Hom}_{F \cdot D}(V, S)$ from $\mathcal{M}(F \cdot D)$ to $\mathcal{M}(\Pi(F))$.

The pair of functors $\mathcal{U}$ and $\mathcal{V}$, therefore, are the analogues for the differential Galois case of the corresponding functors in the polynomial Galois case. And we will see that, as in the polynomial Galois case, both $\mathcal{V}(\cdot)$ and $\mathcal{U}(\cdot)$ are equivalences.

We begin by describing the $\Pi(F)$-module structure of S, and for this we now fix the following notation:

Notation 3.1 *Let Π denote $\Pi(F)$, the differential Galois group of the Picard–Vessiot closure E of F, and let Π^0 denote its identity component and $\overline{\Pi} = \Pi/\Pi^0$ the profinite quotient.*

We denote the algebraic closure of F by $\overline{F}$. We regard $\overline{F}$ as emdedded in S, where it is a Π submodule and, since also $\overline{F} = S^{\Pi^0}$, a $\overline{\Pi}$ module. Therefore, when we need to regard $\overline{F}$ as a trivial Π module we will denote it $\overline{F}_t$.

Proposition 3.2 *In Notation* (3.1),

1. $\overline{F}_t \otimes_F S \cong \overline{F}_t \otimes_C C[\Pi]$ *as $\overline{F}_t$ algebras and Π modules.*

2. $S \cong \overline{F} \otimes_C C[\Pi^0]$ *as $\overline{F}$ algebras and Π^0 modules.*

Proof Statement (1) is the infinite version of Kolchin's Theorem, [9, Thm. 5.12, p.67]. Since E is also a Picard–Vessiot closure of $\overline{F}$, whose corresponding ring is S as an $\overline{F}$ algebra, statement (2) is Kolchin's Theorem as well. □

We can analyze the functor $\mathcal{V} : \mathcal{M}(\Pi) \to \mathcal{M}(F \cdot D)$ using the structural description of the preceding theorem: since $\mathcal{V}(U) = \mathrm{Hom}_\Pi(U, S)$ we have

$$\begin{aligned} \mathcal{V}(U) = \mathrm{Hom}_\Pi(U, S) &= (\mathrm{Hom}_{\Pi^0}(U, S))^{\overline{\Pi}} \\ &= (\mathrm{Hom}_{\Pi^0}(U, \overline{F} \otimes_C C[\Pi^0]))^{\overline{\Pi}} \\ &= (\overline{F} \otimes_C \mathrm{Hom}_{\Pi^0}(U, C[\Pi^0]))^{\overline{\Pi}} \qquad (\mathcal{V} \text{ factor}) \end{aligned}$$

(For the third equality of ($\mathcal{V}$ factor) we used the isomorphism of Proposition (3.2)(2), and for the final equality of ($\mathcal{V}$ factor), we used the fact that U was finite dimensional.)

Using the decomposition ($\mathcal{V}$ factor), it is a simple matter to see that $\mathcal{V}$ is exact:

Proposition 3.3 *The contrafunctor $\mathcal{V} : \mathcal{M}(\Pi) \to \mathcal{M}(F \cdot D)$ is exact. Moreover, $\mathcal{V}(U)$ is F finite dimensional with $\dim_F(\mathcal{V}(U)) = \dim_C(U)$.*

Proof In ($\mathcal{V}$ factor), we have factored $\mathcal{V}$ as the composition of four functors: first the forgetful functor from Π modules to Π^0 modules, then $U \mapsto \mathrm{Hom}_{\Pi^0}(U, C[\Pi^0])$, $(\cdot) \mapsto \overline{F} \otimes_C (\cdot)$, and $(\cdot) \mapsto (\cdot)^{\overline{\Pi}}$. The first of these is obviously exact. For exactness of the second, we use that $C[\Pi^0]$ is an injective Π^0 module (in fact, for any finite dimensional algebraic Π^0 module W the map

$\mathrm{Hom}_{\Pi^0}(W, C[\Pi^0]) \to (W)^*$ by evaluation at the identity is a C isomorphism to the C linear dual of W). The third functor is also obviously exact. Since our modules are over a field of characteristic zero, taking invariants by a profinite group is also exact, and hence the final functor is exact as well.

To compute dimensions, we note that $\dim_F(\mathcal{V}(U)) = \dim_{\overline{F}}(\mathcal{V}(U) \otimes_F \overline{F})$, then that $\mathcal{V}(U) \otimes_F \overline{F} = \mathrm{Hom}_\Pi(U, S) \otimes_F \overline{F} = \mathrm{Hom}_\Pi(U, \overline{F}_t \otimes_F S)$, and by Proposition (3.2)(1), this latter is $\mathrm{Hom}_\Pi(U, \overline{F}_t \otimes_C C[\Pi]) = \mathrm{Hom}_\Pi(U, C[\Pi]) \otimes_C \overline{F} = U^* \otimes_C \overline{F}$, which has the same dimension over $\overline{F}$ as U does over C. □

Now we turn to the functor $\mathcal{U} = \mathrm{Hom}_{F \cdot D}(\cdot, S)$, and we will see that it also is exact and preserves dimensions. For both of these, we will need a few comments about cyclic $F \cdot D$ modules:

Remark 3.4 An $F \cdot D$ module W is *cyclic*, generated by x, if W is the smallest $F \cdot D$ submodule of W containing x. For any $F \cdot D$ module V and any element $x \in V$, the F span of its derivatives $\sum_{i \geq 0} FD^i(x)$ is a cyclic $F \cdot D$ module, generated by x. If $n = \dim_F(V)$, and $\{D^0x, D^1x, \ldots, D^{k-1}x\}$ is a maximal linearly independent set, then there are elements $\alpha_i \in F$ with $D^kx + \alpha_{k-1}D^{k-1}x + \cdots + \alpha_0 D^0 x = 0$. Note that $k \leq n$. We refer to the differential operator $L = Y^{(k)} + \alpha_{k-1}Y^{(k-1)} + \cdots + \alpha_0 Y$ as the *differential operator corresponding to* x in V.

Now we turn to exactness of $\mathcal{U}$

Proposition 3.5 *The contrafunctor* $\mathcal{U} : \mathcal{M}(F \cdot D) \to \mathcal{M}(\Pi)$ *is exact.*

Proof Since $\mathcal{U}$, being a "Hom into" functor, is right exact, what we need to show is that it carries monomorphisms $V_1 \to V_2$ into epimorphisms. We can assume that the monomorphism is an inclusion and that V_2 is generated over V_1 by a single element x (that is, that V_2 is the sum of V_1 and the cyclic submodule of V_2 generated by x.) We suppose given an $F \cdot D$ morphism $T_1 : V_1 \to S$. We consider the symmetric algebras over F on V_1 and V_2, which we denote $F[V_1]$ and $F[V_2]$. The D operators on the V_i extend to derivations of the $F[V_i]$, and T_1 extends to a differential homomorphism $h : F[V_1] \to S$. We have $F[V_1] \subset F[V_2]$ (this is split as a extension of polynomial algebras over F), and $F[V_2]$ is generated over $F[V_1]$ as a differential algebra by x, which is denoted $F[V_2] = F[V_1]\{x\}$. We will also use h for the extension of h to the quotient field E of S. Let P be the kernel of h, let $\overline{F[V_1]} = F[V_1]/P$ and let $\overline{F[V_2]} = F[V_2]/PF[V_2]$. (Since $PF[V_2]$ is a differential ideal, this latter is a differential algebra.) If $\overline{x}$ denotes the image of x in $\overline{F[V_2]}$, then $\overline{F[V_2]} = \overline{F[V_1]}\{\overline{x}\}$. Now we extend scalars to E:

$$R = E \otimes_{\overline{F[V_1]}} \overline{F[V_1]}\{\overline{x}\}.$$

Note that R is finitely generated as an algebra over E. This implies that if Q is any maximal differential ideal of R, then the quotient field K of R/Q is a differential field extension of E with the same constant field C [6, Cor. 1.18, p. 11]. By construction, K is generated over E as a differential field by the image y of $\overline{x}$. Now x, and hence $\overline{x}$ and y, is the zero of a linear differential operator L of order k, the operator corresponding to x defined above in Remark (3.4). On the other hand, E already contains a Picard–Vessiot extension of F for L, and hence a full set of zeros of L (that is, of dimension k over C). Since K has no new constants, the zero y of L must belong to this set and hence $y \in E$. But this then implies $K = E$. Thus we have a differential F algebra homomorphism $f : F[V_2] \to \overline{F[V_2]} \to R \to R/Q \to E$, and by construction f restricted to $F[V_1]$ is h. Moreover, the image y of x lies in S (since it satisfies a differential equation over F) and hence f has image in S. Finally, the restriction T_2 of f to V_2 is an $F \cdot D$ morphism from V_2 to S extending $T_1 : V_1 \to S$. It follows that $\mathcal{U}$ is left exact, as desired. □

Using exactness, we can also show how $\mathcal{U}$ preserves dimension:

Proposition 3.6 *$\mathcal{U}(V)$ is C finite dimensional with $dim_C(\mathcal{U}(V)) = dim_F(V)$.*

Proof By Proposition (3.5), $\mathcal{U}$ is exact, and since dimension is additive on exact sequences, we can reduce to the case that the $F \cdot D$ module V is cyclic with generator x. Then, by Remark (3.4), V has F basis $\{D^0x, D^1x, \ldots, D^{k-1}x\}$ and corresponding linear operator $L = Y^{(k)} + \alpha_{k-1}Y^{(k-1)} + \cdots + \alpha_0 Y$. Then an $F \cdot D$ morphism $V \to S$ is determined by the image of x, which is an element of S sent to zero by L, and every such element of S determines a morphism. Thus $\mathcal{U}(V)$ is the zeros of L in S, which is the same as the zeros of L in E. Since E contains a Picard–Vessiot extension of F for L, and hence a complete set of solutions, we have $\dim_C(\mathcal{U}(V)) = \dim_C(L^{-1}(0)) = k = \dim_F(V)$. □

Both $\mathcal{U}$ and $\mathcal{V}$ involve a "duality" into S, and hence a "double duality" which we now record, and use to prove that the functors are equivalences.

Theorem 3.7

1. *The function $V \to \mathcal{V}(\mathcal{U}(V)) = Hom_\Pi(Hom_{F \cdot D}(V, S), S)$ by $v \mapsto \hat{v}$, $\hat{v}(T) = T(v)$ is an $F \cdot D$ isomorphism natural in V.*

2. *The function $U \to \mathcal{U}(\mathcal{V}(U)) = Hom_{F \cdot D}(Hom_\Pi(U, S), S)$ by $u \mapsto \hat{u}$, $\hat{u}(\phi) = \phi(u)$ is a Π isomorphism natural in U.*

In particular, $\mathcal{U}$ and $\mathcal{V}$ are category equivalences between the categories $\mathcal{M}(F \cdot D)$ and $\mathcal{M}(\Pi(F))$.

Proof We leave to the reader to check that the maps $v \mapsto \hat{v}$ and $u \mapsto \hat{u}$ are well defined and natural in V and U. To see that they are isomorphisms, we use the fact that $\mathcal{V} \circ \mathcal{U}$ and $\mathcal{U} \circ \mathcal{V}$ are exact to reduce to the case of checking the isomorphism for non–zero simple modules, and then use the fact that $\mathcal{V} \circ \mathcal{U}$ and $\mathcal{U} \circ \mathcal{V}$ preserve dimension to reduce to showing that both double dual maps are non–zero. For (1), this means that there is a non–zero $T \in \mathrm{Hom}_{F \cdot D}(V, S) = \mathcal{U}(V)$. But since $\dim_C(\mathcal{U}(V)) = \dim_F(V) \neq 0$, this holds. For (2), this means that there is a non–zero $\phi \in \mathrm{Hom}_\Pi(U, S) = \mathcal{V}(U)$. Since $\dim_F(\mathcal{V}(U)) = \dim_C(U) \neq 0$, this holds as well. Thus the theorem is proved. □

Theorem (3.7) tells us that the category of $\Pi(F)$ modules is equivalent to the category of $F \cdot D$ modules. As we mentioned above, the proalgebraic group $\Pi(F)$ can be recovered from its category of modules $\mathcal{M}(\Pi(F))$ by the Tannaka Duality. We review this construction briefly: a *tensor automorphism* of $\mathcal{M}(\Pi(F))$ is a family of vector space automorphisms $\sigma_U, U \in |\mathcal{M}(\Pi(F))|$, one for each object in $\mathcal{M}(\Pi(F))$, such that

1. For any $\Pi(F)$ homomorphism $\phi : U \to U'$ we have $\sigma_{U'}\phi = \phi\sigma_U$, and

2. For any $\Pi(F)$ modules U and U', we have $\sigma_{U \otimes U'} = \sigma_U \otimes \sigma_{U'}$

An example of a tensor automorphism is $\mathrm{Id}_U, U \in |\mathcal{M}(\Pi(F))$.

The composition of tensor automorphisms are tensor automorphisms (composition of $\sigma_U, U \in |\mathcal{M}(\Pi(F))|$. and $\tau_U, U \in |\mathcal{M}(\Pi(F))|$ is $\sigma_U\tau_U, U \in |\mathcal{M}(\Pi(F))|$) and so are inverses, and the tensor automorphism $\mathrm{Id}_U, U \in |\mathcal{M}(\Pi(F))|$ is an identity for composition. Thus the tensor automorphisms form a group, denoted $\mathrm{Aut}_\otimes(\Pi(F))$. For notational convenience, we will denote the element of $\mathrm{Aut}_\otimes(\Pi(F))$ given by $\sigma_U, U \in |\mathcal{M}(\Pi(F))|$ simply as σ

If $U \in |\mathcal{M}(\Pi(F))|$, and $u \in U$ and $f \in U^*$, then we can define a function $m_{u,f}$ on $\mathrm{Aut}_\otimes(\Pi(F))$ by $m_{u,f}(\sigma) = f(\sigma_U(u))$. The C algebra of all such functions is denoted $C[\mathrm{Aut}_\otimes(\Pi(F))]$. One shows, as part of Tannaka Duality, that $C[\mathrm{Aut}_\otimes(\Pi(F))]$ is the coordinate ring of a proalgebraic group structure on $\mathrm{Aut}_\otimes(\Pi(F))$.

For any $g \in \Pi(F)$ and any $U \in |\mathcal{M}(\Pi(F))|$, let $L(g)_U$ denote the left action of g on U. Then $L(g)_U, U \in |\mathcal{M}(\Pi(F))|$ is a tensor automorphism, and $L : \Pi(F) \to \mathrm{Aut}_\otimes(\Pi(F))$ is a group homomorphism. Tannaka duality proves that L is actually a group isomorphism (of proalgebraic groups). Thus the proalgebraic group $\Pi(F)$ is recovered from the category $\mathcal{M}(\Pi(F))$ modules. (This procedure works for any proalgebraic group.)

Because of the importance of the tensor product in the Tannaka Duality, we record how the tensor products in $\mathcal{M}(F \cdot D)$ and $\mathcal{M}(\Pi)$ interact with the functors $\mathcal{U}$ and $\mathcal{V}$.

Proposition 3.8 *There are natural (and coherent) isomorphisms*

1. $\mathcal{V}(U_1) \otimes_F \mathcal{V}(U_2) \to \mathcal{V}(U_1 \otimes_C U_2)$, *and*

2. $\mathcal{U}(V_1) \otimes_C \mathcal{U}(V_2) \to \mathcal{U}(V_1 \otimes_F V_2)$.

Proof The map in (1) is defined as follows: if $\phi_i \in \mathrm{Hom}_\Pi(U_i, S)$, then $\phi_1 \otimes_F \phi_2$ is sent to the function in $\mathrm{Hom}_\Pi(U_1 \otimes_C U_2, S)$ given by $u_1 \otimes u_2 \mapsto \phi_1(u_1)\phi_2(u_2)$. Since $\mathcal{V}(U_1) \otimes_F \mathcal{V}(U_2)$ and $\mathcal{V}(U_1 \otimes_C U_2)$ have the same dimension over F (namely that of $U_1 \otimes_C U_2$ over C), to see that the map is an isomorphism it suffices to see that it is injective. To that end, we tensor it over F with $\overline{F}_t$. Then we consider successively:

$$(\mathrm{Hom}_\Pi(U_1, S) \otimes_F \mathrm{Hom}_\Pi(U_2, S)) \otimes_F \overline{F}_t \to \mathrm{Hom}_\Pi(U_1 \otimes_C U_2, S) \otimes_F \overline{F}_t$$

we distribute $\overline{F}_t$ over the tensors and inside the Hom's

$$(\mathrm{Hom}_\Pi(U_1, S) \otimes_F \overline{F}_t) \otimes_{\overline{F}_t} (\mathrm{Hom}_\Pi(U_2, S)) \otimes_F \overline{F}_t) \to \mathrm{Hom}_\Pi(U_1 \otimes_C U_2, S \otimes_F \overline{F}_t)$$

then we apply Proposition (3.2) (1)

$$\mathrm{Hom}_\Pi(U_1, \overline{F}_t \otimes_C C[\Pi]) \otimes_{\overline{F}_t} \mathrm{Hom}_\Pi(U_2, \overline{F}_t \otimes_C C[\Pi])) \to \mathrm{Hom}_\Pi(U_1 \otimes_C U_2, \overline{F}_t \otimes_C C[\Pi])$$

and finally use that spaces of Π maps into $C[\Pi]$ are duals

$$(\mathrm{Hom}_C(U_1, \overline{F}_t) \otimes_{\overline{F}_t} \mathrm{Hom}_C(U_2, \overline{F}_t) \to \mathrm{Hom}_C(U_1 \otimes_C U_2, \overline{F}_t).$$

And this final map is, of course, an isomorphism. This proves (1).

The map in (2) is defined similarly: if $T_i \in \mathrm{Hom}_{F \cdot D}(V_i, S)$ then $T_1 \otimes_F T_2$ is sent to the function in $\mathrm{Hom}_{F \cdot D}(V_1 \otimes V_2, S)$ given by $v_1 \otimes v_2 \mapsto T_1(v_1)T_2(v_2)$. To prove (2), we may assume that $V_i = \mathcal{V}(U_i)$, so we are trying to show that $\mathcal{U}(\mathcal{V}(U_1)) \otimes_C \mathcal{U}(\mathcal{V}(U_2)) \to \mathcal{U}(\mathcal{V}(U_1) \otimes_F \mathcal{V}(U_2))$ is an isomorphism. Note that the domain is, by Theorem (3.7) (2), $U_1 \otimes_C U_2$. On the other hand, if we apply $\mathcal{V}$ to

(1), then we have an isomorphism $\mathcal{U}(\mathcal{V}(U_1 \otimes_C U_2)) \to \mathcal{U}(\mathcal{V}(U_1) \otimes_F \mathcal{V}(U_2))$. Here again, by Theorem (3.7) (2) the domain is $U_1 \otimes_C U_2$. It is a simple matter to check that both maps are the same, and hence conclude (2). $\square$

4 Conclusion

We try to set some of the above in perspective. We consider the problem of understanding the proalgebraic differential Galois group $\Pi(F)$ of a Picard–Vessiot closure of F. By Tannaka Duality, $\Pi(F)$ is determined by and recoverable from its category $\mathcal{M}(\Pi(F))$ of finite dimensional over C algebraic modules – the tensor product over C in $\mathcal{M}(\Pi(F))$ being an essential part of the structure. The (anti)equivalence $\mathcal{U}$ and its inverse $\mathcal{V}$ show that the category $\mathcal{M}(F \cdot D)$ of finite dimensional F spaces with an endomorphism compatible with the derivation on F is (anti)equivalent to the category $\mathcal{M}(\Pi(F))$. In other words, we might say that every finite dimensional $F \cdot D$ module has a "secret identity" as a $\Pi(F)$ module (more appropriately, perhaps, a "dual secret identity", since the equivalences are contravariant). And this identification includes converting tensors over F of $F \cdot D$ modules into tensors over C for $\Pi(F)$ modules. It follows, at least in principle, that the group $\Pi(F)$ is determined by, and determines, the category of $F \cdot D$ modules. So everything that could be learned about F from the group $\Pi(F)$ can be learned by studying $F \cdot D$ modules.

We would also like to tie this observation about differential Galois theory with our earlier discussion of polynomial Galois theory. In that case, we began by considering sets of solutions of polynomial equations in isolation, that is, simply as finite sets, and then found that the structure necessary to tell these "disembodied sets of solutions" from unstructured finite sets was an action of the Galois group of the separable closure. In the same way, modules for the differential Galois group are like "disembodied sets of solutions" for differential equations. But unlike the situation with the polynomial equations, where duality with respect to the closure leads from solution sets to extension fields (actually finite products of extension fields), in the differential case duality with respect to the closure lead from solution spaces to $F \cdot D$ modules, which are more like "disembodied differential equations" (see Remark (3.4)) than extensions. There is a way to pass from $F \cdot D$ modules to extensions (we saw some of this construction in the proof of Proposition (3.5)): for an $F \cdot D$ module V, we form the F symmetric algebra $F[V] = S_F[V]$. This F algebra has a derivation extending that of F, and if we mod out by a maximal differential ideal Q we obtain a differential F integral domain whose quotient field has the same constants as F. One can then show that this domain embeds in a Picard–Vessiot closure of F [8, Prop. 13], and in particular into a Picard–Vessiot extension of F. Different choices of Q are possible, of course. But each arises from a differential F algebra homomorphism from $F[V]$ to the Picard–Vessiot closure of F and hence from differential F algebra homomorphisms $F[V] \to S$. (Since these latter correspond to $F \cdot D$ module homomorphisms $V \to S$, we see our functor $\mathcal{U}$.) One should regard the whole collection of these homomorphisms, or at least all their images, as the corresponding object to the (finite product of) field extensions in the polynomial case.

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Weak Categories in Additive 2-Categories with Kernels

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To my parents Ilídio and Bertina.

Abstract. We introduce a notion of weak category, define additive 2-categories and describe weak categories in them. We make this description more explicit in the case of the additive 2-category of morphisms of abelian groups. In particular we present internal bicategories in the category of abelian groups as presheaf categories.

1 Introduction

Consider the notion of monoidal category as an internal structure in $(\mathrm{Cat}, \times, 1)$. More generally consider it in an abstract 2-category, not necessarily Cat. For the notion emerging in this way we use the name *weak monoid.* Table 1 describes the notion of monoid and weak monoid (for the cases where it is applicable) in some concrete examples of ambient categories.

Table 1

Ambient Category	Monoids	Weak Monoids
Set	ordinary monoids	N/A
$\mathcal{O}$-graphs(Set)	ordinary categories (objects are the elements of $\mathcal{O}$)	N/A
Cat	strict monoidal categories	monoidal categories
$\mathcal{O}$-graphs(Cat)	double categories (the vertical structure is $\mathcal{O}$)	*Weak Categories*
$\mathcal{O}$-graphs(Cat) ($\mathcal{O}$ discrete)	2-categories (objects are the elements of $\mathcal{O}$)	bicategories (objects are the elements of $\mathcal{O}$)

The term *weak category* appears in this way as a generalization of double category and bicategory.

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In this work, after giving explicit description of weak monoids and weak categories, we analyze internal weak categories in additive 2-categories with kernels. We also introduce the notion of additive 2-category by defining 2-Ab-category and more generally 2-V-category where V is a monoidal category.

An example of additive 2-category with kernels is Mor(Ab) and we show that a weak category in Mor(Ab) is completely determined by four abelian groups A_1, A_0, B_1, B_0, together with four group homomorphisms $\partial, \partial', k_1, k_0$, such that the following square is commutative

$$\begin{array}{ccc} A_1 & \xrightarrow{\partial} & A_0 \\ {\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle k_0} \\ B_1 & \xrightarrow{\partial'} & B_0 \end{array}$$

and three more group homomorphisms

$$\lambda, \rho : A_0 \longrightarrow A_1 \ , \ \eta : B_0 \longrightarrow A_1,$$

such that

$$\begin{aligned} k_1\lambda &= 0 = k_1\rho, \\ k_1\eta &= 0. \end{aligned}$$

This result generalizes at the same time the description of internal double categories and of internal bicategories in Ab. Let the morphisms λ, ρ, η become zero morphisms; then we obtain the known description of internal double categories in Ab (for a similar description of (strict) n-categories see e.g. [5],[2],[6] and references there). Taking instead B_1 to be the trivial group, we obtain the description of internal bicategories in Ab [3].

2 Weak categories

This section begins with the formal definition of weak monoid and shows how the notion of weak category can be regarded as a weak monoid. Nevertheless, to consider the category of all weak categories (see [4]), an explicit definition of weak category is required. Last part of the section gives an explicit definition of weak category.

2.1 Weak monoids. An ordinary monoid in a monoidal category $(M, \Box, \mathbf{1})$ is a diagram

$$C\Box C \xrightarrow{m} C \xleftarrow{e} \mathbf{1}$$

(see [1]) in M such that the following diagrams are commutative

$$\begin{array}{ccc} C\Box C\Box C & \xrightarrow{1\Box m} & C\Box C \\ {\scriptstyle m\Box 1}\downarrow & & \downarrow{\scriptstyle m} \\ C\Box C & \xrightarrow{m} & C \end{array}, \qquad (2.1)$$

$$\begin{tikzcd} C \arrow[r, "e\Box 1"] \arrow[dr] & C\Box C \arrow[d, "m"] & C \arrow[l, "1\Box e"'] \arrow[dl] \\ & C & \end{tikzcd} \tag{2.2}$$

In the monoidal category (Cat,$\times$, 1), the monoids are precisely the strict monoidal categories, whereas in the monoidal category ($\mathcal{O}$-Graphs, $\times_{\mathcal{O}}, \mathcal{O} = \mathcal{O}$) the monoids are all categories with the fixed set $\mathcal{O}$ of objects.

Consider a monoidal category $(M, \Box, \mathbf{1})$, in which M is a 2-category and $\Box$ is a 2-bifunctor. Replacing the commutativity of the diagrams (2.1) and (2.2) by the existence of suitable 2-cells satisfying the usual coherence conditions for monoidal categories (see [1]), we obtain the notion of *weak monoid.*

Definition 1 A weak monoid in a monoidal category $(M, \Box, \mathbf{1})$ (where M is a 2-category and $\Box$ is a 2-bifunctor) is a diagram of the form

$$C\Box C \xrightarrow{m} C \xleftarrow{e} \mathbf{1}$$

together with 2-cells

$$\begin{aligned} \alpha &: m\,(1\Box m) \longrightarrow m\,(m\Box 1)\,, \\ \lambda &: m\,(1\Box e) \longrightarrow 1, \\ \rho &: m\,(e\Box 1) \longrightarrow 1, \end{aligned}$$

that are isomorphisms satisfying the identity

$$\lambda \circ e = \rho \circ e,$$

and the commutativity of the following diagrams[1]

$$\begin{tikzcd} \cdot \arrow[rr, "m\circ(1\Box\alpha)"] \arrow[d, "\alpha\circ(1\Box 1\Box m)"'] & & \cdot \arrow[d, "\alpha\circ(1\Box m\Box 1)"] \\ \cdot \arrow[dr, "\alpha\circ(m\Box 1\Box 1)"'] & & \cdot \arrow[dl, "m\circ(\alpha\Box 1)"] \\ & \cdot & \end{tikzcd}\ , \tag{2.3}$$

$$\begin{tikzcd} \cdot \arrow[r, "\alpha\circ(1\Box e\Box 1)"] \arrow[dr, "m\circ(1\Box\lambda)"'] & \cdot \arrow[d, "m\circ(\rho\Box 1)"] \\ & \cdot \end{tikzcd}\ . \tag{2.4}$$

In the monoidal category (Cat,$\times$, 1) a weak monoid is precisely a monoidal category (not necessarily strict). A weak category (see Table 1) is obtained as a particular case of a weak monoid by considering a weak monoid in the monoidal category of ($\mathcal{O}$-Graphs(Cat), $\times_{\mathcal{O}}, \mathcal{O} = \mathcal{O}$) where $\mathcal{O}$-Graphs(Cat) is the category of

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[1] The reader not familiar with internal constructions may think as if the object C had elements a, b, c, d, and write ab for $m\,(a, b)$ in order to obtain

$$\begin{tikzcd} a(b(cd)) \arrow[rr, "a\alpha"] \arrow[d, "\alpha_{a,b,cd}"'] & & a((bc)d) \arrow[d, "\alpha_{a,bc,d}"] \\ (ab)(cd) \arrow[dr, "\alpha_{ab,c,d}"'] & & (a(bc))d \arrow[dl, "\alpha d"] \\ & ((ab)c)d & \end{tikzcd}\ , \quad \begin{tikzcd} a(eb) \arrow[r, "\alpha"] \arrow[dr, "a\lambda"'] & (ae)b \arrow[d, "\rho b"] \\ & ab \end{tikzcd}\ .$$

internal $\mathcal{O}$-graphs in Cat. In this case we have the notion of weak monoid written as

$$\begin{array}{ccccc} C \times_{\mathcal{O}} C & \xrightarrow{m} & C & \xleftarrow{e} & \mathcal{O} \\ {\scriptstyle c\pi_1}\downarrow\downarrow{\scriptstyle d\pi_2} & & {\scriptstyle c}\downarrow\downarrow{\scriptstyle d} & & \| \\ \mathcal{O} & = & \mathcal{O} & = & \mathcal{O} \end{array}$$

with

$$\begin{aligned} de &= 1_{\mathcal{O}} = ce, \\ dm &= d\pi_2 \ , \ cm = c\pi_1, \end{aligned}$$

and the commutativity of the diagrams for associativity and identities replaced by natural isomorphisms α, λ and ρ satisfying the usual coherence conditions. If $\mathbb{X}$ is a 2-category, then a weak monoid in $(\mathcal{O}\text{-Graphs}(\mathbb{X}), \times_{\mathcal{O}}, \mathcal{O} = \mathcal{O})$ is a weak category in $\mathbb{X}$.

2.2 Weak categories. For simplicity we introduce the notion of weak category in several steps. First we define the notion of *precategory,* which is just an internal reflexive graph with composition. Next we define precategory with associativity (up to isomorphism) and call it *associative precategory.* Afterwards we define *associative precategory with identity,* an associative precategory with (up to isomorphism) left and right identities.

With respect to coherence conditions we specify the usual *pentagon* and *triangle* (which generalize 2.3 and 2.4) but also consider an intermediate coherence condition (that we call *mixed* coherence condition). The mixed coherence condition is important since (in an additive 2-category with kernels) an associative precategory with identity, satisfying the triangle and the mixed coherence conditions, completely determines the structure of weak category.

Finally we will define the notion of weak category by saying that it is an associative precategory with identity satisfying the pentagon and the triangle coherence conditions.

Definition 2 An internal *precategory* in a category $\mathbb{C}$ is a diagram in $\mathbb{C}$ of the form

$$C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \underset{c}{\overset{d}{\underset{\longleftarrow e}{\rightrightarrows}}} C_0$$

with

$$\begin{aligned} de &= 1_{C_1} = ce, && (2.5) \\ dm &= d\pi_2 \ , \ cm = c\pi_1 && (2.6) \end{aligned}$$

and where $C_1 \times_{C_0} C_1$ is defined via the pullback diagram

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle c} \\ C_1 & \xrightarrow[d]{} & C_0 \end{array}.$$

Definition 3 An internal *associative precategory,* in a 2-category $\mathbb{C}$, is a system

$$(C_0, C_1, m, d, e, c, \alpha),$$

where (C_0, C_1, m, d, e, c) is a precategory, (internal to $\mathbb{C}$) and

$$\alpha : m\,(1 \times_{C_0} m) \longrightarrow m\,(m \times_{C_0} 1)$$

is an isomorphism with

$$d \circ \alpha = 1_{d\pi_3}, c \circ \alpha = 1_{c\pi_1}. \tag{2.7}$$

Definition 4 An internal *associative precategory with identity*, in a 2-category $\mathbb{C}$, is a system

$$(C_0, C_1, m, d, e, c, \alpha, \lambda, \rho)\,,$$

where $(C_0, C_1, m, d, e, c, \alpha)$ is an associative precategory, and

$$\lambda : m\,\langle ec, 1\rangle \longrightarrow 1_{C_1}\ ,\ \rho : m\,\langle 1, ed\rangle \longrightarrow 1_{C_1}$$

are isomorphisms with

$$\begin{aligned} d \circ \lambda &= 1_d = d \circ \rho, \\ c \circ \lambda &= 1_c = c \circ \rho, \\ \lambda \circ e &= \rho \circ e. \end{aligned} \tag{2.8}$$

Definition 5 An internal *associative precategory with coherent identity*, in a 2-category $\mathbb{C}$, is a system

$$(C_0, C_1, m, d, e, c, \alpha, \lambda, \rho)\,,$$

forming an associative precategory with identity and satisfying the *triangle* and the *mixed* coherence conditions

$$(m \circ (\rho \times 1)) \cdot (\alpha \circ (1 \times \langle ec, 1\rangle)) = m \circ (1 \times \lambda)\,, \tag{2.9}$$

$$\rho \cdot (m \circ \langle \lambda, 1_{ed}\rangle) \cdot (\alpha \circ \langle ec, 1, ed\rangle) = \lambda \cdot (m \circ \langle 1_{ec}, \rho\rangle)\,. \tag{2.10}$$

Definition 6 An internal *weak category* in the 2-category $\mathbb{C}$ is a system

$$(C_0, C_1, m, d, e, c, \alpha, \lambda, \rho)\,,$$

forming an associative precategory with identity and satisfying the *triangle* and the *pentagon* coherence conditions

$$(m \circ (\rho \times 1)) \cdot (\alpha \circ (1 \times \langle ec, 1\rangle)) = m \circ (1 \times \lambda)\,,$$

$$\begin{aligned} &(\alpha \circ (m \times 1 \times 1)) \cdot (\alpha \circ (1 \times 1 \times m)) \\ &\quad = (m \circ (\alpha \times 1)) \cdot (\alpha \circ (1 \times m \times 1)) \cdot (m \circ (1 \times \alpha))\,. \end{aligned} \tag{2.11}$$

If the 2-cells α, λ, ρ were identities, then this would become nothing but the definition of internal category in $\mathbb{C}$. On the other hand, if we let the object C_0 be terminal, then the notion of internal monoidal category is obtained.

In the case where $\mathbb{C}$ is Cat, if the 2-cells are identities we get the definition of a double category; if the category C_0 is discrete (has only objects and the identity morphism for each object) then the definition of bicategory is obtained. More generally, if $\mathbb{C}$ is the category of internal categories in some category $\mathbb{X}$, i.e., $\mathbb{C} = \mathrm{Cat}(\mathbb{X})$ then we obtain the definition of double category in $\mathbb{X}$ on the one hand, and the definition of internal bicategory in $\mathbb{X}$ on the other hand.

In what follows, after defining additive 2-categories, we will give a complete description of the above structures inside (=internal to) them.

3 Additive 2-categories

In order to define additive 2-category we need the notion of 2-Ab-category. To do so, we give the general notion of a 2-V-category, a 2-category enriched in a monoidal category V.

3.1 2-V-categories. Let $\mathbb{V} = (\mathbb{V}, \square, 1)$ be a monoidal category and O a fixed set of objects. A $\mathbb{V}$-category over the set of objects O is given by a system

$$(H, \mu, \varepsilon)$$

where H is a family of objects[2] of $\mathbb{V}$,

$$H = (H(A,B) \in \mathbb{V})_{A,B \in O},$$

μ is a family of morphisms of $\mathbb{V}$

$$\mu = (\mu_{A,B,C} : H(A,B) \square H(B,C) \longrightarrow H(A,C))_{A,B,C \in O}$$

and ε is another family of morphisms of $\mathbb{V}$

$$\varepsilon = (\varepsilon_A : 1 \longrightarrow H(A,A))_{A \in O},$$

such that for every $A, B, C, D \in O$, the following diagrams commute

$$\begin{tikzcd} H(A,B) \square H(B,C) \square H(C,D) \arrow[r, "1_{H(A,B)} \square \mu_{B,C,D}"] \arrow[d, "\mu_{A,B,C} \square 1_{H(C,D)}"'] & H(A,B) \square H(B,D) \arrow[d, "\mu_{A,B,D}"] \\ H(A,C) \square H(C,D) \arrow[r, "\mu_{A,C,D}"] & H(A,D) \end{tikzcd}, \tag{3.1}$$

$$\begin{tikzcd} H(A,B) \arrow[r, "\epsilon_A \square 1"] \arrow[dr] & H(A,A) \square H(A,B) \arrow[d, "\mu_{A,A,B}"] \\ & H(A,B) \end{tikzcd}, \quad \begin{tikzcd} H(A,B) \square H(B,B) \arrow[d, "\mu_{A,B,B}"'] & H(A,B) \arrow[l, "1 \square \epsilon_B"'] \arrow[dl] \\ H(A,B) & \end{tikzcd}. \tag{3.2}$$

A morphism φ between two $\mathbb{V}$-categories over the set of objects O

$$(H, \mu, \varepsilon) \xrightarrow{\varphi} (H', \mu', \varepsilon')$$

is a family of morphisms of $\mathbb{V}$

$$\varphi = (\varphi_{A,B} : H(A,B) \longrightarrow H'(A,B))_{A,B \in O}$$

such that for every $A, B, C \in O$ the following diagrams are commutative

$$\begin{tikzcd} H(A,A) \arrow[r, "\varphi_{A,A}"] & H'(A,A) \\ 1 \arrow[u, "\epsilon_A"] \arrow[ur, "\epsilon'_A"'] & \end{tikzcd}, \tag{3.3}$$

[2]The object $H(A,B) \in \mathbb{V}$ represents $\hom(A,B)$ of the $\mathbb{V}$-category that is being defined.

$$\begin{array}{ccc} H(A,B)\,\square H(B,C) & \xrightarrow{\varphi_{A,B}\square\varphi_{B,C}} & H'(A,B)\,\square H'(B,C) \\ \mu_{A,B,C}\downarrow & & \downarrow \mu'_{A,B,C} \\ H(A,C) & \xrightarrow{\varphi_{A,C}} & H'(A,C) \end{array} \quad . \tag{3.4}$$

Defining composition in the usual way, the category of all $\mathbb{V}$-categories over the set of objects O, denoted by $(\mathbb{V}, O)$-Cat, can be formed.

Definition 7 A *2-$\mathbb{V}$-category* over the set of objects O is an internal category in the category $(\mathbb{V}, O)$-Cat.

A 2-*Ab*-category is obtained by considering the monoidal category $\mathbb{V}=(Ab,\otimes,Z)$.

3.2 2-Ab-categories. Following the previous definition, a 2-Ab-Category over the set of objects O, is an internal category in the category (Ab, O)-Cat, that is, a diagram of the form

$$C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \underset{\xrightarrow[c]{}}{\overset{\xrightarrow{d}}{\xleftarrow{e}}} C_0$$

satisfying the usual axioms for a category. In order to analyze the definition, it is convenient to think of the object C_0 as an ordinary Ab-category (not given in terms of hom objects) and to think of C_1 as given by a system $C_1 = (H, \mu, \varepsilon)$ (see previous section). Since m, d, e, c are morphisms between Ab-categories, for each two objects A, B of C_0 (note that the objects of C_0 are by definition the elements of O), we have the following diagram in the category of abelian groups

$$H(A,B) \times_{\hom(A,B)} H(A,B) \xrightarrow{m} H(A,B) \underset{\xrightarrow[c]{}}{\overset{\xrightarrow{d}}{\xleftarrow{e}}} \hom(A,B).$$

Using the well-known equivalence Cat(Ab)$\sim$Mor(Ab), the diagram can be presented as

$$\ker d_{A,B} \oplus \ker d_{A,B} \oplus \hom(A,B) \xrightarrow{m} \ker d_{A,B} \oplus \hom(A,B) \underset{\xrightarrow[(D\ 1)]{}}{\overset{\xrightarrow{(0\ 1)}}{\xleftarrow{\binom{0}{1}}}} \hom(A,B)$$

with $m = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The group homomorphism $D : \ker d_{A,B} \longrightarrow \hom(A,B)$ sends each 2-cell with zero domain to its codomain.

Applying the commutativity of (3.4) to the cases $\varphi = d, e, c$ we conclude that the horizontal composition is completely determined by the composition of the 2-cells in $\ker d$ and by the horizontal composition of each element in $\ker d$ with left and right identity 2-cells. In fact a 2-cell $\tau^* : f \longrightarrow g$ with $f, g : A \longrightarrow B$ may be decomposed into the sum

$$\tau^* = \tau + 1_f \ ,$$

where $\tau \in \ker d_{A,B}$ and $D(\tau) = g - f$ $(\tau : 0 \longrightarrow g - f)$. The horizontal composition (in C_1) of $\sigma^* : f' \longrightarrow g'$ $(: B \longrightarrow C)$ and $\tau^* : f \longrightarrow g$ $(: A \longrightarrow B)$ is given by

$$\mu(\tau^*, \sigma^*) = \sigma^* \circ \tau^* = (\sigma + 1_{f'}) \circ (\tau + 1_f)$$

and, since horizontal composition is bilinear, we obtain the following formula

$$\sigma^* \circ \tau^* = (\sigma \circ \tau + \sigma \circ 1_f + 1_{f'} \circ \tau) + 1_{f'f}.$$

Also, by condition (3.4) applied to $\varphi = c$, the homomorphism D must satisfy the following conditions

$$\begin{aligned} D(\tau \circ \sigma) &= D(\tau) D(\sigma), \\ D(\tau \circ 1_f) &= D(\tau) f, \\ D(1_g \circ \tau) &= gD(\tau). \end{aligned}$$

Moreover, requiring the commutativity of (3.4) for $\varphi = m$ is the same as to require the four middle interchange law. Consider a diagram of the form

$$\begin{tikzcd} A \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=4, "\Downarrow\tau^*"] \arrow[r, phantom, shift right=4, "\Downarrow\tau^{*\prime}"] & B \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=4, "\Downarrow\sigma^*"] \arrow[r, phantom, shift right=4, "\Downarrow\sigma^{*\prime}"] & C \end{tikzcd}$$

in C_1, the four middle interchange law states that

$$(\sigma^{*\prime} \cdot \sigma^*) \circ (\tau^{*\prime} \cdot \tau^*) = (\sigma^{*\prime} \circ \tau^{*\prime}) \cdot (\sigma^* \circ \tau^*).$$

Using the formulas obtained above, we have

$$\begin{aligned} ((\sigma' + \sigma + 1_f) \circ (\tau' + \tau + 1_g)) &= ((\sigma' \circ \tau' + \sigma' \circ 1_{g'} + 1_{f'} \circ \tau') + 1_{f'g'}) \\ &\quad \cdot ((\sigma \circ \tau + \sigma \circ 1_g + 1_f \circ \tau) + 1_{fg}), \end{aligned}$$

which extends to

$$\begin{aligned} &\sigma' \circ \tau' + \sigma \circ \tau' + 1_f \circ \tau' + \sigma' \circ \tau + \sigma \circ \tau + 1_f \circ \tau + \sigma' \circ 1_g + \sigma \circ 1_g + 1_{fg} \\ &= ((\sigma' \circ \tau' + \sigma' \circ 1_{g'} + 1_{f'} \circ \tau' + \sigma \circ \tau + \sigma \circ 1_g + 1_f \circ \tau) + 1_{fg}), \end{aligned}$$

and then becomes

$$\sigma \circ \tau' + 1_f \circ \tau' + \sigma' \circ \tau + \sigma' \circ 1_g = \sigma' \circ 1_{g'} + 1_{f'} \circ \tau'.$$

By substituting

$$g' = D(\tau) + g \ , \ f' = D(\sigma) + f \ ,$$

in the formula above we obtain

$$\sigma \circ \tau' + \sigma' \circ \tau = \sigma' \circ 1_{D(\tau)} + 1_{D(\sigma)} \circ \tau'$$

which is equivalent to

$$\sigma \circ \tau = \sigma \circ 1_{D(\tau)} = 1_{D(\sigma)} \circ \tau.$$

Finally, by the commutativity of (3.2) we have

$$\begin{aligned} \tau \circ 1_A &= \tau \\ 1_C \circ \sigma &= \sigma. \end{aligned}$$

We may summarize the above calculations in the following proposition.

Proposition 1 *Giving a 2-Ab-category is the same as to give the following data:*

- *An Ab-category $\mathbb{A}$;*
- *An abelian group $K(A, B)$, for each pair of objects A, B of $\mathbb{A}$;*

- *A group homomorphism* $D_{A,B} : K(A,B) \longrightarrow \hom_{\mathbb{A}}(A,B)$*, for each pair of objects* A, B *of* $\mathbb{A}$*;*
- *Associative and bilinear laws of composition*

$$g\tau \ , \ \sigma\tau \ , \ \sigma f \in K\left(A,C\right)$$

for each $\tau \in K\left(A,B\right), \sigma \in K\left(B,C\right), f \in \hom_{\mathbb{A}}\left(A,B\right), g \in \hom_{\mathbb{A}}\left(B,C\right)$ *with* A, B, C *objects of* $\mathbb{A}$*, satisfying the following conditions*

$$\begin{aligned} \tau 1_A &= \tau, \\ 1_B \tau &= \tau, \end{aligned} \tag{3.5}$$

$$\begin{aligned} D\left(\sigma\tau\right) &= D\left(\sigma\right) D\left(\tau\right), \\ D\left(\sigma f\right) &= D\left(\sigma\right) f, \\ D\left(g\tau\right) &= gD\left(\tau\right), \end{aligned} \tag{3.6}$$

$$\sigma\tau = \sigma D\left(\tau\right) = D\left(\sigma\right)\tau. \tag{3.7}$$

The data given in the above proposition determines a 2-category structure in the Ab-category $\mathbb{A}$. Given two morphisms $f, g : A \longrightarrow B$ of $\mathbb{A}$, a 2-cell from f to g is a pair (τ, f) with τ in $K(A,B)$ and $D\left(\tau\right) = g - f$. Note that $K\left(A,B\right)$ plays the role of $\ker d_{A,B}$.

The vertical composition is given by the formula

$$(\sigma, g) \cdot (\tau, f) = (\sigma + \tau, f)$$

whereas the horizontal composition is given by

$$(\tau', f') \circ (\tau, f) = (\tau'\tau + \tau' f + f'\tau \ , \ f' f) \, .$$

We always write the three different compositions $g\tau, \sigma\tau, \sigma f$ as justaposition, because it is clear from the context. We also use small letters like f, g, h, k to denote the morphisms of $\mathbb{A}$ and small greek letters, like $\alpha, \lambda, \rho, \eta$ to denote the elements of K. Sometimes the same greek letter is used to denote the element of K and the 2-cell itself, e.g. $\alpha = (\alpha, m\left(1 \times m\right))$.

Definition 8 An additive 2-category is a 2–Ab-category with:

- a zero object;
- all binary biproducts;

In the next section simple properties of additive 2-categories are presented.

3.3 Properties of additive 2-categories. Let $\mathbb{A}$ be an additive 2-category (with K and D as above). As is well known, in an additive category (see [1]), a morphism between iterated biproducts is described as a matrix of its components and composition is just the product of matrices. In an additive 2-category the same is true for the 2-cells since we are able to compose them with the projections and the injections of the biproducts. This means that if we have

$$\tau \in K\left(A_1 \oplus A_2, B_1 \oplus B_2\right),$$

then we can write

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$$

with

$$\tau_{ij} \in K\left(A_j, B_i\right).$$

Let us recall:

Proposition 2 *A split epi $X \xrightarrow{u} Y$ (with splitting $Y \xrightarrow{v} X$) in an additive category with kernels is isomorphic to*

$$\ker u \oplus Y \underset{i}{\overset{p}{\rightleftarrows}} Y$$

where $p = (0\ ,\ 1)$ and $i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Proposition 3 *Let $X \times_Y Z$ be the object of a pullback diagram in an additive category with kernels, where u is a split epi, with splitting v, as in the following diagram*

$$\begin{tikzcd} X \times_Y Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[d, "w"] \\ X \arrow[r, "u"] & Y \end{tikzcd}.$$

Then $X \times_Y Z \cong \ker u \oplus Z$ and the pullback diagram becomes

$$\begin{tikzcd} \ker u \oplus Z \arrow[r, "(0\ 1)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & w \end{smallmatrix}\right)"'] & Z \arrow[d, "w"] \\ \ker u \oplus Y \arrow[r, "(0\ 1)"] & Y \end{tikzcd}.$$

In the following section we will describe the notion of weak category in an additive 2-category with kernels.

4 Weak categories in additive 2-categories with kernels

Let $\mathbb{A}$ be an additive 2-category with kernels. We will identify $\mathbb{A}$ with the data $(\mathbb{A}, K, D)$ of Proposition 1. When it is clear from the context, we will refer to a 2-cell

$$(\tau, f) : f \longrightarrow f + D(\tau)$$

simply by τ.

4.1 Precategories. (See Definition 2).

Proposition 4 *An internal precategory in $\mathbb{A}$ is completely determined by four morphisms of $\mathbb{A}$,*

$$\begin{aligned} k &: A \longrightarrow B, \\ f, g &: A \longrightarrow A, \\ h &: B \longrightarrow A, \end{aligned}$$

with

$$kf = k = kg\ ,\ kh = 0, \tag{4.1}$$

and is given (up to an isomorphism) by

$$A \oplus A \oplus B \xrightarrow{m} A \oplus B \underset{\xrightarrow[(k\ 1)]{}}{\overset{\xrightarrow{(0\ 1)}}{\xleftarrow{\binom{0}{1}}}} B, \tag{4.2}$$

where $m = \begin{pmatrix} f & g & h \\ 0 & 0 & 1 \end{pmatrix}$.

Proof Since the morphism $d : C_1 \longrightarrow C_0$ in Definition 2 is a split epi, using Proposition 2 we conclude that the object C_1 is of the form $A \oplus B$ (considering A as the kernel of d and $C_0 = B$). This means that the underlying reflexive graph of our precategory is of the form

$$A \oplus B \underset{\xrightarrow[(k\ 1)]{}}{\overset{\xrightarrow{(0\ 1)}}{\xleftarrow{\binom{0}{1}}}} B.$$

The object $C_1 \times_{C_0} C_1$ is (by Proposition 3) isomorphic to

$$A \oplus A \oplus B$$

and the projections π_1 and π_2 are given by the diagram

$$A \oplus B \xleftarrow{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & k & 1 \end{smallmatrix}\right)} A \oplus A \oplus B \xrightarrow{\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} A \oplus B.$$

The composition $m : A \oplus A \oplus B \longrightarrow A \oplus B$ is a morphism satisfying $dm = d\pi_2$, which means that

$$m = \begin{pmatrix} f & g & h \\ 0 & 0 & 1 \end{pmatrix}$$

with $f, g : A \longrightarrow A$ and $h : B \longrightarrow A$ arbitrary morphisms of $\mathbb{A}$. Nevertheless, the condition $cm = c\pi_1$ yields

$$kf = k = kg\ ,\ kh = 0.$$

□

4.2 Associative precategories. In order to analyze the 2-cell

$$\alpha : m\,(1 \times_{C_0} m) \longrightarrow m\,(m \times_{C_0} 1)$$

(see Definition 3), the morphisms $m\,(1 \times_{C_0} m)$ and $m\,(m \times_{C_0} 1)$ have to be described. Having in mind (by Proposition 4) that $C_1 \times_{C_0} C_1$ is of the form $A \oplus A \oplus B$ and, using Proposition 3, we conclude that $C_1 \times_{C_0} C_1 \times_{C_0} C_1$ is of the form

$$A \oplus A \oplus A \oplus B.$$

The projections for $C_1 \times_{C_0} (C_1 \times_{C_0} C_1)$ are given as in the diagram

$$A \oplus B \xleftarrow{\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & k & k & 1 \end{smallmatrix}\right)} A \oplus (A \oplus A \oplus B) \xrightarrow{(0\ 1)} (A \oplus A \oplus B)$$

and the projections for $(C_1 \times_{C_0} C_1) \times_{C_0} C_1$ are given by

$$A \oplus A \oplus B \xleftarrow{p} A \oplus A \oplus A \oplus B \xrightarrow{\left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)} A \oplus B$$

with

$$p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \end{pmatrix}.$$

The reader may appreciate checking that

$$1 \times_{C_0} m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f & g & h \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$m \times_{C_0} 1 = \begin{pmatrix} f & g & hk & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using matrix multiplication we have that

$$m\,(1 \times_{C_0} m) = \begin{pmatrix} f & gf & g^2 & gh+h \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$m\,(m \times_{C_0} 1) = \begin{pmatrix} f^2 & fg & fhk+g & fh+h \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The isomorphism

$$(\alpha, m\,(1 \times m)) : m\,(1 \times m) \longrightarrow m\,(m \times 1)$$

has α in $K\,(A \oplus A \oplus A \oplus B, A \oplus B)$ and

$$D\,(\alpha) = m\,(m \times 1) - m\,(1 \times m)\,. \tag{4.3}$$

Since α must satisfy

$$d \circ \alpha = 1_{d\pi_3},$$

which may be written as[3]

$$(d\alpha, d\pi_3) = (0, d\pi_3)\,,$$

we conclude that $d\alpha = 0$. Having in mind that $d = (0\ 1)$ and α is a 2×4 matrix, we have

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\alpha_1, \alpha_2, \alpha_3 \in K\,(A, A)$ and $\alpha_0 \in K\,(B, A)$. Similarly, from $c \circ \alpha = 1_{c\pi_1}$, we conclude that $c\alpha = 0$. Furthermore, since $c = (k\ 1)$ we have

$$k\alpha_i = 0\ ,\ i = 0, 1, 2, 3.$$

In order to satisfy condition (4.3), we must also have

$$\begin{aligned} D\,(\alpha_1) &= f^2 - f, \\ D\,(\alpha_2) &= fg - gf, \\ D\,(\alpha_3) &= fhk + g - g^2, \\ D\,(\alpha_4) &= fh - gh. \end{aligned}$$

We are now ready to establish the following:

---

[3]Note that for any morphism $\varphi : A \longrightarrow A'$, the 2-cell 1_φ is of the form $(0, \varphi)$ and so, $1_{d\pi_3}$ is of the form $(0, d\pi_3)$. The composite $d \circ \alpha$ is of the form $(0, d) \circ (\alpha, m\,(1 \times m)) = (d\alpha, dm\,(1 \times m)) = (d\alpha, d\pi_3)$.

Proposition 5 *An internal associative precategory in* $\mathbb{A}$ *is completely determined by morphisms*

$$\begin{aligned} k &: A \longrightarrow B, \\ f,g &: A \longrightarrow A, \\ h &: B \longrightarrow A, \end{aligned}$$

with

$$kf = k = kg \ , \ kh = 0,$$

and objects $\alpha_1, \alpha_2, \alpha_3 \in K(A,A)$, $\alpha_0 \in K(B,A)$ *with*

$$k\alpha_i = 0 \ , \ i = 0,1,2,3$$

$$\begin{aligned} D(\alpha_1) &= f^2 - f, \\ D(\alpha_2) &= fg - gf, \\ D(\alpha_3) &= fhk + g - g^2, \\ D(\alpha_4) &= fh - gh. \end{aligned}$$

4.3 Associative precategories with identity. In order to analyze the 2-cells for the left and right identities (see Definition 4), we have to describe the morphisms $m\langle ec, 1\rangle$ and $m\langle 1, ed\rangle$ from C_1 to C_1. Proposition 3 yields

$$\langle ec, 1\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} , \ \langle 1, ed\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$m\langle ec, 1\rangle = \begin{pmatrix} g & h \\ 0 & 1 \end{pmatrix} , \ m\langle 1, ed\rangle = \begin{pmatrix} f & h \\ 0 & 1 \end{pmatrix}.$$

Since $(\lambda, m\langle ec, 1\rangle) : m\langle ec, 1\rangle \longrightarrow 1$ is a 2-cell from $A \oplus B$ to $A \oplus B$, we conclude that λ is in $K(A \oplus B, A \oplus B)$ and

$$D(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} g & h \\ 0 & 1 \end{pmatrix}, \tag{4.4}$$

while ρ is in $K(A \oplus B, A \oplus B)$ and

$$D(\rho) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} f & h \\ 0 & 1 \end{pmatrix}. \tag{4.5}$$

In order to satisfy conditions (2.8) λ and ρ must be of the form

$$\begin{aligned} \lambda &= \begin{pmatrix} \lambda_1 & \lambda_0 \\ 0 & 0 \end{pmatrix}, \\ \rho &= \begin{pmatrix} \rho_1 & \rho_0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

with $\lambda_1, \rho_1 \in K(A,A)$ and $\lambda_0, \rho_0 \in K(B,A)$ such that

$$\begin{aligned} k\lambda_1 &= 0 = k\rho_1, \\ k\lambda_0 &= 0 = k\rho_0. \end{aligned} \tag{4.6}$$

From the condition

$$\lambda \circ e = \rho \circ e,$$

we conclude that $\lambda_0 = \rho_0$. To simplify notation a new letter, η, is introduced to denote λ_0 and ρ_0. In this way, instead of having $\lambda_0, \rho_0 \in K(B,A)$ and one condition $\lambda_0 = \rho_0$ we simply have $\eta \in K(B,A)$. Since λ_0 and ρ_0 are not used anymore, we will write λ and ρ instead of λ_1 and ρ_1 respectively.

With this new notation, conditions (4.4) and (4.5) become

$$\begin{pmatrix} D(\lambda) & D(\eta) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} g & h \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} D(\rho) & D(\eta) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} f & h \\ 0 & 1 \end{pmatrix}.$$

This means that λ and ρ completely determine the morphisms f, g, h and we have

$$\begin{aligned} g &= 1 - D(\lambda), \\ f &= 1 - D(\rho), \\ h &= -D(\eta). \end{aligned}$$

Next, we show that the conditions (4.1) are satisfied with f, g, h given as above. Since

$$kf = k - kD(\rho),$$

and $kD(\rho) = D(k\rho)$ (by condition (3.6)) and $k\rho = 0$, we have

$$kf = k.$$

The same argument shows that $k = kg$ and $kh = 0$, since $k\lambda = 0$ and $k\eta = 0$.

This suggests the following description of associative precategories with identity in an additive 2-category with kernels:

Proposition 6 *An associative precategory with identity in an additive 2-category with kernels is completely determined by a morphism*

$$A \xrightarrow{k} B$$

together with

$$\begin{aligned} \alpha_1, \alpha_2, \alpha_3, \lambda, \rho &\in K(A,A), \\ \alpha_0, \eta &\in K(B,A), \end{aligned}$$

subject to the following conditions

$$k\alpha_i = 0\ ,\ i = 0,1,2,3$$

$$k\lambda = 0 = k\rho,\ k\eta = 0,$$

$$\begin{aligned} D(\alpha_1) &= f^2 - f, \\ D(\alpha_2) &= fg - gf, \\ D(\alpha_3) &= fhk + g - g^2, \\ D(\alpha_0) &= fh - gh, \end{aligned} \tag{4.7}$$

where f, g, h are defined as follows:

$$\begin{aligned} g &= 1 - D(\lambda), \\ f &= 1 - D(\rho), \\ h &= -D(\eta). \end{aligned} \tag{4.8}$$

4.4 Associative precategories with coherent identity. An associative precategory with coherent identity is an associative precategory with identity (see previous section) where the triangle and mixed coherent conditions are satisfied (see Definition 5). We proceed using the description of associative precategory with identity given as in the previous section to describe the triangle and the mixed coherence conditions in additive 2-categories with kernels.

4.4.1 *Triangle coherence condition.* In order to analyze the triangle coherence condition

$$(m \circ (\rho \times 1)) \cdot (\alpha \circ (1 \times \langle ec, 1 \rangle)) = m \circ (1 \times \lambda)$$

the 2-cells $\rho \times_{C_0} 1$ and $1 \times_{C_0} \lambda$, have to be described. Since they are elements in $K(A \oplus A \oplus B, A \oplus A \oplus B)$, the reader is invited to show that

$$\rho \times_{C_0} 1 = \begin{pmatrix} \rho & \eta k & \eta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 1 \times_{C_0} \lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & \eta \\ 0 & 0 & 0 \end{pmatrix}.$$

We have already seen that

$$\langle ec, 1 \rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ so } (1 \times \langle ec, 1 \rangle) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The definition of horizontal composition in an additive 2-category yields

$$\begin{aligned} m \circ (\rho \times 1) &= \begin{pmatrix} f\rho & f\eta k & f\eta \\ 0 & 0 & 0 \end{pmatrix}, \\ m \circ (1 \times \lambda) &= \begin{pmatrix} 0 & g\lambda & g\eta \\ 0 & 0 & 0 \end{pmatrix}, \\ \alpha \circ (1 \times \langle ec, 1 \rangle) &= \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Finally, the coherence condition may be written as

$$\begin{pmatrix} f\rho & f\eta k & f\eta \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & g\lambda & g\eta \\ 0 & 0 & 0 \end{pmatrix},$$

or, equivalently, as

$$\begin{aligned} \alpha_1 &= -f\rho, \\ \alpha_3 &= g\lambda - f\eta k, \\ \alpha_0 &= g\eta - f\eta. \end{aligned}$$

The components $\alpha_1, \alpha_3, \alpha_0$ are completely determined. In the next section we show that the component α_2 is also determined by the mixed coherence condition.

4.4.2 *Mixed coherence condition.* Consider the mixed coherence condition

$$\rho \cdot (m \circ \langle \lambda, 1_{ed} \rangle) \cdot (\alpha \circ \langle ec, 1, ed \rangle) = \lambda \cdot (m \circ \langle 1_{ec}, \rho \rangle).$$

We have already seen that

$$\langle 1, ed \rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}. \text{ Thus } \langle ec, 1, ed \rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Furthermore, we have

$$\langle \lambda, 1_{ed} \rangle = \begin{pmatrix} \lambda & \eta \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\langle 1_{ec}, \rho \rangle = \begin{pmatrix} 0 & 0 \\ \rho & \eta \\ 0 & 0 \end{pmatrix}.$$

Using the definition of horizontal composition we obtain

$$\begin{aligned} m \circ \langle \lambda, 1_{ed} \rangle &= \begin{pmatrix} f\lambda & f\eta \\ 0 & 0 \end{pmatrix}, \\ \alpha \circ \langle ec, 1, ed \rangle &= \begin{pmatrix} \alpha_2 & \alpha_0 \\ 0 & 0 \end{pmatrix}, \\ m \circ \langle 1_{ec}, \rho \rangle &= \begin{pmatrix} g\rho & g\eta \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and the mixed coherence condition may be written as

$$\begin{pmatrix} \rho & \eta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f\lambda & f\eta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \alpha_0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda & \eta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} g\rho & g\eta \\ 0 & 0 \end{pmatrix},$$

or, equivalently, as

$$\begin{aligned} \alpha_2 &= \lambda + g\rho - \rho - f\lambda, \\ \alpha_0 &= \eta + g\eta - \eta - f\eta. \end{aligned}$$

Therefor the 2-cell α is completely determined and it is a straightforward calculation checking that $k\alpha_i = 0$, $i = 0, 1, 2, 3$, and that conditions (4.7) are satisfied.

Hence, we have:

Proposition 7 *An associative precategory with coherent identity in an additive 2-category with kernels is completely determined by a morphism*

$$A \xrightarrow{k} B,$$

together with

$$\begin{aligned} \lambda, \rho &\in K(A, A), \\ \eta &\in K(B, A), \end{aligned}$$

subject to the conditions

$$\begin{aligned} k\lambda &= 0, \\ k\rho &= 0, \\ k\eta &= 0. \end{aligned}$$

It is given (up to an isomorphism) by

$$A \oplus A \oplus B \xrightarrow{m} A \oplus B \underset{(k\ 1)}{\overset{(0\ 1)}{\rightrightarrows}} B, \quad \text{with } B \xrightarrow{\binom{0}{1}} A \oplus B,$$

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda = \begin{pmatrix} \lambda & \eta \\ 0 & 0 \end{pmatrix}, \rho = \begin{pmatrix} \rho & \eta \\ 0 & 0 \end{pmatrix},$$

where

$$m = \begin{pmatrix} f & g & h \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{aligned} g &= 1 - D(\lambda), \\ f &= 1 - D(\rho), \\ h &= -D(\eta), \end{aligned}$$

$$\begin{aligned} \alpha_1 &= \rho^2 - \rho, \\ \alpha_2 &= \rho\lambda - \lambda\rho, \\ \alpha_3 &= \lambda - \lambda^2 - f\eta k, \\ \alpha_0 &= \rho\eta - \lambda\eta. \end{aligned}$$

4.5 Weak categories. In this section we show that the pentagon coherence condition does not add new restrictions on the data involved in Proposition 7, i.e. the description of associative precategory with coherent identity is in fact the description of a weak category in an additive 2-category with kernels.

In order to analyze the pentagon coherence condition

$$\begin{aligned} &(\alpha \circ (m \times 1 \times 1)) \cdot (\alpha \circ (1 \times 1 \times m)) \\ &\quad = (m \circ (\alpha \times 1)) \cdot (\alpha \circ (1 \times m \times 1)) \cdot (m \circ (1 \times \alpha)), \end{aligned}$$

we need some preliminary calculations. Namely, all the arrows in the expression have to be described.

To describe the arrow $m \times_{C_0} 1 \times_{C_0} 1$, we have to analyze its domain

$$(C_1 \times_{C_0} C_1) \times_{C_0} C_1 \times_{C_0} C_1$$

and codomain

$$C_1 \times_{C_0} C_1 \times_{C_0} C_1.$$

Using the results obtained in the previous sections, we have that the domain, together with its three projections, is given by

$$\begin{array}{ccccc} A \oplus A \oplus B & \xleftarrow{\pi_1} & A \oplus A \oplus A \oplus A \oplus B & \xrightarrow{\pi_3} & A \oplus B \\ & & \downarrow{\scriptstyle \pi_2} & & \\ & & A \oplus B & & \end{array},$$

where

$$\begin{aligned} \pi_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & k & k & 1 \end{pmatrix}, \\ \pi_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k & 1 \end{pmatrix}, \\ \pi_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The codomain, together with its projections, is given by

$$\begin{array}{ccccc} A \oplus B & \xleftarrow{\pi_1} & A \oplus A \oplus A \oplus B & \xrightarrow{\pi_3} & A \oplus B \\ & & \downarrow{\scriptstyle \pi_2} & & \\ & & A \oplus B & & \end{array},$$

where

$$\begin{aligned}
\pi_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & k & k & 1 \end{pmatrix}, \\
\pi_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \end{pmatrix}, \\
\pi_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Having in mind that

$$\begin{aligned}
m &= \begin{pmatrix} f & g & h \\ 0 & 0 & 1 \end{pmatrix} : A \oplus A \oplus B \longrightarrow A \oplus B, \\
1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : A \oplus B \longrightarrow A \oplus B,
\end{aligned}$$

we obtain

$$(m \times 1 \times 1) = \begin{pmatrix} f & g & hk & hk & h \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By similar calculations we also have

$$(1 \times m \times 1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & f & g & hk & h \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$(1 \times 1 \times m) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & f & g & h \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We remark that $\alpha \times 1$ is in fact an abbreviation of $\alpha \times_{C_0} 1_{1_{C_1}}$, where $1_{1_{C_1}}$ is the identity 2-cell of the arrow 1_{C_1}. So, it is the pair $(0, 1_{C_1})$ with 0 in $K(A \oplus B, A \oplus B)$.

The domain of $\alpha \times 1$ is given (together with its two projections) as in the diagram

$$A \oplus A \oplus A \oplus B \xleftarrow{\pi_1} A \oplus A \oplus A \oplus A \oplus B \xrightarrow{\pi_2} A \oplus B,$$

where

$$\begin{aligned}
\pi_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k & 1 \end{pmatrix}, \\
\pi_2 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

and the codomain is given as in

$$A \oplus B \xleftarrow{\pi_1} A \oplus A \oplus B \xrightarrow{\pi_2} A \oplus B,$$

where

$$\begin{aligned}\pi_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 1 \end{pmatrix},\\ \pi_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

Hence, we obtain

$$\alpha \times 1 = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 k & \alpha_0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, we have get

$$1 \times \alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now that we have all ingredients of our calculation, we begin the main part.

On the one hand, we have to describe

$$\alpha\,(m \times 1 \times 1) + \alpha\,(1 \times 1 \times m)$$

and the result is

$$\begin{pmatrix} \alpha_1 f + \alpha_1 & \alpha_1 g + \alpha_2 & \alpha_1 hk + \alpha_2 + \alpha_3 f & \alpha_1 hk + \alpha_3 + \alpha_3 g & \alpha_1 h + \alpha_0 + \alpha_3 h + \alpha_0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.9}$$

On the other hand, we need

$$(m\,(\alpha \times 1)) + (\alpha\,(1 \times m \times 1)) + (m\,(1 \times \alpha))$$

and the result is

$$\begin{pmatrix} f\alpha_1 + \alpha_1 & f\alpha_2 + \alpha_2 f + g\alpha_1 & f\alpha_3 + \alpha_2 g + g\alpha_2 & * & ** \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.10}$$

where $*$ is $f\alpha_0 k + \alpha_2 hk + \alpha_3 + g\alpha_3$ and $**$ is $f\alpha_0 + \alpha_2 h + \alpha_0 + g\alpha_0$.

To check whether (4.9) and (4.10) are equal is the same as checking whether the following identities hold

$$\begin{aligned}\alpha_1 f &= f\alpha_1 && (4.11)\\ \alpha_1 g + \alpha_2 &= f\alpha_2 + \alpha_2 f + g\alpha_1 && (4.12)\\ \alpha_1 hk + \alpha_2 + \alpha_3 f &= f\alpha_3 + \alpha_2 g + g\alpha_2 && (4.13)\\ \alpha_1 hk + \alpha_3 g &= f\alpha_0 k + \alpha_2 hk + g\alpha_3 && (4.14)\\ \alpha_1 h + \alpha_3 h + \alpha_0 &= f\alpha_0 + \alpha_2 h + g\alpha_0. && (4.15)\end{aligned}$$

We will use

$$\begin{aligned} f &= 1 - D(\rho)\ ,\ g = 1 - D(\lambda)\ ,\ h = -D(\eta),\\ \alpha_1 &= \rho^2 - \rho\ ,\ \alpha_2 = \rho\lambda - \lambda\rho,\\ \alpha_3 &= \lambda - \lambda^2 - \eta k + \rho\eta k,\\ \alpha_0 &= \rho\eta - \lambda\eta.\end{aligned}$$

(see Proposition 7).

The condition (4.11) holds since we have $\rho f = f\rho$.

The condition (4.12) is equivalent to

$$\alpha_1 - \alpha_1\lambda + \alpha_2 = \alpha_2 - \rho\alpha_2 + \alpha_2 - \alpha_2\rho + \alpha_1 - \lambda\alpha_1,$$

which simplifies to

$$-\left(\rho^2-\rho\right)\lambda=-\rho\alpha_2+\alpha_2-\alpha_2\rho-\lambda\left(\rho^2-\rho\right)$$

and then becomes

$$-\rho^2\lambda+\rho\lambda=-\rho^2\lambda+\rho\lambda\rho+\rho\lambda-\lambda\rho-\rho\lambda\rho+\lambda\rho^2-\lambda\rho^2+\lambda\rho$$

which is a trivial condition.

Moreover, the condition (4.13) is equivalent to

$$\alpha_1hk+\alpha_2+\alpha_3-\alpha_3\rho=\alpha_3-\rho\alpha_3+\alpha_2-\alpha_2\lambda+\alpha_2-\lambda\alpha_2,$$

which extends to

$$\left(\rho^2-\rho\right)(-\eta)\,k-\alpha_3\rho=-\rho\alpha_3-(\rho\lambda-\lambda\rho)\,\lambda+\rho\lambda-\lambda\rho-\lambda\,(\rho\lambda-\lambda\rho)\,,$$

and also to

$$\begin{aligned}&\left(\rho^2-\rho\right)(-\eta)\,k-\left(\lambda-\lambda^2-\eta k+\rho\eta k\right)\rho\\&\qquad=-\rho\left(\lambda-\lambda^2-\eta k+\rho\eta k\right)-(\rho\lambda-\lambda\rho)\,\lambda+\rho\lambda-\lambda\rho-\lambda\,(\rho\lambda-\lambda\rho)\,.\end{aligned}$$

Since $k\rho=0$, this condition is also trivial.

The condition (4.14) is equivalent to

$$\alpha_1hk+\alpha_3-\alpha_3\lambda=f\alpha_0k+\alpha_2hk+\alpha_3-\lambda\alpha_3,$$

and then to

$$\begin{aligned}&\left(\rho^2-\rho\right)(-\eta)\,k-\left(\lambda-\lambda^2-\eta k+\rho\eta k\right)\lambda\\&\qquad=(\rho\eta-\lambda\eta)\,k-\rho\,(\rho\eta-\lambda\eta)\,k+(\rho\lambda-\lambda\rho)\,(-\eta)\,k-\lambda\left(\lambda-\lambda^2-\eta k+\rho\eta k\right).\end{aligned}$$

Since $k\lambda=0$ it is trivial again.

The condition (4.15) is equivalent to

$$\alpha_1h+\alpha_3h=-\rho\alpha_0+\alpha_2h+\alpha_0-\lambda\alpha_0,$$

or

$$\begin{aligned}&-\rho^2\eta+\rho\eta-\lambda\eta+\lambda^2\eta+\eta k\eta-\rho\eta k\eta\\&\qquad=-\rho\,(\rho\eta-\lambda\eta)-\rho\lambda\eta+\lambda\rho\eta+\rho\eta-\lambda\eta-\lambda\,(\rho\eta-\lambda\eta)\,.\end{aligned}$$

Since $k\eta=0$, the condition is trivial.

Finally, we obtain:

Proposition 8 *An associative precategory with coherent identity in an additive 2-category with kernels is a weak category.*

5 Examples

In this section we consider internal weak categories in Ab and Mor(Ab) that are examples of additive 2-categories with kernels.

5.1 Abelian groups. According to Proposition 1, taking $\mathbb{A}$=Ab and $D = id : \hom_{\mathrm{Ab}}(A,B) \longrightarrow \hom_{\mathrm{Ab}}(A,B)$, the category Ab of abelian groups is an example of an additive 2-category.

The data describing a weak category in Ab consists of four morphisms of abelian groups

$$A \xrightarrow{k} B\ ,\ \lambda,\rho : A \longrightarrow A\ ,\ \eta : B \longrightarrow A$$

subject to the conditions

$$\begin{aligned} k\lambda &= 0 = k\rho, \\ k\eta &= 0. \end{aligned}$$

This information can be used to construct the corresponding weak category with the objects being the elements of B, the morphisms pairs $(a,b) \in A \oplus B$

$$b \xrightarrow{(a,b)} k(a) + b,$$

and the composition

$$b \xrightarrow{(a,b)} k(a) + b \xrightarrow{(a',k(a)+b)} k(a'+a) + b$$

given by

$$(a', k(a)+b)(a,b) = (a' - \rho(a') + a - \lambda(a) - \eta(b), b).$$

For every three composable morphisms

$$b \xrightarrow{(a,b)} k(a) + b = b' \xrightarrow{(a',b')} k(a') + b' = b'' \xrightarrow{(a'',b'')} k(a'') + b'',$$

the 2-cell

$$\alpha : (a'',b'')((a',b')(a,b)) \longrightarrow ((a'',b'')(a',b'))(a,b),$$

is given by

$$\begin{aligned} &\alpha(a'',a',a,b) \\ &= \left((\rho^2 - \rho)(a'') + (\rho\lambda - \lambda\rho)(a') + \left(\lambda - \lambda^2 - \eta k + \rho\eta k\right)(a) + (\rho\eta - \lambda\eta)(b)\right). \end{aligned}$$

The 2-cells λ and ρ for one morphism (a,b)

$$\begin{aligned} \lambda &: (0,b')(a,b) \longrightarrow (a,b) \\ \rho &: (a,b)(0,b) \longrightarrow (a,b) \end{aligned}$$

are given by

$$\begin{aligned} \lambda(a,b) &= (\lambda a + \eta b, 0), \\ \rho(a,b) &= (\rho a + \eta b, 0). \end{aligned}$$

Note that if $B = 0$, we obtain what we called a weak monoid. If the morphisms λ, ρ, η are the zero morphisms, then we obtain an internal category in Ab (which is well known to be just a group homomorphism).

5.2 Morphisms of Abelian groups. The category Mor(Ab) of morphisms of abelian groups is the category where the objects are morphisms of Ab, say

$$A=\left(A_1 \xrightarrow{\partial} A_0\right).$$

The arrows are pairs of morphisms of Ab $(f_1,f_0):A\longrightarrow B$ such that the following square is commutative

$$\begin{tikzcd} A_1 \arrow[r, "\partial"] \arrow[d, "f_1"'] & A_0 \arrow[d, "f_0"] \\ B_1 \arrow[r, "\partial"] & B_0 \end{tikzcd} .$$

For each two arrows $f=(f_1,f_0)$ and $g=(g_1,g_0)$ from A to B, a 2-cell from f to g is a pair $(\tau,f):f\longrightarrow g$ where $\tau:A_0\longrightarrow B_1$ is a homomorphism of abelian groups with

$$\begin{aligned} \tau\partial &= g_1-f_1 \\ \partial\tau &= g_0-f_0. \end{aligned}$$

In order to be able to see that this category is an example of an additive 2-category we note that, with respect to objects and arrows, it is in fact an additive category. Now, for each pair of objects, A and B, we define the abelian group $K(A,B)$ as

$$K(A,B)=\hom_{Ab}(A_0,B_1)$$

and the homomorphism D as

$$D(\tau)=(\tau\partial,\partial\tau).$$

It can be shown that D satisfies all the conditions in (3.6) and (3.7) if we define the following laws of composition

$$\begin{aligned} g\tau &= g_1\tau, \\ \sigma\tau &= \sigma\partial\tau, \\ \sigma f &= \sigma f_0, \end{aligned}$$

for every $\tau\in K(A,B),\sigma\in K(B,C),f\in\hom(A,B),g\in\hom(B,C)$.

