

EXERCISES ON ADJUNCTIONS

PETER J. HAINE

Notation. We denote an adjunction by an opposing pair of arrows: $F : C \rightleftarrows D : G$. In this notation, F will always be the left adjoint and G the right adjoint.

Exercise 1. Let $F : C \rightleftarrows D : G$ be an adjunction. Prove that the right adjoint G is fully faithful if and only if the counit $\varepsilon : FG \Rightarrow \text{id}_D$ is a natural isomorphism. Notice that in this case the left adjoint F is essentially surjective on objects. Dually, F is fully faithful if and only if the unit $\eta : \text{id}_C \Rightarrow GF$ is a natural isomorphism.

Exercise 2. Let C and D be categories and suppose that we are given adjunctions

$$\begin{array}{ccc}
 & L & \\
 & \curvearrowright & \\
 C & \xrightarrow{U} & D \\
 & \curvearrowleft & \\
 & R &
 \end{array}$$

Prove that L is fully faithful if and only if R is fully faithful.