APPENDIX 3: AN OVERVIEW OF CHOW GROUPS

We review in this appendix some basic definitions and results that we need about Chow groups. For details and proofs we refer to [Ful98]. In particular, we discuss the action of correspondences on cohomology, as well as the specialization map for families of cycles.

1. Operations on Chow groups I: push-forward and pull-back

In the first four sections, unless explicitly mentioned otherwise, we work over a fixed field k. All schemes are assumed separated and of finite type over k.

Definition 1.1. For every scheme X and every nonnegative integer p, the *group of p-cycles* $Z_p(X)$ is the free Abelian group on the set of all closed, irreducible, p-dimensional subsets of X. Given such a subset V of X, we denote by $[V]$ the corresponding element of $Z_p(X)$. We put

$$Z_*(X) := \bigoplus_{p \geq 0} Z_p(X).$$

A cycle $\sum_i n_i[V_i]$ is *effective* if all n_i are non-negative. We say that a cycle $\sum_i n_i[V_i]$ is supported on a closed subscheme Y if all V_i are subsets of Y.

Example 1.2. If Y is a closed subscheme of X, of pure dimension p, then we define

$$[Y] := \sum_W \ell(\mathcal{O}_{Y,W})[W] \in Z_p(X),$$

where the sum is over the irreducible components of Y.

Example 1.3. Suppose that Y is an integral subscheme of X, of dimension $p+1$. If φ is a nonzero rational function on Y, then we put

$$\mathrm{div}(\varphi) = \mathrm{div}_Y(\varphi) := \sum_W \mathrm{ord}_W(\varphi)[W],$$

where the sum is over the irreducible closed subsets W of Y of codimension 1, and where $\mathrm{ord}_W(\varphi)$ is defined as follows. We consider the local ring $R_W := \mathcal{O}_{Y,W}$ and write $\varphi = \frac{a}{b}$, with $a, b \in R_W$; in this case

$$\mathrm{ord}_W(\varphi) = \ell\big(R_W/(a)\big) - \ell\big(R_W/(b)\big)$$

(if R_W is a DVR, then ord_W is the valuation associated to R_W). It is easy to see that $\mathrm{ord}_W(\varphi)$ is independent of the choice of a and b, and there are only finitely many W such that $\mathrm{ord}_W(\varphi) \neq 0$. Moreover, we have $\mathrm{ord}_W(\varphi\psi) = \mathrm{ord}_W(\varphi) + \mathrm{ord}_W(\psi)$ for all nonzero rational functions φ and ψ on V.

Definition 1.4. Two p-cycles are *rationally equivalent* if their difference lies in the subgroup generated by the p-cycles of the form $\mathrm{div}(\varphi)$, where φ is a nonzero rational function on a $(p+1)$-dimensional integral subscheme of X. We write $\sim_{\mathrm{rat}}$ for the rational equivalence relation. The quotient of $Z_p(X)$ by the subgroup of p-cycles rationally equivalent to 0 is the pth *Chow group* $\mathrm{CH}_p(X)$. We put

$$\mathrm{CH}_*(X) := \bigoplus_{p \geq 0} \mathrm{CH}_p(X).$$

Example 1.5. If $\dim(X) = p$, then $\mathrm{CH}_p(X) = Z_p(X)$ is the free Abelian group on the set of irreducible components of X of maximal dimension.

Example 1.6. If X is a normal n-dimensional variety, then $Z_{n-1}(X)$ is the group of Weil divisors on X and $\mathrm{CH}_{n-1}(X)$ is the Class group of X.

Remark 1.7. It is good to keep in mind that Chow groups have similarities with the *homology groups* of a complex algebraic variety (see Remark 3.7 for a precise comparison). However, they are much more subtle: for example, even the group $\mathrm{CH}_0(X)$ contains a lot of information.

The first operation on Chow groups that we discuss is the *proper push-forward*. Let $f\colon X \to Y$ be a proper morphism. We first define this at the level of cycles. Given an irreducible, closed subset V of X, of dimension p, we define $f_*([V])$ to be $\deg\big(V/f(V)\big)[f(V)] \in Z_p(Y)$ if $\dim(V) = \dim\big(f(V)\big)$, and $f_*([V]) = 0$, otherwise. We extend this by linearity to a graded homomorphism $f_*\colon Z_p(X) \to Z_p(Y)$, which then induces $f_*\colon Z_*(X) \to Z_*(Y)$.

One can show (see [Ful98, Proposition 1.4]) that if φ is a nonzero rational function on an integral subscheme V of X, then $f_*\big(\mathrm{div}(\varphi)\big) = 0$ if $\dim\big(f(V)\big) < \dim(V)$, and

$$f_*\big(\mathrm{div}(\varphi)\big) = \mathrm{div}\big(\mathrm{Norm}(\varphi)\big)$$

otherwise, where the norm corresponds to the finite field extension $k(V)/k\big(f(V)\big)$. As a consequence, we see that we get an induced homomorphism $f_*\colon \mathrm{CH}_p(X) \to \mathrm{CH}_p(Y)$.

It is straightforward to check that if $g\colon Y \to T$ is another proper morphism, then $(g \circ f)_* = g_* \circ f_*$ as a map $Z_p(X) \to Z_p(T)$. We deduce that the same equality holds between the corresponding morphisms $\mathrm{CH}_p(X) \to \mathrm{CH}_p(T)$.

Example 1.8. If X is a complete scheme over k, then by applying the above definition to the structural morphism $X \to \mathrm{Spec}(k)$, we obtain the map

$$\deg\colon \mathrm{CH}_0(X) \to \mathrm{CH}_0\big(\mathrm{Spec}(k)\big) \simeq \mathbf{Z}.$$

It follows from definition that this maps $\sum_i n_i[q_i]$ to $\sum_i n_i \cdot \deg\big(k(q_i)/k\big)$, where the q_i are closed points in X.

We now turn to the *flat pull-back*. Suppose that $f\colon X \to Y$ is a flat morphism of relative dimension d. The condition on relative dimension means that all fibers of f have pure dimension d (note that if $f\colon X \to Y$ is a flat morphism, with X pure-dimensional and Y irreducible, then f has relative dimension equal to $\dim(X) - \dim(Y)$). This condition implies that if V is a closed subscheme of Y of pure dimension p, then

$f^{-1}(V)$ is a closed subscheme of X of pure dimension $p+d$. One defines a homomorphism $f^*\colon Z_p(Y) \to Z_{p+d}(X)$ given by $f^*([V]) = [f^{-1}(V)]$ for every integral subscheme V of Y. More generally, it is not hard to see that the same formula holds if V is any closed subscheme of Y of pure dimension p. This immediately implies that if $g\colon W \to X$ is a flat morphism of relative dimension e, then $(f \circ g)^* = g_* \circ f_*$ as homomorphisms $Z_p(Y) \to Z_{p+d+e}(W)$.