A weak category C in the additive 2-category of Mor(Ab) is determined by a commutative square

$$\begin{tikzcd} A_1 \arrow[r, "\partial"] \arrow[d, "k_1"'] & A_0 \arrow[d, "k_0"] \\ B_1 \arrow[r, "\partial'"] & B_0 \end{tikzcd}$$

together with three morphisms

$$\begin{aligned} \lambda,\rho &: A_0\longrightarrow A_1, \\ \eta &: B_0\longrightarrow A_1, \end{aligned}$$

satisfying the conditions

$$\begin{aligned} k_1\lambda &= 0=k_1\rho, \\ k_1\eta &= 0. \end{aligned}$$

The objects of C are pairs (b,d) with $b\in B_0, d\in B_1$. The morphisms are of the form

$$\begin{pmatrix} b & x \\ d & y \end{pmatrix}$$

with $b\in B_0, d\in B_1, x\in A_0, y\in A_1$.

A weak category in Mor(Ab) may also be viewed as a structure with objects, vertical arrows, horizontal arrows and squares, in the following way

$$\begin{array}{ccc} b & \xrightarrow{(b,x)} & b+k_0(x) \\ \binom{b}{d}\Big\downarrow & \begin{pmatrix} b & x \\ d & y \end{pmatrix} & \Big\downarrow\binom{b+k_0(x)}{d+k_1(y)} \\ b+\partial'(d) & \xrightarrow[(b+\partial'(d),x+\partial(y))]{} & * \end{array},$$

where $*$ stands for $b+\partial'(d)+k_0(x+\partial(y)) = b+k_0(x)+\partial'(d+k_1(y))$.

The horizontal composition between squares is given by

$$\begin{pmatrix} b+k_0(x) & x' \\ d+k_1(y) & y' \end{pmatrix} \circ \begin{pmatrix} b & x \\ d & y \end{pmatrix} = \begin{pmatrix} b & f_0(x')+g_0(x)+h_0(b) \\ d & f_1(y')+g_1(y)+h_1(d) \end{pmatrix},$$

where

$$\begin{aligned} f_0 &= 1-\partial\rho, \\ g_0 &= 1-\partial\lambda, \\ h_0 &= -\partial\eta, \end{aligned}$$

$$\begin{aligned} f_1 &= 1-\rho\partial, \\ g_1 &= 1-\lambda\partial, \\ h_1 &= -\eta\partial. \end{aligned}$$

For each three horizontal arrows

$$b \xrightarrow{(b,x)} b' \xrightarrow{(b',x')} b'' \xrightarrow{(b'',x'')} b''+k_0(x'')$$

with $b' = b+k_0(x)$ and $b'' = b'+k_0(x')$ the isomorphism for associativity is given by

$$\begin{array}{ccc} b & \xrightarrow{(b'',x'')\circ((b',x')\circ(b,x))} & b''+k_0(x'') \\ \binom{b}{0}\Big\downarrow & \begin{pmatrix} b & f_0(x'')+g_0(z)+h_0(b) \\ 0 & \alpha_1(x'')+\alpha_2(x')+\alpha_3(x)+\alpha_0(b) \end{pmatrix} & \Big\downarrow\binom{b''+k_0(x'')}{0} \\ b & \xrightarrow[((b'',x'')\circ(b',x'))\circ(b,x)]{} & b''+k_0(x'') \end{array}$$

where $z = f_0(x')+g_0(x)+h_0(b)$ and

$$\begin{aligned} \alpha_1 &= -f_1\rho, \\ \alpha_2 &= \lambda+g_1\rho-\rho-f_1\lambda, \\ \alpha_3 &= g_1\lambda-f_1\eta k_0, \\ \alpha_0 &= g_1\eta-f_1\eta. \end{aligned}$$

The left and right isomorphisms are given, respectively, by

$$\begin{array}{ccccc} b & & \xrightarrow{(b+k_0(x),0)\circ(b,x)} & & b+k_0(x) \\ \binom{b}{0}\Big\downarrow & & \begin{pmatrix} b & g_0(x)+h_0(b) \\ 0 & \lambda(x)+\eta(b) \end{pmatrix} & & \Big\downarrow\binom{b+k_0(x)}{0} \\ b & & \xrightarrow[(b,x)]{} & & b+k_0(x) \end{array}$$

and

$$\begin{array}{ccccc} b & & \xrightarrow{(b,x)\circ(b,0)} & & b+k_0(x) \\ \binom{b}{0}\Big\downarrow & & \begin{pmatrix} b & f_0(x)+h_0(b) \\ 0 & \rho(x)+\eta(b) \end{pmatrix} & & \Big\downarrow\binom{b+k_0(x)}{0} \\ b & & \xrightarrow[(b,x)]{} & & b+k_0(x) \end{array}.$$

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Dendrotopic Sets

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What is before the reader has arisen from research towards the thesis [7] on *weak higher-dimensional categories*. The thesis, to be more precise, will concern one of the many definitions of this notion that have been proposed so far, namely the one put forth by M. Makkai in his yet unpublished paper [5]. The present work concerns only the underlying geometric structures, which he calls *multitopic sets*.

The story of this definition begins with the work of J. Baez and J. Dolan, a good summary of which can be found in [1]. These authors suggest to define weak finite-dimensional categories as *opetopic sets* having universal fillers for certain configurations of cells. The striking feature of this definition is that the usually explicit algebraic operations — composition, associativity, coherence and so on — are implicit in the geometric structure. The aesthete, however, will find two minor flaws: the concept of universality cannot be generalized to the infinite-dimensional case, because it is defined by a reversed induction; and cells possess a non-geometric excess structure (an arbitrary total order on the source facets), which can be traced back to the definition of opetopic sets via *(sorted) operads* (whence the prefix 'ope-').

Both flaws have been eliminated by Makkai. As for the first, the object approach to universality is superseded by a represented-functor approach; the need for a notion of universality in the next dimension disappears. As for the second, symmetric operads are superseded by their non-symmetric counterparts, the so-called *multicategories* (whence the prefix 'multi-').

Multitopic sets were defined in the three-part paper [4]. The definition follows algebraic ideas: a multitopic set gives rise to a free multicategory in each dimension, which in turn allows for cells in the next dimension to be constructed in a globular fashion. In the more recent account [3], it is shown that an alternative definition is available: multitopic sets are precisely the *many-to-one computads*. Here 'computad' refers to a freely generating subset of a strict ∞-category, and 'many-to-one' indicates that this subset is closed under the target operations.

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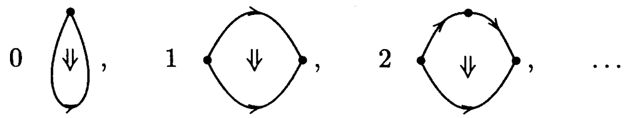
The present account brings yet a third definition, under the more descriptive name '*dendrotopic set*'. It is briefer than the first and the most elementary of the three. It entirely follows geometric ideas.

Dendrotopic sets are built from *dendrotopes*, just as, say, globular sets are built from globes. Dendrotopes in turn are oriented polytopes satisfying certain simple conditions. Here a *polytope* is determined by the incidence structure of its faces. A 0-dendrotope is just a point,

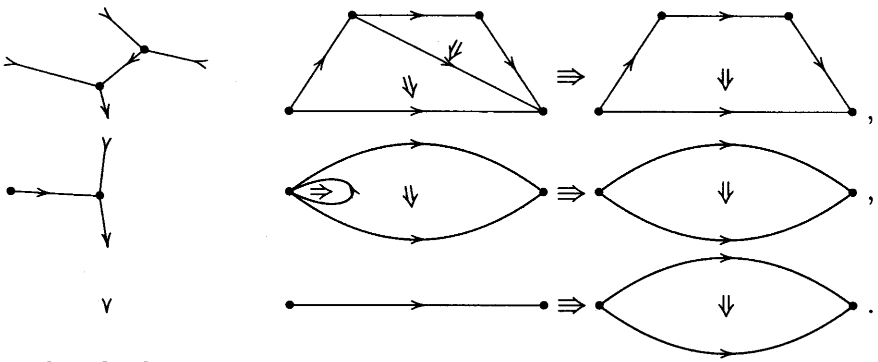
•.

A 1-dendrotope is just a directed line segment from a 0-dendrotope to a 0-dendrotope,

(So far the same can be said about globes.) A 2-dendrotope is a surface suspended from a path of 1-dendrotopes to a single 1-dendrotope; it is determined by the length of the path.



A 3-dendrotope is a solid filling the space from a tree of 2-dendrotopes to a single 2-dendrotope; it is determined by the bare planar tree. For example,



In general, a dendrotope is an oriented polytope whose facets are dendrotopes as well; furthermore

(Δ1) the target consists of a single facet,
(Δ2) the source inherits a tree structure (whence the prefix 'dendro-'),
(Δ3) each 3-codimensional face is a ridge of the target facet precisely once.

The paper is self-contained, to the point that it should be accessible to the general mathematical public. Only a few remarks are directed specifically at category theorists. Part I defines the notion of a dendrotopic set and related ones (with the latter accounting for much of its size). Part II shows that dendrotopic sets are precisely the many-to-one computads, whence they are precisely the multitopic sets as well.

Part I. The Definition

The definition of 'dendrotopic set' will be presented in three steps. The first two introduce notions which are interesting enough to deserve being named: *propolytopic sets* (Section 2) and *oriented propolytopic sets* (Section 4). The former can be specialized to obtain the notion of a *polytopic set*, that is, a combinatorial structure built from polytopes. This is done in Section 3, whose chief aim it is to compare the present concept of polytope with the standard one, as defined for example in the monograph [6]. Note that the concept considered standard here is *not* the one according to which a polytope is (the face lattice of) the convex hull of finitely many points in affine space.

In fact two definitions of 'dendrotopic set' will be presented, the first officially (Section 5), the second in the form of a theorem (Section 6). The first is the natural and practical one: the existence part of axiom ($\Delta 3$) is omitted. The second is the elegant one: the entire axiom ($\Delta 2$) is omitted. (In either case the omitted piece is a consequence of the remaining ones.) Part I is rounded off by the result that every finite arrangement of dendrotopes as a tree has the potential of being the source of another dendrotope (Section 7).

Section 1 supplies two auxiliary concepts.

Section 1. Higher-Dimensional Numbers and Hemigraphs

This preliminary section provides two new, yet basic notions for later reference. *Higher-dimensional numbers* will naturally enumerate the source facets of a dendrotope. The incidence structure of the overall source in codimensions 1 and 2 will be a particular kind of *hemigraph* (namely, a tree).

A *list of length* ℓ is simply an ℓ-tuple. We shall use angular brackets to denote lists. Thus, the list with entries $x_0, \ldots, x_{\ell-1}$ is denoted by $\langle x_0, \ldots, x_{\ell-1}\rangle$. Concatenation of lists is denoted by $+$; the empty list is denoted by 0. We shall differentiate between the element x and the list $\langle x\rangle$.

The lists with entries in a given set X form the monoid X^* freely generated by it. The product is $+$, and the neutral element is 0. Generators are inserted by means of the mapping $x \mapsto \langle x\rangle$. The smallest such monoid that is closed under the insertion of generators consists of the *higher-dimensional numbers*.

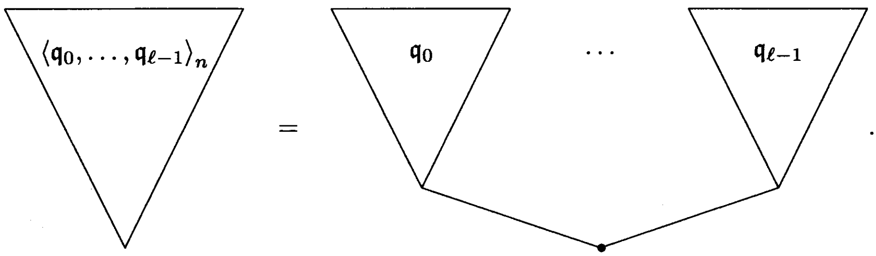
A more explicit description can be given as follows. For $n \geq 0$, we define an *n-dimensional number* inductively to be a list of $(n-1)$-dimensional numbers. The understanding is that -1-dimensional numbers do not exist. Thus, denoting the set of n-dimensional numbers by $\hat{\mathbf{N}}_n$, we have

$$\hat{\mathbf{N}}_{-1} = \varnothing \qquad \text{and} \qquad \hat{\mathbf{N}}_n = \hat{\mathbf{N}}^*_{n-1} \quad (n \geq 0).$$

The only 0-dimensional number is the empty list 0, and 1-dimensional numbers are lists $\langle 0, \ldots, 0\rangle$. We can identify each of the latter with its length, so that $\hat{\mathbf{N}}_1$ becomes precisely the monoid $\mathbf{N}$ of natural numbers. A simple inductive argument shows that we have a chain of monoids $\hat{\mathbf{N}}_0 \subseteq \hat{\mathbf{N}}_1 \subseteq \cdots$. The union $\hat{\mathbf{N}}_\infty$ is the monoid of all higher-dimensional numbers.

The inclusions $\hat{\mathbf{N}}_{n-1} \subseteq \hat{\mathbf{N}}_n$ are convenient as of now, but it will sometimes be important exactly which dimension a number is considered in. We keep track of these dimensions by means of subscripts. We write $\langle \mathfrak{q}_0, \ldots, \mathfrak{q}_{\ell-1}\rangle_n$ for the specifically n-dimensional number whose entries are the specifically $(n-1)$-dimensional numbers $\mathfrak{q}_0, \ldots, \mathfrak{q}_{\ell-1}$. In the case $\ell = 0$ we write 0_n.

The concept of an n-dimensional number is equivalent to the concept of an *n-stage tree* considered by M. Batanin in [2]. In vivid terms, such a tree is an isomorphism class of finite planar rooted trees with all heights $\leq n$. (Here the term 'tree' is used in its standard meaning. A slightly refined notion of tree is introduced further below.) The proof of this equivalence is suggested by the diagrammatic equation



Despite the common goal of defining weak higher-dimensional categories, the present use of higher-dimensional numbers has no evident similarities to Batanin's use of trees.

Before we introduce hemigraphs, we recall some related notions.

A *(directed) graph* $\mathfrak{G}$ consists of two sets $\mathfrak{G}_0$ and $\mathfrak{G}_1$ and two mappings δ_-, $\delta_+ : \mathfrak{G}_1 \rightrightarrows \mathfrak{G}_0$. The set $\mathfrak{G}_0$ contains the *vertices* and the set $\mathfrak{G}_1$ contains the *arrows* of the graph. Given an arrow F, the vertices $F\delta_-$ and $F\delta_+$ are its *source* and its *target*. We may visualize F thus:

$$F\delta_- \bullet \xrightarrow{F} \bullet F\delta_+.$$

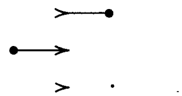
A *path of length* ℓ from (= *starting* at = having *source*) a vertex A to (= *ending* at = having *target*) a vertex B consists of $\ell + 1$ vertices $A = A_0$, A_1, ... , $A_{\ell-1}$, $A_\ell = B$ and ℓ arrows $F_{1/2}$, ... , $F_{\ell-1/2}$ satisfying the *consecutiveness conditions* $F_i\delta_\eta = A_{i+\eta/2}$ ($i \in \{1/2, \ldots, \ell - 1/2\}$, $\eta \in \mathbf{Z}^\times$). (We take the signs $-$ and $+$ to be abbreviations for the numbers -1 and $+1$, which in turn form the group $\mathbf{Z}^\times$ of unit integers.) If $\ell > 0$ and $A = B$ we speak of a *cycle*. We slight the question of equality of cycles, which will not be of importance to us.

An *undirected graph* $\mathfrak{G}$ is a graph together with a direction-reversing involution fixing each vertex, but no arrow. That is, undirectedness is expressed by a mapping $\chi : \mathfrak{G}_1 \to \mathfrak{G}_1$ such that for each arrow F we have $F\chi \neq F$, $F\chi^2 = F$ and $F\chi\delta_\eta = F\delta_{-\eta}$. The orbits of χ are the *edges* of $\mathfrak{G}$. Given an edge $\{F, F\chi\}$, the vertices $F\delta_-$ and $F\delta_+$ are its *extremities*. We may visualize $\{F, F\chi\}$ thus:

$$F\delta_- \bullet \overset{\{F, F\chi\}}{\longrule} \bullet F\delta_+.$$

There is an evident way of specifying the structure of an undirected graph by listing vertices, edges, and the two extremities of each edge. (Specifying the extremities of all edges amounts to specifying a mapping of the set of edges into the set of unordered pairs of vertices.) We can in particular associate to a directed graph an underlying undirected graph, declaring each arrow to become an edge and its source and target to become its extremities. A path in this underlying structure is a *zigzag* in the original graph. Conversely, we can impose a direction on an undirected graph by choosing a representative arrow for each edge.

Hemigraphs are defined in the same way as graphs, the only difference being that the source and target mappings are allowed to be partial. Hence an arrow may lack a source or a target or both. These cases can be visualized thus:



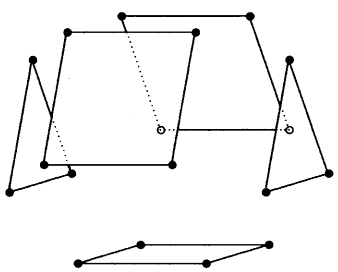
In this context it makes sense to allow paths to start or end at an arrow; lengths are measured naturally in half steps. For example, a path of length $3/2$ from a vertex A to an arrow G consists of two vertices A and B and two arrows F and G with $F\delta_-$ defined and equal to A and both $F\delta_+$ and $G\delta_-$ defined and equal to B.

A *tree* to us is a hemigraph with a distinguished arrow Ω, called *root*, such that from each vertex or arrow there is a unique path to Ω. The length of this path is the *height* of the vertex or arrow. (Hence the heights of arrows are natural numbers and the heights of vertices are natural numbers plus $1/2$.) Necessary conditions for a hemigraph to be a tree with root Ω are that each vertex has precisely one outgoing arrow and that Ω is the only arrow without a target. A hemigraph satisfying these conditions will be called *locally treelike* with *root* Ω. Conversely, in a locally treelike hemigraph the component containing the root is a tree. Each of the other components contains a *forward-infinite path*, that is: vertices A_0, A_1, ... and arrows $F_{1/2}$, $F_{3/2}$, ... satisfying the consecutiveness conditions. We may hence define trees equivalently by local treelikeness plus either connectedness or absence of forward-infinite paths. Now a forward-infinite path either contains a vertex twice, in which case the intermediate vertices and arrows form a cycle, or indicates that the hemigraph itself is infinite. Therefore a finite hemigraph is a tree if and only if it is locally treelike and cycle-free.

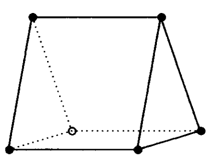
Use of the term 'tree' invokes a baggage of botanical vocabulary. For example, a *leaf* is an arrow without a source. This notion should be contrasted with that of a *stump*, which is a vertex without an incoming arrow. A tree is *empty* if it consists of the root only; that is: if the root is a leaf. In the other case the root has a source, called the *stem* of the tree. (Thus the stem is the only vertex with height $1/2$.)

Section 2. Propolytopic Sets

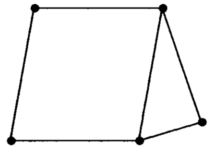
A polytope P can be constructed as follows. First take a finite family of polytopes of the previous dimension.



They become the *facets* of P. They have facets of their own, and thus give rise to a family of "facet facets". Now arrange the "facet facets" into pairs and glue them accordingly.



The joints become the *ridges* of P. Finally fill the interior of the (ideally) hollow figure that has arisen.



A formalization of the choices to be made in this construction leads to the definition of a *propolytopic set.*

Definition. A *propolytopic set* $\mathfrak{P}$ provides for each integer n the data (Π0), (Π1) and (Π2) below.

(Π0) a set $\mathfrak{P}_n$. We call the elements of $\mathfrak{P}_n$ the *cells of dimension* n of $\mathfrak{P}$, the n*-cells* for short.

(Π1) to each n-cell P, a finite family $\langle\, Pq \mid q \in P\mathrm{I}\,\rangle$ of $(n-1)$-cells. We call $q \in P\mathrm{I}$ a *facet location* of P and Pq its *resident.*

A *chamber of depth* d underneath an n-cell P, a d*-chamber* for short, is a d-tuple $\langle p^1, \ldots, p^d\rangle$ in which, for certain cells $P = P^0$, P^1, ..., P^d, the entry p^{i+1} is a facet location of P^i with resident P^{i+1}. The $(n-d)$-cell $P^d = Pp^1 \cdots p^d$ will be called the *floor* of the chamber. The set of all d-chambers underneath P will be denoted by $P\mathrm{I}^d$.

(Π2) to each n-cell P, a partition $P\Diamond$ of $P\mathrm{I}^2$ into 2-element subsets of 2-chambers with common floor. We call $\{\langle q_0, r_0\rangle, \langle q_1, r_1\rangle\} \in P\Diamond$ a *ridge location* of P and $Pq_0r_0 = Pq_1r_1$ its *resident.*

Instead of '$\{\langle q_0, r_0\rangle, \langle q_1, r_1\rangle\} \in P\Diamond$' we shall write more compactly '$P : \frac{q_1\ r_1}{q_0\ r_0}$'. Being given a partition into 2-element subsets is equivalent to being given an involution without fixed element; viewed as such, $P\Diamond$ will be called the *conjugation* underneath P. Consequently the relationship $\frac{q_1\ r_1}{q_0\ r_0}$ can be rendered by saying that $\langle q_0, r_0\rangle$ and $\langle q_1, r_1\rangle$ are *conjugates* of each other.

Cells of dimensions ≥ 0 will be called *proper*, cells of dimension < 0 will be called *improper*. A propolytopic set is *plain* if all of its cells are proper. In this case of course 0-cells have no facet locations and 1-cells have no ridge locations. Any propolytopic set has a *plain part* (that is: underlying plain propolytopic set), obtainable by removing improper cells and the locations where they reside. While our interest in propolytopic sets will be restricted to their plain parts, improper cells will serve a purpose in avoiding low-dimensional exceptions to general statements.

Examples. The plain propolytopic sets characterized below will be referred to as the *classical polytopic sets.*

- In a *globular set* each cell of dimension ≥ 1 has precisely 2 facet locations δ_- and δ_+, and each cell of dimension ≥ 2 has the 2 ridge locations

$$\frac{\delta_-\ \delta_\zeta}{\delta_+\ \delta_\zeta} \qquad (\zeta \in \mathbf{Z}^\times).$$

- In a *simplicial set (without degeneracies!)* each cell of dimension $n \geq 1$ has precisely $n+1$ facet locations δ_0, ..., δ_n, and each cell of dimension $n \geq 2$ has the $(n+2)(n+1)/2$ ridge locations

$$\frac{\delta_j\ \delta_{i-1}}{\delta_i\ \ \delta_j} \qquad (0 \leq j < i \leq n).$$

- In a *cubical set (without degeneracies!)* each cell of dimension $n \geq 1$ has precisely $2n$ facet locations $\delta_{0,-}, \ldots, \delta_{n-1,-}, \delta_{0,+}, \ldots, \delta_{n-1,+}$, and each cell of dimension $n \geq 2$ has the $2(n+1)n$ ridge locations

$$\frac{\delta_{j,\zeta}\ \delta_{i-1,\eta}}{\delta_{i,\eta}\ \ \delta_{j,\zeta}} \quad (0 \leq j < i < n;\ \eta, \zeta \in \mathbf{Z}^{\times}).$$

Facets and ridges are instances of *faces*, which we are now going to introduce in full generality.

A *panel of depth d* underneath an n-cell P, a *d-panel* for short, is a $(d-1)$-tuple

$$\langle p^1, \ldots, p^{\tau-1}, \frac{p_1^\tau\ p_1^{\tau+1}}{p_0^\tau\ p_0^{\tau+1}}, p^{\tau+2}, \ldots, p^d\rangle \tag{1}$$

in which, for certain cells $P = P^0, P^1, \ldots, P^{\tau-1}, P^{\tau+1}, \ldots, P^d$, the entry p^{i+1} is a facet location of P^i with resident P^{i+1} and the entry $\frac{p_1^\tau\ p_1^{\tau+1}}{p_0^\tau\ p_0^{\tau+1}}$ is a ridge location of $P^{\tau-1}$ with resident $P^{\tau+1}$. The number τ will be referred to as the *type* of the panel. We say that (1) *separates* the two chambers

$$\langle p^1, \ldots, p^{\tau-1}, p_j^\tau, p_j^{\tau+1}, p^{\tau+2}, \ldots, p^d\rangle \quad (j \in \{0, 1\}).$$

The $(n-d)$-cell P^d will be called the *floor* of the panel.

The *undirected depth-d chamber graph* $P\Gamma^d$ underneath P is defined as follows. Its vertices are the d-chambers and its edges are the d-panels underneath P. The extremities of a panel are the two chambers it separates. A path in this graph will also be called a *gallery of depth d* underneath P, a *d-gallery* for short. (The fancy terminology — 'chamber/panel/gallery' — goes back to Bourbaki and has been adapted for current needs.) Note that all the chambers and panels along a gallery have the same floor. This $(n-d)$-cell will be called the *floor* of the gallery.

We display a gallery of depth d and length ℓ as an (often somewhat repetitious) $(\ell+1)$-by-d matrix. Each row represents a chamber, and each two consecutive rows represent a panel, with the facet-location entries doubled and the ridge-location entry marked by a horizontal line as before. Thus the generic panel (1), viewed as a gallery of length 1, takes the following appearance:

$$\begin{array}{ccccccc} p^1 & \cdots & p^{\tau-1} & \underline{p_1^\tau\ p_1^{\tau+1}} & p^{\tau+2} & \cdots & p^d \\ p^1 & \cdots & p^{\tau-1} & p_0^\tau\ p_0^{\tau+1} & p^{\tau+2} & \cdots & p^d \end{array}.$$

Definition. The components of $P\Gamma^d$ are called the *d-face locations* of of P. Here '*d*-face' is short for 'face of *codimension d*'. We write $P\Pi^d$ for the set of all d-face locations of P.

Thus each face location is represented by (= contains) a chamber, and two chambers represent the same face location if and only if they are connected by a gallery. It follows that all representatives of a face location f of P have the same floor. This cell is called the *resident* of f and denoted by Pf.

Examples. Let $0 < d \leq n$.

- Each n-cell in a globular set has precisely 2 d-face locations δ_η^d, where η ranges over the set of signs. Here each d-face location δ_η^d has 2^{d-1} representative chambers $\langle \delta_{(1)\theta}, \ldots, \delta_{(d-1)\theta}, \delta_\eta\rangle$, where θ ranges over the set of mappings $\{1, \ldots, d-1\} \longrightarrow \mathbf{Z}^{\times}$.

- Each n-cell in a simplicial set has precisely $\binom{n+1}{d}$ d-face locations δ^d_Λ, where Λ ranges over the set of d-element subsets of $\{0,\ldots,n\}$. Here each d-face location δ^d_Λ has $d!$ representative chambers $\langle\delta_{(1)\bar\lambda},\ldots,\delta_{(d)\bar\lambda}\rangle$, where λ ranges over the set of bijective mappings $\{1,\ldots,d\} \xrightarrow{\sim} \Lambda$ and

$$(i)\bar\lambda = (i)\lambda - \sum_{\substack{j<i\\(j)\lambda<(i)\lambda}} 1.$$

- Each n-cell in a cubical set has precisely $2^d\binom{n}{d}$ d-face locations $\delta^d_{\Lambda,\theta}$, where Λ ranges over the set of d-element subsets of $\{0,\ldots,n-1\}$ and θ ranges over the set of mappings $\Lambda \longrightarrow \mathbf{Z}^\times$. Here each d-face location $\delta^d_{\Lambda,\theta}$ has $d!$ representative chambers $\langle\delta_{(1)\bar\lambda,(1)\lambda\theta},\ldots,\delta_{(d)\bar\lambda,(d)\lambda\theta}\rangle$, where λ ranges over the set of bijective mappings $\{1,\ldots,d\} \xrightarrow{\sim} \Lambda$ and $\bar\lambda$ is defined as before.

A *d-face* of an n-cell P is an $(n-d)$-cell P' together with a d-face location f such that $Pf = P'$. This almost empty definition is intended to solve the following linguistic puzzle: any one face is the resident of a face location (and hence a cell), two different faces are residents of two different face locations (but not necessarily different cells). Any term describing either a face location or the associated face will also be used to describe the other.

Let P' be an u-face of P with location f, and let v be a natural number. For every u-chamber $\langle p^1,\ldots,p^u\rangle$ representing f, we have a canonical embedding

$$(\langle p^1,\ldots,p^u\rangle + ?\,) : P'\Gamma^v \longrightarrow P\Gamma^{u+v}$$

of graphs, which induces a mapping $P'\Pi^v \longrightarrow P\Pi^{u+v}$ of the component sets. This mapping does not depend on the representative chamber, as the reader can easily verify. We denote the image of g under it by fg.

The depth-0 chamber graph underneath P contains but a single vertex, the empty list 0. The represented location is called *trivial* and denoted by 1; its resident is P itself. For $d > 0$, the depth-d chamber graph is, as a combinatorist would put it, $(d-1)$-*regular*, meaning that each vertex links precisely $d-1$ edges. In fact, each of the possible types $1, \ldots, d-1$ occurs precisely once among the panels surrounding a chamber. We can readily classify the possible components for small depths.

1. Each component of $P\Gamma^1$ consists of one vertex and no edge. The three phrases 'q is a facet location of P with resident Q', '$\langle q\rangle$ is a 1-chamber underneath P with floor Q' and 'q is a 1-face location of P with resident Q' evidently mean the same. We identify $P\mathrm{I}$ and $P\Pi^1$ accordingly.
2. Each component of $P\Gamma^2$ consists of two vertices and one connecting edge. The three phrases '$\frac{q_1\ r_1}{q_0\ r_0}$ is a ridge location of P with resident R', '$\langle\frac{q_1\ r_1}{q_0\ r_0}\rangle$ is a 2-panel underneath P with floor R' and '$q_0r_0 = q_1r_1$ is a 2-face location of P with resident R (while $\langle q_0,r_0\rangle \neq \langle q_1,r_1\rangle$)' evidently mean the same. We identify $P\Diamond$ and $P\Pi^2$ accordingly.

3. Each component of $P\Gamma^3$ is a cycle. Its length is even, since panel types alternate between 1 and 2. Its general form is

$$\begin{array}{ccc} \cline{2-3} q_0 & r_{2\ell-1} & s_{\ell-1} \\ \cline{1-2} q_{\ell-1} & r_{2\ell-2} & s_{\ell-1} \\ \cline{2-3} \vdots & \vdots & \vdots \\ \cline{2-3} q_1 & r_1 & s_0 \\ \cline{1-2} q_0 & r_0 & s_0 \\ \cline{2-3} \end{array},$$

where the two dotted lines together signify a single type-2 panel.

By now the category-theoretically educated reader will have noticed that a propolytopic set is nothing but a special kind of category presentation, introduced here with a weird terminology. Cells are the vertices/objects, facet locations are the arrows of the underlying graph (generators), ridge locations are the commutativity conditions (relators), and face locations in general are the morphisms of the presented category. Disagreement may arise about the direction of arrows/morphisms; the author decrees that they point upwards with respect to dimension. Thus a face location f of P is a morphism $f : Pf \twoheadrightarrow P$. (Traditionalists will be satisfied to see that the notation for composition, introduced in the previous paragraph but one, exhibits the confusing reversal of order that they have become so accustomed to. On a more serious note, the upward direction reflects the view that a face location is an inclusion of its resident by its owner. This view is made more precise in the following section, where *geometric realizations* are explained.) Dimensions amount to a *grading* of the underlying graph, by which we provisionally mean a homomorphism into the graph whose vertices as well as arrows are the integers, with the arrow n pointing from the vertex n to the vertex $n + 1$. All said, we can alternatively define a propolytopic set to be a category presentation with its underlying graph graded, satisfying the following two conditions.

- Each vertex has finitely many incoming arrows.
- The paths occurring in the commutativity conditions have length 2, and each length-2 path thus occurs precisely once. (A commutativity condition is taken to be an unordered pair of paths agreeing in source and target.)

A presentation with these properties can be recovered from the abstract category that it gives rise to. We can hence also define a propolytopic set as a special kind of category. Hints to this effect are given towards the end of the following section.

If the author has decided against the "categorical" approach outlined here, he did so for the following reasons. Firstly, he wanted to keep this work as elementary as possible. Secondly, making the graph structure explicit may have led to an overuse of graph terminology. And thirdly, hardly any feature of propolytopic sets discussed beyond Section 3 occurs in the context of category presentations at large. The reader may feel free to skip the parts that become trivial by translation into category-theoretic language.

Section 3. Polytopic Sets

Under a connectedness assumption, a propolytopic set may drop the prefix 'pro-', and its cells are deemed *polytopes*. In fact, the notion of 'polytope' thus defined is *almost* the standard (abstract) one. Arguably the former improves the latter, by allowing for multiple face incidences.

The section opens with three fundamental constructions on a propolytopic set $\mathfrak{P}$. Category theorists will recognize them as yielding slice categories, "coslice" categories and categories of factorizations.

Let A be an α-cell in $\mathfrak{P}$. The propolytopic set $A\mathfrak{P}$ is defined as follows. An n-cell of $A\mathfrak{P}$ consists of an n-cell P of $\mathfrak{P}$ and an $(\alpha - n)$-face location a of A with resident P. A face location of $(P; a)$ in $A\mathfrak{P}$ with resident $(P'; a')$ consists of a such face location f of P in $\mathfrak{P}$ with resident P' as satisfies $af = a'$. We sometimes identify $A\mathfrak{P}$ with A and therefore call the former a *cell* as well.

Let B be a β-cell in $\mathfrak{P}$. The propolytopic set $\mathfrak{P}B$ is defined as follows. An n-cell of $\mathfrak{P}B$ consists of a $(\beta+n+1)$-cell P of $\mathfrak{P}$ and an $(n+1)$-face location b of P with resident B. (The dimension shift is dictated by geometric considerations.) A face location of $(P; a)$ in $\mathfrak{P}B$ with resident $(P'; a')$ consists of a such face location f of P in $\mathfrak{P}$ with resident P' as satisfies $fb' = b$. We call $\mathfrak{P}B$ the *link* about B.

The third construction combines the other two. Let F be a face location of an α-cell A of $\mathfrak{P}$ with resident a β-cell B. The link about $(B; F)$ of the cell A is isomorphic to the cell $(A; F)$ of the link about B. Let us denote either of the two by $A\mathfrak{P}_F B$. The propolytopic set $A\mathfrak{P}_F B$ can also be defined directly as follows. An n-cell of $A\mathfrak{P}_F B$ consists of a $(\beta + n + 1)$-cell P of $\mathfrak{P}$ and two face locations, a of A with resident P (and codimension $\alpha - \beta - n - 1$) and b of P with resident B (and codimension $n + 1$), such that $ab = F$. A face location of $(P; a, b)$ in $A\mathfrak{P}_F B$ with resident $(P'; a', b')$ consists of a such face location f of P in $\mathfrak{P}$ with resident P' as satisfies $af = a'$ and $fb' = b$. We call $A\mathfrak{P}_F B$ the *link* about F.

By a *co-d-face* of an n-cell P we mean the link about an $(n-d+1)$-face location of P. A *cofacet* is a co-1-face, and a *coridge* is a co-2-face.

The three constructions yield the principal examples for the three notions to be introduced next: a cell is a *propolytope*, the link about a cell is a *polytopic set*, and the link about a face location is a *polytope*.

A *propolytope of dimension n*, an *n-propolytope* for short, is a propolytopic set with an n-cell $\top$ that has each cell as a face precisely once. The same definition with 'propolytope/propolytopic' replaced by any similar noun/adjective pair is implicit whenever the adjective as an attribute to 'set' refers to a variant of a propolytopic set. Thus it is clear what is meant by the terms '*globe*', '*simplex*', '*cube*', and it will be clear what is meant by the term '*polytope*' as soon as the following statement has been issued.

Definition. A *polytopic set* is a propolytopic set with a -1-cell $\perp$ that is a face of each cell precisely once.

It follows that each 0-cell has precisely one facet location, which we shall denote by ε, and that each 1-cell has precisely one ridge location. The first observation implies that the facet locations t of a 1-cell S are in one-to-one correspondence with the 2-chambers $\langle t, \varepsilon \rangle$ underneath S, whence the second observation implies that S has precisely two facet locations.

A propolytopic set $\mathfrak{P}$ each of whose 1-cells has precisely two facets will be called *augmentable*. In this situation the 0-cells and 1-cells of $\mathfrak{P}$ are in an obvious manner the vertices and edges of an undirected graph. We can modify any such $\mathfrak{P}$ (the case of interest being the one where $\mathfrak{P}$ is plain) by affixing a -1-cell $\perp$, for each 0-cell a facet location ε with resident $\perp$ and for each 1-cell with facet locations t_0 and t_1 a ridge location $\frac{t_1\ \varepsilon}{t_0\ \varepsilon}$. This process is known as *trivial augmentation*. Thus, any polytopic set can be obtained from a certain augmentable plain propolytopic set (namely, its plain part) by trivial augmentation.

Examples. Classical polytopic sets as defined in the previous section are not polytopic sets as defined here, but they become polytopic sets by trivial augmentation.

Why trivially augmented classical polytopic sets are indeed polytopic sets can be seen most conveniently by taking the following alternative approach to the new concept, which also has the advantage of being more intuitive.

We now clarify the geometric content of the notions discussed in Section 2. Let $\mathfrak{P}$ be a propolytopic set. We assign to each cell P a topological space $P\Gamma$, to be called its *(geometric) realization*, and to each of its facet locations q a continuous mapping $q\Gamma : Pq\Gamma \twoheadrightarrow P\Gamma$. If P is improper, its realization is the empty space. The case of a proper cell is handled by induction on its dimension. The induction step was illustrated at the beginning of Section 2. It is divided into three smaller steps. In the first, we form the topological sum $P\Gamma^{\bullet\bullet}$ of the realizations $Pq\Gamma$ of all facets. In the second, we form the quotient space $P\Gamma^{\bullet}$ of $P\Gamma^{\bullet\bullet}$ that arises from identifying corresponding image points under the two mappings $r_j\Gamma : P\frac{q_1\ r_1}{q_0\ r_0}\Gamma \twoheadrightarrow Pq_j\Gamma$ ($j \in \{0,1\}$), for all ridges. We call $P\Gamma^{\bullet}$ the *boundary* of the realization of P. In the third step, we form the (unreduced) cone $P\Gamma$ over $P\Gamma^{\bullet}$. (The cone is obtained from the cylinder by collapsing one of the two frontiers to a point; it contains at least this point.) The mapping $q\Gamma$ is the composite

$$Pq\Gamma \twoheadrightarrow P\Gamma^{\bullet\bullet} \twoheadrightarrow P\Gamma^{\bullet} \twoheadrightarrow P\Gamma$$

of summand inclusion, projection, base inclusion.

If P has dimension n and $P\Gamma^{\bullet}$ is an $(n-1)$-sphere, we call P *spherical.* In this case $P\Gamma$ itself is an n-ball, while otherwise it is not even an n-manifold with the indicated boundary. Note that for $n > 0$, the space $P\Gamma^{\bullet}$ is a (closed) $(n-1)$-manifold if and only if all facets and cofacets of P are spherical.

Let us call the propolytopic set $\mathfrak{P}$ *spherical* if all of its proper cells are. Ideally sphericality is implied by additional axioms (and in this sense the present use of the term 'cell' is justified). Making it a separate axiom would be useless, since it is too hard to verify. All classical polytopic sets are well known to be spherical.

In low dimensions, the following picture emerges.

0. Let T be a 0-cell. Then $T\Gamma^{\bullet}$ is empty, whence T is automatically spherical.
1. Let S be a 1-cell. Then $S\Gamma^{\bullet}$ is a discrete space, with each point corresponding to a facet of S. Hence S is spherical if and only if it has precisely two facets.
2. Assume all 1-cells of $\mathfrak{P}$ spherical (that is, assume $\mathfrak{P}$ augmentable), and let R be a 2-cell. Then $R\Gamma^{\bullet}$ is a disjoint union of circles. Hence R is spherical if and only if $R\Gamma^{\bullet}$ is connected.

More generally, for every cell of dimension ≥ 2 sphericality implies connectedness of the boundary of the geometric realization. This property in turn is equivalent to connectedness of the combinatorial structures considered next.

Let P be a cell of $\mathfrak{P}$. The *undirected dual graph* $P\mathrm{D}$ of P is defined as follows. Vertices are the facet locations of P; edges are the ridge locations of P. The extremities of a ridge location $\frac{q_1\ r_1}{q_0\ r_0}$ are q_0 and q_1.

The condition we have alluded to appears in the following result.

Proposition 1. *Let $\mathfrak{P}$ be a propolytopic set with precisely one improper cell, in dimension -1. Then $\mathfrak{P}$ is a polytopic set if and only if the dual graphs of all proper cells of $\mathfrak{P}$ are connected.*

Proof. We only show sufficiency. The necessity proof reaches its climax with the observation that the first entries of the type-1 panels of a gallery are the edges of a path in the dual graph.

Suppose the dual graphs of all proper cells of $\mathfrak{P}$ are connected, and let P be an n-cell of $\mathfrak{P}$. We show by induction on n that P has precisely one $(n+1)$-face location. The result is obvious for $n=-1$, so let us suppose that $n \geq 0$.

Since PD is connected, it contains a vertex q. By the existence part of the induction hypothesis, Pq has an n-face location f. Now qf is an $(n+1)$-face location of P, and this settles the existence part of the claim.

Now consider an arbitrary $(n+1)$-face location of P. Being represented by a chamber, it can be written $q'f'$, where q' is a facet location of P and f' is an n-face location of Pq'. Since the graph PD is connected, it contains a path

$$q=q_0 \underset{\frac{q_1\ r'_0}{q_0\ r_0}}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet} q_1 \quad \cdots \quad q_{\ell-1} \underset{\frac{q_\ell\ r'_{\ell-1}}{q_{\ell-1}\ r_{\ell-1}}}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet} q_\ell = q'.$$

(If $n = 0$ then $\ell = 0$.) By the existence part of the induction hypothesis, the resident of $\frac{q_{i+1}\ r'_i}{q_i\ \ r_i}$ has an $(n-1)$-face location g_i. By the uniqueness part of the induction hypothesis, applied to Pq_i, we have $r'_{i-1}g_{i-1} = r_i g_i$, where the reader substitutes 'f' or 'f'' for an undefined expression '$r'_{-1}g_{-1}$' or '$r_\ell g_\ell$'. The situation is summarized in the commutative diagram (containing some abbreviations with obvious meaning)

$$\begin{tikzcd}
 & & & & P & & & & \\
Q_0 \arrow[urrrr, two heads, "q=q_0"] & & Q_1 \arrow[urr, two heads, "q_1"'] & & \cdots & & Q_{\ell-1} \arrow[ull, two heads, "q_{\ell-1}"] & & Q_\ell \arrow[ullll, two heads, "q_\ell=q'"'] \\
 & R_0 \arrow[ul, two heads, "r_0"] \arrow[ur, two heads, "r'_0"'] & & & \cdots & & & R_{\ell-1} \arrow[ul, two heads, "r_{\ell-1}"'] \arrow[ur, two heads, "r'_{\ell-1}"] & \\
 & & & & \bot \arrow[uullll, two heads, "f"] \arrow[ulll, two heads, "g_0"'] \arrow[urrr, two heads, "g_{\ell-1}"] \arrow[uurrrr, two heads, "f'"'] & & & &
\end{tikzcd},$$

which can be read

$$qf = q_0r_0g_0 = q_1r'_0g_0 = \cdots = q_{\ell-1}r_{\ell-1}g_{\ell-1} = q_\ell r'_{\ell-1}g_{\ell-1} = q'f'.$$

This settles the uniqueness part of the claim. □

In an augmentable plain propolytopic set, the dual graph of a 0-cell is empty, and the dual graph of a 1-cell consists of two vertices. Trivial augmentation adds a vertex to the former and an edge to the latter, making them connected; the dual graphs of all other cells remain unchanged. Consequently trivial augmentation makes a spherical plain propolytopic set — the classical polytopic sets being examples — into a (spherical) polytopic set.

The section closes with a (very condensed) comparison of "our" polytopes with the standard ones. The attributes 'present-type' and 'standard' will make the distinction clear; the bare term 'polytope' is reserved for a notion unifying the other two. Namely, *we pretend for the remainder of this section that the word 'finite' has been omitted from item* (Π1) *and that the subsequent text has been changed accordingly.* These changes do not concern the main ideas and results; the most significant one accounts for the fact that a 3-face location (as a graph) can now be a path infinite in both directions (rather than a cycle of even length).

It may be interesting to note that the new polytopes are still modest in size: a simple inductive argument using the connectedness result from Proposition 1 shows that they are countable, that is, the cells of each of them together have countably many facet locations.

The result of our comparison will be as follows: *present-type polytopes are precisely the finite polytopes; standard polytopes are precisely the monic polytopes.* Here a polytope is called *monic* if for any two of its cells one is a face of the other at most once. While the first statement is obviously valid, the second needs an explanation. We start with a definition of standard polytopes, as given with more explicitness in [6].

A *standard polytope* $\mathscr{P}$ *of dimension* n is a (partially) ordered set satisfying conditions (i)–(iv) below.

(i) $\mathscr{P}$ has a smallest element $\perp$ and a largest element $\top$.

(ii) All maximal chains in $\mathscr{P}$ have length $n+1$.

Here the *length* of a chain is the number of its elements minus one. For $B < A$ in $\mathscr{P}$, we consider the segments

$$[B, A] = \{\, P \mid B \leq P \leq A \,\} \qquad \text{and} \qquad]B, A[= \{\, P \mid B < P < A \,\}.$$

as ordered subsets of $\mathscr{P}$. Condition (ii) implies:

(ii)* All maximal chains in $[B, A]$ have the same, finite length.

We denote this length by $d_{B,A}$.

(iii) If $d_{B,A} > 2$, then $]B, A[$ is connected.

(iv) If $d_{B,A} = 2$, then $]B, A[$ contains precisely two elements.

It turns out that polytopes in general can be given the same definition, with only a few words and symbols changed. To achieve this end, we define a *multiordered set* to be a category in which all isomorphisms and all endomorphisms are identities, that is, a category generated by a cycle-free graph. The terms 'element', 'smallest', 'largest' will stand for 'object', 'initial', 'terminal'. A *chain* is a subcategory which is a totally ordered set. For a non-identity morphism $F : B \longrightarrow A$, we denote by $[F]$ the category of factorizations and by $]F[$ the category of proper factorizations ('proper' excluding the two identities as factors) of F.

Now a polytope is (after interpretation as a category and up to isomorphism) exactly a multiordered set $\mathscr{P}$ satisfying conditions (i)–(iv) above, with the symbol string 'B, A' replaced by 'F'. Let us see why this is true, without going through the details of an actual proof. First, let us call the number d_F the *codimension* of F. The two parts of condition (i) evidently correspond, respectively, to the definitions of polytopic sets and propolytopes as particular propolytopic sets. The remaining conditions, with (ii)* replacing (ii) to account for the potential absence of $\perp$ or $\top$, characterize propolytopic sets. More precisely, they imply that $\mathscr{P}$ has a presentation in which the generators are the 1-codimensional morphisms (condition (ii)), the relators are pairs of length-2 paths of generators (length ≥ 2: condition (ii); length ≤ 2: condition (iii)), and each length-2 path of generators occurs in precisely one relator (condition (iv)).

We have considered three different notions of 'polytope'. The differences can be well illustrated in dimension 2. Here a polytope is a *polygon*, more precisely the *ℓ-gon*, where ℓ is the number of its facets. The possible values for ℓ are the integers ≥ 1 and ∞. The ∞-gon (= *apeirogon*) is not a present-type polytope, for it is not finite. The 1-gon (= *monogon*) is not a standard polytope, for it is (ironically) not monic: its only vertex is twice a facet of its only edge.

Section 4. Orientation

An important feature of ∞-categorical diagrams has not yet been accounted for: the arrows. An arrow distinguishes the boundary of a cell into a source and a target. For the present — and most other — purposes it is enough to know how the facets are distributed over the two parts. This information is in each case encoded by a sign: '$-$' for 'source' and '$+$' for 'target'. Now for any given ridge, only half of the possible combinations of signs about it can actually occur. Either the ridge is properly contained in either the source or the target; then the two facets have opposite (matching) directions relative to it.

Or the ridge is part of the common boundary of the overall source and the overall target; then the two facets have the same direction relative to it.

There finally is a third, degenerate case, which agrees in signs with the first. Thus, of the four signs involved, an odd number is $-$ and an odd number is $+$.

Definition. An *orientation* on a propolytopic set provides the data (O1) below and satisfies condition (O2) below.

(O1) for each facet location q, a sign $\{\!\!\{q\}\!\!\}$. The set of facet locations of sign η of a cell P is denoted by $P\mathrm{I}_\eta$.

To a chamber $\langle p^1, \ldots, p^d\rangle$, we associate the sign

$$\{\!\!\{p^1, \ldots, p^d\}\!\!\} = \{\!\!\{p^1\}\!\!\} \cdots \{\!\!\{p^d\}\!\!\}.$$

Put differently: a chamber is positive or negative according to whether an even or an odd number of its entries is negative. (The adjectives 'positive' and 'negative' apply in the obvious way.) The set of all d-chambers of sign η underneath a cell P is denoted by $P\mathrm{I}^d_\eta$.

(O2) *Sign-flip axiom.* For each ridge location $\{\langle q_0, r_0\rangle, \langle q_1, r_1\rangle\}$, we have

$$\{\!\!\{q_0, r_0\}\!\!\} \neq \{\!\!\{q_1, r_1\}\!\!\}.$$

Hence we may view $P\Diamond$ as a one-to-one correspondence between $P\mathrm{I}^2_+$ and $P\mathrm{I}^2_-$.

If an orientation is specified, the line notation for ridge locations serves the additional purpose of keeping track of signs. To this end we decree: the negative chamber always goes above the line; the positive chamber always goes below the line. Thus if we write $\frac{q_1\ r_1}{q_0\ r_0}$, we imply that $\{\!\!|q_1, r_1|\!\!\} = -$ and $\{\!\!|q_0, r_0|\!\!\} = +$. For the sake of flexibility, the two chambers may be permuted, as long as the sign of the permutation is indicated. Thus if we write $\eta\frac{q_1\ r_1}{q_0\ r_0}$, we imply that $\{\!\!|q_1, r_1|\!\!\} = -\eta$ and $\{\!\!|q_0, r_0|\!\!\} = +\eta$.

Examples. The classical polytopic sets can be oriented. In fact, each of them carries a particular orientation suggested by the indices denoting facet locations:

- in a globular set, $\{\!\!|\delta_\eta|\!\!\} = \eta$.
- in a simplicial set, $\{\!\!|\delta_i|\!\!\} = -^i$.
- in a cubical set, $\{\!\!|\delta_{i,\eta}|\!\!\} = -^i\eta$.

The displays of Section 2 showing the corresponding ridge locations now have to be accompanied by a sign each, so as to read $\zeta\frac{\delta_-\ \delta_\zeta}{\delta_+\ \delta_\zeta}$ in the globular case, $-^{i+j}\frac{\delta_j\ \delta_{i-1}}{\delta_i\ \delta_j}$ in the simplicial case and $-^{i+j}\eta\zeta\frac{\delta_{j,\zeta}\ \delta_{i-1,\eta}}{\delta_{i,\eta}\ \delta_{j,\zeta}}$ in the cubical case.

Let $\mathfrak{P}$ be an oriented propolytopic set (that is, a propolytopic set together with an orientation). We take the chamber graphs $X\Gamma^d$ of $\mathfrak{P}$ to be *directed* as follows. Note that the two chambers separated by a panel carry opposite signs. (Hence chamber graphs are *bipartite*, as a combinatorist would put it.) A type-τ panel

$$\langle p^1, \ldots, p^{\tau-1}, \frac{p^\tau_-\ p^{\tau+1}_-}{p^\tau_+\ p^{\tau+1}_+}, p^{\tau+2}, \ldots, p^d\rangle \tag{2}$$

points towards that one of its adjacent chambers that agrees in sign with its initial $(\tau-1)$-subchamber $\langle p^1, \ldots, p^{\tau-1}\rangle$. We then use the term 'gallery' only to refer to directed paths (that is, paths in the directed chamber graphs). Thus we may no more say that each two chambers representing the same face location are connected by a gallery; but of course they remain connected by a zigzag in the chamber graph.

Let us take a closer look at the case $d = 3$. The components of 3-chamber graphs can no longer be described as cycles; rather, they are closed zigzags. The directions along these zigzags, however, are not distributed entirely arbitrarily. Type-1 panels point from negative to positive chambers (this is true for all depths) and so determine a preferred direction along the zigzag. A type-2 panel points in the preferred direction if and only if its first entry is negative. Thus, a 3-face location whose chambers have only negative first entries remains a cycle; a 3-face location whose chambers have only positive first entries becomes a zigzag with alternating directions.

The rules spelled out above for the line notation of ridge incidences extend accordingly to the matrix notation of galleries. Thus, galleries usually run downwards; a change of direction must be indicated by its sign. The generic panel (2), viewed as a gallery of length 1, can be written

$$\begin{array}{cccccccc} p^1 & \cdots & p^{\tau-1} & p^\tau_{-\eta} & p^{\tau+1}_{-\eta} & p^{\tau+2} & \cdots & p^d \\ \hline p^1 & \cdots & p^{\tau-1} & p^\tau_{+\eta} & p^{\tau+1}_{+\eta} & p^{\tau+2} & \cdots & p^d \end{array}$$

or

$$\eta\left\{\begin{array}{cccccccc} p^1 & \cdots & p^{\tau-1} & p^\tau_- & p^{\tau+1}_- & p^{\tau+2} & \cdots & p^d \\ \hline p^1 & \cdots & p^{\tau-1} & p^\tau_+ & p^{\tau+1}_+ & p^{\tau+2} & \cdots & p^d \end{array}\right.,$$

where η is the sign of the final $(d-\tau-1)$-subchamber $\langle p^{\tau+2}, \ldots, p^d\rangle$.

The choice of direction may be a bit surprising. An explanation is given in [7], where the present author shows that in the dendrotopic case, the chosen directions induce a total order on the chambers representing a common face location. (Thus two such chambers are still connected by a path — rather than by a mere zigzag —, but only in one of the two possible ways.) A favourable argument is the fact that the canonical embedding $(\langle p^1, \ldots, p^u \rangle + \,?\,)$ remains a homomorphism of graphs (that is, preserves the chosen directions), independent of the sign of $\langle p^1, \ldots, p^u \rangle$. (Of course this argument may be countered by the observation that the dual statement is wrong: if we consider graphs of chambers and panels with common floors, then the analogous embedding $(\,?\, + \langle p'^1, \ldots, p'^v \rangle)$ preserves or reverses direction according to whether $\langle p'^1, \ldots, p'^v \rangle$ is positive or negative.)

Use of the term 'orientation' seems sufficiently justified by the following fact. For a spherical polytopic set $\mathfrak{P}$, providing an orientation on $\mathfrak{P}$ is equivalent to providing an orientation on the geometric realization of each proper cell of $\mathfrak{P}$. The sphericality assumption is actually redundant, if one agrees that for the purpose of orienting an arbitrary space, points where the manifold axiom of local euclidianness is violated may be cut off.

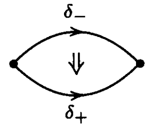
The above statement refers to a specific one-to-one correspondence. The forward mapping can be described inductively as follows. Suppose $\mathfrak{P}$ is oriented. For a cell P of dimension 0, the orientation on the point $P\Gamma$ is the sign of the only facet location of P. For a cell P of dimension > 0, the orientation on the manifold $P\Gamma$ is determined by the orientation induced on its boundary $P\Gamma^\bullet$, which in turn is determined by demanding that for each facet location q of P, the canonical mapping $Pq\Gamma \twoheadrightarrow P\Gamma^\bullet$ preserve or reverse orientation according to whether q is positive or negative. The sign-flip axiom ensures that the orientations thus described on the facets match up on the ridges.

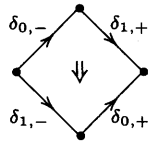
The idea that an individual cell carries one of two possible orientations leads to a construction that applies to oriented propolytopic sets in general. Let Δ be a bipartition of the set of all cells in an oriented propolytopic set $\mathfrak{P}$. We can modify $\mathfrak{P}$ to obtain an oriented propolytopic set $\mathfrak{P}_\Delta$ by reversing the signs of those facet locations $q \in P\mathrm{I}$ for which P and Pq belong to different parts of Δ. For instance, if one part of Δ comprises all cells of dimension $\geq n$, then the sign reversal occurs precisely for the facet locations of n-cells (duality!). When applying this construction to a polytopic set, the part of Δ containing $\perp$ is the one where cell orientations remain unchanged.

It may counter intuition, or be unwanted for concrete reasons, that the points of an (although) oriented polytopic set carry orientations. In this case we may demand axiomatically that $\perp$ not occur as a negative facet. Put affirmatively, this means that the only facet location ε of each 0-cell is positive. The sign-flip axiom then yields that the two facet locations of each 1-cell have opposite signs. Any oriented polytopic set $\mathfrak{P}$ has a variant with "disoriented" points, namely $\mathfrak{P}_\Delta$, where one part of Δ comprises the 0-cells for which ε is negative.

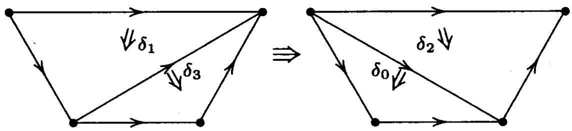
An *oriented* propolytopic set $\mathfrak{P}$ will be called *augmentable* if each of its 1-cells has precisely one facet of each sign. In this situation the 0-cells and 1-cells of $\mathfrak{P}$ are in an obvious manner the vertices and arrows of a directed graph. For any such $\mathfrak{P}$, the process of trivial augmentation works as it did in the absence of orientations; the affixed facet locations ε are taken to be positive.

A polytope P can be represented by drawing the realizations of all its facets Pf as they appear under the mappings $f\Gamma$. An orientation can — in the cases of interest for us — be represented by adding arrows, one for each $>$ 0-dimensional face, pointing from the totality of negative facets to the totality of positive facets and having as many "tails" as the dimension indicates. Here are a picture of a 2-globe and a picture of a 2-cube.





If the dimension of the polytope exceeds the dimension of the paper, source and target of its arrow may be drawn separately in a "double Schlegel diagram". Here is a picture of a 3-simplex.



Similar pictures occur in the theory of higher-dimensional categories, where they may be viewed as freely generating subsets of strict ∞-categories (see Sections 1 and 3 in Part II). Let us for this and the following two paragraphs call such a gadget an *(∞-categorical) diagram scheme.* (The artificial term 'computad' is the standard.) The following two questions naturally arise.

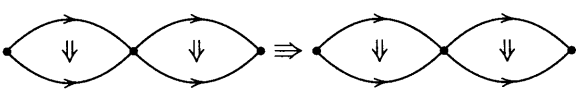
(i) Which oriented polytopic sets can be viewed as ∞-categorical diagram schemes?

(ii) Which ∞-categorical diagram schemes can be viewed as oriented polytopic sets?

A partial answer to both these questions is given by Theorem 10 of Part II. This work is far from pursuing general answers, but a few remarks may be of interest to the reader.

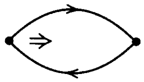
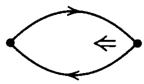
As for (i), it is not difficult to come up with certain geometric conditions satisfied by the classical polytopic sets and apparently sufficient in order for a polytopic set to be a diagram scheme. The envisioned result has already been obtained in two instances. In [2], Batanin explicitly constructs the cells of the free ∞-category associated to a globular set as trees of the kind mentioned in Section 1, labelled in a certain way by the generators. In [8], R. Street gives an implicit description of the free ∞-category associated to a simplicial set, and more generally of the ∞-category associated to what may be described as a presentation by simplices. The question has been fully answered under assumptions of global cycle freeness, reducing the ∞-categorical operations to plain set-theoretic unions. The best-known work of this kind is Street's [9] on his *parity complexes.*

As for (ii), two shortcomings of the notion of a polytopic set ought to be stressed. First, the "gluing" step in the construction of a polytope is restricted to codimension 2. Thus, for example, in an attempt to interpret the ∞-categorical diagram



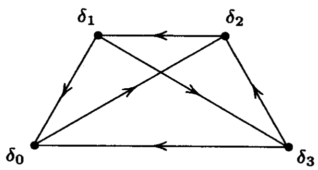
as a 3-(pro-)polytope, the middle one of the three 0-dimensional faces would break into two. (Parity complexes, by contrast, are subject to no such restriction.) Second, the "face pick" step in the construction of a polytope is restricted to codimension 1. Thus, for example, the ∞-categorical diagram

showing a 2-cell looping about a degenerate 1-cell, cannot be interpreted as a 2-(pro-)polytope at all. Apart from that, an oriented polytope lacking facets of one of the two signs may be ∞-categorically underdetermined: compare the two diagrams

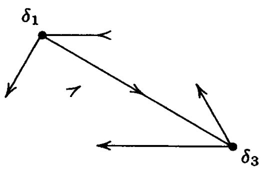
 and .

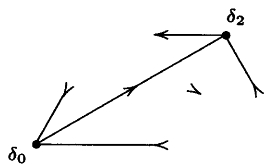
This last problem will not arise for dendrotopic sets: here a >0-dimensional cell lacking negative facets will have precisely one ridge.

As have been the chamber graphs, so will be the dual graphs of a propolytopic set in presence of an orientation: *directed*. We let a ridge location $\frac{q_- \; r_-}{q_+ \; r_+}$ point from the facet location q_- to the facet location q_+. The following diagram shows the directed dual graph of a 3-simplex.



We shall be not so much interested in the dual graph itself than in certain two substructures (in fact, only in one of the two). We define the *sign-η dual hemigraph* PD_η of P as follows. Vertices are the facet locations of P of sign η; arrows are the ridge locations of P. A source or a target of an arrow is its source or its target in the dual graph if possible, and otherwise does not exist. In brief, PD_η is obtained from PD by removing the vertices of sign $-\eta$. Note that in general this operation leaves isolated arrows behind. (It does so for a purpose.) The following two diagrams show the negative and the positive dual hemigraphs of a 3-simplex.





Section 5. Dendrotopic Sets

The time has come to state the main definition of this work.

Definition. Let $\mathfrak{P}$ be an oriented propolytopic set with precisely one improper cell, $\perp$. Let $\perp$ have dimension -1 and not occur as a negative facet. Then $\mathfrak{P}$ is a *dendrotopic set* provided conditions $(\Delta 1)$, $(\Delta 2)$ and $(\Delta 3)_!$ below are satisfied.

($\Delta 1$) Each cell of dimension ≥ 0 has precisely one positive facet. The associated location will be denoted by ω.

It follows that in the negative dual graph of a cell P of dimension ≥ 1, each vertex q has precisely one outgoing arrow, namely $q\omega$, and there is precisely one arrow without target, namely $\omega\omega$. Thus PD_- is locally treelike with root $\omega\omega$.

($\Delta 2$) *Tree axiom.* The negative dual hemigraph of each cell of dimension ≥ 1 is a tree with root $\omega\omega$.

By the preceding remark, this condition can be expressed equivalently (assuming ($\Delta 1$)) as either connectedness or cycle freeness of the (finite) hemigraphs in question.

$(\Delta 3)_!$ *Normalization, uniqueness part.* Each 3-face location is represented by at most one panel of the form $\langle \omega, ** \rangle$.

We call *normal* such a panel, as well as the two chambers it separates. (Hence the name 'normalization' for the axiom.) The latter can be referred to individually as the positive normal chamber and the negative normal chamber.

The qualifier 'uniqueness part' suggests that we shall consider an existence part as well. In fact it occurs in the following proposition, which will be proven at the end of this section.

Proposition 2. *Any dendrotopic set satisfies condition* $(\Delta 3)_¡$ *below.*

$(\Delta 3)_¡$ *Normalization, existence part.* Each 3-face location is represented by at least one panel of the form $\langle \omega, ** \rangle$.

We may more generally define a *normal panel* to be one of the form

$$\langle \omega, \ldots, \omega, ** \rangle.$$

This is done in [7], where the corresponding existence and uniqueness statements are proven for all depths/codimensions ≥ 2.

We know that a 3-face location (as a graph) is a closed zigzag with a preferred direction, given by its type-1 representatives. The normal panel, say $\langle \omega, \frac{r_- \; s_-}{r_+ \; s_+} \rangle$, is the only one that opposes the preferred direction. If we remove it, we are left with a gallery from $\langle \omega, r_-, s_- \rangle$ to $\langle \omega, r_+, s_+ \rangle$, which we call the *Hamilton gallery* of the location. We can display the entire location as its Hamilton gallery, with the normal panel indicated by two dotted lines:

$$\begin{array}{ccc} \omega & r_- & s_- \\ \hline * & * & * \\ \vdots & \vdots & \vdots \\ * & * & * \\ \hline \omega & r_+ & s_+ \end{array} .$$

Let us take a closer look at low dimensions.

0. Let T be a 0-cell. Since $\perp$ does not occur as a negative facet, ω is the only facet location of T. We denote it also by ε. Here is a picture of T.

1. Let S be a 1-cell. All 2-chambers underneath S are of the form $\langle *, \varepsilon \rangle$. Hence the negative facet locations of S are precisely the solutions of $S : \frac{* \; \varepsilon}{\omega \; \varepsilon}$, of which there is of course precisely one. We denote the sign-η facet location of

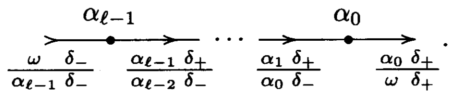
S by δ_η (so that $\delta_+ = \omega$). The negative dual hemigraph of S automatically takes the form

$$\frac{\delta_- \ \varepsilon}{\delta_+ \ \varepsilon} \Big\downarrow^{\bullet\,\delta_-} \quad .$$

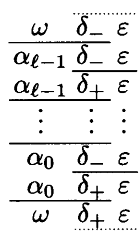
Thus the tree axiom is redundant for dimension 1. Here is a picture of S.

$$\delta_- \bullet\!\longrightarrow\!\bullet\, \delta_+$$

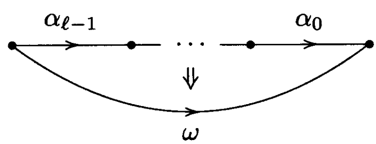
2. Let R be a 2-cell. Each vertex s of its negative dual hemigraph has precisely one incoming arrow, namely $s\delta_-$. Therefore $R\mathrm{D}_-$, being a tree, consists of a single branch



(The naming of the vertices is explained further below in this section.) The 3-chamber graph underneath R automatically takes the form



Thus the normalization axiom is redundant for dimension 2. Here is a picture of R.



3. Let Q be a 3-cell. The incoming arrows of each vertex r of $Q\mathrm{D}_-$ can be listed $r\alpha_0, \ldots, r\alpha_{\ell_r-1}$; they thus carry a total order. These orders make $Q\mathrm{D}_-$ into a *planar tree* and thus induce a total order on the $\ell_\omega = 1+\sum_r(\ell_r - 1)$ leafs. The leafs in turn can be listed $\omega\alpha_0, \ldots, \omega\alpha_{\ell_\omega-1}$; they thus again carry a total order. The two orders agree, as one can show by using the normalization property.

Examples. Every globular set becomes a dendrotopic set by trivial augmentation.

Let us relate the above conditions to some of the notions discussed previously. In the following statements we assume that $\mathfrak{P}$ satisfies the hypothesis of the Definition along with condition (Δ1).

- If $\mathfrak{P}$ has property (Δ2), then $\mathfrak{P}$ is a polytopic set. This can easily be checked in view of Proposition 1.
- If $\mathfrak{P}$ has properties (Δ2) and $(\Delta 3)_!$, then $\mathfrak{P}$ is spherical. This should intuitively be clear; the major ingredients of the proof can be found in that of Theorem 3. The theorem itself will tell us that under assumption of $(\Delta 3)_!$, conditions (Δ2) and $(\Delta 3)_¡$ are equivalent.

The converses of these two statements are false. In fact, there are oriented spherical 3-polytopes satisfying ($\Delta 1$), but none of the other conditions. For instance, consider the oriented 3-propolytopes with one facet location q_η of each sign η both of whose residents are dendrotopes with two negative facets. There are (up to isomorphism) six of them, all of which are polytopes. The two with a ridge location $\frac{q_-\ \omega}{q_+\ \omega}$ are not of interest here. (One is a dendrotope, the other satisfies ($\Delta 2$) and $(\Delta 3)_{\mathrm{i}}$ and has a realization whose boundary is a torus.) The remaining four meet the description.

Let X be an $(n+1)$-cell, and let p' be a negative facet location of X. We are going to assign to p' an n-dimensional number $|p'|$, which we shall call its *index*. The definition is by a double induction, first on the dimension of X, then on the height of p' in the tree $X\mathrm{D}_-$. If $\frac{p'\ \omega}{\omega\ \omega}$, we put $|p'| = 0$. If $\frac{p'\ \omega}{p\ \ q}$ with $\langle p, q\rangle \neq \langle \omega, \omega\rangle$, then p is below p' in $X\mathrm{D}_-$ and we put $|p'| = |p| + \langle |q| \rangle$. We may think of $|p'|$ as encoding the route along which to climb the tree $X\mathrm{D}_-$ in order to reach the vertex p'.

Before we proceed, we introduce an order $\leq^1$ on the set of higher-dimensional numbers. We decree that $\mathfrak{p}_0 \leq^1 \mathfrak{p}_1$ if and only if $\mathfrak{p}_0$ is an initial segment of $\mathfrak{p}_1$, that is, $\mathfrak{p}_1 = \mathfrak{p}_0 + \mathfrak{p}'$. Note that for two facet locations p_0 and p_1 of X, we have $|p_0| \leq^1 |p_1|$ if and only if there is a path from p_1 to p_0 in the negative dual hemigraph of X. We have thus represented the order induced by the tree $X\mathrm{D}_-$ on its vertex set $X\mathrm{I}_-$. We can conclude in particular that different negative facet locations of X have different indices. We may hence write, for example, $p = \alpha_{|p|}$. In this way, the negative facet locations at a given cell bear *a posteori* names $\alpha_{\mathfrak{p}}$, with the indices $\mathfrak{p}$ forming a downward-closed set of higher-dimensional numbers. In this regard dendrotopic sets become roughly similar to simplicial sets, for which the facet locations of a given cell bear *a priori* names δ_i, with the indices i forming a downward-closed set of natural numbers.

For an arbitrary positive 2-chamber $\langle p, q\rangle$ underneath the $(n+1)$-cell X, let us put

$$|p, q| = \begin{cases} 0 & \text{if } \langle p, q\rangle = \langle \omega, \omega\rangle, \\ |p| + \langle |q| \rangle & \text{otherwise.} \end{cases}$$

We call this n-dimensional number the *index* of $\langle p, q\rangle$. Expressed with the new notation, the definition of the index of $p' \in X\mathrm{I}_-$ becomes

$$|p'| = |p, q| \quad \text{if} \quad \frac{p'\ \omega}{p\ \ q}.$$

Thus, each vertex of $X\mathrm{D}_-$ carries the same index as the positive representative of its outgoing arrow. By contrast, the positive representatives of the leafs have indices that do not occur elsewhere in $X\mathrm{D}_-$.

We are almost set to prove Proposition 2. To do so, we need another order, denoted by $\leq^2$, of higher-dimensional numbers, defined to be lexicographical with respect to $\leq^1$. More explicitly, we decree that $\mathfrak{p}_0 \leq^2 \mathfrak{p}_1$ if and only if either $\mathfrak{p}_0$ is an initial segment of $\mathfrak{p}_1$ or $\mathfrak{p}_i = \mathfrak{p} + \langle \mathfrak{q}_i \rangle + \mathfrak{p}'_i$ with $\mathfrak{q}_0 <^1 \mathfrak{q}_1$. (Usually lexicographical orders are constructed with respect to total orders, in which case they are total themselves. The reader may want to convince himself of the redundancy of the totality assumption.)

Proof of Proposition 2. Let us define the *index* of a non-normal negative 3-chamber $\langle p,q,r\rangle$ to be

$$|p,q,r| = |p| + \langle |q,r| \rangle.$$

(Note that since p is negative, $\langle q,r\rangle$ is positive.) We show that for two such chambers successive in $X\Gamma^3$ (that is, with a length-2 gallery from the first to the second), the index of the first is strictly bigger than the index of the second with respect to $\leq^2$. Put more formulaicly, the assertion states that if

$$\begin{array}{ccc} p' & q'' & r' \\ \hline p & q' & r' \\ \hline p & q & r \end{array} \qquad \text{with } p,\, p' \neq \omega,$$

then $|p',q'',r'| >^2 |p,q,r|$. Consider the sign of r', which determines the signs of q' and q''. If $r' \neq \omega$, then $q' = \omega$ and $q'' \neq \omega$, and so

$$|p',q'',r'| = |p'| + \langle |q''| + \langle |r'| \rangle\rangle >^2 |p'| + \langle |q''|, |q,r| \rangle = |p| + \langle |q,r| \rangle = |p,q,r|.$$

If $r' = \omega$, then $q' \neq \omega$ and $q'' = \omega$, and so

$$|p',q'',r'| = |p'| + \langle 0 \rangle >^2 |p'| = |p| + \langle |q'| \rangle = |p| + \langle |q,r| \rangle = |p,q,r|.$$

If there were a 3-face location without a normal chamber, it would be a (directed) cycle. But along this cycle, the indices of negative vertices would strictly decrease, and this is impossible. □

Section 6. Normalization

The existence part of the normalization condition, proven to be satisfied in all dendrotopic sets, can in fact replace the tree condition as an axiom. The present section is devoted to proving this fact.

Theorem 3. *Let $\mathfrak{P}$ be an oriented propolytopic set with precisely one improper cell, $\perp$. Let $\perp$ have dimension -1 and not occur as a negative facet. Then $\mathfrak{P}$ is a dendrotopic set if and only if it satisfies conditions* (Δ1) *above and* (Δ3) *below.*

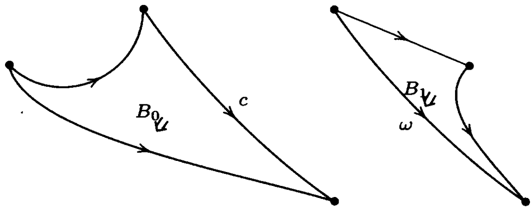
(Δ3) *Normalization.* Each 3-face location is represented by exactly one normal panel.

Proof. Let us temporarily call a propolytopic set satisfying the conditions *normalizing*. Thus we have to show that in a normalizing propolytopic set $\mathfrak{N}$, the tree condition (Δ2) is satisfied. To this end, it suffices to show that the negative dual hemigraph of each cell A of dimension ≥ 1 is cycle-free. We do so by induction on the dimension n of A.

By a previous observation, the statement is automatically true for $n = 1$. So let us consider the case $n \geq 2$. We proceed by another induction, this time on the number ℓ of negative facets of A.

If $\ell = 0$, then AD_- consists of the arrow $\omega\omega$ only and is hence clearly cycle-free. In order to handle the general case, we reduce the problem by means of the following two constructions. (The diagrams will illustrate an example with $n = 3$.)

Claim A. *Let B_0 and B_1 be two $(n-1)$-cells of $\mathfrak{N}$, and let c be a negative facet location of B_0 with resident C equal to the positive facet of B_1.*



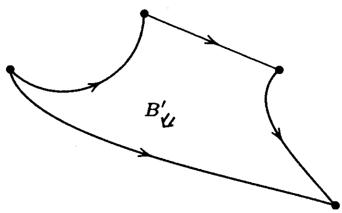
We obtain a new normalizing propolytopic set $\mathfrak{N}'$ by adding to $\mathfrak{N}$ an $(n-1)$-cell B' with incidence structure prescribed as follows. We let the facets of B' be those of B_0 except the one with location c and those of B_1 except the one with location ω. (Without loss of generality disjointness of $B_0\mathrm{I}$ and $B_1\mathrm{I}$ is assumed. The signs are kept, so that the ω of B' is the ω of B_0.) The ridge locations of B' are defined so that (with the evident subscript convention)

$$B' : \frac{r_0^-\ s_0^-}{r_0^+\ s_0^+} \quad \textit{if and only if}$$

$$\textit{either}\quad B_0 : \frac{r_0^-\ s_0^-}{r_0^+\ s_0^+} \quad \textit{or} \quad \left\{\begin{array}{l} B_0 : \dfrac{r_0^-\ s_0^-}{c\quad s} \\[2mm] B_1 : \dfrac{\omega\quad s}{\omega\quad \omega} \\[2mm] B_0 : \dfrac{c\quad \omega}{r_0^+\ s_0^+} \end{array}\right\} \quad \textit{for some } s \in C\mathrm{I}_-;$$

$$B' : \{\!\{s_1\}\!\}\frac{r_1\ s_1}{r_0\ s_0} \quad \textit{if and only if} \quad \left\{\begin{array}{l} B_1 : \{\!\{s_1\}\!\}\dfrac{r_1\ s_1}{\omega\quad s} \\[2mm] B_0 : \{\!\{s_1\}\!\}\dfrac{c\quad s}{r_0\ s_0} \end{array}\right\} \quad \textit{for some } s \in C\mathrm{I}_{\{\!\{s_1\}\!\}};$$

$$B' : \frac{r_1^-\ \ \omega}{r_1^+\ s_1^+} \quad \textit{if and only if} \quad B_1 : \frac{r_1^-\ \ \omega}{r_1^+\ s_1^+}.$$



Proof. We can describe $B'\Diamond$ more vividly as follows. Think of a ridge location of B_0 or B_1 as a domino, with its two representative 2-chambers being the faces. The face $\langle c, s\rangle$ of a B_0-domino is taken to match the face $\langle \omega, s\rangle$ of a B_1-domino. Now form the longest possible rows of dominoes. We can think of each row as a ridge location of B', with its two unmatched faces being the representative 2-chambers. Indeed all possible rows are mentioned in the statement of the claim, as a consequence of the outer induction hypothesis: a ridge location $B_0 : \frac{c\ \omega}{c\ s}$ would be a loop in $B_0\mathrm{D}_-$. This alternative description conveniently shows that we have indeed a one-to-one correspondence between $B'\mathrm{I}^2_-$ and $B'\mathrm{I}^2_+$.

