2022年度集中講義 (数学特別講義)

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Overview

The main topics of this course are the following:

- Zeta function and cohomology
- Motivic complexes and its $\mathbb{Z}/p^n\mathbb{Z}$-variants
- Galois cohomology and Selmer groups of Bloch-Kato
- Étale cohomology of arithmetic schemes

In this course, a *ring* means a commutative ring with unity, and a *scheme* means a locally ringed space which is everywhere locally isomorphic to some affine scheme $\mathrm{Spec}(R)$ (R is a ring), i.e., a pre-scheme in the sense of [GD1]. A *variety over a field k* means an integral scheme which is separated of finite type over $\mathrm{Spec}(k)$.

Contents

1 Weil cohomology and congruence zeta function

Let F be a field, and let K be a field of characteristic 0, where F is independent of K and $\mathrm{ch}(F)$ is arbitrary. Let $\mathbf{SmProj}(F)$ be the category whose objects are smooth projective geometrically integral varieties over F and whose morphisms are morphisms over F. A *Weil cohomology theory on* $\mathbf{SmProj}(F)$ is a contravariant functor

$$H^*(-) : \mathbf{SmProj}(F)^{\mathrm{op}} \longrightarrow \left\{ \begin{matrix} \text{graded commutative} \\ K\text{-algebras} \end{matrix} \right\}$$

with the data (D0)–(D2) which satisfy the axioms (A1)–(A6) below:

(D0) **(Tate twist)** A one-dimensional K-vector space $K(1)$

(D1) **(Trace isomorphisms)** For each $X \in \mathrm{Ob}(\mathbf{SmProj}(F))$, an isomorphism

$$\mathrm{Tr}_X : H^{2d}(X)(d) \xrightarrow{\simeq} K,$$

where $d := \dim X$ and $H^i(X)(r) := H^i(X) \otimes_K K(1)^{\otimes r}$ for $r \geqq 0$.

(D2) **(Cycle class maps)** For each $X \in \mathrm{Ob}(\mathbf{SmProj}(F))$ and for each $q \geqq 0$, a homomorphism

$$\mathrm{cyc}_X : Z^q(X) \longrightarrow H^{2q}(X)(q),$$

where $Z^q(X)$ denotes the group of algebraic cycles on X of codimension q.

(A1) **(Finiteness)** For any $X \in \mathrm{Ob}(\mathbf{SmProj}(F))$ and any $q \geqq 0$, $H^q(X)$ is finite-dimensional over K, and vanishes unless $0 \leqq q \leqq 2\dim X$.

(A2) **(Künneth formula)** For any $X, Y \in \mathrm{Ob}(\mathbf{SmProj}(F))$, the following map is bijective:

$$H^*(X) \otimes_K H^*(Y) \xrightarrow{\simeq} H^*(X \times Y), \qquad \alpha \otimes \beta \mapsto \mathrm{pr}_1^*(\alpha) \cup \mathrm{pr}_2^*(\beta),$$

Here $\cup$ denotes the cup product, i.e., the product structure of $H^*(X \times Y)$.

(A3) **(Poincaré duality)** For any $X \in \mathrm{Ob}(\mathbf{SmProj}(F))$ of dimension d and any $q \geqq 0$, the following pairing given by cup product and trace map is non-degenerate:

$$H^q(X) \times H^{2d-q}(X)(d) \longrightarrow K, \qquad (\alpha, \beta) \mapsto \mathrm{Tr}_X(\alpha \cup \beta).$$

(A4) **(Rational equivalence)** For any $X \in \mathrm{Ob}(\mathbf{SmProj}(F))$, we have $\mathrm{cyc}_X(Z^q(X)_{\mathrm{rat}}) = 0$, where .$Z^q(X)_{\mathrm{rat}}$ denotes the subgroup of $Z^q(X)$ consisting of the cycles which are rationally equivalent to 0.

(A5) **(Functoriality)** For any morphism $f : X \to Y$ in $\mathbf{SmProj}(F)$, we have

$$\mathrm{cyc}_X \circ f^* = f^* \circ \mathrm{cyc}_Y, \qquad \mathrm{cyc}_Y \circ f_* = f_* \circ \mathrm{cyc}_X,$$

where f^* on the left hand side denotes the pull-back of Chow groups modulo rational equivalence $\mathrm{CH}^q(Y) \to \mathrm{CH}^q(X)$; f_* on the right hand side denotes the dual of f^* under the Poincaré duality for $H^*(X)$ and $H^*(Y)$.

(A6) **(Multiplicativity)** For any $X, Y \in \mathrm{Ob}(\mathbf{SmProj}(F))$, $z \in Z^q(X)$ and any $w \in Z^r(Y)$, $\mathrm{cyc}_X(z) \otimes \mathrm{cyc}_Y(w)$ corresponds to $\mathrm{cyc}_{X \times Y}(z \boxtimes w)$ under the Künneth isomorphism in (A2), where $z \boxtimes w$ denotes the outer product of z and w.

(A7) **(Normalization)** If $X = \mathrm{Spec}(k)$, then $\mathrm{cyc}_X(X) = 1$ and $\mathrm{Tr}_X(1) = 1$.

Example 1.1 (ℓ-adic étale cohomology) Let F be an arbitrary field, and let ℓ be a prime number different from $\mathrm{ch}(F)$. Fix a separable closure $\overline{F}$ of F and put $X_{\overline{F}} := X \otimes_F \overline{F}$ for $X \in \mathrm{Ob}(\mathbf{SmProj}(F))$. Then the ℓ-adic étale cohomology

$$H^*(X) := H^*_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_{n \geqq 1} H^*_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/\ell^n\mathbb{Z}) \qquad (X \in \mathrm{Ob}(\mathbf{SmProj}(F)))$$

with $\mathbb{Q}_\ell(1) := \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_{n \geqq 1} \mu_{\ell^n}(\overline{F})$ yields a Weil cohomology theory.

Example 1.2 (crystalline cohomology) Let F be a perfect field of characteristic $p > 0$. Let $W(F)$ be the ring of Witt vectors over F, and let K_0 be the fraction field of $W(F)$. Then the crystalline cohomology

$$H^*(X) := H^*_{\text{crys}}(X/W(F)) \otimes_{W(F)} K_0 \qquad (X \in \mathrm{Ob}(\mathbf{SmProj}(F))),$$

with $K_0(1) := K_0$ yields a Weil cohomology theory.

From the axioms of Weil cohomology theory, one can deduce the following formula by formal computations, which plays a fundamental role in the study of the zeta function of projective smooth varieties over finite fields:

Theorem 1.3 (Lefschetz trace formula) *Let F and K be as above, and let $H^*(-)$ be a Weil cohomology theory on $\mathbf{SmProj}(F)$. Let X be a d-dimensional variety which belongs to $\mathbf{SmProj}(F)$, and let $f : X \to X$ be a morphism over F. Then we have*

$$\deg(\Delta \cdot \Gamma_f) = \sum_{i=0}^{2d} (-1)^i \cdot \mathrm{Tr}(f^* \,|\, H^i(X)),$$

where Δ denotes the diagonal of $X \times X$, and $\Gamma_f \subset X \times X$ denotes the graph of f.

Proof. See e.g. [SS1] §12.7.c, [Y] Theorem 1.75. □

Definition 1.4 Let p be a prime number, and let q be a power of p. For a scheme X of finite type over $\mathbb{F}_q$, we define the *congruence zeta function* $Z(X/\mathbb{F}_q, t)$ of $X/\mathbb{F}_q$ as the exponential of the generating function of $\mathbb{F}_{q^n}$-valued points of X for $n \geqq 1$:

$$Z(X/\mathbb{F}_q, t) := \exp\left(\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n \right) \in \mathbb{Q}[[t]].$$

As a direct consequence of the Lefschetz trace formula, we have the following:

Corollary 1.5 *Let $H^*(-)$ be a Weil cohomology theory on $\mathbf{SmProj}(\mathbb{F}_q)$ with coefficients in the field K. Then for any $X \in \mathrm{Ob}(\mathbf{SmProj}(\mathbb{F}_q))$, we have*

$$Z(X/\mathbb{F}_q, t) = \frac{P^1(X/\mathbb{F}_q, t)_{H^*} \cdots P^{2d-1}(X/\mathbb{F}_q, t)_{H^*}}{P^0(X/\mathbb{F}_q, t)_{H^*}\, P^2(X/\mathbb{F}_q, t)_{H^*} \cdots P^{2d}(X/\mathbb{F}_q, t)_{H^*}} \qquad (d := \dim X)$$

in $K[[t]]$. Here for each $i = 0, 1, \ldots, 2d$, $P^i(X/\mathbb{F}_q, t)_{H^*}$ is defined as

$$P^i(X/\mathbb{F}_q, t)_{H^*} := \det(1 - \mathrm{Fr}_q^* \cdot t \,|\, H^i(X)) \in K[t],$$

and Fr_q denotes the Frobenius morphism $X \to X$ over $\mathbb{F}_q$.

Proof. By the Lefschetz trace formula over $\mathbb{F}_{q^n}$, we have

$$\#X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2d} (-1)^i \cdot \mathrm{Tr}\big((\mathrm{Fr}_q^*)^n \big| H^i(X)\big).$$

On the other hand, we have

$$\det(E - tA)^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{Tr}(A^n)}{n} t^n\right) \quad \text{in} \quad K[[t]]$$

for any square matrix A with entries in K. One obtains the assertion from these facts. □

Theorem 1.6 (Deligne/Katz-Messing) *For any $X \in \mathrm{Ob}(\mathbf{SmProj}(\mathbb{F}_q))$ and any $0 \leqq i \leqq 2\dim(X)$, the polynomial $P^i(X/\mathbb{F}_q, t)_{H^*} \in K[t]$ is independent of the Weil cohomology theory $H^*(-)$ satisfying weak Lefschetz, and lies in $\mathbb{Z}[t]$. Moreover, the reciprocal zeros of $P^i(X/\mathbb{F}_q, t) := P^i(X/\mathbb{F}_q, t)_{H^*}$ have complex absolute value $q^{i/2}$.*

2 Cohomology and zeta values

Throughout this section, let X be a scheme of finite type over $\mathbb{Z}$.

Definition 2.1 We define the *zeta function* $\zeta(X, s)$ of the scheme X as the Euler product

$$\zeta(X, s) := \prod_{x \in X_0} \frac{1}{1 - \mathscr{N}(x)^{-s}} .$$

Here X_0 denotes the set of closed points of X, and $\mathscr{N}(x)$ denotes the order of the residue field $\kappa(x)$ for each $x \in X_0$.

Example 2.2 When $X = \mathrm{Spec}(O_K)$, the spectrum of the integer ring O_K of a number field K, then we have $\zeta(X, s) = \zeta_K(s)$, the *Dedekind zeta function*.

Proposition 2.3 (1) *$\zeta(X, s)$ converges absolutely for $\mathrm{Re}(s) > \dim X$. In particular, it does not have zeros there.*

(2) *$\zeta(X, s)$ is meromorphically continued to $\mathrm{Re}(s) > \dim X - \frac{1}{2}$, and has a pole of order m at $s = \dim X$. Here m denotes the number of the irreducible components of X which have dimension $\dim X$.*

Proof. See [Se1] §1.3∼§1.4. □

Exercise 1 *Let X be a scheme of finite type over $\mathbb{F}_q$. Then show that*

$$\zeta(X, s) = Z(X/\mathbb{F}_q, q^{-s}) \quad \textit{for} \quad \mathrm{Re}(s) > \dim X.$$

Definition 2.4 For each positive integer m invertible on X, let $\mu_m = \mu_{m,X}$ be the étale sheaf of m-th roots of unity on X. For a prime number ℓ invertible on X and an integer $r \geqq 0$, put

$$H^*(X, \mathbb{Z}_\ell(r)) := \varprojlim_{n \geqq 1} H^*_{\text{ét}}(X, \mu_{\ell^n}^{\otimes r}).$$

If X is a smooth scheme over $\mathrm{Spec}(\mathbb{F}_p)$, then we put

$$H^*(X, \mathbb{Z}_p(r)) := \varprojlim_{n \geqq 1} H^{*-r}_{\text{ét}}(X, W_n\Omega^r_{X,\log}),$$

where $W_n\Omega^r_{X,\log}$ denotes the étale subsheaf of logarithmic part of the Hodge-Witt sheaf $W_n\Omega^r_X$ (cf. [I]).

The following theorem is a special case of a theorem of Milne [Mi] Theorem 0.1.

Theorem 2.5 *Let X be a proper smooth geometrically integral variety over $\mathbb{F}_q$. Then we have*

$$\lim_{s\to d}(1-q^{d-s})\zeta(X,s)=\chi(X,\mathscr{O}_X,d)\prod_{i=1}^{2d-1}(\#H^i(X,\widehat{\mathbb{Z}}(d)))^{(-1)^i}\cdot\#(H^{2d}(X,\widehat{\mathbb{Z}}(d))_{\mathrm{tors}}),$$

$$\zeta(X,r)=\chi(X,\mathscr{O}_X,r)\prod_{i=1}^{2d+1}(\#H^i(X,\widehat{\mathbb{Z}}(r)))^{(-1)^i}\qquad(r>d,\ r\in\mathbb{Z}),$$

where $H^*(X,\widehat{\mathbb{Z}}(r))$ and $\chi(X,\mathscr{O}_X,r)$ are defined as follows:

$$H^i(X,\widehat{\mathbb{Z}}(r)):=\prod_{\ell:\,prime}H^i(X,\mathbb{Z}_\ell(r)),$$

$$\chi(X,\mathscr{O}_X,r):=\prod_{i,j\geqq 0}(\#H^j(X,\Omega_X^i))^{(r-i)\cdot(-1)^{i+j}}\qquad(\textit{Milne's correcting factor})$$

Theorem 2.5 is based on the finiteness of $H^i(X,\widehat{\mathbb{Z}}(r))$ ($i\geqq 0$, $r\geqq d$, $(i,r)\neq(2d,d),(2d+1,d)$) and $H^{2d}(X,\widehat{\mathbb{Z}}(d))_{\mathrm{tors}}$, which is a consequence of Theorem 1.6 and a theorem of Gabber [Ga].