One can show (see [Ful98, Theorem 1.7]) that if $f\colon X \to Y$ is flat of relative dimension d, then we obtain induced homomorphisms

$$f^*\colon \mathrm{CH}_p(Y) \to \mathrm{CH}_{p+d}(X)$$

and by taking the direct sum of these we obtain $f^*\colon \mathrm{CH}_*(Y) \to \mathrm{CH}_*(X)$.

An important special case of this construction is given by the restriction to an open subset.

Example 1.9. If Y is a closed subscheme of X, with complement U, and $i\colon Y \hookrightarrow X$ and $j\colon U \to X$ are the inclusions, it follows directly from definitions that for every p, we have an exact sequence

$$Z_p(Y) \xrightarrow{i_*} Z_p(X) \xrightarrow{j^*} Z_p(U) \longrightarrow 0.$$

It is straightforward to deduce that we obtain an exact sequence

$$\mathrm{CH}_p(Y) \xrightarrow{i_*} \mathrm{CH}_p(X) \xrightarrow{j^*} \mathrm{CH}_p(U) \longrightarrow 0.$$

Example 1.10. For every X and every positive integer n, the projection $\pi\colon X \times \mathbf{A}^n \to X$ induces a surjective map

$$\pi^*\colon \mathrm{CH}_{p-n}(X) \to \mathrm{CH}_p(X \times \mathbf{A}^n) \quad \text{for all} \quad p.$$

Indeed, arguing by induction on n, we see that it is enough to treat the case $n = 1$. Moreover, using the exact sequence in Example 1.9, it is easy to see that if the assertion holds for a closed subscheme Y of X and for $X \smallsetminus Y$, then it also holds for X. Using this and induction on $\dim(X)$, it follows that we may assume that X is an affine variety.

Consider a p-dimensional subvariety V of $X \times \mathbf{A}^1$ and let W be the closure of $\pi(V)$ in X. If $\dim(W) = p - 1$, then it is clear that $V = \pi^{-1}(W)$, hence $[V]$ lies in the image of π^*. Suppose now that $\dim(W) = p$, hence V is a codimension 1 subvariety of $W \times \mathbf{A}^1$. Let $R = \mathcal{O}(W)$ and $K = \mathrm{Frac}(R)$. If P is the prime ideal in $R[t]$ corresponding to V, then $P \cap R = \{0\}$, hence $P \cdot K[t]$ is a prime ideal in $K[t]$ that can be written as $(f) \cdot K[t]$ for some nonzero $f \in R[t]$. In this case we have

$$\mathrm{div}(f) - [V] = \sum_{i=1}^r n_i[\pi^{-1}(W_i)]$$

for some $(p-1)$-dimensional subvarieties $W_1, \ldots, W_r$ of W and some nonnegative integers $n_1, \ldots, n_r$, which implies that $[V]$ lies in the image of π^*. This completes the proof of our assertion.

In particular, by taking X to be a point, we see that $\mathrm{CH}_p(\mathbf{A}^n) = 0$ for all $p < n$; of course, we have $\mathrm{CH}_n(\mathbf{A}^n) \simeq \mathbf{Z}$.

Example 1.11. Let us compute the Chow groups of $\mathbf{P}^n$. For every $n \geq 1$, we have a closed embedding $i \colon \mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^n$, with $\mathbf{P}^n \smallsetminus \mathbf{P}^{n-1} \simeq \mathbf{A}^n$. It follows from Examples 1.9 and 1.10 that the induced map

$$i_* \colon \mathrm{CH}_q(\mathbf{P}^{n-1}) \to \mathrm{CH}_q(\mathbf{P}^n)$$

is surjective for all $q \neq n$. Arguing by induction on n, we deduce that for every $q \leq n$, if L_q is a linear subspace of $\mathbf{P}^n$ of dimension q, then $[L_q]$ generates $\mathrm{CH}_q(\mathbf{P}^n)$. In fact, it is a free generator. This is clear when $q = n$ and it is well-known when $q = n - 1$. Suppose now that $q \leq n - 2$ and

$$d[L_q] = \sum_{i=1}^{r} n_i \cdot \mathrm{div}(\varphi_i),$$

for suitable subvarieties V_i of $\mathbf{P}^n$ of dimension $q + 1$ and nonzero rational functions φ_i on V_i. Let Y be the union of the V_i and consider a linear subspace W of $\mathbf{P}^n$ of dimension $n - q - 2$ such that $W \cap Y = \emptyset$. In this case, the projection with center W induces a proper morphism $f \colon Y \to \mathbf{P}^{q+1}$. As we have discussed, since $d[L_q] \sim_{\mathrm{rat}} 0$ on Y, we have $f_*(d[L_q]) \sim_{\mathrm{rat}} 0$ on $\mathbf{P}^n$. However, we have $f_*(d[L_q]) = d[H]$, where H is a hyperplane in $\mathbf{P}^{q+1}$. This gives $d = 0$, proving our assertion.

For a more general result, describing the Chow groups of a projective bundle on X in terms of the Chow groups of X, see Proposition 2.1 below.

Example 1.12. Let $f \colon X \to Y$ be a morphism of schemes, with Y integral, of dimension d. If we consider the generic fiber X_η as a scheme over the function field $k(Y)$ of Y, then it follows easily from definitions that for every p, we have

$$A_p(X_\eta) \simeq \varinjlim_U A_{p+d}(U),$$

where U runs over the open subsets of X, and where for $U \subseteq V$, the map $A_{p+d}(V) \to A_{p+d}(U)$ is the restriction.

Example 1.13. If $f \colon X \to Y$ is a finite flat morphism of degree d (that is, $f_*(\mathcal{O}_X)$ is a locally free $\mathcal{O}_Y$-module of rank d), then

$$f_*\big(f^*(\alpha)\big) = d\alpha \quad \text{for every} \quad \alpha \in \mathrm{CH}_*(Y).$$

The two operations that we discussed so far are related by the following compatibility property: given a Cartesian diagram

$$\begin{array}{ccc} W & \overset{v}{\longrightarrow} & X \\ \scriptstyle g \downarrow & & \downarrow \scriptstyle f \\ Z & \underset{u}{\longrightarrow} & X, \end{array}$$

with f proper and u flat, of relative dimension d, we have

(1) $\qquad\qquad u^* \circ f_* = g_* \circ v^* \quad \text{as morphisms} \quad \mathrm{CH}_p(X) \to \mathrm{CH}_{p+d}(Z).$

For a proof, see [Ful98, Proposition 1.7].