The only part of the claim that remains difficult to check is that the normalization condition is satisfied underneath B'. We do so by noting that $B'\Gamma^3$ can be obtained from $B_0\Gamma^3$ by replacing every occurrence of a type-2 panel with first entry c, the separated chambers included,

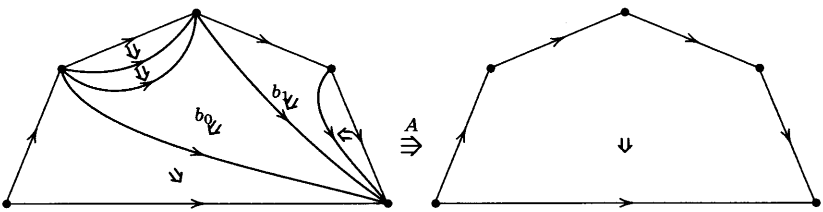
$$\begin{array}{ccc} * & * & * \\ \hline \left.\begin{array}{ccc} c & s_- & t_- \\ \hline c & s_+ & t_+ \end{array}\right\} \text{(this part)} \\ \hline * & * & * \end{array}$$

by the non-normal segment of the corresponding 3-face location of B_1.

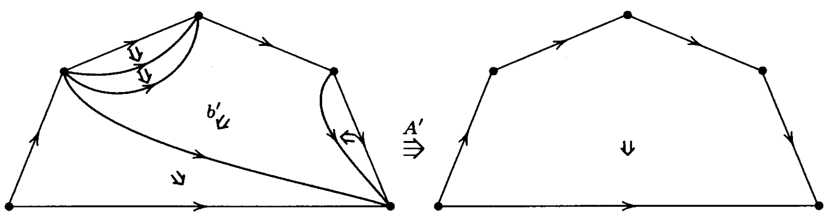
$$\begin{array}{ccc} \omega & s_- & t_- \\ \hline * & * & * \\ \vdots & \vdots & \vdots \\ * & * & * \\ \hline \omega & s_+ & t_+ \end{array} \Bigg\} \text{(this part)} \qquad \square$$

Claim B. *Let b_0 and b_1 be two different negative facet locations of A, and let c be a negative facet location of Ab_0 with*

$$A : \frac{b_1 \; \omega}{b_0 \; c}. \tag{3}$$



We obtain a new normalizing propolytopic set $\mathfrak{N}''$ by adding two cells to $\mathfrak{N}$: first B' of dimension $n-1$ as in Claim A for $B_i = Ab_i$; then A' of dimension n with incidence structure prescribed as follows. We let the facets of A' be those of A, minus the two with locations b_0 and b_1, plus B' with negative location b'. The ridge locations of A' are defined to be the same as those of A, with the understanding that b' replaces both b_0 and b_1 and that (3) is absent.



Proof. Again the only difficult part is to check the normalization condition. We do so by noting that $A'\Gamma^3$ can be obtained from $A\Gamma^3$ by replacing segments

$$\begin{array}{ccc} \hline b_0 & r_0^- & s_0^- \\ \hline b_0 & r_0^+ & s_0^+ \\ \hline \end{array} \quad \text{or} \quad \begin{array}{ccc} \hline b_0 & r_0^- & s_0^- \\ \hline b_0 & c & s \\ \hline b_1 & \omega & s \\ \hline b_1 & \omega & \omega \\ \hline b_0 & c & \omega \\ \hline b_0 & r_0^+ & s_0^+ \\ \hline \end{array} \quad \text{by} \quad \begin{array}{ccc} \hline b' & r_0^- & s_0^- \\ \hline b' & r_0^+ & s_0^+ \\ \hline \end{array},$$

$$\{\!\!\{s\}\!\!\}\begin{cases}\overline{b_1\ \underline{r_1\ s_1}}\\ \underline{b_1\ \omega\ \ s}\\ \overline{b_0\ \ c\ \ s}\\ \underline{b_0\ r_0}\ s_0\end{cases}\quad\text{by}\quad\{\!\!\{s\}\!\!\}\begin{cases}\overline{b'\ \underline{r_1\ s_1}}\\ \underline{b'\ r_0}\ s_0\end{cases},$$

$$\begin{array}{c}\overline{b_1\ r_1^-\ \underline{s_1^-}}\\ \underline{b_1\ r_1^+}\ s_1^+\end{array}\quad\text{by}\quad\begin{array}{c}\overline{b'\ r_1^-\ \underline{s_1^-}}\\ \underline{b'\ r_1^+}\ s_1^+\end{array}. \qquad \square$$

Back to the theorem. In the inner induction step, we analyse the situation of Claim B in a way similar to its proof, but with regard to negative dual hemigraphs rather than depth-3 chamber graphs, and in the opposite direction. Our findings will be the following. We can obtain AD_- from $A'D_-$ by "stretching out" the vertex b' to become the arrow (3) with extremities b_0 and b_1. The outgoing arrow of b' becomes the outgoing arrow of b_0, while an incoming arrow of b' becomes an incoming arrow of either b_0 or b_1. It is clear that no essentially new cycle is created by this modification.

The crucial part of the proof, however, is to make sure that Claim B can actually be applied when needed. This is where the following result comes in. In order to state it conveniently, let us agree to write

$$A : \frac{q\ \ \omega}{\beta_q\ \gamma_q} \qquad (q \in A\mathrm{I}_-).$$

Claim C. *If there is $b \in A\mathrm{I}_-$ such that $\langle \beta_b, \gamma_b\rangle \neq \langle \omega, \omega\rangle$, then there is $b_1 \in A\mathrm{I}_-$ such that $\langle \beta_{b_1}, \gamma_{b_1}\rangle \neq \langle \omega, \omega\rangle$ and $\beta_{b_1} \neq b_1$.*

Proof. Let us assume the contrary. Pick b as indicated, whence $\beta_b = b$. By the outer induction hypothesis, AbD_- is a tree with root $\omega\omega$, whence there is a path

$$\gamma_b = c_h \xrightarrow[\frac{c_h\ \ \omega}{c_{h-1}\ s_{h-1}}]{} c_{h-1} \longrightarrow \cdots \xrightarrow[\frac{c_1\ \omega}{c_0\ s_0}]{} c_0 \xrightarrow[\frac{c_0\ \omega}{\omega\ \omega}]{} .$$

We have $h > 0$, for otherwise there would be a cycle

$$\begin{array}{c}\overline{b\ \omega\ \omega}\\ \underline{b\ c_0\ \omega}\end{array}$$

underneath A, contradicting the existence part of the normalization property. For each $i < h$, the positive 2-chamber $\langle b, c_i\rangle$ is not the conjugate of any $\langle q, \omega\rangle$ with $\{\!\!\{q\}\!\!\} = -$, since otherwise by assumption $q = \beta_q = b$, but $\gamma_q = c_i \neq c_h = \gamma_b$. We can hence select $r_i \in A\omega\mathrm{I}_-$ by demanding that $A : \frac{\omega\ r_i}{b\ c_i}$.

If $i < h - 1$, there is a gallery

$$\begin{array}{c}\underline{\omega\ r_{i+1}}\ \omega\\ \underline{b\ \overline{c_{i+1}\ \omega}}\\ \underline{b\ \ c_i}\ \ s_i\\ \omega\ \ r_i\ \ s_i\end{array}$$

underneath A. By the uniqueness part of the normalization property, $A\omega : \frac{r_{i+1}\ \omega}{r_i\ \ s_i}$. There also is a gallery

$$\begin{array}{lll}
\omega & r_0 & \omega \\ \hline
b & c_0 & \omega \\ \hline
b & \omega & \omega \\ \hline
b & c_h & \omega \\ \hline
b & c_{h-1} & s_{h-1} \\ \hline
\omega & r_{h-1} & s_{h-1}
\end{array}$$

underneath A. For the same reason as before, $A\omega : \frac{r_0 \quad \omega}{r_{h-1}\ s_{h-1}}$. We have thus constructed a cycle in $A\omega\mathrm{D}_-$,

$$\cdots \xrightarrow[\frac{r_1\ \omega}{r_0\ s_0}]{} \underset{r_0}{\bullet} \xrightarrow[\frac{r_0\quad \omega}{r_{h-1}\ s_{h-1}}]{} \underset{r_{h-1}}{\bullet} \longrightarrow \cdots,$$

contradicting the outer induction hypothesis. $\square$

Now consider the case $\ell = 1$. Let us denote the only vertex of the negative dual hemigraph of A by b. The outgoing arrow of b must be $\frac{b\ \omega}{\omega\ \omega}$, since by Claim C, $\beta_b = b$ is impossible. But this arrow has no target in $A\mathrm{D}_-$, whence there is no cycle.

Finally consider the general case $\ell > 1$. As there is at most one $b \in A\mathrm{I}_-$ with $A : \frac{b\ \omega}{\omega\ \omega}$, the hypothesis of Claim C is satisfied. We can hence pick $b_1 \in A\mathrm{I}_-$ as in the conclusion. We then apply Claim B with $\langle b_0, c\rangle = \langle \beta_{b_1}, \gamma_{b_1}\rangle$. The resulting n-cell A' of the extension $\mathfrak{N}''$ has $\ell - 1$ negative facets, whence by inner induction hypothesis, $A'\mathrm{D}_-$ is cycle-free. It follows that $A\mathrm{D}_-$ is cycle-free as well, as we have seen above. $\square$

Section 7. Cell Trees

An $(n+1)$-cell in a dendrotopic set has a positive half boundary, which is an n-cell, and a negative half boundary, which is an *n-cell tree*. An n-cell tree can be viewed as a partitioned cell of sorts and has a boundary of its own. An n-cell and an n-cell tree whose two boundaries agree together are potentially the positive and the negative halves of the boundary of an $(n+1)$-cell. Somewhat hidden in this statement are the globular laws.

Let $\mathfrak{D}$ be a dendrotopic set, and let P be a cell of $\mathfrak{D}$. The *boundary* $P^\bullet$ of P consists of the set $P^\bullet\mathrm{I} = P\mathrm{I}$ of its facet locations, the mappings associating a sign $\{\!|q|\!\}$ and a resident $P^\bullet q = Pq$ to each $q \in P\mathrm{I}$, and the conjugation $P^\bullet\diamondsuit = P\diamondsuit$. In brief, the boundary of P is the collection of the atomic data that depend directly on P, except for the element P itself.

A *frame* is a collection of data that has the potential of being the boundary of a cell. To be precise, without reiterating the entire definition of 'dendrotopic set', we define an n-frame of $\mathfrak{D}$ to be the boundary of an n-cell of a dendrotopic set obtained by extending $\mathfrak{D}$ by just this cell. We usually denote a frame as we would denote such a cell. The boundary of an (actual) n-cell of $\mathfrak{D}$ is an example of an n-frame of $\mathfrak{D}$.

We identify two frames if they can be made equal by renaming their facet locations. (That is, we shall be dealing with frames-of-old and frames-of-new, the latter being isomorphism classes of the former.) Explicitly, given two n-frames P_0

and P_1, the equation $P_0 = P_1$ will mean that there is a one-to-one correspondence $P_0\mathrm{I} \rightleftarrows P_1\mathrm{I}$ with the following properties: whenever $q_0 \leftrightarrows q_1$, we have $P_0 q_0 = P_1 q_1$ and $\{\!|q_0|\!\} = \{\!|q_1|\!\}$; whenever $q_0^\eta \leftrightarrows q_1^\eta$ ($\eta \in \mathbf{Z}^\times$), we have $P_0 : \frac{q_0^- \; r^-}{q_0^+ \; r^+}$ if and only if $P_1 : \frac{q_1^- \; r^-}{q_1^+ \; r^+}$. From these properties it follows that whenever $q_0 \leftrightarrows q_1$ and $\{\!|q_0|\!\}$, $\{\!|q_1|\!\} = -$, we have also $|q_0| = |q_1|$. We conclude that there exists at most one correspondence as stated. (That is, there is at most one isomorphism between two frames-of-old. In particular, all automorphisms are trivial.) We conclude further that every frame will be identified with one for which $q = |q|$ for all negative facet locations. (Readers who doubt the soundness of our identification can replace this paragraph by an additional axiom, demanding that each negative facet location agree with its index and that each positive facet location agree with some element they have chosen once and for all. They then have to revise all the constructions of frames yet to come.) Thus the frames of $\mathfrak{D}$ will form a set within the universe of discourse.

We may ascribe a geometric realization $P\Gamma$ to an n-frame P. Namely, we may take $P\Gamma$ to be the boundary of the geometric realization of P as a cell in a defining extension of $\mathfrak{D}$. This space is an $(n-1)$-sphere. Thus the prefix 'n-' does not denote geometric dimension. The reason is that we always think of a frame as a cell-to-be. The most general notion of a cell-to-be is to be introduced next.

We inductively define a dendrotopic set $\mathfrak{D}^\dagger$ extending $\mathfrak{D}$ by demanding that those n-cells of $\mathfrak{D}^\dagger$ that are not already in $\mathfrak{D}$ are precisely the n-frames of $\mathfrak{D}^\dagger$, each being its own boundary. (Note that the n-frames of a dendrotopic set, here $\mathfrak{D}^\dagger$, are precisely the n-frames of the dendrotopic subset consisting of cells of dimensions up to $n-1$.) We call the cells of $\mathfrak{D}^\dagger$ the *frameworks* of $\mathfrak{D}$. The terminology for cells will be applied for frameworks accordingly.

A face of a framework may be either a cell or not; we call it (and hence the associated location as well) *complete* or *incomplete* accordingly. Of course all faces of a cell are cells and hence complete. It follows that for a framework P and locations f of P and g of Pf, completeness of f implies completeness of fg, whence (contraposition) incompleteness of fg implies incompleteness of f.

A framework with no incomplete face location is a cell. A framework whose only incomplete face location is the trivial one, 1, is a frame. A framework whose only incomplete locations are 1 and the positive facet location, ω, will be called an *outniche*. (The term 'frame' is adopted from Baez and Dolan's pioneering work. What we call an outniche, these authors would call a punctured frame; what they would call a niche, we should call an *inniche* given an opportunity.)

Definition. Let $n \geq 0$. An *n-cell tree* P conveys the following data:

- an $(n-1)$-cell $P{\uparrow}\omega\omega$, called the *root ridge* of P;
- a finite family $\langle\, P{\uparrow}p \mid p \in P{\uparrow}\mathrm{I}_- \,\rangle$ of n-cells, called the *facets* of P;
- a tree $P{\uparrow}\mathrm{D}_-$ with vertex set $P{\uparrow}\mathrm{I}_-$ and arrow set

$$P{\uparrow}\mathrm{I}_+^2 = \{\langle \omega, \omega\rangle\} \cup \{\, \langle p, q\rangle \mid p \in P{\uparrow}\mathrm{I}_-;\ q \in P{\uparrow}p\mathrm{I}_- \,\}$$

such that

— the arrow $\langle \omega, \omega\rangle$ is the root, and in the case $n = 0$ not a leaf;
— the target of an arrow $\langle p, q\rangle \neq \langle \omega, \omega\rangle$ is p;
— if p' is the source of an arrow $\langle p, q\rangle$ — a situation we render by $P{\uparrow} : \frac{p' \; \omega}{p \; q}$ — then $P{\uparrow}p'\omega = P{\uparrow}pq$.

A *facet location* of an n-cell tree P is an element of $P{\uparrow}\mathrm{I}_-$. We identify two n-cell trees if they can be made equal by renaming their facet locations.

For $n = 0$, the tree $P{\uparrow}\mathrm{D}_-$ has a single vertex δ_-. Thus, a 0-cell tree P is given by a 0-cell $P{\uparrow}\delta_-$. For $n > 0$, the tree $P{\uparrow}\mathrm{D}_-$ either is empty or has a stem α_0. In the former case we call P itself *empty*. In the latter case the path from a vertex $p \neq \alpha_0$ to the root reaches the stem via an arrow $\langle \alpha_0, \beta_p \rangle$, for a certain $\beta_p \in P{\uparrow}\alpha_0\mathrm{I}_-$. For a given negative facet location q_0 of $P{\uparrow}\alpha_0$, we can form a new n-cell tree $P_{(q_0)}$ by putting

$$P_{(q_0)}{\uparrow}\omega\omega = P{\uparrow}\alpha_0 q_0, \qquad P_{(q_0)}{\uparrow}\mathrm{I}_- = \{\, p \in P{\uparrow}\mathrm{I}_- \mid \beta_p = q_0 \,\}, \qquad P_{(q_0)}{\uparrow}p = P{\uparrow}p$$

and $P_{(q_0)}{\uparrow} : \frac{p' \;\, \omega}{p \;\, q}$ if and only if either $\langle p, q\rangle \neq \langle \omega, \omega\rangle$ and $P{\uparrow} : \frac{p' \;\, \omega}{p \;\, q}$ or $\langle p, q\rangle = \langle \omega, \omega\rangle$ and $P{\uparrow} : \frac{p' \;\;\; \omega}{\alpha_0 \; q_0}$. A converse construction is also available and leads to the following result. For $n > 0$, an n-cell tree P with root ridge Q is given either by the information that $P{\uparrow}\mathrm{I}_- = \varnothing$ or by an n-cell $P{\uparrow}\alpha_0$ with $P{\uparrow}\alpha_0\omega = Q$ and an n-cell tree $P_{(q)}$ with root ridge $P{\uparrow}\alpha_0 q$ for each $q \in P{\uparrow}\alpha_0\mathrm{I}_-$. This statement is a recursive definition of n-cell trees.

The notation makes clear that every $(n+1)$-cell X has an underlying n-cell tree, which we call the *negative half boundary* of X and denote by $X\alpha$. (There is only a minor subtlety: an arrow $\frac{p_- \; q_-}{p_+ \; q_+}$ of $X\mathrm{D}_-$ occurs in the guise of its positive representative $\langle p_+, q_+\rangle$ in $X\alpha{\uparrow}\mathrm{D}_-$.)

What has been said for $(n+1)$-cells is equally true for $(n+1)$-frames and $(n+1)$-outniches: they have negative half boundaries which are n-cell trees. An arbitrary $(n+1)$-framework has a negative half boundary which is an n-framework tree (that is, an n-cell tree in the dendrotopic set $\mathfrak{D}^\dagger$ of frameworks of $\mathfrak{D}$). Roughly speaking, what distinguishes $(n+1)$-outniches from n-cell trees is the presence of positive facets (which are n-frames). But, as the following theorem tells us, a positive facet is implicit in the structure of a cell tree.

Theorem 4. *Let $n \geq 0$. Each n-cell tree is the negative half boundary of a unique $(n+1)$-outniche.*

From here we can immediately derive the more general result that every n-framework tree is the negative half boundary of a unique $(n+1)$-framework with incomplete face locations 1 and ω.

Proof. Let P be an n-cell tree. We are going to construct an $(n+1)$-outniche X with negative half boundary P. The uniqueness part of the claim can eventually be verified by inspection: the outniche X has to be constructed in the way stated.

We let the negative facet locations of $X\omega$ be the leafs of $P{\uparrow}\mathrm{D}_-$, each written ${}^pq$ rather than $\langle p, q\rangle$. In the case $n = 0$ the root $\langle \omega, \omega\rangle$ is the only arrow of $P{\uparrow}\mathrm{D}_-$, but by definition not a leaf, whence $X\omega$ has no negative facet location, as required. We complete the definition of conjugation underneath X by putting $\frac{\omega \;\, {}^pq}{p \;\, q}$, whence we have to let $X\omega{}^pq = Xpq$. We have thus constructed all of the chamber graph $X\Gamma^3$, except for its normal panels. They are determined by the normalization condition on X. Thus conjugation underneath $X\omega$ is also defined. We now have to examine whether $X\omega$ meets the dendrotopic axioms.

First we verify the tree property. For the cell tree P, we define indices $|p|$ $(p \in P{\uparrow}\mathrm{I}_-)$ and $|p, q|$ $(\langle p, q\rangle \in P{\uparrow}\mathrm{I}^2_+)$ and even $|p, q, r|$ $(p \in P{\uparrow}\mathrm{I}_-;\ \langle q, r\rangle \in P{\uparrow}p\mathrm{I}^2_+)$

just as we have done for cells. We now show that whenever we have an arrow in $X\omega\mathrm{D}_-$ possessing source and target,

$$X\omega : \frac{{}^{p'}q' \;\; \omega}{{}^{p}q \;\; r}, \tag{4}$$

indices decrease from source to target, $|p',q'| >^2 |p,q|$; from here it follows that the negative dual hemigraph of $X\omega$ has no cycles. Consider the Hamilton gallery underneath X that has led to the definition of (4),

$$\begin{array}{ccc} \omega & {}^{p'}q' & \omega \\ \hline p' & q' & \omega \\ \hline * & * & * \\ \vdots & \vdots & \vdots \\ * & * & * \\ \hline p & q & r \\ \hline \omega & {}^{p}q & r \end{array}.$$

By the argument employed in the proof of Proposition 2, the indices of non-normal negative chambers decrease along this gallery. In particular, a comparison of the third chamber from above with the second chamber from below yields $|p',q'| \geq^2 |p,q,r|$. Since $|p,q,r| >^2 |p,q|$, the result follows.

Now we verify the uniqueness part of the normalization property. Let $\langle r^0_\eta, s^0_\eta\rangle$ $(\eta \in \mathbf{Z}^\times)$ be two different 2-chambers underneath $X\omega\omega$. Suppose that there is a zigzag from $\langle \omega, r^0_-, s^0_-\rangle$ to $\langle \omega, r^0_+, s^0_+\rangle$ in $X\omega\Gamma^3$; we want to show that $\langle r^0_\eta, s^0_\eta\rangle$ are conjugates of each other. If we disallow backtracking, the only vertices along the zigzag where the direction changes are normal chambers. Hence we may as well assume that the zigzag is a path, that is, a gallery underneath $X\omega$. Via the canonical embedding $(\langle\omega\rangle + \,?\,) : X\omega\Gamma^3 \twoheadrightarrow X\Gamma^4$ we obtain a gallery from $\langle \omega, \omega, r^0_-, s^0_-\rangle$ to $\langle \omega, \omega, r^0_+, s^0_+\rangle$ underneath X. Unravelling the definition of $X\omega\Diamond$, we can replace each arrow of the form

$$\text{normal panel} + \langle s\rangle$$

by a path of the form

$$\text{Hamilton gallery} + \langle s\rangle.$$

There is hence a gallery from $\langle \omega, \omega, r^0_-, s^0_-\rangle$ to $\langle \omega, \omega, r^0_+, s^0_+\rangle$ containing no type-2 panel with first entry ω. Of all those galleries we pick a shortest one, say $\mathscr{G}$.

The gallery $\mathscr{G}$ contains no chamber of the form $\langle \omega, {}^{p}q, *, *\rangle$: if there were one, $\mathscr{G}$ would have a subgallery

$$\begin{array}{cccc} p & q & r_- & s_- \\ \hline \omega & {}^{p}q & r_- & s_- \\ \hline \omega & {}^{p}q & r_+ & s_+ \\ \hline p & q & r_+ & s_+ \end{array},$$

which could be replaced by the shorter

$$\begin{array}{cccc} p & q & r_- & s_- \\ \hline p & q & r_+ & s_+ \end{array}.$$

Now pick a negative facet location p' which has $\leq^1$-maximal index among those occurring as first entries of chambers of $\mathscr{G}$. There is a subgallery $\langle p'\rangle + \mathscr{G}'$ of $\mathscr{G}$ bordered by type-1 panels. Since the previous and the next chamber can be of neither of the forms $\langle p'', \omega, *, *\rangle$ with p'' negative (maximality of p') or $\langle \omega, {}^{p'}q', *, *\rangle$

(just excluded), it is of the form $\langle p, q, *, * \rangle$, where $P{\uparrow} : \frac{p'\ \omega}{p\ \ q}$. It follows that source and target of $\mathscr{G}'$ are normal chambers $\langle \omega, r_\eta, s_\eta \rangle$:

$$\begin{array}{cccc} \underline{p} & \underline{q} & r_- & s_- \\ p' & \underline{\omega} & \underline{r_-} & \underline{s_-} \\ p' & * & * & * \\ \vdots & \vdots & \vdots & \vdots \\ p' & \underline{*} & \underline{*} & \underline{*} \\ \underline{p'} & \underline{\omega} & r_+ & s_+ \\ p & q & r_+ & s_+ \end{array} \left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \\ a \end{array}}\right\} = \langle p' \rangle + \mathscr{G}' . \tag{5}$$

(The long lines indicate uncertainty as to the types of the panels they symbolize.) Since the normalization condition is satisfied underneath Xp', the 2-chambers $\langle r_\eta, s_\eta \rangle$ underneath $Xp'\omega = Xpq$ are conjugate. We can hence replace the subgallery (5) of $\mathscr{G}$ by the shorter

$$\begin{array}{cccc} p & q & \underline{r_-} & \underline{s_-} \\ p & q & r_+ & s_+ \end{array} ;$$

a contradiction. Hence the implicit assumption that a negative first entry occurs at all among the chambers of $\mathscr{G}$ was wrong. We have thus ruled out all occurrences of chambers with either of the first two entries negative: all chambers of $\mathscr{G}$ are of the form $\langle \omega, \omega, *, * \rangle$. Since type-2 panels with first entry ω are also ruled out, $\mathscr{G}$ consists of nothing but a type-3 panel:

$$\begin{array}{cccc} \omega & \omega & \underline{r^0_-} & \underline{s^0_-} \\ \omega & \omega & r^0_+ & s^0_+ \end{array} .$$

Thus indeed $\langle r^0_\eta, s^0_\eta \rangle$ are conjugates of each other. $\square$

The frame $X\omega$ constructed in this proof will be called the *boundary* of P and denoted by $P^\bullet$. Note that for an $(n+1)$-cell X, the uniqueness part of the theorem yields

$$X\alpha^\bullet = X\omega^\bullet . \tag{6}$$

Let us denote by $\mathfrak{D}^{1*}_n$ the set of n-cell trees of $\mathfrak{D}$. A single n-cell P can be viewed as an n-cell tree with a single facet location δ_-: put $P{\uparrow}\delta_- = P$. Thus we obtain an inclusion $\mathfrak{D}_n \subseteq \mathfrak{D}^{1*}_n$. The boundary of the cell P is independent of the way it is viewed: there is an $(n+1)$-frame X with two facets $X\delta_\eta = P$ ($\eta \in \mathbf{Z}^\times$) and ridge locations $\{\!\{q\}\!\} \frac{\delta_-\ q}{\delta_+\ q}$ ($q \in P\mathrm{I}$).

Now let P be an n-cell tree. If $n \geq 1$, we put $P\omega = P^\bullet\omega$ ($= P{\uparrow}\omega\omega$) and $P\alpha = P^\bullet\alpha$. If $n \geq 2$, we can conclude from (6) that $P\alpha\omega = P\omega\omega$ and $P\alpha\alpha = P\omega\alpha$. Thus, we have constructed a globular set $\mathfrak{D}^{1*}$ with positive facet locations ω and negative facet locations α. The situation may be summarized by the partially commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathfrak{D}^{1*}_{n+1} \arrow[r, "\alpha"] \arrow[dr, "\omega" near start] & \mathfrak{D}^{1*}_n \arrow[r, dash] \arrow[dr] & \cdots \arrow[r] & \mathfrak{D}^{1*}_1 \arrow[r, "\alpha"] \arrow[dr, "\omega" near start] & \mathfrak{D}^{1*}_0 . \\ \cdots \arrow[r] \arrow[ur] & \mathfrak{D}_{n+1} \arrow[r, "\omega"'] \arrow[u, hook] \arrow[ur, "\alpha"' near start] & \mathfrak{D}_n \arrow[r, dash] \arrow[u, hook] & \cdots \arrow[r] \arrow[ur] & \mathfrak{D}_1 \arrow[r, "\omega"'] \arrow[u, hook] \arrow[ur, "\alpha"' near start] & \mathfrak{D}_0 \arrow[u, equal] \end{tikzcd}$$

Part II. Correctness

The task ahead is to show that that the definition of dendrotopic sets is correct, in the sense that it is equivalent to the Hermida–Makkai–Power definition of multitopic sets. There is bound to be a direct proof of this fact; the proof presented here involves a detour via ∞-categories. More precisely, it is shown (in Section 5) that the Harnik–Makkai–Zawadowski characterization is valid here too: dendrotopic sets are precisely the freely generating subsets, closed under the target operation, of strict ∞-categories. To this end, a lemma is proven (in Section 4) which seems to be of interest in its own right. To keep the exposition self-contained, definitions of strict ∞-categories (Section 1) and their freedom (Section 3) are supplied. The elegant way to represent the n-cells of the ∞-category to be generated by a dendrotopic set is as certain $(n+1)$-frameworks of the latter. These frameworks are called *roofs*; they are introduced in Section 2.

Section 1. Strict ∞-Categories

We discuss various ways to define the term 'strict ∞-category' (about which there can be no dispute, contrary to the 'weak' situation).

Put into more familiar terms, a globular set $\mathfrak{G}$ is given by the following items:

- for each $n \geq 0$, the set $\mathfrak{G}_n$ of n-cells;
- for each $n \geq 1$ and $\eta \in \mathbf{Z}^\times$, the *facet operator* ${}_n\delta_\eta : \mathfrak{G}_n \twoheadrightarrow \mathfrak{G}_{n-1}$, $A \mapsto A\delta_\eta$;
- (*globular law*) for each $n \geq 2$, $\zeta \in \mathbf{Z}^\times$ and $A \in \mathfrak{G}_n$, satisfaction of the equation $A\delta_+\delta_\zeta = A\delta_-\delta_\zeta$.

It follows that the expression '$A\delta_{\theta^{(1)}} \cdots \delta_{\theta^{(u)}}$', whenever defined, takes a value independent of $\theta^{(1)}, \ldots, \theta^{(u-1)}$. This value will be denoted by $A\delta^u_{\theta^{(u)}}$. We are thus led to yet another definition. A globular set is given by the following items: for each $n \geq 0$,

- the set $\mathfrak{G}_n$ of n-cells,
- the *face operators* ${}_n\delta^u_\eta : \mathfrak{G}_n \twoheadrightarrow \mathfrak{G}_{n-u}$, $A \mapsto A\delta^u_\eta$ $(0 < u \leq n;\ \eta \in \mathbf{Z}^\times)$,
- (*generalized globular law*) satisfaction of the equations

$$A\delta^u_\eta\delta^v_\zeta = A\delta^{u+v}_\zeta \quad (\,0 < u, v;\ u+v \leq n;\ \eta, \zeta \in \mathbf{Z}^\times;\ A \in \mathfrak{G}_n\,).$$

In view of the "enumeration" below, the generalized globular law will also be referred to by the symbol $(\delta\delta2)^*$.

A *subset* $\mathfrak{g}$ of a globular set $\mathfrak{G}$ consists of a subset $\mathfrak{g}_n$ of each $\mathfrak{G}_n$. A subset $\mathfrak{g}$ of $\mathfrak{G}$ is called *sign-η semiglobular* if $n > 0$ and $A \in \mathfrak{g}_n$ imply $A\delta_\eta \in \mathfrak{g}_{n-1}$. In absence of further qualification the sign is understood to be $+$. For instance, at the end of Part I we have seen how a dendrotopic set $\mathfrak{D}$ is viewed as a semiglobular subset of the globular set $\mathfrak{D}^{1*}$ of its cell trees. A *globular* subset of $\mathfrak{G}$ is one that is both positively and negatively semiglobular.

A *system of constants* for $\mathfrak{G}$ consists of maps $\sigma^u_n : \mathfrak{G}_{n-u} \twoheadrightarrow \mathfrak{G}_n$ $(n \geq u > 0)$, written $A' \mapsto A'\sigma^u$, or just $A' \mapsto A'\sigma$ in the case $u = 1$, satisfying the equations

$$(\sigma\delta1) \qquad A'\sigma\delta_\eta = A',$$

$$(\sigma\delta2) \qquad A''\sigma^{v+1}\delta_\eta = A''\sigma^v.$$

We can immediately infer validity of the more general equations

$$(\sigma\delta 2)^* \qquad A''\sigma^{u+v}\delta^u_\eta = A''\sigma^v,$$
$$(\sigma\delta 1)^* \qquad A'\sigma^u\delta^u_\eta = A',$$
$$(\delta\sigma 2)^* \qquad A'\sigma^u\delta^{u+v}_\zeta = A'\delta^v_\zeta.$$

In view of imminent developments, we call $A'\sigma^u$ the *u-identity* of A'. A cell can be the u-identity of at most one other cell. In fact, if $A = A'\sigma^u$, then by $(\sigma\delta 1)^*$ we have $A' = A\delta^u_\eta$, independent of $\eta \in \mathbf{Z}^\times$. In this case we denote the cell A' suggestively by $A\delta^u$.

We say that two n-cells A and B of a globular set $\mathfrak{G}$ are *u-consecutive*, in symbols $A \circ^u B$, if $A\delta^u_+ = B\delta^u_-$. We allow $\circ$ as a shorthand notation for $\circ^1$. The set of all pairs $\langle A, B\rangle$ of u-consecutive n-cells of $\mathfrak{G}$ will be denoted by $\mathfrak{G}_n \times^u \mathfrak{G}_n$.

A *system of binary operations* for $\mathfrak{G}$ consists of maps $\mu^u_n : \mathfrak{G}_n \times^u \mathfrak{G}_n \to \mathfrak{G}_n$ $(n \geq u > 0)$, written $\langle A, B\rangle \mapsto A \cdot^u B$, or just $\langle A, B\rangle \mapsto A \cdot B$ in the case $u = 1$, satisfying the equations

$$(\mu\delta 1) \qquad (A \cdot B)\delta_- = A\delta_- \quad\text{and}\quad (A \cdot B)\delta_+ = B\delta_+ \quad (A \circ B),$$
$$(\mu\delta 2) \qquad (A \cdot^{v+1} B)\delta_\eta = A\delta_\eta \cdot^v B\delta_\eta \qquad (A \circ^{v+1} B).$$

(Note that $A \circ^{v+1} B$ implies $A\delta_\eta \circ^v B\delta_\eta$.) We can immediately infer validity of the more general equations

$$(\mu\delta 2)^* \qquad (A \cdot^{u+v} B)\delta^u_\eta = A\delta^u_\eta \cdot^v B\delta^u_\eta \qquad (A \circ^{u+v} B),$$
$$(\mu\delta 1)^* \qquad (A \cdot^u B)\delta^u_- = A\delta^u_- \quad\text{and}\quad (A \cdot^u B)\delta^u_+ = B\delta^u_+ \quad (A \circ^u B),$$
$$(\delta\mu 2)^* \qquad (A \cdot^u B)\delta^{u+v}_\zeta = A\delta^{u+v}_\zeta = B\delta^{u+v}_\zeta \qquad (A \circ^u B).$$

In view of imminent developments, we call $A \cdot^u B$ the *u-composite* of A and B.

Definition. A *(strict) ∞-category* $\mathfrak{K}$ is a globular set with a system of constants and a system of binary operations, satisfying the following equations.

$$(\sigma\mu 1)^* \qquad A\delta^u_-\sigma^u \cdot^u A = A = A \cdot^u A\delta^u_+\sigma^u,$$
$$(\mu\mu 1)^* \qquad (A \cdot^u B) \cdot^u C = A \cdot^u (B \cdot^u C) \qquad (A \circ^u B \circ^u C),$$
$$(\mu\sigma 2)^* \qquad (A' \cdot^v B')\sigma^u = A'\sigma^u \cdot^{u+v} B'\sigma^u \qquad (A' \circ^v B'),$$
$$(\mu\mu 2)^* (A \cdot^{u+v} C) \cdot^u (B \cdot^{u+v} D) = (A \cdot^u B) \cdot^{u+v} (C \cdot^u D) \qquad \begin{pmatrix} A & \circ^u & B \\ \circ^{u+v} & & \circ^{u+v} \\ C & \circ^u & D \end{pmatrix}.$$

(In $(\sigma\mu 1)^*$, the composites exist because of $(\sigma\delta 1)^*$. In $(\mu\mu 1)^*$, the outer composites exist because of $(\mu\delta 1)^*$. In $(\mu\sigma 2)^*$, the right-hand composite exists because of $(\delta\sigma 2)^*$. In $(\mu\mu 2)^*$, the outer composites exist because of $(\mu\delta 2)^*$ and $(\delta\mu 2)^*$.)

The equations $(**1)^*$ state that we have a category $\mathfrak{K}^u_n$ with object set $\mathfrak{K}_{n-u}$, morphism set $\mathfrak{K}_n$ and structure maps ${}_n\delta^u_-$, ${}_n\delta^u_+$, σ^u_n, μ^u_n. The equations $(\delta * 2)^*$ state that for each $\eta \in \mathbf{Z}^\times$ we have a functor of $\mathfrak{K}^u_n$ into the discrete category $\mathfrak{K}_{n-u-v}$ with object map ${}_{n-u}\delta^v_\eta$ and morphism map ${}_n\delta^{u+v}_\eta$. It follows that we have a category $\mathfrak{K}^u_n \times^{u+v} \mathfrak{K}^u_n$ with object set $\mathfrak{K}_{n-u} \times^v \mathfrak{K}_{n-u}$, morphism set $\mathfrak{K}_n \times^{u+v} \mathfrak{K}_n$ and structure maps ${}_n\delta^u_- \times^{u+v} {}_n\delta^u_-$, ${}_n\delta^u_+ \times^{u+v} {}_n\delta^u_+$, $\sigma^u_n \times^{u+v} \sigma^u_n$, $\mu^u_n \times^{u+v} \mu^u_n$. The equations $(\mu * 2)^*$ state that we have a functor of $\mathfrak{K}^u_n \times^{u+v} \mathfrak{K}^u_n$ into $\mathfrak{K}^u_n$ with object map μ^v_{n-u} and morphism map μ^{u+v}_n.

By a standard argument (Eckmann–Hilton), the n-cells A of $\mathfrak{K}$ with $A\delta^u_- = A' = A\delta^u_+$ for some fixed $A' = A''\sigma^v$ form a commutative monoid. The operations are equally those of $\mathfrak{K}^u_n$ and those of $\mathfrak{K}^{u+v}_n$. For the neutral element in particular we find that

$$(\sigma\sigma 2)^* \qquad A''\sigma^v\sigma^u = A''\sigma^{u+v}.$$

Let still $\mathfrak{G}$ be a globular set. A system of constants for $\mathfrak{G}$ satisfying $(\sigma\sigma 2)^*$ is given by maps $\sigma_n : \mathfrak{G}_{n-1} \twoheadrightarrow \mathfrak{G}_n$ $(n > 0)$ satisfying $(\sigma\delta 1)$. Such a system makes $\mathfrak{G}$ into a *globular set with degeneracies*. (The phrase 'with degeneracies' stresses the analogy with the other classical polytopic sets. Usually the term 'reflexive' is employed.) We are thus led to a slightly slimmer version of the Definition: an ∞-category $\mathfrak{K}$ is precisely a globular set with degeneracies that bears a system of binary operations, satisfying $(\sigma\mu 1)^*$, $(\mu\mu 1)^*$, $(\mu\mu 2)^*$ and

$$(\mu\sigma 2) \qquad (A' \cdot^v B')\sigma = A'\sigma \cdot^{v+1} B'\sigma \quad (A' \circ^v B').$$

Example. It has been mentioned that higher-dimensional numbers occur in the work of Batanin in the guise of trees. They in fact form an ∞-category $\hat{\mathbf{N}}$, which we are now going to describe. The n-cells of $\hat{\mathbf{N}}$ are the n-dimensional numbers. We define the facet operators by putting

$$\langle \mathfrak{q}_0, \ldots, \mathfrak{q}_{\ell-1}\rangle_n \delta_\eta = \begin{cases} 0_0 & \text{if } n = 1, \\ \langle \mathfrak{q}_0\delta_\eta, \ldots, \mathfrak{q}_{\ell-1}\delta_\eta\rangle_{n-1} & \text{if } n > 1. \end{cases}$$

Note that ${}_n\delta_+$ and ${}_n\delta_-$ have exactly the same effect. We define the degeneracies by putting

$$\langle \mathfrak{r}_0, \ldots, \mathfrak{r}_{\ell-1}\rangle_{n-1}\sigma = \langle \mathfrak{r}_0\sigma, \ldots, \mathfrak{r}_{\ell-1}\sigma\rangle_n.$$

The consecutive pairs can be characterized as follows: we always have

$$\langle \mathfrak{q}_0, \ldots, \mathfrak{q}_{\ell-1}\rangle_n \circ^n \langle \mathfrak{q}'_0, \ldots, \mathfrak{q}'_{\ell'-1}\rangle_n,$$

and for $d < n$ we have

$$\langle \mathfrak{q}_0, \ldots, \mathfrak{q}_{\ell-1}\rangle_n \circ^d \langle \mathfrak{q}'_0, \ldots, \mathfrak{q}'_{\ell'-1}\rangle_n \text{ if and only if } \ell = \ell' \text{ and } \mathfrak{q}_i \circ^d \mathfrak{q}'_i \ (0 \le i < \ell).$$

We define the composition operators by putting

$$\langle \mathfrak{q}_0, \ldots, \mathfrak{q}_{\ell-1}\rangle_n \cdot^d \langle \mathfrak{q}'_0, \ldots, \mathfrak{q}'_{\ell'-1}\rangle_n = \begin{cases} \langle \mathfrak{q}_0, \ldots, \mathfrak{q}_{\ell-1}, \mathfrak{q}'_0, \ldots, \mathfrak{q}'_{\ell'-1}\rangle_n & \text{if } d = n, \\ \langle \mathfrak{q}_0 \cdot^d \mathfrak{q}'_0, \ldots, \mathfrak{q}_{\ell-1} \cdot^d \mathfrak{q}'_{\ell-1}\rangle_n & \text{if } d < n. \end{cases}$$

Often the one-set approach to defining strict ∞-categories (and more generally strict ω-categories; see [8]) is taken. Here each cell of $\mathfrak{K}$ is identified with all its identities. To avoid confusion, the notation for the category operations ${}_n\delta^u_\eta$ and μ^u_n provides the object dimension $n - u$, while the object codimension u can be omitted. Thus either side of $(\delta\sigma 2)^*$ is written, say, $A'\delta_{n'',\eta}$, and either side of $(\mu\sigma 2)^*$ is written, say, $A' \cdot_{n''} B'$. Here $n'' = n' - v$, where n' is the dimension of the "primed" cells. The one-set approach is impractical for our purposes, but it may increase the reader's appreciation of the Example.

The axioms of a system of constants or binary operations prescribe precisely the boundaries of identities or composites. In the axioms of an ∞-category, the boundaries of the equated cells agree automatically. These two statements have to be taken with care: they refer to an inductive construction of an ∞-category. To clarify them, we have to go a bit farther afield.

Let n be an integer. Given any concept $\mathscr{C}$ derived in this paper from that of a propolytopic set, the attribute '*n-dimensional*' will be used to name the corresponding concept $\mathscr{C}_{\leq n}$ that can be defined in place of $\mathscr{C}$ by omitting all references to data in dimensions $> n$. Any model $\mathfrak{X}$ of the concept $\mathscr{C}$ has an underlying model $\mathfrak{X}_{\leq n}$ of the concept $\mathscr{C}_{\leq n}$. We call $\mathfrak{X}_{\leq n}$ the *n-skeleton* of $\mathfrak{X}$. Given a model $\mathfrak{O}$ of $\mathscr{C}_{\leq n}$, we say that $\mathfrak{X}$ is a model *over* $\mathfrak{O}$ if $\mathfrak{X}_{\leq n} = \mathfrak{O}$. Analogous statements apply to the passage from $\mathscr{C}_{\leq N}$ to $\mathscr{C}_{\leq n}$ for $n \leq N < \infty$. We write $\mathfrak{X}_{<N}$ instead of $\mathfrak{X}_{\leq N-1}$.

A model of $\mathscr{C}$ can be constructed inductively by constructing its skeleta. More precisely, a model $\mathfrak{X}$ of $\mathscr{C}$ is given by a sequence consisting of one model $\mathfrak{X}_{\leq N}$ of each $\mathscr{C}_{<N}$ and satisfying the condition that $\mathfrak{X}_{\leq N}$ is over $\mathfrak{X}_{<N}$. Thus, in order to understand models of $\mathscr{C}$, it suffices to understand models of some $\mathscr{C}_{<N_0}$ as well as models of $\mathscr{C}_{\leq N}$ over fixed models of $\mathscr{C}_{<N}$ for all $N \geq N_0$.

It is high time to be more concrete. Let $N \geq -1$. An N-dimensional ∞-category is simply called an *N-category*. To be sure, here is a somewhat more explicit definition. First, an N-dimensional globular set $\mathfrak{X}$ consists of sets $\mathfrak{X}_0, \ldots,$ $\mathfrak{X}_N$ and mappings ${}_1\delta_+, \ldots, {}_n\delta_+, {}_1\delta_-, \ldots, {}_n\delta_-$ satisfying the globular law. An N-category $\mathfrak{K}$ is an N-dimensional globular set together with mappings σ_n^u and μ_n^u $(0 < u \leq n \leq N)$ satisfying the laws of an ∞-category. We furthermore define a *pre-N-category* $\mathfrak{k}$ to be an N-dimensional globular set together with mappings σ_n^u and μ_n^u $(0 < u \leq n < N)$ satisfying the laws of an ∞-category. Every N-category has an underlying pre-N-category (drop the mappings σ_N^u and μ_N^u), and every pre-N-category has an underlying $(N-1)$-category, its $(N-1)$-skeleton.

Let $N \geq 0$ and let $\mathfrak{O}$ be an $(N-1)$-dimensional globular set. A *(globular) N-frame* a of $\mathfrak{O}$ provides for each integer u with $0 < u \leq N$ and each sign η an $(N-u)$-cell $a\delta_\eta^u$, where

$$a\delta_\eta^u \delta_\zeta = a\delta_\zeta^{u+1} \quad (0 < u < N;\ \eta, \zeta \in \mathbf{Z}^\times).$$

If $N = 0$, there is precisely one N-frame (which provides no information). If $N = 1$, an N-frame a is just a pair of 0-cells $a\delta_+$ and $a\delta_-$. If $N \geq 2$, an N-frame a is given by two $(N-1)$-cells $a\delta_+$ and $a\delta_-$ with $a\delta_+\delta_\zeta = a\delta_-\delta_\zeta$.

An N-dimensional globular set $\mathfrak{G}$ over $\mathfrak{O}$ is given by an N-cell set $\mathfrak{G}_N$ and face operators ${}_N\delta_\eta^u : \mathfrak{G}_N \twoheadrightarrow \mathfrak{O}_{N-u}$ satisfying the generalized globular laws. A particular N-dimensional globular set $\mathfrak{O}[\mathbf{1}]$ over $\mathfrak{O}$ can be obtained by taking $\mathfrak{O}[\mathbf{1}]_N$ to be the set of N-frames of $\mathfrak{O}$ and ${}_N\delta_\eta^u$ as suggested by the notation for frames. Back to the general case. Having a family of maps ${}_N\delta_\eta^u : \mathfrak{G}_N \twoheadrightarrow \mathfrak{O}_{N-u}$ satisfying the generalized globular laws is equivalent to having a single map $\mathfrak{G}_N \twoheadrightarrow \mathfrak{O}[\mathbf{1}]_N$. Indeed, given the former, we can assign to each N-cell A of $\mathfrak{G}$ its *boundary* $A^\bullet$, defined by $A^\bullet\delta_\eta^u = A\delta_\eta^u$. (We have constructed $\mathfrak{O}[\mathbf{1}]_N$ as a certain projective limit in the category of sets.) But any map can be replaced by the family of its fibres. An N-cell A of $\mathfrak{G}$ with boundary a will also be called a *filler* of a. The set of all fillers of a will be denoted by $\mathfrak{G}_a$. Thus, an N-dimensional globular set $\mathfrak{G}$ over $\mathfrak{O}$ is given by a family of disjoint sets $\mathfrak{G}_a$ $(a \in \mathfrak{O}[\mathbf{1}]_N)$. Taking this last statement as a definition, we can drop the disjointness condition without gain of generality.

Now suppose that $\mathfrak{D}$ is an $(N-1)$-category. The previous paragraph can be repeated with the 'N-dimensional globular set $\mathfrak{G}$' replaced by a 'pre-N-category $\mathfrak{k}$'. Now an N-category $\mathfrak{K}$ over $\mathfrak{D}$ is given by pre-N-category over $\mathfrak{D}$ together with identity operators σ^u_N and composition operators μ^u_N satisfying the relevant laws. In the case of $\mathfrak{D}[\mathbf{1}]$ there is a unique way to define such operators $\sigma^u_N = \sigma^{u\bullet}_N$ and $\mu^u_N = \mu^{u\bullet}_N$: the u-identity of an $(N-u)$-cell is determined by the laws of a system of constants; the u-composite of two u-consecutive N-frames is determined by the laws for a system of binary operations. One can verify that all the laws of an ∞-category are thus satisfied. Back to the general case. Having a map $\sigma^u_N : \mathfrak{D}_{N-u} \twoheadrightarrow \mathfrak{K}_N$ satisfying the relevant laws is equivalent to having maps (constants) $\sigma^u_{A'} : 1 \twoheadrightarrow \mathfrak{K}_{A'\sigma^{u\bullet}}$, one for each $(N-u)$-cell A' of $\mathfrak{D}$. Having a map $\mu^u_N : \mathfrak{K}_N \cdot^u \mathfrak{K}_N \twoheadrightarrow \mathfrak{K}_N$ satisfying the relevant laws is equivalent to having maps (binary operations) $\mu^u_{a,b} : \mathfrak{K}_a \times \mathfrak{K}_b \twoheadrightarrow \mathfrak{K}_{a\cdot^u b}$, one for each pair of u-consecutive N-frames a and b of $\mathfrak{D}$. The remaining laws can similarly be broken down for frames. They assert commutativity of the following diagrams (with the obvious conditions on the frames).

$(\sigma\mu 1)^*$

$$\begin{tikzcd}
& \mathfrak{K}_a \arrow[dl, "\sigma^u_{a\delta^u_-}\times\mathfrak{K}_a"'] \arrow[dr, "\mathfrak{K}_a\times\sigma^u_{a\delta^u_+}"] \arrow[dd, "\mathfrak{K}_a"] & \\
\mathfrak{K}_{a\delta^u_-\sigma^u}\times\mathfrak{K}_a \arrow[dr, "\mu^u_{a\delta^u_-\sigma^{u\bullet},a}"'] & & \mathfrak{K}_a\times\mathfrak{K}_{a\delta^u_+\sigma^u} \arrow[dl, "\mu^u_{a,a\delta^u_+\sigma^{u\bullet}}"] \\
& \mathfrak{K}_{a\delta^u_-\sigma^{u\bullet}\cdot^u a = a = a\cdot^u a\delta^u_+\sigma^{u\bullet}} &
\end{tikzcd}$$

$(\mu\mu 1)^*$

$$\begin{tikzcd}
& \mathfrak{K}_a\times\mathfrak{K}_b\times\mathfrak{K}_c \arrow[dl, "\mu^u_{a,b}\times\mathfrak{K}_c"'] \arrow[dr, "\mathfrak{K}_a\times\mu^u_{b,c}"] & \\
\mathfrak{K}_{a\cdot^u b}\times\mathfrak{K}_c \arrow[dr, "\mu^u_{a\cdot^u b,c}"'] & & \mathfrak{K}_a\times\mathfrak{K}_{b\cdot^u c} \arrow[dl, "\mu^u_{a,b\cdot^u c}"] \\
& \mathfrak{K}_{(a\cdot^u b)\cdot^u c = a\cdot^u(b\cdot^u c)} &
\end{tikzcd}$$

$(\mu\sigma 2)^*$

$$\begin{tikzcd}
1 \arrow[r, "\sigma^u_{A'}\times\sigma^u_{B'}"] \arrow[d, "\sigma^u_{A'\cdot^v B'}"'] & \mathfrak{K}_{A'\sigma^{u\bullet}}\times\mathfrak{K}_{B'\sigma^{u\bullet}} \arrow[dl, "\mu^u_{A'\sigma^{u\bullet},B'\sigma^{u\bullet}}"] \\
\mathfrak{K}_{(A'\cdot^v B')\sigma^{u\bullet} = A'\sigma^{u\bullet}\cdot^{u+v}B'\sigma^{u\bullet}} &
\end{tikzcd}$$

$(\mu\mu 2)^*$

$$\begin{tikzcd}
& \mathfrak{K}_a\times\mathfrak{K}_c\times\mathfrak{K}_b\times\mathfrak{K}_d \simeq \mathfrak{K}_a\times\mathfrak{K}_b\times\mathfrak{K}_c\times\mathfrak{K}_d \arrow[dl, "\mu^{u+v}_{a,c}\times\mu^{u+v}_{b,d}"'] \arrow[dr, "\mu^u_{a,b}\times\mu^u_{c,d}"] & \\
\mathfrak{K}_{a\cdot^{u+v}c}\times\mathfrak{K}_{b\cdot^{u+v}d} \arrow[dr, "\mu^u_{a\cdot^{u+v}c,\,b\cdot^{u+v}d}"'] & & \mathfrak{K}_{a\cdot^u b}\times\mathfrak{K}_{c\cdot^u d} \arrow[dl, "\mu^{u+v}_{a\cdot^u b,\,c\cdot^u d}"] \\
& \mathfrak{K}_{(a\cdot^{u+v}c)\cdot^u(b\cdot^{u+v}d) = (a\cdot^u b)\cdot^{u+v}(c\cdot^u d)} &
\end{tikzcd}$$

(Thus the N-categories over a fixed $(N-1)$-category $\mathfrak{D}$ form an $\mathfrak{D}[\mathbf{1}]_N$-sorted variety.)

Section 2. Roofs

We construct the ∞-category generated by a dendrotopic set. The cells of the former will be certain frameworks of the latter, called *roofs*.

Let $\mathfrak{D}$ be a dendrotopic set. We want to define those finite configurations of cells in $\mathfrak{D}$ which have an overall direction in every dimension. By their appearance we may call them *cell forests*. Each n-cell forest is equipped with a finite family of n-cells, its facets. A 0-cell forests consists of just a single facet. For $n > 0$, an n-cell forest has an underlying $(n-1)$-cell forest, to each of whose facets it associates an n-cell tree rooted there. Its facets are the facets of these trees.

This description, while being very natural, turns out to be cumbersome. An important piece of data is shielded from direct access: the leafs of the n-cell trees that constitute an n-cell forest form an $(n-1)$-cell forest of their own. To have all the relevant structure at hand, we collect the data of an n-cell forest into an $(n+1)$-framework of a particular kind.

Definition. Let $n \geq 0$, and let X be an $(n+1)$-framework of a dendrotopic set $\mathfrak{D}$. We call X an $(n+1)$-*roof* under the circumstances described inductively as follows. If $n = 0$, the negative facet is a cell, the positive facet is not. If $n \geq 1$, there is precisely one negative facet that is not a cell, and this facet is an n-roof.

Let us temporarily denote the incomplete negative facet location of a roof by τ. In order fully to appreciate the definition, the reader may need the following supplementary information.

Proposition 1. *Let X be an $(n+1)$-roof. If $n \geq 1$, then $X\omega$ is an n-roof (too). If $n \geq 2$, then*

$$X : \frac{\tau\ \omega}{\omega\ \omega} \qquad \textit{and} \qquad X : \frac{\omega\ \tau}{\tau\ \tau}. \tag{1}$$

Proof. We use induction on n.

First consider the case $n = 1$. The floor of $\langle p, \delta_\zeta \rangle \in X\mathrm{I}^2$ is a cell if either $p \notin \{\omega, \tau\}$ or $p = \tau$ and $\zeta = -$. The floor of $\langle \tau, \delta_+ \rangle$ is not a cell. Hence the conjugate of this negative 2-chamber is the only remaining positive 2-chamber, namely $\langle \omega, \delta_+ \rangle$, and so $X\omega\delta_+ = X\tau\delta_+$ is not a cell. Now the conjugate of the other remaining 2-chamber $\langle \omega, \delta_- \rangle$ must be among those 2-chambers whose floors are cells, and so $X\omega\delta_-$ is a cell. Thus $X\omega$ is a roof.

Now consider the case $n \geq 2$. The floor of $\langle p, q \rangle \in X\mathrm{I}^2$ is a cell if either $p \notin \{\omega, \tau\}$ or $p = \tau$ and $q \notin \{\omega, \tau\}$. The floors of $\langle \tau, \omega \rangle$ and $\langle \tau, \tau \rangle$ are roofs, but these two 2-chambers cannot be conjugates of each other because of their common negative first entry. Hence the conjugate of the former, which is negative, is the only remaining positive 2-chamber, namely $\langle \omega, \omega \rangle$ — this proves the left part of (1) —, and the conjugate of the latter, which is positive, is among the remaining negative 2-chambers, namely $\langle \omega, q \rangle$ ($q \in X\omega\mathrm{I}_-$) — this means $X : \frac{\omega\ q_0}{\tau\ \tau}$ for some negative facet location q_0 of $X\omega$. In particular, $X\omega q_0 = X\tau\tau$ is a roof. Now let q be any other negative facet location of $X\omega$. The conjugate of $\langle \omega, q \rangle$ must be among those 2-chambers whose floors are cells, and so $X\omega q$ is a cell. Thus $X\omega$ is a roof with $\tau = q_0$. This last equation settles the right part of (1). □

Let us come back to the introductory remarks about cell forests. The negative half boundary of an $(n+1)$-roof is an n-framework tree with the following property. If $n = 0$, its only facet location is complete; if $n \geq 1$, it has precisely one incomplete

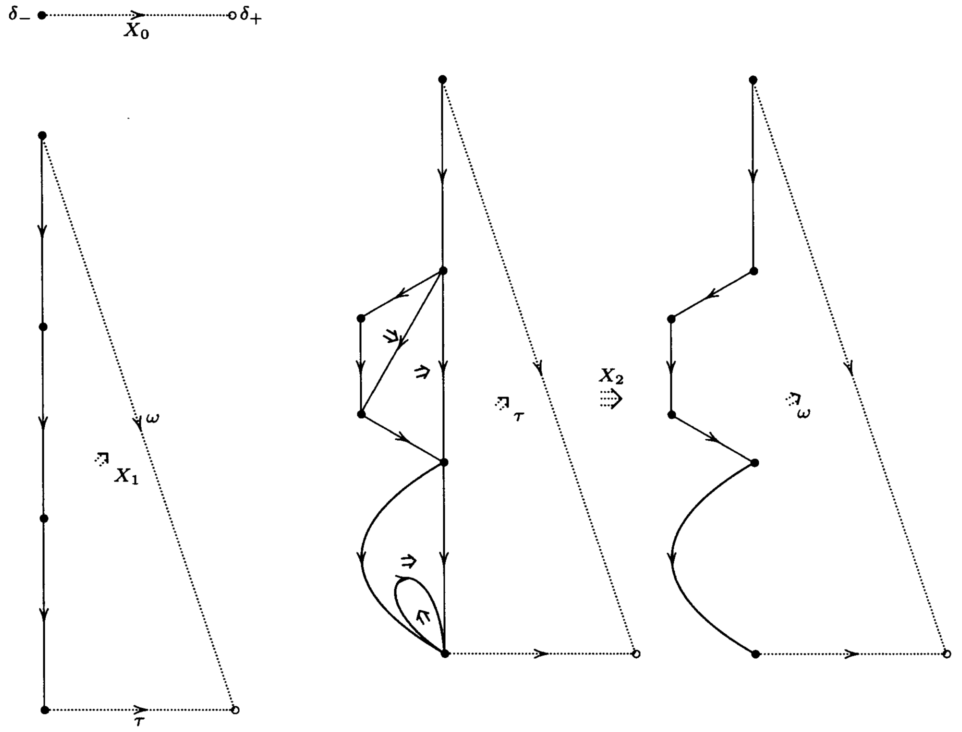


Figure. A 1-roof X_0, a 2-roof X_1 with $X_1\tau = X_0$, a 3-roof X_2 with $X_2\tau = X_1$. (The unfilled circle represents the 0-frame; dotted lines mark incomplete > 0-frameworks.)

facet location, whose resident is a roof. Let us call such a framework tree *roofed*, for lack of a better word. By an adapted version of Theorem 4 of Part I, every roofed n-framework tree is the negative half boundary of exactly one $(n+1)$-roof.

Examining the proof of Proposition 1 again, we find that it in fact shows $X : \frac{\tau\ \omega}{\omega\ \omega}$ for all $n \geq 1$. Thus the incomplete negative facet location of any roof has index 0. We conclude that for roofed framework trees, the roof is the stem. Hence, according to the recursive definition of cell trees, a roofed n-framework tree P consists of an n-roof $P{\uparrow}\tau$ and, for each complete facet location q of $P{\uparrow}\tau$, an n-cell tree $P_{(q)}$ with root $P{\uparrow}\tau q$. (The n-cell tree $P_{(\tau)}$, for $n \geq 2$, is automatically empty, as shown by the left part of (1).) This observation leads to an inductive definition of roofs directly via cell trees, namely the one given above, with the term 'n-cell forest' replacing '$(n+1)$-roof'.

The incomplete facet locations of roofs of dimensions ≥ 1 exhibit a certain symmetry. It will be convenient to express this symmetry in the notation: let us write $\tau = \partial_+$ and $\omega = \partial_-$. Note that ∂_η has the sign $-\eta$, opposite to what the index indicates. The reason becomes clear from the forest point of view: the resident of τ stands for the target of a forest (consisting of its roots), and the resident of ω stands for the source of a forest (consisting of its leafs). Now (1) can be instantly recognized as the globular relationship

$$X : \eta \frac{\partial_- \; \partial_\eta}{\partial_+ \; \partial_\eta}.$$

This tells us that we can define a globular set $\mathfrak{D}^*$ as follows: an n-cell A of $\mathfrak{D}^*$ is an $(n+1)$-roof $A^\natural$ of $\mathfrak{D}$, and the sign-η facet of A, for $n \geq 1$, is specified by

$$A\delta_\eta{}^\natural = A^\natural \partial_\eta.$$

We now want to add degeneracies and a system of binary operations to $\mathfrak{D}^*$. As was remarked before, the boundary of an identity $A'\sigma$ or a composite $A \cdot^u B$ is determined by the operands via the axioms. That is, if we write $X = A'\sigma^\natural$ or $X = (A \cdot^u B)^\natural$, we do not need to specify the facets $X\partial_\eta$. (We may mention them nevertheless, for the convenience of the reader.) The incidence structure of incomplete faces (in particular the incomplete 2-face and 3-face locations) are determined by X being a roof. Thus, we can restrict our attention to the complete part of the boundary. To this end, the subscript '${}_\mathfrak{D}$' will denote the restriction of any set of face locations to the complete ones.

Let $n \geq 1$, and let A' be an $(n-1)$-cell of $\mathfrak{D}^*$. We want to construct the 1-identity $A'\sigma$ of A'. That is, we want to construct the $(n+1)$-roof $A'\sigma^\natural = X$. We let X have no complete facet. The incomplete ones are $X\partial_\eta = A'^\natural$. We let the complete ridge locations of X be

$$\frac{\partial_- \; q}{\partial_+ \; q} \quad \left(q \in A'^\natural \, \mathrm{I}_\mathfrak{D}\right).$$

Then the complete 3-face locations of X are

$$\begin{array}{c} \partial_- \; q_- \; r_- \\ \hline \partial_+ \; q_- \; r_- \\ \hline \partial_+ \; q_+ \; r_+ \\ \hline \partial_- \; q_+ \; r_+ \end{array} \quad \left(\tfrac{q_- \; r_-}{q_+ \; r_+} \in A'^\natural \Diamond_\mathfrak{D} \right).$$

For $u > 1$, the facets of the u-identity $A'\sigma^u$ of A' are identities themselves. In particular, $A'\sigma^{u\natural}\partial_+ = A'\sigma^{u-1\natural}$ has no complete facets. From the cell-tree approach it follows that given an n-roof Y without complete facets, there is only one $(n+1)$-roof X with $X\partial_+ = Y$. Hence whenever $A\delta_+ = A'\sigma^{u-1}$, we can conclude that $A = A'\sigma^u$.

Let $u \geq 1$, and let A and B be two u-consecutive n-cells. Let us denote by A' the intermediate $(n-u)$-cell $A\delta_+^u = B\delta_-^u$. We want to construct the u-composite $A \cdot^u B$ of A and B. That is, we want to construct the $(n+1)$-roof $(A \cdot^u B)^\natural = X$. This construction has to be carried out by induction on u. In all cases, we let the complete facets of X be those of $A^\natural$ and those of $B^\natural$.

In the case $u = 1$, the incomplete facets of X are $X\partial_- = A^\natural\partial_-$ and $X\partial_+ = B^\natural\partial_+$. We let the complete ridge locations of X be those of $A^\natural$ and those of $B^\natural$, except that

$$A^\natural : \frac{p_- \; q_-}{\partial_+ \; q} \quad \text{and} \quad B^\natural : \frac{\partial_- \; q}{p_+ \; q_+}$$

are replaced by

$$X : \frac{p_- \; q_-}{p_+ \; q_+}$$

for each $q \in A'^{\natural}\,\mathrm{I}_{\mathfrak{D}}$. Let us say that the chambers $\langle \partial_+, q\rangle$ of $A^{\natural}$ and $\langle \partial_-, q\rangle$ of $B^{\natural}$ *cancel* each other. The construction is an instance of the one discussed in Claim A in the proof of Theorem 3 of Part I (with $A^{\natural} : \partial_+$ and $B^{\natural} : \partial_-$ here playing the roles of $B_0 : c$ and $B_1 : \omega$ there). Note that the ridge location $\eta\frac{\partial_-\ \partial_\eta}{\partial_+\ \partial_\eta}$ of X replaces the corresponding ones of $A^{\natural}$ and $B^{\natural}$.

In the case $u > 1$, the incomplete facets of X are $X\partial_\eta = (A\delta_\eta \cdot^{u-1} B\delta_\eta)^{\natural}$. We let the complete ridge locations of X be those of $A^{\natural}$ and those of $B^{\natural}$. Then the complete 3-face locations of X are also those of $A^{\natural}$ and those of $B^{\natural}$, with the following exceptions in the case $u = 2$: here for each $r \in A'^{\natural}\,\mathrm{I}_{\mathfrak{D}}$, the two locations

$$A^{\natural} : \{\!\!\{r\}\!\!\} \left\{ \begin{array}{l} \begin{array}{ccc} \partial_- & \partial_+ & r \\ \partial_+ & \partial_+ & r \end{array} \\ \left.\begin{array}{ccc} \partial_+ & q_{+,+} & r_{+,+} \\ * & * & * \\ \vdots & \vdots & \vdots \\ * & * & * \\ \partial_- & q_{-,+} & r_{-,+} \end{array}\right\} = \mathscr{G}_+ \end{array}\right. \qquad \text{and} \qquad B^{\natural} : \{\!\!\{r\}\!\!\} \left\{ \begin{array}{l} \left.\begin{array}{ccc} \partial_- & q_{-,-} & r_{-,-} \\ * & * & * \\ \vdots & \vdots & \vdots \\ * & * & * \\ \partial_+ & q_{+,-} & r_{+,-} \end{array}\right\} = \mathscr{G}_- \\ \begin{array}{ccc} \partial_+ & \partial_- & r \\ \partial_- & \partial_- & r \end{array} \end{array}\right.$$

are relinked and thus replaced by

$$X : \{\!\!\{r\}\!\!\} \left\{ \begin{array}{l} \left.\begin{array}{ccc} \partial_- & q_{-,-} & r_{-,-} \\ * & * & * \\ \vdots & \vdots & \vdots \\ * & * & * \\ \partial_+ & q_{+,-} & r_{+,-} \end{array}\right\} = \mathscr{G}_- \\ \left.\begin{array}{ccc} \partial_+ & q_{+,+} & r_{+,+} \\ * & * & * \\ \vdots & \vdots & \vdots \\ * & * & * \\ \partial_- & q_{-,+} & r_{-,+} \end{array}\right\} = \mathscr{G}_+ \end{array}\right. .$$

For $u > 1$, the cell-tree approach yields a different way to describe $(A \cdot^u B)^{\natural} = X$. By induction hypothesis, we know that $X\partial_+\mathrm{I}_{\mathfrak{D}}$ is the disjoint union of $A^{\natural}\,\mathrm{I}_{\mathfrak{D}}$ and $B^{\natural}\,\mathrm{I}_{\mathfrak{D}}$. For each $q \in A^{\natural}\partial_+\mathrm{I}_{\mathfrak{D}}$ we have put $X\alpha_{(q)} = A^{\natural}\alpha_{(q)}$, and for each $q \in B^{\natural}\partial_+\mathrm{I}_{\mathfrak{D}}$ we have put $X\alpha_{(q)} = B^{\natural}\alpha_{(q)}$. This description does not, however, directly reveal the fact that $X\partial_- = (A\delta_- \cdot^{u-1} B\delta_-)^{\natural}$.

We now want to verify that $\mathfrak{D}^*$, including the structure just introduced, is indeed an ∞-category. We have to consider four equations of cells in $\mathfrak{D}^*$, each of which can be equivalently expressed as an equation of frameworks in $\mathfrak{D}$. Following the construction steps above, we easily obtain a common descriptions of both — or in case of $(\sigma\mu 1)^*$ all three — sides of each of the latter equations. We only give this description here. According to an earlier remark, the globular boundaries in the original equations agree as a consequence of the axioms on identities and composites. Hence we can again restrict our attention to the complete parts of the dendrotopic boundaries.

$(\sigma\mu 1)^*$ $(A\delta_-^u\sigma^u \cdot^u A)^{\natural} = A^{\natural} = (A \cdot^u A\delta_+^u\sigma^u)^{\natural}$. The complete facets as well as the complete ridge locations are precisely those of $A^{\natural}$. To evaluate the left-hand side ($\eta = -$) and the right-hand side ($\eta = +$) in the case $u = 1$, note that $A\delta_\eta\sigma^{\natural} : \frac{\partial_-\ q}{\partial_+\ q}$ ($q \in A\delta_\eta{}^{\natural}\,\mathrm{I}_{\mathfrak{D}}$) act as neutral elements for cancellation.

($\mu\mu$1)* $((A \cdot^u B) \cdot^u C)^\natural = (A \cdot^u (B \cdot^u C))^\natural$. The complete facets are precisely those of $A^\natural$, $B^\natural$, $C^\natural$. The same goes for the complete ridge locations, with the following exceptions in the case $u = 1$. Let us denote by A' the cell $A\delta_+ = B\delta_-$ and by B' the cell $B\delta_+ = C\delta_-$. For each $q \in A'^\natural \mathrm{I}_{\mathfrak{D}}$ the chambers $\langle \partial_+, q\rangle$ underneath $A^\natural$ and $\langle \partial_-, q\rangle$ underneath $B^\natural$ cancel each other, and so do for each $q \in B'^\natural \mathrm{I}_{\mathfrak{D}}$ the chambers $\langle \partial_+, q\rangle$ underneath $B^\natural$ and $\langle \partial_-, q\rangle$ underneath $C^\natural$.

($\mu\sigma$2) $(A \cdot^u B)\sigma^\natural = (A\sigma \cdot^{u+1} B\sigma)^\natural$. There are no complete facets. The complete ridge locations are precisely $\frac{\partial_- \; q}{\partial_+ \; q}$ for all complete facet locations q of $A^\natural$ and of $B^\natural$.

($\mu\mu$2)* $((A \cdot^u B) \cdot^{u+v} (C \cdot^u D))^\natural = ((A \cdot^{u+v} C) \cdot^u (B \cdot^{u+v} D))^\natural$. The complete facets are precisely those of $A^\natural$, $B^\natural$, $C^\natural$, $D^\natural$. The same goes for the complete ridge locations, with the following exceptions in the case $u = 1$. Let us denote by A' the cell $A\delta_+ = B\delta_-$ and by C' the cell $C\delta_+ = D\delta_-$. For each $q \in A'^\natural \mathrm{I}_{\mathfrak{D}}$ the chambers $\langle \partial_+, q\rangle$ underneath $A^\natural$ and $\langle \partial_-, q\rangle$ underneath $B^\natural$ cancel each other, and so do for each $q \in C'^\natural \mathrm{I}_{\mathfrak{D}}$ the chambers $\langle \partial_+, q\rangle$ underneath $C^\natural$ and $\langle \partial_-, q\rangle$ underneath $D^\natural$. (Note once again that $(A \cdot^{v+1} C)\delta_+ = A' \cdot^v C' = (B \cdot^{v+1} D)\delta_-$.)

The achievement of this section ought to be stated for the record.

Proposition 2. *Let $\mathfrak{D}$ be a dendrotopic set. The roofs of $\mathfrak{D}$ form an ∞-category $\mathfrak{D}^*$ as defined above.*

Not surprisingly, a quick comparison of the ∞-category construction of the Hermida–Makkai–Zawadowski paper [3] with the present one uncovers many parallels, hidden not least by different terminologies. The translations

pasting diagram = cell tree,

indeterminate = incomplete framework,

good pasting diagram (in the superstructure with indeterminates) = roofed framework tree

suggest themselves. Of course these "equations" gain formal meaning only after the concepts of 'multitopic set' and of 'dendrotopic set' have been suitably related. Then the following assessments, delivered here without proofs, can be made.

The first equation clearly holds. The second equation holds too, but to evaluate the left-hand side one has to go beyond the cited paper, which (in its present form) is silent on how many 0-dimensional indeterminates there are. In avoiding such objects, the authors unnecessarily give special treatment to the case of dimension 1. Here they take a "good pasting diagram" to be a pasting diagram in the original structure, while a "roofed framework tree" is a cell tree extended by incomplete root and stem. Thus the third equation fails, even though the steps in the inductive definitions of the two sides amount to the same.

Section 3. Freedom

We say what it means for a strict ∞-category to be *free*. Two concepts of an n-category being freely generated by a pre-n-category naturally arise, and we show that they are equivalent.

Oddly enough, we have so far been discussing only objects, not their accompanying *morphisms*. With the introduction of the latter, some essentially category-theoretical arguments will enter the scene.

The objects defined in Section 1 have obvious (homo-)morphisms, sending cells to cells of the same dimension and preserving the operators. Here the term 'preservation' takes its usual sense (meaning "on the nose"). The morphisms of globular sets are also called *globular maps*, the morphisms of ∞-categories are also called ∞-*functors*, and so on. Terminology and notation for morphisms follows that for objects as far as possible.

To be explicit in at least one case, here is what we mean by a globular map $\Phi : \mathfrak{G} \longrightarrow \mathfrak{H}$: it consists of maps $\Phi_n : \mathfrak{G}_n \longrightarrow \mathfrak{H}_n$, written $A \longmapsto A\Phi$, that satisfy the equations $A\Phi\delta_\eta = A\delta_\eta\Phi$. Just as we can construct a globular set $\mathfrak{G}$ inductively by specifying its filler sets $\mathfrak{G}_a$, we can construct a globular map $\Phi : \mathfrak{G} \longrightarrow \mathfrak{H}$ inductively by specifying its filler maps $\Phi_a : \mathfrak{G}_a \longrightarrow \mathfrak{H}_{a\Phi}$. (Note that an $(N-1)$-dimensional globular map sends N-frames to N-frames.)

There is an even more obvious way to obtain identity morphisms and composite morphisms. As a result, we have a notion of *isomorphism*. The identity for an object is denoted in the same way as the object itself. The composite of two morphisms is denoted by writing them next to each other, first the one to be carried out first, second the one to be carried out second.

There are two sensible concepts of an N-category being freely generated by a pre-N-category $\mathfrak{k}$. We call an N-category $\mathfrak{K}$ and a pre-N-functor $e : \mathfrak{k} \longrightarrow \mathfrak{K}$ *universal* if for every N-category $\mathfrak{L}$ and every pre-N-functor $f : \mathfrak{k} \longrightarrow \mathfrak{L}$, there is a unique N-functor $F : \mathfrak{K} \longrightarrow \mathfrak{L}$ such that

$$f = eF. \tag{2}$$

$$\begin{tikzcd}
\mathfrak{k} \arrow[r, "e"] \arrow[dr, "f"'] & \mathfrak{K} \arrow[d, dashed, "F"] \\
& \mathfrak{L}
\end{tikzcd}$$

Now write $\mathfrak{O}$ for the $(N-1)$-skeleton of $\mathfrak{k}$. In order to avoid overuse of the word 'over', let us agree that the prefix 'N-' and the qualifier 'over $\mathfrak{O}$' together can be replaced by the prefix '$\mathfrak{O}$-'. Thus, $\mathfrak{k}$ is a pre-$\mathfrak{O}$-category. We call an $\mathfrak{O}$-category $\mathfrak{K}$ and a pre-$\mathfrak{O}$-functor $e : \mathfrak{k} \longrightarrow \mathfrak{K}$ *universal over* $\mathfrak{O}$ if for every $\mathfrak{O}$-category $\mathfrak{L}$ and every pre-$\mathfrak{O}$-functor $f : \mathfrak{k} \longrightarrow \mathfrak{L}$, there is a unique $\mathfrak{O}$-functor $F : \mathfrak{K} \longrightarrow \mathfrak{L}$ such that (2). Of course we could have omitted the explicit requirement that F be over $\mathfrak{O}$, since it is implicit in (2).