Remark 2.6 In [Mi] Theorem 0.1, Milne describes the behavior of $\zeta(X,s)$ at $s\to r$ ($r\in\mathbb{Z}$) by $H^*(X,\widehat{\mathbb{Z}}(r))$ and $\chi(X,\mathscr{O}_X,r)$, assuming the projectivity of X over $\mathbb{F}_q$ and the 1-*semi-simplicity conjecture* on the action of the Frobenius element $\varphi_q\in\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_q)$ on

$$H^{2r}(\overline{X},\mathbb{Q}_\ell(r))=H^{2r}(\overline{X},\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell}\mathbb{Q}_\ell(1)^{\otimes r}\qquad(\overline{X}:=X\otimes_{\mathbb{F}_q}\overline{\mathbb{F}_q})$$

for all prime number ℓ, including p (see also [T2] §3 (d)). This 1-semi-simplicity conjecture obviously holds true for any $r\geqq d$ under the properness of X over $\mathbb{F}_q$, so do the assertions in [Mi] Theorem 0.1 under the same assumption.

Definition 2.7 (Deligne cohomology) Assume that X is *flat* over $\mathbb{Z}$, and that $X_{\mathbb{Q}}:=X\otimes_{\mathbb{Z}}\mathbb{Q}$ is *smooth* over $\mathbb{Q}$. We write $X(\mathbb{C})$ for the set of $\mathbb{C}$-valued points of X over $\mathbb{Z}$, and write $X(\mathbb{C})^{\mathrm{an}}$ for the complex analytic variety associated with $X(\mathbb{C})$. For a subring $A\subset\mathbb{R}$ and $r\geqq 0$, let $A(r)_{\mathscr{D}}$ be the following complex of sheaves on $X(\mathbb{C})^{\mathrm{an}}$:

$$A(r)_{\mathscr{D}}\ :\ (2\pi\sqrt{-1})^r\cdot A\longrightarrow\mathscr{O}_{X(\mathbb{C})^{\mathrm{an}}}\xrightarrow{d}\Omega^1_{X(\mathbb{C})^{\mathrm{an}}}\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{r-1}_{X(\mathbb{C})^{\mathrm{an}}},$$

where $(2\pi\sqrt{-1})^r\cdot A$ is placed in degree 0. Then we define

$$H^*_{\mathscr{D}}(X_{/\mathbb{C}},A(r)):=H^*(X(\mathbb{C})^{\mathrm{an}},A(r)_{\mathscr{D}}),$$
$$H^*_{\mathscr{D}}(X_{/\mathbb{R}},A(r)):=H^*_{\mathscr{D}}(X_{/\mathbb{C}},A(r))^+,$$

where $^+$ means the fixed part by the complex conjugation, acting on both $X(\mathbb{C})^{\mathrm{an}}$ and $A(r)_{\mathscr{D}}$.

Example 2.8 There is a cartesian diagram of sheaves on $X(\mathbb{C})^{\mathrm{an}}$

$$\begin{tikzcd} 2\pi\sqrt{-1}\cdot\mathbb{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{O}_{X(\mathbb{C})^{\mathrm{an}}} \arrow[d, two heads, "\exp"] \\ 0 \arrow[r] & \mathscr{O}^{\times}_{X(\mathbb{C})^{\mathrm{an}}}, \end{tikzcd}$$

which implies that we have $\mathbb{Z}(1)_{\mathscr{D}}\cong\mathscr{O}^{\times}_{X(\mathbb{C})^{\mathrm{an}}}[-1]$ in $D^b(X(\mathbb{C})^{\mathrm{an}})$.

Setting 2.9 In the rest of this section, we put $X := \mathrm{Spec}(O_K)$, the spectrum of the integer ring of a number field K. We often write n for $[K : \mathbb{Q}]$. In this case, $X(\mathbb{C})$ is exactly the set of the ring homomorphisms $\tau : O_K \to \mathbb{C}$, i.e., consists of distinct n points. We write r_1 (resp. r_2, h, R, D) for the number of the real (resp. number of complex places, class number, regulator, discriminant) of K, and write w for the number of roots of unity in K.

By the isomorphism $\mathbb{Z}(1)_{\mathscr{D}} \cong \mathscr{O}^{\times}_{X(\mathbb{C})^{\mathrm{an}}}[-1]$ in Example 2.8, there is a commutative diagram

$$\begin{tikzcd} & \Big(\prod_{\tau\in X(\mathbb{C})} \mathbb{C}\Big)^{+} \arrow[dl, "\beta"'] \arrow[d, "\exp"] \\ H^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1)) \arrow[r, "\simeq", "\alpha"'] & \Big(\prod_{\tau\in X(\mathbb{C})} \mathbb{C}^{\times}\Big)^{+}. \end{tikzcd} \tag{2.1}$$

The isomorphism $\mathbb{Z}(1)_{\mathscr{D}} \cong \mathscr{O}^{\times}_{X(\mathbb{C})^{\mathrm{an}}}[-1]$ also yields a natural homomorphism, called *the regulator map*

$$\mathrm{reg}^{i,1}_{\mathscr{D}} : H^{i-1}(X_{\mathrm{zar}}, \mathscr{O}_X^{\times}) \longrightarrow H^i_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1)).$$

for each $i \geqq 1$. Note that $\mathrm{reg}^{i,1}_{\mathscr{D}}$ is injective for $i = 1$ and zero otherwise. When we take the kernel $\widetilde{H}^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1))$ of the trace map

$$\mathrm{Tr}_X : H^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1)) \longrightarrow \mathbb{R}, \qquad (z_\tau)_{\tau\in X(\mathbb{C})} \mapsto \sum_{\tau\in X(\mathbb{C})} \ln|z_\tau|,$$

then $\mathrm{reg}^{1,1}_{\mathscr{D}}(O_K^{\times})$ is a discrete cocompact subgroup of $\widetilde{H}^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1))$. Moreover, its covolume under a natural Haar measure defined in Definition 2.10 below involves several important invariants of the number field K (see Proposition 2.11 below).

We define the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on the $\mathbb{C}$-vector space $K \otimes_{\mathbb{Q}} \mathbb{C}$ as

$$\sigma(a \otimes z) := a \otimes \sigma(z) \qquad (a \in K,\ z \in \mathbb{C},\ \sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})).$$

There is a canonical $\mathbb{C}$-linear isomorphism

$$K \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod_{\tau\in X(\mathbb{C})} \mathbb{C}, \qquad a \otimes z \longmapsto (\tau(a)z)_\tau,$$

which is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant and induces an isomorphism of $\mathbb{R}$-vector spaces

$$K \otimes_{\mathbb{Q}} \mathbb{R} \cong \Big(\prod_{\tau\in X(\mathbb{C})} \mathbb{C}\Big)^{+}. \tag{2.2}$$

This isomorphism and the map β in (2.1) define a natural continuous and open homomorphism from $K \otimes_{\mathbb{Q}} \mathbb{R} \to H^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1))$, whose image is a connected open subgroup of index 2^{r_1}. Let $\mathfrak{m}_0$ be the Haar measure on $H^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1))$ with respect to the lattice O_K of $K \otimes_{\mathbb{Q}} \mathbb{R}$. We construct a Haar measure $\mathfrak{m}_1$ on $\widetilde{H}^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1))$ from $\mathfrak{m}_0$ and the Lebesgue measure λ_1 on $\mathbb{R}$ as follows.

Definition 2.10 We put $n := [K : \mathbb{Q}] = \#X(\mathbb{C})$, and fix a continuous section of Tr_X

$$s : \mathbb{R} \longrightarrow H^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1)), \qquad x \mapsto \alpha^{-1}((\exp(x/n))_\tau),$$

which yields a homeomorphism of topological groups

$$H^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1)) \cong \widetilde{H}^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1)) \times \mathbb{R}.$$

Under this topological direct decomposition, we define

$$\mathfrak{m}_1(V) := \frac{\mathfrak{m}_0(V \times s(Z))}{\lambda_1(Z)} \tag{2.3}$$

for any Borel subset $V \subset \widetilde{H}^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1))$, where Z on the right hand side is an auxiliary bounded (and non-empty) open subset of $\mathbb{R}$.

Exercise 2 *Check that the value on the right hand side of* (2.3) *is independent of the choice of a bounded open subset $Z \neq \emptyset$, and that $\mathfrak{m}_1$ is a Haar measure on $\widetilde{H}^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1))$.*

Now we are ready to state the following Proposition 2.11 concerning the reduced regulator map

$$\widetilde{\mathrm{reg}}^{1,1}_{\mathscr{D}} : O_K^\times \longrightarrow \widetilde{H}^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1)).$$

Proposition 2.11 *Let $\mathfrak{m}_2$ be the quotient Haar measure of $\mathfrak{m}_1$ on* $\mathrm{Coker}(\widetilde{\mathrm{reg}}^{1,1}_{\mathscr{D}})$. *Then*

$$\mathfrak{m}_2(\mathrm{Coker}(\widetilde{\mathrm{reg}}^{1,1}_{\mathscr{D}})) = \frac{2^{r_1}(2\pi)^{r_2} R}{w\sqrt{|D|}}.$$

Proof. See e.g. [Sa4], Appendix A. □

By this formula, the classical class number formula is reorganized as follows:

Corollary 2.12 *We have*

$$\operatorname*{Res}_{s=1} \zeta_K(s) = \frac{\mathfrak{m}_2(\mathrm{Coker}(\widetilde{\mathrm{reg}}^{1,1}_{\mathscr{D}}))}{\#\mathrm{Ker}(\widetilde{\mathrm{reg}}^{1,1}_{\mathscr{D}})} \cdot \frac{\#\mathrm{Ker}(\mathrm{reg}^{2,1}_{\mathscr{D}})}{\#\mathrm{Coker}(\mathrm{reg}^{2,1}_{\mathscr{D}})},$$

where $\mathrm{reg}^{2,1}_{\mathscr{D}}$ *denotes* $\mathrm{Pic}(X) \to 0$.

Remark 2.13 The isomorphism (2.2) is in fact the comparison isomorphism

$$H^0_{\mathrm{dR}}(K/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \cong \left(\prod_{\tau \in X(\mathbb{C})} H^0_{\mathrm{sing}}(X(\mathbb{C})^{\mathrm{an}}, \mathbb{C}) \right)^+$$

between the algebraic de Rham cohomology of $\mathrm{Spec}(K)$ and the singular cohomology of $X(\mathbb{C})^{\mathrm{an}}$. In short, the measure $\mathfrak{m}_1$ on $\widetilde{H}^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1))$ is determined by the lattice O_K of $H^0_{\mathrm{dR}}(K/\mathbb{Q})$, and Proposition 2.11 computes the covolume of $\widetilde{\mathrm{reg}}^{1,1}_{\mathscr{D}}(O_K^\times)$ concerning $\mathfrak{m}_1$.

Exercise 3 *Using classical facts on number fields, show that*

$$H^i_{\text{ét}}(\mathrm{Spec}(O_K), \mathbb{G}_{\mathrm{m}}) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r'} & (i = 2) \\ \mathbb{Q}/\mathbb{Z} & (i = 3) \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1} & (i \geqq 4,\ \textit{even}) \\ 0 & (i \geqq 5,\ \textit{odd}), \end{cases}$$

where r_1 denotes the number of real places of K, and $r' := \max\{r_1 - 1, 0\}$.

3 Étale motivic complexes

Let X be a noetherian regular scheme. We introduce here a collection of axioms (L0)–(L7) due to Lichtenbaum [Li1], [Li3] concerning a family $\{\mathbb{Z}(r)\}_{r \geqq 0}$ of complexes of étale sheaves on X. This family $\{\mathbb{Z}(r)\}_{r \geqq 0}$ satisfying the following (L0)–(L7) and good candidates for them are both called *étale motivic complexes on X*.

(L0) $\mathbb{Z}(0) = \mathbb{Z}$, $\mathbb{Z}(1) = \mathbb{G}_{\mathrm{m}}[-1]$.

(L1) **(acyclicity)** For $r \geqq 2$, $\mathbb{Z}(r)$ is acyclic outside of $[1, r]$.

(L2) **(Hilbert's Theorem 90)** Let $\epsilon : X_{\text{ét}} \to X_{\text{zar}}$ be the natural continuous map of sites. Then the Zariski sheaf $R^{r+1}\epsilon_*\mathbb{Z}(r)$ is zero for any $r \geqq 0$.

(L3) **(Kummer theory)** Let m be a positive integer which is invertible on X. Then there exists a distinguished triangle

$$\mathbb{Z}(r) \xrightarrow{\times m} \mathbb{Z}(r) \longrightarrow \mu_m^{\otimes r} \longrightarrow \mathbb{Z}(r)[1] \quad \text{in} \quad D(X_{\text{ét}})$$

for any $r \geqq 0$.