Remark 1.14. If K/k is a field extension, then for every scheme X over k, we have a homomorphism

$$Z_p(X) \to Z_p(X_K), \quad [V] \to [V_K],$$

where for a scheme Y over k, we put $Y_K = Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$. It is easy to see that this induces a homomorphism $\mathrm{CH}_p(X) \to \mathrm{CH}_p(X_K)$. If the field extension is finite, then this map can be interpreted as the flat pull-back corresponding to the finite flat morphism $X_K \to X$.

It is easy to see, using the definitions, that if $\overline{k}$ is an algebraic closure of k, then we have an isomorphism

$$\mathrm{CH}_p(X_{\overline{k}}) \simeq \varinjlim_{k'} \mathrm{CH}_p(X_{k'}),$$

where the direct limit is over the subextensions of $\overline{k}/k$ of finite type over k.

Remark 1.15. Given two schemes X and Y, one can define an *exterior product*

$$Z_p(X) \otimes_{\mathbf{Z}} Z_q(Y) \to Z_{p+q}(X \times Y)$$

that maps $[V] \otimes [W]$ to $[V \times W]$. It is straightforward to see that this induces a homomorphism

$$\mathrm{CH}_p(X) \otimes \mathrm{CH}_q(Y) \to \mathrm{CH}_{p+q}(X \times Y), \quad \alpha \otimes \beta \to \alpha \times \beta.$$

This operation satisfies obvious compatibilities with proper push-forward and flat pull-back.

2. Operations on Chow groups II: Chern classes

This section is not really required for the applications to rationality questions, but we include it for completeness. For details and proofs, we refer to [Ful98, Chapters 2,3]. Given a locally free sheaf $\mathcal{E}$ of rank r on X, the Chern classes of $\mathcal{E}$ are defined as homomorphisms

$$\mathrm{CH}_*(X) \to \mathrm{CH}_*(X), \quad \alpha \to c_i(\mathcal{E}) \cap \alpha$$

for $1 \leq i \leq r$, mapping each $\mathrm{CH}_p(X)$ to $\mathrm{CH}_{p-i}(X)$ (the notation is justified by the analogy with the topological setting, where Chern classes live in cohomology and act on homology via cap product).

We begin by discussing the case of a line bundle L. In this case, the Chern class map

$$\mathrm{CH}_p(X) \to \mathrm{CH}_{p-1}(X), \quad \alpha \to c^1(L) \cap \alpha$$

is a group homomorphism characterized by the following properties:

i) If $L = \mathcal{O}_X(D)$, for an effective Cartier divisor D, and $V \subseteq X$ is an irreducible, closed subset of dimension p, such that V is not contained in the support of D, then
$$c^1(L) \cap [V] = [D \cap V] \in \mathrm{CH}_{p-1}(X).$$

ii) If $L_1, L_2 \in \mathrm{Pic}(X)$, then
$$c^1(L_1 \otimes L_2) \cap \alpha = c^1(L_1) \cap \alpha + c^1(L_2) \cap \alpha.$$

iii) If $f\colon X \to Y$ is a proper morphism, then for every $L \in \operatorname{Pic}(Y)$ and every $\alpha \in \operatorname{CH}_p(X)$, we have

$$f_*\big(c^1(f^*(L)) \cap \alpha\big) = c^1(L) \cap f_*(\alpha).$$

iv) If $f\colon X \to Y$ is flat, of relative dimension n, then for every $L \in \operatorname{Pic}(Y)$ and every $\alpha \in \operatorname{CH}_p(Y)$, we have

$$f^*\big(c^1(L) \cap \alpha\big) = c^1(f^*(L)) \cap f^*(\alpha).$$

v) If $L, L' \in \operatorname{Pic}(X)$, then for every $\alpha \in \operatorname{CH}_p(X)$, we have

$$c^1(L) \cap \big(c^1(L') \cap \alpha\big) = c^1(L') \cap \big(c^1(L) \cap \alpha\big) \in \operatorname{CH}_{p-2}(X).$$

First Chern classes can be used to describe the Chow groups of a projective bundle. One should compare this formula with that describing the cohomology of a projective bundle in Corollary 4.2 in Appendix 1 (note, however, that the Chow groups behave like the homology, not the cohomology, of a topological space, see Remark 3.7 below).

Proposition 2.1. *If $\mathcal{E}$ is a locally free sheaf on X, of rank r, and $\pi\colon \mathbf{P}(\mathcal{E}) \to X$ is the corresponding projective bundle, then for every p we have an isomorphism*

$$\bigoplus_{i=0}^{r-1} \operatorname{CH}_{p-i}(X) \simeq \operatorname{CH}_p\big(\mathbf{P}(\mathcal{E})\big), \quad (\alpha_0, \ldots, \alpha_{r-1}) \to \sum_{j=0}^{r-1} c^1\big(\mathcal{O}(1)\big)^{r-1-j} \cap \pi^*(\alpha_j).$$

Remark 2.2. It follows from the above proposition that the map

$$\operatorname{CH}_0(X) \to \operatorname{CH}_0\big(\mathbf{P}(\mathcal{E})\big), \quad u \to c_1\big(\mathcal{O}(1)\big)^{r-1} \cap \pi^*(\alpha)$$

is an isomorphism. It is not hard to see that the inverse map is given by π_*.

Given a locally free sheaf $\mathcal{E}$ on X, of rank r, it follows from the description of $\operatorname{CH}_*\big(\mathbf{P}(\mathcal{E})\big)$ in the above proposition that there are unique homomorphisms $c_i(\mathcal{E})\colon \operatorname{CH}_p(X) \to \operatorname{CH}_{p-i}(X)$, with $0 \le i \le r$, such that we have the following identity in $\operatorname{CH}_*\big(\mathbf{P}(\mathcal{E})\big)$:

$$c_1\big(\mathcal{O}(1)\big)^r \cap \pi^*(\alpha) = \sum_{i=1}^{r} (-1)^{i-1} c_1\big(\mathcal{O}(1)\big)^{r-i} \cap \pi^*\big(c_i(\mathcal{E}) \cap \alpha\big) = 0 \quad \text{for every} \quad \alpha \in \operatorname{CH}_*(X),$$

where $\pi\colon \mathbf{P}(\mathcal{E}) \to X$ is the projection. It is convenient to make the convention that $c_0(\mathcal{E})$ is the identity map and $c_i(\mathcal{E}) = 0$ for $i > r$.