The two concepts are equivalent in the following sense.

Proposition 3. *If $(\mathfrak{K}; e)$ is universal over $\mathfrak{O}$, then $(\mathfrak{K}; e)$ is universal.*

Proposition 4. *If $(\mathfrak{K}; e)$ is universal, then $e_{<N} : \mathfrak{k}_{<N} \longrightarrow \mathfrak{K}_{<N}$ is an isomorphism of $(N-1)$-categories. (Thus, when we identify $\mathfrak{K}_{<N}$ with $\mathfrak{O} = \mathfrak{k}_{<N}$ accordingly, $(\mathfrak{K}; e)$ becomes universal over $\mathfrak{O}$.)*

Readers familiar with category theory know that these statements are consequences of the fact that in the commutative triangle of categories and functors

$$\begin{tikzcd}
N\text{-categories} \arrow[rr] \arrow[dr] & & \text{pre-}N\text{-categories}\ , \arrow[dl] \\
& (N-1)\text{-categories} &
\end{tikzcd}$$

the two lateral ones are fibrations, and the top one preserves cartesian liftings. Of course this fact itself would still deserve a proof. We use a weaker fact instead, which suffices to obtain the intended conclusions in a categorical manner.

Lemma 5. *Let $g : \mathfrak{k} \to \mathfrak{M}$ be a pre-N-functor into an N-category $\mathfrak{M}$. There is a decomposition of g into a pre-$\mathfrak{O}$-functor $f : \mathfrak{k} \to \mathfrak{L}$ and an N-functor $H : \mathfrak{L} \to \mathfrak{M}$ with the following universal property. For any decomposition of g into a pre-$\mathfrak{O}$-functor $f' : \mathfrak{k} \to \mathfrak{L}'$ and an N-functor $H' : \mathfrak{L}' \to \mathfrak{M}$ there is a unique $\mathfrak{O}$-functor $D : \mathfrak{L}' \to \mathfrak{L}$ such that $f'D = f$ and $DH = H'$.*

$$\begin{tikzcd}
& \mathfrak{L}' \arrow[d, dashed, "D"'] \arrow[ddr, "H'"] & \\
& \mathfrak{L} \arrow[dr, "H"] & \\
\mathfrak{k} \arrow[uur, "f'"] \arrow[ur, "f"] \arrow[rr, "g"'] & & \mathfrak{M}
\end{tikzcd}$$

Again we could have omitted the explicit requirement that the fill-in N-functor (in this case D) be over $\mathfrak{O}$.

Proof. The straight-forward translation of the conditions into constructions is left to the reader. The only slightly difficult step is the specification of the set $\mathfrak{L}_N$ and the map $H_N : \mathfrak{L}_N \to \mathfrak{M}_N$. We demand (and have to demand) that for each N-frame a of $\mathfrak{L}_{<N} = \mathfrak{k}_{<N}$, the filler map $H_a : \mathfrak{L}_a \to \mathfrak{M}_{aH = ag}$ be bijective. □

Proof of Proposition 3. Suppose $(\mathfrak{K}; e)$ is universal over $\mathfrak{O}$. Let $\mathfrak{M}$ be an arbitrary N-category, and let $g : \mathfrak{k} \to \mathfrak{L}$ be a pre-N-functor. We pick a decomposition $g = fH$ as in the lemma. By universality of $(\mathfrak{K}; e)$, there is a unique $\mathfrak{O}$-functor $F : \mathfrak{K} \to \mathfrak{L}$ such that $eF = f$. Now $G = FH$ is an N-functor $\mathfrak{K} \to \mathfrak{M}$ such that $eG = g$.

$$\begin{tikzcd}
\mathfrak{k} \arrow[rr, "e"] \arrow[dr, "f"] \arrow[ddrr, "g"'] & & \mathfrak{K} \arrow[dl, "F"'] \arrow[dd, "G"] \\
& \mathfrak{L} \arrow[dr, "H"] & \\
& & \mathfrak{M}
\end{tikzcd}$$

Conversely, suppose that $G' : \mathfrak{K} \to \mathfrak{M}$ is an N-functor such that $eG' = g$. By universality of the decomposition, there is an $\mathfrak{O}$-functor $F' : \mathfrak{K} \to \mathfrak{L}$ such that $eF' = f$ and $F'H = G$. Of these two equations, the former yields $F' = F$ by uniqueness of F, whence the latter yields $G' = G$. □

Proof of Proposition 4. Suppose $(\mathfrak{K}; e)$ is universal. We pick a decomposition of e into $e^0 : \mathfrak{k} \to \mathfrak{K}^0$ and $E : \mathfrak{K}^0 \to \mathfrak{K}$ as in the lemma. By universality of $\mathfrak{K}$, there is an N-functor $E^0 : \mathfrak{K} \to \mathfrak{K}^0$ such that $eE^0 = e^0$.

$$\begin{tikzcd} \mathfrak{k} \arrow[r, "e"] \arrow[dr, "e^0"'] & \mathfrak{K} \arrow[d, shift left, "E^0"] \\ & \mathfrak{K}^0 \arrow[u, shift left, "E"] \end{tikzcd}$$

Now $eE^0E = e^0E = e$, whence by universality of e the endomorphism E^0E of $\mathfrak{K}$ is the identity. It follows that $E^0EE^0 = E^0$. Also $e^0EE^0 = eE^0 = e^0$,

$$\begin{tikzcd} & \mathfrak{K}^0 \arrow[d, shift right, "\mathfrak{K}^0"'] \arrow[d, shift left, "EE^0"] \arrow[ddr, "E"] & \\ & \mathfrak{K}^0 \arrow[dr, "E"] & \\ \mathfrak{k} \arrow[uur, "e^0"] \arrow[ur, "e^0"] \arrow[rr, "e"'] & & \mathfrak{K} \end{tikzcd}$$

whence by universality of the decomposition the endomorphism EE^0 of $\mathfrak{K}^0$ is the identity as well. Thus E and E^0 are isomorphisms. Applying this result to the $(N-1)$-skeleton of the situation, we find that $e_{<N} = E_{<N}$ is also an isomorphism. (It is furthermore clear that $(\mathfrak{K}^0; e^0)$ is universal over $\mathfrak{O}$.) $\square$

Now let us view things from the perspective of a given $\mathfrak{O}$-category $\mathfrak{K}$. Providing a pre-$\mathfrak{O}$-functor $e : \mathfrak{k} \to \mathfrak{K}$ reduces to providing a map $e_N : \mathfrak{k}_N \to \mathfrak{K}_N$. Indeed, the boundaries for the N-cells of $\mathfrak{k}$ are determined by the homomorphy requirement. If e_N is moreover injective, we can identify $\mathfrak{k}_N$ with a subset of $\mathfrak{K}_N$. The following result shows that the assumption is satisfied if $(\mathfrak{K}; e)$ is universal (generally or over $\mathfrak{O}$).

Proposition 6. *Let $(\mathfrak{K}; e)$ be universal over $\mathfrak{O}$. Then $e_N : \mathfrak{k}_N \to \mathfrak{K}_N$ is injective.*

Proof. We construct an $\mathfrak{O}$-category $\mathfrak{O}[\mathbf{N}]$ as follows. We let the filler set of each N-frame of $\mathfrak{O}$ be $\mathbf{N}$, the set of natural numbers. In dimension N, we let every identity be zero and every composite be the sum of the operands. The equations of an ∞-category clearly hold. (An analogous construction can be carried out for any commutative monoid in place of $\mathbf{N}$.)

Now let a be an N-frame in $\mathfrak{O}$, and let A_0, $A_1 \in \mathfrak{k}_a$ such that $A_0e = A_1e$. We define a pre-$\mathfrak{O}$-functor $\chi^{(A_1)} : \mathfrak{k} \to \mathfrak{O}[\mathbf{N}]$ by sending A_1 to $1 \in \mathfrak{O}[\mathbf{N}]_a$ and all other elements $A \in \mathfrak{k}_N$ to 0. By universality of e, there is an $\mathfrak{O}$-functor $X^{(A_1)} : \mathfrak{K} \to \mathfrak{O}[\mathbf{N}]$ such that $eX^{(A_1)} = \chi^{(A_1)}$. Now $A_0\chi^{(A_1)} = A_0eX^{(A_1)} = A_1eX^{(A_1)} = A_1\chi^{(A_1)} = 1$, whence $A_0 = A_1$. $\square$

We say that a subset of $\mathfrak{K}_N$ *freely generates $\mathfrak{K}$ over $\mathfrak{O}$* if $\mathfrak{K}$ together with the pre-$\mathfrak{O}$-functor arising from the inclusion is universal over $\mathfrak{O}$. We can now submit this section to its main purpose.

Definition. Let $\mathfrak{K}$ be an ∞-category, and let $\mathfrak{k}$ be a subset of $\mathfrak{K}$. We say that $\mathfrak{k}$ *freely generates* $\mathfrak{K}$ if for each $n \geq 0$ the subset $\mathfrak{k}_n$ of $\mathfrak{K}_n$ freely generates $\mathfrak{K}_{\leq n}$ over $\mathfrak{K}_{<n}$.

Section 4. Factoriality and Exact Size Functions

We prove an important lemma, called Proposition 8, with the help of which we can recognize certain ∞-categories as being freely generated by a semiglobular subset.

Throughout this section, we fix an ∞-category $\mathfrak{K}$. For convenience we put $A\delta^0_\eta = A$. (Note, however, that the generalized globular law $A\delta^u_\eta\delta^v_\zeta = A\delta^{u+v}_\zeta$ cannot be extended to the case $v = 0$.)

We call $\mathfrak{K}$ *(positively) factorial* if for each of its cells C the following two conditions are satisfied. Let us write $C\delta_+ = C'$.

(i) If $C' = C'\delta\sigma$, then $C = C\delta^2\sigma^2$.

(ii) Whenever $C' = A' \cdot^v B'$, there are unique cells A and B such that $A\delta_+ = A'$, $B\delta_+ = B'$ and $A \cdot^{v+1} B = C$.

By induction on u we can infer, respectively, the following more general conditions. Here we write $C\delta^u_+ = C'$.

(i)* If $C' = C'\delta\sigma$, then $C = C\delta^{u+1}\sigma^{u+1}$.

(ii)* Whenever $C' = A' \cdot^v B'$, there are unique cells A and B such that $A\delta^u_+ = A'$, $B\delta^u_+ = B'$ and $A \cdot^{u+v} B = C$.

A *size function* on $\mathfrak{K}$ assigns to each cell A a natural number $|A|$, called the *size* of A, such that the following two conditions are satisfied.

(iii) For each cell A', we have $|A'\sigma| = 0$.

(iv) Whenever $A \circ^u B$, we have $|A \cdot^u B| = |A| + |B|$.

Thus a size function provides for each natural number n an n-functor $\mathfrak{K}_{\leq n} \to \mathfrak{K}_{<n}[\mathbf{N}]$ (see the proof of Proposition 6) over $\mathfrak{K}_{<n}$.

In presence of a size function, we may proof facts on n-cells A by $(n+1)$-fold induction: first on $|A|$, then on $|A\delta_+|$, and so on, and finally on $|A\delta^n_+|$. We refer to this proof method as *δ_+-induction on A*. To be more systematic, we may express this method as induction on the set $\mathfrak{K}_n$ with respect to the (strict) well-founding order $\prec$ according to which

$$A \prec B \quad \text{if and only if} \quad \langle |A|, |A\delta_+|, \ldots, |A\delta^n_+|\rangle <_{\text{lex}} \langle |B|, |B\delta_+|, \ldots, |B\delta^n_+|\rangle,$$

where $<_{\text{lex}}$ stands for (strict) lexicographical order. Note that if $C = A \cdot^u B$, then existence of a natural number $i < u$ with $|A\delta^i_+| < |C\delta^i_+|$ (or, equivalently, $|B\delta^i_+| > 0$) implies $A \prec C$, and similarly existence of a natural number $j < u$ with $|B\delta^j_+| < |C\delta^j_+|$ (or, equivalently, $|A\delta^j_+| > 0$) implies $B \prec C$.

We keep considering a size function on $\mathfrak{K}$. We call an n-cell A of $\mathfrak{K}$ *(positively) simple* if $|A|$, $|A\delta_+|$, ... , $|A\delta^n_+| = 1$. Note that the simple cells form a semiglobular subset of $\mathfrak{K}$. The size function will be called *(positively) exact* if for every cell A such that $A\delta_+$ is simple, the following two conditions are satisfied.

(v) If $|A| = 0$, then $A = A\delta\sigma$.

(vi) If $|A| > 1$, then there are unique cells A_+ and A_0 such that A_0 is simple and $A = A_+ \cdot A_0$.

(These statements imply that every 0-cell is simple.) If $\mathfrak{K}$ is also factorial we can infer, respectively, that for every $u > 0$ and for every cell A such that $A\delta^u_+$ is simple, the following more general conditions are satisfied.

(v)* If $|A\delta^{u-1}_+| = 0$, then $A = A\delta^u\sigma^u$.

(vi)* If $|A\delta^{u-1}_+| > 1$, then there are unique cells A_+ and A_0 such that $A_0\delta^{u-1}_+$ is simple and $A = A_+ \cdot^u A_0$.

We refer to the cells A_+ and A_0 of condition (vi)* as forming the *principal decomposition* of A. Note that A_+, $A_0 \prec A$. Note also that the codimension u that gives rise to the principal decomposition does indeed depend only on A. In general, the minimal number u for which $A\delta_+^u$ is simple will be called the *complexity* of A. Thus a cell is simple if and only if its complexity is 0.

In order to deal conveniently with degeneracies in the main lemma, we prove the following preliminary lemma. In the application we have in mind, its conclusion is evidently satisfied.

Lemma 7. *Let $\mathfrak{K}$ be a factorial strict ∞-category with an exact size function, and let A be a cell in $\mathfrak{K}$. If $|A| = 0$, then $A = A\delta\sigma$.*

As a consequence, we have $A = A\delta^u\sigma^u$ if and only if $|A\delta_+^{u-1}| = 0$. No simplicity assumption is needed.

Proof. Suppose $|A| = 0$; we want to show that $A = A'\sigma$ for some cell A'. We do so by δ_+-induction on A. The complexity k of A is clearly > 0. If $|A\delta_+^{k-1}| = 0$, then $A = A\delta^k\sigma^k$, in particular $A = A\delta\sigma$. So let us suppose that $|A\delta_+^{k-1}| > 1$, whence $k > 1$.

Let $A = A_+ \cdot^k A_0$ be the principal decomposition. Then $|A_+| + |A_0| = |A| = 0$ and hence $|A_+|$, $|A_0| = 0$. By induction hypothesis, we have $A_+ = A'_+\sigma$ and $A_0 = A'_0\sigma$, say. From $A_+ \circ^k A_0$ we infer $A'_+ \circ^{k-1} A'_0$ and then put $A' = A'_+ \cdot^{k-1} A'_0$. Now $A = A'_+\sigma \cdot^k A'_0\sigma = A'\sigma$. □

If A is simple, we can put $A_+ = A\delta_-\sigma$ and $A_0 = A$, thus obtaining a decomposition $A_+ \cdot A_0$ of A that looks very much like the one in condition (vi). In fact, — supposing exactness and factoriality — Lemma 7 implies that it is the only decomposition $A_+ \cdot A_0$ of A with A_0 being simple. This observation yields a slightly generalized version of property (vi)*: whenever $A\delta_+^u$ is simple and $|A\delta_+^{u-1}| \geq 1$, there are unique cells A_+ and A_0 such that $A_0\delta_+^{u-1}$ is simple and $A = A_+ \cdot^u A_0$.

We have arrived at the main lemma itself.

Proposition 8. *A factorial strict ∞-category with an exact size function is freely generated by the simple cells.*

Proof. Fix $n \geq 0$. We write $\mathfrak{k}_n$ for the set of simple n-cells of $\mathfrak{K}$, which gives rise to a pre-n-category $\mathfrak{k}$ over $\mathfrak{K}_{<n}$. Let φ be a pre-n-functor of $\mathfrak{k}$ into an n-category $\mathfrak{L}$. We show that there is a unique n-functor Φ extending φ to all of $\mathfrak{K}_{\leq n}$. (We directly show the stronger of the two universal properties because it makes the algebraic manipulations clearer — though slightly longer — without causing actual extra work.)

First we put $\Phi_{<n} = \varphi_{<n}$. Then we consider an n-cell A of $\mathfrak{K}$. By δ_+-induction on A we define an n-cell $A\Phi$ of $\mathfrak{L}$ with $A\Phi\delta_\eta = A\delta_\eta\Phi$ as follows. If A is simple, we put $A\Phi = A\varphi$. Otherwise A has complexity $u > 0$. If $A = A\delta^u\sigma^u = A\delta\sigma$, we put $A\Phi = A\delta^u\Phi\sigma^u = A\delta\Phi\sigma$. If A has a principal decomposition $A_+ \cdot^u A_0$, we use the induction hypothesis in order to put $A\Phi = A_+\Phi \cdot^u A_0\Phi$. The reader will find no difficulty in verifying the condition on the boundary of $A\Phi$.

We have thus defined a pre-n-functor $\Phi : \mathfrak{K} \twoheadrightarrow \mathfrak{L}$. Moreover every n-functor $\mathfrak{K} \twoheadrightarrow \mathfrak{L}$ extending $\varphi : \mathfrak{k} \twoheadrightarrow \mathfrak{L}$ clearly has to agree with Φ, showing that the uniqueness part of the universal property is satisfied. All that is left to be verified is that Φ is indeed an n-functor.

Let $C = C'\sigma \in \mathfrak{K}_n$; we want to show that $C\Phi = C'\Phi\sigma$. We do so by δ_+-induction on C. The complexity k of C is clearly > 0. If $C = C\delta^k\sigma^k$, then we obtain directly $C\Phi = C\delta\Phi\sigma = C'\sigma\delta\Phi\sigma = C'\Phi\sigma$. Otherwise $k > 1$, and C has a principal decomposition $C_+ \cdot^k C_0$. Let us put $C'_i = C_i\delta_+$ $(i \in \{0,+\})$. Then $C' = C\delta_+ = C'_+ \cdot^{k-1} C'_0$ and hence $C = C'_+\sigma \cdot^k C'_0\sigma$, where $C'_0\sigma\delta_+^{k-1} = C_0\delta_+^{k-1}$ is simple. Since the principal decomposition is unique, we can conclude that $C_i = C'_i\sigma$ $(i \in \{0,+\})$. (This argument can be simplified slightly by using Lemma 7.) Now we obtain

$$\begin{aligned}
C\Phi &= C_+\Phi \cdot^k C_0\Phi && \text{(definition of } C\Phi\text{)}\\
&= C'_+\sigma\Phi \cdot^k C'_0\sigma\Phi\\
&= C'_+\Phi\sigma \cdot^k C'_0\Phi\sigma && \text{(induction hypothesis)}\\
&= (C'_+\Phi \cdot^{k-1} C'_0\Phi)\sigma\\
&= (C'_+ \cdot^{k-1} C'_0)\Phi\sigma\\
&= C'\Phi\sigma.
\end{aligned}$$

Let $C = A \cdot^u B \in \mathfrak{K}_n$; we want to show that $C\Phi = A\Phi \cdot^u B\Phi$. We do so by δ_+-induction on C. The case that either $A = A\delta^u\sigma^u = B\delta_-^u\sigma^u$ or $B = B\delta^u\sigma^u = A\delta_+^u\sigma^u$ is easy and left to the reader. We may henceforth suppose that neither A nor B is a u-identity. Lemma 7 allows us to conclude that $|A\delta^{u-1}|$, $|B\delta^{u-1}| \geq 1$, whence we further have $|C\delta^{u-1}| \geq 2$. The complexity k of C is consequently $\geq u$. If we had $C = C\delta^k\sigma^k$, then $C\delta_+^{u-1} = C\delta^u\sigma$, and hence $|C\delta^{u-1}| = 0$, contradicting a previous statement. Thus C has a principal decomposition $C_+ \cdot^k C_0$.

Note that $B\delta_+^k = C\delta_+^k$ is simple. If we had $|B\delta_+^{k-1}| = 0$, then $B = B\delta^k\sigma^k$ by Lemma 7, and in particular $B = B\delta^u\sigma^u$, contradicting our assumption. We therefore have $|B\delta_+^{k-1}| \geq 1$, whence there is some decomposition $B_+ \cdot^k B_0$ of B with $B_0\delta^{k-1}$ being simple.

First consider the case $k = u$. Here $C = A \cdot^u B_+ \cdot^u B_0$, and since $B_0\delta^{u-1}$ is simple, uniqueness of the principal decomposition yields $C_0 = B_0$ and $C_+ = A \cdot^u B_+$. We conclude that

$$\begin{aligned}
C\Phi &= C_+\Phi \cdot^u C_0\Phi && \text{(definition of } C\Phi\text{)}\\
&= (A\Phi \cdot^u B_+\Phi) \cdot^u C_0\Phi && \text{(induction hypothesis for } C_+\text{)}\\
&= A\Phi \cdot^u (B_+\Phi \cdot^u B_0\Phi)\\
&= A\Phi \cdot^u B\Phi && \text{(induction hypothesis for } B\text{).}
\end{aligned}$$

Now consider the case $k > u$. Note that $A\delta_+^u = B\delta_-^u = B_+\delta_-^u \cdot^{k-u} B_0\delta_-^u$, whence by factoriality we have $A = A_+ \cdot^k A_0$ with $A_i\delta_+^u = B_i\delta_-^u$ $(i \in \{0,+\})$. It follows that $C = (A_+ \cdot^k A_0) \cdot^u (B_+ \cdot^k B_0) = (A_+ \cdot^u B_+) \cdot^k (A_0 \cdot^u B_0)$. Since $(A_0 \cdot^u B_0)\delta_+^{k-1} = B_0\delta_+^{k-1}$ is simple, uniqueness of the principal decomposition yields $C_0 = A_0 \cdot^u B_0$ and $C_+ = A_+ \cdot^u B_+$. We conclude that

$$\begin{aligned}
C\Phi &= C_+\Phi \cdot^k C_0\Phi && \text{(definition of } C\Phi\text{)}\\
&= (A_+\Phi \cdot^u B_+\Phi) \cdot^k (A_0\Phi \cdot^u B_0\Phi) && \text{(induction hypothesis for } C_+ \text{ and } C_0\text{)}\\
&= (A_+\Phi \cdot^k A_0\Phi) \cdot^u (B_+\Phi \cdot^k B_0\Phi)\\
&= A\Phi \cdot^u B\Phi && \text{(induction hypothesis for } A \text{ and } B\text{).} \quad \square
\end{aligned}$$

Example. The ∞-category $\hat{\mathbf{N}}$ of higher-dimensional numbers is factorial. It can be equipped with an exact size function, namely by putting

$$|0_0| = 1; \qquad |\langle \mathfrak{q}_0, \ldots, \mathfrak{q}_{\ell-1} \rangle_n| = |\mathfrak{q}_0| + \cdots + |\mathfrak{q}_{\ell-1}| \quad (n > 0).$$

There is precisely one simple cell $\mathfrak{e}_n$ in each dimension n. We can define $\mathfrak{e}_n$ inductively by putting

$$\mathfrak{e}_0 = 0_0; \qquad \mathfrak{e}_n = \langle \mathfrak{e}_{n-1} \rangle_n \quad (n > 0).$$

Since δ_+ and δ_- act in the same way, factoriality and exactness hold not only positively, but also negatively (in the obvious sense), and the simple cells are the same in both instances.

Section 5. The Makkai Equivalence

We show that dendrotopic sets are (essentially) precisely the freely generating semiglobular subsets of strict ∞-categories. Thus, according to [3], the concept of a dendrotopic set introduced here is equivalent to the Makkai concept of a multitopic set.

We are going to show that the ∞-category $\mathfrak{D}^*$ generated by a dendrotopic set $\mathfrak{D}$ meets the assumptions of Proposition 8, with the cells of $\mathfrak{D}$ playing the role of the simple cells.

We first show that $\mathfrak{D}^*$ is factorial. To this end, let C be a cell of $\mathfrak{D}^*$, and let $C\delta_+ = C'$. In terms of frameworks of $\mathfrak{D}$, this means that $C^\natural \partial_+ = C'^\natural$.

(i) Suppose that $C' = C'\delta\sigma$; we want to show that $C = C'\sigma$. Since $C'^\natural = C'\delta\sigma^\natural$ has no complete facet, there is precisely one special framework with ∂_+-facet $C'^\natural$. Since $C^\natural \partial_+ = C'^\natural$ as well as $C'\sigma^\natural \partial_+ = C'^\natural$, we can conclude that $C^\natural = C'\sigma^\natural$.

(ii) Let $C' = A' \cdot^v B'$; we want to show that there uniquely exist two cells A and B such that $A\delta_+ = A'$, $B\delta_+ = B'$ and $C = A \cdot^{v+1} B$. Since $C'^\natural = (A' \cdot^v B')^\natural$, the complete facets of $C'^\natural$ are the complete facets of $A'^\natural$ and $B'^\natural$. Assuming $A^\natural \partial_+ = A'^\natural$ and $B^\natural \partial_+ = B'^\natural$, the frameworks $A^\natural$ and $B^\natural$ are determined by the cell trees $A^\natural \alpha_{(q)}$ $(q \in A'^\natural \mathrm{I}_{\mathfrak{D}})$ and $B^\natural \alpha_{(q)}$ $(q \in B'^\natural \mathrm{I}_{\mathfrak{D}})$, and assuming further $C^\natural = (A \cdot^{v+1} B)^\natural$, these cell trees are given by $A^\natural \alpha_{(q)} = C^\natural \alpha_{(q)}$ and $B^\natural \alpha_{(q)} = C^\natural \alpha_{(q)}$, respectively. Along the same lines we find a construction of $A^\natural$ and $B^\natural$.

Now we want to put an exact size function on $\mathfrak{D}^*$. To this end, we let $|A|$ be the number of complete facets of $A^\natural$. The homomorphy conditions (iii) and (iv) are clearly satisfied. Before we proceed to show exactness, we want to understand the structure of the critical cells.

Let A be a simple n-cell of $\mathfrak{D}^*$. The corresponding roof $A^\natural$ has precisely one complete facet, whose location we shall denote by ι. If $n = 0$, then $A^\natural : \iota = \delta_-$. Suppose now that $n > 0$. Then $A^\natural : \frac{\iota \ \omega}{\partial_+ \ *}$ is complete, and since $A\delta_+$ itself is simple, the yet unknown complete facet location is the respective ι:

$$A^\natural : \frac{\iota \quad \omega}{\partial_+ \quad \iota}. \tag{3}$$

(Hence, the complete facet location of the roof corresponding to a simple n-cell is $\mathfrak{e}_n$; see the Example in the previous section.) As a consequence we have the identity

$$A\delta_+{}^\natural \iota = A^\natural \iota\omega, \tag{4}$$

which states that the assignment $A \mapsto A^\natural \iota$ preserves positive facets. The complete ridge locations of $A^\natural$ apart from (3) have to be of the form $\frac{\partial_- \; *}{\iota \;\; *}$. Thus $A^\natural \diamondsuit$ induces a one-to-one correspondence between the complete facet locations of $A^\natural \partial_- = A\delta_-{}^\natural$ on the one hand and the negative facet locations of $A^\natural \iota$ on the other hand. We express this correspondence by an identification according to the rule

$$A^\natural : \frac{\partial_- \;\; q}{\iota \;\;\; q}.$$

If the dimension of A is 1, we thus identify the two facet locations already both known as δ_-. Suppose now that the dimension of A is > 1. By removing the type-2 panels with first entry ∂_- or ι, each complete 3-face location of $A^\natural$ can be decomposed into an "upper" gallery

$$\begin{array}{ccc} \partial_- & \partial_- & r \\ \cline{1-2} \partial_+ & \partial_- & r \\ \cline{2-3} \partial_+ & \iota & r \\ \cline{1-2} \iota & \omega & r \end{array} \qquad \big(r \in A\delta_+{}^\natural \partial_- \mathrm{I}_{\mathfrak{D}} = A\delta_+{}^\natural \iota \mathrm{I}_-\big) \qquad \text{or}$$

$$\begin{array}{ccc} \partial_- & q & \omega \\ \cline{1-2} \iota & q & \omega \end{array} \qquad \big(q \in A^\natural \partial_- \mathrm{I}_{\mathfrak{D}} = A^\natural \iota \mathrm{I}_-\big)$$

and a "lower" gallery

$$\begin{array}{ccc} \iota & \omega & \omega \\ \cline{1-3} \partial_+ & \iota & \omega \\ \cline{2-3} \partial_+ & \partial_+ & \iota \\ \cline{1-2} \partial_- & \partial_+ & \iota \end{array} \qquad \text{or} \qquad \begin{array}{ccc} \iota & q & r \\ \cline{1-2} \partial_- & q & r \end{array} \qquad \big(q \in A^\natural \partial_- \mathrm{I}_{\mathfrak{D}} = A^\natural \iota \mathrm{I}_-;\; r \neq \omega\big).$$

Thus $A^\natural \Gamma^3$ induces a one-to-one correspondence between the complete ridge locations of $A^\natural \partial_- = A\delta_-{}^\natural$ on the one hand and the ridge locations of $A^\natural \iota$ on the other hand. The best way to put this result is the following. First note that since A is simple, so is $A\delta_+^2 = A\delta_-\delta_+$. Therefore the complete part of the boundary of $A\delta_-{}^\natural$ makes up precisely the $(n-1)$-cell tree $A\delta_-{}^\natural \alpha_{(\iota)}$. What we have shown here is that this cell tree is precisely the negative half boundary of $A^\natural \iota$:

$$A\delta_-{}^\natural \alpha_{(\iota)} = A^\natural \iota \alpha. \tag{5}$$

The assignment $A \mapsto A^\natural \iota$ is a bijection of the set of simple n-cells of $\mathfrak{D}^*$ onto the set of n-cells of $\mathfrak{D}$. We prove this statement by induction on n. Let P be an n-cell in $\mathfrak{D}$; we want to show that there is a unique simple n-cell A in $\mathfrak{D}^*$ such that $A^\natural \iota = P$. Our reasoning for uniqueness will also produce a construction. If $n = 0$, then the cell A is determined by $A^\natural \iota = A^\natural \delta_-$, which is required to be P. Now suppose that $n > 0$. Then the cell A is determined by $A^\natural \iota$ and $A\delta_+$. The former is required to be P; the latter has to be a simple $(n-1)$-cell A' such that $A'^\natural \iota = A^\natural \partial_+ \iota = P\omega$. By induction hypothesis, there is only one such cell A'.

We now want to understand the structure of those cells of $\mathfrak{D}^*$ whose positive facets are simple. Let A be such a cell. Then $A\delta_+{}^\natural$ has precisely one complete facet location ι. Hence A determines and is determined by the cell tree $A^\natural \alpha_{(\iota)}$, which we shall abbreviate by $A^\natural \iota$ as well. If A itself is simple, then the cell $A^\natural \iota$ defined earlier can be viewed as a one-facet cell tree, which is precisely the cell tree $A^\natural \iota$ defined here. For all $n \geq 0$, the assignment $A \mapsto A^\natural \iota$ is evidently a bijection of the set of n-cells of $\mathfrak{D}^*$ with complexity ≤ 1 onto the set of n-cell trees of $\mathfrak{D}$.

Using the new notation, we may rewrite the identity (5) as follows:

$$A\delta_-{}^\natural\iota = A^\natural\iota\alpha. \tag{6}$$

This result also holds if the dimension of A is 1, since then $A^\natural : \frac{\partial_-\ \delta_-}{\iota\ \ \delta_-}$. We now want to generalize the identities (4) and (6) by proving them for all cells A of dimension ≥ 1 and complexity ≤ 1 (not just the simple ones). The anticipated result can also be phrased as follows.

Proposition 9. *The assignment $A \mapsto A^\natural\iota$ is an isomorphism of the globular subset of $\mathfrak{D}^*$ constituted by cells of complexity ≤ 1 onto the globular set $\mathfrak{D}^{1*}$ of cell trees of $\mathfrak{D}$.*

Proof. We have to show that in the case in question, the two equations (4) and (6) are satisfied. As for the first, we find on both sides the root of the cell tree $A^\natural\alpha_{(\iota)}$. As for the second, we take recourse to a trick, to avoid the tiresome fiddling with 3-face locations.

The problem remains unchanged if new cells are added to $\mathfrak{D}$. We may hence work with the $(n+1)$-outniche X corresponding to the n-cell tree $A^\natural\iota$ as if it were an actual cell of $\mathfrak{D}$. Thus, without loss of generality we pick an $(n+1)$-cell X with $X\alpha = A^\natural\iota$. There is a corresponding simple cell F in $\mathfrak{D}^*$. The following calculations will make use of the results (4) and (6) for the simple cells F and $F\delta_+$. First note that since

$$F\delta_-{}^\natural\iota = F^\natural\iota\alpha = X\alpha = A^\natural\iota,$$

we have $F\delta_- = A$. We conclude that

$$A\delta_-{}^\natural\iota = F\delta_-\delta_-{}^\natural\iota = F\delta_+\delta_-{}^\natural\iota = F\delta_+{}^\natural\iota\alpha = F^\natural\iota\omega\alpha = F^\natural\iota\alpha\alpha = F\delta_-{}^\natural\iota\alpha = A^\natural\iota\alpha.$$

□

We henceforth view $\mathfrak{D}^{1*}$ as a globular subset of $\mathfrak{D}^*$ as suggested by the proposition. Given the earlier identification of dendrotopic cells with one-facet cell trees, we view in particular $\mathfrak{D}$ as a semiglobular subset of $\mathfrak{D}^*$.

We now show that the given size function is exact. To this end, let A be a cell of $\mathfrak{D}^*$ for which $A\delta_+$ is simple.

(v) Suppose that $|A| = 0$; we want to show that $A = A\delta\sigma$. The result follows from the fact that both $A^\natural$ and $A\delta_+\sigma^\natural$ have the ∂_+-facet $A\delta_+{}^\natural$ and no complete facet.

(vi) Suppose that $|A| > 1$; we want to show that there uniquely exist two cells A_+ and A_0 such that A_0 is simple and $A = A_+ \cdot A_0$. We show uniqueness by assuming the conditions and then expressing A_0 and A_+ in terms of A. The complete facet location ι of $A_0{}^\natural$ corresponds to a facet location ι' of $A^\natural$ determined by the requirement that

$$A^\natural : \frac{\iota'\ \ \omega}{\partial_+\ \ \iota}. \tag{7}$$

In turn A_0 is determined by $A_0{}^\natural\iota = A^\natural\iota'$. Now $A_+{}^\natural\partial_+ = A_0{}^\natural\partial_-$, and the complete facets of $A_+{}^\natural$ are those of $A^\natural$ except the one with location ι'. The complete ridge locations of $A_+{}^\natural$ are the ones of $A^\natural$ except (7) and except for the replacement of each $\frac{p_-\ q_-}{\iota'\ \ q}$ with $\frac{p_-\ q_-}{\partial_+\ q}$ (corresponding to a ridge location $\frac{\partial_-\ q}{\iota\ \ q}$ of $A_0{}^\natural$). Thus A_+ is also determined. To show existence, we first note that since $|A| > 1$, there is in particular some complete facet location ι' of $A^\natural$ such that (7). Now we construct A_0 and A_+ along the lines of the uniqueness proof and convince ourselves that $A_+{}^\natural$ has the tree property.

We have arrived at the main theorem of this work.

Theorem 10. *Let $\mathfrak{K}$ be a strict ∞-category, and let $\mathfrak{k}$ be a subset of $\mathfrak{K}$. The following three conditions are equivalent.*

(i) *$\mathfrak{k}$ is semiglobular and freely generates $\mathfrak{K}$.*

(ii) *$\mathfrak{K}$ is factorial and carries an exact size function with respect to which $\mathfrak{k}$ is the set of simple cells.*

(iii) *$(\mathfrak{K};\mathfrak{k})$ is isomorphic to $(\mathfrak{D}^*;\mathfrak{D})$ for a dendrotopic set $\mathfrak{D}$.*

As for condition (iii), the notion of an isomorphism is the obvious one: an invertible ∞-functor for the ∞-categories inducing a bijective mapping for the distinguished subsets.

Proof. The implication of (i) by (ii) is Proposition 8. The implication of (ii) by (iii) has just been demonstrated. We hence know that every dendrotopic set generates a free ∞-category. It remains to show the implication of (iii) by (i).

So suppose that $\mathfrak{k}$ is semiglobular and freely generates $\mathfrak{K}$. We want to construct a dendrotopic set $\mathfrak{D}$ and an isomorphism $\Phi : (\mathfrak{K};\mathfrak{k}) \twoheadrightarrow (\mathfrak{D}^*;\mathfrak{D})$. We do so by inductively constructing the skeleta $\mathfrak{D}_{\leq n}$ and $\Phi_{\leq n}$ of $\mathfrak{D}$ and Φ. So let us suppose that we have an isomorphism $\Phi_{<n} : (\mathfrak{K}_{<n};\mathfrak{k}_{<n}) \twoheadrightarrow (\mathfrak{D}^*{}_{<n};\mathfrak{D}_{<n})$ of $(n-1)$-categories with distinguished subsets. We let $\mathfrak{D}_n$ contain one copy $A\varphi$ of each $A \in \mathfrak{k}_n$. We now have to specify a boundary for each $A\varphi$ so that the assignment $A \mapsto A\varphi$ specifies a pre-n-functor φ over $\Phi_{<n}$. Then the desired isomorphism $\Phi_{\leq n}$ of N-categories arises due to the universal properties of the respective inclusions $\mathfrak{k}_n \subseteq \mathfrak{K}_n$ and $\mathfrak{D}_n \subseteq \mathfrak{D}^*{}_n$.

We explicitly treat the case $n \geq 2$; the constructions are even simpler in the cases $n = 1$ and $n = 0$. Since $\mathfrak{k}$ is semiglobular, $A\delta_+$ is in $\mathfrak{k}_{n-1}$, and is hence sent to an $(n-1)$-cell Q_+ in $\mathfrak{D}_{<n}$ by $\Phi_{<n}$. Similarly, $A\delta_-\delta_+ = A\delta_+\delta_+ \in \mathfrak{k}_{n-2}$, hence $A\delta_-\delta_+\Phi \in \mathfrak{D}_{n-2}$, and we conclude that $A\delta_-$ is sent to an $(n-1)$-cell tree Q_- in $\mathfrak{D}_{<n}$. Preservation of the globular operations in dimension $n-1$ ensures that $Q_+^\bullet = Q_-^\bullet$, so that we can form an n-frame P of $\mathfrak{D}_{<n}$ by putting $P\omega = Q_+$ and $P\alpha = Q_-$. We take this frame to be the boundary of $A\varphi$. Then φ clearly preserves the globular operations in dimension n. $\square$

The theorem can be extended to assert the equivalence of three (ordinary) categories:

(i) the category of ∞-categories with freely generating semiglobular subsets,

(ii) the category of factorial ∞-categories with exact size functions,

(iii) the category of dendrotopic sets.

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On Factorization Systems and Admissible Galois Structures

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Abstract. Full reflective subcategories of congruence-modular regular categories which are closed under subobjects and quotients are admissible in the sense of G. Janelidze's categorical Galois theory, with respect to regular epimorphisms.

1 Introduction

As defined in [3], a *Galois structure* Γ in a category $\mathbb{C}$ with finite limits, is an adjunction $(I, H, \eta, \varepsilon) : \mathbb{C} \to \mathbb{X}$ together with two classes of morphisms (called fibrations) F and Φ in $\mathbb{C}$ and $\mathbb{X}$, respectively, such that

1. F and Φ are pullback stable and pullbacks along morphisms of F or Φ exist,
2. F and Φ are closed under composition and contain all isomorphisms,
3. $I(F) \subseteq \Phi$ and $H(\Phi) \subseteq F$.

Γ is *admissible* if for every object C in $\mathbb{C}$ and every fibration $\phi : X \to I(C)$ in $\mathbb{X}$, the composite of canonical morphisms $I(C \times_{HI(C)} H(X)) \to IH(X) \to X$ is an isomorphism.

When the adjunction is a full reflection, admissibility is equivalent to preservation by I of pullbacks of the form

$$\begin{array}{ccc} C & \xrightarrow{t} & X \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \varphi\in\Phi} \\ B & \xrightarrow[\eta_B]{} & I(B) \end{array} \tag{1.1}$$

with X in $\mathbb{X}$, which coincides with semi-left exactness with respect to the class Φ.

In [4] it is shown that full reflective subcategories of congruence-modular exact categories closed under subobjects and quotients, are admissible with respect to regular epimorphisms. Here, we show that the same result holds when the

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exactness assumption is relaxed to regularity. In fact, we show that modularity of $\mathcal{E}-Quot(A)$ for each A in $\mathbb{C}$, implies admissibility of the Galois structure $(I, H, \eta, \varepsilon) : \mathbb{C} \to \mathbb{X}$, $(\mathcal{E}, \mathcal{E} \cap mor\mathbb{X})$ when $\mathbb{X}$ is a full reflective subcategory of $\mathbb{C}$, closed under $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients, $(\mathcal{E}, \mathcal{M})$ is a stable factorization system and $\mathcal{E}$ is contained in the class of epimorphisms.

We will use the following terminology and notation, for any factorization system $(\mathcal{E}, \mathcal{M})$ in a category $\mathbb{C}$ with finite limits:

- an $\mathcal{M}$-subobject of an object A in $\mathbb{C}$ is an isomorphism class of morphisms in $\mathcal{M}$ with codomain A;
- an $\mathcal{E}$-quotient of an object A in $\mathbb{C}$ is an isomorphism class of morphisms in $\mathcal{E}$ with domain A, the set of which we denote by $\mathcal{E}-Quot(A)$.

$\mathcal{E} - Quot(A)$ is preordered by $[s] \leq [r]$ if $s = t \circ r$ for some morphism t, with greatest element 1_A; suprema of pairs exist and the supremum of $\{[p], [q]\}$ (denoted $[p] \vee [q]$) is the $\mathcal{E}$ factor of the $(\mathcal{E}, \mathcal{M})$ factorization of (p, q); the infimum of $\{[p], [q]\}$ (denoted $[p] \wedge [q]$) exists whenever a pushout of p and q exists and these conditions are equivalent if $\mathcal{E}$ is contained in the class of epimorphisms.

The properties of factorization systems that we use can be found in [2].

2 Full reflective subcategories closed under $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients, for a factorization system $(\mathcal{E}, \mathcal{M})$

Let $\mathbb{C}$ be a category with finite limits, and $(\mathcal{E},\mathcal{M})$ any factorization system in $\mathbb{C}$.

Let $\mathbb{X}$ be a full replete reflective subcategory of $\mathbb{C}$, and let H denote the inclusion functor and I the reflector. Since $\mathbb{X}$ is full and $\mathbb{C}$ has finite limits, $\mathbb{X}$ has finite limits, and these are exactly the same as in $\mathbb{C}$.

It is well known (see e.g. [1], Theorem 16.8) that

Proposition 2.1 *$\mathbb{X}$ is closed under $\mathcal{M}$-subobjects iff for each C in $\mathbb{C}$, $\eta_C \in \mathcal{E}$.*

Corollary 2.2 *If $\mathbb{X}$ is closed under $\mathcal{M}$-subobjects, for any f in $\mathcal{E}$, $If \in \mathcal{E}$.*

Proof Let $f : A \to B$ be in $\mathcal{E}$. From Proposition 2.1 and, as $\mathcal{E}$ is closed under composition, $\eta_A \in \mathcal{E}$ and $If \circ \eta_A = \eta_B \circ f \in \mathcal{E}$. Then, If is in $\mathcal{E}$, since $(\mathcal{E}, \mathcal{M})$ is a factorization system. □

Lemma 2.3 *If $\mathbb{X}$ is closed under both $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients, for each C in $\mathbb{C}$, η_C is an epimorphism.*

Proof : Let $f, g : I(C) \to B$ be such that $f \circ \eta_C = g \circ \eta_C$ and assume these factorize as $m \circ e$; let f and g factorize as $f = m_1 \circ e_1$ and $g = m_2 \circ e_2$ as in the commutative diagram:

$$\begin{tikzcd} C \arrow[rr, "e"] \arrow[d, "\eta_C"'] & & A \arrow[d, "m"] \\ I(C) \arrow[r, "e_1"] \arrow[dr, "e_2"'] & E_1 \arrow[r, "m_1"] & B \\ & E_2 \arrow[ur, "m_2"'] & \end{tikzcd}$$

Since $\mathbb{X}$ is closed under $\mathcal{M}$-subobjects, we know that η_C is in $\mathcal{E}$. Then there are isomorphisms $i : E_1 \to A$ and $j : E_2 \to A$ such that

$$i \circ e_1 \circ \eta_C = e \quad \text{and} \quad m \circ i = m_1, \qquad j \circ e_2 \circ \eta_C = e \quad \text{and} \quad m \circ j = m_2.$$

As $\mathbb{X}$ is closed under $\mathcal{E}$-quotients, A is in $\mathbb{X}$. Therefore, by the universal property of the adjunction, the unique morphism w satisfying $Hw \circ \eta_C = e$, must be $i \circ e_1 = j \circ e_2$. But then, $f = m_1 \circ e_1 = m \circ i \circ e_1 = m \circ j \circ e_2 = m_2 \circ e_2 = g$, and so, η_C is an epimorphism. $\square$

Proposition 2.4 *If $\mathbb{X}$ is closed under $\mathcal{M}$-subobjects then, $\mathbb{X}$ is closed under $\mathcal{E}$-quotients if and only if for each $f : A \to B$, $f \in \mathcal{E}$, the diagram*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & I(A) \arrow[d, "If"] \\ B \arrow[r, "\eta_B"'] & I(B) \end{tikzcd}$$

is a pushout.

Proof If diagrams as in the statement are pushouts, in particular they are so for A in $\mathbb{X}$ and $f \in \mathcal{E}$. Then η_B is an isomorphism, since η_A is, and therefore B is in $\mathbb{X}$.

Conversely, assume that $\mathbb{X}$ is closed under $\mathcal{E}$-quotients. Let $f : A \to B$ be in $\mathcal{E}$, and suppose that $u : I(A) \to D$ and $v : B \to D$ are morphisms such that $u \circ \eta_A = v \circ f$. Consider the factorizations $u = m \circ q$, $v = m' \circ q'$ with $m, m' \in \mathcal{M}$ and $q, q' \in \mathcal{E}$. We have then the commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \arrow[dr, "q'"] & \\ I(A) \arrow[r, "If"'] \arrow[dr, "q"'] & I(B) \arrow[r, dotted, "g"] & Q' \arrow[dl, dotted, "i"] \arrow[dd, "m'"] \\ & Q \arrow[dr, "m"'] & \\ & & D \end{tikzcd}$$

As $\eta_A, \eta_B \in \mathcal{E}$ and $\mathcal{E}$ is closed under composition, $q \circ \eta_A$, $\eta_B \circ f \in \mathcal{E}$. By the uniqueness of the factorization there exists an isomorphism $i : Q' \to Q$ such that $i \circ q' \circ f = q \circ \eta_A$ and $m \circ i = m'$. Then, as $\mathbb{X}$ is closed under $\mathcal{E}$-quotients, Q is in $\mathbb{X}$ and therefore Q' is in $\mathbb{X}$. By the universal property, there is then a morphism $g : I(B) \to Q'$ satisfying $q' = g \circ \eta_B$. Hence, $m' \circ g \circ \eta_B = m' \circ q' = v$, and

$$u \circ \eta_A = v \circ f = m' \circ q' \circ f = m' \circ g \circ \eta_B \circ f = m' \circ g \circ If \circ \eta_A.$$

By the Lemma, as η_A is an epimorphism, $m' \circ g \circ If = u$. To show uniqueness, suppose that $w \circ \eta_B = v$; then $w \circ \eta_B = m' \circ g \circ \eta_B$, and as η_B is an epimorphism, $w = m' \circ g$. $\square$

Corollary 2.5 *Let $\mathbb{C}$ be a category with finite limits, and $\mathbb{X}$ a full, replete and reflective subcategory of $\mathbb{C}$ with the reflection given by $(I, H, \eta, \varepsilon) : \mathbb{C} \to \mathbb{X}$. Let $(\mathcal{E}, \mathcal{M})$ be a stable factorization system in $\mathbb{C}$. Then, $(I, H, \eta, \varepsilon) : \mathbb{C} \to \mathbb{X}$ together with the pair of classes of morphisms $(\mathcal{E}, \mathcal{E} \cap \mathrm{mor}\mathbb{X})$ is a Galois structure.*

Theorem 2.6 *Let $(\mathcal{E},\mathcal{M})$ be a stable factorization system in a category $\mathbb{C}$ with finite limits, with $\mathcal{E}$ contained in the class of epimorphisms. Let $\mathbb{X}$ be a full subcategory of $\mathbb{C}$, reflective through $(I,H,\eta,\varepsilon):\mathbb{C}\to\mathbb{X}$, and closed under $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients.*

Then, if for each $C\in\mathbb{C}$ and each $s,t\in\mathcal{E}$, with $t\leq\eta_C$

$$\eta_C\wedge(s\vee t)=(\eta_C\wedge s)\vee t,$$

the Galois structure $(I,H,\eta,\varepsilon):\mathbb{C}\to\mathbb{X}$, $(\mathcal{E},\mathcal{E}\cap mor\mathbb{X})$ is admissible.

Proof Since ε_X is an isomorphism for each X in $\mathbb{X}$, we only need to show that under the stated conditions, for a pullback as in (1.1) It is an isomorphism. Note that as $\mathcal{E}$ is pullback stable and $\eta_B\in\mathcal{E}$, both s and t are in $\mathcal{E}$. As X is in $\mathbb{X}$, by the universal property, there exists $\beta:I(C)\to X$ such that $t=\beta\circ\eta_C$. Then, $\varepsilon_X\circ It=\varepsilon_X\circ IH\beta\circ I\eta_C=\beta\circ\varepsilon_{I(C)}\circ I\eta_C=\beta$ and therefore, It is invertible if and only if so is β; since $\eta_C\in\mathcal{E}$ is an epimorphism, this happens precisely when in $\mathcal{E}-Quot(C)$, the classes $[\eta_C]$ and $[t]$, coincide. In the diagram

$$\begin{array}{ccccccc}
C & \xrightarrow{\eta_C} & I(C) & \xrightarrow{It} & I(X) & \xrightarrow{\varepsilon_X} & X \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle Is} & & \downarrow{\scriptstyle I\varphi} & & \downarrow{\scriptstyle \varphi} \\
B & \xrightarrow[\eta_B]{} & I(B) & \xrightarrow[I\eta_B]{} & I(B) & \xrightarrow[\varepsilon_{IB}]{} & I(B)
\end{array}$$

the square on the left is a pushout by assumption, so that $\eta_C\wedge s$ exists. Since the exterior diagram is a pullback, the morphism $(s,t):C\to B\times_{I(B)}X$, induced by s and t, is in $\mathcal{M}$, and therefore $s\vee t=1_C$. Also,

$$s\wedge\eta_C=Is\circ\eta_C=\varphi\circ\varepsilon_X\circ It\circ\eta_C=\varphi\circ t,$$

and thus $\eta_C\wedge s\leq t$. Hence, $t=(\eta_C\wedge s)\vee t\leq\eta_C\wedge(s\vee t)=\eta_C\wedge 1_C=\eta_C$.

Therefore, β is invertible if and only if $(\eta_C\wedge s)\vee t=\eta_C\wedge(s\vee t)$. □

Corollary 2.7 *If $\mathbb{C}$ is regular and for each object C in $\mathbb{C}$, $Cong(C)$ is a modular lattice, any full and reflective subcategory of $\mathbb{C}$ closed under mono-subobjects and regular epi-quotients is admissible with respect to regular epimorphisms.*

3 Examples

In the category of structures for a first order language regular epis are the strong surjective homomorphisms that is, surjective homomorphisms $h:A\longrightarrow B$, such that, for each natural n, and each n-ary predicate symbol P, $h^n(P^A)\subseteq P^B$; varieties are regular categories (not necessarily exact).

1. A variety of Ω-groups with relations is congruence modular. Any of its full and reflective subcategories, closed under subobjects and strong homomorphic images is, by Corollary 2.7, admissible with respect to the class of strong surjective homomorphisms. Examples of such are axiomatizable subclasses of varieties of Ω-groups with relations, whose axioms, besides those that characterize the variety, are equivalent to either universal atomic sentences or strict universal Horn sentences $\forall x_1...\forall x_n(\theta_1\wedge...\wedge\theta_k\to\delta)$ which satisfy:

i) for each $1\leq i\leq k$, θ_i is of the form $Px_1^i...x_{l_i}^i$ where P is a predicate symbol and the x_j^i's are distinct variables

ii) for each $1\leq i,j\leq k$, $i\neq j$, θ_i and θ_j have disjoint sets of variables,

iii) δ is of the form $t_1\approx t_2$.

2. Let $\mathbb{X}$ be a subvariety of a variety $\mathbb{V}$, in a *relational* language. The pair $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ is the class of surjective homomorphisms and $\mathcal{M}$ the class of substructure monomorphisms, is a factorization system in $\mathbb{V}$ ([5]). $(\mathcal{E}, \mathcal{M})$ together with the reflection $(I, H, \eta, \varepsilon) : \mathbb{V} \to \mathbb{X}$, $\eta_C = id_C$, $\varepsilon_X = 1_X$, is a Galois structure. Assume that $\mathbb{X}$ is axiomatizable by (universal atomic) $T' = T \cup \Sigma$ where T is an axiomatization of $\mathbb{V}$ and Σ consists of sentences with atomic part $Px_1...x_n$ where $x_i \neq x_j$ for $i \neq j$, that is to say, the interpretation in a structure A of each predicate symbol P in Σ is A^k where k is the arity of P (*). Then, for $s, t \in \mathcal{E}$, $t : C \to B$ with $t \leq \eta_C$, B is in $\mathbb{X}$ and it follows that $\eta_C \wedge (s \vee t) = (\eta_C \wedge s) \vee t$, so that, by Theorem 2.6, the above Galois structure is admissible. In fact, condition (*) is necessary ([5]).

3. Consider the adjunction $\mathbb{C}at \rightleftarrows Preord$ formed by the inclusion of the category *Preord* of preorders into the category $\mathbb{C}at$ of small categories, and its left adjoint. We define the fibrations in $\mathbb{C}at$ as all full functors bijective on objects, and accordingly the fibrations in *Preord* are just the isomorphisms. The admissibility of this Galois structure follows from Theorem 2.6; however, since the isomorphisms are involved, this case is in fact trivial. The same is true if we replace *Preord* with the category of (partially) ordered sets. Note that those admissibilities also hold for the *(All functors, Isomorphisms)* factorization system, as shown in [6].

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Hopf-Galois and Bi-Galois Extensions

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CONTENTS

1 Introduction

Hopf-Galois extensions were introduced by Chase and Sweedler [8] (in the commutative case) and Kreimer and Takeuchi [25] (in the case of finite dimensional Hopf algebras) by axioms directly generalizing those of a Galois extension of rings, replacing the action of a group on the algebra by the coaction of a Hopf algebra H; the special case of an ordinary Galois extension is recovered by specializing H to be the dual of a group algebra. Hopf-Galois extensions also generalize strongly graded algebras (here H is a group algebra) and certain inseparable field extensions (here the Hopf algebra is the restricted envelope of a restricted Lie algebra, or, in more general cases, generated by higher derivations). They comprise twisted group rings $R * G$ of a group G acting on a ring R (possibly also twisted by a cocycle), and similar constructions for actions of Lie algebras. If the Hopf algebra involved is the coordinate ring of an affine group scheme, faithfully flat Hopf-Galois extensions are precisely the coordinate rings of affine torsors or principal homogeneous spaces. By analogy, Hopf-Galois extensions with Hopf algebra H the coordinate ring of a quantum group can be considered as the noncommutative analog of a principal homogeneous space, with a quantum group as its structure group. Apart from this noncommutative-geometric interpretation, and apart from their role as a unifying

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language for many examples of good actions of things on rings, Hopf-Galois extensions are frequently used as a tool in the investigation of the structure of Hopf algebras themselves.

In this paper we try to collect some of the basic facts of the theory of Hopf-Galois extensions and (see below) bi-Galois extensions, offering alternative proofs in some instances, and proving new facts in very few instances.

In the first part we treat Hopf-Galois extensions and discuss various properties by which they can, to some extent, be characterized. After providing the necessary definitions, we first treat the special case of cleft extensions, repeating (with some more details) a rather short proof from [38] of their characterization, due to Blattner, Cohen, Doi, Montgomery, and Takeuchi [15, 5, 6]. Cleft extensions are the same as crossed products, which means that they have a combinatorial description that specializes in the case of cocommutative Hopf algebras to a cohomological description in terms of Sweedler cohomology [46].

In Section 2.3 we prove Schneider's structure theorem for Hopf modules, which characterizes faithfully flat Hopf-Galois extensions as those comodule algebras A that give rise to an equivalence of the category of Hopf modules $\mathcal{M}_A^H$ with the category of modules of the ring of coinvariants under the coaction of H. The structure theorem is one of the most ubiquitous applications of Hopf-Galois theory in the theory of Hopf algebras. We emphasize the role of faithfully flat descent in its proof.

A more difficult characterization of faithfully flat Hopf-Galois extensions, also due to Schneider, is treated in Section 2.4. While the definition of an H-Galois extension A of B asks for a certain canonical map $\beta\colon A \otimes_B A \to A \otimes H$ to be bijective, it is sufficient to require it to be surjective, provided we work over a field and A is an injective H-comodule. When we think of Hopf-Galois extensions as principal homogeneous spaces with structure quantum group, this criterion has a geometric meaning. We will give a new proof for it, which is more direct than that in [44]. The new proof has two nice side-effects: First, it is more parallel to the proof that surjectivity of the canonical map is sufficient for finite-dimensional Hopf algebras (in fact so parallel that we prove the latter fact along with Schneider's result). Secondly, it yields without further work the fact that an H-Galois extension A/B that is faithfully flat as a B-module is always projective as a B-module [1].

Section 2.5 treats (a generalized version of) a characterization of Hopf-Galois extensions due to Ulbrich: An H-Galois extension of B is (up to certain additional conditions) the same thing as a monoidal functor ${}^H\mathcal{M} \to {}_B\mathcal{M}_B$ from the monoidal category of H-comodules to the category of B-bimodules.

In Section 2.6 we deal with another characterization of Hopf-Galois extensions by monoidal functors: Given any H-comodule algebra A with coinvariants B, we can define a monoidal category ${}_A\mathcal{M}_A^H$ of Hopf bimodules (monoidal with the tensor product over A), and a *weak* monoidal functor from this to the category of B-bimodules. Again up to some technical conditions, the functor is monoidal if and only if A is an H-Galois extension of B.

In Section 2.8 we show how to characterize Hopf-Galois extensions without ever mentioning a Hopf algebra. The axioms of a torsor we give here are a simplified

[1] Since the present paper was submitted, the new proof has been developed further in joint work with H.-J. Schneider, in particular to also prove some results on Q-Galois extensions for a quotient coalgebra and one sided module of H; this type of extensions will not be considered in the present paper.

variant of axioms recently introduced by Grunspan. A crucial ingredient in the characterization is again the theory of faithfully flat descent.

The second part of the paper deals with bi-Galois objects. This means, first of all, that we restrict our attention to Galois extensions of the base ring k rather than of an arbitrary coinvariant subring. Contrary, as it were, to the theory of torsors that can do without any Hopf algebras, the theory of bi-Galois extensions exploits the fact that any Hopf-Galois object has *two* rather than only one Hopf algebra in it. More precisely, for every H-Galois extension A of k there is a uniquely determined second Hopf algebra L such that A is a left L-Galois extension of A and an L-H-bicomodule. We will give an account of the theory and several ways in which the new Hopf algebra L can be applied. Roughly speaking, this may happen whenever there is a fact or a construction that depends on the condition that the Hopf algebra H be cocommutative (which, in terms of bi-Galois theory, means $L \cong H$). If this part of the cocommutative theory does not survive if H fails to be cocommutative, then maybe L can be used to replace H. Our approach will stress a very general universal property of the Hopf algebra L in an L-H-Galois extension. Several versions of this were already used in previous papers, but the general version we present here appears to be new. The construction of L was invented in the commutative case by Van Oystaeyen and Zhang to repair the failing of the fundamental theorem of Galois theory for Hopf-Galois extensions. We will discuss an application to the computation of Galois objects over tensor products, and to the problem of reducing the Hopf algebra in a Hopf-Galois object to a quotient Hopf algebra (here, however, L arises because of a lack of commutativity rather than cocommutativity). Perhaps the most important application is that bi-Galois extensions classify monoidal category equivalences between categories of comodules over Hopf algebras.

Some conventions and background facts can be found in an appendix. Before starting, however, let us point out a general notational oddity: Whenever we refer to an element $\xi \in V \otimes W$ of the tensor product of two modules, we will take the liberty to "formally" write $\xi = v \otimes w$, even if we know that the element in question is not a simple tensor, or, worse, has to be chosen from a specific submodule that is not even generated by simple tensors. Such formal notations are of course widely accepted under the name Sweedler notation for the comultiplication $\Delta(c) = c_{(1)} \otimes c_{(2)} \in C \otimes C$ in a coalgebra C, or $\delta(v) = v_{(0)} \otimes v_{(1)}$ for a right comodule, or $\delta(v) = v_{(-1)} \otimes v_{(0)}$ for a left comodule.

For a coalgebra C and a subspace $V \subset C$ we will write $V^+ = V \cap \mathrm{Ker}(\varepsilon)$. C^{cop} denotes the coalgebra C with coopposite comultiplication, A^{op} the algebra A with opposite multiplication. Multiplication in an algebra A will be denoted by $\nabla\colon A \otimes A \to A$.

2 Hopf-Galois theory

2.1 Definitions. Throughout this section, H is a k-bialgebra, flat over k. A (right) H-comodule algebra A is by definition an algebra in the monoidal category of right H-comodules, that is, a right H-comodule via $\delta\colon A \ni a \mapsto a_{(0)} \otimes a_{(1)}$ and an algebra, whose multiplication $\nabla\colon A \otimes A \to A$ is a colinear map, as well as the unit $\eta\colon k \to A$. These conditions mean that the unit $1_A \in A$ is a coinvariant element, $1_{(0)} \otimes 1_{(1)} = 1 \otimes 1$, and that $\delta(xy) = x_{(0)}y_{(0)} \otimes x_{(1)}y_{(1)}$ holds for all $x, y \in A$. Equivalently, A is an algebra and an H-comodule in such a way that the comodule structure is an algebra homomorphism $\delta\colon A \to A \otimes H$. For any H-comodule M we let $M^{\operatorname{co} H} := \{m \in M | \delta(m) = m \otimes 1\}$ denote the subset of H-coinvariants. It is straightforward to check that $A^{\operatorname{co} H}$ is a subalgebra of A.

Definition 2.1.1 The right H-comodule algebra A is said to be an H-Galois extension of $B := A^{\operatorname{co} H}$, if the Galois map

$$\beta\colon A \underset{B}{\otimes} A \ni x \otimes y \mapsto xy_{(0)} \otimes y_{(1)} \in A \otimes H$$

is a bijection. More precisely we should speak of a right H-Galois extension; it is clear how a left H-Galois extension should be defined.

We will use the term "(right) Galois object" as shorthand for a right H-Galois extension A of k which is a faithfully flat k-module.

The first example that comes to mind is the H-comodule algebra H itself:

Example 2.1.2 Let H be a bialgebra. Then H is an H-comodule algebra, with $H^{\operatorname{co} H} = k$. The Galois map $\beta\colon H \otimes H \to H \otimes H$ is the map $T(id)$, where

$$T\colon \operatorname{Hom}(H, H) \to \operatorname{End}_{H-}^{-H}(H \otimes H)$$

is the anti-isomorphism from Lemma 4.4.1. Thus, H is a Hopf algebra if and only if the identity on H is convolution invertible if and only if the Galois map is bijective if and only if H is an H-Galois extension of k.

The notion of a Hopf-Galois extension serves to unify various types of extensions. These are recovered as we specialize the Hopf algebra H to one of a number of special types:

Example 2.1.3 Let A/k be a Galois field extension, with (finite) Galois group G. Put $H = k^G$, the dual of the group algebra. Then A is an H-Galois extension of k. Bijectivity of the Galois map $A \otimes A \to A \otimes H$ is a consequence of the independence of characters.

The definition of a Galois extension A/k of commutative rings in [9] requires (in one of its many equivalent formulations) precisely the bijectivity of the Galois map $A \otimes A \to A \otimes k^G$, beyond of course the more obvious condition that k be the invariant subring of A under the action of a finite subgroup G of the automorphism group of A. Thus Hopf-Galois extensions of commutative rings are direct generalizations of Galois extensions of commutative rings.

Example 2.1.4 Let $A = \bigoplus_{g \in G} A_g$ be a k-algebra graded by a group G. Then A is naturally an H-comodule algebra for the group algebra kG, whose coinvariant subring is $B = A_e$, the homogeneous component whose degree is the neutral element. The Galois map $A \otimes_B A \to A \otimes H$ is surjective if and only if $A_g A_h = A_{gh}$ for all $g, h \in G$, that is, A is strongly graded [10, 52]. As we shall see in Corollary 2.4.9, this condition implies that A is an H-Galois extension of B if k is a field.

We have seen already that a bialgebra H is an H-Galois extension of k if and only if it is a Hopf algebra. The following more general observation is the main result of [34]; we give a much shorter proof that is due to Takeuchi [51].

Lemma 2.1.5 *Let H be a k-flat bialgebra, and A a right H-Galois extension of $B := A^{\operatorname{co} H}$, which is faithfully flat as k-module. Then H is a Hopf algebra.*

Proof H is a Hopf algebra if and only if the map $\beta_H \colon H \otimes H \ni g \otimes h \mapsto gh_{(1)} \otimes h_{(2)}$ is a bijection. By assumption the map $\beta_A \colon A \otimes_B A \ni x \otimes y \mapsto xy_{(0)} \otimes y_{(1)} \in A \otimes H$ is a bijection. Now the diagram

$$\begin{tikzcd}
A \otimes_B A \otimes_B A \arrow[r, "A\otimes_B \beta_A"] \arrow[d, "\beta_A \otimes_B A"'] & A \otimes_B A \otimes H \arrow[dd, "\beta_A \otimes H"] \\
(A \otimes H) \otimes_B A \arrow[d, "(\beta_A)_{13}"'] & \\
A \otimes H \otimes H \arrow[r, "A \otimes \beta_H"'] & A \otimes H \otimes H
\end{tikzcd}$$

commutes, where $(\beta_A)_{13}$ denotes the map that applies β_A to the first and third tensor factor, and leaves the middle factor untouched. Thus $A \otimes \beta_H$, and by faithful flatness of A also β_H, is a bijection. □

The Lemma also shows that if A is an H-Galois extension and a flat k-module, then A^{op} is never an H^{op}-Galois extension, unless the antipode of H is bijective. On the other hand (see [44]):

Lemma 2.1.6 *If the Hopf algebra H has bijective antipode and A is an H-comodule algebra, then A is an H-Galois extension if and only if A^{op} is an H^{op}-Galois extension.*

Proof The canonical map $A^{\mathrm{op}} \otimes_{B^{\mathrm{op}}} A^{\mathrm{op}} \to A^{\mathrm{op}} \otimes H^{\mathrm{op}}$ identifies with the map $\beta' \colon A \otimes_B A \to A \otimes H$ given by $\beta'(x \otimes y) = x_{(0)} y \otimes x_{(1)}$. One checks that the diagram

$$\begin{tikzcd}
A \otimes_B A \arrow[r, "\beta"] \arrow[dr, "\beta'"'] & A \otimes H \arrow[d, "\alpha"] \\
& A \otimes H
\end{tikzcd}$$

commutes, where $\alpha \colon A \otimes H \ni a \otimes h \mapsto a_{(0)} \otimes a_{(1)} S(h)$ is bijective with $\alpha^{-1}(a \otimes h) = a_{(0)} \otimes S^{-1}(h) a_{(1)}$. □

Lemma 2.1.7 *Let A be an H-Galois extension of B. For $h \in H$ we write $\beta^{-1}(1 \otimes h) =: h^{[1]} \otimes h^{[2]}$. For $g, h \in H$, $b \in B$ and $a \in A$ we have*

$$h^{[1]} h^{[2]}{}_{(0)} \otimes h^{[2]}{}_{(1)} = 1 \otimes h \tag{2.1.1}$$

$$h^{[1]} \otimes h^{[2]}{}_{(0)} \otimes h^{[2]}{}_{(1)} = h_{(1)}{}^{[1]} \otimes h_{(1)}{}^{[2]} \otimes h_{(2)} \tag{2.1.2}$$

$$h^{[1]}{}_{(0)} \otimes h^{[2]} \otimes h^{[1]}{}_{(1)} = h_{(2)}{}^{[1]} \otimes h_{(2)}{}^{[2]} \otimes S(h_{(1)}) \tag{2.1.3}$$

$$h^{[1]} h^{[2]} = \varepsilon(h) 1_A \tag{2.1.4}$$

$$(gh)^{[1]} \otimes (gh)^{[2]} = h^{[1]} g^{[1]} \otimes g^{[2]} h^{[2]} \tag{2.1.5}$$

$$b h^{[1]} \otimes h^{[2]} = h^{[1]} \otimes h^{[2]} b \tag{2.1.6}$$

$$a_{(0)} a_{(1)}{}^{[1]} \otimes a_{(1)}{}^{[2]} = 1 \otimes a \tag{2.1.7}$$

We will omit the proof, which can be found in [45, (3.4)].

Definition 2.1.8 Let H be a Hopf algebra, and A an H-Galois extension of B. The Miyashita-Ulbrich action of H on the centralizer A^B of B in A is given by $x \leftharpoonup h = h^{[1]}xh^{[2]}$ for $x \in A^B$ and $h \in H$.

The expression $h^{[1]}xh^{[2]}$ is well-defined because $x \in A^B$, and it is in A^B again because $h^{[1]} \otimes h^{[2]} \in (A \otimes_B A)^B$. The following properties of the Miyashita-Ulbrich action can be found in [52, 16] in different language.

Lemma 2.1.9 *The Miyashita-Ulbrich action makes A^B an object of $\mathcal{YD}^H_H$, and thus the weak center of the monoidal category $\mathcal{M}^H$ of right H-comodules. A^B is the center of A in the sense of Definition 4.2.1.*

Proof It is trivial to check that A^B is an subcomodule of A. It is a Yetter-Drinfeld module by (2.1.3) and (2.1.2). Now the inclusion $A^B \hookrightarrow A$ is central in the sense of Definition 4.2.1, since $a_{(0)}(x \leftharpoonup a_{(1)}) = a_{(0)}{a_{(1)}}^{[1]}x{a_{(1)}}^{[2]} = xa$ for all $a \in A$ and $x \in A^B$ by (2.1.7). Finally let us check the universal property in Definition 4.2.1: Let V be a Yetter-Drinfeld module, and $f\colon V \to A$ an H-colinear map with $a_{(0)}f(v \leftharpoonup a_{(1)}) = f(v)a$ for all $v \in V$ and $a \in A$. Then we see immediately that f takes values in A^B. Moreover, we have $f(v) \leftharpoonup h = h^{[1]}f(v)h^{[2]} = h^{[1]}{h^{[2]}}_{(0)}f(v \leftharpoonup {h^{[2]}}_{(1)}) = f(v \leftharpoonup h)$ for all $h \in H$ by (2.1.1). $\square$

Much of the "meaning" of the Miyashita-Ulbrich action can be guessed from the simplest example $A = H$. Here we have $h^{[1]} \otimes h^{[2]} = S(h_{(1)}) \otimes h_{(2)} \in H \otimes H$, and thus the Miyashita-Ulbrich action is simply the adjoint action of H on itself.

2.2 Cleft extensions and crossed products. Throughout the section, H is a k-bialgebra.

Definition 2.2.1 Let B be a k-algebra. A map $\rightharpoonup\colon H \otimes B \to B$ is a measuring if $h \rightharpoonup (bc) = (h_{(1)} \rightharpoonup b)(h_{(2)} \rightharpoonup c)$ and $h \rightharpoonup 1 = 1$ hold for all $h \in H$ and $b, c \in B$.

Let H be a bialgebra, and B an algebra. A crossed product $B\#_\sigma H$ is the structure of an associative algebra with unit $1\#1$ on the k-module $B\#_\sigma H := B \otimes H$, in which multiplication has the form

$$(b\#g)(c\#h) = b(g_{(1)} \rightharpoonup c)\sigma(g_{(2)} \otimes h_{(1)})\#g_{(3)}h_{(2)}$$

for some measuring $\rightharpoonup\colon H \otimes B \to B$ and some linear map $\sigma\colon H \otimes H \to B$.

We have quite deliberately stated the definition without imposing any explicit conditions on σ. Such conditions are implicit, however, in the requirement that multiplication be associative and have the obvious unit. We have chosen the definition above to emphasize that the explicit conditions on σ are never used in our approach to the theory of crossed products. They are, however, known and not particularly hard to derive:

Proposition 2.2.2 *Let H be a bialgebra, $\rightharpoonup\colon H \otimes B \to B$ a measuring, and $\sigma\colon H \otimes H \to B$ a k-linear map. The following are equivalent:*

1. *$A = B\#H := B \otimes H$ is an associative algebra with unit $1\#1$ and multiplication*

$$(b\#g)(c\#h) = b(g_{(1)} \rightharpoonup c)\sigma(g_{(2)} \otimes h_{(1)})\#g_{(3)}h_{(2)}.$$

2. (a) *$\rightharpoonup$ is a twisted action, that is $(g_{(1)} \rightharpoonup (h_{(1)} \rightharpoonup b))\sigma(g_{(2)} \otimes h_{(2)}) = \sigma(g_{(1)} \otimes h_{(1)})(g_{(2)}h_{(2)} \rightharpoonup b)$ and $1 \rightharpoonup b = b$ hold for all $g, h \in H$ and $b \in B$.*

(b) *σ is a two-cocycle, that is* $(f_{(1)} \rightharpoonup \sigma(g_{(1)} \otimes h_{(1)}))\sigma(f_{(2)} \otimes g_{(2)}h_{(2)}) = \sigma(f_{(1)} \otimes g_{(1)})\sigma(f_{(2)} \otimes g_{(2)} \otimes h)$ *and* $\sigma(h \otimes 1) = \sigma(1 \otimes h) = 1$ *hold for all* $f, g, h \in H$.

Not only are the conditions on σ known, but, more importantly, they have a cohomological interpretation in the case where H is cocommutative and B is commutative. In this case a twisted action is clearly simply a module algebra structure. Sweedler [46] has defined cohomology groups $H^\bullet(H, B)$ for a cocommutative bialgebra H and commutative H-module algebra B, and it turns out that a convolution invertible map σ as above is precisely a two-cocycle in this cohomology. Sweedler's paper also contains the construction of a crossed product from a two-cocycle, and the fact that his second cohomology group classifies cleft extensions (which we shall define below) by assigning the crossed product to a cocycle. Group cohomology with coefficients in the unit group of B as well as (under some additional conditions) Lie algebra cohomology with coefficients in the additive group of B are examples of Sweedler cohomology, and the cross product costruction also has precursors for groups (twisted group rings with cocycles, which feature in the construction of elements of the Brauer group from group cocycles) and Lie algebras. Thus, the crossed product construction from cocycles can be viewed as a nice machinery producing (as we shall see shortly) Hopf-Galois extensions *in the case of cocommutative Hopf algebras and commutative coinvariant subrings.* In the general case, the equations do not seem to have any reasonable cohomological interpretation, so while cleft extensions remain an important special class of Hopf-Galois extensions, it is rarely possible to construct them by finding cocycles in some conceptually pleasing way.

We now proceed to prove the characterization of crossed products as special types of comodule algebras, which is due to Blattner, Cohen, Doi, Montgomery, and Takeuchi:

Definition 2.2.3 Let A be a right H-comodule algebra, and $B := A^{\operatorname{co} H}$.

1. A is cleft if there exists a convolution invertible H-colinear map $j\colon H \to A$ (also called a cleaving).
2. A normal basis for A is an H-colinear and B-linear isomorphism $\psi\colon B \otimes H \to A$.

If $\tilde{\jmath}$ is a cleaving, then $\tilde{\jmath}(1)$ is a unit in B, and thus $j(h) = \tilde{\jmath}(1)^{-1}\tilde{\jmath}(h)$ defines another cleaving, which, moreover, satisfies $j(1) = 1$.

It was proved by Doi and Takeuchi [15] that A is H-Galois with a normal basis if and only if it is cleft, and in this case A is a crossed product $A \cong B\#_\sigma H$ with an invertible cocycle $\sigma\colon H \otimes H \to B$. Blattner and Montgomery [6] have shown that crossed products with an invertible cocycle are cleft.

Clearly a crossed product is always an H-comodule algebra with an obvious normal basis.