(L4) **(p-Kummer theory)** Let p be a prime number, and assume that X is over $\mathbb{F}_p$. Then there exists a distinguished triangle

$$\mathbb{Z}(r) \xrightarrow{\times p^n} \mathbb{Z}(r) \longrightarrow W_n\Omega^r_{X,\log}[-r] \longrightarrow \mathbb{Z}(r)[1] \quad \text{in} \quad D(X_{\text{ét}})$$

for any $r \geqq 0$ and $n \geqq 1$.

(L5) **(Products)** For each $r, r' \geqq 0$, there exists a product morphism

$$\mathbb{Z}(r) \otimes^{\mathbb{L}} \mathbb{Z}(r') \longrightarrow \mathbb{Z}(r+r') \quad \text{in} \quad D(X_{\text{ét}}).$$

(L6) **(Connection with K-theory)** The q-th cohomology sheaf $\mathscr{H}^q(\mathbb{Z}(r))$ is isomorphic to the étale sheafification of the presheaf

$$U \in \mathrm{Ob}(\text{Ét}/X) \longmapsto \mathrm{gr}^r_\gamma K_{2r-q}(U)$$

up to torsion involving primes $\leqq r-1$. Here $K_{2r-q}(U)$ denotes the $(2r-q)$-th algebraic K-group associated with the category of vector bundles over U, cf. [Q]; γ means the γ-filtration, cf. [So]. Moreover, $\mathscr{H}^r(\mathbb{Z}(r))$ is isomorphic to the étale sheafification of the presheaf

$$U \in \mathrm{Ob}(\text{Ét}/X) \longmapsto K^M_r(\Gamma(U, \mathscr{O}_U)),$$

where for a ring R, $K^M_r(R)$ denotes the r-th Milnor K-group of R.

(L7) **(Purity)** Let $i : Z \hookrightarrow X$ be a locally closed immersion with Z regular and of pure codimension c. Then there exists a canonical isomorphism

$$i_* : \mathbb{Z}(r-c)_Z[-2c] \xrightarrow{\simeq} \tau_{\leqq r+c} Ri^!\mathbb{Z}(r)_X \quad \text{in} \quad D(Z_{\text{ét}}).$$

In his paper [Li2], Lichtenbaum constructed a candidate of $\mathbb{Z}(2)$ using algebraic K-groups.

Theorem 3.1 *Lichtenbaum's $\mathbb{Z}(2)$ satisfies*

(L1) *by definition,* (L2) *for any X up to 2-torsion,*

(L3) *for any X smooth of finite type over a field, and any odd m invertible on X,*

(L4) *for any X smooth of finite type over a field of characteristic $p \geqq 3$,*

(L5) *for any X,* (L6) *for any X smooth of finite type over a field,*

(L7) *for any X smooth of finite type over a field, and any Z with $c = 1$, up to 2-torsion.*

Proof. See [Li2], [Li3] for details. □

There are other strong candidates of $\mathbb{Z}(r)$ for $r \geqq 2$:

- $\mathbb{Z}(r)$ using Bloch's cycle complex $z^r(-,*)$ ([Bl2]). See Definition 3.3 below.
- $\mathbb{Z}(r)$ for smooth schemes of finite type over a field [SV]. See also Remark 3.4 below.

See Exercise 4 below for (L0) for Bloch's $\mathbb{Z}(r)$. (L0) for Suslin-Voevodsky's $\mathbb{Z}(r)$ is straightforward.

Theorem 3.2 *For any X smooth of finite type over a field, Bloch's $\mathbb{Z}(r)$ and Suslin-Voevodsky's $\mathbb{Z}(r)$ agree in $D(X_{\text{ét}})$, and satisfy*

(L1) *for degrees $> r$,* (L2)–(L5), (L7) *for any Z with $c = 1$.*

The former half of (L6) *holds for any X smooth of finite type over a field up to torsion, and the latter half holds for the same X over an infinite field.*

Proof. See [V1] for the comparison of the two candidates over a field. (L1) for degrees $> r$ follows from [SV] Lemma 3.2, or the Gersten conjecture for higher Chow groups [Bl2] Theorem 10.1. See [GL2] for (L3) under the Bloch-Kato conjecture for norm-residue homomorphisms, which has been proved in [V2], [V3]. See [GL1] for (L4); (L5) is obvious for Suslin-Voevodsky's $\mathbb{Z}(r)$. The assertions on the former half (resp. the latter half) of (L6) is due to Bloch [Bl2] Theorem 9.1 (resp. Nesterenko-Suslin-Totato [NS], [To] and Kerz [Ke] Theorem 1.1). See [Ge] Theorem 1.2 (2), (1) for (L2) and (L7). $\square$

In what follows, we review the definitions of Bloch's cycle complex and $\mathbb{Z}(r)$, briefly.

Definition 3.3 (Bloch's $z^r(U,*)$ and $\mathbb{Z}(r)$) For each integer $q \geqq 0$, put

$$\varDelta^q := \mathrm{Spec}(\mathbb{Z}[t_0, t_1, \dots, t_q]/(t_0 + t_1 + \cdots + t_q - 1)).$$

A *face* of $\varDelta^q$ of codimension $c \geqq 1$ is a closed subscheme defined as

$$t_{i_1} = t_{i_2} = \cdots = t_{i_c} = 0 \quad \text{for some } \ 0 \leqq i_1 < i_2 < \ldots < i_c \leqq q.$$

When $c = 1$, we often identify the face $\{t_i = 0\}$ with the closed immersion $\varDelta^{q-1} \hookrightarrow \varDelta^q$ given by

$$t_j \longmapsto \begin{cases} t_j & (0 \leqq j < i) \\ t_{j-1} & (i < j \leqq q). \end{cases}$$

For a noetherian uni-codimensional (e.g. integral) scheme U, let $z^r(U,q)$ be the free abelian group generated by the set of the integral closed subschemes $V \subset U \times \varDelta^q$ of codimension r which meet all faces of $U \times \varDelta^q$ *properly*, that is, for any face $F \subset \varDelta^q$ and any irreducible component T of $V \times_{U \times \varDelta^q} (U \times F)$, we have

$$\mathrm{codim}_{U \times \varDelta^q}(T) \geqq \mathrm{codim}_{U \times \varDelta^q}(V) + \mathrm{codim}_{\varDelta^q}(F).$$

For each face $\partial_i : \{t_i = 0\} \hookrightarrow \varDelta^q$ of codimension 1 ($i = 0, 1, \dots, q$), we define the *coface map*

$$\partial_i^* : z^r(U,q) \longrightarrow z^r(U,q-1)$$

as the pull-back of algebraic cycles along the effective Cartier divisor

$$\mathrm{id}_X \times \partial_i : \{t_i = 0\} \times X \hookrightarrow X \times \varDelta^q.$$

Taking the alternating sum

$$d_q := \sum_{i=0}^{q} (-1)^i \cdot \partial_i^* : z^r(U,q) \longrightarrow z^r(U,q-1)$$

we obtain *Bloch's cycle complex* $z^r(U,*) = ((z^r(U,q))_{q\geqq 0}, (d_q)_{q\geqq 1})$, which is in fact a complex of abelian groups. We define Bloch's $\mathbb{Z}(r)$ on $X_{\text{ét}}$ (resp. on X_{zar}) by the assignment

$$\begin{aligned} U \in \mathrm{Ob}(\text{Ét}/X) &\longmapsto z^r(U,*)[-2r] \\ (\text{resp. } U \subset X \text{ (open)} &\longmapsto z^r(U,*)[-2r]), \end{aligned}$$

which is a complex of abelian sheaves on $X_{\text{ét}}$ (resp. on X_{zar}).

Exercise 4 *Show that Bloch's $\mathbb{Z}(r)$ satisfies* (L0) *for any regular noetherian scheme X.*

Remark 3.4 In [Ge], Geisser proves that for X smooth of finite type over a Dedekind ring, Bloch's $\mathbb{Z}(r)$ satisfies

(L1) for degrees $> r$, (L2), (L3), (L7) for any Z with $c = 1$.

In [CD] §11, Cisinski and Deglise construct a candidate of $H^*_{\text{zar}}(X, \mathbb{Z}(r))$ for any regular scheme X of finite dimension, generalizing Voevodsky's construction.

4 Finite-coefficient variant of Lichtenbaum's axioms

Setting 4.1 Let $\mathfrak{O}$ be a Dedekind domain, and let K be its fraction field. Let p be a prime number, and suppose that $0 \subsetneqq p\mathfrak{O} \subsetneqq \mathfrak{O}$. Let X be an integral regular scheme which is flat of finite type over $B := \mathrm{Spec}(\mathfrak{O})$ and assume that

($*_1$) *the divisor $Y := (X \otimes_{\mathbb{Z}} \mathbb{F}_p)_{\text{red}}$ has normal crossings on X.*

Let ι and j be as follows:

$$X[p^{-1}] \xhookrightarrow{\ j\ } X \xhookleftarrow{\ \iota\ } Y.$$

We will often write π for the structure morphism $X \to B$.

For a point $x \in X$, we often write ι_x for the natural map $x \hookrightarrow X$ (more precisely, $\{x\} \hookrightarrow X$) and write $R\iota_x^!$ for $i_x^* R\iota_Z^!$, where Z denotes the Zariski closure of $\{x\}$ in X and i_x (resp. ι_Z) denotes the natural map $x \hookrightarrow Z$ (resp. the closed immersion $Z \hookrightarrow X$). Note that $R\iota_x^!$ is *not the right adjoint* of $R\iota_{x*}$ unless $Z = \{x\}$, i.e., x is a closed point of X.

4.1 Axioms and a solution

We introduce here a collection of axioms (T1)–(T5) on a family $\{\mathfrak{T}_n(r)\}_{r\geqq 0}$ of complexes of étale $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on X.

(T1) **(Trivialization)** There is an isomorphism $t : j^*\mathfrak{T}_n(r) \cong \mu_{p^n}^{\otimes r}$.

(T2) **(Acyclicity)** $\mathfrak{T}_n(r)$ is concentrated in $[0, r]$, i.e., the q-th cohomology sheaf is zero unless $0 \leqq q \leqq r$.

(T3) **(Purity)** For a locally closed regular subscheme $\iota_Z : Z \hookrightarrow X$ of characteristic p and of codimension $c\,(\geqq 1)$, there is a Gysin isomorphism

$$W_n\Omega^{r-c}_{Z,\log}[-r-c] \xrightarrow{\ \simeq\ } \tau_{\leqq r+c} R\iota_Z^! \mathfrak{T}_n(r) \quad \text{in } D^b(Z_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}).$$

(T4) **(Compatibility)** For any two points $x, y \in X$ satisfying $\mathrm{ch}(x) = p$, $x \in \overline{\{y\}}$ and $c := \mathrm{codim}_X(x) = \mathrm{codim}_X(y) + 1$, the connecting homomorphism

$$\delta^{\text{loc}} : R^{r+c-1}\iota_{y*}(R\iota_y^! \mathfrak{T}_n(r)) \longrightarrow R^{r+c}\iota_{x*}(R\iota_x^! \mathfrak{T}_n(r))$$

in localization theory agrees with the sheafified boundary map of Galois cohomology groups due to Kato ([KCT])

$$\partial^{\mathrm{val}} : \left\{ \begin{array}{ll} R^{r-c+1}\iota_{y*}\mu_{p^n}^{\otimes r-c+1} & (\mathrm{ch}(y)=0) \\ \iota_{y*}W_n\Omega_{y,\log}^{r-c+1} & (\mathrm{ch}(y)=p) \end{array} \right\} \longrightarrow \iota_{x*}W_n\Omega_{x,\log}^{r-c}$$

up to a sign depending only on $(\mathrm{ch}(y), c)$, via the Gysin isomorphisms for ι_y and ι_x. Here the Gysin isomorphism for ι_y with $\mathrm{ch}(y) = 0$ is defined by the isomorphism t in **T1** and Deligne's cycle class in $R^{2c-2}\iota_y^!\mu_{p^n}^{\otimes c-1}$.

(T5) **(Products)** There is a unique morphism

$$\mathfrak{T}_n(r) \otimes^{\mathbb{L}} \mathfrak{T}_n(r') \longrightarrow \mathfrak{T}_n(r+r') \quad \text{in } D^-(X_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$$

that extends the natural isomorphism $\mu_{p^n}^{\otimes r} \otimes \mu_{p^n}^{\otimes r'} \cong \mu_{p^n}^{\otimes r+r'}$ on $X[p^{-1}]$.