The fundamental property of Chern classes is the Whitney formula: given an exact sequence of locally free sheaves

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0,$$

we have

$$c_q(\mathcal{E}) = \sum_{i+j=q} c_i(\mathcal{E}') c_j(\mathcal{E}'').$$

In particular, if $\mathcal{E}$ admits a filtration by locally free sheaves such that the successive quotients are line bundles, we compute the Chern classes of $\mathcal{E}$ as symmetric functions of the (first) Chern classes of these quotients. Moreover, by successively using Proposition 2.1 one sees that for every locally free sheaf $\mathcal{E}$, there is a smooth morphism $f\colon Y \to X$, of

certain relative dimension, such that $f^* \colon \mathrm{CH}_*(X) \to \mathrm{CH}_*(Y)$ is injective and such that $f^*(\mathcal{E})$ admits a filtration by locally free sheaves such that the successive quotients are line bundles. This allows proving the basic properties of Chern classes of arbitrary vector bundles by reducing to the case when we have such filtrations. While we do not go into details, we mention that there are various compatibilities between Chern classes and the proper push-forward and flat pull-back operations discussed in the previous section.

3. Operations on Chow groups III: Gysin maps

The more subtle operations on Chow groups are provided by the Gysin maps. Given a Cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow{\scriptstyle f} \\ X & \stackrel{i}{\longrightarrow} & Y, \end{array}$$

in which i is a regular embedding of codimension d (this means that i is a closed immersion, defined locally by a regular sequence of length d), then we have a *refined Gysin map*

$$i^! \colon \mathrm{CH}_*(Y') \to \mathrm{CH}_*(X'), \quad \text{with} \quad i^!\big(\mathrm{CH}_p(Y')\big) \subseteq \mathrm{CH}_{p-d}(X').$$

One easy case is when V is a closed subscheme of Y' such that $V \times_Y X \hookrightarrow V$ is again a regular embedding of codimension d, in which case

$$i^!([V]) = [V \times_Y X].$$

The general definition is more involved (see [Ful98, Chapter 6]). Note that unlike the proper push-forward and flat pull-back, this is not induced by a map at the level of cycles. When $Y' = Y$ and $f = 1_Y$, then this is the *Gysin map* associated to i. In the following remarks we list some basic properties of this definition (for proofs and further properties, see *loc. cit.*).

Remark 3.1. Consider a commutative diagram with Cartesian squares:

$$\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ X' & \stackrel{j}{\longrightarrow} & Y' \\ \downarrow & & \downarrow \\ X & \stackrel{i}{\longrightarrow} & Y. \end{array}$$

If β is proper then we have

$$i^! \circ \beta_* = \alpha_* \circ i^! \colon \mathrm{CH}_p(Y'') \to \mathrm{CH}_{p-d}(X') \quad \text{for all} \quad p \in \mathbf{Z}.$$

Similarly, if β is flat, of relative dimension m, then

$$i^! \circ \beta^* = \alpha^* \circ i^! \colon \mathrm{CH}_p(Y') \to \mathrm{CH}_{p+m-d}(X'') \quad \text{for all} \quad p \in \mathbf{Z}.$$

We also note that if j is a regular embedding of codimension d, too, then

$$i^! = j^! \colon \mathrm{CH}_p(Y'') \to \mathrm{CH}_{p-d}(X'') \quad \text{for all} \quad p \in \mathbf{Z}.$$

Remark 3.2. The construction of refined Gysin maps is functorial in the following sense: given a commutative diagram with Cartesian squares

$$
\begin{array}{ccccc}
X' & \longrightarrow & Y' & \longrightarrow & Z' \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\ i\ } & Y & \xrightarrow{\ j\ } & Z,
\end{array}
$$

in which i and j are regular embeddings of codimensions d and e, respectively (in which case $j \circ i$ is a regular embedding of codimension $d + e$), then

$$(j \circ i)^! = i^! \circ j^! \colon \mathrm{CH}_p(Z') \to \mathrm{CH}_{p-d-e}(X') \quad \text{for all} \quad p \in \mathbf{Z}.$$

Remark 3.3. Refined Gysin maps commute, in the following sense: consider a commutative diagram

$$
\begin{array}{ccccc}
X' & \longrightarrow & Y' & \longrightarrow & Z' \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle j} \\
X & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow & & \downarrow & & \\
U & \xrightarrow{\ i\ } & V, & &
\end{array}
$$

in which i and j are regular embeddings of codimensions d and e, respectively. In this case, we have

$$i^! \circ j^! = j^! \circ i^! \colon \mathrm{CH}_p(Y) \to \mathrm{CH}_{p-d-e}(X') \quad \text{for all} \quad p \in \mathbf{Z}.$$

Remark 3.4. If $i \colon X \hookrightarrow Y$ is a regular closed embedding of codimension d, and $N = N_{X/Y}$ is the normal sheaf (note that this is locally free, of rank d), then we have the *self-intersection formula*

$$i^!\big(i_*(\alpha)\big) = c_d(N) \cap \alpha \quad \text{for all} \quad \alpha \in \mathrm{CH}_*(X).$$

More generally, given a Cartesian diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\ i'\ } & Y' \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\ i\ } & Y,
\end{array}
$$

we have

$$i^! i'_*(\alpha) = c_d\big(g^*(N)\big) \cap \alpha \in \mathrm{CH}_{p-d}(X') \quad \text{for all} \quad \alpha \in \mathrm{CH}_p(X').$$

A related formula is the following: suppose that $\mathcal{E}$ is a locally free sheaf of rank d on Y and $X = Z(s)$, where s is a section of E, such that $i \colon X \hookrightarrow Y$ is a regular embedding of codimension d (this holds, for example, if Y is a Cohen-Macaulay variety and X has codimension d in Y). In this case, we have

$$i_*\big(i^!(\alpha)\big) = c_d(\mathcal{E}) \cap \alpha \quad \text{for all} \quad \alpha \in \mathrm{CH}_*(Y).$$

The Gysin map can be used to define a ring structure on the Chow group $\mathrm{CH}_*(X)$ when X is smooth. Note that if X is a smooth n-dimensional variety, then the diagonal map $\Delta\colon X \hookrightarrow X \times X$ is a regular embedding of codimension d. Given $\alpha \in \mathrm{CH}_p(X)$ and $\beta \in \mathrm{CH}_q(X)$, one defines

$$\alpha \cdot \beta := \Delta^!(\alpha \times \beta) \in \mathrm{CH}_{p+q-n}(X).$$

In this case, it is convenient to put $\mathrm{CH}^p(X) := \mathrm{CH}_{n-p}(X)$, so that the above *intersection product* gives maps

$$\mathrm{CH}^p(X) \otimes_{\mathbf{Z}} \mathrm{CH}^q(X) \to \mathrm{CH}^{p+q}(X).$$

One can show that $\mathrm{CH}^*(X) = \bigoplus_{p=0}^{n} \mathrm{CH}^p(X)$ is a graded, commutative ring, with unit given by $[X] \in \mathrm{CH}^0(X) = \mathrm{CH}_n(X)$; see [Ful98, Chapter 8.3] for the proof.