Lemma 2.2.4 *Assume that the H-comodule algebra A has a normal basis $\psi\colon B \otimes H \to A$ satisfying $\psi(1 \otimes 1) = 1$. Then A is isomorphic (via ψ) to a crossed product.*

Proof In fact we may as well assume $B \otimes H = A$ as B-modules and H-comodules. Define $h \rightharpoonup b = (B \otimes \varepsilon)((1 \otimes h)(b \otimes 1))$ and $\sigma(g \otimes h) = (B \otimes \varepsilon)((1 \otimes$

$g)(1\otimes h))$. Since multiplication is H-colinear, we find

$$\begin{aligned}(1\otimes g)(c\otimes 1) &= (B\otimes\varepsilon\otimes H)(B\otimes\Delta)((1\otimes g)(c\otimes 1))\\ &= (B\otimes\varepsilon\otimes H)((1\otimes g_{(1)})(b\otimes 1)\otimes g_{(2)}) = g_{(1)}\rightharpoonup b\otimes g_{(2)},\end{aligned}$$

$$\begin{aligned}(1\otimes g)(1\otimes h) &= (B\otimes\varepsilon\otimes H)(B\otimes\Delta)((1\otimes g)(1\otimes h))\\ &= (B\otimes\varepsilon\otimes H)((1\otimes g_{(1)})(1\otimes h_{(1)})\otimes g_{(2)}h_{(2)}) = \sigma(g_{(1)}\otimes h_{(1)})\otimes g_{(2)}h_{(2)},\end{aligned}$$

and finally

$$\begin{aligned}(b\otimes g)(c\otimes h) &= (b\otimes 1)(1\otimes g)(c\otimes 1)(1\otimes h) = (b\otimes 1)(g_{(1)}\rightharpoonup c\otimes g_{(2)})(1\otimes h)\\ &= b(g_{(1)}\rightharpoonup c)\sigma(g_{(2)}\otimes h_{(1)})\otimes g_{(3)}h_{(2)}.\end{aligned}$$

□

To prove the remaining parts of the characterization, we will make heavy use of the isomorphisms T_A^C from Lemma 4.4.1, for various choices of algebras A and coalgebras C.

Lemma 2.2.5 *Let $j\colon H\to A$ be a cleaving. Then there is a normal basis $\psi\colon B\otimes H\to A$ with $j=\psi(\eta_B\otimes H)$. If $j(1)=1$, then $\psi(1\otimes 1)=1$.*

Proof We claim that $\psi\colon B\otimes H\ni b\otimes h\mapsto bj(h)\in A$ is a normal basis.

Since the comodule structure $\delta\colon A\to A\otimes H$ is an algebra map, δj is convolution invertible. Moreover $\delta j=(j\otimes H)\Delta$ by assumption. For $a\in A$, we have

$$\begin{aligned}&T^H_{A\otimes H}(\delta j)(\delta(a_{(0)}j^{-1}(a_{(1)}))\otimes a_{(2)}) = T^H_{A\otimes H}(\delta j)T^H_{A\otimes H}(\delta j^{-1})(\delta(a_{(0)})\otimes a_{(1)})\\ &= a_{(0)}j^{-1}(a_{(1)})j(a_{(2)})\otimes a_{(3)}\otimes a_{(4)} = T^H_{A\otimes H}((j\otimes H)\Delta)(a_{(0)}j^{-1}(a_{(1)})\otimes 1\otimes a_{(2)}),\end{aligned}$$

hence $\delta(a_{(0)}j^{-1}(a_{(1)}))\otimes a_{(2)} = a_{(0)}j^{-1}(a_{(1)})\otimes 1\otimes a_{(2)}$, and further $\delta(a_{(0)}j^{-1}(a_{(1)})) = a_{(0)}j^{-1}(a_{(1)})\otimes 1$. Thus $A\ni a\mapsto a_{(0)}j^{-1}(a_{(1)})\otimes a_{(2)}\in B\otimes H$ is well defined and easily checked to be an inverse for ψ. □

Lemma 2.2.6 *Let A be an H-comodule algebra with a normal basis. Put $B:=A^{\operatorname{co}H}$. The following are equivalent:*

1. *A is an H-Galois extension of B.*
2. *A is cleft.*

Proof We can assume that $A=B\#_\sigma H$ is a crossed product, and that $j(h)=1\otimes h$.

The map $\alpha\colon B\otimes H\otimes H\to A\otimes_B A$ with $\alpha(b\otimes g\otimes h)=b\otimes g\otimes 1\otimes h$ is an isomorphism. For $b\in B$ and $g,h\in H$ we have

$$\beta_A\alpha(b\otimes g\otimes h)=\beta_A(b\otimes g\otimes j(h))=(b\otimes g)j(h_{(1)})\otimes h_{(2)}=T_A^H(j)(b\otimes g\otimes h),$$

that is $\beta_A\alpha=T_A^H(j)$. In particular, β_A is an isomorphism if and only if $T_A^H(j)$ is, if and only if j is convolution invertible. □

In particular, if B is faithfully flat over k, then cleft extensions can only occur if H is a Hopf algebra. If this is the case, we find:

Theorem 2.2.7 *Let H be a Hopf algebra and A a right H-comodule algebra with $B:=A^{\operatorname{co}H}$. The following are equivalent:*

1. *A is H-cleft.*
2. *A is H-Galois with a normal basis.*

3. *A is isomorphic to a crossed product $B\#_\sigma H$ such that the cocycle $\sigma\colon H\otimes H\to B$ is convolution invertible.*

Proof We have already shown that under any of the three hypotheses we can assume that $A\cong B\#_\sigma H=B\otimes H$ is a crossed product, with $j(h)=1\otimes h$, and we have seen that (1) is equivalent to (2), even if H does not have an antipode.

Now, for $b\in B$, $g,h\in H$ we calculate

$$\begin{aligned}T_A^H(j)(b\otimes g\otimes h)&=bj(g)j(h_{(1)})\otimes h_{(2)}=b\sigma(g_{(1)}\otimes h_{(1)})\otimes g_{(2)}h_{(2)}\otimes h_{(3)}\\&=(B\otimes\beta_H)(b\sigma(g_{(1)}\otimes h_{(1)})\otimes g_{(2)}\otimes h_{(2)})=(B\otimes\beta_H)T_B^{H\otimes H}(\sigma)(b\otimes g\otimes h)\end{aligned}$$

that is, $T_A^H(j)=(B\otimes\beta_H)T_B^{H\otimes H}(\sigma)$. Since we assume that β_H is a bijection, we see that j is convolution invertible if and only if $T_A^H(j)$ is bijective, if and only if $T_B^{H\otimes H}(\sigma)$ is bijective, if and only if σ is convolution invertible. □

The reader that has seen the proof of (3)⇒(1) in [6] may be worried that we have lost some information: In [6] the convolution inverse of j is given explicitly, while we only seem to have a rather roundabout existence proof. However, we see from our arguments above that

$$j^{-1}=(T_A^H)^{-1}\left(T_B^{H\otimes H}(\sigma^{-1})(B\otimes\beta_H^{-1})\right),$$

that is,

$$\begin{aligned}j^{-1}(h)&=(A\otimes\varepsilon)T_B^{H\otimes H}(\sigma^{-1})(B\otimes\beta_H^{-1})(1\otimes 1\otimes h)\\&=(A\otimes\varepsilon)T_B^{H\otimes H}(\sigma^{-1})(1\otimes S(h_{(1)})\otimes h_{(2)})\\&=(A\otimes\varepsilon)(\sigma^{-1}(S(h_{(2)})\otimes h_{(3)})\otimes S(h_{(1)})\otimes h_{(4)})\\&=\sigma^{-1}(S(h_{(2)})\otimes h_{(3)})\#S(h_{(1)}).\end{aligned}$$

2.3 Descent and the structure of Hopf modules.

Definition 2.3.1 Let A be a right H-comodule algebra. A Hopf module $M\in\mathcal{M}_A^H$ is a right A-module in the monoidal category of H-comodules. That is, M is a right H-comodule and a right A-module such that the module structure is an H-colinear map $M\otimes A\to M$. This in turn means that $\delta(ma)=m_{(0)}a_{(0)}\otimes m_{(1)}a_{(1)}$ holds for all $m\in M$ and $a\in A$.

For any comodule algebra, one obtains a pair of adjoint functors between the category of Hopf modules and the category of modules over the coinvariant subalgebra.

Lemma 2.3.2 *Let H be a k-flat Hopf algebra, A a right H-comodule algebra, and $B=A^{\operatorname{co}H}$. Then the functor*

$$\mathcal{M}_A^H\ni M\mapsto M^{\operatorname{co}H}\in\mathcal{M}_B$$

is right adjoint to

$$\mathcal{M}_B\ni N\mapsto N\underset{B}{\otimes}A\in\mathcal{M}_A^H$$

Here, both the A-module and H-comodule structures of $N\otimes_B A$ are induced by those of A. The unit and counit of the adjunction are

$$N\ni n\mapsto n\otimes 1\in(N\underset{B}{\otimes}A)^{\operatorname{co}H}$$

$$M^{\operatorname{co}H}\underset{B}{\otimes}A\ni m\otimes a\mapsto ma\in M$$

If the adjunction in the Lemma is an equivalence, then we shall sometimes say that the structure theorem for Hopf modules holds for the extension. A theorem of Schneider [44] characterizes faithfully flat Hopf-Galois extensions as those comodule algebras for which the adjunction above is an equivalence. The proof in [44] uses faithfully flat descent; we rewrite it to make direct use of the formalism of faithfully flat descent of modules that we recall in Section 4.5. This approach was perhaps first noted in my thesis [32], though it is certainly no surprise; in fact, one of the more prominent special cases of the structure theorem for Hopf modules over Hopf-Galois extensions that is one direction of the characterization goes under the name of Galois descent.

Example 2.3.3 Let A/k be a Galois field extension with Galois group G. A comodule structure making an A-vector space into a Hopf module $M \in \mathcal{M}_A^{kG}$ is the same as an action of the Galois group G on M by semilinear automorphisms, i.e. in such a way that $\sigma \cdot (am) = \sigma(a)(\sigma \cdot m)$ holds for all $m \in M$, $a \in A$ and $\sigma \in G$. Galois descent (see for example [23]) says, most of all, that such an action on M forces M to be obtained from a k-vector space by extending scalars. This is (part of) the content of the structure theorem for Hopf modules.

Remark 2.3.4 Let A be an H-comodule algebra; put $B := A^{\mathrm{co}\,H}$. As a direct generalization of the Galois map $\beta\colon A \otimes_B A \to A \otimes H$, we have a right A-module map

$$\beta_M\colon M \underset{B}{\otimes} A \ni m \otimes a \mapsto ma_{(0)} \otimes a_{(1)} \in M. \otimes H.$$

which is natural in $M \in \mathcal{M}_A$. Of course, the Galois map is recovered as $\beta = \beta_A$. Note that β_M can be identified with $M \otimes_A \beta_A$, so that all β_M are bijective once β_A is bijective.

Lemma 2.3.5 *Let A be a right H-comodule algebra, and $B := A^{\mathrm{co}\,H}$.*
For each descent data $(M, \theta) \in \mathcal{D}(A \downarrow B)$, the map

$$\delta := \left(M \xrightarrow{\theta} M \underset{B}{\otimes} A \xrightarrow{\beta_M} M \otimes H\right)$$

is a right H-comodule structure on M making $M \in \mathcal{M}_A^H$.
Thus, we have defined a functor $\mathcal{D}(A \downarrow B) \to \mathcal{M}_A^H$.
If A is an H-Galois extension of B, then the functor is an equivalence.

Proof Let $\theta\colon M \to M \otimes_B A$ be a right A-module map, and $\delta := \beta_M \theta$. Of course δ is a right A-module map, so that M is a Hopf module if and only if it is a comodule.

Now we have the commutative diagrams

$$\begin{tikzcd}
M \arrow[r, "\theta"] \arrow[dr, "\delta"'] & M \otimes_B A \arrow[r, "\theta \otimes_B A"] \arrow[d, "\beta_M"] \arrow[drr, "\delta \otimes_B A"] & M \otimes_B A \otimes_B A \arrow[r, phantom, ""] & \\
& M \otimes H \arrow[drr, "\delta \otimes H"'] & & (M \otimes H) \otimes_B A \arrow[d, "\beta_{M \otimes H}"] \\
& & & M \otimes H \otimes H
\end{tikzcd}$$

using naturality of β with respect to the right A-module map δ, and

$$
\begin{tikzcd}
M \arrow[r, "\theta"] \arrow[dr, "\delta"'] & M \otimes_B A \arrow[r, "\eta_2"] \arrow[d, "\beta_M"] \arrow[dr, "M\otimes\eta\otimes_B A"] & M \otimes_B A \otimes_B A \arrow[d, "\beta_M \otimes_B A"] \\
 & M \otimes H \arrow[dr, "M\otimes\Delta"'] & (M \otimes H) \otimes_B A \arrow[d, "\beta_{M\otimes H}"] \\
 & & M \otimes H \otimes H
\end{tikzcd}
$$

using $\beta_{M\otimes H}(m\otimes 1\otimes a) = (m\otimes 1)a_{(0)}\otimes a_{(1)} = ma_{(0)}\otimes a_{(1)}\otimes a_{(2)} = (M\otimes\Delta)\beta_M(m\otimes a)$. Moreover

$$
\begin{tikzcd}
M \arrow[r, "\theta"] \arrow[dr, "\delta"'] & M \otimes_B A \arrow[r, "\mu"] \arrow[d, "\beta_M"] & M \\
 & M \otimes H \arrow[ur, "M\otimes\varepsilon"'] &
\end{tikzcd}
$$

also commutes. Thus, if θ is a descent data, then δ is a comodule.

Conversely, if β is bijective, then the natural transformation β_M is an isomorphism. In particular the formula $\delta = \beta_M\theta$ defines a bijective correspondence between A-module maps $\theta\colon M \to M \otimes_B A$ and $\delta\colon M \to M_{\bullet} \otimes H_{\bullet}$. The same diagrams as above show that δ is a comodule structure if and only if θ is a descent data. $\square$

Schneider's structure theorem for Hopf modules is now an immediate consequence of faithfully flat descent:

Corollary 2.3.6 *The following are equivalent for an H-comodule algebra A:*

1. *A is an H-Galois extension of $B := A^{\operatorname{co} H}$, and faithfully flat as left B-module.*
2. *The functor $\mathcal{M}_B \ni N \mapsto N \otimes_B A \in \mathcal{M}_A^H$ is an equivalence.*

Proof (1)$\Rightarrow$(2): We have established an equivalence $\mathcal{D}(A \downarrow B) \to \mathcal{M}_A^H$, and it is easy to check that the diagram

$$
\begin{tikzcd}
\mathcal{D}(A \downarrow B) \arrow[rr, "\sim"] \arrow[dr, "(-)^{\theta}"'] & & \mathcal{M}_A^H \arrow[dl, "(-)^{\operatorname{co} H}"] \\
 & \mathcal{M}_B &
\end{tikzcd}
\tag{2.3.1}
$$

commutes. Thus the coinvariants functor is an equivalence by faithfully flat descent.

(2)$\Rightarrow$(1): Since $(-)^{\operatorname{co} H}\colon \mathcal{M}_A^H \to \mathcal{M}_B$ is an equivalence, and $\beta\colon A \otimes_B A^{\bullet}_{\bullet} \to A_{\bullet} \otimes H^{\bullet}_{\bullet}$ is a Hopf module homomorphism, β is an isomorphism if and only if $\beta^{\operatorname{co} H}$ is. But

$$A \cong A \underset{B}{\otimes} A^{\operatorname{co} H} \xrightarrow{\beta^{\operatorname{co} H}} A \otimes H^{\operatorname{co} H} \cong A$$

is easily checked to be the identity. Thus, A is an H-Galois extension of B. It is faithfully flat since $(-) \otimes_B A\colon \mathcal{M}_B \to \mathcal{M}_A^H$ is an equivalence. $\square$

To shed some further light on the connection between descent data and the Galois map, it may be interesting to prove a partial converse to Lemma 2.3.5:

Proposition 2.3.7 *Let H be a bialgebra, A a right H-comodule algebra, and $B = A^{\mathrm{co}\,H}$.*

If the natural functor $\mathcal{D}(A \downarrow B) \to \mathcal{M}_A^H$ is an equivalence, then the Galois map $\beta\colon A \otimes_B A \to A \otimes H$ is surjective.

If, moreover, A is flat as left B-module, then A is an H-Galois extension of B.

Proof By assumption there is an A-module map $\theta = \theta_M\colon M \to M \otimes_B A$, natural in $M \in \mathcal{M}_A^H$, such that the H-comodule structure of M is given by $\delta_M = \beta_M \theta_M$.

Specializing $M = V^{\boldsymbol{\cdot}} \otimes A^{\boldsymbol{:}}_{\boldsymbol{\cdot}}$ for $V \in \mathcal{M}^H$, we obtain a natural A-module map $\tilde{\theta}_V\colon V \otimes A \to V \otimes A \otimes_B A$, which, being an A-module map, is determined by $\phi_V\colon V \to V \otimes A \otimes_B A$. Finally, since ϕ_V is natural, it has the form

$$\phi_V(v) = v_{(0)} \otimes {v_{(1)}}^{[1]} \otimes {v_{(1)}}^{[2]}$$

for the map $\gamma\colon H \ni h \mapsto h^{[1]} \otimes h^{[2]} \in A \otimes_B A$ defined by $\gamma = (\varepsilon \otimes A \otimes_B A)\phi_H$. In particular we have $\tilde{\theta}_V(v \otimes a) = v_{(0)} \otimes {v_{(1)}}^{[1]} \otimes {v_{(1)}}^{[2]}a$, and hence, specializing $V = H$ and $a = 1$:

$$\begin{aligned} h_{(1)} \otimes 1 \otimes h_{(2)} &= \delta_{H \otimes A}(h \otimes 1) \\ &= \beta_{H \otimes A}\theta_{H \otimes A}(h \otimes 1) \\ &= \beta_{H \otimes A}(h_{(1)} \otimes {h_{(2)}}^{[1]} \otimes {h_{(2)}}^{[2]}) \\ &= h_{(1)} \otimes {h_{(2)}}^{[1]} {h_{(2)}}^{[2]}{}_{(0)} \otimes {h_{(2)}}^{[2]}{}_{(1)} \\ &= h_{(1)} \otimes \beta_A(h^{[1]} \otimes h^{[2]}) \end{aligned}$$

for all $h \in H$, and thus $\beta(ah^{[1]} \otimes h^{[2]}) = a \otimes h$ for all $a \in A$.

If A is left B-flat, then $\theta_M(m) \in M^\theta \otimes_B A \subset M^{\mathrm{co}\,H} \otimes_B A$ implies, in particular, that $a_{(0)} \otimes {a_{(1)}}^{[1]} \otimes {a_{(1)}}^{[2]} \in (A \otimes A)^{\mathrm{co}\,H} \otimes_B A$, and thus $a_{(0)}{a_{(1)}}^{[1]} \otimes {a_{(1)}}^{[2]} \in B \otimes A$. Hence $\beta^{-1}(a \otimes h) = ah^{[1]} \otimes h^{[2]}$ is actually (not only right) inverse to β by the calculation $\beta^{-1}\beta(x \otimes y) = \beta^{-1}(xy_{(0)} \otimes y_{(1)}) = xy_{(0)}{y_{(1)}}^{[1]} \otimes {y_{(1)}}^{[2]} = x \otimes y_{(0)}{y_{(1)}}^{[1]}{y_{(1)}}^{[2]} = x \otimes y$. □

2.4 Coflat Galois extensions. A faithfully flat H-Galois extension is easily seen to be a faithfully coflat H-comodule:

Lemma 2.4.1 *Let H be a k-flat Hopf algebra, and A an H-Galois extension of B. If A_B is faithfully flat and A is a faithfully flat k-module, then A is a faithfully coflat H-comodule.*

Proof If A_B is flat, then we have an isomorphism, natural in $V \in {}^H\mathcal{M}$:

$$A \underset{B}{\otimes} (A \underset{H}{\Box} V) \cong (A \underset{B}{\otimes} A) \underset{H}{\Box} V \cong (A \otimes H) \underset{H}{\Box} V \cong A \otimes V.$$

If A_B is faithfully flat and A is faithfully flat over k, then it follows that the functor $A \Box_H -$ is exact and reflects exact sequences. □

The converse is trivial if $B = k$, for then any (faithfully) coflat comodule is a (faithfully) flat k-module by the definition we chose for coflatness. This is not at all clear if B is arbitrary. However, it is true if k is a field. In this case much more can be said. Schneider [44] has proved that a coflat H-comodule algebra A is already a faithfully flat (on either side) Hopf-Galois extension if we only assume that the Galois map is surjective, and the antipode of H is bijective. We will give a different proof of this characterization of faithfully flat Hopf-Galois extensions. Like the original, it is based on Takeuchi's result that coflatness and injectivity coincide for comodules if k is a field, and on a result of Doi on injective comodule algebras (for which, again, we will give a slightly different proof). Our proof of Schneider's criterion will have a nice byproduct: In the case that k is a field and the Hopf algebra H has bijective antipode, every faithfully flat H-Galois extension is a projective module (on either side) over its coinvariants.

Before going into any details, let us comment very briefly on the algebro-geometric meaning of Hopf-Galois extensions and the criterion. If H is the (commutative) Hopf algebra representing an affine group scheme G, A the algebra of an affine scheme X on which H acts, and Y the affine scheme represented by $A^{\mathrm{co}\,H}$, then A is a faithfully flat H-Galois extension of B if and only if the morphism $X \to Y$ is faithfully flat, and the map $X \times G \to X \times_Y X$ given on elements by $(x, g) \mapsto (x, xg)$ is an isomorphism of affine schemes. This means that X is an affine scheme with an action of G and a projection to the invariant quotient Y which is locally trivial in the faithfully flat topology (becomes trivial after a faithfully flat extension of the base Y). This is the algebro-geometric version of a principal fiber bundle with structure group G, or a G-torsor [11]. If we merely require the canonical map $A \otimes A \to A \otimes H$ to be surjective, this means that we require the map $X \times G \to X \times X$ given by $(x, g) \mapsto (x, xg)$ to be a closed embedding, or that we require the action of G on X to be free. Thus, the criterion we are dealing with in this section says that under the coflatness condition on the comodule structure freeness of the action is sufficient to have a principal fiber bundle. Note in particular that surjectivity of the canonical map is trivial in the case where H is a quotient Hopf algebra of a Hopf algebra A (or G is a closed subgroup scheme of an affine group scheme X), while coflatness in this case is a representation theoretic condition (the induction functor is exact). See [44] for further literature.

For the rest of this section, we assume that k is a field.

We start by an easy and well-known observation regarding projectivity of modules over a Hopf algebra.

Lemma 2.4.2 *Let H be a Hopf algebra and $M, P \in {}_H\mathcal{M}$ with P projective. Then $.P \otimes .M \in {}_H\mathcal{M}$ is projective. In particular, H is semisimple if and only if the trivial H-module is projective.*

Proof The second statement follows from the first, since every module is its own tensor product with the trivial module. The diagonal module $.H \otimes .M$ is free by the structure theorem for Hopf modules, or since

$$.H \otimes M \ni h \otimes m \mapsto h_{(1)} \otimes h_{(2)}m \in .H \otimes .M$$

is an isomorphism. Since any projective P is a direct summand of a direct sum of copies of H, the general statement follows. $\square$

If H has bijective antipode, then in the situation of the Lemma also $M \otimes P$ is projective.

For our proof, we will need the dual variant. To prepare, we observe:

Lemma 2.4.3 *Let C be a coalgebra and $M \in \mathcal{M}^C$. Then M is injective if and only if it is a direct summand of $V \otimes C^{\bullet} \in \mathcal{M}^C$ for some $V \in \mathcal{M}_k$. In particular, if M is injective, then so is every $V \otimes M^{\bullet} \in \mathcal{M}^C$ for $V \in \mathcal{M}_k$.*

Lemma 2.4.4 *Let H be a Hopf algebra and $M, I \in \mathcal{M}^H$ with I injective. Then $M^{\bullet} \otimes I^{\bullet} \in \mathcal{M}^H$ is injective.*

Proof Since I is a direct summand of some $V \otimes H^{\bullet}$, it is enough to treat the case $I = H$. But then

$$M^{\bullet} \otimes H^{\bullet} \ni m \otimes h \mapsto m_{(0)} \otimes m_{(1)}h \in M \otimes H^{\bullet}$$

is a colinear bijection, and $M \otimes H^{\bullet}$ is injective. □

We come to a key property of comodule algebras that are injective comodules, which is due to Doi [13]:

Proposition 2.4.5 *Let H be a Hopf algebra and A an H-comodule algebra that is an injective H-comodule. Then every Hopf module in $\mathcal{M}^H_A$ is an injective H-comodule. If H has bijective antipode, then also every Hopf module in ${}_A\mathcal{M}^H$ is an injective H-module.*

Proof Let $M \in \mathcal{M}^H_A$. Since the module structure $\mu\colon M^{\bullet} \otimes A^{\bullet} \to M$ is H-colinear, and splits as a colinear map via $M \ni m \mapsto m \otimes 1 \in M \otimes A$, the comodule M is a direct summand of the diagonal comodule $M \otimes A$. The latter is injective, since A is. The statement on Hopf modules in ${}_A\mathcal{M}^H$ follows since H^{op} is a Hopf algebra and we can identify ${}_A\mathcal{M}^H$ with $\mathcal{M}^{H^{\mathrm{op}}}_{A^{\mathrm{op}}}$. □

Lemma 2.4.6 *The canonical map $\beta_0\colon A \otimes A \to A \otimes H$ is a morphism of Hopf modules in ${}_A\mathcal{M}^H$ if we equip its source and target with the obvious left A-module structures, the source with the comodule structure coming from the left tensor factor, and its target with the comodule structure given by $(a \otimes h)_{(0)} \otimes (a \otimes h)_{(1)} = a_{(0)} \otimes h_{(2)} \otimes a_{(1)}S(h_{(1)})$. The latter can be viewed as a codiagonal comodule structure, if we first endow H with the comodule structure restricted along the antipode. Thus we may write briefly that*

$$\beta_0\colon {}_{\bullet}A^{\bullet} \otimes A \to {}_{\bullet}A^{\bullet} \otimes H^S$$

is a morphism in ${}_A\mathcal{M}^H$.

Proposition 2.4.7 *Let H be a Hopf algebra, and A a right H-comodule algebra; put $B := A^{\mathrm{co}\,H}$. Assume there is an H-comodule map $\gamma\colon H^S \to A^{\bullet} \otimes A$ such that $\beta(\gamma(h)) = 1 \otimes h$ for all $h \in H$ (where we abuse notations and also consider $\gamma(h) \in A \otimes_B A$).*

Then the counit $M^{\mathrm{co}\,H} \otimes_B A \to M$ of the adjunction in Lemma 2.3.2 is an isomorphism for every $M \in \mathcal{M}^H_A$. Its inverse lifts to a natural transformation $M \to M^{\mathrm{co}\,H} \otimes A$ (with the tensor product over k).

In particular A is an H-Galois extension of B, and a projective left B-module.

Proof We shall write $\gamma(h) =: h^{[1]} \otimes h^{[2]}$. This is to some extent an abuse of notations, since the same symbol was used for the map $H \to A \otimes_B A$ induced by the inverse of the canonical map in a Hopf-Galois extension. However, the abuse is not so bad, because in fact the map we use in the present proof will turn out to induce that inverse. Our assumptions on γ read ${h_{(2)}}^{[1]} \otimes {h_{(2)}}^{[2]} \otimes S(h_{(1)}) = {h^{[1]}}_{(0)} \otimes h^{[2]} \otimes {h^{[1]}}_{(1)}$ and $h^{[1]}{h^{[2]}}_{(0)} \otimes {h^{[2]}}_{(1)} = 1 \otimes h \in A \otimes H$ for all $h \in H$. The latter implies in particular that $h^{[1]}h^{[2]} = \varepsilon(h)1_A$.

It follows for all $m \in M \in \mathcal{M}_A^H$ that $m_{(0)}m_{(1)}{}^{[1]} \otimes m_{(1)}{}^{[2]} \in M^{\mathrm{co}\,H} \otimes A$; indeed $\rho(m_{(0)}m_{(1)}{}^{[1]}) \otimes m_{(1)}{}^{[2]} = m_{(0)}m_{(3)}{}^{[1]} \otimes m_{(1)}S(m_{(2)}) \otimes m_{(3)}{}^{[2]} = m_{(0)}m_{(1)}{}^{[1]} \otimes 1 \otimes m_{(1)}{}^{[2]}$.

Now we can write down the natural transformation $\psi\colon M \ni m \mapsto m_{(0)}m_{(1)}{}^{[1]} \otimes m_{(1)}{}^{[2]} \in M^{\mathrm{co}\,H} \otimes A$, and define $\vartheta\colon M \to M^{\mathrm{co}\,H} \otimes_B A$ as the composition of ψ with the canonical surjection.

We claim that ϑ is inverse to the adjunction map $\phi\colon M^{\mathrm{co}\,H} \otimes_B A \to M$. Indeed

$$\phi\vartheta(m) = \phi(m_{(0)}m_{(1)}{}^{[1]} \otimes m_{(1)}{}^{[2]}) = m_{(0)}m_{(1)}{}^{[1]}m_{(1)}{}^{[2]} = m$$

and

$$\vartheta\phi(n \otimes a) = \phi^{-1}(na) = na_{(0)}a_{(1)}{}^{[1]} \otimes a_{(1)}{}^{[2]} = n \otimes a_{(0)}a_{(1)}{}^{[1]}a_{(1)}{}^{[2]} = n \otimes a,$$

using that $a_{(0)}a_{(1)}{}^{[1]} \otimes a_{(1)}{}^{[2]} \in B \otimes A$.

Since the adjunction map is an isomorphism, A is an H-Galois extension of B.

The instance $\psi_A\colon A \ni a \mapsto a_{(0)}a_{(1)}{}^{[1]} \otimes a_{(2)}{}^{[2]} \in B \otimes A$ of ψ splits the multiplication map $B \otimes A \to A$, so that A is a direct summand of $B \otimes A$ as left B-module, and hence a projective B-module. □

Corollary 2.4.8 *Let H be a Hopf algebra and A a right H-comodule algebra such that that the canonical map $\beta_0\colon A \otimes A \to A \otimes H$ is a surjection.*

Assume in addition that $\beta_0\colon A^{\bullet} \otimes A \to A^{\bullet} \otimes H^S$ splits as a comodule map for the indicated H-comodule structures. Then A is a right H-Galois extension of B and a projective left B-module.

In particular, the assumption can be verified in the following cases:

1. *H is finite dimensional.*
2. *A is injective as H-comodule, and H has bijective antipode.*

Proof First, if β_0 splits as indicated via a map $\alpha\colon A^{\bullet} \otimes H^S \to A^{\bullet} \otimes A$ with $\beta_0\alpha = id$, then the composition

$$\gamma = \left(H \xrightarrow{\eta\otimes H} A \otimes H \xrightarrow{\alpha} A \otimes A\right)$$

satisfies the assumptions of Proposition 2.4.7.

If A is an injective comodule, and H has bijective antipode, then every Hopf module in ${}_A\mathcal{M}^H$ is an injective comodule by Proposition 2.4.5. Thus the (kernel of the) Hopf module morphism β_0 splits as a comodule map. Finally, if H is finite dimensional, then we take the view that β_0 should split as a surjective H^*-module map. But H^S is projective as H^*-module, and hence $A \otimes H^S$ is projective as well, and thus the map splits. □

As a corollary, we obtain Schneider's characterization of faithfully flat Hopf-Galois extensions from [44] (and in addition projectivity of such extensions).

Corollary 2.4.9 *Let H be a Hopf algebra with bijective antipode over a base field k, A a right H-comodule algebra, and $B := A^{\mathrm{co}\,H}$. The following are equivalent:*

1. *The Galois map $A \otimes A \to A \otimes H$ is onto, and A is injective as H-comodule.*
2. *A is an H-Galois extension of B, and right faithfully flat as B-module.*
3. *A is an H-Galois extension of B, and left faithfully flat as B-module.*

In this case, A is a projective left and right B-module.

Proof We already know from the beginning of this section that 2⇒1. Assume 1. Then Corollary 2.4.8 implies that A is Galois and a projective left B-module, and that the counit of the adjunction in Lemma 2.3.2 is an isomorphism. By Corollary 2.3.6 it remains to prove that the unit $N \to (N \otimes_B A)^{\mathrm{co}\,H}$ is also a bijection for all $N \in \mathcal{M}_B$. But $N \otimes_B A$ is defined by a coequalizer

$$N \otimes B \otimes A \rightrightarrows N \otimes A \to N \underset{B}{\otimes} A \to 0,$$

which is a coequalizer in the category $\mathcal{M}_A^H$. Since every Hopf module is an injective comodule, every short exact sequence in $\mathcal{M}_A^H$ splits colinearly, so the coinvariants functor $\mathcal{M}_A^H \to \mathcal{M}_B$ is exact, and applying it to the coequalizer above we obtain a coequalizer

$$N \otimes B \otimes A \rightrightarrows N \otimes B \to (N \underset{B}{\otimes} A)^{\mathrm{co}\,H}$$

which says that $(N \otimes_B A)^{\mathrm{co}\,H} \cong N \otimes_B B \cong N$.

The equivalence of 1 and 3 is proved by applying that of 1 and 2 to the H^{op} comodule algebra A^{op}. □

2.5 Galois extensions as monoidal functors. In this section we prove the characterization of Hopf-Galois extensions as monoidal functors from the category of comodules due to Ulbrich [53, 54]. We are somewhat more general in allowing the invariant subring to be different from the base ring. In this general setting, we have proved one direction of the characterization in [35], but the proof is really no different from Ulbrich's. Some details of the reverse direction (from functors to extensions) are perhaps new. It will turn out that in fact suitably exact *weak* monoidal functors on the category of comodules are the same as comodule algebras, while being monoidal rather than only weak monoidal is related to the Galois condition.

Proposition 2.5.1 *Let H be a bialgebra, and $A \in \mathcal{M}^H$ coflat.*
If A is an H-comodule algebra, then

$$\xi\colon (A \underset{H}{\square} V) \otimes (A \underset{H}{\square} W) \ni (x \otimes v) \otimes (y \otimes w) \mapsto xy \otimes v \otimes w \in A \underset{H}{\square} (V \otimes W)$$

and $\xi_0\colon k \ni \alpha \mapsto 1 \otimes \alpha \in A \square_H k$ define the structure of a weak monoidal functor on $A \square_H -\colon {}^H\mathcal{M} \to \mathcal{M}_k$.

Conversely, every weak monoidal functor structure on $A \square_H -$ has the above form for a unique H-comodule algebra structure on A.

Proof The first claim is easy to check. For the second, given a monoidal functor structure ξ, define multiplication on A as the composition

$$A \otimes A \cong (A \underset{H}{\square} H) \otimes (A \underset{H}{\square} H) \xrightarrow{\xi} A \underset{H}{\square} (H \otimes H) \xrightarrow{A \square \nabla} A \underset{H}{\square} H \cong A.$$

By naturality of ξ in its right argument, applied to $\Delta\colon H \to {}^{\bullet}H \otimes H$, we have a commutative diagram

$$\begin{tikzcd}
(A \square_H H) \otimes (A \square_H H) \arrow[r, "\xi"] \arrow[d, "A\square_H H \otimes A \square_H \Delta"'] & A \square_H (H \otimes H) \arrow[d, "A\square_H (H\otimes\Delta)"] \\
(A \square_H H) \otimes (A \square_H (H \otimes H)) \arrow[d] & A \square_H (H \otimes H \otimes H) \arrow[d] \\
(A \square_H H) \otimes (A \square_H H) \otimes H \arrow[r, "\xi\otimes H"] & (A \square_H (H \otimes H)) \otimes H
\end{tikzcd}$$

In other words, $\xi\colon (A \square_H H) \otimes (A \square_H H^{\bullet}) \to A \square_H (H \otimes H^{\bullet})$ is an H-comodule map with respect to the indicated structures. Similarly (though a little more complicated to write), $\xi\colon (A \square_H H^{\bullet}) \otimes (A \square_H H) \to A \square_H (H^{\bullet} \otimes H)$ is also colinear, and from both we deduce that $\xi\colon (A \square_H H^{\bullet}) \otimes (A \square_H H^{\bullet}) \to A \square_H (H^{\bullet} \otimes H^{\bullet})$ is colinear. Hence the muliplication on A is colinear. Associativity of multiplication follows from coherence of ξ, so that A is a comodule algebra. $\square$

Corollary 2.5.2 *Let H be a bialgebra, A a right H-comodule algebra, and $\iota\colon B \to A^{\mathrm{co}\,H}$ a subalgebra.*

Then for each $V \in {}^H\mathcal{M}$ we have $A \square_H V \in {}_B\mathcal{M}_B$ with bimodule structure induced by that of A (induced in turn by ι). The weak monoidal functor structure in Proposition 2.5.1 induces a weak monoidal functor structure on $A \square_H (-)\colon {}^H\mathcal{M} \to {}_B\mathcal{M}_B$, which we denote again by

$$\xi\colon (A \underset{H}{\square} V) \underset{B}{\otimes} (A \underset{H}{\square} W) \ni x \otimes v \otimes y \otimes w \mapsto xy \otimes v \otimes w \in A \underset{H}{\square} (V \otimes W)$$

and $\xi_0\colon B \ni b \mapsto b \otimes 1 \in A \square_H k$. If A is a (faithfully) coflat H-comodule, the functor is (faithfully) exact

Every exact weak monoidal functor ${}^H\mathcal{M} \to {}_B\mathcal{M}_B$ commuting with arbitrary direct sums, for a k-algebra B, has this form.

Proof Again, it is not hard to verify that every comodule algebra A and homomorphism ι gives rise to a weak monoidal functor as stated. For the converse, note that a weak monoidal functor ${}^H\mathcal{M} \to {}_B\mathcal{M}_B$ can be composed with the weak monoidal underlying functor ${}_B\mathcal{M}_B \to \mathcal{M}_k$ to yield a weak monoidal functor ${}^H\mathcal{M} \to \mathcal{M}_k$. The latter is exact by assumption, so has the form $V \mapsto A \square_H V$ for some coflat H-comodule A by Lemma 4.3.3, and A is an H-comodule algebra by Proposition 2.5.1. One ingredient of the weak monoidal functor structure that we assume to exist is a B-B-bimodule map $\xi_0\colon B \to A \square_H k$ with $\xi_0(1) = 1$, which has the form $\xi_0(b) = \iota(b) \otimes 1$ for some map $\iota\colon B \to A^{\mathrm{co}\,H}$ that also satisfies $\iota(b) = 1$. By coherence of the weak monoidal functor, the left B-module structure of $A \square_H V$, which is also one of the coherence isomorphisms of the monoidal category of B-B-bimodules, is given by

$$B \underset{B}{\otimes} (A \underset{H}{\square} V) \xrightarrow{\xi_0 \otimes id} (A \underset{H}{\square} k) \underset{B}{\otimes} (A \underset{H}{\square} V) \xrightarrow{\xi} A \underset{H}{\square} V.$$

Thus $b \cdot (x \otimes v) = \iota(b)x \otimes v$ holds for all $b \in B$ and $x \otimes v \in A \square_H V$. If we specialize $V = H$ and use the isomorphism $A \square_H H$, we see that ι is an algebra homomorphism, and for general V we see that $A \square_H V$ has the claimed B-B-bimodule structure. $\square$

Theorem 2.5.3 *Let H be a k-flat Hopf algebra, and B a k-algebra.*

1. *Every exact monoidal functor $\mathcal{F}\colon {}^H\mathcal{M} \to {}_B\mathcal{M}_B$ that commutes with arbitrary colimits has the form $\mathcal{F}(V) = A \Box_H V$ for some right coflat H-Galois extension A of B, with monoidal functor structure given as in Corollary 2.5.2.*
2. *Assume that A is a right faithfully flat H-Galois extension of B. Then the weak monoidal functor $A \Box_H$ — as in Corollary 2.5.2 is monoidal.*

If we assume that k is a field, and H has bijective antipode, then a Hopf-Galois extension is coflat as H-comodule if and only if it is faithfully flat as right (or left) B-module. Also, if k is arbitrary, then a Hopf-Galois extension of k is faithfully coflat as H-comodule if and only if it is faithfully flat as k-module. Thus we have:

Corollary 2.5.4 *Let H be a Hopf algebra and B a k-algebra. Assume either of the following conditions:*

1. *k is a field and the antipode of H is bijective.*
2. *$B = k$.*

Then Corollary 2.5.2 establishes a bijective correspondence between exact monoidal functors ${}^H\mathcal{M} \to {}_B\mathcal{M}_B$ and faithfully flat H-Galois extensions of B.

Closing the section, let us give two curious application of the monoidal functor associated to a Galois object.

If H is a Hopf algebra, then any $V \in {}^H\mathcal{M}$ that is a finitely generated projective k-module has a right dual object in the monoidal category ${}^H\mathcal{M}$. Monoidal functors preserve duals. Thus, whenever A is a right faithfully flat H-Galois extension of B, the B-bimodule $A \Box_H V$ will have a right dual in the monoidal category of B-bimodules. This in turn means that $A \Box_H V$ is finitely generated projective as a left B-module. We have proved:

Corollary 2.5.5 *Let A be a right H-Galois extension of B and a right faithfully flat B-module. Then for every $V \in {}^H\mathcal{M}$ which is a finitely generated projective k-module, the left B-module $A \Box_H V$ is finitely generated projective. If H has bijective antipode, the right B-module $A \Box_H V$ is also finitely generated projective.*

The corollary (which has other proofs as well) has a conceptual meaning when we think of A as a principal fiber bundle with structure quantum group H. Then $A \Box_H V$ is analogous to the module of sections in an associated vector bundle with fiber V, and it is of course good to know that such a module of sections is projective, in keeping with the classical Serre-Swan theorem.

Definition 2.5.6 Let H be a k-flat Hopf algebra, and B a k-algebra. We define $\mathrm{Gal}_B(H)$ to be the set of all isomorphism classes of H-Galois extensions of B that are faithfully flat as right B-modules and (faithfully) flat as k-modules. We write $\mathrm{Gal}(H) = \mathrm{Gal}_B(H)$.

Proposition 2.5.7 $\mathrm{Gal}_B(-)$ *is a contravariant functor. For a Hopf algebra map $f\colon F \to H$ between k-flat Hopf algebras, the map* $\mathrm{Gal}_B(f)\colon \mathrm{Gal}_B(H) \to \mathrm{Gal}_B(F)$ *maps the isomorphism class of A to that of $A \Box_H F$.*

Proof In fact, f defines an exact monoidal functor ${}^F\mathcal{M} \to {}^H\mathcal{M}$, which composes with the monoidal functor $A \Box_H (-)\colon {}^H\mathcal{M} \to {}_B\mathcal{M}_B$ defined by A to give the functor $(A \Box_H F) \Box_F$ —, since $A \Box_H V = A \Box_H (F \Box_F V) \cong (A \Box_H F) \Box_F V$ by k-flatness of A. This implies that $A \Box_H F$ is a right F-Galois extension of B.

It is faithfully flat on the right since A is, and for any left B-module M we have $A \otimes_B (A \square_H F) \otimes_B N \cong ((A \otimes_B A) \square_H F) \otimes_B N \cong (A \otimes H \square_H F) \otimes_B N \cong A \otimes_B N \otimes H$. $\square$

2.6 Hopf bimodules. Let A be an H-comodule algebra. Since A is an algebra in the monoidal category of H-comodules, we can consider the category of bimodules over A in the monoidal category $\mathcal{M}^H$. Such a bimodule $M \in {}_A\mathcal{M}_A^H$ is an A-bimodule fulfilling both Hopf module conditions for a Hopf module in $\mathcal{M}_A^H$ and ${}_A\mathcal{M}^H$. By the general theory of modules over algebras in monoidal categories, the category ${}_A\mathcal{M}_A^H$ is a monoidal category with respect to the tensor product over A. Now without further conditions, taking coinvariants gives a *weak* monoidal functor:

Lemma 2.6.1 *Let A be an H-comodule algebra, and let $B \subset A^{\mathrm{co}\,H}$ be a subalgebra. Then*

$${}_A\mathcal{M}_A^H \ni M \mapsto M^{\mathrm{co}\,H} \in {}_B\mathcal{M}_B$$

is a weak monoidal functor with structure maps

$$\xi_0 \colon M^{\mathrm{co}\,H} \underset{B}{\otimes} N^{\mathrm{co}\,H} \ni m \otimes n \mapsto m \otimes n \in (M \underset{A}{\otimes} N)^{\mathrm{co}\,H}$$

and $\xi_0 \colon B \to A^{\mathrm{co}\,H}$ the inclusion.

The proof is straightforward. The main result of this section is that the functor from the Lemma is monoidal rather than only weak monoidal if and only if A is an H-Galois extension. The precise statement is slightly weaker:

Proposition 2.6.2 *Let H be a Hopf algebra, A a right H-comodule algebra, and $B := A^{\mathrm{co}\,H}$.*

If A is a left faithfully flat H-Galois extension of B, then the weak monoidal functor from Lemma 2.6.1 is monoidal.

Conversely, if the weak monoidal functor from Lemma 2.6.1 is monoidal, then the counit of the adjunction 2.3.2 is an isomorphism, and in particular, A is an H-Galois extension of B.

Proof If A is a left faithfully flat H-Galois extension of B, then ξ is an isomorphism if and only if $\xi \otimes_B A$ is. But via the isomorphisms

$$M \underset{A}{\otimes} N \cong M^{\mathrm{co}\,H} \underset{B}{\otimes} A \underset{A}{\otimes} N \cong M^{\mathrm{co}\,H} \underset{B}{\otimes} N \cong M^{\mathrm{co}\,H} \underset{B}{\otimes} N^{\mathrm{co}\,H} \underset{B}{\otimes} A$$

and $(M \otimes_A N)^{\mathrm{co}\,H} \otimes_B A \cong M \otimes_A N$, the map $\xi \otimes_B A \colon M^{\mathrm{co}\,H} \otimes_B N^{\mathrm{co}\,H} \otimes_B A \to (M \otimes_A N)^{\mathrm{co}\,H} \otimes_B A$ identifies with the identity on $M \otimes_A N$.

Conversely, if ξ is an isomorphism, we can specialize $N := .A. \otimes .H^{\bullet}_{\bullet} \in {}_A\mathcal{M}_A^H$. We have $N^{\mathrm{co}\,H} \cong A$. In ${}_A\mathcal{M}^H$ we have an isomorphism

$$.A^{\bullet} \otimes H^{\bullet} \cong .A \otimes .H^{\bullet}; a \otimes h \mapsto a_{(0)} \otimes a_{(1)}h.$$

Thus we have an isomorphism

$$M \underset{A}{\otimes} N \cong M \underset{A}{\otimes} (.A^{\bullet} \otimes H^{\bullet}) \cong M^{\bullet} \otimes H^{\bullet}; m \otimes a \otimes h \mapsto ma_{(0)} \otimes S(a_{(1)})h.$$

composing with $M^{\bullet} \otimes H^{\bullet} \ni m \otimes h \mapsto m_{(0)} \otimes m_{(1)}h \in M \otimes H^{\bullet}$ yields the isomorphism

$$M \underset{A}{\otimes} N \cong M \otimes H^{\bullet}; m \otimes a \otimes h \mapsto m_{(0)}a \otimes m_{(1)}h.$$

Thus we find that

$$M^{\mathrm{co}\,H} \underset{B}{\otimes} A \cong M^{\mathrm{co}\,H} \underset{B}{\otimes} N^{\mathrm{co}\,H} \xrightarrow{\xi} (M \underset{A}{\otimes} N)^{\mathrm{co}\,H} \cong (M \otimes H^{\bullet})^{\mathrm{co}\,H} \cong M$$

maps $m \otimes a$ to ma, hence is the adjunction counit in question. $\square$

2.7 Reduction. We have already seen that $\mathrm{Gal}_B(—)$ is a functor. In particular, we have a map $\mathrm{Gal}_B(Q) \to \mathrm{Gal}_B(H)$ for any (suitable) quotient Hopf algebra Q of H. In this section we will be concerned with the image and fibers of this map. The question has a geometric interpretation when we think of Galois extensions as principal fiber bundles: It is then the question under what circumstances a principal bundle with structure group G can be reduced to a principal bundle whose structure group is a prescribed subgroup of G.

The results in this section were proved first in [37] for the case of conormal quotients Q (i.e. normal subgroups, when we think of principal homogeneous spaces). The general case was obtained in [20, 21]. The proof we give here was essentially given in [43]; we rewrite it here with (yet) more emphasis on its background in the theory of algebras in monoidal categories. We begin with a Theorem of Takeuchi [49] on Hopf modules for a quotient of a Hopf algebra. We prove a special case in a new way here, which we do not claim to be particularly natural, but which only uses category equivalences that we have already proved above.

Theorem 2.7.1 *Let H be a k-flat Hopf algebra, and $H \to Q$ a quotient Hopf algebra of H which is also k-flat and has bijective antipode. Assume that H is a left Q-Galois extension of $K := {}^{\mathrm{co}\,Q}H$, and faithfully flat as left as well as right K-module.*

Then $\mathcal{M}_K^H \ni M \mapsto M/MK^+ \in \mathcal{M}^Q$ is a category equivalence. The inverse equivalence maps $N \in \mathcal{M}^Q$ to $N \,\square_Q\, H$ with the K-module and H-comodule structures induced by those of H.

Remark 2.7.2 As we learned in Corollary 2.4.9, our list of requirements on the quotient $H \to Q$ is fulfilled if k is a field, Q has bijective antipode, and H is a coflat left Q-comodule.

Proof By the structure theorem for Hopf modules over a Hopf-Galois extension, we have an equivalence $\mathcal{F}\colon \mathcal{M}_K \to {}^Q\mathcal{M}_H$ given by $\mathcal{F}(N) = N \otimes_K H$, with quasi-inverse $\mathcal{F}(M) = {}^{\mathrm{co}\,Q}M$. We claim that $\mathcal{F}$ induces an equivalence $\hat{\mathcal{F}}\colon \mathcal{M}_K^H \to {}^Q\mathcal{M}_H^H$. Indeed, if $N \in \mathcal{M}_K^H$, then $N \otimes_K H$ is an object of ${}^Q\mathcal{M}_H^H$ when endowed with the diagonal right H-module structure (which is well-defined since K is an H-comodule subalgebra of H). Conversely, if $M \in {}^Q\mathcal{M}_H^H$, then ${}^{\mathrm{co}\,Q}M$ is a right H-subcomodule of M and in this way a Hopf module in $\mathcal{M}_K^H$. It is straightforward to check that the adjunction morphisms for $\mathcal{F}$ and $\mathcal{F}^{-1}$ are compatible with these additional structures.

Next, we have the category equivalence $\mathcal{M}_H^H \cong \mathcal{M}_k$, which induces an equivalence $\mathcal{G}\colon {}^Q\mathcal{M} \to {}^Q\mathcal{M}_H^H$. Indeed, if $V \in {}^Q\mathcal{M}$, then $V \otimes H \in {}^Q\mathcal{M}_H^H$ with the codiagonal left Q-comodule structure, and conversely, if $M \in {}^Q\mathcal{M}_H^H$, then $M^{\mathrm{co}\,H}$ is a left Q-subcomodule of M. Now consider the composition

$$\mathcal{T} := \left(\mathcal{M}^Q \to {}^Q\mathcal{M} \xrightarrow{\mathcal{G}} {}^Q\mathcal{M}_H^H \xrightarrow{\hat{\mathcal{F}}^{-1}} \mathcal{M}_K^H\right)$$

where the first functor is induced by the inverse of the antipode. We have $\mathcal{T}(V) = {}^{\mathrm{co}\,Q}(V^{S^{-1}} \otimes H) = V \,\square_Q\, H$. We leave it to the reader to check that the module and comodule structure of $\mathcal{T}(V)$ are indeed those induced by H.

Since it is, finally, easy to check that the $\mathcal{M}_K^H \ni M \to M/MK^+ \in \mathcal{M}^Q$ is left adjoint to $\mathcal{T}$, it is also its quasi-inverse. $\square$

Let H and Q be k-flat Hopf algebras, and $\nu\colon H \to Q$ a Hopf algebra map. Then $K := {}^{\mathrm{co}\,Q}H$ is stable under the right adjoint action of H on itself defined by $x \leftharpoonup h = S(h_{(1)})xh_{(2)}$, since for $x \in K$ we have $\nu((x \leftharpoonup h)_{(1)}) \otimes (x \leftharpoonup h)_{(2)} = \nu(S(h_{(2)})x_{(1)}h_{(3)}) \otimes S(h_{(1)})x_{(2)}h_{(4)} = \nu(S(h_{(2)})h_{(3)}) \otimes S(h_{(1)})xh_{(4)} = 1 \otimes S(h_{(1)})xh_{(2)}$ for all $h \in H$. Thus K is a subalgebra of the (commutative) algebra H in the category $\mathcal{YD}_H^H$ of right-right H-Yetter-Drinfeld modules, which in turn is the center of the monoidal category $\mathcal{M}^H$ of right H-comodules.

As a corollary, the category $\mathcal{M}_K^H$ is equivalent to the monoidal subcategory $\mathcal{S} \subset {}_K\mathcal{M}_K^H$ of symmetric bimodules in $\mathcal{M}^H$, that is, the category of those $M \in {}_K\mathcal{M}_K^H$ for which $xm = m_{(0)}(x \leftharpoonup m_{(1)})$ holds for all $m \in M$ and $x \in K$. The equivalence is induced by the underlying functor ${}_K\mathcal{M}_K^H \to \mathcal{M}_K^H$. Since now the source and target of the equivalence in Theorem 2.7.1 are monoidal functors, the following Theorem answers an obvious question:

Theorem 2.7.3 *The category equivalence from Theorem 2.7.1 is a monoidal category equivalence with respect to the isomorphisms*

$$\xi\colon (V \underset{Q}{\square} H) \underset{K}{\otimes} (W \underset{Q}{\square} H) \ni v \otimes g \otimes w \otimes h \mapsto v \otimes w \otimes gh \in (V \otimes W) \underset{Q}{\square} H$$

Proof We have already seen (with switched sides) in Section 2.5 that ξ makes $(\text{—}) \square_Q H\colon \mathcal{M}^Q \to {}_K\mathcal{M}_K$ a monoidal functor. Quite obviously, $V \square_Q H$ has the structure of a right H-comodule in such a way that $V \square_Q H \in {}_K\mathcal{M}_K^H$, and ξ is an H-comodule map. Thus, we have a monoidal functor $(\text{—}) \square_Q H\colon \mathcal{M}^Q \to {}_K\mathcal{M}_K^H$. For $t = v \otimes h \in V \square_Q H$ we have $xt = v \otimes xh = v \otimes h_{(1)}(x \leftharpoonup h_{(2)}) = t_{(0)}(x \leftharpoonup t_{(1)})$, so that the monoidal functor $(\text{—}) \square_Q H$ takes values in the subcategory $\mathcal{S}$. Observe, finally, that it composes with the underlying functor to $\mathcal{M}_K^H$ to give the equivalence of categories from Theorem 2.7.1. From the commutative triangle

$$\begin{tikzcd}
\mathcal{M}^Q \arrow[rr, "(\text{—})\square_Q H"] \arrow[dr, "(\text{—})\square_Q H"'] & & \mathcal{S} \arrow[dl, "\mathcal{U}"] \\
& \mathcal{M}_K^H &
\end{tikzcd}$$

of functors, in which we already know the slanted arrows to be equivalences, we deduce that the top arrow is an equivalence. □

Corollary 2.7.4 *Assume the hypotheses of Theorem 2.7.1.*

The categories of right Q-comodule algebras, and of algebras in the category $\mathcal{M}_K^H$ are equivalent. The latter consists of pairs (A, f) in which A is a right H-comodule algebra, and $f\colon K \to A$ is a right H-comodule algebra map satisfying $f(x)a = a_{(0)}f(x \leftharpoonup a_{(1)})$ for all $x \in K$ and $a \in A$.

Suppose given an algebra $A \in \mathcal{M}_K^H$, corresponding to an algebra $\overline{A} \in \mathcal{M}^Q$. Then the categories of right A-modules in $\mathcal{M}_K^H$ and of right $\overline{A}$-modules in $\mathcal{M}^Q$ are equivalent. The former is the category of right A-modules in $\mathcal{M}_K^H$, so we have a *commutative diagram of functors*

$$\begin{tikzcd} \mathcal{M}^Q_{\overline{A}} \arrow[rr, "(-)\square_Q H"] \arrow[dr, "(-)^{\mathrm{co}\,H}"'] & & \mathcal{M}^H_A \arrow[dl, "(-)^{\mathrm{co}\,Q}"] \\ & \mathcal{M}_B & \end{tikzcd}$$

where $B = A^{\mathrm{co}\,H} \cong \overline{A}^{\mathrm{co}\,Q}$.

Corollary 2.7.5 *Assume the hypotheses of Theorem 2.7.1. Then we have a bijection between*

1. *isomorphism classes of left faithfully flat* Q*-Galois extensions of* B*, and*
2. *equivalences of pairs* (A, f) *in which* A *is a left faithfully flat* H*-Galois extension of* B*, and* $f\colon K \to A^B$ *is a homomorphism of algebras in* $\mathcal{YD}^H_H$*. Here, two pairs* (A, f) *and* (A', f') *are equivalent if there is a* B*-linear* H*-comodule algebra map* $t\colon A \to A'$ *such that* $tf = f'$*.*

Proof We know that Q-comodule algebras $\overline{A}$ correspond to isomorphism classes of pairs (A, f) in which A is an H-comodule algebra and $f\colon K \to A$ an H-comodule algebra map which is central in the sense of Definition 4.2.1. Since faithfully flat Galois extensions are characterized by the structure theorem for Hopf modules, see Corollary 2.3.6, the diagram in Corollary 2.7.4 shows that $\overline{A}$ is faithfully flat Q-Galois if and only if A is faithfully flat H-Galois. But if A is faithfully flat H-Galois, then every central H-comodule algebra map factors through a Yetter-Drinfeld module algebra map to A^B by Lemma 2.1.9. $\square$

The preceding corollary can be restated as follows:

Corollary 2.7.6 *Assume the hypotheses of Theorem 2.7.1. Consider the map* $\pi\colon \mathrm{Gal}_B(Q) \to \mathrm{Gal}_B(H)$ *given by* $\pi(\overline{A}) = A \,\square_Q\, H$*, and let* $A \in \mathrm{Gal}_B(H)$*. Then*

$$\pi^{-1}(A) \cong \mathrm{Alg}^{-H}_{-H}(K, A^B) / \mathrm{Aut}^H_B(A),$$

where $\mathrm{Aut}^H_B(A)$ *acts on* $\mathrm{Alg}^{-H}_{-H}(K, A^B)$ *by composition.*

2.8 Hopf Galois extensions without Hopf algebras. Cyril Grunspan [19] has revived an idea that appears to have been known in the case of commutative Hopf-Galois extensions (or torsors) for a long time, going back to a paper of Reinhold Baer [1]: It is possible to write down axioms characterizing a Hopf-Galois extension without mentioning a Hopf algebra.

This approach to (noncommutative) Hopf-Galois extensions begins in [19] with the definition of a quantum torsor (an algebra with certain additional structures) and the proof that every quantum torsor gives rise to two Hopf algebras over which it is a bi-Galois extension of the base field. The converse was proved in [41]: Every Hopf-Galois extension of the base field is a quantum torsor in the sense of Grunspan. Then the axioms of a quantum torsor were simplified in [42] by showing that a key ingredient of Grunspan's definition (a certain endomorphism of the torsor) is actually not needed to show that a torsor is a Galois object. The simplified version of the torsor axioms admits a generalization to general Galois extensions (not only of the base ring or field).

Definition 2.8.1 Let B be a k-algebra, and $B \subset T$ an algebra extension, with T a faithfully flat k-module. The centralizer $(T \otimes_B T)^B$ of B in the (obvious) B-B-bimodule $T \otimes_B T$ is an algebra by $(x \otimes y)(a \otimes b) = ax \otimes yb$ for $x \otimes y, a \otimes b \in (T \otimes_B T)^B$.

A B-torsor structure on T is an algebra map $\mu\colon T \to T \otimes (T \otimes_B T)^B$; we denote by $\mu_0\colon T \to T \otimes T \otimes_B T$ the induced map, and write $\mu_0(x) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)}$.

The torsor structure is required to fulfill the following axioms:

$$x^{(1)}x^{(2)} \otimes x^{(3)} = 1 \otimes x \in T \underset{B}{\otimes} T \tag{2.8.1}$$

$$x^{(1)} \otimes x^{(2)}x^{(3)} = x \otimes 1 \in T \otimes T \tag{2.8.2}$$

$$\mu(b) = b \otimes 1 \otimes 1 \quad \forall b \in B \tag{2.8.3}$$

$$\mu(x^{(1)}) \otimes x^{(2)} \otimes x^{(3)} = x^{(1)} \otimes x^{(2)} \otimes \mu(x^{(3)}) \in T \otimes T \underset{B}{\otimes} T \otimes T \underset{B}{\otimes} T \tag{2.8.4}$$

Note that (2.8.4) makes sense since μ is a left B-module map by (2.8.3).

Remark 2.8.2 If $B = k$, then the torsor axioms simplify as follows: They now assume the existence of an algebra map $\mu\colon T \to T \otimes T^{\mathrm{op}} \otimes T$ such that the diagrams

$$\begin{tikzcd} T \arrow[r, "\mu"] \arrow[d, "\mu"] & T \otimes T^{\mathrm{op}} \otimes T \arrow[d, "T\otimes T^{\mathrm{op}}\otimes\mu"] \\ T \otimes T^{\mathrm{op}} \otimes T \arrow[r, "\mu\otimes T^{\mathrm{op}}\otimes T"] & T \otimes T^{\mathrm{op}} \otimes T \otimes T^{\mathrm{op}} \otimes T \end{tikzcd}$$

$$\begin{tikzcd} & T \arrow[dl, "T\otimes\eta"'] \arrow[d, "\mu"] \arrow[dr, "\eta\otimes T"] & \\ T \otimes T & T \otimes T \otimes T \arrow[l, "T\otimes\nabla"] \arrow[r, "\nabla\otimes T"'] & T \otimes T \end{tikzcd}$$

commute.

The key observation is now that a torsor provides a descent data. Here we use left descent data, i.e. certain S-linear maps $\theta\colon M \to S \otimes_R M$ for a ring extension $R \subset S$ and a left S-module M, as opposed to the right descent data in Section 4.5. For a left descent data $\theta\colon M \to S \otimes_R M$ from S to R on a left S-module M we will write ${}^{\theta}M := \{m \in M | \theta(m) = 1 \otimes m\}$.

Lemma 2.8.3 *Let T be a B-torsor. Then a descent data D from T to k on $T \otimes_B T$ is given by $D(x \otimes y) = xy^{(1)} \otimes y^{(2)} \otimes y^{(3)}$. It satisfies $(T \otimes D)\mu(x) = x^{(1)} \otimes 1 \otimes x^{(2)} \otimes x^{(3)}$.*

Proof Left T-linearity of D is obvious. We have

$$\begin{aligned}(T \otimes D)\mu(x) &= x^{(1)} \otimes D(x^{(2)} \otimes x^{(3)}) \\ &= x^{(1)} \otimes (\nabla \otimes T \otimes T)(x^{(2)} \otimes \mu(x^{(3)})) \\ &= (T \otimes \nabla \otimes T \otimes T)(\mu(x^{(1)}) \otimes x^{(2)} \otimes x^{(3)}) \\ &= x^{(1)} \otimes 1 \otimes x^{(2)} \otimes x^{(3)}\end{aligned}$$

and thus

$$\begin{aligned}(T\otimes D)D(x\otimes y) &= xy^{(1)}\otimes D(y^{(2)}\otimes y^{(3)})\\ &= xy^{(1)}\otimes 1\otimes y^{(2)}\otimes y^{(3)}\\ &= (T\otimes\eta\otimes T\otimes T)D(x\otimes y).\end{aligned}$$

Finally $(\nabla\otimes T\otimes T)D(x\otimes y) = xy^{(1)}y^{(2)}\otimes y^{(3)} = x\otimes y$. □

Note that $D(T\otimes_B T)\subset T\otimes(T\otimes_B T)^B$. Since T is faithfully flat over k, then faithfully flat descent implies that ${}^D(T\otimes_B T)\subset(T\otimes_B T)^B$.

Theorem 2.8.4 *Let T be a B-torsor, and assume that T is a faithfully flat right B-module.*

Then $H := {}^D(T\otimes_B T)$ is a k-flat Hopf algebra. The algebra structure is that of a subalgebra of $(T\otimes_B T)^B$, the comultiplication and counit are given by

$$\begin{aligned}\Delta(x\otimes y) &= x\otimes y^{(1)}\otimes y^{(2)}\otimes y^{(3)},\\ \varepsilon(x\otimes y) &= xy\end{aligned}$$

for $x\otimes y\in H$. The algebra T is an H-Galois extension of B under the coaction $\delta\colon T\to T\otimes H$ given by $\delta(x)=\mu(x)$.

Proof H is a subalgebra of $(T\otimes_B T)^B$ since for $x\otimes y, a\otimes b\in H$ we have

$$\begin{aligned}D((x\otimes y)(a\otimes b)) &= D(ax\otimes yb)\\ &= ax(yb)^{(1)}\otimes(yb)^{(2)}\otimes(yb)^{(3)}\\ &= axy^{(1)}b^{(1)}\otimes b^{(2)}y^{(2)}\otimes y^{(3)}b^{(3)}\\ &= ab^{(1)}\otimes b^{(2)}x\otimes yb^{(3)}\\ &= 1\otimes ax\otimes yb\\ &= 1\otimes(x\otimes y)(a\otimes b).\end{aligned}$$

To see that the coaction δ is well-defined, we have to check that the image of μ is contained in $T\otimes H$, which is, by faithful flatness of T, the equalizer of

$$T\otimes T\otimes_B T \underset{T\otimes\eta\otimes T\otimes_B T}{\overset{T\otimes D}{\rightrightarrows}} T\otimes T\otimes T\otimes_B T\,.$$

But $(T\otimes D)\mu(x) = (T\otimes\eta\otimes T\otimes_B T)\mu(x)$ was shown in Lemma 2.8.3. Since μ is an algebra map, so is the coaction δ, for which we employ the usual Sweedler notation $\delta(x)=x_{(0)}\otimes x_{(1)}$. Note that (2.8.3) implies that $\delta(b)=b\otimes 1$ for all $b\in B$; in other words, δ is left B-linear.

The Galois map $\beta\colon T\otimes_B T\to T\otimes H$ for the coaction δ is given by $\beta(x\otimes y) = xy_{(0)}\otimes y_{(1)} = xy^{(1)}\otimes y^{(2)}\otimes y^{(3)} = D(x\otimes y)$. Thus it is an isomorphism by faithfully flat descent, Theorem 4.5.2. It follows that H is faithfully flat over k.

Since δ is left B-linear,

$$\Delta_0\colon T\underset{B}{\otimes}T\ni x\otimes y\mapsto x\otimes y^{(1)}\otimes y^{(2)}\otimes y^{(3)}\in T\underset{B}{\otimes}T\otimes H$$

is well-defined. To prove that Δ is well-defined, we need to check that $\Delta_0(H)$ is contained in $H\otimes H$, which, by faithful flatness of H, is the equalizer of

$$T\otimes_B T\otimes H \underset{\eta\otimes T\otimes T\otimes H}{\overset{D\otimes H}{\rightrightarrows}} T\otimes T\otimes_B T\otimes H\,.$$

Now for $x \otimes y \in H$ we have

$$\begin{aligned}(D \otimes H)\Delta_0(x \otimes y) &= (D \otimes H)(x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)})\\ &= xy^{(1)(1)} \otimes y^{(1)(2)} \otimes y^{(1)(3)} \otimes y^{(2)} \otimes y^{(3)}\\ &= xy^{(1)} \otimes y^{(2)} \otimes \mu(y^{(3)})\\ &= (T \otimes T \otimes \mu)D(x \otimes y)\\ &= (T \otimes T \otimes \mu)(1 \otimes x \otimes y)\\ &= 1 \otimes \Delta_0(x \otimes y)\end{aligned}$$

Δ is an algebra map since μ is, and coassociativity follows from the coassociativity axiom of the torsor T.

For $x \otimes y \in H$ we have $xy \otimes 1 = xy^{(1)} \otimes y^{(2)}y^{(3)} = 1 \otimes xy$, whence $xy \in k$ by faithful flatness of T. Thus, ε is well-defined. It is straightforward to check that ε is an algebra map, that it is a counit for Δ, and that the coaction δ is counital. Thus, H is a bialgebra.

We may now write the condition $\delta(b) = b \otimes 1$ for $b \in B$ simply as $B \subset T^{\mathrm{co}\,H}$. Conversely, $x \in T^{\mathrm{co}\,H}$ implies $x \otimes 1 = x^{(1)}x^{(2)} \otimes x^{(3)} = 1 \otimes x \in T \otimes_B T$, and thus $x \in B$ by faithful flatness of T as a B-module. Since we have already seen that the Galois map for the H-extension $B \subset T$ is bijective, T is an H-Galois extension of B, and from Lemma 2.1.5 we deduce that H is a Hopf algebra. □

Lemma 2.8.5 *Let H be a k-faithfully flat Hopf algebra, and let T be a right faithfully flat H-Galois extension of $B \subset T$. Then T is a B-torsor with torsor structure*

$$\mu\colon T \ni x \mapsto x_{(0)} \otimes x_{(1)}{}^{[1]} \otimes x_{(1)}{}^{[2]} \in T \otimes (T \underset{B}{\otimes} T)^B,$$

where $h^{[1]} \otimes h^{[2]} = \beta^{-1}(1 \otimes h) \in T \otimes_B T$, with $\beta\colon T \otimes_B T \to T \otimes H$ the Galois map.

3 Hopf-bi-Galois theory

3.1 The left Hopf algebra. Let A be a faithfully flat H-Galois object. Then A is a torsor by Lemma 2.8.5 By the left-right switched version of Theorem 2.8.4, there exists a Hopf algebra $L := L(A, H)$ such that A is a left L-Galois extension of k. Moreover, since the torsor structure $\mu\colon A \to A \otimes A^{\mathrm{op}} \otimes A$ is right H-colinear, we see that A is an L-H-bicomodule.

Definition 3.1.1 An L-H-bi-Galois object is a k-faithfully flat L-H-bicomodule algebra A which is simultaneously a left L-Galois object and a right H-Galois object.

We have seen that every right H-Galois object can be endowed with a left L-comodule algebra structure making it an L-H-bi-Galois object. We shall prove uniqueness by providing a universal property shared by every L that makes a given H-Galois object into an L-H-bi-Galois object.

Proposition 3.1.2 *Let H and L be k-flat Hopf algebras, and A an L-H-bi-Galois object.*

Then for all $n \in \mathbb{N}$ and k-modules V, W we have a bijection

$$\Phi := \Phi_{V,W,n}\colon \operatorname{Hom}(V \otimes L^{\otimes n}, W) \cong \operatorname{Hom}^{-H}(V \otimes A^{\otimes n}, W \otimes A)$$

(where $A^{\otimes n}$ carries the codiagonal comodule structure), given by $\Phi(f)(v\otimes x_1\otimes\ldots\otimes x_n) = f(v\otimes x_{1(-1)}\otimes\ldots\otimes x_{n(-1)})\otimes x_{1(0)}\cdot\dots\cdot x_{n(0)}$.

In particular, for every k-module we have the universal property that every right H-colinear map $\phi\colon A\to W\otimes A$ factors uniquely in the form $\phi=(f\otimes A)\delta_\ell$ as in the diagram

$$\begin{tikzcd} A \arrow[r, "\delta_\ell"] \arrow[dr, "\phi"'] & L\otimes A \arrow[d, "f\otimes A"] & L \arrow[d, "f"] \\ & W\otimes A & W \end{tikzcd}$$

Proof Note first that the left Galois map $\beta_\ell\colon A^\bullet\otimes A^\bullet_\bullet\to L\otimes A^\bullet_\bullet$ is evidently a map of Hopf modules in $\mathcal{M}^H_A$ with the indicated structures. We deduce that for any $M\in\mathcal{M}^H$ we have

$$M^\bullet\otimes L^{\otimes m}\otimes A^\bullet_\bullet\cong M^\bullet\otimes L^{\otimes(m-1)}\otimes A^\bullet\otimes A^\bullet_\bullet\cong M^\bullet\otimes A^\bullet\otimes L^{\otimes(m-1)}\otimes A^\bullet_\bullet$$

in $\mathcal{M}^H_A$, and hence by induction

$$V\otimes L^{\otimes n}\otimes A^\bullet_\bullet\cong V\otimes(A^\bullet)^{\otimes n}\otimes A^\bullet_\bullet\in\mathcal{M}^H_A$$

We can now use the structure theorem for Hopf modules, Corollary 2.3.6, to compute

$$\begin{aligned}\operatorname{Hom}(V\otimes L^{\otimes n},W)&\cong\operatorname{Hom}^{-H}_{-A}(V\otimes L^{\otimes n}\otimes A^\bullet_\bullet,W\otimes A^\bullet_\bullet)\\ &\cong\operatorname{Hom}^{-H}_{-A}(V\otimes(A^\bullet)^{\otimes n}\otimes A^\bullet_\bullet,W\otimes A^\bullet_\bullet)\cong\operatorname{Hom}^{-H}(V\otimes A^{\otimes n},W\otimes A).\end{aligned}$$

We leave it to the reader to verify that the bijection has the claimed form. □

Corollary 3.1.3 *Let A be an L-H-bi-Galois object, B a k-module, $f\colon L\to B$, and $\lambda=\Phi(f)\colon A\to B\otimes A$.*

1. *Assume B is a coalgebra. Then f is a coalgebra map if and only if λ is a comodule structure.*
2. *Assume B is an algebra. Then f is an algebra map if and only if λ is.*
3. *In particular, assume B is a bialgebra. Then f is a bialgebra map if and only if λ is a comodule algebra structure. In particular, the bialgebra L in an L-H-bi-Galois object is uniquely determined by the H-Galois object A.*

Proof We have $\Delta f=(f\otimes f)\Delta\colon L\to B\otimes B$ if and only if $\Phi(\Delta f)=\Phi((f\otimes f)\Delta)\colon A\to B\otimes B\otimes A$. But $\Phi(\Delta f)(a)=(\Delta\otimes A)\lambda$, and $\Phi((f\otimes f)\Delta)(a)=(f\otimes f)\Delta(a_{(-1)})\otimes a_{(0)}=f(a_{(-2)})\otimes f(a_{(-1)})\otimes a_{(0)}=f(a_{(-1)})\otimes\lambda(a_{(0)})=(B\otimes\lambda)\lambda(a)$, proving (1).

We have $\nabla(f\otimes f)=f\nabla\colon L\otimes L\to B$ if and only if $\Phi(\nabla(f\otimes f))=\Phi(f\nabla)\colon A\otimes A\to B\otimes A$. But $\Phi(\nabla(f\otimes f))(x\otimes y)=f(x_{(-1)})f(y_{(-1)})\otimes x_{(0)}y_{(0)}=\lambda(x)\lambda(y)$ and $\Phi(f\nabla)(x\otimes y)=f(x_{(-1)}y_{(-1)})\otimes x_{(0)}y_{(0)}=\lambda(xy)$, proving (2).

(3) is simply a combination of (1) and (2), since L as a bialgebra is uniquely determined once it fulfills a universal property for bialgebra maps. □

Corollary 3.1.4 *Let A be an L-H-bi-Galois object. Then*

$$\operatorname{Alg}(L,k)\ni\varphi\mapsto(a\mapsto\varphi(a_{(-1)})a_{(0)})\in\operatorname{Aut}^{-H}(A)$$

is an isomorphism from the group of algebra maps from L to k (i.e. the group of grouplikes of L^* if L is finitely generated projective) to the group of H-colinear algebra automorphisms of A.

If H is cocommutative, then every H-Galois object is trivially an H-H-bi-Galois object, so:

Corollary 3.1.5 *If H is cocommutative and A is an L-H-bi-Galois object, then $L \cong H$.*

It is also obvious that $L(H,H) = H$. There is a more general important case in which $L(A,H)$ can be computed in some sense (see below, though), namely that of cleft extensions:

Proposition 3.1.6 *Let $A = k\#_\sigma H$ be a crossed product with invertible cocycle σ. Then $L(A,H) = H$ as coalgebras, while multiplication in $L(A,H)$ is given by*

$$g \cdot h = \sigma(g_{(1)} \otimes h_{(1)}) g_{(2)} h_{(2)} \sigma^{-1}(g_{(3)} \otimes h_{(3)}).$$

We will say that $L(A,H) := H^\sigma$ is a cocycle double twist of H. The construction of a cocycle double twist is dual to the construction of a Drinfeld twist [17], and was considered by Doi [14]. We have said already that the isomorphism $L(k\#_\sigma H, H) = H^\sigma$ computes the left Hopf algebra in case of cleft extensions in some sense. In applications, this may rather be read backwards: Cocycles in the non-cocommutative case are not easy to compute for lack of a cohomological interpretation, while it may be easier to guess a left Hopf algebra from generators and relations of A. In this sense the isomorphism may be used to compute the Hopf algebra H^σ helped by the left Hopf algebra construction. This is quite important in the applications we will cite in Section 3.2.

We will give a different proof from that in [33] of Proposition 3.1.6. It has the advantage not to use the fact that H^σ is a Hopf algebra — checking the existence of an antipode is in fact one of the more unpleasant parts of the construction.