The axioms (T1)–(T3) and (T5) are $\mathbb{Z}/p^n\mathbb{Z}$-analogue of (L1)–(L5) and (L7); (T4) is not among Lichtenbaum's axioms, but a natural property to be satisfied. Concerning these axioms, we have the following fundamental result:

Theorem 4.2 ([SH], [Sa5]) *If $\pi : X \to B$ is log smooth around Y, then there exists a family $\{\mathfrak{T}_n(r)\}_{r\geqq 0}$ of objects in $D^b(X_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ satisfying* (T1)–(T5). *Moreover, for each $r \geqq 0$, the pair $(\mathfrak{T}_n(r), t)$ of $\mathfrak{T}_n(r)$ and t of* (T1) *satisfying* (T2)–(T4) *is unique up to a unique isomorphism in $D^b(X_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$.*

The complexes $\{\mathfrak{T}_n(r)\}_{r\geqq 0}$ in Theorem 4.2 are functorial in the following sense:

Theorem 4.3 ([SH], [Sa5]) *Let $X \to B$ be as in Theorem 4.2. Let $\mathfrak{O}'$ be a Dedekind domain which is flat over $\mathfrak{O}$, and let X' be an integral regular scheme flat of finite type over $B' := \mathrm{Spec}(\mathfrak{O}')$ such that $Y' := (X' \otimes_{\mathbb{Z}} \mathbb{F}_p)_{\mathrm{red}}$ has normal crossing on X' and such that $\pi' : X' \to B'$ is log smooth around Y'. Let $f : X' \to X$ be an arbitrary morphism, and let $\psi : X'[p^{-1}] \to X[p^{-1}]$ be the induced morphism. Put $c := \dim(X_K) - \dim(X'_{K'})$, where K' denotes* $\mathrm{Frac}(\mathfrak{O}')$ *and we put $X_K := X \otimes_{\mathfrak{O}} K$ and $X'_{K'} := X' \otimes_{\mathfrak{O}'} K'$. Then*:

(T6) **(Contravariant functoriality)** *There exists a unique morphism*

$$f^\sharp : f^*\mathfrak{T}_n(r)_X \longrightarrow \mathfrak{T}_n(r)_{X'} \quad \textit{in} \quad D^b(X'_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$$

that extends the natural isomorphism $\psi^*\mu_{p^n}^{\otimes r} \cong \mu_{p^n}^{\otimes r}$ on $(X'[p^{-1}])_{\text{ét}}$.

(T7) **(Covariant functoriality)** *If f is separated of finite type, then there exists a unique morphism*

$$\mathrm{tr}_f : Rf_!\mathfrak{T}_n(r-c)_{X'}[-2c] \longrightarrow \mathfrak{T}_n(r)_X \quad \textit{in} \ D^b(X_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$$

that extends the push-forward map $\mathrm{tr}_\psi : R\psi_!\mu_{p^n}^{\otimes r-c}[-2c] \to \mu_{p^n}^{\otimes r}$ on $(X[p^{-1}])_{\text{ét}}$.

Remark 4.4 $f^\sharp$ in (T6) is *not an isomorphism* in genral, unless f is étale. We *do not need the log smoothness* of X nor X' for the existence of tr_f in (T7) with $r \geqq \dim X$.

4.2 Construction of $\mathfrak{T}_n(r)$

We start the construction of $\mathfrak{T}_n(r)$ with the following straight-forward observation, where we do not need the *log-smoothness around* Y. For a point $x \in X$, let i_x be the natural map $x \hookrightarrow X$.

Lemma 4.5 *Assume that there exists an object $\mathfrak{T}_n(r) \in D^b(X_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ satisfying* (T1)–(T4)*. Then*:

(1) *There is an exact sequence of sheaves on $X_{\text{ét}}$*

$$R^r j_*\mu_{p^n}^{\otimes r} \longrightarrow \bigoplus_{y \in Y^0} \iota_{y*} W_n\Omega^{r-1}_{y,\log} \longrightarrow \bigoplus_{x \in Y^1} \iota_{x*} W_n\Omega^{r-2}_{x,\log}, \tag{4.1}$$

where each arrow arises from the boundary maps of Galois cohomology groups.

(2) *There is a distinguished triangle in $D^b(X_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ of the form*

$$\iota_*\nu^{r-1}_{Y,n}[-r-1] \xrightarrow{\ g\ } \mathfrak{T}_n(r) \xrightarrow{\ t'\ } \tau_{\leqq r} Rj_*\mu_{p^n}^{\otimes r} \xrightarrow{\ \sigma\ } \iota_*\nu^{r-1}_{Y,n}[-r]. \tag{4.2}$$

Here t' is induced by t of (T1) *and the acyclicity property* (T2); *$\nu^{r-1}_{Y,n}$ denotes the sheaf on $Y_{\text{ét}}$ defined as the kernel of the second arrow in* (4.1) (*restricted onto Y*), *and σ denotes the morphism induced by the exact sequence* (4.1).

Proof. Consider a localization distinguished triangle

$$\mathfrak{T}_n(r) \xrightarrow{\ j^*\ } Rj_*j^*\mathfrak{T}_n(r) \xrightarrow{\ \delta^{\text{loc}}_{U,Z}\ } \iota_*R\iota^!\mathfrak{T}_n(r)[1] \xrightarrow{\ \iota_*\ } \mathfrak{T}_n(r)[1]. \tag{4.3}$$

We have $t : j^*\mathfrak{T}_r(n) \cong \mu_{p^n}^{\otimes r}$ by (T1). On the other hand, one has

$$\tau_{\leqq r}(\iota_*R\iota^!\mathfrak{T}_n(r)[1]) \cong \iota_*\nu^{r-1}_{Y,n}[-r]$$

by (T3) and (T4). The map of cohomology sheaves at degree r of $\delta^{\text{loc}}_{U,Z}$ looks like

$$R^r j_*\mu_{p^n}^{\otimes r} \longrightarrow \iota_*\nu^{r-1}_{Y,n}, \tag{4.4}$$

which is compatible with Kato's boundary maps up to a sign by (T4). Thus the sequence (4.1) must be a complex and we obtain the morphism σ of (4.2). Finally by (T2), the map (4.4) must be surjective, which implies the exactness of (4.1) and that we obtain the triangle (4.2) by truncating and shifting the triangle (4.3) suitably. □

In view of Lemma 4.5, the next step is to show the following proposition without assuming the existence of $\mathfrak{T}_n(r)$, where we do not need the log-smoothness around Y yet:

Proposition 4.6 *The sequence* (4.1) *is exact étale locally on X.*

Proof. The assertion that the sequence (4.1) is a complex follows from a result of Kato [KCT] Proposition 1.7, and then one can check the exactness of (4.1) using the Gersten conjecture for logarithmic Hodge-Witt sheaves for regular schemes in characteristic p ([GS], [Sh]). See [SH] Lemma 3.2.4 and the first part of Theorem 3.4.2 for details. □

By Proposition 4.6, there exists a morphism

$$\sigma : \tau_{\leqq r} Rj_*\mu_{p^n}^{\otimes r} \longrightarrow \iota_*\nu^{r-1}_{Y,n}[-r] \quad \text{in} \quad D^b(X_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$$

and the induced homomorphism of cohomology sheaves

$$\mathscr{H}^r(\sigma) : R^r j_*\mu_{p^n}^{\otimes r} \longrightarrow \iota_*\nu^{r-1}_{Y,n}$$

is surjective.

Definition 4.7 For $n \geqq 1$ and $r \geqq 0$, we define the desired complex $\mathfrak{T}_n(r)$ as that fitting into a distinguished triangle of the same form as (4.2):

$$\iota_* \nu_{Y,n}^{r-1}[-r-1] \xrightarrow{g} \mathfrak{T}_n(r) \xrightarrow{t'} \tau_{\leqq r} Rj_* \mu_{p^n}^{\otimes r} \xrightarrow{\sigma} \iota_* \nu_{Y,n}^{r-1}[-r]. \tag{4.5}$$

$\mathfrak{T}_n(r)$ satisfies (T1) (resp. (T2)) by definition (resp. the surjectivity of $\mathscr{H}^r(\sigma)$). Since

$$\mathrm{Hom}_{D^b(X_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})}(\mathfrak{T}_n(r), \iota_* \nu_{Y,n}^{r-1}[-r-1]) = 0$$

for the reason of degrees, the pair $(\mathfrak{T}_n(r), t')$ is unique up to a unique isomorphism in $D^b(X_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ and g is determined by $(\mathfrak{T}_n(r), t')$. See also Exercise 5 (3) below. If the residue fields of $\mathfrak{O}$ of characteritic p are perfect, then we have

$$\mathfrak{T}_n(r) \cong Rj_* \mu_{p^n}^{\otimes r} \quad \text{for any } r \geqq \dim X_K + 1$$

Exercise 5 *Let $\mathscr{A}$ be an abelian category with enough injective objects, and let $\mathscr{N}_1 \xrightarrow{f} \mathscr{N}_2 \xrightarrow{g} \mathscr{N}_3 \xrightarrow{h} \mathscr{N}_1[1]$ be a distinguished triangle in $D^-(\mathscr{A})$. Show the following*:

(1) *Let $i : \mathscr{K} \to \mathscr{N}_2$ be a morphism with $g \circ i = 0$, and assume $\mathrm{Hom}_{D^-(\mathscr{A})}(\mathscr{K}, \mathscr{N}_3[-1]) = 0$. Then there exists a unique morphism $i' : \mathscr{K} \to \mathscr{N}_1$ that i factors through.*

(2) *Let $p : \mathscr{N}_2 \to \mathscr{K}$ be a morphism with $p \circ f = 0$ and suppose $\mathrm{Hom}_{D^-(\mathscr{A})}(\mathscr{N}_1[1], \mathscr{K}) = 0$. Then there exists a unique morphism $p' : \mathscr{N}_3 \to \mathscr{K}$ that p factors through.*

(3) *Assume that $\mathrm{Hom}_{D^-(\mathscr{A})}(\mathscr{N}_2, \mathscr{N}_1) = 0$. Then relatively to a morphism $h : \mathscr{N}_3 \to \mathscr{N}_1[1]$, the triple $(\mathscr{N}_2, f, g)$ is unique up to a unique isomorphism, and f is determined by the pair $(\mathscr{N}_2, g)$.*

4.3 Proof of (T3)–(T7)

In our proof of (T3)–(T7), the following fact plays and essential role:

Theorem 4.8 *For any $r \geqq 0$, the sheaf $R^r j_* \mu_{p^n}^{\otimes r}$ on $X_{\text{ét}}$ is generated by the image of the symbol map*

$$\left(j_* \mathscr{O}_{X[p^{-1}]}^{\times}\right)^{\otimes r} \longrightarrow R^r j_* \mu_{p^n}^{\otimes r}.$$

Proof. See [BK1] Corollary 6.1.1, [H] Theorem 1.6 (1) and [SS2] Theorem 1.1 (see also [Sa5] Remark 2.4). □

Proof of (T5), (T6). Put $U^1\mathscr{O}_X^{\times} := \mathrm{Ker}(\mathscr{O}_X^{\times} \to \iota_* \mathscr{O}_Y^{\times})$ (in the étale topology). We define a filtration

$$0 \subset U^1 R^r j_* \mu_{p^n}^{\otimes r} \subset FR^r j_* \mu_{p^n}^{\otimes r} \subset R^r j_* \mu_{p^n}^{\otimes r}$$

on the sheaf $R^r j_* \mu_{p^n}^{\otimes r}$ as

$U^1 R^r j_* \mu_{p^n}^{\otimes r} :=$ the subsheaf generated étale locally by symbols of the form $\{a, b_1, \ldots, b_{r-1}\}$ with $a \in U^1\mathscr{O}_X^{\times}$ and $b_j \in j_* \mathscr{O}_{X[p^{-1}]}^{\times}$,

$FR^r j_* \mu_{p^n}^{\otimes r} :=$ the subsheaf generated étale locally by $U^1 R^r j_* \mu_{p^n}^{\otimes r}$ and the symbols $\{a_1, a_2, \ldots, a_r\}$ with $a_j \in \mathscr{O}_X^{\times}$.

We have $R^r j_* \mu_{p^n}^{\otimes r} / FR^r j_* \mu_{p^n}^{\otimes r} \cong \iota_* \nu_{Y,n}^{r-1}$ by Theorem 4.8 and [SH] Theorem 3.4.2. Hence

$$\mathscr{H}^r(\mathfrak{T}_n(r)) \cong FR^r j_* \mu_{p^n}^{\otimes r}. \tag{4.6}$$

Now (T5) follows from (4.6) for $r = r, r'$ and the following diagram:

$$\begin{tikzcd}
\mathfrak{T}_n(r) \otimes^{\mathbb{L}} \mathfrak{T}_n(r') \arrow[r, "t\otimes t"] \arrow[d, dotted, "\text{Exercise 5 (1)}"'] \arrow[rrd, bend left, "0 \text{ by } (4.6)"] & \tau_{\leqq r} Rj_*\mu_{p^n}^{\otimes r} \otimes^{\mathbb{L}} \tau_{\leqq r'} Rj_*\mu_{p^n}^{\otimes r'} \arrow[d, "\text{product}"] & \\
\mathfrak{T}_n(r+r') \arrow[r, "t"] & \tau_{\leqq r+r'} Rj_*\mu_{p^n}^{\otimes r+r'} \arrow[r, "\sigma"] & \nu_{Y,n}^{r+r'-1}[-r-r']
\end{tikzcd}$$