Remark 3.5. If $\mathcal{E}$ is a locally free sheaf on the smooth variety X, then the maps given by the Chern classes of $\mathcal{E}$ satisfy

$$c_i(\mathcal{E}) \cap \alpha = (c_i(\mathcal{E}) \cap [X]) \cdot \alpha.$$

Because of this, it is often the case that in this setting one defines the ith Chern class of $\mathcal{E}$ as $c^i(\mathcal{E}) \cap [X] \in \mathrm{CH}^i(X)$, with the corresponding Chern class map being given by multiplication with this element of $\mathrm{CH}^i(X)$.

Remark 3.6. One case in which computing the intersection $\alpha \cdot \beta$ is easy is that when α and β are classes of cycles that intersect properly. One says that two cycles $u \in Z_p(X)$ and $v \in Z_q(X)$ *intersect properly* if $u = \sum_i a_i[V_i]$ and $v = \sum_j b_j[W_j]$ such that $\dim(V_i \cap W_j) = p + q - n$ for all i and j. If α and β are the classes of u and v, respectively, in the Chow group of X, then

$$\alpha \cdot \beta = \sum_{i,j} a_i b_j [V_i \cap W_j] \in \mathrm{CH}_{p+q-n}(X).$$

When X is quasi-projective, this gives all products. Indeed, the Moving Lemma (see [Ful98, Chapter 11.4]) says that given any two cycles u and v on X, one can find a cycle u' rationally equivalent to u such that u' and v intersect properly. This approach was used classically to define multiplication on $\mathrm{CH}^*(X)$. The downside is that it is not clear that the definition is independent of the choice of u'. The approach in [Ful98] circumvents this issue by defining Gysin maps via the deformation to the normal cone.

Suppose now that we only consider schemes that admit closed embeddings in smooth varieties over the ground field (for example, all smooth varieties and all quasi-projective varieties satisfy this condition). In particular, every morphism $f\colon X \to Y$ admits a factorization $X \overset{i}{\hookrightarrow} W \overset{p}{\longrightarrow} Y$, with p smooth and i a closed embedding. The morphism f is a *complete intersection morphism* of relative dimension d if it has such a factorization with p smooth, of some relative dimension n, and i a regular embedding of codimension $n - d$. In this case, given any Cartesian diagram

$$
\begin{array}{ccccc}
X' & \longrightarrow & W' & \overset{p'}{\longrightarrow} & Y' \\
\downarrow & & \downarrow & & \downarrow \\
X & \overset{i}{\longrightarrow} & W & \overset{p}{\longrightarrow} & Y,
\end{array}
$$

we define

$$f^! = i^! \circ (p')^* \colon \operatorname{CH}_q(Y') \to \operatorname{CH}_{q+n}(W') \to \operatorname{CH}_{q+d}(X')$$

for every $q \in \mathbf{Z}$. One can show that the definition is independent of the factorization of f. In particular, if f is either a regular embedding or a flat morphism, we recover the definition of the refined Gysin map, respectively, of flat pull-back. Moreover, the definition is functorial: if $g \colon Y \to Z$ is a locally complete intersection morphism of relative dimension e, then $g \circ f$ is a locally complete intersection morphism of relative dimension $d + e$ and we have $f^! \circ g^! = (g \circ f)^!$. For all these properties, see [Ful98, Chapter 6.6].

In particular, this applies to any morphism $f \colon X \to Y$ between smooth varieties, since every such morphism is locally complete intersection, of relative dimension $\dim(X) - \dim(Y)$; indeed, it is enough to consider the factorization $X \hookrightarrow X \times Y \to Y$ by the graph of f. In this case, it is customary to write f^* for $f^!$. We thus obtain homomorphisms

$$f^* \colon \operatorname{CH}^p(Y) \to \operatorname{CH}^p(X)$$

which, when f is flat, agree with the flat pull-back. In general, the induced map $\operatorname{CH}^*(Y) \to \operatorname{CH}^*(X)$ is a ring homomorphism. Moreover, if f is proper, then we have the projection formula

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta) \quad \text{for} \quad \alpha \in \operatorname{CH}^*(Y), \beta \in \operatorname{CH}^*(X).$$

For the proofs of these statements, see [Ful98, Chapter 8.3].

Remark 3.7. If X is a complete algebraic variety over $\mathbf{C}$, then by taking the cohomology class of a subvariety we get a map

$$\operatorname{cl} \colon Z_p(X) \to H_{2p}(X, \mathbf{Z}), \quad \sum_{i=1}^{r} n_i [V_i] \to \sum_{i=1}^{r} n_i \eta_{V_i}.$$

On can show that this induces a *cycle class map*

$$\operatorname{cl} \colon \operatorname{CH}_p(X) \to H_{2p}(X, \mathbf{Z}).$$

This is compatible with push-forward: note that every morphism $f \colon X \to Y$ of complete varieties is proper and we have

$$f_*\big(\operatorname{cl}(\alpha)\big) = \operatorname{cl}\big(f_*(\alpha)\big) \quad \text{for all} \quad \alpha \in \operatorname{CH}_*(X).$$

Similarly, this is compatible with flat pull-back and arbitrary pull-back in the case of morphisms between smooth varieties. For all these facts, and more, see [Ful98, Chapter 19].

More generally, one can define cycle class maps for arbitrary complex varieties

$$\operatorname{CH}_p(X) \to H_{2p}^{\operatorname{BM}}(X),$$

where $H_*^{\operatorname{BM}}(X)$ is the Borel-Moore homology of X. We do not pursue this since we have not discussed Borel-Moore homology.

4. Correspondences

In this section we discuss the action of correspondences on Chow groups. For some of the details, we refer to [Ful98, Chapter 16]. More generally, correspondences act on any cohomology theory H^* that comes with cycle class maps $A^* \to H^*$; we only discuss this for singular cohomology.

We work over a fixed ground field k and assume that all varieties are smooth over k.

Definition 4.1. Given smooth varieties X and Y, a *correspondence from X to Y*, written $\alpha \colon X \vdash Y$, is an element $\alpha \in \mathrm{CH}^*(X \times Y)$.

Given a correspondence $\alpha \colon X \vdash Y$, using the isomorphism $X \times Y \simeq Y \times X$, $(x, y) \to (y, x)$, we obtain from α its *transpose* $\alpha' \colon Y \vdash X$. It is clear that $(\alpha')' = \alpha$.