Proof of Proposition 3.1.6 We will not check here that H^σ is a bialgebra. Identify $A = k\#_\sigma H = H$, with multiplication $g \circ h = \sigma(g_{(1)} \otimes h_{(1)}) g_{(2)} h_{(2)}$. Then it is straightforward to verify that comultiplication in H is an H^σ-comodule algebra structrure $A \to H^\sigma \otimes A$ which, of course, makes A an H^σ-H-bicomodule algebra. One may now finish the proof by appealing to Lemma 3.2.5 below, but we will stay more elementary. We shall verify that H^σ fulfills the universal property of $L(A,H)$. Of course it does so as a coalgebra, since the left coaction is just the comultiplication of H. Thus a B-H-bicomodule algebra structure $\lambda\colon A \to B \otimes A$ gives rise to a unique coalgebra map $f\colon H^\sigma \to B$ by $f(h) = (B \otimes \varepsilon)\lambda(h) = h_{(-1)}\varepsilon(h_{(0)})$. We have to check that f is an algebra map:

$$\begin{aligned}
f(g \cdot h) &= \sigma(g_{(1)} \otimes h_{(1)}) f(g_{(2)} h_{(2)}) \sigma^{-1}(g_{(3)} \otimes h_{(3)}) = f(g_{(1)} \circ h_{(1)}) \sigma^{-1}(g_{(2)} \otimes h_{(2)}) \\
&= (B \otimes \varepsilon)(\lambda(g_{(1)})\lambda(h_{(1)})) \sigma^{-1}(g_{(2)} \otimes h_{(2)}) \\
&= g_{(1)(-1)} h_{(1)(-1)} \varepsilon(g_{(1)(0)} \circ h_{(1)(0)}) \sigma^{-1}(g_{(2)} \otimes h_{(2)}) \\
&= g_{(-1)} h_{(-1)} \varepsilon(g_{(0)(1)} \circ g_{(0)(1)}) \sigma^{-1}(g_{(0)(2)} \otimes h_{(0)(2)}) = g_{(-1)} h_{(-1)} \varepsilon(g_{(0)} \cdot h_{(0)}) \\
&= g_{(-1)} \varepsilon(g_{(0)}) h_{(-1)} \varepsilon(h_{(0)}) = f(g) f(h) \qquad \square
\end{aligned}$$

Remark 3.1.7 Let A be an L-H-bi-Galois object. The left Galois map $\beta_\ell\colon A^\bullet \otimes A^\bullet \to L \otimes A^\bullet$ is right H-colinear as indicated, and thus induces an isomorphism $(A \otimes A)^{\operatorname{co} H} \cong (L \otimes A)^{\operatorname{co} H} \cong L$, where the coinvariants of $A \otimes A$ are taken with respect to the codiagonal comodule structure. Let us check that the isomorphism is an algebra map to a subalgebra of $A \otimes A^{\mathrm{op}}$: If $x \otimes y, x' \otimes y' \in A \otimes A$

are such that $x_{(-1)} \otimes x_{(0)}y = \ell \otimes 1$ and $x'_{(-1)} \otimes x'_{(0)}y = \ell' \otimes 1$ for $\ell, \ell' \in L$, then $\beta_\ell(xx' \otimes y'y) = x_{(-1)}x'_{(-1)} \otimes x_{(0)}x'_{(0)}y'y = x_{(-1)}\ell' \otimes x_{(0)}y = \ell\ell' \otimes 1$.

3.2 Monoidal equivalences and the groupoid of bi-Galois objects. Let H be a Hopf algebra, and A an L-H-bi-Galois object. Then the monoidal functor $(A \square_H -, \xi)$ considered in Section 2.5 also defines a monoidal functor $A \square_H -: {}^H\mathcal{M} \to {}^L\mathcal{M}$. If B is an H-R-bi-Galois object, then $A \square_H B$ is an L-H-bicomodule algebra, and since the functor ${}^R\mathcal{M} \ni V \mapsto (A \square_H B) \square_R V \in {}^L\mathcal{M}$ is the composition of the two monoidal functors $(B \square_R -)$ and $A \square_H -$, it is itself monoidal, so that $A \square_H B$ is an R-Galois object by Corollary 2.5.4. By symmetric arguments, $A \square_H B$ is also a left L-Galois and hence an L-R-bi-Galois object. Thus, without further work, we obtain:

Corollary 3.2.1 *k-flat Hopf algebras form a category BiGal when we define a morphism from a Hopf algebra H to a Hopf algebra L to be an isomorphism class of L-H-bi-Galois objects, and if we define the composition of bi-Galois objects as their cotensor product.*

On the other hand we can define a category whose objects are Hopf algebras, and in which a morphism from H to L is an isomorphism class of monoidal functors ${}^H\mathcal{M} \to {}^L\mathcal{M}$.

A functor from the former category to the latter is described by assigning to an L-H-bi-Galois object A the functor $A \square_H -: {}^H\mathcal{M} \to {}^L\mathcal{M}$.

The purpose of the Corollary was to collect what we can deduce without further effort from our preceding results. The following Theorem gives the full information:

Theorem 3.2.2
1. *The category BiGal is a groupoid; that is, for every L-H-bi-Galois object A there is an H-L-bi-Galois object A^{-1} such that $A \square_H A^{-1} \cong L$ as L-L-bicomodule algebras and $A^{-1} \square_L A \cong H$ as H-H-bicomodule algebras.*
2. *The category BiGal is equivalent to the category whose objects are all k-flat Hopf algebras, and in which a morphism from H to L is an isomorphism class of monoidal category equivalences ${}^H\mathcal{M} \to {}^L\mathcal{M}$.*

If k is a field, there is a short conceptual proof for the Theorem, in which the second claim is proved first, and the first is an obvious consequence. If k is arbitrary, there does not seem to be a way around proving the first claim first. This turns out to be much easier if we assume all antipodes to be bijective. We will sketch all approaches below, but we shall comment first on the main application of the result.

Definition 3.2.3 Let H, L be two k-flat Hopf algebras. We call H and L monoidally Morita-Takeuchi equivalent if there is a k-linear monoidal equivalence ${}^H\mathcal{M} \to {}^L\mathcal{M}$.

Since the monoidal category structure of the comodule category of a Hopf algebra is one of its main features, it should be clear that monoidal Morita-Takeuchi equivalence is an interesting notion of equivalence between two Hopf algebras, weaker than isomorphy. Theorem 3.2.2 immediately implies:

Corollary 3.2.4 *For two k-flat Hopf algebras H and L, the following are equivalent:*

1. *H and L are monoidally Morita-Takeuchi equivalent.*
2. *There exists an L-H-bi-Galois object.*

3. *There is a k-linear monoidal category equivalence* $\mathcal{M}^H \to \mathcal{M}^L$.

As a consequence of Corollary 3.2.4 and Proposition 3.1.6, Hopf algebras are monoidally Morita-Takeuchi equivalent if they are cocycle double twists of each other (one should note, though, that it is quite easy to give a direct proof of this fact). Conversely, if H is a finite Hopf algebra over a field k, then every H-Galois object is cleft. Thus every Hopf algebra L which is monoidally Morita-Takeuchi equivalent to H is a cocycle double twist of H.

In many examples constructing bi-Galois objects has proved to be a very practicable way of constructing monoidal equivalences between comodule categories. This is true also in the finite dimensional case over a field. The reason seems to be that it is much easier to construct an associative algebra with nice properties, than to construct a Hopf cocycle (or, worse perhaps, a monoidal category equivalence). I will only very briefly give references for such applications: Nice examples involving the representation categories of finite groups were computed by Masuoka [27]. In [28] Masuoka proves that certain infinite families of non-isomorphic pointed Hopf algebras collapse under monoidal Morita-Takeuchi equivalence. That paper also contains a beautiful general mechanism for constructing Hopf bi-Galois objects for quotient Hopf algebras of a certain type. This was applied further, and more examples of families collapsing under monoidal Morita-Takeuchi equivalence were given, in Daniel Didt's thesis [12]. Bichon [4] gives a class of infinite-dimensional examples that also involve non-cleft extensions.

Now we return to the proof of Theorem 3.2.2. First we state and prove (at least sketchily) the part that is independent of k and any assumptions on the antipode.

Lemma 3.2.5 *Let L and H be k-flat Hopf algebras. Then every k-linear equivalence $\mathcal{F}\colon {}^H\mathcal{M} \to {}^L\mathcal{M}$ has the form $\mathcal{F}(V) = A \,\square_H\, V$ for some L-H-bi-Galois object A.*

More precisely, every exact k-linear functor $\mathcal{F}\colon {}^H\mathcal{M} \to {}^L\mathcal{M}$ commuting with arbitrary colimits has the form $\mathcal{F}(V) = A \,\square_H\, V$ for an L-H-bicomodule algebra that is an H-Galois object, and if $\mathcal{F}$ is an equivalence, then A is an L-Galois object.

Proof Let B be a k-flat bialgebra, and $\mathcal{F}\colon {}^H\mathcal{M} \to {}^L\mathcal{M}$ an exact functor commuting with colimits. We already know that the composition $\mathcal{F}_0\colon {}^H\mathcal{M} \to \mathcal{M}_k$ of $\mathcal{F}$ with the underlying functor has the form $\mathcal{F}_0(V) = A \,\square_H\, V$ for an H-Galois object A. It is straightforward to check that $\mathcal{F}$ has the form $\mathcal{F}(V) = A \,\square_H\, V$ for a suitable L-comodule algebra structure on A making it an L-H-bicomodule algebra (just take the left L-comodule structure of $A = A \,\square_H\, H = \mathcal{F}_0(A)$, and do a few easy calculations). Conversely, every B-H-bicomodule algebra structure on A for some flat bialgebra B lifts $\mathcal{F}_0$ to a monoidal functor $\mathcal{G}\colon {}^H\mathcal{M} \to {}^B\mathcal{M}$. If $\mathcal{F}$ is an equivalence, we can fill in the dashed arrow in the diagram

$$\begin{tikzcd} {}^H\mathcal{M} \arrow[r, "\mathcal{F}"] \arrow[dr, "\mathcal{G}"'] & {}^L\mathcal{M} \arrow[d, dashed] \arrow[dr, "\mathcal{U}"] & \\ & {}^B\mathcal{M} \arrow[r, "\mathcal{U}"] & \mathcal{M}_k \end{tikzcd}$$

by a monoidal functor. To see this, simply note that every L-module is by assumption naturally isomorphic to one of the form $A \,\square_H\, V$ with $V \in {}^H\mathcal{M}$, and thus it is also a B-module. Now a monoidal functor ${}^L\mathcal{M} \to {}^B\mathcal{M}$ that commutes with the underlying functors has the form ${}^f\mathcal{M}$ for a unique bialgebra map $f\colon L \to B$. We have shown that L has the universal property characterizing the left Hopf algebra $L(A, H)$. $\square$

Now what is left of the proof of Theorem 3.2.2 is to provide a converse to Lemma 3.2.5.

In the case that k is a field, we can argue by the general principles of reconstruction theory for quantum groups, which also go back to work of Ulbrich [55]; see e.g. [31]. Assume given an H-Galois object A. The restriction $A \square_H -\colon {}^H\mathcal{M}_{\mathrm{f}} \to \mathcal{M}_k$ of the functor $A \square_H -$ to the category of finite-dimensional H-comodules takes values in finite dimensional vector spaces (see Corollary 2.5.5). Thus there exists a Hopf algebra L such that the functor factors over an equivalence ${}^H\mathcal{M}_{\mathrm{f}} \to {}^L\mathcal{M}_{\mathrm{f}}$; by the finiteness theorem for comodules this also yields an equivalence ${}^H\mathcal{M} \to {}^L\mathcal{M}$. By Lemma 3.2.5 we see that this equivalence comes from an L-H-bi-Galois structure on A, and in particular that cotensoring with a bi-Galois extension A is an equivalence ${}^H\mathcal{M} \to {}^{L(A,H)}\mathcal{M}$.

The general technique of reconstruction behind this proof is to find a Hopf algebra from a monoidal functor $\omega\colon \mathcal{C} \to \mathcal{M}_k$ by means of a coendomorphism coalgebra construction. More generally, one can construct a cohomomorphism object $\mathrm{cohom}(\omega, \nu)$ for every pair of functors $\omega, \nu\colon \mathcal{C} \to \mathcal{M}_k$ taking values in finite dimensional vector spaces. Ulbrich in fact reconstructs a Hopf-Galois object from a monoidal functor ${}^H\mathcal{M}_{\mathrm{f}} \to \mathcal{M}_k$ by applying this construction to the monoidal functor in question on one hand, and the underlying functor on the other hand. It is clear that the left Hopf algebra of a Hopf-Galois object A can be characterized as the universal Hopf algebra reconstructed as a coendomorphism object from the functor $A \square_H -$. Bichon [3] has taken this further by reconstructing a bi-Galois object, complete with both its Hopf algebras, from a pair of monoidal functors $\omega, \nu\colon \mathcal{C} \to \mathcal{M}_k$ taking values in finite dimensional vector spaces. He also gives an axiom system (called a Hopf-Galois system, and extended slightly to be symmetric by Grunspan [19]) characterizing the complete set of data arising in a bi-Galois situation: An algebra coacted upon by two bialgebras, and in addition another bicomodule algebra playing the role of the inverse bi-Galois extension.

In the case where k is not a field, reconstruction techniques as the ones used above are simply not available, and we have to take a somewhat different approach. If we can show that <u>*BiGal*</u> is a groupoid, then the rest of Theorem 3.2.2 follows: The inverse of the functor $A \square_H - {}^H\mathcal{M} \to {}^L\mathcal{M}$ can be constructed as $A^{-1} \square_L\colon {}^L\mathcal{M} \to {}^H\mathcal{M}$ when A^{-1} is the inverse of A in the groupoid <u>*BiGal*</u>.

Now let A be an L-H-bi-Galois object. By symmetry it is enough to find a right inverse for A. For this in turn it is enough to find some left H-Galois object B such that $A \square_H B \cong L$ as left L-comodule algebras. For B is an H-R-bi-Galois object for some Hopf algebra R, and $A \square_H B$ is then an L-R-bi-Galois object. But if $A \square_H B \cong L$ as left L-comodule algebra, then $R \cong L$ by the uniqueness of the right Hopf algebra in the bi-Galois extension L. More precisely, there is an automorphism of the Hopf algebra L such that $A \square_H B \cong L^f$, where L^f has the right L-comodule algebra structure induced along f. But then $A \square_H (B^{f^{-1}}) \cong L$ as L-bicomodule algebras.

We already know that $L \cong (A \otimes A)^{\mathrm{co}\,H}$, a subalgebra of $A \otimes A^{\mathrm{op}}$. From the way the isomorphism was obtained in Remark 3.1.7, it is obviously left L-colinear, with the left L-comodule structure on $(A \otimes A)^{\mathrm{co}\,H}$ induced by that of the left tensor factor A. Thus it finally remains to find some left H-Galois object B such that

$A \,\square_H\, B \cong (A \otimes A)^{\operatorname{co} H}$. If the antipode of H is bijective, we may simply take $B := A^{\mathrm{op}}$, with the left comodule structure $A \ni a \mapsto S^{-1}(a_{(1)}) \otimes a_{(0)}$. If the antipode of H is not bijective, we can take $B := (H \otimes A)^{\operatorname{co} H}$, where the coinvariants are taken with respect to the diagonal comodule structure, the algebra structure is that of a subalgebra of $H \otimes A^{\mathrm{op}}$, and the left H-comodule structure is induced by that of H. By contrast to the case where the antipode is bijective, it is not entirely trivial to verify that B is indeed a left H-Galois object. We refer to [33] for details at this point. However, it is easy to see that the obvious isomorphism $A \,\square_H\, B = A \,\square_H\, (H \otimes A)^{\operatorname{co} H} \cong (A \,\square_H\, H \otimes A)^{\operatorname{co} H} \cong (A \otimes A)^{\operatorname{co} H}$ is a left L-comodule algebra map.

3.3 The structure of Hopf bimodules. Let A be an L-H-bi-Galois object. We have studied already in Section 2.6 the monoidal category ${}_A\mathcal{M}_A^H$ of Hopf bimodules, which allows an underlying functor to the category $\mathcal{M}_k$ which is monoidal. The result of this section is another characterization of the left Hopf algebra L: It is precisely that Hopf algebra for which we obtain a commutative diagram of monoidal functors

$$\begin{tikzcd} {}_A\mathcal{M}_A^H \arrow[rr, "\sim"] \arrow[dr, "(-)^{\operatorname{co} H}"'] & & {}_L\mathcal{M} \arrow[dl] \\ & \mathcal{M}_k & \end{tikzcd} \tag{3.3.1}$$

in which the top arrow is an equivalence, and the unmarked arrow is the underlying functor.

Theorem 3.3.1 *Let A be an L-H-bi-Galois object. Then a monoidal category equivalence ${}_L\mathcal{M} \to {}_A\mathcal{M}_A^H$ is defined by sending $V \in {}_L\mathcal{M}$ to $V \otimes A$ with the obvious structure of a Hopf module in $\mathcal{M}_A^H$, and the additional left A-module structure $x(v \otimes y) = x_{(-1)} \cdot v \otimes x_{(0)} y$. The monoidal functor structure is given by the canonical isomorphism $(V \otimes A) \otimes_A (V \otimes A) \cong V \otimes W \otimes A$.*

Proof We know that every Hopf module in $\mathcal{M}_A^H$ has the form $V \otimes A$ for some k-module V. It remains to verify that left A-module structures on $V \otimes A$ making it a Hopf module in ${}_A\mathcal{M}_A^H$ are classified by left L-module structures on V. A suitable left A-module structure is a colinear right A-module map $\mu\colon A \otimes V \otimes A \to V \otimes A$, and such maps are in turn in bijection with colinear maps $\sigma\colon A \otimes V \to V \otimes A$. By the general universal property of L, such maps σ are in turn classified by maps $\mu_0\colon L \otimes V \to V$ through the formula $\sigma(a \otimes v) = a_{(-1)} \cdot v \otimes a_{(0)}$, with $\mu_0(\ell \otimes v) =: \ell \cdot v$. Now it only remains to verify that μ_0 is an L-module structure if and only if μ, which is now given by $\mu(x \otimes v \otimes y) = x_{(-1)} \cdot v \otimes x_{(0)} y$, is an A-module structure. We compute

$$x(y(v \otimes z)) = x(y_{(-1)} \cdot v \otimes y_{(0)} z) = x_{(-1)} \cdot (y_{(-1)} \cdot v) \otimes x_{(0)} y_{(0)} z$$
$$(xy)(v \otimes z) = (xy)_{(-1)} \cdot v \otimes (xy)_{(0)} z = (x_{(-1)} y_{(-1)}) \cdot v \otimes x_{(0)} y_{(0)} z$$

so that the associativity of μ and μ_0 is equivalent by another application of the universal property of L. We skip unitality.

We have seen that the functor in consideration is well-defined and an equivalence. To check that it is monoidal, we should verify that the canonical isomorphism

$f\colon V\otimes W\otimes A\to (V\otimes A)\otimes_A (W\otimes A)$ is a left A-module map for $V,W\in L$. Indeed

$$\begin{aligned}f(x(v\otimes w\otimes y)) &= f(x_{(-1)}\cdot(v\otimes w)\otimes x_{(0)}y) = f(x_{(-2)}\cdot v\otimes x_{(-1)}\cdot w\otimes x_{(0)}y)\\ &= x_{(-2)}\cdot v\otimes 1\otimes x_{(-1)}\cdot w\otimes x_{(0)}y = x_{(-1)}\cdot v\otimes 1\otimes x_{(0)}(w\otimes y)\\ &= x_{(-1)}\cdot v\otimes x_{(0)}\otimes (w\otimes y) = x(v\otimes 1)\otimes 1\otimes y = xf(v\otimes w\otimes y)\end{aligned}$$

for all $x,y\in A$, $v\in V$ and $w\in W$. □

Corollary 3.3.2 *Let A be an L-H-bi-Galois object. Then there is a bijection between isomorphism classes of*

1. *Pairs (T,f), where T is an H-comodule algebra, and $f\colon A\to T$ is an H-comodule algebra map, and*
2. *L-module algebras R*

It is given by $R:=T^{\operatorname{co}H}$, and $T:=R\#A:=R\otimes A$ with multiplication given by $(r\#x)(s\#y)=r(x_{(-1)}\cdot s)\#x_{(0)}y$.

Note in particular that every T as in (1) is a left faithfully flat H-Galois extension of its coinvariants.

Remark 3.3.3 If A is a faithfully flat H-Galois extension of B, then ${}_A\mathcal{M}_A^H$ is still a monoidal category, and the coinvariants functor is still a monoidal functor to ${}_B\mathcal{M}_B$. It is a natural question whether there is still some L whose modules classify Hopf modules in the same way as we have shown in this section for the case $B=k$, and whether L is still a Hopf algebra in any sense. This was answered in [35] by showing that $L=(A\otimes A)^{\operatorname{co}H}$ still yields a commutative diagram (3.3.1), and that L now has the structure of a $\times_B$-bialgebra in the sense of Takeuchi [48]. These structures have been studied more recently under the name of quantum groupoids or Hopf algebroids. They have the characteristic property that modules over a $\times_B$-bialgebra still form a monoidal category, so that it makes sense to say that (3.3.1) will be a commutative diagram of monoidal functors. The $\times_B$-bialgebra L can step in in some cases where the left Hopf algebra L is useful, but $B\neq k$. Since the axiomatics of $\times_B$-bialgebras are quite complicated, we will not pursue this matter here.

3.4 Galois correspondence. The origin of bi-Galois theory is the construction in [18] of certain separable extensions of fields that are Hopf-Galois with more than one possibility for the Hopf algebra. The paper [18] also contains information about what may become of the classical Galois correspondence between subfields and subgroups in this case. In particular, there are examples of classically Galois field extensions that are also H-Galois in such a way that the quotient Hopf algebras of H correspond one-to-one to the *normal* intermediate fields, that is, to the intermediate fields that are stable under the coaction of the dual Hopf algebra k^G of the group algebra of the Galois group. Van Oystaeyen and Zhang [56] then constructed what we called $L(A,H)$ above for the case of commutative A (and H), and proved a correspondence between quotients of $L(A,H)$ and H-costable intermediate fields in case A is a field. The general picture was developed in [33, 36]. We will not comment on the proof here, but simply state the results.

Theorem 3.4.1 *Let A be an L-H-bi-Galois object for k-flat Hopf algebras L,H with bijective antipodes.*

A bijection between

- *coideal left ideals $I \subset L$ such that L/I is k-flat and L is a faithfully coflat left (resp. right) L/I-comodule, and*
- *H-subcomodule algebras $B \subset A$ such that B is k-flat and A is a faithfully flat left (resp. right) B-module*

is given as follows: To a coideal left ideal $I \subset L$ we assign the subalgebra $B := {}^{\mathrm{co}\,A}L/I$. To an H-subcomodule algebra $B \subset A$ we assign the coideal left ideal $I \subset L \cong (A \otimes A)^{\mathrm{co}\,H}$ such that $L/I \cong (A \otimes_B A)^{\mathrm{co}\,H}$.

Let $I \subset L$ and $B \subset A$ correspond to each other as above. Then

1. *I is a Hopf ideal if and only if B is stable under the Miyashita-Ulbrich action of H on A.*
2. *I is stable under the left coadjoint coaction of L on itself if and only if B is stable under the coaction of L on A.*
3. *I is a conormal Hopf ideal if and only if B is stable both under the coaction of L and the Miyashita-Ulbrich action of H on A.*

As the special case $A = H$, the result contains the quotient theory of Hopf algebras, that is, the various proper Hopf algebra analogs of the correspondence between normal subgroups and quotient groups of a group. See [50, 26].

3.5 Galois objects over tensor products. Let H_1, H_2 be two Hopf algebras. If both H_i are cocommutative, then $\mathrm{Gal}(H_i)$ are groups under cotensor product, as well as $\mathrm{Gal}(H_1 \otimes H_2)$. If both H_i are also commutative, then we have the subgroups of these three groups consisting of all commutative Galois extensions. If, in particular, we take both H_i to be the duals of group algebras of abelian groups, then $H_1 \otimes H_2$ is the group algebra of the direct sum of those two groups, and the groups of commutative Galois objects are the Harrison groups. It is an old result that the functor "Harrison group" is additive. This means that $\mathrm{Har}(H_1 \otimes H_2) \cong \mathrm{Har}(H_1) \oplus \mathrm{Har}(H_2)$ as (abelian) groups. The same result holds true unchanged if we consider general commutative and cocommutative Hopf algebras. However, the same is not true for the complete Gal(—) groups. A result of Kreimer [24] states very precisely what is true instead: For two commutative cocommutative finitely generated projective Hopf algebras, we have an isomorphism of abelian groups

$$\mathrm{Gal}(H_1) \oplus \mathrm{Gal}(H_2) \oplus \mathrm{Hopf}(H_2, H_1^*) \to \mathrm{Gal}(H_1 \otimes H_2),$$

where $\mathrm{Hopf}(H_2, H_1^*)$ denotes the set of all Hopf algebra maps from H_2 to H_1^*, which is a group under convolution because H_2 is cocommutative and H_1^* is commutative. The assumption that both Hopf algebras H_i are commutative is actually not necessary. One can also drop the assumption that they be finitely generated projective, if one replaces the summand $\mathrm{Hopf}(H_2, H_1^*)$ by the group (under convolution) $\mathrm{Pair}(H_2, H_1)$ of all Hopf algebra pairings between H_2 and H_1; this does not change anything if H_1 happens to be finitely generated projective.

One cannot, however, get away without the assumption of cocommutativity: First of all, of course, we do not have any groups in the case of general H_i. Secondly, some of the information in the above sequence does survive on the level of pointed sets, but not enough to amount to a complete description of $\mathrm{Gal}(H_1 \otimes H_2)$.

As we will show in this section (based on [39]), bi-Galois theory can come to the rescue to recover such a complete description. Instead of pairings between the Hopf algebras H_i, one has to take into account pairings between the left Hopf algebras L_i in certain H_i-Galois objects.

Lemma 3.5.1 *Let H_1, H_2 be two k-flat Hopf algebras, and A a right H-comodule algebra for $H = H_1 \otimes H_2$.*

We have $A_1 := A^{\operatorname{co} H_2} \cong A \,\square_H\, H_1$ and $A_2 := A^{\operatorname{co} H_1} \cong A \,\square_H\, H_2$.

A is an H-Galois object if and only if A_i is an H_i-Galois object for $i = 1, 2$.

If this is the case, then multiplication in A induces an isomorphism $A_1 \# A_2 \to A$, where the algebra structure of $A_1 \# A_2$ is a smash product as in Corollary 3.3.2 for some L_2-module structure on A_1, where $L_2 := L(A_2, H_2)$; the H-comodule structure is the obvious one.

Proof It is straightforward to check that $A^{\operatorname{co} H_i} \cong A \,\square_H\, H_j$ for $i \neq j$. We know from Proposition 2.5.7 that A_i are Hopf-Galois objects if A is one.

Now assume that A_i is a faithfully flat H_i-Galois extension of k for $i = 1, 2$. By Corollary 3.3.2 we know that multiplication in A induces an isomorphism $A_1 \# A_2 \to A$ for a suitable L_2-module algebra structure on A_1. We view the Galois map $A \otimes A \to A \otimes H$ as a map of Hopf modules in $\mathcal{M}^{H_2}_{A_2}$. Its H_2-coinvariant part is the map $A \otimes A_1 \to A \otimes H_1$ given by $x \otimes y \mapsto xy_{(0)} \otimes y_{(1)}$, which we know to be a bijection. Thus the canonical map for A is a bijection, and A is faithfully flat since it is the tensor product of A_1 and A_2. $\square$

To finish our complete description of $H_1 \otimes H_2$-Galois objects, we need two more consequences from the universal property of the left Hopf algebra:

Lemma 3.5.2 *Let A be an L-H-bi-Galois object, and let R be an L-module algebra and F-comodule algebra for some k-flat bialgebra F. Then $R \# A$ as in Corollary 3.3.2 is an $F \otimes H$-comodule algebra if and only if it is an F-comodule algebra, if and only if R is an L-F-dimodule in the sense that $(\ell \cdot r)_{(0)} \otimes (\ell \cdot r)_{(1)} = \ell \cdot r_{(0)} \otimes r_{(1)}$ holds for all $r \in R$ and $\ell \in L$.*

Proof Clearly $R \# A$ is an $F \otimes H$-comodule algebra if and only if it is an F-comodule algebra, since we already know it to be an H-comodule algebra.

Now (ignoring the unit conditions) $R \# A$ is an F-comodule algebra if and only if

$$r_{(0)}(x_{(-1)} \cdot s)_{(0)} \# x_{(0)} y \otimes r_{(1)}(x_{(-1)} \cdot s)_{(1)}$$
$$r_{(0)}(x_{(-1)} \cdot s_{(0)}) \# x_{(0)} y \otimes r_{(1)} s_{(1)}$$

agree for all $r, s \in R$ and $x, y \in A$. By the universal property of L, this is the same as requiring

$$r_{(0)}(\ell \cdot s)_{(0)} \otimes r_{(1)}(\ell \cdot s)_{(1)} = r_{(0)}(\ell \cdot s_{(0)}) \otimes r_{(1)} s_{(1)}$$

for all $r, s \in R$ and $\ell \in A$, which in turn is the same as requiring the dimodule condition for R. $\square$

Lemma 3.5.3 *Let A be an L-H-bi-Galois object, B a k-module, and $\mu\colon B \otimes A \to A$ an H-colinear map. Then $\mu = \Phi(\tau)$, that is, $\mu(b \otimes a) = \tau(b \otimes a_{(-1)}) a_{(0)}$ for a unique $\tau\colon B \otimes L \to A$. Moreover,*

1. *Assume that B is a coalgebra. Then μ is a measuring if and only if $\tau(b \otimes \ell m) = \tau(b_{(1)} \otimes \ell)\tau(b_{(2)} \otimes m)$ and $\tau(b \otimes 1) = \varepsilon(b)$ hold for all $b \in C$ and $\ell, m \in L$.*
2. *Assume that B is an algebra. Then μ is a module structure if and only if $\tau(bc \otimes \ell) = \tau(b \otimes \ell_{(2)})\tau(c \otimes \ell_{(1)})$ and $\tau(1 \otimes \ell) = \varepsilon(\ell)$ hold for all $b, c \in B$ and $\ell \in L$.*

3. *Assume that B is a bialgebra. Then μ makes A a B-module algebra if and only if τ is a skew pairing between B and L, in the sense of the following definition:*

Definition 3.5.4 Let B and L be two bialgebras. A map $\tau\colon B\otimes L\to k$ is called a skew pairing if

$$\tau(b\otimes \ell m)=\tau(b_{(1)}\otimes\ell)\tau(b_{(2)}\otimes m),\qquad \tau(b\otimes 1)=\varepsilon(b)$$
$$\tau(bc\otimes\ell)=\tau(b\otimes\ell_{(2)})\tau(c\otimes\ell_{(1)}),\qquad \tau(1\otimes\ell)=\varepsilon(\ell)$$

hold for all $b,c\in B$ and $\ell,m\in L$. Note that if B is finitely generated projective, then a skew pairing is the same as a bialgebra morphism $L^{\mathrm{cop}}\to B^*$

Proof We write $\mu(b\otimes a)=b\cdot a$. We have $\mu=\Phi(\tau)$ for $\tau\colon B\otimes L\to k$ as a special case of the universal properties of L.

If B is a coalgebra, then

$$b\cdot(xy)=\tau(b\otimes x_{(-1)}y_{(-1)})x_{(0)}y_{(0)}$$
$$(b_{(1)}\cdot x)(b_{(2)}\cdot y)=\tau(b_{(1)}\otimes x_{(-1)})\tau(b_{(2)}\otimes y_{(-1)})x_{(0)}y_{(0)}$$

are the same for all $b\in B$, $x,y\in A$ if and only if $\tau(b\otimes\ell m)=\tau(b_{(1)}\otimes\ell)\tau(b_{(2)}\otimes m)$ for all $b\in B$, $\ell,m\in L$, by the universal property again. We omit treating the unit condition for a measuring

If B is an algebra then

$$(bc)\cdot x=\tau(bc\otimes x_{(-1)})x_{(0)}$$
$$b\cdot c\cdot x=\tau(c\otimes x_{(-1)})b\cdot x_{(0)}=\tau(c\otimes x_{(-2)})\tau(b\otimes x_{(-1)})x_{(0)}$$

agree for all $b,c\in B$, $x\in A$ if and only if $\tau(bc\otimes\ell)=\tau(c\otimes\ell_{(1)})\tau(b\otimes\ell_{(2)})$ holds for all $b,c\in B$ and $\ell\in L$. Again, we omit treating the unit condition for a module structure.

Since a module algebra structure is the same as a measuring that is a module structure, we are done. □

Now we merely need to put together all the information obtained so far to get the following theorem.

Theorem 3.5.5 *Let H_1,H_2 be two k-flat Hopf algebras, and put $H=H_1\otimes H_2$. The map*

$$\pi\colon\ \mathrm{Gal}(H_1\otimes H_2)\to\mathrm{Gal}(H_1)\times\mathrm{Gal}(H_2);A\mapsto(A\mathop{\square}_{H}H_1,A\mathop{\square}_{H}H_2)$$

is surjective. For $A_i\in\mathrm{Gal}(H_i)$ let $L_i:=L(A_i,H_i)$. The Hopf algebra automorphism groups of L_i act on the right on the set of all skew pairings between L_1 and L_2. We have a bijection

$$\mathrm{SPair}(L_1,L_2)/\operatorname{CoInn}(L_1)\times\operatorname{CoInn}(L_2)\to\pi^{-1}(A_1,A_2),$$

given by assigning to the class of a skew pairing τ the algebra $A_1\#_\tau A_2:=A_1\otimes A_2$ with multiplication $(r\#x)(s\#y)=r\tau(s_{(-1)}\otimes x_{(-1)})s_{(0)}\#x_{(0)}y$.

In particular, we have an exact sequence

$$\operatorname{CoInn}(H_1)\times\operatorname{CoInn}(H_2)\to\mathrm{SPair}(H_2,H_1)\to\mathrm{Gal}(H_1\otimes H_2)\to\mathrm{Gal}(H_1)\times\mathrm{Gal}(H_2)$$

Proof Since $\pi(A_1 \otimes A_2) = (A_1, A_2)$, the map π is onto. Fix $A_i \in \mathrm{Gal}(H_i)$. Then the inverse image of $A_1 \otimes A_2$ under π consists of all those H-Galois objects A for which $A \square_H H_i \cong A_i$. By the discussion preceding the theorem, every such A has the form $A = A_1 \# A_2$, with multiplication given by an L_2-module algebra structure on A_1, which makes A_1 an L_2-H_1-dimodule, and is thus given by a skew pairing between L_1 and L_2.

Assume that for two skew pairings τ, σ we have an isomorphism $f\colon A_1 \#_\tau A_2 \to A_1 \#_\sigma A_2$. Then f has the form $f = f_1 \otimes f_2$ for automorphisms f_i of the H_i-comodule algebra A_i, which are given by $f_i(x) = u_i(x_{(-1)})x_{(0)}$ for algebra maps $u_i\colon L_i \to k$. The map f is an isomorphism of algebras if and only if $f((1\#x)(r\#1)) = f(1\#x)f(r\#1)$ for all $r \in A_1$ and $x \in A_2$. Now

$$
\begin{aligned}
f((1\#x)(r\#1)) &= \tau(r_{(-1)} \otimes x_{(-1)}) f(r_{(0)} \# x_{(0)}) \\
&= \tau(r_{(-2)} u_1(r_{(-1)}) \otimes x_{(-2)} u_2(x_{(-1)})) r_{(0)} \# x_{(0)}
\end{aligned}
$$

and on the other hand

$$f(1\#x)f(r\#1) = u_1(r_{(-2)}) u_2(x_{(-2)}) \tau(r_{(-1)} \otimes x_{(-1)}) r_{(0)} \# x_{(0)}$$

These two expressions are the same for all r, x if and only if τ and σ agree up to composition with $\mathrm{coinn}(u_1) \otimes \mathrm{coinn}(u_2)$, by yet another application of the universal properties of L_1 and L_2. □

3.6 Reduction. We take up once again the topic of reduction of the structure group, or the question of when an H-Galois extension reduces to a Q-Galois extension for a quotient Hopf algebra Q of H. We treated the case of a general base B of the extension in Section 2.7. Here, we treat some aspects that are more or less special to the case of a trivial coinvariant subring k, and involve the left Hopf algebra L.

We start by a simple reformulation of the previous results, using Corollary 3.1.4:

Corollary 3.6.1 *Assume the hypotheses of Theorem 2.7.1. Consider the map* $\pi\colon \mathrm{Gal}(Q) \to \mathrm{Gal}(H)$ *given by* $\pi(\overline{A}) = A \square_Q H$, *and let* $A \in \mathrm{Gal}(H)$. *Then* $\pi^{-1}(A) \cong \mathrm{Hom}_{-H}^{-H}(K, A)/\mathrm{Alg}(L, k)$, *where* $L = L(A, H)$.

The criterion we have given above for reducibility of the structure quantum group (i.e. the question when an H-Galois extension comes from a Q-Galois extension) is "classical" in the sense that analogous results are known for principal fiber bundles: If we take away the Miyashita-Ulbrich action on A^B, which is a purely noncommutative feature, we have to find a colinear algebra map $K \to A$, which is to say an equivariant map from the principal bundle (the spectrum of A) to the coset space of the structure group under the subgroup we are interested in. Another criterion looks even simpler in the commutative case: According to [11, III §4, 4.6], a principal fiber bundle, described by a Hopf-Galois extension A, can be reduced if and only if the associated bundle with fiber the coset space of the subgroup in the structure group admits a section. In our terminology, this means that there is an algebra map $(A \otimes K)^{\mathrm{co}\,H} \to B$ of the obvious map $B \to A$; alternatively, one may identify the associated bundle with $A^{\mathrm{co}\,Q} \cong (A \otimes K)^{\mathrm{co}\,H}$, see below. As it turns out, this criterion can be adapted to the situation of general Hopf-Galois extensions as well. In the noncommutative case, there are, again, extra requirements on the map $A^{\mathrm{co}\,Q} \to B$. In fact suitable such conditions were spelled out in [7, Sec.2.5], although the formulas there seem to defy a conceptual interpretation. As it turns out, the extra conditions can be cast in a very simple form using the left Hopf algebra L: The relevant map $A^{\operatorname{co} Q} \to B$ should simply be L-linear with respect to the Miyashita-Ulbrich action of L. Since the result now involves the left Hopf algebra, it can only be formulated like that in the case $B = k$; we note, however, that the result, as well as its proof, is still valid for the general case — one only has to take the $\times_B$-bialgebra L, see Remark 3.3.3, in place of the ordinary bialgebra L.

Theorem 3.6.2 *Assume the hypotheses of Theorem 2.7.1, and let A be an L-H-bi-Galois object. $A^{\operatorname{co} Q} \subset A$ is a submodule with respect to the Miyashita-Ulbrich action of L on A.*

The following are equivalent:

1. *$A \in \operatorname{Gal}(H)$ is in the image of* $\operatorname{Gal}(Q) \ni \overline{A} \mapsto \overline{A} \square_Q H \in \operatorname{Gal}(H)$.
2. *There is an L-module algebra map $A^{\operatorname{co} Q} \to k$.*

Proof The inverse of the Galois map of the left L-Galois extension A maps L to $(A \otimes A)^{\operatorname{co} H}$, so it is straightforward to check that $A^{\operatorname{co} Q}$ is invariant under the Miyashita-Ulbrich action of L.

We have an isomorphism $\theta_0\colon A \to (A \otimes H)^{\operatorname{co} H}$ with $\theta(a) = a_{(0)} \otimes S(a_{(1)})$ and $\theta_0^{-1}(a \otimes h) = a\varepsilon(h)$. One checks that $\theta_0(a) \in (A \otimes K)^{\operatorname{co} H}$ if and only if $a \in A^{\operatorname{co} Q}$, so that we have an isomorphism $\theta\colon A^{\operatorname{co} Q} \to (A \otimes K)^{\operatorname{co} H}$ given by $\theta(a) = a_{(0)} \otimes S(a_{(1)})$. It is obvious that θ is an isomorphism of algebras, if we regard $(A \otimes K)^{\operatorname{co} H}$ as a subalgebra of $A \otimes K^{\mathrm{op}}$, but we would like to view this in a more complicated way: Since K is an algebra in the center of $\mathcal{M}^H$, we can endow $A \otimes K$ with the structure of an algebra in $\mathcal{M}^H$ by setting $(a \otimes x)(b \otimes y) = ab_{(0)} \otimes (x \leftharpoonup b_{(1)})y$. If $x \otimes a \in (A \otimes K)^{\operatorname{co} H}$, then $(a \otimes x)(b \otimes y) = ab_{(0)} \otimes (x \leftharpoonup b_{(1)})y = ab_{(0)} \otimes y_{(0)}(x \leftharpoonup b_{(1)}y_{(1)}) = ab \otimes yx$, so $(A \otimes K)^{\operatorname{co} H}$ is a subalgebra of $A \otimes K^{\mathrm{op}}$.

Now the obvious map $A \to A \otimes K$ is an H-colinear algebra map, so $A \otimes K$ is an algebra in the monoidal category ${}_A\mathcal{M}_A^H$ and hence $(A \otimes K)^{\operatorname{co} H}$ is an L-module algebra by Corollary 3.3.2. Writing $\ell^{(1)} \otimes \ell^{(2)}$ for the preimage of $\ell \otimes 1$ under the Galois map for the left L-Galois extension A, we can compute the relevant L-module structure as $\ell \triangleright (a \otimes x) = \ell^{(1)}(a \otimes x)\ell^{(2)} = \ell^{(1)}a\ell^{(2)}{}_{(0)} \otimes x \leftharpoonup \ell^{(2)}{}_{(1)}$. It is immediate that θ^{-1} is L-linear.

Now we have a bijection between H-colinear maps $f\colon K \to A$ and H-colinear and left A-linear maps $\hat{f}\colon A \otimes K \to A$ given by $f(x) = \hat{f}(1 \otimes x)$ and $\hat{f}(a \otimes x) = af(x)$. Let us check that f is a right H-module algebra map if and only if $\hat{f}$ is an A-ring map in $\mathcal{M}^H$. First, assume that f is an H-module algebra map. Then

$$
\begin{aligned}
\hat{f}((a \otimes x)(b \otimes y)) &= \hat{f}(ab_{(0)} \otimes (x \leftharpoonup b_{(1)})y) \\
&= ab_{(0)}f(x \leftharpoonup b_{(1)})f(y) \\
&= ab_{(0)}(f(x) \leftharpoonup b_{(1)})f(y) \\
&= af(x)bf(y) = \hat{f}(a \otimes x)\hat{f}(b \otimes y),
\end{aligned}
$$

and $\hat{f}(1 \otimes a) = a$. Conversely, assume that $\hat{f}$ is an A-ring morphism in $\mathcal{M}^H$. Then f is trivially an algebra map, and

$$
f(x)a = \hat{f}(1 \otimes x)\hat{f}(a \otimes 1) = \hat{f}((1 \otimes x)(a \otimes 1)) = \hat{f}(a_{(0)} \otimes x \leftharpoonup a_{(1)}) = a_{(0)}f(x \leftharpoonup a_{(1)})
$$

for all $a \in A$ and $x \in K$ implies that f is H-linear.

Finally, we already know that A-ring morphisms $\hat{f}\colon A \otimes K \to A$ in $\mathcal{M}^H$, that is, algebra maps in ${}_A\mathcal{M}^H_A$, are in bijection with L-module algbra maps $g\colon (A \otimes K)^{\operatorname{co} H} \to k$. $\square$

If we want to reduce the right Hopf algebra in an L-H-bi-Galois extension, it is of course also a natural question what happens to the left Hopf algebra in the process:

Lemma 3.6.3 *Assume the situation of Theorem 2.7.1. Let $\overline{A}$ be a Q-Galois object, and A the corresponding H-Galois object. Then for any $V \in \mathcal{M}^H$ the map $(V \otimes A)^{\operatorname{co} H} \to (V \otimes \overline{A})^{\operatorname{co} Q}$ induced by the surjection $A \to \overline{A}$ is an isomorphism.*

Proof It is enough to check that $\alpha\colon (V \otimes A)^{\operatorname{co} H} \to (V \otimes \overline{A})^{\operatorname{co} Q}$ is bijective after tensoring with $\overline{A}$. We compose $\alpha \otimes \overline{A}$ with the isomorphism $(V \otimes \overline{A})^{\operatorname{co} Q} \otimes \overline{A} \to V \otimes \overline{A}$ from the structure theorem for Hopf modules in $\mathcal{M}^Q_{\overline{A}}$, and have to check that

$$(V \otimes A)^{\operatorname{co} H} \otimes \overline{A} \to (V \otimes \overline{A})^{\operatorname{co} Q} \otimes \overline{A} \to V \otimes \overline{A}; \quad v \otimes a \otimes \overline{b} \mapsto v \otimes \overline{ab}$$

is bijective. But this is the image under the equivalence $\mathcal{M}^H_K \to \mathcal{M}^Q$ of the isomorphism

$$(V \otimes A)^{\operatorname{co} H} \otimes A \to V \otimes A; \quad v \otimes a \otimes b \to v \otimes ab$$

from the structure theorem for Hopf modules in $\mathcal{M}^H_A$. $\square$

Theorem 3.6.4 *Assume the situation of Theorem 2.7.1, let A be an H-Galois object, $f\colon K \to A$ a Yetter-Drinfeld algebra map, and $\overline{A} = A/Af(K^+)$ the corresponding Q-Galois object.*

Using the identification $L := L(A,H) = (A \otimes A)^{\operatorname{co} H}$, the left Hopf algebra of $\overline{A}$ is given by

$$L(\overline{A}, Q) = (A \underset{K}{\otimes} A)^{\operatorname{co} H},$$

where the K-module structure of A is induced via f.

Proof We have to verify that the surjection $A \to \overline{A}$ induces an isomorphism $(A \otimes_K A)^{\operatorname{co} H} \to (\overline{A} \otimes \overline{A})^{\operatorname{co} Q}$. Since $(—)^{\operatorname{co} H}\colon \mathcal{M}^H_A \to \mathcal{M}_k$ is an equivalence, this amounts to showing that we have a coequalizer

$$(A \otimes K \otimes A)^{\operatorname{co} H} \rightrightarrows (A \otimes A)^{\operatorname{co} H} \to (\overline{A} \otimes \overline{A})^{\operatorname{co} Q} \to 0.$$

Using Lemma 3.6.3 this means a coequalizer

$$(A \otimes K \otimes \overline{A})^{\operatorname{co} Q} \rightrightarrows (A \otimes \overline{A})^{\operatorname{co} Q} \to (\overline{A} \otimes \overline{A})^{\operatorname{co} Q} \to 0.$$

Since $(—)^{\operatorname{co} Q}\colon \mathcal{M}^Q_{\overline{A}} \to \mathcal{M}_k$ is an equivalence, we may consider this before taking the Q-coinvariants, when it is just the definition of $\overline{A} = A/Af(K^+)$ tensored with $\overline{A}$. $\square$

4 Appendix: Some tools

4.1 Monoidal category theory. A monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \Phi, I, \lambda, \rho)$ consists of a category $\mathcal{C}$, a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a natural isomorphism $\Phi\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, an object I, and natural isomorphisms $\lambda\colon I \otimes X \to X$ and $\rho\colon X \otimes I \to X$, all of which are coherent. This means that all diagrams that one can compose from Φ (which rearranges brackets), λ, ρ (which cancel instances of the unit object I) and their inverses commute. By Mac Lane's coherence theorem, it is actually enough to ask for one pentagon of Φ's, and one triangle with λ, ρ, and Φ, to commute in order that all diagrams commute. A monoidal category is called strict if Φ, λ, and ρ are identities.

The easiest example of a monoidal category is the category $\mathcal{M}_k$ of modules over a commutative ring, with the tensor product over k and the canonical isomorphisms expressing associativity of tensor products. Similarly, the category ${}_R\mathcal{M}_R$ of bimodules over an arbitrary ring R is monoidal with respect to the tensor product over R. We are interested in monoidal category theory because of its very close connections with Hopf algebra theory. If H is a bialgebra, then both the category of, say, left H-modules, and the category of, say, right H-comodules have natural monoidal category structures. Here, the tensor product of $V, W \in {}_H\mathcal{M}$ (resp. $V, W \in \mathcal{M}^H$) is $V \otimes W$, the tensor product over k, equipped with the diagonal module structure $h(v \otimes w) = h_{(1)}v \otimes h_{(2)}w$ (resp. the codiagonal comodule structure $\delta(v \otimes w) = v_{(0)} \otimes w_{(0)} \otimes v_{(1)}w_{(1)}$. The unit object is the base ring k with the trivial module (resp. comodule) structure induced by the counit ε (resp. the unit element of H). Since the associativity and unit isomorphisms in all of these examples are "trivial", it is tempting never to mention them at all, practically treating all our examples as if they were strict monoidal categories; we will do this in all of the present paper. In fact, this sloppiness is almost justified by the fact that every monoidal category is monoidally equivalent (see below) to a strict one. For the examples in this paper, which are categories whose objects are sets with some algebraic structure, the sloppiness is even more justified [40].

A weak monoidal functor $\mathcal{F} = (\mathcal{F}, \xi, \xi_0)\colon \mathcal{C} \to \mathcal{D}$ consists of a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$, a natural transformation $\xi\colon \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$ and a morphism $\xi_0\colon \mathcal{F}(I) \to I$ making the diagrams

$$\begin{tikzcd}
(\mathcal{F}(X) \otimes \mathcal{F}(Y)) \otimes \mathcal{F}(Z) \arrow[r, "\xi\otimes 1"] \arrow[d, "\Phi"'] & \mathcal{F}(X \otimes Y) \otimes \mathcal{F}(Z) \arrow[r, "\xi"] & \mathcal{F}((X \otimes Y) \otimes Z) \arrow[d, "\mathcal{F}(\Phi)"] \\
\mathcal{F}(X) \otimes (\mathcal{F}(Y) \otimes \mathcal{F}(Z)) \arrow[r, "1\otimes\xi"'] & \mathcal{F}(X) \otimes \mathcal{F}(Y \otimes Z) \arrow[r, "\xi"'] & \mathcal{F}(X \otimes (Y \otimes Z))
\end{tikzcd}$$

commute and satisfying

$$\mathcal{F}(\lambda)\xi(\xi_0 \otimes id) = \lambda\colon I \otimes \mathcal{F}(X) \to \mathcal{F}(X)$$
$$\mathcal{F}(\rho)\xi(id \otimes \xi_0) = \rho\colon \mathcal{F}(X) \otimes I \to \mathcal{F}(X).$$

A standard example arises from a ring homomorphism $R \to S$. The restriction functor ${}_S\mathcal{M}_S \to {}_R\mathcal{M}_R$ is a weak monoidal functor, with $\xi\colon M \otimes_R N \to M \otimes_S N$ for $M, N \in {}_S\mathcal{M}_S$ the canonical surjection.

A monoidal functor is a weak monoidal functor in which ξ and ξ_0 are isomorphisms. Typical examples are the underlying functors ${}_H\mathcal{M} \to \mathcal{M}_k$ and $\mathcal{M}^H \to \mathcal{M}_k$

for a bialgebra H. In this case, the morphisms ξ, ξ_0 are even identities; we shall say that we have a strict monoidal functor.

A prebraiding for a monoidal category $\mathcal{C}$ is a natural transformation $\sigma_{XY}\colon X\otimes Y\to Y\otimes X$ satisfying

$$\sigma_{X,Y\otimes Z}=(Y\otimes\sigma_{XZ})(\sigma_{XY}\otimes Z)\colon X\otimes Y\otimes Z\to Y\otimes Z\otimes X$$
$$\sigma_{X\otimes Y,Z}=(\sigma_{XZ}\otimes Y)(X\otimes\sigma_{YZ})\colon X\otimes Y\otimes Z\to Z\otimes X\otimes Y$$
$$\sigma_{XI}=\sigma_{IX}=id_X$$

A braiding is a prebraiding that is an isomorphism. A symmetry is a braiding with $\sigma_{XY}=\sigma_{YX}^{-1}$. The notion of a symmetry captures the properties of the monoidal category of modules over a commutative ring. For the topological flavor of the notion of braiding, we refer to Kassel's book [22]. We call a (pre)braided category a category with a (pre)braiding.

The (weak) center construction produces a (pre)braided monoidal category from any monoidal category: Objects of the weak center $\mathcal{Z}_0(\mathcal{C})$ are pairs $(X,\sigma_{X,\_})$ in which $X\in\mathcal{C}$, and $\sigma_{XY}\colon X\otimes Y\to Y\otimes X$ is a natural transformation satisfying

$$\sigma_{X,Y\otimes Z}=(Y\otimes\sigma_{XZ})(\sigma_{XY}\otimes Z)\colon X\otimes Y\otimes Z\to Y\otimes Z\otimes X$$

for all $Y,Z\in\mathcal{C}$, and $\sigma_{XI}=id_X$. The weak center is monoidal with tensor product $(X,\sigma_{X,\_})\otimes(Y,\sigma_{Y,\_})=(X\otimes Y,\sigma_{X\otimes Y,\_})$, where

$$\sigma_{X\otimes Y,Z}=(\sigma_{XZ}\otimes Y)(X\otimes\sigma_{YZ})\colon X\otimes Y\otimes Z\to Z\otimes X\otimes Y$$

for all $Z\in\mathcal{C}$, and with neutral element $(I,\sigma_{I,\_})$, where $\sigma_{IZ}=id_Z$. The weak center is prebraided with the morphism σ_{XY} as the prebraiding of X and Y. The center $\mathcal{Z}(\mathcal{C})$ consists of those objects $(X,\sigma_{X\_})\in\mathcal{Z}_0(\mathcal{C})$ in which all σ_{XY} are isomorphisms.

The main example of a (pre)braided monoidal category which we use in this paper is actually a center. Let H be a Hopf algebra. The category $\mathcal{Z}_0(\mathcal{M}^H)$ is equivalent to the category $\mathcal{YD}_H^H$ of right-right Yetter-Drinfeld modules, whose objects are right H-comodules and right H-modules V satisfying the condition

$$(v\leftharpoonup h)_{(0)}\otimes(v\leftharpoonup h)_{(1)}=v_{(0)}\leftharpoonup h_{(2)}\otimes S(h_{(1)})v_{(1)}h_{(3)}$$

for all $v\in V$ and $h\in H$. A Yetter-Drinfeld module V becomes an object in the weak center by

$$\sigma_{VW}(v\otimes w)=w_{(0)}\otimes v\leftharpoonup w_{(1)}$$

for all $v\in V$ and $w\in W\in\mathcal{M}^H$. It is an object in the center if and only if H has bijective antipode, in which case $\sigma_{VW}^{-1}(w\otimes v)=v\leftharpoonup S^{-1}(w_{(1)})\otimes w_{(0)}$.

4.2 Algebras in monoidal categories. At some points in this paper we have made free use of the notion of an algebra within a monoidal category, modules over it, and similar notions. In this section we will spell out (without the easy proofs) some of the basic facts. It is possible that the notion of center that we define below is new.

Let $\mathcal{C}$ be a monoidal category, which we assume to be strict for simplicity. An algebra in $\mathcal{C}$ is an object A with a multiplication $\nabla\colon A\otimes A\to A$ and a unit $\eta\colon I\to A$ satisfying associativity and the unit condition that $A\cong A\otimes I\xrightarrow{A\otimes\eta}A\otimes A\xrightarrow{\nabla}A$ (and a symmetric construction) should be the identity. It should be clear what a morphism of algebras is. A left A-module in $\mathcal{C}$ is an object M together with a module structure $\mu\colon A\otimes M\to M$ which is associative and fulfills an obvious unit condition. It is clear how to define right modules and bimodules in a monoidal category.

An algebra in the (pre)braided monoidal category $\mathcal{C}$ is said to be commutative if $\nabla_A = \nabla_A\sigma\colon A \otimes A \to A$. Obviously we can say that a subalgebra B in A (or an algebra morphism $\iota\colon B \to A$) is central in A if

$$\nabla_A(\iota \otimes A) = \nabla_A\sigma_{AA}(\iota \otimes A) = \nabla_A(A \otimes \iota)\sigma_{BA}\colon B \otimes A \to A.$$

Since the last notion needs only the braiding between B and A to be written down, it can be generalized as follows:

Definition 4.2.1 Let A be an algebra in $\mathcal{C}$. A morphism $f\colon V \to A$ in $\mathcal{C}$ from an object $V \in \mathcal{Z}_0(\mathcal{C})$ is called central, if $\nabla_A(f \otimes A) = \nabla_A(A \otimes f)\sigma_{VA}\colon V \otimes A \to A$.

A center of A is a couniversal central morphism $c\colon C \to A$, that is, an object $C \in \mathcal{Z}_0(\mathcal{C})$ with a morphism $c\colon C \to A$ in $\mathcal{C}$ such that every central morphism $f\colon V \to A$ factors through a morphism $g\colon V \to C$ in $\mathcal{Z}_0(\mathcal{C})$.

It is not clear whether a center of an algebra A exists, or if it does, if it is a subobject in $\mathcal{C}$, though this is true in our main application Lemma 2.1.9. However, the following assertions are not hard to verify:

Remark 4.2.2 Let A be an algebra in $\mathcal{C}$, and assume that A has a center (C, c).

1. Any center of A is isomorphic to C.
2. C is a commutative algebra in $\mathcal{Z}_0(\mathcal{C})$.
3. If R is an algebra in $\mathcal{Z}_0(\mathcal{C})$ and $f\colon R \to A$ is central and an algebra morphism in $\mathcal{C}$, then its factorization $g\colon R \to C$ is an algebra morphism.

Let A and B be algebras in the prebraided monoidal category $\mathcal{C}$. Then $A \otimes B$ is an algebra with multiplication

$$A \otimes B \otimes A \otimes B \xrightarrow{A \otimes \sigma_{BA} \otimes B} A \otimes A \otimes B \otimes B \xrightarrow{\nabla_A \otimes \nabla_B} A \otimes B.$$

Again, this is also true if we merely assume B to be an algebra in the weak center of $\mathcal{C}$.

If B is a commutative algebra in the weak center of $\mathcal{C}$, then every right B-module M has a natural left B-module structure

$$B \otimes M \xrightarrow{\sigma_{BM}} M \otimes B \xrightarrow{\mu} M$$

which makes it a B-B-bimodule.

Provided that the category $\mathcal{C}$ has coequalizers, one can define the tensor product of a right A-module M and a left A-module N by a coequalizer

$$M \otimes A \otimes N \rightrightarrows M \otimes N \to M \underset{A}{\otimes} N.$$

If M is an L-A-bimodule, and N is an A-R-bimodule, then $M \otimes_A N$ is an L-R-bimodule provided that tensoring on the left with L and tensoring on the right with R preserves coequalizers. The extra condition is needed to show, for example, that $L \otimes (M \otimes_A N) \to M \otimes_A N$ is well-defined, using that $L \otimes (M \otimes_A N) \cong (L \otimes M) \otimes_A N$, which relies on $L \otimes —$ preserving the relevant coequalizer.

Some more technicalities are necessary to assure that the tensor product of three bimodules is associative. Assume given in addition an S-R-bimodule T such that $T\otimes$— and $S\otimes$— preserve coequalizers. Since colimits commute with colimits, $T\otimes_S$ — also preserves coequalizers, and we have in particular a coequalizer

$$T\underset{S}{\otimes}(M\otimes A\otimes N)\rightrightarrows T\underset{S}{\otimes}(M\otimes N)\to T\underset{S}{\otimes}(M\underset{A}{\otimes}N).$$

To get the desired isomorphism

$$T\underset{L}{\otimes}(M\underset{A}{\otimes}N)\cong(T\underset{L}{\otimes}M)\underset{A}{\otimes}N,$$

we need to compare this to the coequalizer

$$(T\underset{S}{\otimes}M)\otimes A\otimes N\rightrightarrows(T\underset{S}{\otimes}M)\otimes N\to(T\underset{S}{\otimes}M)\underset{A}{\otimes}N,$$

which can be done if we throw in the extra condition that the natural morphism $(T\otimes_S M)\otimes X\to T\otimes_S(M\otimes X)$ is an isomorphism for all $X\in\mathcal{C}$.

4.3 Cotensor product. To begin with, the cotensor product of comodules is nothing but a special case of the tensor product of modules in monoidal categories: A k-coalgebra C is an algebra in the opposite of the category of k-modules, so the cotensor product of a right C-comodule M and a left C-comodule N (two modules in the opposite category) is defined by an equalizer

$$0\to M\underset{C}{\square}N\to M\otimes N\rightrightarrows M\otimes C\otimes N.$$

We see that the cotensor product of a B-C-bicomodule M and a C-D-bicomodule N is a B-D-bicomodule provided that B and C are flat k-modules. Since flatness of C is even needed to make sense of equalizers within the category of C-comodules, it is assumed throughout this paper that all coalgebras are flat over k.

A right C-comodule V is called C-coflat if the cotensor product functor $V\square_C$ —: ${}^C\mathcal{M}\to\mathcal{M}_k$ is exact. Since $V\square_C(C\otimes W)=V\otimes W$ for any k-module W, this implies that V is k-flat. If V is k-flat, it is automatic that $V\square_C$ — is left exact. Also, $V\square_C$ — commutes with (infinite) direct sums. From this we can deduce

Lemma 4.3.1 *If V is a coflat right C-comodule, then for any k-module X and any left C-comodule W the canonical map $(V\square_C W)\otimes X\to V\square_C(W\otimes X)$ is a bijection.*

In particular, if D is another k-flat coalgebra, W is a C-D-bicomodule, and U is a left D-comodule, then cotensor product is associative:

$$(V\underset{C}{\square}W)\underset{D}{\square}U\cong V\underset{C}{\square}(W\underset{D}{\square}U).$$

Proof The second claim follows from the first and the discussion at the end of the preceding section. For the first, observe first that cotensor product commutes with direct sums, so that the canonical map is bijective with a free module $k^{(I)}$ in place of X. Now we choose a presentation $k^{(I)}\to k^{(J)}\to X\to 0$ of X. Since $V\square_C$ — commutes with this coequalizer, we see that the canonical map for X is also bijective. □

It is a well-known theorem of Lazard that a module is flat if and only if it is a direct limit of finitely generated projective modules. It is well-known, moreover, that a finitely presented module is flat if and only if it is projective. If k is a field, and C a k-coalgebra, every C-comodule is the direct limit of its finite dimensional subcomodules. Thus the following remarkable characterization of Takeuchi [47, A.2.1] may seem plausible (though of course far from obvious):

Theorem 4.3.2 *Let k be a field, C a k-coalgebra, and V a C-comodule. Then V is coflat if and only if C is injective (that is, an injective object in the category of comodules).*

We refer to [47] for the proof.

In Section 2.5 we have made use of a comodule version of Watts' theorem (which, in the original, states that every right exact functor between module categories is tensor product by a bimodule). For the sake of completeness, we prove the comodule version here:

Lemma 4.3.3 *Let C be a k-flat coalgebra, and $\mathcal{F}\colon {}^C\mathcal{M} \to \mathcal{M}_k$ an exact additive functor that commutes with arbitrary direct sums.*

Then there is an isomorphism $\mathcal{F}(M) \cong A \,\square_C\, M$, natural in $M \in {}^C\mathcal{M}$, for some comodule $A \in \mathcal{M}^C$ which is k-flat and C-coflat.

Proof We first observe that $\mathcal{F}$ is an $\mathcal{M}_k$-functor. That is to say, there is an isomorphism $\mathcal{F}(M \otimes V) \cong \mathcal{F}(M) \otimes V$, natural in $V \in \mathcal{M}_k$, which is coherent (which is to say, the two obvious composite isomorphisms $\mathcal{F}(M \otimes V \otimes W) \cong \mathcal{F}(M) \otimes V \otimes W$ coincide, and $\mathcal{F}(M \otimes k) \cong \mathcal{F}(M) \otimes k$ is trivial). We only sketch the argument: To construct $\zeta\colon \mathcal{F}(M) \otimes V \to \mathcal{F}(M \otimes V)$, choose a presentation $k^{(I)} \xrightarrow{p} k^{(J)} \to V$. The map p can be described by a column-finite matrix, which can also be used to define a morphism $\hat{p}\colon \mathcal{F}(M)^{(I)} \to \mathcal{F}(M)^{(J)}$, which has both $\mathcal{F}(M) \otimes V$ and (since $\mathcal{F}$ commutes with cokernels) $\mathcal{F}(M \otimes V)$ as its cokernel, whence we get an isomorphism ζ between them. Clearly ζ is natural in M. Naturality in V is proved along with independence of the presentation: Let $k^{(K)} \to k^{(L)} \to W$ be a presentation of another k-module W, and $f\colon V \to W$. By the Comparison Theorem for projective resolutions, f can be lifted to a pair of maps $f_1\colon k^{(J)} \to k^{(L)}$ and $f_2\colon k^{(I)} \to k^{(K)}$. Since the maps of free k-modules can be described by matrices, they give rise to a diagram

$$\begin{array}{ccc} \mathcal{F}(M)^{(I)} & \longrightarrow & \mathcal{F}(M)^{(J)} \\ \downarrow & & \downarrow \\ \mathcal{F}(M)^{(K)} & \longrightarrow & \mathcal{F}(M)^{(L)} \end{array}$$

which can be filled to the right both by $\mathcal{F}(M) \otimes f\colon \mathcal{F}(M) \otimes V \to \mathcal{F}(M) \otimes W$ and by $\mathcal{F}(M \otimes f)\colon \mathcal{F}(M \otimes V) \to \mathcal{F}(M \otimes W)$. If $W = V$ and $f = id$, this proves independence of ζ of the resolution, and for a general choice of W and f it proves naturality of ζ. Coherence is now easy to check.

The rest of our claim is now Pareigis' version [30, Thm. 4.2] of Watts' theorem [57]. For completeness, we sketch the proof: Put $A := \mathcal{F}(C)$. Then A is a C-comodule via

$$A = \mathcal{F}(C) \xrightarrow{\mathcal{F}(\Delta)} \mathcal{F}({}^{\bullet}C \otimes C) \cong \mathcal{F}(C) \otimes C = A \otimes C.$$

The functors $\mathcal{F}$ and $A \,\square_C$ — are isomorphic, since for $M \in {}^C\mathcal{M}$ we have $M \cong C \,\square_C\, M$, that is, we have an equalizer

$$M \to {}^{\bullet}C \otimes M \rightrightarrows {}^{\bullet}C \otimes C \otimes M,$$

which is preserved by $\mathcal{F}$, and hence yields an equalizer

$$\mathcal{F}(M) \to A \otimes M \rightrightarrows A \otimes C \otimes M.$$

$\square$

Finally, let us note the following associativity between tensor and cotensor product:

Lemma 4.3.4 *Let C be a coalgebra, A an algebra, M a right A-module, N a left A-module and right C-comodule satisfying the dimodule condition $(am)_{(0)} \otimes (am)_{(1)} = am_{(0)} \otimes m_{(1)}$ for all $a \in M$ and $c \in C$; finally let W be a left C-comodule. There is a canonical map*

$M \otimes_A (N \,\square_C\, V) \to (M \otimes_A N) \,\square_C\, V$, given by $m \otimes (n \otimes v) \mapsto (m \otimes n) \otimes v$.

If M is flat as A-module, or V is coflat as left C-comodule, then the canonical map is a bijection.

In fact, if M is flat, then $M \otimes_A$ — preserves the equalizer defining the cotensor product. If V is coflat, we may argue similarly using Lemma 4.3.1.

4.4 Convolution and composition. Let C be a k-coalgebra and A a k-algebra. The convolution product

$$f * g = \nabla_A(f \otimes g)\Delta_C \colon C \to a$$

defined for any two k-linear maps $f, g\colon C \to A$ is ubiquitous in the theory of coalgebras and bialgebras. It makes $\mathrm{Hom}(C, A)$ into an algebra, with the k-dual C^* as a special case. A lemma due to Koppinen (see [29, p.91] establishes a correspondence of convolution with composition (which, of course, is an even more ubiquitous operation throughout all of mathematics):

Lemma 4.4.1 *Let C be a k-coalgebra, and A a k-algebra. Then*

$$T = T_A^C\colon \mathrm{Hom}(C, A) \ni \varphi \mapsto (a \otimes c \mapsto a\varphi(c_{(1)}) \otimes c_{(2)}) \in \mathrm{End}_{A-}^{-C}(A \otimes C)$$

is an anti-isomorphism of k-algebras, with inverse given by $T^{-1}(f) = (A \otimes \varepsilon_C) f(\eta_A \otimes C)$.

In particular, $\varphi\colon C \to A$ is invertible with respect to convolution if and only if $T(\varphi)$ is bijective.

The assertions are straightforward to check. Let us point out that bijectivity of T is a special case of the following Lemma, which contains the facts that $A \otimes V$ is the free A-module over the k-module V, and $W \otimes C$ is the cofree C-comodule over the k-module W:

Lemma 4.4.2 *Let A be a k-algebra, C a k-coalgebra, and V a right C-comodule, and W a left A-module. Then we have an isomorphism*

$$\tilde{T}\colon \mathrm{Hom}(V, W) \ni \varphi \mapsto (a \otimes v \mapsto a\varphi(v_{(0)}) \otimes v_{(1)}) \in \mathrm{Hom}_{A-}^{-C}(A \otimes V, W \otimes C)$$

4.5 Descent. In this section we very briefly recall the mechanism of faithfully flat descent for extensions of noncommutative rings. This is a very special case of Beck's theorem; a reference is [2].

Definition 4.5.1 Let $\eta\colon R \subset S$ be a ring extension. A (right) descent data from S to R is a right S-module M together with an S-module homomorphism $\theta\colon M \to M \otimes_R S$ (also called a descent data on the module M) making the diagrams

$$\begin{tikzcd} M \arrow[r, "\theta"] \arrow[d, "\theta"] & M \otimes_R S \arrow[d, "\theta \otimes_R S"] \\ M \otimes_R S \arrow[r, "M \otimes_R \eta \otimes_R S"] & M \otimes_R S \otimes_R S \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\theta"] \arrow[dr, equal] & M \otimes_R S \arrow[d, "m"] \\ & M \end{tikzcd}$$

commute (where m is induced by the S-module structure of M). Descent data (M,θ) from S to R form a category $\mathcal{D}(S\downarrow R)$ in an obvious way.

If N is a right R-module, then the induced S-module $N\otimes_R S$ carries a natural descent data, namely the map $\theta\colon N\otimes_R S\ni n\otimes s\mapsto n\otimes 1\otimes s\in N\otimes_R S\otimes_R S$. This defines a functor from $\mathcal{M}_R$ to the category of descent data from S to R.

Theorem 4.5.2 (Faithfully flat descent) *Let $\eta\colon R\subset S$ be an inclusion of rings, such that S is faithfully flat as a left R-module.*

Then the canonical functor from $\mathcal{M}_R$ to the category of descent data from S to R is an equivalence of categories. The inverse equivalence maps a descent data (M,θ) to

$$M^\theta := \{m\in M|\theta(m)=m\otimes 1\}.$$

In particular, for every descent data (M,θ), the map $f\colon (M^\theta)\otimes_R S\ni m\otimes s\mapsto ms\in M$ is an isomorphism with inverse induced by θ, i.e. $f^{-1}(m)=\theta(m)\in M^\theta\otimes_R S\subset M\otimes_R S$.

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Extension Theory in Mal'tsev Varieties

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Abstract. The paper provides a brief survey of extension theory for Mal'tsev varieties based on centrality and monadic cohomology. Extension data are encoded in the form of a *seeded simplicial map*. Such a map yields an extension if and only if it is unobstructed. Second cohomology groups classify extensions, and third cohomology groups classify obstructions.

1 Introduction

Extension theory for Mal'tsev varieties was developed in [9, Chapter 6], generalising earlier treatment of special cases such as groups (cf. [4], [7]), commutative algebras (cf. [1]), loops [6], and other "categories of interest" [8]. Because of renewed attention being paid to the topic, a brief survey of the theory appears timely. With the exception of parts of Section 2, the context throughout the paper is that of a Mal'tsev variety $\mathfrak{V}$. An *extension* of a $\mathfrak{V}$-algebra R is considered as a $\mathfrak{V}$-algebra T equipped with a congruence α such that R is isomorphic to the quotient T^α of T by the congruence α.

The essential properties of centrality in Mal'tsev varieties are recalled in Section 2. Section 3 describes the "seeded simplicial maps" which provide a concise encoding of the raw material required for constructing an extension (analogous to the "abstract kernels" of [7]). Section 4 gives a brief, algebraic description of the rudiments of monadic cohomology, culminating in the Definition 4.2 of the obstruction of a seeded simplicial map as a cohomology class. Theorem 5.2 then shows that a seeded simplicial map yields an extension if and only if it is unobstructed. The final section discusses the classification of extensions by second cohomology groups, and of obstructions by third cohomology groups. Against this background, the pessimism expressed in [5] ("To classify the extensions ... is too big a project to admit of a reasonable answer") appears unwarranted.

For concepts and conventions not otherwise explained in the paper, readers are referred to [10]. In particular, note the general use of postfix notation, so that

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composites are read in natural order from left to right. For a binary relation ρ on a set X, and an element x of X, write $x^\rho = \{y \mid (x,y) \in \rho\}$.

2 Centrality in Mal'tsev varieties

Recall that a variety $\mathfrak{V}$ of universal algebras is a *Mal'tsev variety* if there is a derived ternary *parallelogram* operation P such that the identities

$$(x,x,y)P = y = (y,x,x)P$$

are satisfied. Equivalently, the relation product of two congruences is commutative, and thus agrees with their join. Moreover, reflexive subalgebras of direct squares are congruences [9, Proposition 143].

Consider two congruences γ and β on a general algebra, not necessarily in a Mal'tsev variety. Then γ is said *to centralise* β if there is a congruence $(\gamma|\beta)$ on β, called a *centreing congruence*, such that the following conditions are satisfied [9, Definition 211]:

(C0): $(x,y)\ (\gamma|\beta)\ (x',y') \Rightarrow x\ \gamma\ x'$;
(C1): $\forall (x,y) \in \beta,\ \pi^0 : (x,y)^{(\gamma|\beta)} \to x^\gamma; (x',y') \mapsto x'$ bijects;
(C2): **(RR):** $\forall (x,y) \in \gamma,\ (x,x)\ (\gamma|\beta)\ (y,y)$;
(RS): $(x,y)\ (\gamma|\beta)\ (x',y') \Rightarrow (y,x)\ (\gamma|\beta)\ (y',x')$;
(RT): $(x,y)\ (\gamma|\beta)\ (x',y')$ and $(y,z)\ (\gamma|\beta)\ (y',z') \Rightarrow (x,z)\ (\gamma|\beta)\ (x',z')$.

Example 2.1 Suppose that A is a (not necessarily associative) ring. For congruences β and γ on A, consider the ideals $B = 0^\beta$ and $C = 0^\gamma$. Then γ centralises β iff $BC + CB = \{0\}$ [9, pp.27-8].

In a Mal'tsev variety $\mathfrak{V}$, centreing congruences are unique [9, Proposition 221]. Moreover, for each congruence α on an algebra A in $\mathfrak{V}$, there is a unique largest congruence $\eta(\alpha)$, called the *centraliser* of α, which centralises α [9, 228]. Note that $\alpha \circ \eta(\alpha)$ centralises $\alpha \cap \eta(\alpha)$ [9, Corollary 227].

If R is a member of a variety $\mathfrak{V}$ of universal algebras, then following Beck [2] the *category of R-modules* is the category of abelian groups in the slice category $\mathfrak{V}/R$. For example, if A is an algebra in a Mal'tsev variety $\mathfrak{V}$ having nested congruences $\beta \leq \gamma$ such that γ centralises β, then $\beta^{(\gamma|\beta)} \to A^\gamma$ is an A^γ-module. Indeed, given $a_1\,\beta\,a_0\,\gamma\,b_0\,\beta\,b_2$ in A, one has

$$(a_0,a_1)^{(\gamma|\beta)} + (b_0,b_1)^{(\gamma|\beta)} = (a_0,a_3)^{(\gamma|\beta)}$$

for a_2 given by $(a_0,a_2)(\gamma|\beta)(b_0,b_2)$ using (C1) and then for a_3 given similarly by $(a_0,a_1)(\gamma|\beta)(a_2,a_3)$.

3 Seeded simplicial maps

The data used for the construction of extensions are most succinctly expressed in terms of simplicial maps. These are described using the direct algebraic approach of [9], to which the reader is referred for fuller detail. Compare also [3].

Let ε_n^i be the operation which deletes the $(i+1)$-th letter from a non-empty word of length n. Let δ_n^i be the operation which repeats the $(i+1)$-th letter in a non-empty word of length n. These operations, for all positive integers n and natural numbers $i < n$, generate (the morphisms of) a category Δ called the *simplicial category*. A *simplicial object* B^* in $\mathfrak{V}$ is (the image of) a functor from Δ to $\mathfrak{V}$. A *simplicial map* is (the set of components of) a natural transformation between such functors. Generically, the morphisms of a simplicial object B^* are denoted by their preimages in Δ, namely as $\varepsilon_n^i : B^n \to B^{n-1}$ and $\delta_n^i : B^n \to B^{n+1}$. (In other words, one treats simplicial objects as heterogeneous algebras in $\mathfrak{V}$.)

Given $(\theta^0, \dots, \theta^{n-1}) \in \mathfrak{V}(X,Y)^n$, the *simplicial kernel* $\ker(\theta^0, \dots, \theta^{n-1})$ is the largest subalgebra K of the power X^{n+1} for which the θ^i and the restrictions of the projections from the power respectively model the identities satisfied by the simplicial ε_n^i and ε_{n+1}^i. For example, the simplicial kernel of a single $\mathfrak{V}$-morphism $\theta^0 : X \to Y$ is $K = \{(x_0, x_1) \in X^2 \mid x_0\theta^0 = x_1\theta^0\}$, namely the usual kernel of θ^0, which models the single simplicial identity $\varepsilon_2^0\varepsilon_1^0 = \varepsilon_2^1\varepsilon_1^0$ by $\pi^0\theta^0 = \pi^1\theta^0$ for $\pi^i : K \to X; (x_0, x_1) \mapsto x_i$. Similarly, one has

$$\ker(\theta^0, \theta^1) = \{(x_0, x_1, x_2) \in X^3 \mid x_0\theta^0 = x_1\theta^0, x_1\theta^1 = x_2\theta^1, x_2\theta^0 = x_0\theta^1\},$$

on which for instance $\pi^2\theta^0 = \pi^0\theta^1$ models $\varepsilon_3^2\varepsilon_2^0 = \varepsilon_3^0\varepsilon_2^1$ by virtue of the condition $x_2\theta^0 = x_0\theta^1$.