(T6) also follows from (4.6) and a similar argument. □

Proof of (T3). Let Z be a closed subscheme of Y of pure codimension, and let $\iota_Z : Z \hookrightarrow X$ be the natural closed immersion. For $s \geqq 0$, let $C_{n,Z}^s$ be the Gersten complex on $Z_{\text{ét}}$:

$$\bigoplus_{z\in Z^0} i_{z*}W_n\Omega_{z,\log}^s \xrightarrow{(-1)^{s-1}\partial^{\text{val}}} \bigoplus_{z\in Z^1} i_{z*}W_n\Omega_{z,\log}^{s-1} \xrightarrow{(-1)^{s-1}\partial^{\text{val}}} \bigoplus_{z\in Z^2} i_{z*}W_n\Omega_{z,\log}^{s-2} \xrightarrow{(-1)^{s-1}\partial^{\text{val}}} \cdots,$$

where i_z denotes the natural map $z \hookrightarrow Z$ for each $z \in Z$; Z^q denotes the set of the points on Z of codimension q for each $q \geqq 0$, and the first term is placed in degree 0. Now put $c := \text{codim}_X(Z)$. To prove (T3), we consider a composite morphism

$$\iota_{Z*} : \iota_{Z*}\nu_{Z,n}^{r-c}[-r-c] \xrightarrow{\gamma} \iota_*\nu_{Y,n}^{r-1}[-r-1] \xrightarrow{g} \mathfrak{T}_n(r) \quad \text{in} \quad D^b(X_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}),$$

where we define the left arrow γ as the following zig-zag of complex homomorphisms

$$\iota_{Z*}\nu_{Z,n}^{r-c}[-r-c] \longrightarrow \iota_{Z*}C_{n,Z}^{r-c}[-r-c] \longrightarrow \iota_*C_{n,Y}^{r-1}[-r-1] \xleftarrow{\text{qis}} \iota_*\nu_{Y,n}^{r-1}[-r-1]$$

where we have used a result of Gros-Suwa [GS] to verify that the most right arrow is a quasi-isomorphism [Sa1] Corollary 2.2.5 (1). Since γ induces an isomorphism

$$\nu_{Z,n}^{r-c}[-r-c] \cong \tau_{\leqq r+c}(R\iota_Z^!\iota_*\nu_{Y,n}^{r-1}[-r-1]) \quad \text{in} \quad D^b(Z_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$$

by [Sa1] Theorem 2.4.2, it remains to check that

$$\tau_{\leqq r+c}(\iota_Z^!\iota_*\nu_{Y,n}^{r-1}[-r-1]) \cong \tau_{\leqq r+c}\iota_Z^!\mathfrak{T}_n(r) \tag{4.7}$$

via $R\iota_Z^!(g)$. To prove (4.7), it is enough to show that

$$\tau_{\leqq r+c}R\iota_Z^!(\tau_{\leqq r}Rj_*\mu_{p^n}^{\otimes r}) = 0. \tag{4.8}$$

(4.8) is reduced to the $n = 1$ case by a distinguished triangle

$$\tau_{\leqq r}Rj_*\mu_{p^{n-1}}^{\otimes r} \longrightarrow \tau_{\leqq r}Rj_*\mu_{p^n}^{\otimes r} \longrightarrow \tau_{\leqq r}Rj_*\mu_p^{\otimes r} \longrightarrow (\tau_{\leqq r}Rj_*\mu_{p^{n-1}}^{\otimes r})[1],$$

and then further reduced to showing that

$$\tau_{\leqq r+c-1}R\iota_Z^!(\tau_{\geqq r+1}Rj_*\mu_p^{\otimes r}) = 0 \tag{4.9}$$

by a distinguished triangle

$$\tau_{\leqq r}Rj_*\mu_p^{\otimes r} \longrightarrow Rj_*\mu_p^{\otimes r} \longrightarrow \tau_{\geqq r+1}Rj_*\mu_p^{\otimes r} \longrightarrow (\tau_{\leqq r}Rj_*\mu_p^{\otimes r})[1],$$

Finally the vanishing (4.9) is due to Hagihara [SH] Theorem A.2.6 (see also [Sa5] Proof of Proposition 2.6). □

Proof of (T4), (T7). See [SH] §6 and [Sa5] Lemma 2.8 for the proof of (T4). The property (T4) is a key ingredient of (and closely related to) the proof of (T7). (T7) is proved mainly in the following three steps (cf. Exercise 5 (1)):

Step 1. Show the existence (and the uniqueness) of tr_f, when $f : Z \to X$ is isomorphic to the projective space $\mathbb{P}^m_X \to X$. This step is in fact a part of the final step of the proof of (T4). See [SH] Lemma 6.4.1 for details.

Step 2. Show that

$$\mathrm{Hom}_{D^b(X_{\text{ét}},\mathbb{Z}/p^n\mathbb{Z})}(Rf_!\mathfrak{T}_n(r-c)_Z[-2c], \iota_*R\iota^!\mathfrak{T}_n(r)_X) = 0.$$

See [SH] (7.2.1) and [Sa5] Proof of Proposition 2.9 (1) for details.

Step 3. Show that the composite morphism

$$Rf_!\mathfrak{T}_n(r-c)_Z[-2c] \xrightarrow{Rj_*(\mathrm{tr}_\psi)} Rj_*\mu_{p^n}^{\otimes r} \xrightarrow{\delta^{\mathrm{loc}}} \iota_*R\iota^!\mathfrak{T}_n(r)_X[1]$$

is zero in $D^b(X_{\text{ét}},\mathbb{Z}/p^n\mathbb{Z})$. We use Step 1 to prove of Step 3 in the general case. See [SH] (7.2.2) for details. □

Remark 4.9 Assume that the residue field of $\mathfrak{O}$ of characteritic p are perfect. When $r \geqq d := \dim X_K + 1$, one can construct a canonical trace morphism

$$\mathrm{tr}_\pi : R\pi_!\mathfrak{T}_n(r)_X[2(d-1)] \longrightarrow \mathfrak{T}_n(r+1-d)_B$$

using the arguments in [JSS] §5.4; we do not need the log smoothness of π there. It is not so difficult to see the uniqueness of tr_π.

The following relative duality theorem is a consequence of Gabber's absolute purity [FG] and duality results of [JSS] Theorems 4.6.1, 4.6.2, where de Jong's alteration theorem [dJ] plays an important role.

Theorem 4.10 *Assume that $\pi : X \to B$ is separated, and that any residue field of $\mathfrak{O}$ of characteristic p is perfect. Then (without log smoothness assumption) the adjunction morphism of tr_π is an isomorphism for any $r \geqq d = \dim X_K + 1$:*

$$\mathfrak{T}_n(r)_X[2(d-1)] \cong R\pi^!\mathfrak{T}_n(r+1-d)_B \quad \textit{in} \ \ D^+(X_{\text{ét}},\mathbb{Z}/p^n\mathbb{Z}).$$

4.4 Comparison with other complexes

Theorem 4.11 ([Sa3], [Sa5]) *Let $\mathbb{Z}(r)$ be Bloch's $\mathbb{Z}(r)$ considered on $X_{\text{ét}}$. If $\pi : X \to B$ is log smooth around Y, then there exists a canonical morphism*

$$\mathrm{cyc}^r : \mathbb{Z}(r) \otimes \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathfrak{T}_n(r) \quad \textit{in} \quad D^-(X_{\text{ét}},\mathbb{Z}/p^n\mathbb{Z})$$

which agrees with Bloch's cycle morphism [Bl3] *restricted onto $X[p^{-1}]$.*

To construct this cycle morphism, we need the following improvements on $\mathfrak{T}_n(r)$:

- We extend $\mathfrak{T}_n(r)$ to a complex of sheaves on a *big étale site* $\mathscr{C}_{\text{ét}}$ whose underlying category $\mathscr{C}$ is the category of schemes X over B which satisfies $(*_1)$ of Setting 4.1 and which is log smooth over B around Y.

- We introduce $\mathfrak{T}_n(r)$ for $r < 0$ to formulate a *projective bundle formula* correctly.

$$\mathfrak{T}_n(r) := j_! \mathscr{H}om\big(\mu_{p^n}^{\otimes(-r)}, \mathbb{Z}/p^n\big) \quad \text{for } \ r < 0.$$

- Because $\mathfrak{T}_n(r)$ is *not homotopy invariant* for $r \geqq 0$, we introduce a version of the complex $\mathfrak{T}_n(r)$ *with log poles* along a nice divisor $D \subset X$ which is flat over B, and formulate a certain homotopy invariance using this new complex. See Lemma B.5 below.

- We further prove purity of $\mathfrak{T}_n(r)_{(X,D)}$ *along log poles.* See (B9) of Appendix B below.

Let $\epsilon : \mathscr{C}_{\text{ét}} \to \mathscr{C}_{\text{zar}}$ be the natural continuous map of big sites. We apply the framework of Appendix B to the complexes $\{R\epsilon_*\mathfrak{T}_n(r)\}_{r\in\mathbb{Z}}$ on $\mathscr{C}_{\text{zar}}$ to obtain cyc^r of Theorem 4.11 for each $X \in \text{Ob}(\mathscr{C})$, where we need the property (T4) of $\mathfrak{T}(r)$ for $r \geqq 0$ to verify that $\{R\epsilon_*\mathfrak{T}_n(r)\}_{r\in\mathbb{Z}}$ satisfies the axiom (B4).

Exercise 6 *Show that* cyc^r *of Theorem 4.11 is an isomorphism for* $r = 0, 1$.

Conjecture 4.12 ([SH] Conjecture 1.4.1) cyc^r *is an isomorphism for any* $r \geqq 2$.

This conjecture is equivalent to another conjecture that $\mathbb{Z}(r) \otimes \mathbb{Z}/p^n\mathbb{Z}$ *is acyclic at degrees* $> r$. See [Sa3] Remark 7.2 and [Z] Theorem 1.3. If X is smooth over B then this last acyclicity conjecture holds true by Geisser [Ge] Theorem 1.2 (5).

Proposition 4.13 *If* $\mathfrak{O}$ *is a complete d.v.r. and* X *is smooth over* B, *then we have*

$$\iota^*\mathfrak{T}_n(r) \cong \mathcal{S}_n(r) \quad \textit{in} \quad D^-(Y_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$$

for any $0 \leqq r \leqq p-2$, *where the right hand side denotes the syntomic complex of Fontaine-Messing.*

Proof. The assertion follows from a result of Kurihara [Ku] Theorem 1 and the definition of $\mathfrak{T}_n(r)$. $\square$

5 Selmer group of Bloch-Kato

For a profinite group G and a topological G-module M, let $H^*(G, M)$ be the continuous Galois cohomology in the sense of Tate [T3]. For example, $H^0(G, M) = M^G$ and

$$H^1(G, M) = \frac{\big\{\varphi : G \to M \text{ continuous map} \,\big|\, {}^\forall x, {}^\forall y \in G, \varphi(xy) = \varphi(x) + x\cdot\varphi(y)\big\}}{\big\{\varphi : G \to M \text{ continuous map} \,\big|\, {}^\exists a \in M, {}^\forall x \in G, \varphi(x) = x\cdot a - a\big\}}$$

by definition. For a field K, we fix a separable closure $\overline{K}$ of K and put $G_K := \text{Gal}(\overline{K}/K)$. For a topological G_K-module M, we write $H^*(K, M)$ for $H^*(G_K, M)$. In this section, we introduce the Selmer group of Bloch-Kato [BK2] associated with ℓ-adic representations of G_K for local and global fields K.

5.1 Selmer group of local Galois representations

In this subsection, let K be a p-adic field, i.e., a finite field extension of $\mathbb{Q}_p$. Let B_{dR}, B_{st} and B_{crys} be *Fontaine's period rings of de Rham*, *semistable* and *crystalline representations*, respectively [F1], [F2]. Let ℓ be a prime number, and let V be a finite-dimensional $\mathbb{Q}_\ell$-vector space endowed with a continuous $\mathbb{Q}_\ell$-linear G_K-action. Recall that V is called a *de Rham* (resp. *semistable*, *crystalline*) *representation*, if $\ell = p$ and

$$\dim_{\mathbb{Q}_p} V = \dim_K (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

$$\big(\text{resp. } \dim_{\mathbb{Q}_p} V = \dim_{K_0} (B_{\text{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}, \quad \dim_{\mathbb{Q}_p} V = \dim_{K_0} (B_{\text{crys}} \otimes_{\mathbb{Q}_p} V)^{G_K}\big),$$

where K_0 denotes the maximal unramified extension of $\mathbb{Q}_p$ in K.

Definition 5.1 (1) If $\ell \neq p$, then we define

$$H^1_f(K,V) := \mathrm{Ker}(\mathrm{Res} : H^1(K,V) \to H^1(K^{\mathrm{nr}},V)),$$

where K^{nr} denotes the maximal unramified extension of K (in $\overline{K}$).

(2) If $\ell = p$, then we define

$$H^1_f(K,V) := \mathrm{Ker}(H^1(K,V) \to H^1(K, B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)).$$

$\mathrm{DR}(V) := (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$, and $\mathrm{Crys}(V) := (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{G_K}$.

Example 5.2 We have

$$H^1(K,\mathbb{Q}_\ell) = \mathrm{Hom}_{\mathrm{cont}}(G_K,\mathbb{Q}_\ell), \qquad H^1(K,\mathbb{Q}_\ell(1)) \cong \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_{n\geqq 1} K^\times/(K^\times)^{\ell^n}.$$

If $\ell \neq p$, then we have

$$H^1_f(K,\mathbb{Q}_\ell) = \mathrm{Hom}_{\mathrm{cont}}(G_k,\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell, \qquad H^1_f(K,\mathbb{Q}_\ell(1)) = 0.$$

If $\ell = p$, then

$$H^1_f(K,\mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{cont}}(G_k,\mathbb{Q}_p) \cong \mathbb{Q}_p, \qquad H^1_f(K,\mathbb{Q}_p(1)) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n\geqq 1} O_K^\times/(O_K^\times)^{p^n},$$

where O_K denotes the valuation ring of K. The last isomorphism can be explained by an exponential map.