A more interesting operation on correspondences is the composition. Suppose that X, Y, and Z are smooth varieties, and

$$p_{12} \colon X \times Y \times Z \to X \times Y, \ p_{13} \colon X \times Y \times Z \to X \times Z, \ p_{23} \colon X \times Y \times Z \to Y \times Z$$

are the canonical projections. Note that if Y is complete, then p_{13} is proper. It follows that in this case, if $\alpha \colon X \vdash Y$ and $\beta \colon Y \vdash Z$ are correspondences, then we can define

$$\beta \circ \alpha \colon X \vdash Z, \ \beta \circ \alpha = p_{13*}\big(p_{12}^*(\alpha) \cdot p_{23}^*(\beta)\big).$$

Note that if $\alpha \in \mathrm{CH}^k(X \times Y)$ and $\beta \in \mathrm{CH}^\ell(Y \times Z)$, then $\beta \circ \alpha \in \mathrm{CH}^{k+\ell-d}(X \times Z)$, where $d = \dim(Y)$.

The properties in the following proposition follow easily from the general properties of the operations on Chow groups.

Proposition 4.2. *Let X, Y, Z, and W be smooth varieties.*

    i) *The composition of correspondences is associative, that is, given $\alpha \colon X \vdash Y$, $\beta \colon Y \vdash Z$, and $\gamma \colon Z \vdash W$, if Y and Z are complete, then*

$$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha.$$

    ii) *For every $\alpha \colon X \vdash Y$ and $\beta \colon Y \vdash Z$, if X and Y are complete, we have*

$$(\beta \circ \alpha)' = \alpha' \circ \beta'.$$

Example 4.3. Let $f \colon X \to Y$ be a morphism of smooth varieties and let $\alpha \colon X \vdash Y$ be given by the class of the graph Γ_f of f.

    i) For every correspondence $\beta \colon Y \vdash Z$, if Y is complete, then $\beta \circ \alpha = (f \times 1_Z)^*(\beta)$.
    ii) For every correspondence $\gamma \colon W \vdash X$, if X is complete, then $\alpha \circ \gamma = (1_W \times f)_*(\gamma)$.

Let's check for example the second equality. Applying (1) for the Cartesian diagram

$$W \times X \xrightarrow{\ j\ } W \times X \times Y$$
$$\downarrow \qquad\qquad \downarrow p_{23}$$
$$X \xrightarrow{\ i\ } X \times Y,$$

where $i(x) = \big(x, f(x)\big)$ and $j(w, x) = \big(w, x, f(x)\big)$, we obtain

$$p_{23}^*(\alpha) = p_{23}^* i_*([X]) = j_*([W \times X]).$$

By combining this with the projection formula, and using the fact that $p_{12} \circ j = 1_{W \times X}$, we get

$$p_{12}^*(\gamma) \cdot p_{23}^*(\alpha) = p_{12}^*(\gamma) \cdot j_*([W \times X]) = j_* j^* p_{12}^*(\gamma) = j_*(\gamma).$$

Since $p_{13} \circ j = 1_W \times f$, we conclude

$$\alpha \circ \gamma = p_{13*}\big(p_{12}^*(\gamma) \cdot p_{13}^*(\alpha)\big) = p_{13*} j_*(\gamma) = (1_W \times f)_*(\gamma).$$

The proof of i) is similar.

Remark 4.4. If X is a smooth, complete variety, we get a new ring structure on the Abelian group $\mathrm{CH}^*(X \times X)$, in which multiplication is given by $(\alpha, \beta) \to \alpha \circ \beta$. Note that multiplication is associative by Proposition 4.2i) and it follows from Example 4.3 that $[\Delta_X]$ is the identity element.

Correspondences are interesting since they act on Chow groups, as follows. If X and Y are smooth, complete varieties, and we have a correspondence $\alpha \colon X \vdash Y$, then we define

$$\alpha_* \colon \ \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y), \quad \alpha_*(u) = p_{2*}\big(p_1^*(u) \cdot \alpha\big),$$

where $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ are the two projections. We put $\alpha^* = (\alpha')_* \colon \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$.

Note that by definition, we have $\alpha_*(u) = \alpha \circ u$ if we view $u \in \mathrm{CH}^*(X)$ as $u \colon \mathrm{Spec}(k) \vdash X$. Similarly, we have $\alpha^*(v) = v \circ \alpha$ if we view $v \in \mathrm{CH}^*(Y)$ as $v \colon Y \vdash \mathrm{Spec}(k)$. We deduce from the associativity of the composition of correspondences that if $\beta \colon Y \vdash Z$, then we have

$$(\beta \circ \alpha)_* = \beta_* \circ \alpha_* \colon \ \mathrm{CH}^*(X) \to \mathrm{CH}^*(Z) \quad \text{and}$$

$$(\beta \circ \alpha)^* = \alpha^* \circ \beta^* \colon \ \mathrm{CH}^*(Z) \to \mathrm{CH}^*(X).$$

Example 4.5. The push-forward and pull-back operations associated to morphisms are special cases of actions by correspondences. Indeed, if $f \colon X \to Y$ is a morphism between smooth, complete varieties, and we consider the correspondence $[\Gamma_f] \colon X \vdash Y$, where $\Gamma_f \subseteq X \times Y$ is the graph of f, then it follows from Example 4.3 that

$$[\Gamma_f]_* = f_* \quad \text{and} \quad [\Gamma_f]^* = f^*.$$

The following result illustrates the usefulness of correspondences. We use this in Lecture 7.

Theorem 4.6. *If X and Y are stably birational smooth, complete varieties over a field k, then we have an isomorphism*

$$\mathrm{CH}_0(X) \simeq \mathrm{CH}_0(Y).$$

Before giving the proof of the theorem, we make some preparations.

Lemma 4.7. *Let X be a variety over a field k and $z \in Z_0(X)$ be a 0-cycle with support contained in the regular[1] locus X_{reg} of X. If U is a nonempty open subset of X, then there is $z' \in Z_0(X)$ with support in U such that $z \sim_{\mathrm{rat}} z'$.*

Proof. Of course, we may assume that $z = [p]$ for some regular closed point $p \in X \smallsetminus U$. In particular, we have $\dim(X) \geq 1$. In this case, there is a 1-dimensional subvariety C of X, with $p \in C$ a regular point, and such that $C \cap U \neq \emptyset$. After replacing X by C, we may thus assume that $\dim(X) = 1$. We may also assume that $X = \mathrm{Spec}(R)$ is affine: if a curve is not affine, then it is complete, and by Example 1.9, it is enough to prove the assertion when replacing (X, U) by $(X \smallsetminus \{q\}, U \smallsetminus \{q\})$, for some point $q \in U$. Let $\mathfrak{m}$ be the maximal ideal corresponding to p and $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ the maximal ideals corresponding to the points in $X \smallsetminus U$. By assumption, $R_\mathfrak{m}$ is a DVR, and if

$$\varphi \in \mathfrak{m} \smallsetminus \left((\mathfrak{m}^2 R_\mathfrak{m} \cap R) \cup \mathfrak{m}_1 \cup \ldots \cup \mathfrak{m}_r \right)$$

(note that such an element exists by the prime avoidance lemma), then $\mathrm{div}(\varphi) - [p]$ is an (effective) 0-cycle with support in U. This completes the proof of the lemma. $\square$

Proposition 4.8. *Let X be a smooth, n-dimensional, complete variety over k. If V is a proper closed subset of X and $\alpha \in \mathrm{CH}_n(X \times X)$ is the class of an n-cycle supported on $V \times X$, then the map $\alpha_*\colon \mathrm{CH}_0(X) \to \mathrm{CH}_0(X)$ is the zero map.*

Proof. Let $i\colon V \times X \hookrightarrow X \times X$ be the inclusion. By assumption, we can write $\alpha = i_*(\beta)$ for some $\beta \in \mathrm{CH}_n(V \times X)$.