For each positive integer n, removing all operations from Δ that involve words of length greater than n leaves the *simplicial category* Δ_n *truncated at* n. Functors from Δ_n are called *simplicial objects truncated at dimension* n. Truncated simplicial objects may be extended to full simplicial objects by successively tacking on simplicial kernels. In such cases one may omit the epithet "truncated," speaking merely of simplicial objects, even when one has only specified the lower-dimensional part.

Definition 3.1 A simplicial object B^* is said to be *seeded* if:

1. It is truncated at dimension 2;
2. $(\varepsilon_2^0, \varepsilon_2^1) : B^2 \to \ker(\varepsilon_1^0)$ surjects;
3. $\varepsilon_1^0 : B^1 \to B^0$ surjects;
4. $\ker(\varepsilon_2^0 : B^2 \to B^1) = \eta(\ker(\varepsilon_2^1 : B^2 \to B^1))$.

Lemma 3.2 *In a seeded simplicial object B, let C be the equalizer of the pair $(\varepsilon_2^0, \varepsilon_2^1)$. Define V on C by*

$$c\ V\ c' \Leftrightarrow ((c\varepsilon_2^0\delta_1^0, c), (c'\varepsilon_2^0\delta_1^0, c')) \in (\ker\varepsilon_2^0 \circ \ker\varepsilon_2^1 | \ker\varepsilon_2^0 \cap \ker\varepsilon_2^1).$$

Then

$$C^V \to B^0; c^V \mapsto c\varepsilon_2^0\varepsilon_1^0 \tag{3.1}$$

is a module over B^0, isomorphic to $(\ker\varepsilon_2^0 \cap \ker\varepsilon_2^1)^{(\ker\varepsilon_2^0 \circ \ker\varepsilon_2^1 | \ker\varepsilon_2^0 \cap \ker\varepsilon_2^1)}$.

The module (3.1) of Lemma 3.2 is called the module *grown* by the seeded simplicial object B^*. If α is a congruence on a $\mathfrak{V}$-algebra T, then

$$\alpha^{(\eta(\alpha)|\alpha)} \rightrightarrows T^{\eta(\alpha)} \to T^{\alpha\circ\eta(\alpha)} \tag{3.2}$$

is a seeded simplicial object with $\varepsilon_2^i : (t_0, t_1)^{(\eta(\alpha)|\alpha)} \mapsto t_i^{\eta(\alpha)}$, growing the module

$$(\alpha \cap \eta(\alpha))^{(\alpha\circ\eta(\alpha)|\alpha\cap\eta(\alpha))} \to T^{\alpha\circ\eta(\alpha)}; (t_0, t_1)^{(\alpha\circ\eta(\alpha)|\alpha\cap\eta(\alpha))} \mapsto t_0^{\alpha\circ\eta(\alpha)}.$$

The seeded simplicial object (3.2) is said to be *planted* by the congruence α on the algebra T.

Definition 3.3 A simplicial map $p^* : A^* \to B^*$ is said to be *seeded* if the codomain object B^* is seeded in the sense of Definition 3.1, and if $p^0 : A^0 \to B^0$ surjects.

4 Obstructions

Along with the simplicial theory outlined in Section 3, the second tool used for studying extensions of Mal'tsev algebras is monadic cohomology. Once again, full details may be found in [3] and [9]; the summary given here follows the direct approach of the latter reference.

For each $\mathfrak{V}$-algebra A, let AG denote the free $\mathfrak{V}$-algebra over the generating set $\{\{a\} \mid a \in A\}$. Given a $\mathfrak{V}$-algebra R, let $\varepsilon_n^j : RG^n \to RG^{n-1}$ denote the uniquely defined $\mathfrak{V}$-morphism deleting the j-th layer of braces, where $j = 0$ corresponds to the inside layer and $j = n-1$ to the outside. Let $\delta_n^j : RG^n \to RG^{n+1}$ insert the j-th layer of braces. One obtains a simplicial object RG^*, known as the *free resolution* of A. Each RG^n projects to R by a composition

$$\varepsilon_n^0 \dots \varepsilon_1^0 : RG^n \to R. \tag{4.1}$$

An R-module $E \to R$ becomes an RG^n-module by pullback along (4.1). Write $\mathrm{Der}(RG^n, E)$ for the abelian group $\mathfrak{V}/R(RG^n \to R, E \to R)$ of *derivations*. Define *coboundary homomorphisms*

$$d_n : \mathrm{Der}(RG^n, E) \to \mathrm{Der}(RG^{n+1}, E); f \mapsto \sum_{i=0}^{n} (-)^i \varepsilon_{n+1}^i f$$

for each natural number n. For each positive integer n, define

$$\mathrm{H}^n(R, E) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n-1}), \tag{4.2}$$

the so-called n-th *monadic cohomology group of* R *with coefficients in* E. [Note that [3] and [9] use $\mathrm{H}^{n-1}(R, E)$ for (4.2).] The cosets forming (4.2) are known as *cohomology classes.* Elements of $\mathrm{Ker}(d_n)$ are known as *cocycles*, and elements of $\mathrm{Im}(d_{n-1})$ are *coboundaries.*

Lemma 4.1 *Let $p^* : RG^* \to B^*$ be a seeded simplicial map whose codomain grows module M. Pull M from B^0 back to R along p^0. Then*

$$p^3(\varepsilon_3^0, \varepsilon_3^1, \varepsilon_3^2)P^V : RG^3 \to M \tag{4.3}$$

is a cocycle in $\mathrm{Der}(B^3, M)$.

Definition 4.2 The cohomology class of (4.3) is called the *obstruction* of the seeded simplicial map p^*. The simplicial map is said to be *unobstructed* if this class is zero.

Lemma 4.3 *The obstruction of a seeded simplicial map $p^* : RG^* \to B^*$ is uniquely determined by its bottom component $p^0 : R \to B^0$.*

The diagram-chasing proofs of Lemmas 4.1 and 4.3 are given in [9, pp.124–7].

5 Constructing extensions

Definition 5.1 A seeded simplicial map $p^* : RG^* \to B^*$ is said to be *realised* by an algebra T if there is a congruence α on T planting B^* such that p^0 is the natural projection $T^\alpha \to T^{\alpha \circ \eta(\alpha)}$.

Theorem 5.2 *A seeded simplicial map $p^* : RG^* \to B^*$ is unobstructed iff it is realised by an algebra T.*

Proof (Sketch.) "**If:**" Consider the diagram

$$\begin{array}{ccccccc} \Rrightarrow & RG^2 & \Rrightarrow & RG & \to & R & \\ & \downarrow \sigma^2 & & \downarrow \sigma^1 & & \downarrow \sigma^0 & \\ \Rrightarrow & \alpha & \Rrightarrow & T & \to & R & \\ & \downarrow & & \downarrow & & \downarrow p^0 & \\ \Rrightarrow & \alpha^{(\eta(\alpha)|\alpha)} & \Rrightarrow & T^{\eta(\alpha)} & \to & T^{\alpha\circ\eta(\alpha)} & \end{array} \tag{5.1}$$

in which σ^0 is the identity on $R = T^\alpha$, σ^1 is given by the freeness of RG, and σ^2 exists since $\alpha = \ker(T \to R)$. Take p^2, p^1, p^0 to be the composites down the respective columns of (5.1), the second factors of these composites all being natural projections. Writing $\pi^i : \alpha \to T; (t_0, t_1) \mapsto t_i$, one has

$$\begin{aligned} (\varepsilon_3^0\sigma^2, \varepsilon_3^1\sigma^2, \varepsilon_3^2\sigma^2)P\pi^0 &= (\varepsilon_3^0\varepsilon_2^0, \varepsilon_3^1\varepsilon_2^0, \varepsilon_3^2\varepsilon_2^0)P\sigma^1 = \varepsilon_3^2\varepsilon_2^0\sigma^1 \\ &= \varepsilon_3^0\varepsilon_2^1\sigma^1 = (\varepsilon_3^0\varepsilon_2^1, \varepsilon_3^1\varepsilon_2^1, \varepsilon_3^2\varepsilon_2^1)P\sigma^1 = (\varepsilon_2^0\sigma^2, \varepsilon_2^1\sigma^2, \varepsilon_2^2\sigma^2)P\pi^1, \end{aligned}$$

so the obstruction of p^* is the zero element $(\varepsilon_3^0p^2, \varepsilon_3^1p^2, \varepsilon_3^2p^2)P^V$ of the group $\mathrm{Der}(RG^3, (\alpha \cap \eta(\alpha))^{(\alpha\circ\eta(\alpha)|\alpha\cap\eta(\alpha))})$, as required.

"**Only if:**" If p^* is unobstructed, then as shown in [9, p.129], one may assume without loss of generality that (4.3) itself is zero, and not just in the zero cohomology class. Let Q be a pullback in

$$\begin{array}{ccc} Q & \to & RG \\ \downarrow & & \downarrow p_1 \\ B^2 & \underset{\varepsilon_2^1}{\to} & B^1 \end{array}$$

realised, say, by $Q = \{(b, w) \in B^2 \times RG \mid wp^1 = b\varepsilon_2^1\}$. Define a congruence W on Q by $(b, w)\ W\ (b', w')$ iff $w\varepsilon_1^0 = w'\varepsilon_1^0$ and

$$(b, b')\ (\ker \varepsilon_2^1 | \ker \varepsilon_2^0)\ (\{w\}p^2, \{w'\}p^2).$$

Set $T = Q^W$, and take α on T to be the kernel of $T \to R; (b, w)^W \mapsto w\varepsilon_1^0$. For the details of the verification that T realises p^*, with α planting B^*, see [9, pp.129–132]. In particular, note that $\eta(\alpha)$ is the kernel of $T \to B^1; (b, w)^W \mapsto b\varepsilon_2^0$.

□

6 Classifying extensions and obstructions

Let $p^* : RG^* \to B^*$ be a seeded simplicial map whose codomain grows a module $M \to B^0$. Pull M back along $p^0 : R \to B^0$ to an R-module. An extension $\alpha \rightrightarrows T \to R$ is said to be *singular for* p^* if its kernel α is self-centralising, with an R-module isomorphism $\alpha^{(\alpha|\alpha)} \to M$. (Note that the central extensions of [5] form a special case of the singular extensions, in which α centralises all of $T \times T$.) Let p^*S be the set of $\mathfrak{V}/R$-isomorphism classes of extensions that are singular for p^*. This set becomes an abelian group, with the class of the split extension $M \to R$ as zero. The addition operation on p^*S is known as the *Baer sum*. To obtain a representative of the Baer sum of the isomorphism classes of two extensions

$\alpha_i \rightrightarrows T_i \to R$, with module isomorphism $\theta : \alpha_1^{(\alpha_1|\alpha_1)} \to \alpha_2^{(\alpha_2|\alpha_2)}$, take the quotient of the pullback $T_1 \times_R T_2$ by the congruence

$$\{((t_1,t_2),(t'_1,t'_2)) \mid (t_i,t'_i) \in \alpha_i,\ (t_1,t'_1)^{(\alpha_1|\alpha_1)}\theta = (t_2,t'_2)^{(\alpha_2|\alpha_2)}\}.$$

Singular extensions are then classified as follows [9, Theorem 632], cf. [2], [3].

Theorem 6.1 *The groups p^*S and $\mathrm{H}^2(R,M)$ are isomorphic.*

Now assume additionally that the seeded simplicial map $p^* : RG^* \to B^*$ is unobstructed. An extension $\alpha \rightrightarrows T \to R$ is said to be *non-singular for p^** if T realises p^*. Let p^*N denote the set of $\mathfrak{V}/R$-isomorphism classes of extensions that are non-singular for p^*. By Theorem 5.2, p^*N is non-empty. Non-singular extensions are then classified as follows [9, Theorem 634].

Theorem 6.2 *The abelian group p^*S acts regularly on p^*N, so the sets p^*N and $\mathrm{H}^2(R,M)$ are isomorphic.*

Let $\beta \rightrightarrows S \to R$ be singular for p^*, and let $\alpha \rightrightarrows T \to R$ be non-singular for p^*. To obtain a representative for the image of the class of α under the action of the class of β, assuming an R-module isomorphism $\theta : (\alpha \cap \eta(\alpha))^{(\alpha\circ\eta(\alpha)|\alpha\cap\eta(\alpha))} \to \beta^{(\beta|\beta)}$, take the quotient of the pullback $T \times_R S$ by the congruence

$$\{((t,s),(t',s')) \mid (t,t') \in \alpha\cap\eta(\alpha),\ (s',s) \in \beta,\ (t,t')^{(\alpha|\alpha\cap\eta(\alpha))}\theta = (s',s)^{(\beta|\beta)}\}.$$

The final result [9, Theorem 641] shows how obstructions may be classified by elements of the third monadic cohomology groups. Note that for non-trivial R, the hypothesis on R is always satisfied in varieties $\mathfrak{V}$, such as the variety of all groups, where free algebras have little centrality. On the other hand, it is not satisfied, for example, by the three-element group in the variety of commutative Moufang loops.

Theorem 6.3 *Let R be a $\mathfrak{V}$-algebra for which $\eta(\ker(\varepsilon_1^0 : RG \to R)) = \widehat{RG}$. Let $M \to R$ be an R-module, and let $\xi \in \mathrm{H}^3(R,M)$. Then ξ is the obstruction to a seeded simplicial map $p^* : RG^* \to B^*$ whose codomain grows a module that pulls back to $M \to R$ along p^0.*

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On Projective Generators Relative to Coreflective Classes

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Abstract. Projective $\mathcal{E}$-generators, for $\mathcal{E}$ a coreflective class of morphisms, are studied. Under mild conditions, it is shown that, for cocomplete categories $\mathbf{A}$ with a projective $\mathcal{E}$-generator P, the colimit-closure of P is the smallest $\mathcal{E}$-coreflective subcategory of $\mathbf{A}$, and, furthermore, it is premonadic over $\mathbf{Set}$ via the functor $\hom(P,-)$. A variety of examples is given.

Introduction

Several generalizations of the concept of factorization system for morphisms have appeared in the literature; here we work with one of them, the notion of coreflective class (see, e.g., [7], [12], [17], [18]). Pushout-stable coreflective classes of morphisms (in a category $\mathbf{A}$ with pushouts) are just those which, regarded as subcategories[1] of $\mathrm{Mor}(\mathbf{A})$, are coreflective. A significant role is played by the stabilization of a coreflective class $\mathcal{E}$, denoted by $\mathrm{St}(\mathcal{E})$, and defined as being the subclass of $\mathcal{E}$ which consists of all morphisms whose pullbacks along any morphism belong to $\mathcal{E}$ (see [7]). By a projective $\mathcal{E}$-generator it is meant an $\mathcal{E}$-generator which is projective with respect to $\mathrm{St}(\mathcal{E})$. Many examples of categories with a projective $\mathcal{E}$-generator P are given. In all of them projectivity does not hold with respect to the whole $\mathcal{E}$ unless $\mathcal{E}$ coincides with its stabilization. In fact, $\mathrm{St}(\mathcal{E})$ is shown to be precisely the class of all those morphisms to which P is projective.

Subcategories which are "colimit-generated" by a projective $\mathcal{E}$-generator of their cocomplete supercategories are shown to have special properties. Namely, let $\mathbf{A}$ be an $\mathcal{E}$-cocomplete category with pullbacks and $\mathcal{E}$ closed under composition with split epimorphisms. We show that if P is a projective $\mathcal{E}$-generator of $\mathbf{A}$, the colimit closure of P, denoted by $\mathbb{C}(P)$, is the smallest $\mathcal{E}$-coreflective subcategory of $\mathbf{A}$; furthermore, if the stabilization of $\mathcal{E}$ is closed under coequalizers of kernel pairs,

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[1]Throughout, by subcategory we mean a full and isomorphism-closed subcategory.

then P is a regular generator of $\mathbb{C}(P)$, and thus $\mathbb{C}(P)$ is equivalent to a reflective subcategory of the category of Eilenberg-Moore algebras of the monad induced by the functor $\hom(P,-)$, that is, $\mathbb{C}(P)$ is premonadic over **Set**. Analogous results are obtained when (under the cocompleteness of $\mathbf{A}$) the coreflectiveness of $\mathbb{C}(P)$ replaces the $\mathcal{E}$-cocompleteness of $\mathbf{A}$. These properties are illustrated with many examples.

1 Projective $\mathcal{E}$-generators

Definition 1.1 A class $\mathcal{E}$ of epimorphisms of a category $\mathbf{A}$ (closed under composition with isomorphisms) is said to be a *coreflective class* whenever, for each $B \in \mathbf{A}$, the embedding $\mathcal{E}(B) \to B \downarrow \mathbf{A}$, where $\mathcal{E}(B)$ denotes de subcategory of $B \downarrow \mathbf{A}$ whose objects are $\mathcal{E}$-morphisms, is a left adjoint; that is, each $\mathbf{A}$-morphism f has a factorization $m \cdot e$ with $e \in \mathcal{E}$, and such that if $m' \cdot e'$ is another such a factorization of f then there is a (unique) morphism t with $t \cdot e' = e$ and $m \cdot t = m'$. We say that $m \cdot e$ is the *$\mathcal{E}$-factorization of f*. (cf. [12], [18] and [7].)

Remark 1.2 If $\mathbf{A}$ has pushouts and $\mathcal{E}$ is a coreflective class of $\mathcal{A}$, the following facts are easy consequences of the above definition:

1. $\mathcal{E}$ is pushout-stable if and only if the "local coreflections" from $B \downarrow \mathbf{A}$ to $\mathcal{E}(B)$ determine a "global coreflection" from $\mathrm{Mor}(\mathbf{A})$ to $\mathcal{E}$, where $\mathcal{E}$ is regarded as a subcategory of $\mathrm{Mor}(\mathbf{A})$. (The fact that $\mathbf{A}$ has a pushout-stable coreflective class $\mathcal{E}$ means in the terminology of [12] that $\mathbf{A}$ has a locally orthogonal $\mathcal{E}$-factorization.)
2. $\mathcal{E}$ determines a factorization system for morphisms if and only if it is pushout-stable and closed under composition.

We add another property which will play a role throughout:

Lemma 1.3 *If $\mathcal{E}$ is a pushout-stable coreflective class in a category with pushouts, then the following conditions are equivalent:*

(i) Any split epimorphism m which is part of an $\mathcal{E}$-factorization $m \cdot e$ is an isomorphism.

(ii) $\mathcal{E}$ is closed under composition with split epimorphisms from the left.

(iii) $\mathcal{E}$ is closed under composition with split epimorphisms (from the left and from the right).

Proof (i) $\Rightarrow$ (iii): Let $r \cdot s$ be defined with $s \in \mathcal{E}$ and r a split epi, and let $m \cdot e$ be the $\mathcal{E}$-factorization of $r \cdot s$. Then there is t such that $m \cdot t = r$ and, since r is a split epi, so is m, thus m is an isomorphism and $r \cdot s$ belongs to $\mathcal{E}$.

Take $r \cdot s$ with $r \in \mathcal{E}$ and s a split epi, let u be such that $s \cdot u = 1$, and let $m \cdot e$ be the $\mathcal{E}$-factorization of $r \cdot s$. From 1.2.1 and the equality $1 \cdot r = m \cdot e \cdot u$, we get a morphism t such that $mt = 1$ and $tr = eu$. Condition (i) ensures that m is an iso, and $r \cdot s \in \mathcal{E}$.

(ii) $\Rightarrow$ (i): Let $m \cdot e$ be the $\mathcal{E}$-factorization of some f with m a split epi. Then $f \in \mathcal{E}$ and so f has an $\mathcal{E}$-factorization of the form $1 \cdot f$. Therefore, m is iso. □

Remark 1.4 The property of "$\mathcal{E}$ being closed under composition with split epimorphisms" will be used several times along the paper. If $\mathcal{E}$ is part of a proper factorization system, then the property holds. But the converse is not true as it is shown by the following example (G. Janelidze, private communication): Let $\mathbf{A}$ be the ordered set $0 \to 1 \to 2$ regarded as a category. Let $\mathcal{E}$ be the set of all maps in $\mathbf{A}$ except $0 \to 2$. Then $\mathcal{E}$ is a pushout-stable coreflective class closed under composition with split epimorphisms but not closed under composition. Another example of the fact that the closedness of a pushout-stable coreflective class $\mathcal{E}$ under composition with split epimorphisms does not imply that $\mathcal{E}$ is part of a factorization system for morphisms is given in [5], for $\mathcal{E} = \{$regular epimorphisms$\}$.

Definition 1.5 (cf. [6]) An object P is an *$\mathcal{E}$-generator* of the category $\mathbf{A}$ with copowers of P if, for each $A \in \mathbf{A}$, the canonical morphism ε_A from the coproduct $\coprod_{\hom(P,A)} P$ to A belongs to $\mathcal{E}$.

Assumption 1.6 From now on we assume that the category $\mathbf{A}$ has pullbacks and pushouts, and $\mathcal{E}$ is a pushout-stable coreflective class contained in Epi($\mathbf{A}$) which is closed under composition with split epimorphisms.

A morphism is said to be *stably in* $\mathcal{E}$ if its pullback along any morphism belongs to $\mathcal{E}$. The *stabilization of* $\mathcal{E}$ is the class of all morphisms that are stably in $\mathcal{E}$; it is denoted by St($\mathcal{E}$) and it is clearly contained in $\mathcal{E}$ (see [7]).

Lemma 1.7 (see [5]) *$\mathcal{E}$ and St($\mathcal{E}$) are strongly right-cancellable.*

Proof Given $r \cdot s \in \mathcal{E}$, we want to show that $r \in \mathcal{E}$. Let $m \cdot e$ be the $\mathcal{E}$-factorization of r. Then the equality $1 \cdot r \cdot s = m \cdot e \cdot s$ determines, by 1.2.1, the existence of a morphism t such that $mt = 1$ and $t \cdot (r \cdot s) = e \cdot s$. Since m is a split epimorphism and $m \cdot e$ is an $\mathcal{E}$-factorization, then, by 1.3, m is an isomorphism; so $r \in \mathcal{E}$. The strong right-cancellability of St($\mathcal{E}$) follows easily from the strong right-cancellability of $\mathcal{E}$. □

Definition 1.8 An object P is said to be a *projective $\mathcal{E}$-generator* if it is an $\mathcal{E}$-generator which is St($\mathcal{E}$)-projective, that is, for each $f \in$ St($\mathcal{E}$), the function hom(P, f) is surjective.

Notation 1.9 For A an $\mathbf{A}$-object, Proj(A) denotes the class of all $\mathbf{A}$-morphisms f such that A is f-projective. It is easily seen that Proj(A) is pullback-stable.

Assumption 1.10 In the following, besides the assumptions stated in 1.6, we also assume that $\mathbf{A}$ is cocomplete.

Proposition 1.11 *If P is a projective $\mathcal{E}$-generator, then St($\mathcal{E}$) = Proj(P).*

Proof By the assumption on P, the inclusion St($\mathcal{E}$) $\subseteq$ Proj(P) is trivial. It remains to show that if P is $(f : X \to Y)$-projective then $f \in$ St($\mathcal{E}$), i.e., any pullback of f along any morphism belongs to $\mathcal{E}$. Let $(\bar{f} : W \to Z, \bar{g} : W \to X)$ be the pullback of (f, g). Since P is f-projective, any coproduct of P is also f-projective; thus, from the pullback-stability of any class Proj(A), $\bar{f} \in$ Proj($\coprod_{\hom(P,Z)} P$). Let s be a morphism fulfilling $\bar{f} \cdot s = \varepsilon_Z$ and let $m \cdot e$ be the $\mathcal{E}$-factorization of $\bar{f}$. We get the equality $1_Z \cdot \varepsilon_Z = m \cdot e \cdot s$, which, since $\varepsilon_Z \in \mathcal{E}$, implies the existence of some t such that $t \cdot \varepsilon_Z = e \cdot s$ and $m \cdot t = 1_Z$. Then, from 1.3, and in view of 1.6, m is an iso and $\bar{f} \in \mathcal{E}$. □

Corollary 1.12 *If P is a projective $\mathcal{E}$-generator, then it is a projective St($\mathcal{E}$)-generator.*

Several examples of categories with a projective $\mathcal{E}$-generator are described in 2.5 below.

One question arises: When is the stabilization of a corefletive class $\mathcal{E}$ of the form Proj(P), or, at least, when is it of the form Proj$(\mathbf{B})$ for some subcategory $\mathbf{B}$ of $\mathbf{A}$? The next proposition gives a partial answer.

We are going to make use of the following lemma.

Lemma 1.13 *If* $\mathbf{B}$ *is an* $\mathcal{E}$*-coreflective subcategory of* $\mathbf{A}$*, then it is (St(*$\mathcal{E}$*)* $\cap$ *Mono(*$\mathbf{A}$*))-coreflective.*

Proof Let $s_X : S(X) \to X$ be a coreflection of X in $\mathbf{B}$ (with S the coreflector functor). By hypothesis, s_X lies in $\mathcal{E}$; in order to show that it belongs to St$(\mathcal{E})$, let $(\bar{s} : W \to Y, \bar{g} : W \to S(X))$ be the pullback of (s_X, g), for some morphism g. To conclude that $\bar{s} \in \mathcal{E}$, let $m \cdot e$ be the $\mathcal{E}$-factorization of $\bar{s}$. Since $s_X \cdot S(g) = g \cdot s_Y$, there is a unique morphism v such that $\bar{s} \cdot v = s_Y$ and $\bar{g} \cdot v = S(g)$. Then we have $1_Y \cdot s_Y = m \cdot e \cdot v$; since $m \cdot e$ is an $\mathcal{E}$-factorization and $s_Y \in \mathcal{E}$, there is a morphism t such $m \cdot t = 1_Y$, and, taking into account 1.3 and 1.6, we get that $\bar{s} \in \mathcal{E}$. It remains to show that s_X is a monomorphism: Let $a, b : Y \to S(X)$ be such that $s_X \cdot a = s_X \cdot b$; then the equality $s_X \cdot a \cdot s_Y = s_X \cdot b \cdot s_Y$ implies that $a \cdot s_Y = b \cdot s_Y$, and thus, since $\mathcal{E} \subseteq \text{Epi}(\mathbf{A})$, $a = b$. □

Let us recall that, if $\mathcal{F}$ is a class of morphisms of a category $\mathbf{A}$ containing all isomorphisms and closed under composition with isomorphisms, an $\mathcal{F}$-morphism $f : B \to A$ is said to be *$\mathcal{F}$-coessential* whenever any composition $f \cdot g$ belongs to $\mathcal{F}$ only if $g \in \mathcal{F}$. We say that the category $\mathbf{A}$ has *enough $\mathcal{F}$-projectives* if, for each $A \in \mathbf{A}$, there is some $\mathcal{F}$-morphism $f : B \to A$ with B $\mathcal{F}$-projective; if, in addition, f can be chosen to be $\mathcal{F}$-coessential, we say that $\mathbf{A}$ has *$\mathcal{F}$-projective hulls.*

Proposition 1.14 *1. If* $\mathbf{A}$ *has enough St(*$\mathcal{E}$*)-projectives, then St(*$\mathcal{E}$*) = Proj(*$\mathbf{B}$*) for some subcategory* $\mathbf{B}$ *of* $\mathbf{A}$*.*

2. If St($\mathcal{E}$*) = Proj(*$\mathbf{B}$*) for some* $\mathcal{E}$*-coreflective subcategory* $\mathbf{B}$ *of* $\mathbf{A}$*, then* $\mathbf{B}$ *has St(*$\mathcal{E}$*)-projective hulls.*

Proof 1. Let $\mathbf{B}$ consist of all objects of $\mathbf{A}$ which are St$(\mathcal{E})$-projective; clearly St$(\mathcal{E}) \subseteq$ Proj$(\mathbf{B})$. In order to show the converse inclusion, let $f : A \to B$ belong to Proj$(\mathbf{B})$, and let $(\bar{f} : D \to C, \bar{g} : D \to A)$ be the pullback of f and g for some $g : C \to B$. Since $\mathbf{A}$ has enough St$(\mathcal{E})$-projectives, there is some St$(\mathcal{E})$-morphism $q : E \to C$ with $E \in \mathbf{B}$. Now, using with the morphism q the same technique used in the proof of 1.11 with the morphism ε_Z, we get that $\bar{f}$ belongs to $\mathcal{E}$.

2. By Lemma 1.13, each coreflection $s_A : S(A) \to A$ into $\mathbf{B}$ belongs to St$(\mathcal{E})$. It remains to show that s_A is a St$(\mathcal{E})$-coessential morphism. Let g be such that $s_A \cdot g$ belongs to St$(\mathcal{E})$, and let $\bar{g}$ be the pullback of g along any h. By 1.13, s_A is a monomorphism, then $\bar{g}$ is also the pullback of $s_A \cdot g$ along $s_A \cdot h$. Thus $\bar{g} \in \mathcal{E}$, and so g belongs to St$(\mathcal{E})$. □

2 The colimit-closure of a projective $\mathcal{E}$-generator

In this section we study properties of the colimit-closure of a projective $\mathcal{E}$-generator.

Remark 2.1 Let us recall that, for $\mathcal{E}$ a class containing all isomorphisms and closed under composition with isomorphisms, $\mathbf{A}$ is said to be $\mathcal{E}$-cocomplete if every pushout of any $\mathcal{E}$-morphism exists and belongs to $\mathcal{E}$ and any family of morphisms of $\mathcal{E}$ has a cointersection belonging to $\mathcal{E}$. The $\mathcal{E}$-cocompletness of $\mathbf{A}$ implies that $\mathcal{E} \subseteq$ Epi$(\mathbf{A})$ and that $\mathcal{E}$ is a coreflective class (see [17]). On the other hand, it is easily seen that if $\mathbf{A}$ is a cocomplete and $\mathcal{E}$-cowellpowered category with $\mathcal{E}$ a pushout-stable coreflective class then $\mathbf{A}$ is $\mathcal{E}$-cocomplete.

Notation 2.2 $\mathbb{C}(P)$ denotes the colimit-closure of P in $\mathbf{A}$, that is, the smallest subcategory of $\mathbf{A}$ containing P and closed under all colimits in $\mathbf{A}$.

Theorem 2.3 *Let $\mathbf{A}$ be $\mathcal{E}$-cocomplete and let P be a projective $\mathcal{E}$-generator of $\mathbf{A}$. Then $\mathbb{C}(P)$ consists of all $\mathbf{A}$-objects A such that the function $hom(f, A)$ is bijective for each $f \in St(\mathcal{E}) \cap Mono(\mathbf{A})$. Furthermore, $\mathbb{C}(P)$ is the smallest $\mathcal{E}$-coreflective subcategory of $\mathbf{A}$.*

Proof In order to prove that $\mathbb{C}(P)$ is coreflective in $\mathbf{A}$, it suffices to show the solution set condition for every $A \in \mathbf{A}$. Let $A \in \mathbf{A}$ and consider the following family, indexed by I:

$$\mathcal{S}_A = \{e_i : \coprod_{\hom(P,A)} P \to S_i \,|\, e_i \in \mathcal{E},\, S_i \in \mathbb{C}(P) \text{ and } t_i \cdot e_i = \varepsilon_A \text{ for some } t_i\}.$$

Let $(e : \coprod_{\hom(P,A)} P \to S;\ d_i : S_i \to S)$ be the cointersection of the family $\mathcal{S}_A$, and let $t : S \to A$ be the unique morphism such that $t \cdot e = \varepsilon_A$ and $t \cdot d_i = t_i$. We show that any morphism $f : B \to A$ with B in $\mathbb{C}(P)$ factorizes through S. Let $f^\star : \coprod_{\hom(P,B)} P \to \coprod_{\hom(P,A)} P$ be the morphism determined by $f : B \to A$, and let $(\bar{e} : \coprod_{\hom(P,A)} P \to \bar{S};\ \bar{f} : B \to \bar{S})$ be the pushout of $(\varepsilon_B, f^\star)$. Then $\bar{e}$ belongs to $\mathcal{S}_A$: The equality $f \cdot \varepsilon_B = \varepsilon_A \cdot f^\star$ gives rise to the existence of a unique morphism $w : \bar{S} \to A$ such that $w \cdot \bar{e} = \varepsilon_A$ and $w \cdot \bar{f} = f$. The morphism $\bar{e}$ is equal to e_i for some $i \in I$. Then $w \cdot e_i = \varepsilon_A = t \cdot e = t \cdot d_i \cdot e_i$, and since e_i is an epimorphism, $w = t \cdot d_i$. Therefore $f = t \cdot (d_i \cdot \bar{f})$.

It is well known that, being coreflective, $\mathbb{C}(P)$ coincides with the co-orthogonal closure of P in $\mathbf{A}$, that is, $\mathbb{C}(P)$ consists of all those $\mathbf{A}$-objects A such that for any morphism f, $\hom(A, f)$ is a bijection whenever $\hom(P, f)$ is so. Consequently, in order to conclude that $\mathbb{C}(P)$ is the subcategory of $\mathbf{A}$ of those objects A such that the function $\hom(f, A)$ is bijective for each $f \in \mathrm{St}(\mathcal{E}) \cap \mathrm{Mono}(\mathbf{A})$, it suffices to show that

$$\mathrm{St}(\mathcal{E}) \cap \mathrm{Mono}(\mathbf{A}) = \{f \in \mathrm{Mor}(\mathbf{A}) \,|\, \hom(P, f) \text{ is an iso}\}. \tag{2.1}$$

From 1.11, the inclusion "$\subseteq$" is trivial. Let $\hom(P, f)$ be an iso. Then, by 1.11, f belongs to $\mathrm{St}(\mathcal{E})$. In order to conclude that f is a mono, let $a, b : S \to X$ be morphisms such that $fa = fb$. Then for any $t : P \to S$, we have $fat = fbt$, which implies, since $\hom(P, f)$ is an iso, that $at = bt$. Thus $a = b$, because P is a generator.

By 1.11, each coreflection $r_A : RA \to A$ into $\mathbb{C}(P)$ belongs to $\mathrm{St}(\mathcal{E}) \subseteq \mathcal{E}$, because P is r_A-projective. In order to show that $\mathbb{C}(P)$ is the smallest $\mathcal{E}$-coreflective subcategory of $\mathbf{A}$, let $\mathbf{B}$ be another $\mathcal{E}$-coreflective subcategory of $\mathbf{A}$. Let $C \in \mathbb{C}(P)$ and let $s_C : S(C) \to C$ be the coreflection of C in $\mathbf{B}$. By Lemma 1.13, s_C belongs to $\mathrm{St}(\mathcal{E}) \cap \mathrm{Mono}(\mathbf{A})$. Therefore, by the equality (2.1), we get a morphism $t : C \to S(C)$ such that $s_C \cdot t = 1_C$, and thus s_C is an isomorphism. □

Remark 2.4 In the above proof, the only role of the $\mathcal{E}$-cocompleteness of $\mathbf{A}$ is to assure that $\mathbb{C}(P)$ is corefletive in $\mathbf{A}$. So, in Theorem 2.3 (in the presence of the conditions of Assumption 1.10), we can replace "$\mathbf{A}$ is $\mathcal{E}$-cocomplete" by "$\mathbb{C}(P)$ is coreflective".

We have seen that the existence of a projective $\mathcal{E}$-generator P gives a characterization of the stabilization of $\mathcal{E}$, St($\mathcal{E}$) =Proj(P), and, in case **A** is $\mathcal{E}$-cocomplete, it guarantees that P "generates" the smallest $\mathcal{E}$-coreflective subcategory of **A**. In the following we give several examples of this situation.

Examples 2.5 1. For any monadic category **A** over **Set** and $\mathcal{E}$ = RegEpi(**A**), let $P = FS$ for some $S \neq \emptyset$, where F is the corresponding left adjoint. Then P is an $\mathcal{E}$-generator such that $\mathrm{St}(\mathcal{E}) = \mathcal{E} = \mathrm{Proj}(P)$, and $\mathbf{A} = \mathbb{C}(P)$.

We point out that, under the conditions of Theorem 2.3, **A** and $\mathbb{C}(P)$ are identical whenever $\mathrm{St}(\mathcal{E}) \cap \mathrm{Mono}(\mathbf{A}) = \mathrm{Iso}(\mathbf{A})$, since { coreflections of **A** in $\mathbb{C}(P)$ } $\subseteq \mathrm{St}(\mathcal{E}) \cap \mathrm{Mono}(\mathbf{A})$.

2. For **A** = **Cat**, $\mathcal{E}$ the class of extremal epimorphisms of **A**, and $\mathbf{2} = \{0 \to 1\}$ the category given by the ordered set 2, we have that **2** is an $\mathcal{E}$-generator, $\mathrm{St}(\mathcal{E}) = \mathrm{Proj}(\mathbf{2})$, and $\mathcal{E}$ does not coincides with its stabilization (see [8], [10]). The equality $\mathbf{A} = \mathbb{C}(\mathbf{2})$ also holds.
3. Let **PreOrd** be the category whose objects are preordered sets (i.e., sets with a reflexive and transitive binary relation), and whose morphisms are preorder-preserving maps. For $\mathcal{E}$ = { regular epimorphisms } = { extremal epimorphisms }, the object **2** as in the above example is a projective $\mathcal{E}$-generator, in particular $\mathrm{St}(\mathcal{E}) = \mathrm{Proj}(\mathbf{2})$, although $\mathrm{St}(\mathcal{E}) \neq \mathcal{E}$. The colimit closure of **2** coincides with **PreOrd**.

In the following examples, it is more convenient to consider the dual situation. That is, now P is an injective $\mathcal{M}$-cogenerator of **A**, with **A** and $\mathcal{M}$ fulfilling the dual conditions of 1.10. A morphism belongs to St($\mathcal{M}$) whenever its pushout along any morphism lies in $\mathcal{M}$. It holds the equality St($\mathcal{M}$)=Inj(P), and the limit closure of P in **A**, $\mathbb{L}(P)$, is the smallest $\mathcal{M}$-reflective subcategory of **A**.

4. In the category **Set**, the pushout-stable class $\mathcal{M}$ of monomorphisms coincides with Inj(P) for P the $\mathcal{M}$-cogenerator set $\{0,1\}$, and $\mathbb{L}(P) = \mathbf{Set}$.
5. For **A** the category **Top** of topological spaces and continuous maps, let $\mathcal{M}$ = {embeddings}, and let P be the topological space $\{0,1,2\}$ whose only non trivial open is $\{0\}$. Then P is an $\mathcal{M}$-cogenerator and Inj(P)=St($\mathcal{M}$)=$\mathcal{M}$. The subcategory $\mathbb{L}(P)$ is the whole category **Top**.
6. For the category $\mathbf{Top_0}$ of T_0-topological spaces, $\mathcal{M}$ = {embeddings}, the Sierpiński space S is an $\mathcal{M}$-cogenerator which fulfils Inj(S)=St($\mathcal{M}$)=$\mathcal{M}$. Here $\mathbb{L}(S)$ is the subcategory of sober spaces.
7. If **A** is the subcategory of **Top** of all 0-dimensional spaces, and $\mathcal{M}$ consists of all embeddings, then $\mathcal{M} \neq \mathrm{St}(\mathcal{M})$, but again $\mathrm{St}(\mathcal{M}) = \mathrm{Inj}(P)$, where P is the space $\{0,1,2\}$ whose only non trivial opens are $\{0\}$ and $\{1,2\}$. (The morphisms of St($\mathcal{M}$) are just those embeddings $m : X \to Y$ such that for each clopen set G of X there is some clopen H in Y such that $G = m^{-1}(H)$ (see [15]).) We have that $\mathbb{L}(P) = \mathbf{A}$ (by the dual of 2.3). A similar situation occurs for the category of 0-dimensional Hausdorff spaces and $\mathcal{M}$ the class of embeddings, which again is not pushout-stable, if we choose P as being the space $\{0,1\}$ with the discrete topology (see [15]).
8. For **Tych** the category of Tychonoff spaces, the class $\mathcal{M}$ of embeddings is not stable under pushouts, and $\mathrm{St}(\mathcal{M}) = \mathrm{Inj}(I)$, where I is the unit interval,

with the usual topology. The $\mathrm{St}(\mathcal{M})$-morphisms are just the C^*-embeddings (see [15]) and $\mathbb{L}(I)$ is the subcategory of compact Hausdorff spaces.

9. For $\mathbf{Vect}_{\mathbb{K}}$ the category of vector spaces and linear maps over the field $\mathbb{K}$, the class $\mathcal{M}$ of all monomorphisms is stable under pushouts and $\mathbb{K}$ is an injective $\mathcal{M}$-cogenerator.
10. In the category $\mathbf{Ab}$ of abelian groups and homomorphisms of group, let $\mathcal{M}$ be the class of all monomorphisms. It is well known that $\mathbb{Q}/\mathbb{Z}$ (where $\mathbb{Q}$ and $\mathbb{Z}$ are the groups of rational numbers and of integer numbers, respectively) is an $\mathcal{M}$-cogenerator, and it holds that $\mathrm{St}(\mathcal{M}) = \mathcal{M} = \mathrm{Inj}(\mathbb{Q}/\mathbb{Z})$. The limit closure of $\mathbb{Q}/\mathbb{Z}$ is the whole category $\mathbf{Ab}$, taking into account the dual of 2.3 and that in $\mathbf{Ab}$ any monomorphism is regular.
11. In the category of torsion-free abelian groups and homomorphisms of group, for $\mathcal{M}$ the class of all monomorphisms, the group of rational numbers $\mathbb{Q}$ is an $\mathcal{M}$-cogenerator and $\mathrm{St}(\mathcal{M}) = \mathcal{M} = \mathrm{Inj}(\mathbb{Q})$. $\mathbb{L}(\mathbb{Q})$ is the subcategory of torsion-free divisible abelian groups.

Next we are going to study the premonadicity of the colimit-closure of a projective $\mathcal{E}$-generator.

Definition 2.6 A functor $U : \mathbf{A} \to \mathbf{Set}$ is said to be *premonadic* if it is a right-adjoint whose comparison functor of the induced monad is full and faithful.

Remark 2.7 In [13] it is proven that for a faithful right adjoint $U : \mathbf{A} \to \mathbf{X}$ the following are equivalent:
(i) U is premonadic;
(ii) U reflects split coequalizers;
(iii) for morphisms $f : C \to A$ and $g : C \to B$ in $\mathbf{A}$ and $h : UA \to UB$ in $\mathbf{X}$, if $Ug = h \cdot Uf$ and Uf is a split epimorphism, then there is some $h' : A \to B$ in $\mathbf{A}$ such that $Uh' = h$.
The equivalence (i) $\Leftrightarrow$ (iii) was obtained in [2] for $\mathbf{X} = \mathbf{Set}$.

It is clear that each monadic functor is premonadic. It is known that there exists a monadic functor $U : \mathbf{A} \to \mathbf{Set}$ (see e.g. [4] or [11]) if and only if (i) $\mathbf{A}$ has finite limits, (ii) $\mathbf{A}$ is exact, (iii) $\mathbf{A}$ has copowers of P, for P such that (iv) P is a regular generator and (v) P is projective. In this classical case, the assumption (ii) ensures that $\mathrm{St}(\mathcal{E}) = \mathcal{E}$ for $\mathcal{E}$ the class of regular epimorphisms; thus (iv) and (v) mean that P is a projective $\mathcal{E}$-generator in the sense of Definition 1.8.

For any cocomplete category $\mathbf{A}$ and a generator P of $\mathbf{A}$, the functor $U = \hom(P, -)$ is a right adjoint with the counit consisting of all canonical morphisms of the form $\varepsilon_A : \coprod_{\hom(P,A)} P \to A$. It is well known that the comparison functor of the monad induced by U is full and faithful if and only if the morphisms ε_A are regular epimorphisms, that is, P is a regular generator of $\mathbf{A}$. Moreover, it is proven in [1] that full reflective subcategories of monadic categories over $\mathbf{Set}^S$ (for some set S) are just the cocomplete categories with a regular generator. (Here by a regular generator of the category $\mathbf{A}$ is meant a set $\{G_s,\ s \in S\}$ of $\mathbf{A}$-objects such that, for each object A of $\mathbf{A}$, the canonical morphism $\coprod_S \left(\coprod_{\hom(G_s,A)} G_s\right) \to A$ is a regular epimorphism.) As a consequence, if $U = \hom(P, -)$ is premonadic then $\mathbf{A}$ is just $\mathbb{C}(P)$. Indeed, given $A \in \mathbf{A}$, let $\varepsilon_A = \mathrm{coeq}(f, g)$, with $f, g : B \to \coprod_{\hom(P,A)} P$; it is trivially seen that, since ε_B is an epimorphism, ε_A is also the coequalizer of $f \cdot \varepsilon_B$ and $g \cdot \varepsilon_B$. Thus $\mathbb{C}(P)$ is the only colimit-closed subcategory of $\mathbf{A}$ containing

P which is candidate to be equivalent to a reflective subcategory of the category of algebras of the monad induced by U via the corresponding comparison functor. The next theorem (2.11), where projectivity has a relevant role, gives conditions under which such a candidate satisfies that property. The required conditions have a certain parallel with (i)–(v) above for monadic functors.

A significant role is played by the following notion:

Definition 2.8 We say that a class $\mathcal{F}$ of epimorphisms is *saturated* if, for each $f \in \mathcal{F}$, the coequalizer of the kernel pair of f belongs to $\mathcal{F}$.

Lemma 2.9 *If the class $\mathcal{E}$ admits a projective $\mathcal{E}$-generator, then the following two assertions are equivalent:*

1. *$St(\mathcal{E})$ is saturated.*
2. *For each $e \in St(\mathcal{E})$, the unique morphism d such that $d \cdot c = e$, for c the coequalizer of the kernel pair of e, is a monomorphism.*

Proof By Proposition 1.11, $\mathrm{St}(\mathcal{E}) = \mathrm{Proj}(P)$, where P is a projective $\mathcal{E}$-generator. Let $\mathrm{St}(\mathcal{E})$ be saturated, let $e \in \mathrm{St}(\mathcal{E})$, let c be the coequalizer of the kernel pair (u, v) of e, and let d be the unique morphism d such that $d \cdot c = e$. Let a and b be morphisms such that $d \cdot a = d \cdot b$; in order to show that $a = b$, since P is a generator, we may assume without loss of generality that P is the domain of a and b. Consequently, since $c \in \mathrm{St}(\mathcal{E}) = \mathrm{Proj}(P)$, there are $\bar{a}$ and $\bar{b}$ such that $c \cdot \bar{a} = a$ and $c \cdot \bar{b} = b$; then $e \cdot \bar{a} = d \cdot c \cdot \bar{a} = d \cdot a = d \cdot b = d \cdot c \cdot \bar{b} = e \cdot \bar{b}$. Since (u, v) is the kernel pair of e, this implies the existence of a unique morphism t such that $u \cdot t = \bar{a}$ and $v \cdot t = \bar{b}$. Therefore, $a = c \cdot \bar{a} = c \cdot u \cdot t = c \cdot v \cdot t = c \cdot \bar{b} = b$.

Conversely, if d is a monomorphism, for each morphism f with domain P and codomain in the codomain of c, we have some morphism f' such that $e \cdot f' = d \cdot f$, because $e \in \mathrm{St}(\mathcal{E}) = \mathrm{Proj}(P)$. Hence, $d \cdot c \cdot f' = d \cdot f$, and $c \cdot f' = f$, proving that $c \in \mathrm{Proj}(P) = \mathrm{St}(\mathcal{E})$. □

Remark 2.10 If $\mathcal{E} \subseteq \{\text{regular epimorphisms}\}$, $\mathrm{St}(\mathcal{E})$ is trivially saturated, since a regular epimorphism is the coequalizer of its kernel pair. The saturation of $\mathrm{St}(\mathcal{E})$ is also clear when $\mathcal{E}$ is pullback stable and $\mathcal{E}$ contains all regular epimorphisms.

In fact, non-trivial classes $\mathcal{E}$ with saturated $\mathrm{St}(\mathcal{E})$ do exist in everyday categories. Indeed, in all examples of 2.5, for the considered class $\mathcal{E}$ or $\mathcal{M}$, the corresponding stabilization is saturated, except in the second example. In this last case, where $\mathcal{E}$ is the class of extremal epimorphisms in the category **Cat**, $\mathrm{St}(\mathcal{E})$ is not saturated. To see that, consider the functor $F : A \to B$ where $A = \mathbf{2}$ and B is the category with a unique object and with an only non-identity morphism f such that $f \cdot f = f$. Then the coequalizer of the kernel pair of F is $G : A \to C$ where C is the monoid of natural numbers regarded as one-object category, and G sends the non-identity morphism of $\mathbf{2}$ to the generator of this monoid. Clearly the functor G does not belong to $\mathrm{St}(\mathcal{E})$ (see [10]).

Theorem 2.11 *Let P be a projective $\mathcal{E}$-generator of $\mathbf{A}$ and let $St(\mathcal{E})$ be saturated. Then, assuming that $\mathbb{C}(P)$ is coreflective in $\mathbf{A}$, the functor $hom(P, -)$: $\mathbb{C}(P) \to$ **Set** is premonadic, and $\mathbb{C}(P)$ is equivalent to a reflective subcategory of the corresponding category of Eilenberg-Moore algebras.*

Proof Since $\mathrm{hom}(P, -)$ is a right adjoint and $\mathbb{C}(P)$ has coequalizers, we know that the comparison functor is a right adjoint. In order to show that it is full and faithful, it suffices to prove that the counits of the right adjoint $\mathrm{hom}(P, -)$:

$\mathbb{C}(P) \to$ **Set** are regular epimorphisms. For each $B \in \mathbb{C}(P)$, let us consider the corresponding co-unit

$$\varepsilon_B : \coprod_{\hom(P,B)} P \longrightarrow B.$$

Since P is ε_B-projective, ε_B belongs to $\mathrm{St}(\mathcal{E})$, by 1.11. Let (u, v) be the kernel pair of ε_B and let c be the coequalizer of u and v. Since $\mathrm{St}(\mathcal{E})$ is saturated, the morphism d such that $d \cdot c = \varepsilon_B$ is a monomorphism. But $\mathrm{St}(\mathcal{E})$ is strongly right-cancellable, by 1.7, so $d \in \mathrm{St}(\mathcal{E})$. Then $d \in \mathrm{St}(\mathcal{E}) \cap \mathrm{Mono}(\mathbf{A})$ and, thus, by Theorem 2.3 (see also Remark 2.4), there is some morphism t such that $d \cdot t = 1_B$. Therefore d is an isomorphism and ε_B is a regular epimorphism. □

Remark 2.12 In the proof of Theorem 2.11 the saturation of $\mathrm{St}(\mathcal{E})$, that is, the fact that $\mathrm{St}(\mathcal{E})$ is closed under coequalizers of kernel pairs, is crucial for concluding that P is a regular generator of $\mathbb{C}(P)$. All examples of 2.5 fulfil the conditions of the above theorem, except the second one. As seen in 2.10, in this case, the saturation of $\mathrm{St}(\mathcal{E})$ fails. And, curiously, the corresponding functor into **Set** is not premonadic: It is easily seen that the corresponding counits are not necessarily regular epimorphisms (cf. [10]).

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The Monotone-light Factorization for Categories Via Preorders

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Abstract. It is shown that the reflection **Cat** $\rightarrow$ **Preord** of the category of all categories into the category of preorders determines a monotone-light factorization system on **Cat** and that the light morphisms are precisely the faithful functors.

1 Introduction

1.1 Every map $\alpha : A \rightarrow B$ of compact Hausdorff spaces has a factorization $\alpha = me$ such that $m : C \rightarrow B$ has totally disconnected fibres and $e : A \rightarrow C$ has only connected ones. This is known as the classical monotone-light factorization of S. Eilenberg [3] and G. T. Whyburn [7].

Consider now, for an arbitrary functor $\alpha : A \rightarrow B$, the factorization $\alpha = me$ such that m is a faithful functor and e is a full functor bijective on objects. We shall show that this familiar factorization for categories is as well monotone-light, meaning that both factorizations are special and very similar cases of the categorical monotone-light factorization in an abstract category $\mathbb{C}$, with respect to a full reflective subcategory $\mathbb{X}$, as was studied in [1].

It is well known that any full reflective subcategory $\mathbb{X}$ of a category $\mathbb{C}$ gives rise, under mild conditions, to a factorization system $(\mathcal{E}, \mathcal{M})$. Hence, each of the two reflections **CompHaus** $\rightarrow$ **Stone**, of compact Hausdorff spaces into Stone spaces, and **Cat** $\rightarrow$ **Preord**, of categories into preorders, yields its own reflective factorization system.

Moreover, the process of simultaneously stabilizing $\mathcal{E}$ and localizing $\mathcal{M}$, in the sense of [1], was already known to produce a new non-reflective and stable factorization system $(\mathcal{E}', \mathcal{M}^*)$ for the adjunction **CompHaus** $\rightarrow$ **Stone**, which is just the

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(**Monotone, Light**)-factorization mentioned above. But this process does not work in general, the monotone-light factorization for the reflection **CompHaus** $\to$ **Stone** being just one of a few known examples where it does. Nevertheless, we shall prove that the (**Full and Bijective on Objects, Faithful**)-factorization for categories is another instance of a successful simultaneous stabilization and localization.

What guarantees the success is the following pair of conditions, which hold in both cases:

1. the reflection $I : \mathbb{C} \to \mathbb{X}$ has stable units (in the sense of [2]);
2. for each object B in $\mathbb{C}$, there is a monadic extension[1] (E, p) of B such that E is in the full subcategory $\mathbb{X}$.

Indeed, the two conditions 1 and 2 trivially imply that the $(\mathcal{E}, \mathcal{M})$-factorization is locally stable, which is a necessary and sufficient condition for $(\mathcal{E}', \mathcal{M}^*)$ to be a factorization system (cf. the central result of [1]).

Actually, we shall prove that the reflection **Cat** $\to$ **Preord** also has stable units, as the reflection **CompHaus** $\to$ **Stone** was known to have. And, for the reflection **Cat** $\to$ **Preord**, the monadic extension (E, p) of B may be chosen to be the obvious projection from the coproduct $E = \mathbf{Cat}(\mathbf{4}, B) \cdot \mathbf{4}$ of sufficiently many copies of the ordinal number **4**, one copy for each triple of composable morphisms in B. As for **CompHaus** $\to$ **Stone**, it was chosen to be the canonical surjection from the Stone-Čech compactification $E = \beta|B|$ of the underlying set of B.

In both cases these monadic extensions are precisely the counit morphisms of the following adjunctions from **Set**: the unique (up to an isomorphism) adjunction $\mathbf{Cat}(\mathbf{4}, -) \dashv (-) \cdot \mathbf{4} : \mathbf{Set} \to \mathbf{Cat}$ which takes the terminal object 1 to the ordinal number **4**, and the adjunction $|\cdot| \dashv \beta : \mathbf{Set} \to \mathbf{CompHaus}$, where the standard forgetful functor $|\cdot|$ is monadic, respectively.

Notice that this perfect matching exists in spite of the fact that **CompHaus** is an exact category and **Cat** is not even regular[2] (the reader may even extend the analogy, to the characterizations of the classes in the factorization systems involved, by simply making the following naive correspondence between some concepts of spaces and categories: "point"/"arrow"; "connected component"/"hom-set"; "fibre"/"inverse image of an arrow"; "connected"/"in the same hom-set"; "totally disconnected"/"every two arrows are in distinct hom-sets").

1.2 The two reflections may be considered as admissible Galois structures, in the sense of categorical Galois theory, since having stable units implies admissibility.

Therefore, in both cases, for every object B in $\mathbb{C}$, one knows that the full subcategory $TrivCov(B)$ of $\mathbb{C}/B$, determined by the trivial coverings of B (i.e., the morphisms over B in $\mathcal{M}$), is equivalent to $\mathbb{X}/I(B)$.

Moreover, the categorical form of the fundamental theorem of Galois theory gives us even more information on each $\mathbb{C}/B$ using the subcategory $\mathbb{X}$. It states that the full subcategory $Spl(E, p)$ of $\mathbb{C}/B$, determined by the morphisms split by the monadic extension (E, p) of B, is equivalent to the category $\mathbb{X}^{Gal(E,p)}$ of internal actions of the Galois pregroupoid of (E, p).

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[1] It is said that (E, p) is a monadic extension of B, or that p is an effective descent morphism, if the pullback functor $p^* : \mathbb{C}/B \to \mathbb{C}/E$ is monadic.

[2] A monadic extension in **CompHaus** is just an epimorphism, i.e., a surjective mapping, while on **Cat** epimorphisms, regular epimorphisms and monadic extensions are distinct classes.

In fact, conditions 1 and 2 above imply that $Gal(E,p)$ is really an internal groupoid in $\mathbb{X}$ (cf. section 5.3 of [1]).

And, as all the monadic extensions (E,p) of B described above are projective[3], one has in both cases that $Spl(E,p) = Cov(B)$, the full subcategory of $\mathbb{C}/B$ determined by the coverings of B (i.e., the morphisms over B in $\mathcal{M}^*$).

Condition 1 implies as well that any covering over an object which belongs to the subcategory is just a trivial covering.

An easy consequence of this last statement, condition 2, and of the fact that coverings are pullback stable, is that a morphism $\alpha : A \to B$ is a covering over B if and only if, for every morphism $\phi : X \to B$ with X in the subcategory $\mathbb{X}$, the pullback $X \times_B A$ of α along ϕ is also in $\mathbb{X}$.

In particular, when the reflection has stable units, a monadic extension (E,p) as in condition 2 is a covering if and only if the kernel pair of p is in the full subcategory $\mathbb{X}$ of $\mathbb{C}$.

Thus, since the monadic extensions considered for the two cases are in fact coverings, one concludes that $Gal(\mathbf{Cat}(\mathbf{4},B)\cdot\mathbf{4},p)$ and $Gal(\beta|B|,p)$ are not just internal groupoids, but internal equivalence relations in **Preord** and **Stone**, respectively.

In symbols, specifically for the reflection $\mathbf{Cat} \to \mathbf{Preord}$:

- $Faithful(B) \simeq \mathbf{Preord}^{Gal(\mathbf{Cat}(\mathbf{4},B)\cdot\mathbf{4},p)}$, for a general category B, and
- $Faithful(X) \simeq \mathbf{Preord}/X$, when X is a preorder.

As for $\mathbf{CompHaus} \to \mathbf{Stone}$:

- $Light(B) \simeq \mathbf{Stone}^{Gal(\beta|B|,p)}$, for a general compact Hausdorff space B, and
- $Light(X) \simeq \mathbf{Stone}/X$, when X is a Stone space.

1.3 The fact that $Gal(\beta|B|,p)$ is an internal equivalence relation in **Stone** was already stated in [5].

Actually, the Stone spaces constitute what was defined there to be a generalized semisimple class of objects in **CompHaus**, and such that every compact Hausdorff space is a quotient of a Stone space.

In this way, the equivalence $Light(B) \simeq \mathbf{Stone}^{Gal(\beta|B|,p)}$ is just a special case of its main Theorem 3.1. Which can be easily extended to non-exact categories, by using monadic extensions instead of regular epis, and in such a manner that the equivalence $Faithful(B) \simeq \mathbf{Preord}^{Gal(\mathbf{Cat}(\mathbf{4},B)\cdot\mathbf{4},p)}$ is also a special case of it.

Hence, the faithful functors coincide with the locally semisimple coverings[4], $Gal(\mathbf{Cat}(\mathbf{4},B)\cdot\mathbf{4},p)$ is an internal equivalence relation in **Preord**, and the reflection $\mathbf{Cat} \to \mathbf{Preord}$ stands now as an interesting non-exact example of the case studied in [5].

2 The reflection of Cat into Preord has stable units

Consider the adjunction

$$(I,H,\eta,\epsilon) : \mathbf{Cat} \to \mathbf{Preord}, \tag{2.1}$$

where:

---

[3]I.e., for each monadic extension (A,f) of B there exists a morphism $g : E \to A$ with $fg = p$.

[4]On condition that one replaces regular epis by monadic extensions in the Definition 2.1 of [5].

- $H(X)$ is the preordered set X regarded as a category;
- $I(A) = A_0$ is the preordered set of objects a in A, in which $a \leq a'$ if and only if there exists a morphism from a to a';
- $\eta_A : A \to HI(A)$ is the unique functor with $\eta_A(a) = a$ for each object a in A;
- $\epsilon : IH \to 1$ is the identity natural transformation.

The following obvious lemma will be used many times below:

Lemma 2.1 *A commutative diagram*

$$\begin{tikzcd} D \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\ C \arrow[r, "\gamma"] & B \end{tikzcd}$$

in **Cat** *is a pullback square if and only if its object version*

$$\begin{tikzcd} D_0 \arrow[r, "v_0"] \arrow[d, "u_0"] & A_0 \arrow[d, "\alpha_0"] \\ C_0 \arrow[r, "\gamma_0"] & B_0 \end{tikzcd}$$

is a pullback square in **Set***, and also its hom-set version*

$$\begin{tikzcd} Hom_D(d, d') \arrow[r] \arrow[d] & Hom_A(v(d), v(d')) \arrow[d] \\ Hom_C(u(d), u(d')) \arrow[r] & Hom_B(\alpha v(d), \alpha v(d')) \end{tikzcd}$$

for arbitrary objects d and d' in D, where the maps are induced by the arrow functions of the functors above.

Proposition 2.2 *The adjunction 2.1 has stable units in the sense of* [2] *and* [1]*; that is, the functor I :* **Cat** $\to$ **Preord** *preserves every pullback square of the form*

$$\begin{tikzcd} A \times_{H(X)} B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & H(X) \end{tikzcd} .$$

Proof Since the reflector I does not change the sets of objects (i.e., the underlying set of $I(A)$ is the same as the set of objects in A), the underlying sets of the two preorders $I(A \times_{H(X)} B)$ and $I(A) \times_{IH(X)} I(B)$ are both equal to $A_0 \times_{H(X)_0} B_0$.

Moreover, for any pair of objects (a, b) and (a', b') in $A_0 \times_{H(X)_0} B_0$, we observe that:

$(a, b) \leq (a', b')$ in $I(A \times_{H(X)} B) \Leftrightarrow$
there exist two morphisms $f : a \to a'$ in A and $g : b \to b'$ in B
such that $\alpha(f) = \beta(g) \Leftrightarrow$
(since $H(X)$ has no parallel arrows!)
there exist two morphisms $f : a \to a'$ in A and $g : b \to b'$ in $B \Leftrightarrow$
$a \leq a'$ in $I(A)$ and $b \leq b'$ in $I(B) \Leftrightarrow$
$(a, b) \leq (a', b')$ in $I(A) \times_{IH(X)} I(B)$.

□

3 Trivial coverings

Consider the two classes of functors $\mathcal{E}$ and $\mathcal{M}$:

- $\mathcal{E}$ is the class of all functors inverted by I : **Cat** $\to$ **Preord**, i.e., of all morphisms $e : A \to C$ in **Cat** such that $I(e) : I(A) \to I(C)$ is a preorder isomorphism;
- $\mathcal{M}$ is the class of all trivial coverings with respect to the adjunction 2.1, i.e., of all morphisms $m : C \to B$ in **Cat** such that the following diagram

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "m"'] & HI(C) \arrow[d, "HI(m)"] \\ B \arrow[r, "\eta_B"] & HI(B) \end{tikzcd} \tag{3.1}$$

is a pullback square.

The fact that the reflection of **Cat** into **Preord** has stable units is known to imply that

- $(\mathcal{E}, \mathcal{M})$ is a factorization system on **Cat**, and that
- the $(\mathcal{E}, \mathcal{M})$-factorization of an arbitrary functor $\alpha : A \to B$ is given by $\alpha = u \circ \langle \alpha, \eta_A \rangle$ in the commutative diagram of Figure 1, whose square part is a pullback.

Proposition 3.1 *A functor $m : C \to B$ belongs to $\mathcal{M}$ if and only if for every two objects c and c' in C with $Hom_C(c, c')$ nonempty, the map $Hom_C(c, c') \to Hom_B(m(c), m(c'))$ induced by m is a bijection.*

We will also express this by saying that m is a trivial covering with respect to the adjunction 2.1 if and only if m is a faithful and "almost full" functor.

Proof According to Lemma 2.1, the diagram 3.1 is a pullback square in **Cat** if and only if the diagram of Figure 2 and the diagrams of Figure 3, for arbitrary

$$
\begin{tikzcd}
A \arrow[dr, "{\langle \alpha, \eta_A \rangle}"] \arrow[ddr, "\alpha"'] \arrow[drr, "\eta_A"] & & \\
& C \arrow[r, "v"] \arrow[d, "u"] & HI(A) \arrow[d, "HI(\alpha)"] \\
& B \arrow[r, "\eta_B"] & HI(B)
\end{tikzcd}
$$

Figure 1 The $(\mathcal{E}, \mathcal{M})$-factorization of $\alpha : A \to B$

$$
\begin{tikzcd}
C_0 \arrow[r, "1"] \arrow[d, "m_0"] & C_0 \arrow[d, "m_0"] \\
B_0 \arrow[r, "1"] & B_0
\end{tikzcd}
$$

Figure 2 The object version diagram of the adjunction unit $\eta : 1 \to HI$

$$
\begin{tikzcd}
Hom_C(c, c') \arrow[r] \arrow[d] & Hom_{HI(C)}(c, c') \arrow[d] \\
Hom_B(m(c), m(c')) \arrow[r, tail] & Hom_{HI(B)}(m(c), m(c'))
\end{tikzcd}
$$

Figure 3 The arrow function diagrams of the adjunction unit $\eta : 1 \to HI$

objects c and c' in C, whose maps are induced by the arrow functions of the functors in diagram 3.1, are all pullback squares in **Set**.

We then observe:

the functor $m : C \to B$ belongs to $\mathcal{M} \Leftrightarrow$

(since the diagram of Figure 2 is obviously a pullback square)
for every two objects c and c' in C,
the diagram of Figure 3 is a pullback square $\Leftrightarrow$

(if $Hom_C(c, c')$ is empty then $Hom_{HI(C)}(c, c')$ is also empty!)
for every two objects c and c' in C, provided $Hom_C(c, c')$ is nonempty,
the diagram of Figure 3 is a pullback square $\Leftrightarrow$

(if $Hom_C(c, c')$ is nonempty then $Hom_{HI(C)}(c, c') \cong 1 \cong Hom_{HI(B)}(m(c), m(c'))$)
for every two objects c and c' in C, provided $Hom_C(c, c')$ is nonempty,
the induced map $Hom_C(c, c') \to Hom_B(m(c), m(c'))$ is a bijection.

$\square$

Proposition 3.2 *A functor $\alpha : A \to B$ belongs to $\mathcal{E}$ if and only if the following two conditions hold:*

1. *the functor α is bijective on objects;*
2. *for every two objects a and a' in A, if $Hom_B(\alpha(a), \alpha(a'))$ is nonempty then so is $Hom_A(a, a')$.*

Proof The condition 2 reformulated in terms of the functor I becomes:

- *for every two objects a and a' in $I(A)$, $a \leq a'$ in $I(A)$ if and only if $\alpha(a) \leq \alpha(a')$ in $I(B)$.*

Therefore, the two conditions together are satisfied if and only if $I(\alpha)$ is an isomorphism, i.e., α is in $\mathcal{E}$.

□

4 Coverings

Starting from the given factorization system $(\mathcal{E}, \mathcal{M})$, we define in the following manner two new classes $\mathcal{E}'$ and $\mathcal{M}^*$ of functors:

- $\mathcal{E}'$ is the class of all functors $e' : A \to C$ in **Cat** such that every pullback of e' is in $\mathcal{E}$, i.e., $\mathcal{E}'$ is the largest pullback-stable class contained in $\mathcal{E}$;
- $\mathcal{M}^*$ is the class of all coverings with respect to the adjunction 2.1, i.e., of all functors $m^* : C \to B$ in **Cat** such that some pullback of m^* along a monadic extension (E, p) of B is in $\mathcal{M}$.

The next Lemma 4.1 is needed to prove the following Lemma 4.2, from which the characterization of coverings in Proposition 4.3 becomes an easy task.

Lemma 4.1 *Consider a Galois structure $\mathbf{\Gamma} = ((I, H, \eta, \varepsilon), \mathbf{F}, \mathbf{\Phi})$[5] on a category $\mathbb{C}$ with finite limits, such that its trivial coverings are pullback stable and the left adjoint $I : \mathbb{C} \to \mathbb{X}$ preserves every pullback square like the one in Proposition 2.2, provided its right-hand edge is a fibration.*

If a fibration (A, α) over B is a covering and $\varphi : H(X) \to B$ is any morphism in $\mathbb{C}$ with X an object in $\mathbb{X}$, then the pullback $(H(X) \times_B A, \varphi^*(\alpha))$ of (A, α) along φ is a trivial covering of $H(X)$.

Proof The lemma follows immediately from the next two facts:

- coverings are pullback stable, whenever trivial coverings are also so;
- any covering over an object of the form $H(X)$ is a trivial covering, whenever the left adjoint $I : \mathbb{C} \to \mathbb{X}$ preserves the pullback squares as above.

The proofs of which are given in §6.1 and §5.4 of [1], respectively.

□

Lemma 4.2 *A functor $\alpha : A \to B$ in **Cat** is a covering with respect to the adjunction 2.1 if and only if, for every functor $\varphi : X \to B$ over B from any preorder X, the pullback $X \times_B A$ of α along φ is also a preorder.*

Proof Consider the adjunction 2.1 as a Galois structure in which all morphisms are fibrations. One just has to show that

- the adjunction 2.1 satisfies the preceding lemma, and that

[5] In the sense of categorical Galois theory as presented in [4].

- for every category B in **Cat**, there is a monadic extension (X, p) of B with X a preorder.

We already know that the reflection 2.1 has stable units, by Proposition 2.2. Therefore, our reflection is an *admissible* Galois structure[6], in which the trivial coverings are known to be pullback stable.

Thus, we complete the proof by presenting, for each category B in **Cat**, a monadic extension (X, p) of B with X a preorder:[7]

make X the coproduct of all composable triples in B,
and then let p be the obvious projection of X into B.

□

Proposition 4.3 *A functor $\alpha : A \to B$ in* **Cat** *is a covering with respect to the adjunction 2.1 if and only if it is faithful.*

Proof We have:

the functor $\alpha : A \to B$ in **Cat** is a covering $\Leftrightarrow$ (by Lemma 4.2)

for every functor $\varphi : X \to B$ from a preorder X, the pullback $X \times_B A$ is a preorder $\Leftrightarrow$

for every functor $\varphi : X \to B$ from a preorder X, for any (x, a) and (x', a') in $X \times_B A$, $Hom_{X\times_B A}((x, a), (x', a'))$ has at most one element $\Leftrightarrow$

for every functor $\varphi : X \to B$ from a preorder X, if f is the unique morphism from x to x' in X, and if any two morphisms $g : a \to a'$ and $h : a \to a'$ in A are such that $\alpha(g) = \varphi(f) = \alpha(h)$, then $g = h \Leftrightarrow$

the functor $\alpha : A \to B$ is faithful.

□

Proposition 4.4 *A functor $\alpha : A \to B$ belongs to $\mathcal{E}'$ if and only if it is a full functor bijective on objects.*

Proof We have:

a functor $\alpha : A \to B$ belongs to $\mathcal{E}' \Leftrightarrow$
(according to the above definitions of $\mathcal{E}'$ and $\mathcal{E}$)
for every pullback u of α, $I(u)$ is an isomorphism $\Leftrightarrow$
$I(\alpha)$ is an isomorphism and I preserves every pullback of $\alpha \Leftrightarrow$
(according to Proposition 3.2)
α is bijective on objects
and
$Hom_A(a, a')$ is empty if and only if $Hom_B(\alpha(a), \alpha(a'))$ is so,
for arbitrary a and a' in A
and
(by Lemma 2.1, and since the reflector I does not change the sets of objects)
for every pullback

[6] A Galois structure, like the one in Lemma 4.1, is said to be admissible if for every object C in $\mathbb{C}$ and every fibration $\varphi : X \to I(C)$ in $\mathbb{X}$, the composite of canonical morphisms $I(C \times_{HI(C)} H(X)) \to IH(X) \to X$ is an isomorphism.

[7] A monadic extension in **Cat** is just a functor surjective on composable triples.

$$\begin{tikzcd} C \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\ C \arrow[r, "\gamma"] & B \end{tikzcd}$$

of α, the hom-set version

$$\begin{tikzcd} Hom_{I(C\times_B A)}((c,a),(c',a')) \arrow[r] \arrow[d] & Hom_{I(A)}(a,a') \arrow[d] \\ Hom_{I(C)}(c,c') \arrow[r] & Hom_{I(B)}(\alpha(a),\alpha(a')) \end{tikzcd}$$

of its image by I is also a pullback square in **Set**,
for arbitrary objects (c,a) and (c',a') in $C \times_B A \Leftrightarrow$

α is bijective on objects
and
$Hom_{I(A)}(a,a') \cong Hom_{I(B)}(\alpha(a),\alpha(a'))$, for arbitrary a and a' in A
and
for every pullback $C \times_B A$ of α, $Hom_{I(C\times_B A)}((c,a),(c',a')) \cong Hom_{I(C)}(c,c')$
for arbitrary objects (c,a) and (c',a') in $C \times_B A \Leftrightarrow$

α is bijective on objects
and
α is full.

□

Conclusion 4.5 *As follows from the previous results (and the results of* [1]*), $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system. Moreover, Propositions 4.3 and 4.4 also tell us that it is a well-known one.*

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Separable Morphisms of Categories Via Preordered Sets

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Abstract. We give explicit descriptions of separable and purely inseparable morphisms with respect to the reflection **Cat** $\to$ **Preord** of the category of all categories into the category of preorders. It follows that the monotone-light, concordant-dissonant and inseparable-separable factorizations on **Cat** do coincide in this case.

1 Introduction

1.1 The monotone-light factorization. The classical monotone-light factorization of S. Eilenberg [3] and G. T. Whyburn [8], for maps of compact Hausdorff spaces, is a special case of the abstract categorical process studied by A. Carboni, G. Janelidze, G. M. Kelly, and R. Paré [1].

The reflection **CompHaus** $\to$ **Stone**, of compact Hausdorff spaces into Stone spaces, induces a reflective factorization system $(\mathcal{E}, \mathcal{M})$ on **CompHaus**. The $(\mathcal{E}, \mathcal{M})$-factorization of an arbitrary map $\alpha : A \to B$ is given by $\alpha = u \circ \langle \alpha, \eta_A \rangle$ in the commutative diagram of Figure 1, whose square part is a pullback, I is the reflector, H the inclusion functor and η the unit of the adjunction.

Then, by stabilizing $\mathcal{E}$ and localizing $\mathcal{M}$, a new monotone-light factorization system $(\mathcal{E}', \mathcal{M}^*)$ is obtained.

Indeed, the reflection **CompHaus** $\to$ **Stone** has stable units (in the sense of [2]), and for each object B in **CompHaus**, there is an effective descent morphism $p : E \to B$ such that E is in the full subcategory **Stone**: the canonical surjection from the Stone-Čech compactification $E = \beta|B|$ of the underlying set of B. These two conditions trivially imply that the $(\mathcal{E}, \mathcal{M})$-factorization is locally stable, which is a necessary and sufficient condition for $(\mathcal{E}', \mathcal{M}^*)$ to be a factorization system by the central result of [1].

We showed in [9] that the process outlined above for compact Hausdorff spaces is also successful when applied to the reflection **Cat** $\to$ **Preord** of categories into

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$$\begin{tikzcd}
A \arrow[dr, "\langle\alpha,\eta_A\rangle"] \arrow[ddr, "\alpha"'] \arrow[drr, "\eta_A"] & & \\
& C \arrow[r, "v"] \arrow[d, "u"] & HI(A) \arrow[d, "HI(\alpha)"] \\
& B \arrow[r, "\eta_B"] & HI(B)
\end{tikzcd}$$

Figure 1 The $(\mathcal{E},\mathcal{M})$-factorization of $\alpha : A \to B$

preorders, which takes a category to the preorder obtained by identifying all morphisms in the same hom-set.

Hence, the $(Full\ and\ Bijective\ on\ Objects, Faithful)$-factorization of functors can be viewed as another non-trivial example of the monotone-light factorization: **Cat** $\to$ **Preord** has also stable units and for each category B there is an effective descent morphism $p : E \to B$ with E a preorder. Notice that there are not many other examples where this process produces a non-reflective factorization system.

Having stable units is a condition stronger than admissibility (also called semi-left-exactness), and so the reflection **Cat** $\to$ **Preord**, as before **CompHaus** $\to$ **Stone**, provides a new application for categorical Galois theory. In this context the light morphisms are called the coverings, which are classified for each object B with the category of internal actions of the precategory $Gal(E,p)$ (reflection of the equivalence relation associated to the effective descent morphism $p : E \to B$ mentioned above) in the full subcategory:

$$Faithful(B) \simeq \mathbf{Preord}^{Gal(E,p)},$$

for a general category B.

In fact, the reflection **Cat** $\to$ **Preord** is an interesting non-exact example of the case studied in [5], and therefore $Gal(E,p)$ is really an equivalence relation for both adjunctions that we are comparing.

One sees that there is a strong analogy between **CompHaus** $\to$ **Stone** and **Cat** $\to$ **Preord** in what concerns categorical Galois theory and monotone-light factorization.

1.2 The concordant-dissonant factorization. Moreover, both reflections have concordant-dissonant $(Conc, Diss)$ factorization systems, in the sense of [6, §2.11], where $Conc$ is the class $\mathcal{E} \cap RegEpi$ of regular epimorphisms in the left-hand side $\mathcal{E}$ of the reflective factorization system $(\mathcal{E},\mathcal{M})$.

One concludes from Corollary 2.11 in [6] that $(Conc, Diss)$ is a factorization system on **CompHaus**, since **CompHaus** is an exact category and so it has a regular epi-mono factorization system $(RegEpi, Mono)$.