Exercise 7 *Let $\xi \in H^1(K,V)$ correspond to an extension of ℓ-adic representations of G_K*

$$0 \longrightarrow V \longrightarrow E \longrightarrow \mathbb{Q}_\ell \longrightarrow 0.$$

Then show the following:

(1) *If $\ell \neq p$, then ξ belongs to $H^1_f(K,V)$ if and only if the induced sequence*

$$0 \longrightarrow V^{I_K} \longrightarrow E^{I_K} \longrightarrow \mathbb{Q}_\ell^{I_K}(=\mathbb{Q}_\ell) \longrightarrow 0$$

is exact.

(2) *If $\ell = p$, then ξ belongs to $H^1_f(K,V)$ if and only if the induced sequence*

$$0 \longrightarrow \mathrm{Crys}(V) \longrightarrow \mathrm{Crys}(E) \longrightarrow \mathrm{Crys}(\mathbb{Q}_p)(=K_0) \longrightarrow 0$$

is exact.

Exercise 8 *Let G be a profinite group and let N be a closed normal subgroup of G. Let M be a topological G-module, and put*

$$Z^1(N,M) := \{\varphi : N \to M \text{ continuous map} \,|\, {}^\forall x, {}^\forall y \in N, \varphi(xy) = \varphi(x) + x\cdot\varphi(y)\},$$

$$B^1(N,M) := \{\varphi : N \to M \text{ continuous map} \,|\, {}^\exists a \in M, {}^\forall x \in N, \varphi(x) = x\cdot a - a\}.$$

Then show the following:

(1) *For $\varphi \in Z^1(N,M)$ and $g \in G$, define a map $g\cdot\varphi : N \to M$ by*

$$(g\cdot\varphi)(x) := g\cdot(\varphi(g^{-1}xg)).$$

Then $g\cdot\varphi$ belongs to $Z^1(N,M)$, and the map

$$\gamma : G \times Z^1(N,M) \to Z^1(N,M), \quad (g,\varphi) \mapsto g\cdot\varphi$$

defines a left G-action on $Z^1(N,M)$.

(2) *$B^1(N,M)$ is a left G-submodule of $Z^1(N,M)$.*

(3) *N acts trivially on $H^1(N,M)$ via γ, i.e., $H^1(N,M)$ is a left G/N-module.*

Exercise 9 *Let G be a profinite group and let N be a closed normal subgroup of G. Put $\Gamma := G/N$. Let M be a topological G-module. Then show that there is an inflation-restriction exact sequence*

$$0 \longrightarrow H^1(\Gamma, M^N) \xrightarrow{\text{Inf}} H^1(G,M) \xrightarrow{\text{Res}} H^1(N,M)^\Gamma. \tag{5.1}$$

Let k be the residue field of K, and let $I_K = \mathrm{Gal}(\overline{K}/K^{\mathrm{nr}})$ be the inertia subgroup of G_K. We have $G_K/I_K \cong G_k$.

Proposition 5.3 (1) *If $\ell \neq p$, then we have*

$$H^1(k, V^{I_K}) = H^1_f(K,V).$$

(2) *If $\ell = p$, then we have*

$$H^1(k, V^{I_K}) \subset H^1_f(K,V).$$

I learned the following proof of (2) from Kentaro Nakamura.

Proof. (1) follows immediately from the exact sequence (5.1) for $(G,I,M) = (G_K, I_K, V)$. (2) follows from a commutative diagram

$$\begin{tikzcd}
H^1(k, V^{I_K}) \arrow[rr, "\text{Inf}"] \arrow[d, "\text{Inf}"'] \arrow[dr, dotted, "0"] & & H^1(K,V) \arrow[d] \\
H^1(K, V^{I_K}) \arrow[r] & H^1(K, B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} (V^{I_K})) \arrow[r] & H^1(K, B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)
\end{tikzcd}$$

and the fact that the unramified representation is crystalline [FO] Proposition 9.3. $\square$

Theorem 5.4 ([BK2] Proposition 3.8) *Put $V^* := \mathrm{Hom}_{\mathbb{Q}_\ell}(V, \mathbb{Q}_\ell)$. If $\ell = p$, assume that V is a de Rham representation. Then under the non-degenerate pairing*

$$H^1(K,V) \times H^1(K, V^*(1)) \longrightarrow H^2(K, \mathbb{Q}_\ell(1)) \cong \mathbb{Q}_\ell$$

the subspaces $H^1_f(K,V)$ and $H^1_f(K,V^*(1))$ are the exact annihilators of each other, where $V^*(1) := V^* \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(1)$.

Example 5.5 In the case $V = \mathbb{Q}_p$, under the non-degenerate pairing

$$\mathrm{Hom}_{\mathrm{cont}}(G_K, \mathbb{Q}_p) \times \left(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n K^\times/(K^\times)^{p^n}\right) \longrightarrow \mathbb{Q}_p,$$

$\mathrm{Hom}_{\mathrm{cont}}(G_k, \mathbb{Q}_p)$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n O_K^\times/(O_K^\times)^{p^n}$ are the exact annihilators of each other.

5.2 Sketch of Theorem 5.4

We omit the case $\ell \neq p$ and include a sketch of the case $\ell = p$. Note first that $\mathrm{DR}(V^*) \cong \mathrm{DR}(V)^*$ by [F1] 3.10 Théorème (v), so that V^* is also de Rham. Our task is to check

(a) $\dim_{\mathbb{Q}_p} H^1_f(K,V) + \dim_{\mathbb{Q}_p} H^1_f(K, V^*(1)) = \dim_{\mathbb{Q}_p} H^1(K,V)$,

(b) $H^1_f(K,V) \times H^1_f(K,V^*(1))$ goes to 0 under the pairing.

We omit (b) and explain (a) in what follows. There is a short exact sequence of topological G_K-modules

$$0 \longrightarrow \mathbb{Q}_p \xrightarrow{\alpha} B_{\mathrm{crys}} \oplus B^+_{\mathrm{dR}} \xrightarrow{\beta} B_{\mathrm{crys}} \oplus B_{\mathrm{dR}} \longrightarrow 0,$$

where $\alpha(x) = (x, x)$ and $\beta(x, y) = (x - \phi(x), x - y)$ ([BK2] Proposition 1.17), and B_{dR}^{+} denotes the valuation ring of B_{dR}. Tensoring this exact sequence with V and taking continuous Galois cohomology, we obtain the following exact sequence of finite-dimensional $\mathbb{Q}_p$-vector spaces:

$$0 \longrightarrow V^{G_K} \longrightarrow \mathrm{Crys}(V) \oplus \mathrm{DR}(V)^0 \longrightarrow \mathrm{Crys}(V) \oplus \mathrm{DR}(V) \xrightarrow{\delta} H^1_f(K, V) \longrightarrow 0, \quad (5.2)$$

where $\mathrm{DR}(V)^0 := (B_{\mathrm{dR}}^{+} \otimes_{\mathbb{Q}_p} V)^{G_K}$, and we have used the assumption that V is *de Rham* to verify the surjectivity of δ. See [BK2] Lemma 3.8.1 for details, and see also loc. cit. Remark 1.18 for a topological remark. By the exact sequence (5.2), we have

$$\dim_{\mathbb{Q}_p} H^1_f(K, V) = \dim_{\mathbb{Q}_p} V^{G_K} + \dim_{\mathbb{Q}_p}(\mathrm{DR}(V)/\mathrm{DR}(V)^0). \quad (5.3)$$

Applying this formula for $V^*(1)$, we obtain

$$\dim_{\mathbb{Q}_p} H^1_f(K, V^*(1)) = \dim_{\mathbb{Q}_p} V^*(1)^{G_K} + \dim_{\mathbb{Q}_p}(\mathrm{DR}(V^*)/\mathrm{DR}(V^*)^1),$$

where

$$\mathrm{DR}(V^*)^1 := (B_{\mathrm{dR}}^{+} \otimes_{\mathbb{Q}_p} V^*(1))^{G_K}.$$

On the other hand, by Tate's formula, we have

$$\dim_{\mathbb{Q}_p} V^{G_K} - \dim_{\mathbb{Q}_p} H^1(K, V) + \dim_{\mathbb{Q}_p} V^*(1)^{G_K} = -[K : \mathbb{Q}_p] \cdot \dim_{\mathbb{Q}_p} V.$$

Since V is de Rham by assumption, the right hand side agrees with $-\dim_{\mathbb{Q}_p} \mathrm{DR}(V)$. Therefore in order to prove (a), it remains to check that

$$\dim_K \mathrm{DR}(V)^0 + \dim_K \mathrm{DR}(V^*)^1 = \dim_K \mathrm{DR}(V).$$

Indeed, we have

$$\begin{aligned}
\dim_K \mathrm{DR}(V) &= \sum_{i \in \mathbb{Z}} \dim_K H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V) \\
&= \sum_{i \geqq 0} \dim_K H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V) + \sum_{i \leqq -1} \dim_K H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V) \\
&\overset{\star}{=} \sum_{i \geqq 0} \dim_K H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V) + \sum_{i \geqq 1} \dim_K H^0(K, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V^*) \\
&= \dim_K \mathrm{DR}(V)^0 + \dim_K \mathrm{DR}(V^*)^1,
\end{aligned}$$

where the first and the last equality is explained in [BK2] Proof of Lemma 3.8.1.

Exercise 10 *Show the equality $\star$.*

5.3 Selmer group of global Galois representations

In this subsection, let K be a number field, i.e., a finite field extension of $\mathbb{Q}$. Let P be the set of the places of K. For each place v of K, let K_v for the completion of K at v; we fix a K-homomorphism $\overline{K} \to \overline{K_v}$. Let p be a prime number, and let V be a finite-dimensional $\mathbb{Q}_p$-vector space endowed with a continuous $\mathbb{Q}_p$-linear G_K-action. We assume the following:

Condition 5.6 *There exists a finite set S of places of K including $\{v \in P \mid v|p \text{ or } v|\infty\}$ such that V is unramified outside of S, i.e., for any place $v \notin S$, the inertia subgroup I_v of the decomposition group $D_v = \mathrm{Gal}(\overline{K_v}/K_v)$ acts trivially on V.*

Definition 5.7 We fix a finite set S of places of K as in Condition 5.6, and define

$$H^1_f(K, V) := \mathrm{Ker}\left(H^1(G_S, V) \longrightarrow \bigoplus_{v \in S} \frac{H^1(K_v, V)}{H^1_f(K_v, V)} \right).$$

Here $G_S := \mathrm{Gal}(K_S/K)$ with K_S the maximal extension of K which is unramified outside of S. The space $H^1(G_S, V)$ is finite-dimensional over $\mathbb{Q}_p$, so is $H^1_f(K, V)$.

Exercise 11 *Show that $H^1_f(K, V)$ is independent of the choice of S as in Condition 5.6.*

Example 5.8 We have

$$H^1_f(K, \mathbb{Q}_p) = 0, \qquad H^1_f(K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p \otimes O_K^\times.$$

Exercise 12 *Let E be an elliptic curve over K, and let $\mathrm{Sel}(E/K)^{(p)}$ be the p-primary Selmer group:*

$$\mathrm{Sel}(E/K)^{(p)} := \mathrm{Ker}\left(H^1(K, E\{p\}) \longrightarrow \prod_{v \in P} \frac{H^1(K_v, E\{p\})}{E(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right),$$

where $E\{p\}$ denotes the p-primary torsion part of $E(\overline{K})$. Put

$$V_E := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geqq 1} {}_{p^n}E,$$

where ${}_{p^n}E$ denotes the p^n-torsion part of $E(\overline{K})$. Is there a natural map

$$H^1_f(K, V_E) \longrightarrow \mathrm{Sel}(E/K)^{(p)}\,?$$

If so, is the cokernel finite?

Proposition 5.9 ([J] Lemma 4) *Let X_K be a proper smooth variety over K and put $V^i := H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. Let S be the set of the places $v \in P$ which divides p or ∞, or at which X_K has bad reduction. Then the inflation map*

$$\mathrm{Inf} : H^1(G_S, V^i(r)) \longrightarrow H^1(K, V^i(r))$$

is bijective for any (i, r) with $i - 2r \neq -2$. In particular, $H^1(K, V^i(r))$ is finite-dimensional over $\mathbb{Q}_p$ for the same (i, r).

Proof. Consider the inflation-restriction exact sequence

$$0 \longrightarrow H^1(G_S, V^i(r)) \xrightarrow{\mathrm{Inf}} H^1(K, V^i(r)) \xrightarrow{\mathrm{Res}} H^1(K_S, V^i(r))^{G_S}.$$

Noting that $G_{K_S} = \mathrm{Gal}(\overline{K}/K_S)$ is the smallest closed normal subgroup of G_K containing I_v for all $v \notin S$, we have

$$H^1(K_S, V^i(r))^{G_S} = \mathrm{Hom}_{\mathrm{cont}}(G_{K_S}, V^i(r))^{G_S} \overset{\mathrm{Res}}{\hookrightarrow} \prod_{v \notin S} \mathrm{Hom}_{\mathrm{cont}}(I_v, V^i(r))^{\Gamma_v}.$$

where $\Gamma_v := D_v/I_v$. For any $v \notin S$, we have $v \nmid p$ and

$$\mathrm{Hom}_{\mathrm{cont}}(I_v, V^i(r))^{\Gamma_v} \cong \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p(1), V^i(r))^{\Gamma_v} \cong V^i(r-1)^{\Gamma_v},$$

which is zero by Deligne [D] Corollary 3.3.9 and the assumption that $i \neq 2(r-1)$. $\square$

Remark 5.10 By Proposition 5.9, we have

$$H^1_f(K, V^i(r)) = \mathrm{Ker}\left(H^1(K, V^i(r)) \longrightarrow \bigoplus_{v \in S} \frac{H^1(K_v, V^i(r))}{H^1_f(K_v, V^i(r))} \right)$$

if $2r - i - 1 \neq 1$, i.e., $i - 2r \neq -2$. This fact corresponds to the conjecture that $\mathrm{CH}^r(X_K)_{\mathrm{hom}}$ (resp. $\mathrm{CH}^r(X_K, 2r-i-1)$) is finitely generated modulo torsion if $2r - i - 1 = 0$ (resp. $2r - i - 1 \geqq 2$).