Since $\alpha \in \mathrm{CH}_n(X \times X)$, we get induced maps $\alpha_*\colon \mathrm{CH}_p(X) \to \mathrm{CH}_p(X)$ for all p. Given any $u \in \mathrm{CH}_0(X)$, it follows from Lemma 4.7 that we can write $u \sim_{\mathrm{rat}} \sum_{i=1}^r n_i [q_i]$ in $\mathrm{CH}_0(X)$, for some points $q_1, \ldots, q_r \in X \smallsetminus V$ and $n_1, \ldots, n_r \in \mathbf{Z}$.

If $\pi_2\colon X \times X \to X$ is the projection onto the second component, then

$$\alpha_*(u) = \sum_{i=1}^r n_i \cdot \left(i_*(\beta) \right)_*([q_i]) = \sum_{i=1}^r n_i \cdot \pi_{2*}\left([\{q_i\} \times X] \cdot i_*(\beta) \right) = 0$$

since $\left(\{q_i\} \times X \right) \cap \left((X \smallsetminus U) \times X \right) = \emptyset$. $\square$

Proof of Theorem 4.6. We need to prove two things. First, we have an isomorphism

$$\mathrm{CH}_0(X) \simeq \mathrm{CH}_0(X \times \mathbf{P}^n) \quad \text{for every} \quad n \geq 1.$$

---

[1]Recall that a point $p \in X$ is regular if the local ring $\mathcal{O}_{X,p}$ is a regular ring. Since we do not assume that the ground field is perfect, this is a weaker condition than the smoothness of X at p.

This is a consequence of the description of the Chow groups of a projective bundle in Proposition 2.1. Second, and this is the interesting statement, if X and Y are birational smooth, complete varieties, then $\mathrm{CH}_0(X) \simeq \mathrm{CH}_0(Y)$.

Let $U \subseteq X$ and $V \subseteq Y$ be open subsets such that we have an isomorphism $\varphi\colon U \to V$. Let $W \subseteq X \times Y$ be the closure of the graph of φ. Note that if $\alpha = [W] \in \mathrm{CH}^n(X \times Y)$, where $n = \dim(X) = \dim(Y)$, then we have $\alpha_*\colon \mathrm{CH}_0(X) \to \mathrm{CH}_0(Y)$ and $\alpha'_*\colon \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$. We will show that these two maps are mutual inverses.

Note first that we have $\alpha'_* \circ \alpha_* = (\alpha' \circ \alpha)_*$. We also have
$$(\alpha' \circ \alpha)|_{U \times X} = \alpha' \circ [\Gamma_\varphi] = (\varphi \times 1_X)^*(\alpha'),$$
where $\Gamma_\varphi \subseteq U \times Y$ is the graph of φ and the second equality follows from Example 4.3. Note that $\varphi \times 1_X$ is an open immersion and by restricting α' to $U \times X$ via $\varphi \times 1_X$, we obtain the graph of the inclusion $U \hookrightarrow X$. We thus conclude that
$$\big(\alpha' \circ \alpha - [\Delta_X]\big)|_{U \times X} = 0.$$
We deduce from Example 1.9 that if $i\colon (X \smallsetminus U) \times X \hookrightarrow X \times X$ is the inclusion, then
$$\alpha' \circ \alpha = [\Delta_X] + i_*(\beta) \quad \text{for some} \quad \beta \in A_n\big((X \smallsetminus U) \times X\big).$$
Since $[\Delta_X]_*$ is the identity on $\mathrm{CH}_0(X)$ and $\big(i_*(\beta)\big)*$ is the zero map by Proposition 4.8, we conclude that $\alpha'_* \circ \alpha_* = 1_{\mathrm{CH}_0(X)}$. By reversing the roles of X and Y, we see that $\alpha_* \circ \alpha'_* = 1_{\mathrm{CH}_0(Y)}$. This concludes the proof of the theorem. $\qquad\square$

Remark 4.9. Note that the isomorphism in Theorem 4.6 is compatible with the two degree maps to $\mathbf{Z}$. First, when $Y = X \times \mathbf{P}^n$, then the isomorphism $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is given by π_*, where $\pi\colon X \times \mathbf{P}^n \to X$ is the projection (see Remark 2.2); in this case the compatibility is clear. Second, when we have an isomorphism φ between open subsets of X and Y, the isomorphism $\mathrm{CH}_0(X) \to \mathrm{CH}_0(Y)$ is given by $[W]_*$, where $W \subseteq X \times Y$ is the closure of the graph of φ. On the other hand, the degree map $\deg_Y\colon \mathrm{CH}_0(Y) \to \mathbf{Z}$ can be identified with $[Y]_*$, where $[Y]\colon Y \vdash \mathrm{Spec}(k)$. Using the definition of a composition of correspondences, we conclude that if $\pi\colon X \times Y \to X$ is the projection onto the first component, then
$$\deg_Y \circ [W]_* = [Y]_* \circ [W]_* = ([Y] \circ [W])_* = \pi_*([W])_* = [X]_* = \deg_X.$$

We only discussed the action of correspondences on Chow groups, but we have similar actions on other cohomology theories. Let us discuss briefly the case of singular cohomology. We consider smooth, complete complex varieties. Recall that for such a variety X, we have a cycle class map
$$\mathrm{cl}\colon \mathrm{CH}^*(X) \to H^*(X, \mathbf{Z}), \quad \text{with} \quad \mathrm{cl}(\alpha) \in H^{2p}(X, \mathbf{Z}) \quad \text{for} \quad \alpha \in \mathrm{CH}^p(X),$$
which is compatible with pull-back and push-forward maps. In particular, given a correspondence $\alpha\colon X \vdash Y$, we obtain an element $\mathrm{cl}(\alpha) \in H^*(X \times Y, \mathbf{Z})$. If $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ are the projections, then we obtain a group homomorphism
$$\alpha_*\colon H^*(X, \mathbf{Z}) \to H^*(Y, \mathbf{Z}), \quad \alpha_*(u) = p_{2*}\big(p_1^*(u) \cap \mathrm{cl}(\alpha)\big).$$
We also have a group homomorphism in the opposite direction
$$\alpha^* = \alpha'_*\colon H^*(Y, \mathbf{Z}) \to H^*(X, \mathbf{Z}).$$