On the other hand, the existence of an extremal epi-mono factorization system $(ExtEpi, Mono)$ on **Cat** implies that $(\mathcal{E} \cap ExtEpi, Diss)$ is also a factorization system on **Cat** (cf. [1, §3.9] which generalizes Corollary 2.11 in [6]), where $\mathcal{E} \cap ExtEpi$

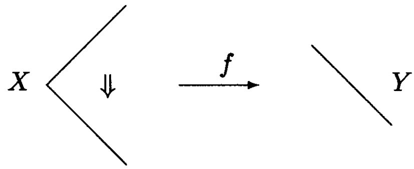

Figure 2 The concordant map $f : X \to Y$

is the class of extremal epis in $\mathcal{E}$.

Finally, remark that the two classes $\mathcal{E} \cap ExtEpi$ and $Conc = \mathcal{E} \cap RegEpi$ coincide on **Cat**.

This follows easily from the known characterizations of extremal epis and regular epis on **Cat** and from Lemma 2.3 below, since in this case the functors in $\mathcal{E}$ are known to be exactly the vertical ones in the sense of Definition 2.1 below (cf. [1]).

So far, the analogy between the two reflections continues.

1.3 The analogy ends. Now notice that for **Cat** $\to$ **Preord** the concordant morphisms are exactly the monotone morphisms, i.e., the full functors bijective on objects (cf. [9]), but for **CompHaus** $\to$ **Stone** it is not so.

Indeed, consider the map in Figure 2 which bends a closed segment in the Euclidean plane through its middle point, identifying in this way its two halves.

It is a concordant map, i.e., a surjection whose fibres are contained in connected components[1], since X has only one component. But it is not monotone, i.e., a map whose fibres are all connected: every point of Y, excepted one of the vertices, has disconnected two-point fibres.

Hence, we have:

- $(\mathcal{E}', \mathcal{M}^*) = (Conc, Diss)$, for **Cat** $\to$ **Preord**;
- $\mathcal{M}^*$ contains strictly the maps in $Diss$, i.e., the maps whose fibres meet the connected components in at most one point, for **CompHaus** $\to$ **Stone**.

1.4 The inseparable-separable factorization of functors. One also knows from [6, §4.1] that

$$Pin \cap RegEpi = Pin^* \subseteq Ins \subseteq Conc \ ,$$

where Ins and Pin are respectively the classes of inseparable and purely inseparable morphisms on **Cat**.

By Proposition 2.5 a functor is in Pin if and only if it is injective on objects. So, one easily concludes that

$$Pin^* = Ins = Conc = \mathcal{E}' \ .$$

[1]Every description, given at this Introduction, of a class of maps of compact Hausdorff spaces, was either taken from [1, §7] or obtained from the Example 5.1 in [6]. At the latter, those descriptions were stated for the reflection of topological spaces into hereditarily disconnected ones, which extends **CompHaus** $\to$ **Stone**.

$$\begin{tikzcd}
& {} \arrow[dr, "m_{\delta_\alpha}"] & & & {} \arrow[dr, "m^*_\alpha"] & \\
A \arrow[ur, "e_{\delta_\alpha}"] \arrow[rr, "\delta_\alpha"'] & & A \times_B A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & A \arrow[ur, "e'_\alpha"] \arrow[rr, "\alpha"'] & & B
\end{tikzcd}$$

Figure 3 The inseparable-separable factorization

$$\begin{tikzcd}
{} \arrow[r, "n"] & {} \arrow[dr, "m_\alpha"] & \\
A \arrow[u, "f"] \arrow[ur, "e_\alpha"'] \arrow[rr, "\alpha"'] & & B
\end{tikzcd}$$

Figure 4 The concordant-dissonant factorization

And, by Theorem 4.4 in [6], the monotone-light factorization on **Cat**, besides being also a concordant-dissonant factorization, is in addition an inseparable-separable factorization:

$$(\mathcal{E}', \mathcal{M}^*) = (Conc, Diss) = (Pin^*, Sep) = (Ins, Sep) \ .$$

Proposition 2.4 gives a direct proof of this fact by stating that the separable morphisms on **Cat** are just the faithful functors, i.e., the light morphisms on **Cat** (cf. [9]).

We see that the reflection of categories into preorders is very well-behaved, in the sense that it equalizes three factorization systems.

1.5 Two procedures for computing the monotone-light factorization of a functor. For an inseparable-separable factorization $m^*_\alpha \cdot e'_\alpha$ of any functor $\alpha : A \to B$ (see Figure 3), e'_α is just the coequalizer of $(u \cdot m_{\delta_\alpha}, v \cdot m_{\delta_\alpha})$, where (u, v) is the kernel pair of α and $m_{\delta_\alpha} \cdot e_{\delta_\alpha}$ is the reflective $(\mathcal{E}, \mathcal{M})$-factorization of the fibred product $\delta_\alpha : A \to A \times_B A$ (see [6, §3.2]).

Hence, one has two procedures for obtaining the monotone-light factorization of a functor via preorders:

- the one just given in Figure 3, corresponding to the inseparable-separable factorization;
- another one in Figure 4, associated with the concordant-dissonant factorization considered above: $\alpha = (m_\alpha \cdot n) \cdot f$, such that $e_\alpha = n \cdot f$ is the extremal epi-mono factorization of e_α, and $\alpha = m_\alpha \cdot e_\alpha$ is the reflective $(\mathcal{E}, \mathcal{M})$-factorization of α.

1.6 The not so good behaviour of monotone-light factorization of maps of compact Hausdorff spaces. As for the reflection **CompHaus** $\to$ **Stone**, the classes Pin and Pin^* are not closed under composition, and so they cannot be part of a factorization system. This was shown in [6, §5.1] with a counterexample. It also follows from that same counterexample that $Ins \neq Pin^*$.[2]

Remark that, as far as separable and dissonant morphisms are concerned, and unlike the reflection **CompHaus** $\to$ **Stone**, the reflection **Top** $\to$ $\mathbf{T_0}$ of topological spaces into T_0 – spaces is analogous to the reflection **Cat** $\to$ **Preord**. Indeed, one has for the reflection **Top** $\to$ $\mathbf{T_0}$ that (see [6, §5.4]):

$$Conc = Ins = Pin^* \quad \text{and} \quad Diss = Sep\ .$$

2 Separable and purely inseparable morphisms

Definition 2.1 Consider the adjunction $(I, H, \eta, \epsilon) : \mathbb{C} \to \mathbb{X}$ and the morphism $\alpha : A \to B$ in $\mathbb{C}$ with kernel pair $(u, v) : A \times_B A \to A$.

We call the morphism α separable with respect to the given adjunction if the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "\delta_\alpha"'] & HI(A) \arrow[d, "HI(\delta_\alpha)"] \\ A \times_B A \arrow[r, "\eta_{A\times_B A}"] & HI(A \times_B A) \end{tikzcd}$$

is a pullback square, where δ_α is the fibred product $\langle 1_A, 1_A \rangle$ of the identity morphism 1_A by itself.

That is, α is a separable morphism if δ_α is a trivial covering[3], with respect to the same adjunction.

Similarly, α is called purely inseparable with respect to the given adjunction if there exists a morphism d making the diagram

$$\begin{tikzcd} A \times_B A \arrow[r, "d"] \arrow[d, "1_{A\times_B A}"'] & HI(A) \arrow[d, "HI(\delta_\alpha)"] \\ A \times_B A \arrow[r, "\eta_{A\times_B A}"] & HI(A \times_B A) \end{tikzcd}$$

a pullback square.

That is, α is a purely inseparable morphism if δ_α is vertical[4], with respect to the same adjunction.

Consider the adjunction

$$(I, H, \eta, \epsilon) : \mathbf{Cat} \to \mathbf{Preord}, \tag{2.1}$$

where:

---

[2]See the last sentence in the previous footnote.
[3]In the sense of categorical Galois theory as presented in [4].
[4]In the sense of [6, §2.4].

- $H(X)$ is the preordered set X regarded as a category;
- $I(A) = A_0$ is the preordered set of objects a in A, in which $a \leq a'$ if and only if there exists a morphism from a to a';
- $\eta_A : A \to HI(A)$ is the unique functor with $\eta_A(a) = a$ for each object a in A;
- $\epsilon : IH \to 1$ is the identity natural transformation.

With respect to the adjunction 2.1, which is certainly a reflection, it is known that a functor α is a trivial covering or vertical exactly when it belongs respectively to the right-hand or left-hand side of the associated reflective factorization system $(\mathcal{E}, \mathcal{M})$ (see [9]).

The next two lemmas were proved in [9].

Lemma 2.2 *A functor $\alpha : A \to B$ is a trivial covering with respect to the adjunction 2.1 if and only if, for every two objects a and a' in A with $Hom_A(a, a')$ nonempty, the map $Hom_A(a, a') \to Hom_B(\alpha(a), \alpha(a'))$ induced by α is a bijection.*

We will also express this by saying that α is a trivial covering with respect to the adjunction 2.1 if and only if α is a faithful and "almost full" functor.

Lemma 2.3 *A functor $\alpha : A \to B$ is vertical with respect to the adjunction 2.1 if and only if the following two conditions hold:*

1. *the functor α is bijective on objects;*
2. *for every two objects a and a' in A, if $Hom_B(\alpha(a), \alpha(a'))$ is nonempty then so is $Hom_A(a, a')$.*

Proposition 2.4 *A functor $\alpha : A \to B$ in* **Cat** *is a separable morphism with respect to the adjunction 2.1 if and only if it is faithful.*

Proof According to Definition 2.1 and Lemma 2.2 one has to show that the functor $\delta_\alpha : A \to A \times_B A$ is faithful and "almost full" if and only if α is faithful.

We observe that for all objects a and a' in A, the map

$$Hom_A(a, a') \to Hom_{A \times_B A}((a, a), (a', a'))$$

induced by δ_α is injective. Hence, this map is bijective if and only if it is surjective.

Since the domain of the map is empty if and only if the codomain is empty, the surjectivity condition for all of these maps amounts to asking that any two morphisms f and g in A with the same domain and codomain and with $\alpha(f) = \alpha(g)$ must coincide, which is to say that α must be a faithful functor.

□

Proposition 2.5 *A functor $\alpha : A \to B$ in* **Cat** *is a purely inseparable morphism with respect to the adjunction 2.1 if and only if its object function is injective.*

Proof According to Definition 2.1 and Lemma 2.3, $\alpha : A \to B$ is purely inseparable if and only if the following two conditions hold:

1. the functor $\delta_\alpha : A \to A \times_B A$ is bijective on objects;
2. for every two objects a and a' in A, if $Hom_{A \times_B A}(\delta_\alpha(a), \delta_\alpha(a'))$ is nonempty then so is $Hom_A(a, a')$.

Of these condition 2 holds trivially, since $\delta_\alpha(a) = (a, a)$ for every object a in A, and the morphisms in $A \times_B A$ are just the ordered pairs (f, g) of morphisms in

A such that $\alpha(f) = \alpha(g)$.

Furthermore, since the object function of δ_α is always injective on objects, it is bijective on objects if and only if it is surjective on objects, which is to say that the functor α must be injective on objects.

$\square$

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Frobenius Algebras in Tensor Categories and Bimodule Extensions

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Introduction

By recent research developments, the notion of tensor category has been recognized as a fundamental language in describing quantum symmetry, which can replace the traditional method of groups for investigating symmetry.

The terminology of tensor category is used here as a synonym of linear monoidal category and hence it has a good affinity with semigroup. One way to incorporate the invertibility axiom of groups is to impose rigidity (or duality) on tensor categories, which will be our main standpoint in what follows.

When a tensor category bears a finite group symmetry inside, it is an interesting problem to produce a new tensor category by taking quotients with respect to this inner symmetry. For quantum symmetries of rational conformal field theory, this kind of constructions are worked out in a direct and individual way with respect to finite cyclic groups.

In our previous works, these specific constructions are organized by interpreting them as bimodule tensor categories for the symmetry of finite groups with a satisfactory duality on bimodule extensions [12]. The construction is afterward generalized to the symmetry of tensor categories governed by finite-dimensional Hopf algebras [13].

We shall present in this paper a further generalization to symmetries described by categorical Frobenius algebras, which are formulated and utilized by J. Fuchs and C. Schweigert for a mathematical description of boundary conditions in conformal field theory [3] (see [5] for earlier studies on categorical Frobenius structures). A similar notion has been introduced under the name of Q-systems by R. Longo in connection with subfactory theory ([6], cf. also [9]). More precisely, a Q-system, if it is algebraically formulated, is equivalent to giving a Frobenius algebra satisfying a certain splitting condition, which is referred to as a special Frobenius algebra according to the terminology in [3].

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Since our viewpoint here is that Q-systems (or special Frobenius algebras) should play the role of group algebras in classical symmetries, we first give an autonomic status to categorical Frobenius algebras as algebraic systems, which enables us to introduce the dual Frobenius algebras without assuming background tensor categories, together with a satisfactory duality on Frobenius algebras.

On the other hand, if Frobenius algebras are realized inside a tensor category $\mathcal{T}$, it is fundamental to consider bimodule extensions of $\mathcal{T}$ and we shall generalize the duality result on bimodule extensions to symmetries specified by categorical Frobenius algebras.

More precisely, given a special Frobenius algebra A realized inside a tensor category $\mathcal{T}$, we show the existence of a natural imbedding of the dual Frobenius algebra B of A into the tensor category ${}_A\mathcal{T}_A$ of A-A bimodules in $\mathcal{T}$. The duality for bimodule extensions is then formulated so that the second bimodule extension ${}_B({}_A\mathcal{T}_A)_B$ of B-B bimodules in ${}_A\mathcal{T}_A$ is naturally isomorphic (monoidally equivalent) to the starting tensor category $\mathcal{T}$.

The author is greatful to A. Masuoka and M. Müger for helpful communications on the subject during the preparation of this article.

Convention: By a tensor category over a field $\mathbb{K}$, we shall mean a $\mathbb{K}$-linear category together with a compatible monoidal structure. If semisimplicty is involved, we assume that $\mathbb{K}$ is an algebraically closed field of zero characteristic.

Since we are primarily interested in the use for quantum symmetry, we shall not discriminate tensor categories as long as they provide the equivalent information; we shall implicitly assume the strictness of associativity as well as the saturation under taking direct sums and subobjects for example.

For basic categorical definitions, we refer to the standard text [8].

1 Monoidal algebras

Let $\mathcal{T}$ be a strict tensor category over a field $\mathbb{K}$ and assume that $\mathrm{End}(I) = \mathbb{K}1_I$ for the unit object I. Given an object X in $\mathcal{T}$, set

$$A_{m,n} = \mathrm{Hom}(X^{\otimes n}, X^{\otimes m})$$

for non-negative integers m, n. The family $\{A_{m,n}\}_{m,n\geq 0}$ is then a **block system of algebra** in the sense that $A = \oplus_{m,n\geq 0} A_{m,n}$ is an algebra satisfying $A_{k,l}A_{m,n} \subset \delta_{l,m}A_{k,n}$ and $A_{0,0} = \mathbb{K}$. Denote the unit of A_n by 1_n.

The tensor product in the category $\mathcal{T}$ defines a bilinear map

$$A_{k,l} \times A_{m,n} \ni f \times g \mapsto f \otimes g \in A_{k+m,l+n}$$

such that

1. the unit 1_0 of A_0 satisfies $1_0 \otimes f = f \otimes 1_0 = f$,
2. the tensor product is associative; $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ and
3. compatible with the composition; $(f \otimes g)(f' \otimes g') = (ff') \otimes (gg')$.

A block system of algebra is called a **monoidal algebra** according to Kazhdan and Wenzl [4] (though they use this terminology in a more restricted meaning) if it is furnished with the operation of taking tensor products which satisfies the above conditions.

Conversely, given a monoidal algebra A, we define a tensor category $\mathcal{A}$ in the following way; objects in $\mathcal{A}$ are parametrized by non-negative integers and the hom-set $\mathrm{Hom}(m, n)$ is the vector space $A_{n,m}$ with the composition of morphisms given by the multiplication in the algebra A. The tensor product operation in $\mathcal{A}$ is the one naturally induced from that of monoidal algebra. If the monoidal algebra A is associated to an object X in a tensor category $\mathcal{T}$, the tensor category $\mathcal{A}$ associated to A is monoidally equivalent to the tensor category generated by X.

If the starting tensor category is semisimple, the monoidal algebra is **locally semisimple** in the sense that for any finite subset F of non-negative integers, the subalgebra $\oplus_{i,j \in F} A_{i,j}$ is semisimple. Conversely, a locally semisimple monoidal algebra A gives rise to a semisimple tensor category $\overline{\mathcal{A}}$ as the Karoubian envelope of $\mathcal{A}$: an object in $\overline{\mathcal{A}}$ is a pair (n, e) of an integer $n \geq 0$ and an idempotent e in A_n with hom-sets defined by

$$\mathrm{Hom}((m, e), (n, f)) = f A_{n,m} e.$$

The operation of tensor product is given by

$$(m, e) \otimes (n, f) = (m + n, e \otimes f)$$

on objects.

A similar construction works for bicategories as well; consider a (strict) bicategory of two objects $\{1, 2\}$ for example and choose objects X, Y in the hom-categories $\mathcal{H}om(2, 1)$, $\mathcal{H}om(1, 2)$ respectively. By using the tensor product notation for the composition in the bicategory, we have the four systems of block algebras

$$\begin{aligned}
A_{m,n} &= \mathrm{Hom}((X \otimes Y)^{\otimes n}, (X \otimes Y)^{\otimes m}), \\
B_{m+1,n+1} &= \mathrm{Hom}((X \otimes Y)^{\otimes n} \otimes X, (X \otimes Y)^{\otimes m} \otimes X), \\
C_{m+1,n+1} &= \mathrm{Hom}(Y \otimes (X \otimes Y)^{\otimes n}, Y \otimes (X \otimes Y)^{\otimes m}), \\
D_{m,n} &= \mathrm{Hom}((Y \otimes X)^{\otimes n}, (Y \otimes X)^{\otimes m})
\end{aligned}$$

(note that $(X \otimes Y)^{\otimes n} \otimes X = X \otimes (Y \otimes X)^{\otimes n}$ are alternating tensor products of X and Y) with the operation of tensor product among them applied in a 2×2-matrix way,

$$\begin{aligned}
A_{m,m'} \otimes B_{n,n'} &\subset B_{m+n,m''+n''}, \\
B_{m,m'} \otimes D_{n,n'} &\subset B_{m+n,m'+n'}, \\
C_{m,m'} \otimes A_{n,n'} &\subset C_{m+n,m'+n'}, \\
D_{m,m'} \otimes C_{n,n'} &\subset C_{m+n,m'+n'}, \\
B_{m,m'} \otimes C_{n,n'} &\subset A_{m+n-1,m'+n'-1}, \\
C_{m,m'} \otimes B_{n,n'} &\subset D_{m+n-1,m'+n'-1},
\end{aligned}$$

which satisfies the associativity and multiplicativity (and the unit condition for tensor products involving A_0 or D_0) exactly as in the definition of monoidal algebra.

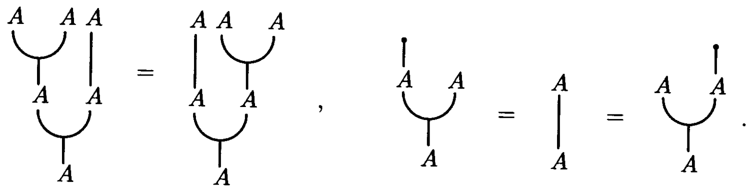
Conversely, given such an algebraic system, we can recover a (two-object) bicategory together with off-diagonal objects X and Y in an obvious way.

We can also talk about isomorphisms of monoidal algebras or their bicategorical counterparts, which exactly correspond to isomorphisms between associated tensor categories or bicategories.

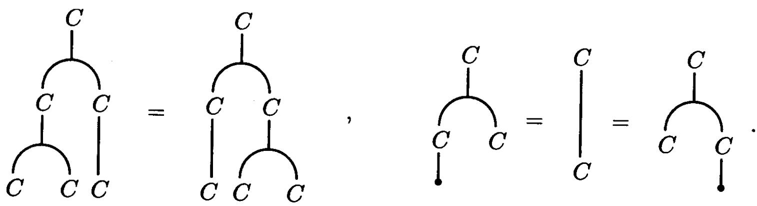
2 Frobenius algebras

It would be just a formal business to formulate axioms of algebraic systems in terms of categorical languages such as monoids or algebras, see [8] for example. Here is a bit more elaborate formulation of Frobenius algebra structure in tensor categories, which we shall describe here, following [3] and [9], mainly to fix the notation with some rewritings of axioms.

Let $\mathcal{T}$ be a tensor category. An **algebra** in $\mathcal{T}$ is a triplet (A, T, δ) with A an object in $\mathcal{T}$, $T \in \mathrm{Hom}(A \otimes A, A)$ and $\delta \in \mathrm{Hom}(I, A)$ satisfying $T(T \otimes 1_A) = T(1_A \otimes T)$, $T(\delta \otimes 1_A) = 1_A = T(1_A \otimes \delta)$, which are graphically denoted in the following way:



By reversing the direction of arrows, a **coalgebra** in $\mathcal{T}$ is a triplet (C, S, ϵ) with $S : C \to C \otimes C$ and $\epsilon : C \to I$ satisfying $(S \otimes 1_C)S = (1_C \otimes S)S$, $(\epsilon \otimes 1_C)S = 1_C = (1_C \otimes \epsilon)S$:



Note that δ and ϵ are uniquely determined by T and S respectively.

A **Frobenius algebra** in $\mathcal{T}$ is, by definition, a quintuplet $(A, S, T, \delta, \epsilon)$ with (A, T, δ) an algebra and (A, S, ϵ) a coalgebra, which satisfies the compatibility condition (st-duality), Fig. 1. The terminology is justified because the axioms turn out to be equivalent to those for ordinary Frobenius algebras if we work with the tensor category of finite-dimensional vector spaces. For an early appearance of categorical Frobenius structures, see [5].

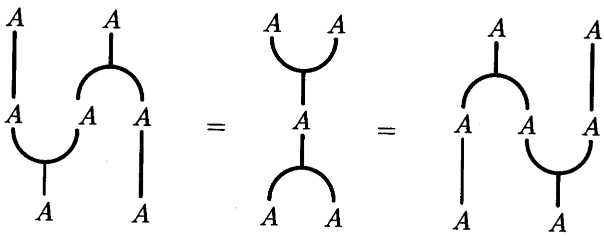
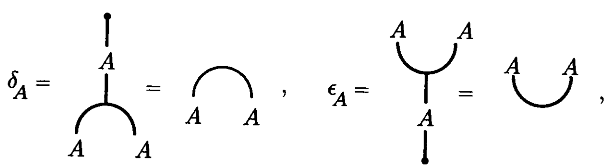
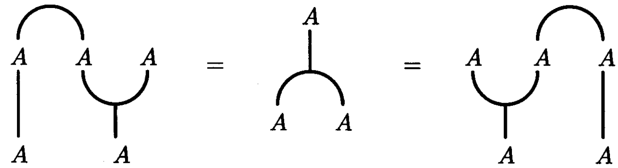


Figure 1

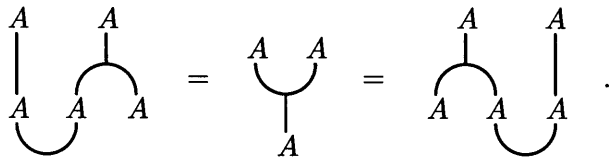
For a Frobenius algebra $(A, S, T, \delta, \epsilon)$, the object A is self-dual with the rigidity pair given by



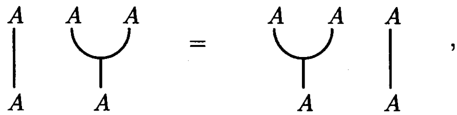
which satisfies the conditions



and



Conversely, given an algebra (A, T, δ) in $\mathcal{T}$ with A a self-dual object and a rigidity copairing $\delta_A : I \to A \otimes A$ fulfilling the **compatibility condition**



we can recover the Frobenius algebra by straightforward arguments so that the above common morphism $A \to A \otimes A$ serves as comultiplication. Since algebra and coalgebra structures are interchangeable with each other in the present context, we have the following characterizations of Frobenius algebra.

Proposition 2.1 *Let A be an object in a tensor category $\mathcal{T}$. Then the following data give the equivalent information on A.*

1. *A Frobenius algebra structure on A.*
2. *An algebra structure (T, δ) on A together with a rigidity copairing $\delta_A : I \to A \otimes A$ satisfying the compatibility condition (A being self-dual particularly).*
3. *A coalgebra structure (S, ϵ) on A together with a rigidity pairing $\epsilon_A : A \otimes A \to I$ satisfying the compatibility condition.*
4. *A pair of morphisms (S, T) satisfying the st-duality and the existence of units and counits.*

It would be worth pointing out here that, in a C*-tensor category $\mathcal{T}$, any coalgebra (A, S, ϵ) is canonically supplemented to a Frobenius algebra (with the coalgebra structure given by taking adjoints of S and ϵ) provided that S is a scalar multiple of an isometry [7].

In what follows, we shall assume that

$$TS = (\text{non-zero scalar})1_A \qquad \text{and} \qquad \epsilon\delta = (\text{non-zero scalar})1_I.$$

Note that the st-duality relation for the pair (S, T) uniquely determines ϵ and δ. For example, if we change (S, T) into $(\lambda S, \mu T)$, then (ϵ, δ) is modified into $(\mu^{-1}\epsilon, \lambda^{-1}\delta)$. Thus, by adjusting scalar multiplications, we may assume that the scalars appearing in TS and $\epsilon\delta$ coincide. If this is the case, we call the pair (S, T) an **algebraic Q-system** (see [6] for the original meaning of Q-systems) and denote the common scalar by d. The associated Frobenius algebra is then referred to as a **special Frobenius algebra** according to [3]. (In [9], the adjective 'strongly separable' is used instead of 'special'.)

A standard model for special Frobenius algebras is the following: Assume that we are given a (strict) bicategory of two objects $\{1,2\}$ and arrange the associated four hom-categories in the matrix form $\begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{21} & \mathcal{H}_{22} \end{pmatrix}$ with $\mathcal{H}_{ij} = \mathcal{H}om(j,i)$.

Choose off-diagonal objects $H \in \mathcal{H}_{12}$ and $H^* \in \mathcal{H}_{21}$ such that H^* is a left and right dual of H at the same time with a (right) rigidity pairing $\epsilon : H^* \otimes H \to I_2$ and a (left) copairing $\delta : I_2 \to H^* \otimes H$. Then $A = H \otimes H^*$ is a Frobenius algebra with multiplication and comultiplication given by $T = 1_H \otimes \epsilon \otimes 1_{H^*}$ and $S = 1_H \otimes \delta \otimes 1_{H^*}$ respectively.

If we further assume the irreducibility of H as well as the existence of unit objects $I_1 \in \mathcal{H}_{11}$ and $I_2 \in \mathcal{H}_{22}$, then A is a special Frobenius algebra.

Remark 2.2

1. If we consider the case of the tensor category of normal *-endomorphisms of an infinite factor, we are reduced to the situation of Q-systems in [6], [7].
2. See [9] for more information on the relationship with the notion of Q-system.

3 Dual systems

Given an algebraic Q-system and objects X, Y in $\mathcal{T}$, we introduce an idempotent operator $E = E_{Y,X} : \mathrm{Hom}(A \otimes X, A \otimes Y) \to \mathrm{Hom}(A \otimes X, A \otimes Y)$ by

$$E(f) = \frac{1}{d}(T \otimes 1_Y)(1_A \otimes f)(S \otimes 1_X),$$

where d is the non-zero scalar associated to the algebraic Q-system.

The following is an easy consequence of graphical computations.

Lemma 3.1 *For $f \in \mathrm{Hom}(A \otimes X, A \otimes Y)$, the following conditions are equivalent.*

1. $E(f) = f$.
2. $f(T \otimes 1_X) = (T \otimes 1_Y)(1_A \otimes f)$.
3. $(S \otimes 1_Y)f = (1_A \otimes f)(S \otimes 1_X)$.

Corollary 3.2 *The image of $\mathrm{End}(A \otimes X)$ under the map E, i.e., $\{f \in \mathrm{End}(A \otimes X); E(f) = f\}$, is a subalgebra of $\mathrm{End}(A \otimes X)$.*

Similarly we can introduce the idempotent operator F associated to the right tensoring of A. We consider the monoidal algebra $\{A_{m,n} = \mathrm{Hom}(A^{\otimes n}, A^{\otimes m})\}_{m,n \geq 0}$ associated with the object A. Set

$$D_{m,n} = \{f \in A_{m+1,n+1}; E(f) = f \text{ and } F(f) = f\}$$

for m, $n \geq 0$ and

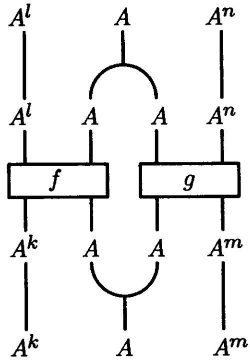
$$B_{m,n} = \{f \in A_{m,n}; F(f) = f\},$$
$$C_{m,n} = \{f \in A_{m,n}; E(f) = f\}$$

for m, $n \geq 1$. Note here that $EF = FE$ on $A_{m+1,n+1}$ by the associativity of S and T.

The above corollary then shows that $\{D_{m,n}\}_{m,n \geq 0}$ is a block system of algebra, i.e., $D_{k,l} D_{m,n} \subset \delta_{l,m} D_{k,n}$, where the product is performed inside the block system of algebra $\bigoplus_{i,j \geq 0} A_{i,j}$. Similarly for $\{B_{m,n}\}_{m,n \geq 1}$ and $\{C_{m,n}\}_{m,n \geq 1}$.

We shall now make $\{D_{k,l}\}$ into a monoidal algebra. Let $f \in D_{k,l}$ and $g \in D_{m,n}$. We define $f\widehat{\otimes}g \in A_{k+m+1,l+n+1}$ by

$$f\widehat{\otimes}g \quad = \quad \frac{1}{d}$$



The following is easily checked by graphical computations.

Lemma 3.3

1. *We have* $f\widehat{\otimes}g \in D_{k+m,l+n}$.
2. *The unit* 1_A *of* $A_{0,0}$ *satisfies* $1_A\widehat{\otimes}f = f\widehat{\otimes}1_A = f$ *for* $f \in D_{m,n}$.
3. *For* $f : V \otimes A \to W \otimes A$, $h : A \otimes X \to A \otimes Y$ *and* $g \in D_{m,n}$ *with* m, $n \geq 0$, *we have* $(f\widehat{\otimes}g)\widehat{\otimes}h = f\widehat{\otimes}(g\widehat{\otimes}h)$.
4. *For* $f \in D_{m,m'}$, $f' \in D_{m',m''}$, $g \in D_{n,n'}$ *and* $g' \in D_{n',n''}$, *we have*

$$(f\widehat{\otimes}g)(f'\widehat{\otimes}g') = (ff')\widehat{\otimes}(gg').$$

The block system $\{D_{m,n}\}$ is now a monoidal algebra by the previous lemma. The construction can be obviously extended to the systems $\{B_{m,n}\}$ and $\{C_{m,n}\}$ so that they give rise to a 2×2-bicategory $\mathcal{B}$:

$$\begin{aligned}
A_{m,m'} \otimes B_{n,n'} &\subset B_{m+n,m'+n'}, \\
B_{m,m'}\widehat{\otimes}D_{n,n'} &\subset B_{m+n,m'+n'}, \\
C_{m,m'} \otimes A_{n,n'} &\subset C_{m+n,m'+n'}, \\
D_{m,m'}\widehat{\otimes}C_{n,n'} &\subset C_{m+n,m'+n'}, \\
B_{m,m'}\widehat{\otimes}C_{n,n'} &\subset A_{m+n-1,m'+n'-1}, \\
C_{m,m'} \otimes B_{n,n'} &\subset D_{m+n-1,m'+n'-1}
\end{aligned}$$

with analogous properties of tensor products for $\{D_{m,n}\}$.

If we denote by H and H^* objects associated to $B_{1,1}$ and $C_{1,1}$ respectively, then A is identified with $H \otimes H^*$ and $D_{m,n} = \mathrm{Hom}((H^* \otimes H)^{\otimes n}, (H^* \otimes H)^{\otimes m})$.

Proposition 3.4 *The bicategory* $\mathcal{B}$ *is rigid. More precisely, the generators* H *and* H^* *are rigid with rigidity pairs given by*

$$\begin{aligned}
&\delta : I \to H \otimes H^* = A, \qquad &T \in D_{0,1} = \mathrm{Hom}(H^* \otimes H, J), \\
&S \in D_{1,0} = \mathrm{Hom}(J, H^* \otimes H), \qquad &\epsilon : H \otimes H^* = A \to I.
\end{aligned}$$

(J denotes the unit object for D.)

Proof The hook identities for these pairs are nothing but the unit and counit identities for T and S respectively. □

The rigidity pairs then induce the Frobenius algebra structure on $H^* \otimes H$ by switching the roles of (δ, ϵ) and (S, T), which is referred to as the **dual Q-system**: the multiplication and comultiplication in $H^* \otimes H$ are given respectively by

$$1 \otimes \epsilon \otimes 1 : H^* \otimes H \otimes H^* \otimes H \to H^* \otimes H, \qquad 1 \otimes \delta \otimes 1 : H^* \otimes H \to H^* \otimes H \otimes H^* \otimes H.$$

Now the following duality for algebraic Q-systems, although obvious, generalizes an operator algebraic result in [6].

Proposition 3.5 *Given an algebraic Q-system (S, T), its bidual Q-system is canonically isomorphic to (S, T).*

4 Bicategory of bimodules

Recall that a morphism $f : X \to Y$ in a category is called a **monomorphism** (**epimorphism** respectively) if $g_j : Z \to X$ ($g_j : Y \to Z$) for $j = 1, 2$ satisfies $fg_1 = fg_2$ ($g_1 f = g_2 f$), then $g_1 = g_2$. A **subobject** of an object Y is a pair (X, j) of an object X and a monomorphism $j : X \to Y$. A subobject $j : X \to Y$ is called a **direct summand** if we can find a morphism $p : Y \to X$ such that $pj = 1_X$.

In what follows, categories are assumed to be linear, have splitting idempotents and be closed under taking direct sums. Given an idempotent $e \in \mathrm{End}(X)$, we denote the associated subobject of X by eX (with e regarded as a monomorphism in $\mathrm{Hom}(eX, X) = \mathrm{End}(X)e$), which is a direct summand of X and we have the obvious identification $eX \oplus (1 - e)X = X$.

Let A be a Frobenius algebra in a tensor category $\mathcal{T}$. By a **left A-module**, we shall mean an object M in $\mathcal{T}$ together with a morphism (called the action) $\lambda : A \otimes M \to M$ satisfying $\lambda(\epsilon \otimes 1_M) = 1_M$ and the commutative diagram

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "1\otimes\lambda"] \arrow[d, "T\otimes 1"'] & A \otimes M \arrow[d, "\lambda"] \\
A \otimes M \arrow[r, "\lambda"'] & M
\end{tikzcd} .$$

The notion of **right A-module** is defined analogously. Let B be another Frobenius algebra. By an A-B bimodule, we shall mean a left A-module M (with the left action $\lambda : A \otimes M \to M$) which is a right B-module (with the right action $\mu : M \otimes B \to M$) at the same time and makes the diagram

$$\begin{tikzcd}
A \otimes M \otimes B \arrow[r, "\lambda\otimes 1"] \arrow[d, "1\otimes\mu"'] & M \otimes B \arrow[d, "\mu"] \\
A \otimes M \arrow[r, "\lambda"'] & M
\end{tikzcd}$$

commutative.

An A-B bimodule based on an object M in $\mathcal{T}$ is simply denoted by ${}_AM_B$. Given another A-B bimodule ${}_AN_B$, a morphism $f : M \to N$ in the category $\mathcal{T}$ is said to be A-B **linear** if the diagram

$$\begin{tikzcd}
A \otimes M \otimes B \arrow[r, "1\otimes f\otimes 1"] \arrow[d] & A \otimes N \otimes B \arrow[d] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

commutes.

The totality of A-B bimodules $\{_AM_B\}$ forms a linear category $_A\mathcal{T}_B$ by

$$\mathrm{Hom}(_AM_B, {}_AN_B) = \{f \in \mathrm{Hom}(M, N); f \text{ is } A\text{-}B \text{ linear}\}.$$

Recall here that we have assumed splitting idempotents in the category $\mathcal{T}$ and the same holds for $_A\mathcal{T}_B$: if $e \in \mathrm{End}(_AM_B)$ is an idempotent, then the A-B action on M induces an A-B action on the subobject eM, i.e., $_A(eM)_B$.

From here on we exclusively deal with Frobenius algebras of algebraic Q-systems, i.e., special Frobenius algebras, and shall introduce the notion of tensor product for bimodules. A more general and categorical construction is available in [3] but we prefer the following less formal description, which enables us to easily check the associativity (the so-called pentagonal relation) of tensor products.

Let X_B and $_BY$ be right and left B-modules with action morphisms ρ and λ respectively. Let $e \in \mathrm{End}(X \otimes Y)$ be an idempotent defined by

$$e = d^{-1}(\rho \otimes \lambda)(1_X \otimes \delta_A \otimes 1_Y),$$

where d is the common scalar for TS and $\epsilon\delta$.

The **module tensor product** $X \otimes_B Y$ is, by definition, the subobject $e(X \otimes Y)$ of $X \otimes Y$ associated to the idempotent e. For bimodules $_AX_B$ and $_BY_C$, e belongs to $\mathrm{End}(_AX \otimes Y_C)$ and hence it induces an A-C bimodule $_AX \otimes_B Y_C$.

Let $_CZ$ be another left C-module and $f \in \mathrm{End}(Y \otimes Z)$ be the idempotent associated to the inner action of C. Then it is immediate to show the commutativity $(e \otimes 1_Z)(1_X \otimes f) = (1_X \otimes f)(e \otimes 1_Z)$ by the compatibility of left and right actions on Y, which enables us to identify

$$(X \otimes_B Y) \otimes_C Z = (e \otimes 1_Z)(1_X \otimes f)(X \otimes Y \otimes Z) = X \otimes_B (Y \otimes_C Z).$$

Moreover, given morphisms $\varphi : {}_AX_B \to {}_AX'_B$ and $\psi : {}_BY_C \to {}_BY'_C$, $\varphi \otimes_B \psi : {}_AX \otimes_B Y_C \to {}_AX' \otimes_B Y'_C$ is defined by

$$\varphi \otimes_B \psi = (\varphi \otimes \psi)e = e'(\varphi \otimes \psi),$$

where $e' \in \mathrm{End}(X' \otimes Y')$ denotes the idempotent associated to the inner action of B on $X' \otimes Y'$. It is also immediate to see the associativity for the tensor product of morphisms:

$$(\phi \otimes_A \varphi) \otimes_B \psi = \phi \otimes_A (\varphi \otimes_B \psi).$$

(More precisely, the identification is through the natural isomorphisms among module tensor products of objects.)

The Frobenius algebra A itself bears the structure of A-A bimodule by the multiplication morphism, which is denoted by $_AA_A$. Given a left A-module $\lambda : A \otimes X \to X$, let $\lambda^* : X \to A \otimes X$ be the associated coaction: $\lambda^* = (1_A \otimes \lambda)(\delta_A \otimes 1_X)$.

Lemma 4.1 *Both of λ and λ^* are A-linear.*

Proof The A-linearity of λ is just the associativity of the action. To see the A-linearity of λ^*, we use the identity

$$(T \otimes 1_A)(1_A \otimes \delta_A) = S = (1_A \otimes T)(\delta_A \otimes 1_A).$$

□

Lemma 4.2 *Let $e \in \mathrm{End}(A \otimes X)$ be the idempotent associated to the inner action of A on $A \otimes X$. Then we have*

$$\lambda\lambda^* = d1_X, \qquad \lambda^*\lambda = de.$$

Proof These follow from simple graphical computations of $\lambda\lambda^*$ and $\lambda^*\lambda$. □

Lemma 4.3 *The action morphism $\lambda : A \otimes X \to X$ induces the A-linear isomorphism $l : A \otimes_A X \to X$ with the inverse given by $d^{-1}\lambda^*$. Likewise a right A-module $\rho : X \otimes A \to X$ induces the isomorphism $r : X \otimes_A A \to X$ with the inverse given by $d^{-1}\rho^*$.*

Here is another useful observation, which is an immediate consequence of definitions.

Lemma 4.4 *Let A be a Frobenius algebra. Then, by the correspondance $(\lambda : A \otimes X \to X) \iff (\lambda^* : X \to A \otimes X)$, there is an equivalence between the category of left A-modules and the category of left A-comodules.*

$$\begin{array}{ccc} A\otimes X & \xrightarrow{1\otimes f} & A\otimes Y \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array} \iff \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ A\otimes X & \xrightarrow[1\otimes f]{} & A\otimes Y \end{array}.$$

Lemma 4.5 *Let X_A be a right A-module and ${}_AY$ be a left A module with the associated isomorphisms $r : X \otimes_A A \to X$ and $l : A \otimes_A Y \to Y$. Then r and l satisfy the triangle identity: $r \otimes_A 1_Y = 1_X \otimes_A l$ on $X \otimes_A A \otimes_A Y$.*

Proof Let $e_X \in \mathrm{End}(X \otimes A)$, $e_Y \in \mathrm{End}(A \otimes Y)$ and $e \in \mathrm{End}(X \otimes Y)$ be idempotents associated to the inner actions of A. We need to show the equality

$$e(\rho \otimes 1_Y)(e_X \otimes 1_Y)(1_X \otimes e_Y) = e(1_X \otimes \lambda)(e_X \otimes 1_Y)(1_X \otimes e_Y).$$

By a graphical computation, we see that

$$d^2(\rho\otimes 1_Y)(e_X\otimes 1_Y)(1_X\otimes e_Y) = d(\rho\otimes\lambda)(1_X\otimes S\otimes 1_Y) = d^2(1_X\otimes\lambda)(e_X\otimes 1_Y)(1_X\otimes e_Y).$$

□

Summarizing the discussions so far, we have

Proposition 4.6 *The family of categories $\{{}_A\mathcal{T}_B\}$ indexed by pairs of special Frobenius algebras forms a bicategory with unit constraints given by l and r in the previous lemma.*

The following is not needed in what follows but enables us to compare our definition with the one in [3].

Lemma 4.7 *The projection $e : X \otimes Y \to X \otimes_B Y$ gives the cokernel of*

$$(\rho \otimes 1_Y - 1_X \otimes \lambda) : X \otimes B \otimes Y \to X \otimes Y.$$

Proof By a graphical computation, we have

$$e(\rho \otimes 1_Y) = (\rho \otimes \lambda)(1_X \otimes S \otimes 1_Y) = e(1_X \otimes \lambda).$$

Conversely, given a morphism $f : X \otimes Y \to Z$ satisfying $f(\rho \otimes 1_Y) = f(1_X \otimes \lambda)$, we can show $ef = f$. □

5 Rigidity in bimodules

The rigidity of categorical modules is considered in [3] under the assumption of a certain 'commutativity' of Frobenius algebras. Although its general validity would be well-known for experts, we shall describe here the relevant points for completeness.

Let A and B be Frobenius algebras in a tensor category $\mathcal{T}$ and ${}_AX_B$ be an A-B bimodule in $\mathcal{T}$. Assume that the object X admits a (left) dual X^* in $\mathcal{T}$ with a rigidity pair given by $\epsilon : X \otimes X^* \to I$ and $\delta : I \to X^* \otimes X$. We can then define the B-A action on X^* as the transposed morphism: consider $B \otimes X^* \to X^*$ and $X^* \otimes A \to X^*$ defined by Fig. 2.

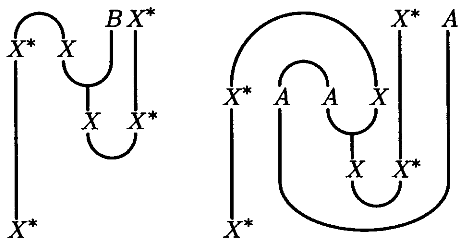
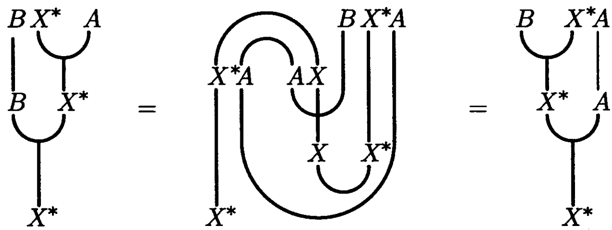


Figure 2

The following is immediate by easy graphical works.

Lemma 5.1 *These in fact define the left and right actions on X^*, which are compatible in the following sense.*



We shall show that the bimodule ${}_BX^*_A$ is a dual object of ${}_AX_B$. To this end, we first introduce morphisms $\epsilon : X \otimes X^* \to A$ and $\delta : B \to X^* \otimes X$ by Fig. 3

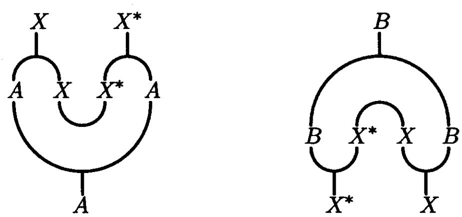


Figure 3

Lemma 5.2 *We have*

$$\epsilon = d_A(1_A \otimes \epsilon_X)((X \to A \otimes X) \otimes 1_{X^*}),$$
$$\delta = d_B(1_{X^*} \otimes (X \otimes B \to X))(\delta_X \otimes 1_B).$$

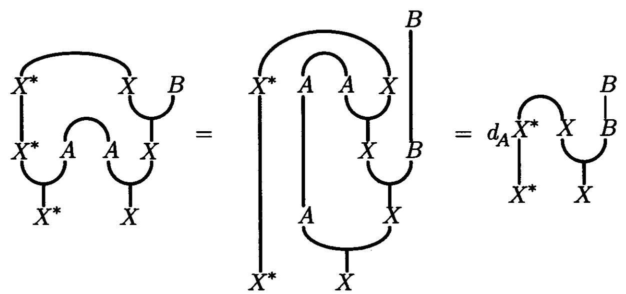
Proof Insert the definition of (co)actions on X^* and compute graphically. □

Lemma 5.3 *The morphism $\epsilon : X \otimes X^* \to A$ is A-A linear whereas the morphism $\delta : B \to X^* \otimes X$ is B-B linear.*

Proof We use the formulas in the above lemma together with the definition of (co)actions on X^*. □

Lemma 5.4 *Let $e_B \in \mathrm{End}(X \otimes X^*)$ and $e_A \in \mathrm{End}(X^* \otimes X)$ be idempotents associated with inner actions. Then ϵ and δ are supported by these idempotents: $\epsilon e_B = \epsilon$ and $e_A \delta = \delta$.*

Proof For example, the equality $e_A\delta = \delta$ is proved by



□

By the above lemmas, we can regard ϵ and δ as defining morphisms ${}_AX \otimes_B X^*_A \to {}_AA_A$ and ${}_BB_B \to {}_BX^* \otimes_A X_B$ respectively.

Lemma 5.5 *The compositions*

$$X \xrightarrow{1 \otimes_B \delta} X \otimes_B X^* \otimes_A X \xrightarrow{\epsilon \otimes_A 1} X,$$
$$X^* \xrightarrow{\delta \otimes_B 1} X^* \otimes_A X \otimes_B X^* \xrightarrow{1 \otimes_A \epsilon} X^*$$

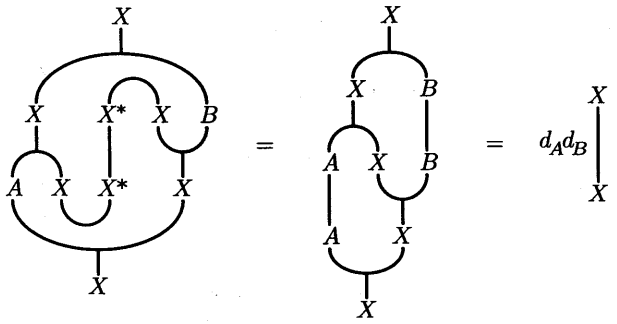
are scalar multiplication of identities by the common scalar ${d_A}^2 d_B$.

Proof By the previous lemma together with Corollary 4.3, we need to compare compositions

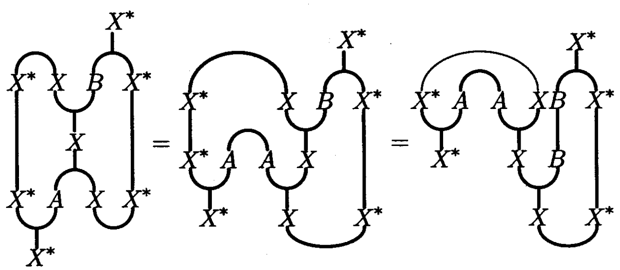
$$X \xrightarrow{d_B^{-1}\rho^*} X \otimes B \xrightarrow{1_X \otimes \delta} X \otimes X^* \otimes X \xrightarrow{\epsilon \otimes 1_X} A \otimes X \xrightarrow{\lambda} X,$$
$$X^* \xrightarrow{d_B^{-1}\lambda^*} B \otimes X^* \xrightarrow{\delta \otimes 1_{X^*}} X^* \otimes X \otimes X^* \xrightarrow{1_{X^*} \otimes \epsilon} X^* \otimes A \xrightarrow{\rho} X^*,$$

where λ denotes one of the left actions $A \otimes X \to X$, $B \otimes X^* \to X^*$ and similarly for ρ, λ^* and ρ^*.

By multiplying d_A^{-1} on both of these compositions, the former is reduced to



whereas the latter is given by



and turns out to be $d_A d_B 1_X$ from the relations in Fig. 4. □

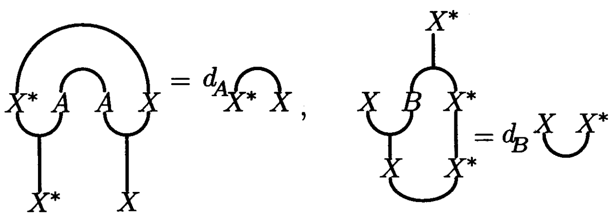


Figure 4

Proposition 5.6 *Let ${}_AX_B$ be a bimodule and assume that X is rigid in $\mathcal{T}$. Then the bimodule ${}_AX_B$ is rigid in the bicategory with the dual bimodule given by ${}_BX^*_A$.*

Proof This is just a paraphrase of the previous lemma. □

Definition 5.7 Given a Frobenius algebra A in a tensor category $\mathcal{T}$, we denote by ${}_A\mathcal{T}_A$ the tensor category of A-A bimodules.

Proposition 5.8 *Given a special Frobenius algebra A in a tensor category $\mathcal{T}$, let B be the dual Frobenius algebra of A. Then the bicategory connecting A and B is generated by the bimodule $H = {}_IA_A$ in $\mathcal{T}$: $H \otimes_A H^* \cong A$ while the Frobenius algebra ${}_AH^* \otimes H_A$ is isomorphic to B.*

Theorem 5.9 (Duality for Tensor Categories) *Given a special Frobenius algebra A in a tensor category $\mathcal{T}$, the dual Frobenius algebra B is canonically realized in the tensor category ${}_A\mathcal{T}_A$ and the tensor category ${}_B({}_A\mathcal{T}_A)_B$ of B-B bimodules in ${}_A\mathcal{T}_A$ is naturally monoidally equivalent to the starting tensor category $\mathcal{T}$.*

Proof By the identification $B = H^* \otimes H$, the object H has the structure of a right B-module in an obvious way and, if we regard this as defining an object M at an off-diagonal corner of a bicategory connecting $\mathcal{T}$ and ${}_B({}_A\mathcal{T}_A)_B$, then it satisfies the imprimitivity condition; $M \otimes_B M^* = I$ (the unit object in $\mathcal{T}$) and $M^* \otimes M = {}_B B_B$ (the unit object in ${}_B({}_A\mathcal{T}_A)_B$). Thus taking adjoint tensor multiplications by M gives rise to a monoidal equivalence of tensor categories in question. □

6 Semisimplicity

An object X is said to be **semisimple** if any subobject is a direct summand and said to be **simple** if there is no non-trivial subobject.

Note that, if $\mathrm{End}(X)$ is finite-dimensional for a semisimple object X, then X is isomorphic to a direct sum of simple objects.

A tensor category is **semisimple** if every object is semisimple.

The following is a direct and simplified version of the proof in [3, §5.4] (cf. also [10]).

Proposition 6.1 *Let A and B be special Frobenius algebras in a tensor category $\mathcal{T}$. An A-B bimodule ${}_AX_B$ is semisimple in ${}_A\mathcal{T}_B$ if the base object X is semisimple in $\mathcal{T}$.*

Proof Let $f : {}_AY_B \to {}_AX_B$ be a monomorphism in ${}_A\mathcal{T}_B$. We first show that f is monomorphic as a morphism in $\mathcal{T}$.

In fact, given a morphism $h : Z \to Y$ in $\mathcal{T}$ such that $fh = 0$, the induced morphism $\widetilde{h} : A \otimes Z \otimes B \to Y$ defined by

$$\widetilde{h} = (A \otimes Y \otimes B \to Y)(1_A \otimes h \otimes 1_B)$$

is A-B linear and satisfies

$$f\widetilde{h} = (A \otimes Y \otimes B \to Y)(1_A \otimes fh \otimes 1_B) = 0$$

by the A-B linearity of f. Since f is assumed to be monomorphic in ${}_A\mathcal{T}_B$, this implies $\widetilde{h} = 0$ and hence

$$h = \widetilde{h}(\delta \otimes 1_Z \otimes \delta) = 0,$$

where δ denotes one of unit morphisms $I \to A$ and $I \to B$ in the Frobenius algebras.

So far, we have proved that $f : Y \to X$ gives a subobject of X. Since X is semisimple by our assumption, we can find a morphism $g : X \to Y$ satisfying $gf = 1_Y$. Let $\widetilde{g} : Y \to X$ be defined by

$$\widetilde{g} = (A \otimes X \otimes B \to X)(1_A \otimes g \otimes 1_B)(Y \to A \otimes Y \otimes B),$$

which is A-B linear as a composition of A-B linear morphisms.

Now the computation

$$\begin{aligned}\widetilde{g}f &= (A \otimes Y \otimes B \to Y)(1_A \otimes g \otimes 1_B)(Y \to A \otimes Y \otimes B)f \\ &= (A \otimes Y \otimes B \to Y)(1_A \otimes gf \otimes 1_B)(Y \to A \otimes Y \otimes B) \\ &= (A \otimes Y \otimes B \to Y)(Y \to A \otimes Y \otimes B) \\ &= d_A d_B 1_Y\end{aligned}$$

shows that ${}_AY_B$ is a direct summand of ${}_AX_B$. □

Corollary 6.2 *Let A and B be special Frobenius algebras in a semisimpte tensor category $\mathcal{T}$. Then the category ${}_A\mathcal{T}_B$ of A-B bimodules in $\mathcal{T}$ is semisimple as well.*

7 Tannaka duals

By the Tannaka dual of a Hopf algebra H, we shall mean the tensor category of finite-dimensional (left) H-modules.

We shall here work with the Tannaka dual $\mathcal{A}$ of a semisimple Hopf algebra H which is realized in a tensor category $\mathcal{T}$, i.e., we are given a faithful monoidal functor $F : \mathcal{A} \to \mathcal{T}$. The notion of $\mathcal{A}$-modules is introduced in [13] in terms of the notion of trivializing isomorphisms.

Let $\mathbb{A}$ be the unit object in the tensor category of $\mathcal{A}$-$\mathcal{A}$ modules in $\mathcal{T}$. Recall that the object $\mathbb{A}$ is isomorphic to

$$\bigoplus_V F(V) \otimes V^*$$

as an object in $\mathcal{T}$. By interchanging left and right actions, the dual object $\mathbb{A}^*$ of $\mathbb{A}$ is an $\mathcal{A}$-$\mathcal{A}$ module in a canonical way, which is isomorphic to the unit object ${}_{\mathcal{A}}\mathbb{A}_{\mathcal{A}}$. We shall give an explicit formula for the isomorphism ${}_{\mathcal{A}}\mathbb{A}^*_{\mathcal{A}} \cong {}_{\mathcal{A}}\mathbb{A}_{\mathcal{A}}$.

Lemma 7.1 *The isomorphism $\mathbb{A}^* \to \mathbb{A}$ given by*

$$\mathbb{A}^* = \bigoplus\nolimits_V F(V^*) \otimes V \xrightarrow{\oplus_V d(V)1} \bigoplus\nolimits_V F(V^*) \otimes V = \mathbb{A}$$

is $\mathcal{A}$-$\mathcal{A}$ linear.

Proof Let us prove the left $\mathcal{A}$-linearity for example. To this end, we first recall that the left action $F(U) \otimes \mathbb{A}^* \to \mathbb{A}^* \otimes U$ on $\mathbb{A}^*$ is given by the composition

$$F(U)\otimes\mathbb{A}^* \to \mathbb{A}^*\otimes U\otimes U^*\otimes\mathbb{A}\otimes F(U)\otimes\mathbb{A}^* \to \mathbb{A}^*\otimes U\otimes\mathbb{A}\otimes F(U^*)\otimes F(U)\otimes\mathbb{A}^* \to \mathbb{A}^*\otimes U.$$

We can check this formula by working on vector spaces: Let

$$\{X \xrightarrow{\ \xi\ } W \otimes U \xrightarrow{\ \xi^*\ } X\}$$

be an irreducible decomposition of $W \otimes U$ and $\{x_l\}$, $\{w_k\}$, $\{u_i\}$ be bases of vector spaces X, W, U with the dual bases indicated by asterisk. Then

$$\begin{aligned}
&F(u) \otimes F(v^*) \otimes v \\
&\quad\mapsto \bigoplus_W \sum_{i,j,k} F(w_j^*) \otimes w_k \otimes u_i \otimes u_i^* \otimes F(w_j) \otimes w_k^* \otimes F(u) \otimes F(v^*) \otimes v \\
&\quad\mapsto \bigoplus_W \sum_{i,j,k} \sum_{X,\xi,l} F(w_j^*) \otimes w_k \otimes u_i \otimes u_i^* \otimes F(x_l) \otimes \xi \otimes w_k^* \otimes F(v^*) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\otimes v\langle x_l^*, \xi^*(w_j \otimes u)\rangle \\
&\quad\mapsto \bigoplus_W \sum F(w_j^*) \otimes w_k \otimes u_i \otimes u_i^* \otimes F(x_l) \otimes \widetilde{\xi}(w_k^*) \otimes F(v^*) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\otimes v\langle x_l^*, \xi^*(w_j \otimes u)\rangle
\end{aligned}$$

($F(x_l)$ and $\widetilde{\xi}(w_k^*)$ being coupled with $F(v^*)$ and $u_i^* \otimes v$ respectively)

$$\mapsto \bigoplus_W \sum F(w_j^*) \otimes w_k \otimes u_i \langle u_i^* \otimes v, \widetilde{\xi}(w_k^*)\rangle \langle v^*, \xi^*(w_j \otimes u)\rangle$$

(letting $X = V$)

$$
\begin{aligned}
&= \bigoplus_W \sum F(w_j^*) \otimes \xi v \langle v^*, \xi^*(w_j \otimes u) \rangle \\
&= \bigoplus_W \sum_\xi F(\widetilde{\xi^*}(u \otimes v^*)) \otimes \xi v \\
&= \bigoplus_W \sum_{\eta: UV^* \to W^*} \frac{d(V)}{d(W)} F(\eta(u \otimes v^*)) \otimes \widetilde{\eta}^* v,
\end{aligned}
$$

where the family $\{W^* \xrightarrow{\eta^*} U \otimes V^* \xrightarrow{\eta} W^*\}$ denotes an irreducible decomposition of $U \otimes V^*$.

Comparing the last expression with the definition of trivialization isomorphism $F(U) \otimes \mathbb{A} \to \mathbb{A} \otimes U$, we see that

$$
\mathbb{A}^* = \oplus_V F(V^*) \otimes V \xrightarrow{\oplus_V d(V) 1} \oplus_V F(V^*) \otimes V = \mathbb{A}
$$

is $\mathcal{A}$-linear. $\square$

The object $A = \mathbb{A}^* \otimes_{\mathcal{A}} \mathbb{A}$ in $\mathcal{T}$ is a Frobenius algebra by the rigidity of ${}_{\mathcal{A}}\mathbb{A}$: the multiplication morphism is given by

$$
A \otimes A = \mathbb{A}^* \otimes_{\mathcal{A}} \mathbb{A} \otimes \mathbb{A}^* \otimes_{\mathcal{A}} \mathbb{A} \xrightarrow{1 \otimes \epsilon \otimes 1} \mathbb{A}^* \otimes_{\mathcal{A}} \mathbb{A} \otimes_{\mathcal{A}} \mathbb{A} = \mathbb{A}^* \otimes_{\mathcal{A}} \mathbb{A} = A.
$$

By the natural identification $\mathbb{A}^* \otimes_{\mathcal{A}} \mathbb{A} = \mathbb{A}$, this can be rewritten as

$$
\mathbb{A} \otimes \mathbb{A} \to \mathbb{A} \otimes \mathbb{A}^* \xrightarrow{\epsilon} \mathbb{A},
$$

where ϵ denotes a rigidity pair for ${}_{\mathcal{A}}\mathbb{A}$ (ϵ being $\mathcal{A}$-$\mathcal{A}$ linear) and is defined by the formula after Corollary 6.2:

$$
\begin{aligned}
\mathbb{A} \otimes \mathbb{A}^* &\xrightarrow{\oplus_X d(X) 1 \otimes \epsilon_X \otimes 1} \oplus_X \mathbb{A} \otimes X \otimes X^* \otimes \mathbb{A}^* \\
&\longrightarrow \oplus_X F(X) \otimes \mathbb{A} \otimes \mathbb{A}^* \otimes X^* \xrightarrow{\oplus_X 1 \otimes \epsilon_{\mathbb{A}} \otimes 1} \oplus_X F(X) \otimes X^* = \mathbb{A}
\end{aligned}
$$

with ϵ_X the ordinary vector space pairing and $\epsilon_{\mathbb{A}}$ the rigidity pairing for the object $\mathbb{A}$ in $\mathcal{T}$ (with the trivial action).

Since ${}_{\mathcal{A}}\mathbb{A}_{\mathcal{A}}$ is identified with ${}_{\mathcal{A}}\mathbb{A}^*_{\mathcal{A}}$ by multiplying the weight $\{d(V)^{-1}\}_V$, the multiplication morphism $\mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ is given by the following process on vectors:

$$
\begin{aligned}
&F(v) \otimes v^* \otimes F(w) \otimes w^* \\
&\quad \mapsto d(W)^{-1} F(v) \otimes v^* \otimes F(w) \otimes w^* \\
&\quad \mapsto \bigoplus_X \frac{d(X)}{d(W)} \sum_i F(v) \otimes v^* \otimes x_i \otimes x_i^* \otimes F(w) \otimes w^* \\
&\quad \mapsto \bigoplus_{X,U} \frac{d(X)}{d(W)} \sum_{\xi,i,j} F(v) \otimes \xi \otimes \langle u_j, \xi^*(v^* \otimes x_i) \rangle u_j^* \otimes x_i^* \otimes F(w) \otimes w^* \\
&\quad \mapsto \bigoplus_{X,U} \frac{d(X)}{d(W)} \sum_{\xi,i,j} F(\widetilde{\xi} v) \otimes \langle u_j, \xi^*(v^* \otimes x_i) \rangle u_j^* \otimes x_i^* \otimes F(w) \otimes w^*
\end{aligned}
$$

(letting $U = W^*$ and $u_j = w_j^*$ for the pairing)

$$\begin{aligned}
&\mapsto \bigoplus_X \frac{d(X)}{d(W)} \sum F(\langle\widetilde{\xi}v\rangle_w) \otimes \langle w_j^*, \xi^*(v^* \otimes x_i)\rangle \langle w_j, w^*\rangle x_i^* \\
&= \bigoplus_X \frac{d(X)}{d(W)} \sum_{\xi,i} F(\xi'(v \otimes w)) \otimes \langle w^*, \xi^*(v^* \otimes x_i)\rangle x_i^* \\
&= \bigoplus_X \frac{d(X)}{d(W)} \sum_{\xi} F(\xi'(v \otimes w)) \otimes {}^t(\xi^*)'(w^* \otimes v^*),
\end{aligned}$$

where the family

$$\left\{U^* \xrightarrow{\ \xi\ } V^* \otimes X \xrightarrow{\ \xi^*\ } U^*\right\}$$

denotes an irreducible decomposition of $V^* \otimes X$ with $\widetilde{\xi}$ and ξ' Frobenius transforms of ξ.

Since

$$\frac{d(X)}{d(W)} \left(X \xrightarrow{(\xi^*)'} V \otimes W \xrightarrow{\ \xi'\ } X\right)$$

gives an irreducible decomposition of $V \otimes W$, we have the following.

Proposition 7.2 *The object $\mathbb{A}$ in $\mathcal{T}$ is an Frobenius algebra by the multiplication morphism*

$$F(v) \otimes v^* \otimes F(w) \otimes w^* \mapsto \bigoplus_U \sum_{\eta: U \to VW} F(\eta^*(v \otimes w)) \otimes {}^t\eta(w^* \otimes v^*)$$

(the family $\left\{U \xrightarrow{\ \eta\ } V \otimes W \xrightarrow{\ \eta^*\ } U\right\}$ being an irreducible decomposition of $V \otimes W$) with the compatible rigidity copairing $\delta_A : \mathbb{A} \otimes \mathbb{A} \to I$ given by the composition

$$\begin{aligned}
\bigoplus_{V,W} F(V) \otimes V^* \otimes F(W) \otimes W^* &\to \bigoplus_V F(V) \otimes V^* \otimes F(V^*) \otimes V \quad (\text{letting } W = V^*) \\
&\xrightarrow{\oplus_V d(V)1} \oplus_V F(V) \otimes V^* \otimes F(V^*) \otimes V \to I,
\end{aligned}$$

where the last morphism is the summation of the canonical pairing

$$F(V) \otimes F(V^*) \otimes V^* \otimes V \to I \otimes \mathbb{C} = I.$$

The associated unit (morphism) is given by the obvious imbedding

$$I \to F(\mathbb{C}) \otimes \mathbb{C} \subset \bigoplus_V F(V) \otimes V^*.$$

Corollary 7.3 *The multiplication morphism $\mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ is associative.*

We have seen so far that Tannaka duals give rise to a special class of Frobenius algebras in a canonical way (a depth two characterization of the class is possible in terms of factorization of Frobenius algebras, see [13]). It is worth pointing out here that a similar computation is carried out in [9, §6] based on the analysis of Hopf algebra strucutres. As can be recognized in the above arguments, our proof is purely categorical with the explicit use of fibre functors.

In what follows, we shall use calligraphic letters, say $\mathcal{A}$, to express Tannaka duals (realized in a tensor category $\mathcal{T}$) with the associated Frobenius algebras denoted by the corresponding boldface letters, say $\mathbb{A}$.

Recall here that Tannaka duals give rise to the bicategory of bimodules, whereas there is a natural notion of bimodule of Frobenius algebras which produces another bicategory.

We shall now construct a monoidal functor Φ, which associate an $\mathbb{A}$-$\mathbb{B}$ bimodule to each $\mathcal{A}$-$\mathcal{B}$ bimodule. For simplicity, consider a left $\mathcal{A}$-bimodule X with the trivialization isomorphisms $\{\phi_V : F(V)\otimes X \to X\otimes V\}$. We choose a representative family $\{V_j\}$ of simple objects in the relevant Tannaka dual and set $\phi_j = \phi_{V_j}$.

The action morphism $\phi : \mathbb{A}\otimes X \to X$ is then introduced by

$$\bigoplus_j \widetilde{\phi}_j : \bigoplus_j F(V_j)\otimes V_j^* \otimes X \to X,$$

where $\widetilde{\phi}_j : F(V_j)\otimes V_j^*\otimes X \to X$ corresponds to ϕ_j under the isomorphism

$$\mathrm{Hom}(F(V_j)\otimes V_j^*\otimes X, X) \cong \mathrm{Hom}(F(V_j)\otimes X, X)\otimes V_j = \mathrm{Hom}(F(V_j)\otimes X, X\otimes V_j).$$

Now the square diagram

$$\begin{array}{ccc} \mathbb{A}\otimes\mathbb{A}\otimes X & \xrightarrow{1\otimes\phi} & \mathbb{A}\otimes X \\ {\scriptstyle \mu\otimes 1}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathbb{A}\otimes X & \xrightarrow[\phi]{} & X \end{array}$$

is commutative if and only if so is the diagram

$$\begin{array}{ccc} \bigoplus_{i,j}\mathrm{Hom}(Y, F(V_i)\otimes F(V_j)\otimes X)\otimes V_i^*\otimes V_j^* & \longrightarrow & \bigoplus_i \mathrm{Hom}(Y, F(V_i)\otimes X)\otimes V_i^* \\ \downarrow & & \downarrow \\ \bigoplus_k \mathrm{Hom}(Y, F(V_k)\otimes X)\otimes V_k^* & \longrightarrow & \mathrm{Hom}(Y, X) \end{array}$$

for any object Y. If we trace the morphisms starting from $f\otimes v_i^*\otimes v_j^*$ for $f : Y \to F(V_i)\otimes F(V_j)\otimes X$ and $v_i^* \in V_i^*$, then the commutativity is reduced to the identity

$$\sum_k \sum_\xi \langle \phi_k(\xi^*\otimes 1_X) f\rangle_{{}^t\xi(v_i^*\otimes v_j^*)} = \langle (\phi_i\otimes 1)(1\otimes\phi_j) f\rangle_{v_i^*\otimes v_j^*},$$

where the family $\{V_k \xrightarrow{\xi} V_i\otimes V_j \xrightarrow{\xi^*} V_k\}$ denotes an irreducible decomposition.

Since the choice of $v_i^* \in V_i^*$ is arbitrary, the above relation is equivalent to

$$\sum_{k,\xi} (1\otimes\xi)\phi_k(\xi^*\otimes 1_X) f = (\phi_i\otimes 1)(1\otimes\phi_j) f$$

for any f or simply

$$\sum_{k,\xi} (1\otimes\xi)\phi_k(\xi^*\otimes 1_X) = (\phi_i\otimes 1)(1\otimes\phi_j),$$

which is exactly the $\mathcal{A}$-module property of X, i.e., the commutativity of the diagram

$$\begin{array}{ccc} F(V_i)\otimes F(V_j)\otimes X & \longrightarrow & F(V_i)\otimes X\otimes V_j \\ \downarrow & & \downarrow \\ \bigoplus_k F(V_k)\otimes X\otimes \begin{bmatrix} V_k \\ V_i\, V_j \end{bmatrix} & \longrightarrow & \bigoplus_k X\otimes V_k\otimes \begin{bmatrix} V_k \\ V_i\, V_j \end{bmatrix} = X\otimes V_i\otimes V_j \end{array}$$

The unitality for the $\mathbb{A}$-action, which says that

$$X = I \otimes X \to A \otimes X \to X$$

is the identity, is reduced to that of the $\mathcal{A}$-action on X.

By summarizing the arguments so far, we have associated a left $\mathbb{A}$-module ${}_{\mathbb{A}}X$ to each $\mathcal{A}$-module ${}_{\mathcal{A}}X$ with the common base object X in $\mathcal{T}$. Moreover, given another ${}_{\mathcal{A}}Y$ with the associated ${}_{\mathbb{A}}Y$, we have the equality

$$\mathrm{Hom}({}_{\mathcal{A}}X, {}_{\mathcal{A}}Y) = \mathrm{Hom}({}_{\mathbb{A}}X, {}_{\mathbb{A}}Y)$$

as subsets of $\mathrm{Hom}(X, Y)$ from our construction.

Thus the correspondance ${}_{\mathcal{A}}X \mapsto {}_{\mathbb{A}}X$ defines a fully faithful functor $\Phi : {}_{\mathcal{A}}\mathcal{T} \to {}_{\mathbb{A}}\mathcal{T}$.

We shall now identify the tensor products. Given a right $\mathcal{A}$-module ${}_{\mathcal{A}}X$ and a left $\mathcal{A}$-module ${}_{\mathcal{A}}Y$ in $\mathcal{T}$ with the trivialization isomorphisms $\phi_V : X \otimes F(V) \to V \otimes X$ and $\psi_V : F(V) \otimes Y \to Y \otimes V$, denote the associated action morphisms of $\mathbb{A}$ by $\phi : X \otimes \mathbb{A} \to X$ and $\psi : Y \otimes \mathbb{A} \to Y$ respectively.

Given a basis $\{v_i\}$ of V, we introduce morphisms $\phi_{V,i} : X \otimes F(V) \to X$ by the relation

$$\phi_V = \sum_i v_i \otimes \phi_{V,i}$$

in the vector space $\mathrm{Hom}(X \otimes F(V), V \otimes X) = V \otimes \mathrm{Hom}(F(V) \otimes X, X)$. Likewise, we define morphisms $\psi_{V,i} : F(V) \otimes Y \to Y$ so that

$$\psi_V = \sum_i \psi_{V,i} \otimes v_i.$$

From the definition of ϕ, $\phi \otimes 1_Y$ is identified with

$$\bigoplus_V \sum_i v_i \otimes \phi_{V,i} \quad \in \quad \bigoplus_V V \otimes \mathrm{Hom}(X \otimes F(V) \otimes Y, X \otimes Y)$$

in the vector space

$$\begin{aligned}\mathrm{Hom}(X \otimes \mathbb{A} \otimes Y, X \otimes Y) &= \bigoplus_V \mathrm{Hom}(X \otimes F(V) \otimes V^* \otimes Y, X \otimes Y) \\ &= \bigoplus_V V \otimes \mathrm{Hom}(X \otimes F(V) \otimes Y, X \otimes Y).\end{aligned}$$

Similarly we have the expression

$$1_X \otimes \psi = \bigoplus_V \sum_i v_i \otimes \psi_{V,i} \quad \text{in} \quad \bigoplus_V V \otimes \mathrm{Hom}(X \otimes F(V) \otimes Y, X \otimes Y).$$

Now the idempotent $p \in \mathrm{End}(X \otimes Y)$ producing the relative tensor product $X \otimes_{\mathbb{A}} Y$ is given by

$$\sum_V \frac{d_{F(V)}}{d_{\mathbb{A}}} \sum_i (\phi_{V,i} \otimes 1_Y)(1_X \otimes \psi^*_{V,i})$$

from the definition of p and the formula for δ_A. Here we denote by $\{\psi^*_{V,i} : Y \to F(V) \otimes Y\}$ the cosystem of $\{\psi_{V,i}\}_i$:

$$\psi_{V,i}\psi^*_{V,j} = \delta_{ij} 1_Y, \qquad \sum_i \psi^*_{V,i}\psi_{V,i} = 1_{F(V)\otimes Y}$$

and $d_{F(V)}$, $d_{\mathbb{A}}$ are quantum dimensions of the objects $F(V)$, $\mathbb{A}$ respectively.

We next derive an explicit formula for the idempotent $\pi(e)$ which is used to define $X \otimes_{\mathcal{A}} Y$. Recall here that π is an algebra homomorphism of the dual Hopf algebra H^* into $\mathrm{End}(X \otimes Y)$ and $e \in H^*$ denotes the counit functional of H.

By using the explicit definition of π in [13], we see that

$$\pi(e) = \sum_V \frac{\dim(V)}{\dim(H)} \sum_i (\phi_{V,i} \otimes 1_Y)(1_X \otimes \psi^*_{V,i}),$$

which is exactly the idempotent p because of $d_{F(V)} = \dim(V)$ and $d_{\mathbb{A}} = \dim(H)$.

Proposition 7.4 *The fully faithful functor $\Phi : {}_{\mathcal{A}}\mathcal{T}_{\mathcal{B}} \to {}_{\mathbb{A}}\mathcal{T}_{\mathbb{A}}$ is monoidal by the equality $X \otimes_{\mathcal{B}} Y = X \otimes_{\mathbb{B}} Y$ in $\mathcal{T}$.*

Proposition 7.5 *The monoidal functor $\Phi : {}_{\mathcal{A}}\mathcal{T}_{\mathcal{B}} \to {}_{\mathbb{A}}\mathcal{T}_{\mathbb{B}}$ is an equivalence of categories, i.e., any $\mathbb{A}$-$\mathbb{B}$ bimodule in $\mathcal{T}$ is isomorphic to $\Phi({}_{\mathcal{A}}X_{\mathcal{B}})$ with ${}_{\mathcal{A}}X_{\mathcal{B}}$ an $\mathcal{A}$-$\mathcal{B}$ bimodule in $\mathcal{T}$.*

Proof Let ${}_{\mathbb{A}}X_{\mathbb{B}}$ be an $\mathbb{A}$-$\mathbb{B}$ bimodule in $\mathcal{T}$. Since the $\mathbb{A}$-$\mathbb{B}$ bimodule ${}_{\mathbb{A}}\mathbb{A}\otimes X \otimes \mathbb{B}_{\mathbb{B}}$ is isomorphic to $\Phi({}_{\mathcal{A}}\mathbb{A} \otimes X \otimes \mathbb{B}_{\mathcal{B}})$ and since the functor is fully faithful, we can find an idempotent $p \in \mathrm{End}({}_{\mathcal{A}}\mathbb{A} \otimes X \otimes \mathbb{B}_{\mathcal{B}})$ such that $\Phi(p)$ induces the relative tensor product $\mathbb{A} \otimes_{\mathbb{A}} X \otimes_{\mathbb{B}} \otimes \mathbb{B}$. Thus ${}_{\mathbb{A}}X_{\mathbb{B}}$ is isomorphic to $\Phi({}_{\mathcal{A}}p(\mathbb{A} \otimes X \otimes \mathbb{B})_{\mathcal{B}})$. □

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