Conjecture 5.11 ([BK2] Conjecture 5.3) *Let X_K be a proper smooth variety over K. Then the p-adic Abel-Jacobi maps*

$$\mathrm{aj}^{2r,r} : \mathrm{CH}^r(X_K)_{\mathrm{hom}} \longrightarrow H^1(K, V^{2r-1}(r))$$

$$\mathrm{aj}^{2r-i,r} : \mathrm{CH}^r(X_K, i) \longrightarrow H^1(K, V^{2r-i-1}(r)) \qquad (i \geqq 1)$$

induce isomorphisms

$$\mathrm{CH}^r(X_K)_{\mathrm{hom}} \otimes \mathbb{Q}_p \cong H^1_f(K, V^{2r-1}(r))$$

$$\mathrm{CH}^r(X_K, i)_{\mathbb{Z}} \otimes \mathbb{Q}_p \cong H^1_f(K, V^{2r-i-1}(r)) \qquad (i \geqq 1),$$

where $\mathrm{CH}^r(X_K, i)_{\mathbb{Z}}$ *denotes the integral part of* $\mathrm{CH}^r(X_K, i) \otimes \mathbb{Q} \cong K_i(X_K)^{(r)}$*, cf.* [Sch].

Conjecture 5.11 extends the Tate conjecture to '*higher extensions*', and the first part of the Tamagawa number conjecture.

5.4 Local-global maps

In this subsection, K remains to be a number field. Let P be the set of all places of K.

Setting 5.12 Let T be a free $\mathbb{Z}_p$-module of finite rank on which G_K acts continuously. Put $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$, and assume

(i) *There exists a finite set S of places of K containing all places dividing $p \cdot \infty$ such that the action of G_K is unramified outside of S.*

Put

$$H^1_f(K, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) := \mathrm{Im}(H^1_f(K, V) \to H^1(K, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)).$$

For each $v \in P$, put

$$H^1_f(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) := \mathrm{Im}(H^1_f(K_v, V) \to H^1(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)),$$

and let $D_v \subset G_K$ be the decomposition group of v, which is dependent on the (fixed) K-homomorphism $\overline{K} \to \overline{K_v}$. Let k_v be the residue field of K_v. Let α and β be as follows:

$$\alpha : \frac{H^1(K, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{H^1_f(K, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)} \longrightarrow \bigoplus_{v \in P} \frac{H^1(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{H^1_f(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)},$$

$$\beta : \ H^2(K, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \bigoplus_{v \in P} H^2(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p),$$

where we have to note that $H^1_f(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) = H^1(k_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ for any $v \notin S$, to verify the well-definedness of α.

Proposition 5.13 ([BK2] Lemma 5.16) $\mathrm{Ker}(\alpha)$ *is finite,* $\mathrm{Coker}(\beta) = 0$*, and* $\mathrm{Coker}(\alpha)$ *and* $\mathrm{Ker}(\beta)$ *are cofinitely generated over $\mathbb{Z}_p$. Assume further that*

(ii) *V is a de Rham representation of D_v at any $v \in S$ with $v|p$.*

(iii) *$V^{D_v} = 0$ for any $v \in S$ with $v \nmid p \cdot \infty$, and $\mathrm{Crys}(V|_{D_v})^{\varphi_v = 1} = 0$ for any $v \in S$ with $v|p$. Here for each $v|p$, $\mathrm{Crys}(V|_{D_v})$ denotes $(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{D_v}$ and φ_v denotes the Frobenius operator.*

(iv) *$V(-1)^{D_v} = 0$ for any $v \in P \setminus S$.*

Then we have

$$\dim_{\mathbb{Q}_p} H^1_f(K, V) = [K : \mathbb{Q}] \cdot \dim_{\mathbb{Q}_p} V - \dim_{\mathbb{Q}_p} \mathrm{DR}(V)^0 - \dim_{\mathbb{Q}_p} V^+$$
$$+ \mathrm{corank}_{\mathbb{Z}_p} \mathrm{Coker}(\alpha) + \mathrm{corank}_{\mathbb{Z}_p} \mathrm{Ker}(\beta),$$

where $\mathrm{DR}(V)^0$ *denotes the direct sum of* $(B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V)^{D_v}$ *for all* $v|p$ *and* V^+ *denotes the direct sum of* V^{D_v} *for all* $v|\infty$*.*

Proof. Take an open subset $U \subset B \smallsetminus S$, and consider restriction homomorphisms

$$\alpha_U : \frac{H^1(U, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{H^1_f(K, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)} \longrightarrow \bigoplus_{v \in P \smallsetminus U} \frac{H^1(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{H^1_f(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)},$$

$$\beta_U : H^2(U, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \bigoplus_{v \in P \smallsetminus U} H^2(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

Note that β_U (hence β) is surjective by Tate duality [T1]. We first claim that we have $\mathrm{Ker}(\alpha_U) = \mathrm{Ker}(\alpha)$ and an exact sequence

$$0 \longrightarrow \mathrm{Coker}(\alpha_U) \longrightarrow \mathrm{Coker}(\alpha) \longrightarrow \mathrm{Ker}(\beta_U) \longrightarrow \mathrm{Ker}(\beta) \longrightarrow 0. \tag{5.4}$$

Indeed, there is a commutative diagram with exact columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\mathrm{Ker}(\alpha_U) \arrow[r] & \dfrac{H^1(U, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{H^1_f(K, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)} \arrow[r, "\alpha_U"] \arrow[d] & \bigoplus_{v \notin U} \dfrac{H^1(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{H^1_f(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)} \arrow[r] \arrow[d] & \mathrm{Coker}(\alpha_U) \\
\mathrm{Ker}(\alpha) \arrow[r] & \dfrac{H^1(K, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{H^1_f(K, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)} \arrow[r, "\alpha"] \arrow[d] & \bigoplus_{v \in P} \dfrac{H^1(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{H^1_f(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)} \arrow[r] \arrow[d] & \mathrm{Coker}(\alpha) \\
 & \bigoplus_{v \in U} H^2_v(U, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \arrow[r, equal, "\sim"] \arrow[d] & \bigoplus_{v \in U} \dfrac{H^1(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{H^1_f(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)} \arrow[d, "0"] & \\
\mathrm{Ker}(\beta_U) \arrow[r] & H^2(U, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \arrow[r, "\beta_U"] \arrow[d] & \bigoplus_{v \notin U} H^2(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \arrow[r] \arrow[d] & 0 \\
\mathrm{Ker}(\beta) \arrow[r] & H^2(K, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \arrow[r, "\beta"] \arrow[d] & \bigoplus_{v \in P} H^2(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \arrow[r] \arrow[d] & 0 \\
 & \bigoplus_{v \in U} H^3_v(U, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \arrow[r, equal, "\sim"] \arrow[d] & \bigoplus_{v \in U} H^2(K_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}$$

We obtain the above claims by a diagram chase on this diagram. Since $H^*_c(U, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is cofinitely generated over $\mathbb{Z}_p$, we see that $\mathrm{Ker}(\alpha_U) = \mathrm{Ker}(\alpha)$ is finite by the definition of $H^1_f(K, V)$ and that $\mathrm{Coker}(\alpha_U)$ and $\mathrm{Ker}(\beta_U)$ (hence $\mathrm{Coker}(\alpha)$ and $\mathrm{Ker}(\beta)$) are cofinitely generated over $\mathbb{Z}_p$. See [BK2] Proof of Lemma 5.16 for the dimension formula. $\square$

Definition 5.14 Let X_K be a proper smooth variety over K. When $T = \widetilde{H}^i(X_{\overline{K}}, \mathbb{Z}_p(r)) := H^i(X_{\overline{K}}, \mathbb{Z}_p(r))/H^i(X_{\overline{K}}, \mathbb{Z}_p(r))\{p\}$, then $\mathrm{Ker}(\alpha)$ is called *the p-Tate-Shafarevich group of the motive $H^i(X_K)(r)$* and often denoted by $Ш^{(p)}(H^i(X_K)(r))$.

Conjecture 5.15 ([BK2] Proposition 5.14 (2)) *Assume that $T = \widetilde{H}^i(X_{\overline{K}}, \mathbb{Z}_p(r))$ with $i - 2r \leqq -3$. Then* $\mathrm{Coker}(\alpha)$ *is finite and isomorphic to* $\mathrm{Hom}_{G_K}(T, \mathbb{Q}_p/\mathbb{Z}_p(1))^\vee$.

By Jannsen [J] p. 337 Theorem 3 (d), this conjeture for T is equivalent to the following conjecture for $T^*(1) = \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p(1))$:

Conjecture 5.16 ([Fl] Conjecture 1.6) *Assume that $T = \widetilde{H}^i(X_{\overline{K}}, \mathbb{Z}_p(r))$ with $i-2r \geqq 0$. Then we have $H^1_f(K, V) = 0$.*

Example 5.17 Conjecture 5.15 holds true for $T = \mathbb{Z}_p(2)$ (Moore 1968, Garland 1971), and $T = \mathbb{Z}_p(r)$ with $r \geqq 3$ (Borel 1974, Soulé 1979, Kahn unpublished).

6 Filtration on the direct image

The sections 6–8 are devoted to the proof of Theorem 8.2 below.

Setting 6.1 Let $\mathfrak{O}, B, K, p, X, Y, \iota$ and j be as in Setting 4.1. In this section we further assume that

($*_2$) *the structure morphism $\pi : X \to B$ is separated and surjective, and any residue field of $\mathfrak{O}$ of characteristic p is perfect.*

We do not assume the log smoothness of π in this section. Unless indicated otherwise, all cohomology groups of schemes are taken over the étale topology.

6.1 The complex $\mathfrak{H}^*(X, \mathfrak{T}_n(r))$

Lemma 6.2 *For any $r \geqq d := \dim X$, we have*

$$R\pi_*\mathfrak{T}_n(r)_X \cong R\mathscr{H}om_{B,\mathbb{Z}/p^n\mathbb{Z}}(R\pi_!\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(1)_B)[2-2d]$$

in $D^+(B_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$, where $\mathfrak{T}_n(s) := j_!\mathscr{H}om\big(\mu_{p^n}^{\otimes(-s)}, \mathbb{Z}/p^n\big)$ for $s < 0$.

Proof. Since $r \geqq d$, there exists a canonical isomorphism

$$\mathfrak{T}_n(r)_X \cong R\mathscr{H}om_{X,\mathbb{Z}/p^n\mathbb{Z}}(\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(d)_X) \quad \text{in} \quad D^+(X_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}),$$

which is obvious if $r = d$. Otherwise, this isomorphism follows from the isomorphism

$$\mathfrak{T}_n(r)_X \cong Rj_*\mu_{p^n}^{\otimes r} \qquad (\text{for } r > d, \text{ by the assumption } (*_2))$$

and the adjunction between $j_!$ and Rj_*. Hence we have

$$\begin{aligned} R\pi_*\mathfrak{T}_n(r)_X &\cong R\pi_*R\mathscr{H}om_{X,\mathbb{Z}/p^n\mathbb{Z}}(\mathfrak{T}_n(d-r)_X, R\pi^!\mathfrak{T}_n(1)_B[2-2d]) \\ &\cong R\mathscr{H}om_{B,\mathbb{Z}/p^n\mathbb{Z}}(R\pi_!\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(1)_B)[2-2d] \end{aligned}$$

in $D^+(B_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ by Theorem 4.10 and the adjunction between $R\pi_!$ and $R\pi_*$. □

Definition 6.3 For each $r \geqq d$ and $i \in \mathbb{Z}$, we define

$$\begin{aligned} \mathfrak{H}^{\leqq i}(X, \mathfrak{T}_n(r)) &:= R\mathscr{H}om_{B,\mathbb{Z}/p^n\mathbb{Z}}(\tau_{\geqq 2(d-1)-i}R\pi_!\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(1)_B)[2-2d], \\ \mathfrak{H}^i(X, \mathfrak{T}_n(r)) &:= R\mathscr{H}om_{B,\mathbb{Z}/p^n\mathbb{Z}}(R^{2(d-1)-i}\pi_!\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(1)_B), \end{aligned}$$

which are objects of $D^+(B_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$. See §4.4 for the definition of $\mathfrak{T}_n(s)$ for $s < 0$.