It follows from the compatibility of the class map with push-forward and pull-back that for every $\alpha\colon X \vdash Y$, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{CH}^*(X) & \xrightarrow{\ \mathrm{cl}\ } & H^*(X, \mathbf{Z}) \\
\alpha_* \downarrow & & \alpha_* \downarrow \\
\mathrm{CH}^*(Y) & \xrightarrow{\ \mathrm{cl}\ } & H^*(Y, \mathbf{Z})
\end{array}
$$

and a similar one for α^*. The following also hold, with similar proofs as in the case of actions on Chow groups:

i) If $\alpha\colon X \vdash Y$ and $\beta\colon Y \vdash W$ are correspondences, then

$$(\beta\circ\alpha)_* = \beta_*\circ\alpha_*\colon H^*(X, \mathbf{Z}) \to H^*(W, \mathbf{Z}) \quad \text{and} \quad (\beta\circ\alpha)^* = \alpha^*\circ\beta^*\colon H^*(W, \mathbf{Z}) \to H^*(X, \mathbf{Z}).$$

ii) If $f\colon X \to Y$ is a morphism and $\Gamma_f \subseteq X \times Y$ is the graph of f, then

$$[\Gamma_f]_* = f_*\colon H^*(X, \mathbf{Z}) \to H^*(Y, \mathbf{Z}) \qquad [\Gamma_f]^* = f^*\colon H^*(Y, \mathbf{Z}) \to H^*(X, \mathbf{Z}).$$

5. THE SPECIALIZATION MAP

So far, we always worked over a ground field k. In fact, as explained in [Ful98, Chapter 20.1], the theory works in a more general framework, in which all schemes are separated and of finite type over a fixed regular scheme S. One important difference is that instead of using dimension when defining the group of cycles, we use *dimension over* S: if V is an integral scheme of finite type over S, we put

$$\dim_S(V) = \mathrm{trdeg}_{K(W)}\big(K(V)\big) - \mathrm{codim}_S(W),$$

where W is the closure of the image of V in S and $K(V)$ and $K(W)$ are the function fields of V and W, respectively. Using the fact that regular local rings are universally catenary, one sees that this definition is well-behaved: if W is a closed integral subscheme of V, then

$$\dim_S(V) = \dim_S(W) + \mathrm{codim}_V(W).$$

Moreover, if $f\colon V \to Z$ is a dominant morphism of integral schemes over S, then

$$\dim_S(V) = \dim_S(Z) + \mathrm{trdeg}_{K(Z)}\big(K(V)\big).$$

With this definition, one takes $Z_p(X/S)$ to be free Abelian group on the set of closed irreducible subsets V of X, with $\dim_S(V) = p$. The quotient of $Z_p(X/S)$ by the rational equivalence relation is denoted $\mathrm{CH}_p(X/S)$.

All operations on Chow groups discussed in §1-3 extend to this relative setting. In particular, we have the following version of Example 1.9.

Suppose that S is a regular scheme, $i\colon Z \hookrightarrow S$ is a regular closed immersion of codimension r, with Z regular, and $j\colon S^\circ \hookrightarrow S$ the open immersion, with $S^\circ = S \setminus Z$.

In this case, given a scheme X over S, we have a commutative diagram with Cartesian squares:

$$\begin{array}{ccccc} X_Z & \xrightarrow{i_X} & X & \xleftarrow{j_X} & X^\circ \\ \downarrow & & \downarrow & & \downarrow \\ Z & \xrightarrow{\ i\ } & S & \xleftarrow{\ j\ } & S^\circ \end{array}$$

and we have an exact sequence

$$\mathrm{CH}_p(X_Z/S) \xrightarrow{i_{X*}} \mathrm{CH}_p(X/S) \xrightarrow{j_X^*} \mathrm{CH}_p(X^\circ/S) \longrightarrow 0.$$

Note that we have

$$\mathrm{CH}_p(X_Z/S) = \mathrm{CH}_{p+r}(X_Z/Z) \quad \text{and} \quad \mathrm{CH}_p(X^\circ/S) = \mathrm{CH}_p(X^\circ/S^\circ).$$

Suppose now that we are in the above setting and that the normal bundle $N_{Z/S}$ is trivial. In this case, it follows from Remark 3.4 that $i^! \circ i_{X*} \colon \mathrm{CH}_p(X_Z/S) \to \mathrm{CH}_{p-r}(X_Z/S)$ is the zero map. It follows from the above exact sequence that we have a *specialization map*

$$\sigma \colon \mathrm{CH}_p(X^\circ/S^\circ) \to \mathrm{CH}_p(X_Z/Z)$$

such that for every irreducible closed subset V of X, we have $\sigma([V \cap X^\circ]) = i^!([V])$.

One can show that this specialization map is compatible with proper push-forward, flat pull-back, and pull-back via lci morphisms. Moreover, if X is smooth over S, then the specialization map gives a ring homomorphism $\mathrm{CH}^*(X^\circ/S^\circ) \to \mathrm{CH}^*(X_Z/Z)$.

Example 5.1. An important example of specialization map is that when $S = \mathrm{Spec}(R)$, for a DVR $(R, \mathfrak{m}, k)$. In this case, if we take $Z = \mathrm{Spec}(k)$, then the normal bundle $N_{Z/X}$ is trivial, hence the above discussion applies. Given a scheme X over R, we put $X_K = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$ for the generic fiber of X, where $K = \mathrm{Frac}(R)$, and $X_k = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ for the special fiber of X, and we get the specialization map

$$\sigma \colon \mathrm{CH}_p(X_K) \to \mathrm{CH}_p(X_k).$$

Note that if V is an irreducible closed subset of X_K and $\overline{V}$ is the closure of V in X, then the closed fiber $\overline{V}_k \hookrightarrow \overline{V}$ is a regular embedding of codimension 1, hence $\sigma([V]) = [\overline{V}_k]$. In particular, we see that in this case the specialization map is induced by a map at the level of cycles

$$\sigma \colon Z_p(X_K) \to Z_p(X_k).$$

References

[Ful98]   W. Fulton, Intersection theory, second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 2, Springer-Verlag, Berlin, 1998. 1, 2, 3, 4, 5, 7, 9, 10, 11, 15