Note that $\mathfrak{H}^i(X,\mathfrak{T}_n(r))$ is *not* the sheaf $R^i\pi_*\mathfrak{T}_n(r)$, but a complex of sheaves. These objects are related by a distinguished triangle of the form

$$\mathfrak{H}^{\leqq i-1}(X,\mathfrak{T}_n(r)) \longrightarrow \mathfrak{H}^{\leqq i}(X,\mathfrak{T}_n(r)) \longrightarrow \mathfrak{H}^i(X,\mathfrak{T}_n(r))[-i] \longrightarrow \mathfrak{H}^{\leqq i-1}(X,\mathfrak{T}_n(r))[1].$$

By Lemma 6.2 and the proper base change theorem (for $R\pi_!$), we have

$$\mathfrak{H}^{\leqq i}(X,\mathfrak{T}_n(r)) \cong \begin{cases} 0 & (i \leqq -1) \\ R\pi_*\mathfrak{T}_n(r)_X & (i \geqq 2(d-1)) \end{cases}$$
$$\mathfrak{H}^i(X,\mathfrak{T}_n(r)) = 0 \qquad \text{unless} \quad 0 \leqq i \leqq 2(d-1).$$

Thus the data $\{\mathfrak{H}^{\leqq i}(X,\mathfrak{T}_n(r))\}_{i\leqq 2(d-1)}$ forms a finite ascending filtration on the complex $\mathfrak{H}^{\leqq 2(d-1)}(X,\mathfrak{T}_n(r)) \cong R\pi_*\mathfrak{T}_n(r)_X$, and yield a spectral sequence

$$E_2^{a,b} = H^a(B,\mathfrak{H}^b(X,\mathfrak{T}_n(r))) \Longrightarrow H^{a+b}(X,\mathfrak{T}_n(r)), \tag{6.1}$$

which relates the étale cohomology of B with coefficients in $\mathfrak{H}^*(X,\mathfrak{T}_n(r))$ to the étale cohomology of X with coefficients in $\mathfrak{T}_n(r)$.

Example 6.4 ([Sa5] Proposition 3.4) Assume that $\pi : X \to B$ is *proper* and that the generic fiber X_K is *geometrically connected over* K. Let $U \subset B[p^{-1}]$ be an open subset for which $X_U = X \times_B U \to U$ is smooth. Then

(1) $\mathfrak{H}^i(X,\mathfrak{T}_n(r))|_U$ is the locally constant constructible sheaf placed in degree 0, associated with the $\pi_1(U,\overline{\eta})$-module $H^i(X_{\overline{K}},\mu_{p^n}^{\otimes r})$, where $\overline{\eta} := \mathrm{Spec}(\overline{K})$.

(2) The trace morphism $\mathrm{tr}_{X/B} : R\pi_*\mathfrak{T}_n(r)_X[2(d-1)] \to \mathfrak{T}_n(r+1-d)_B$ induces an isomorphism
$$\mathfrak{H}^{2(d-1)}(X,\mathfrak{T}_n(r)) \cong \mathfrak{T}_n(r+1-d)_B.$$

6.2 Local structure of $\mathfrak{H}^*(X,\mathfrak{T}_n(r))$

For a closed point $v \in B$, we often write Y_v (resp. $Y_{\overline{v}}$, $X_{\overline{v}}$) for $(X \times_B v)_{\mathrm{red}}$ (resp. $X \times_B \overline{v}$, $X \times_B B^{\mathrm{sh}}_{\overline{v}}$), where $B^{\mathrm{sh}}_{\overline{v}}$ denotes the spectrum of the strict henselization of $\mathfrak{O}_v = \mathscr{O}_{B,v}$ at its maximal ideal.

Proposition 6.5 *Let v be a closed point on B, and let q and m be integers. We write ι_v for the closed immersion $v \hookrightarrow B$ and j_v for the open immersion $B \smallsetminus v \hookrightarrow B$. Assume $r \geqq d$. Then*

(1) *We have $R^q\iota_v^!\mathfrak{H}^i(X,\mathfrak{T}_n(r)) = 0$ unless $q = 2$, and a canonical isomorphism*
$$(R^2\iota_v^!\mathfrak{H}^i(X,\mathfrak{T}_n(r)))_{\overline{v}} \cong H^{i+2}_{Y_{\overline{v}}}(X_{\overline{v}},\mathfrak{T}_n(r)).$$

(2) *We have*
$$(R^q j_{v*}j_v^*\mathfrak{H}^i(X,\mathfrak{T}_n(r)))_{\overline{v}} \cong H^q(I_v, H^i(X_{\overline{K}},\mu_{p^n}^{\otimes r})),$$
which is zero unless $q = 0$ or 1 by the fact that $\mathrm{cd}_p(I_v) = 1$.

Proof. (1) By the adjunction between ι_{v*}, we have

$$\begin{aligned} R\iota_v^!\mathfrak{H}^i(X,\mathfrak{T}_n(r)) &= R\iota_v^! R\mathscr{H}om_{B,\mathbb{Z}/p^n\mathbb{Z}}(R^{2(d-1)-i}\pi_!\mathfrak{T}_n(d-r)_X, \mathfrak{T}_n(1)_B) \\ &\cong R\mathscr{H}om_{v,\mathbb{Z}/p^n\mathbb{Z}}(\iota_v^* R^{2(d-1)-i}\pi_!\mathfrak{T}_n(d-r)_X, R\iota_v^!\mathfrak{T}_n(1)_B) \\ &\cong R\mathscr{H}om_{v,\mathbb{Z}/p^n\mathbb{Z}}(R^{2(d-1)-i}\pi_{Y_v/v!}(\iota_{Y_v}^*\mathfrak{T}_n(d-r)_X), \mathbb{Z}/p^n\mathbb{Z})[-2], \end{aligned}$$

where ι_{Y_v} denotes the closed immersion $Y_v \hookrightarrow X$, and we have used the proper base change theorem for $R\pi_!$ and the purity for $\mathfrak{T}_n(1)_B$ in the last isomorphism. The assertion now follows from the fact that there is a canonical non-degenerate pairing of finite groups

$$H_c^{2(d-1)-i}(Y_{\overline{v}}, \iota_{Y_v}^* \mathfrak{T}_n(d-r)) \times H^{i+2}_{Y_{\overline{v}}}(X_{\overline{v}}, \mathfrak{T}_n(r)) \longrightarrow H^{2d}_{Y_{\overline{v}},c}(X_{\overline{v}}, \mathfrak{T}_n(d)) \xrightarrow{\mathrm{Tr}} \mathbb{Z}/p^n\mathbb{Z}$$

([Sa5] Corollary 2.11) and the fact that $\mathbb{Z}/p^n\mathbb{Z}$ is an injective $\mathbb{Z}/p^n\mathbb{Z}$-module, where the subscript c means the étale cohomology with compact support.

(2) We may assume that B is local with closed point v, without loss of generality. Put $\eta := B \smallsetminus v$, which is the generic point of B. The sheaf $j_v^* R^{2(d-1)-i}\pi_! \mathfrak{T}_n(d-r)_X$ is locally constant on $\eta_{\mathrm{\acute{e}t}}$, and the object

$$j_v^* \mathfrak{H}^i(X, \mathfrak{T}_n(r)) = R\mathscr{H}om_{\eta, \mathbb{Z}/p^n\mathbb{Z}}(j_v^* R^{2(d-1)-i}\pi_! \mathfrak{T}_n(d-r), \mu_{p^n})$$

is isomorphic to the sheaf (on $\eta_{\mathrm{\acute{e}t}}$) associated with $H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes r})$ placed in degree 0 by the Poincaré duality. The assertion follows from this fact. $\square$

Corollary 6.6 *$\mathfrak{H}^i(X, \mathfrak{T}_n(r))$ is concentrated in $[0,2]$, and $R\pi_* \mathfrak{T}_n(r)_X$ is concentrated in $[0, 2d]$.*

6.3 Standard finiteness

Proposition 6.7 *Let m and n be positive integers. Then:*

(1) *There exists a unique morphism $\underline{p}^m : \mathfrak{T}_n(r) \to \mathfrak{T}_{n+m}(r)$ in $D^b(X_{\mathrm{\acute{e}t}})$ that extends the natural inclusion $\mu_{p^n}^{\otimes r} \hookrightarrow \mu_{p^{n+m}}^{\otimes r}$ on $X[p^{-1}]_{\mathrm{\acute{e}t}}$.*

(2) *There exists a unique morphism $\mathscr{R}^m : \mathfrak{T}_{n+m}(r) \to \mathfrak{T}_n(r)$ in $D^b(X_{\mathrm{\acute{e}t}})$ that extends the natural surjection $\mu_{p^{n+m}}^{\otimes r} \twoheadrightarrow \mu_{p^n}^{\otimes r}$ on $X[p^{-1}]_{\mathrm{\acute{e}t}}$.*

(3) *There exists a canonical Bockstein morphism $\delta_{n,m} : \mathfrak{T}_n(r) \to \mathfrak{T}_m(r)[1]$ in $D^b(X_{\mathrm{\acute{e}t}})$ satisfying*

(3.1) *$\delta_{n,m}$ extends the Bockstein morphism $\mu_{p^n}^{\otimes r} \to \mu_{p^m}^{\otimes r}[1]$ in $D^b(X[p^{-1}]_{\mathrm{\acute{e}t}})$ associated with the short exact sequence $0 \to \mu_{p^m}^{\otimes r} \to \mu_{p^{m+n}}^{\otimes r} \to \mu_{p^n}^{\otimes r} \to 0$ on $X[p^{-1}]_{\mathrm{\acute{e}t}}$.*

(3.2) *$\delta_{n,m}$ fits in to an anti-distinguished triangle*

$$\mathfrak{T}_m(r) \xrightarrow{\underline{p}^n} \mathfrak{T}_{m+n}(r) \xrightarrow{\mathscr{R}^m} \mathfrak{T}_n(r) \xrightarrow{\delta_{n,m}} \mathfrak{T}_m(r)[1].$$

Proof. See [SH] Proposition 4.3.1 and [Sa5] Proposition 2.5. $\square$

Setting 6.8 In what follows, we assume that $\mathfrak{O}$ and $K = \mathrm{Frac}(\mathfrak{O})$ satisfy either of the following conditions:

(L) *K is a non-archimedean local field of characteristic 0, i.e., a finite field extension of $\mathbb{Q}_\ell$ for some prime number ℓ, and $\mathfrak{O}$ is the valuation ring of K.*

(G) *K is an algebraic number field, i.e., a finite field extension of $\mathbb{Q}$, and $B = \mathrm{Spec}(\mathfrak{O})$ is an open subset of $\mathrm{Spec}(O_K)$, where O_K denotes the integer ring of K.*

Proposition 6.9 *There is a canonical isomorphism*

$$H^q(B, \mathfrak{H}^i(X, \mathfrak{T}_n(r))) \cong \mathrm{Ext}^q_B(R^{2(d-1)-i}\pi_! \mathfrak{T}_n(d-r), \mathbb{G}_{\mathrm{m}}) \tag{6.2}$$

for any $q, i \geqq 0$, $n \geqq 1$ and $r \geqq d$. Moreover, $H^q(X, \mathfrak{T}_n(r))$ and $H^q(B, \mathfrak{H}^i(X, \mathfrak{T}_n(r)))$ are finite for the same (q, i, n, r).

Proof. The first assertion follows from the definition of $\mathfrak{H}^i(X,\mathfrak{T}_n(r))$ and the canonical isomorphism

$$R\mathscr{H}om_B(\mathbb{Z}/p^n\mathbb{Z},\mathbb{G}_{\mathrm{m}}) \cong \mathfrak{T}_n(1),$$

which is a variant of Exercise 6 for $r=1$. The finiteness of $\mathrm{Ext}^q_B(R^{2(d-1)-i}\pi_!\mathfrak{T}_n(d-r),\mathbb{G}_{\mathrm{m}})$ follows from the constructibility of $R^{2(d-1)-i}\pi_!\mathfrak{T}_n(d-r)$ and the finiteness of Ext-groups in the Artin-Verdier duality [Ma] (2.4). The finiteness of $H^q(X,\mathfrak{T}_n(r))$ follows from the spectral sequence (6.1). □

For $r \geqq d$, we introduce the following groups:

$$H^q(X,\mathbb{Z}_p(r)) := \varprojlim_{n\geqq 1} H^q(X,\mathfrak{T}_n(r)), \qquad H^q(X,\mathbb{Q}_p(r)) := H^q(X,\mathbb{Z}_p(r))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p,$$

$$H^q(X,\mathbb{Q}_p/\mathbb{Z}_p(r)) := \varinjlim_{n\geqq 1} H^q(X,\mathfrak{T}_n(r)),$$

$$H^q(B,\mathfrak{H}^i(X,\mathbb{Z}_p(r))) := \varprojlim_{n\geqq 1} H^q(B,\mathfrak{H}^i(X,\mathfrak{T}_n(r))),$$

$$H^q(B,\mathfrak{H}^i(X,\mathbb{Q}_p(r))) := H^q(B,\mathfrak{H}^i(X,\mathbb{Z}_p(r)))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p,$$

$$H^q(B,\mathfrak{H}^i(X,\mathbb{Q}_p/\mathbb{Z}_p(r))) := \varinjlim_{n\geqq 1} H^q(B,\mathfrak{H}^i(X,\mathfrak{T}_n(r))).$$

Here the transition maps in the definition of $H^q(B,\mathfrak{H}^i(X,\mathbb{Z}_p(r)))$ are defined by the commutative diagram

$$\begin{tikzcd} H^q(B,\mathfrak{H}^i(X,\mathfrak{T}_{n+1}(r))) \arrow[r, dotted] \arrow[d, "(6.2)"', "\cong"] & H^q(B,\mathfrak{H}^i(X,\mathfrak{T}_n(r))) \arrow[d, "(6.2)"', "\cong"] \\ \mathrm{Ext}^q_B(R^{2(d-1)-i}\pi_!\mathfrak{T}_{n+1}(d-r),\mathbb{G}_{\mathrm{m}}) \arrow[r] & \mathrm{Ext}^q_B(R^{2(d-1)-i}\pi_!\mathfrak{T}_n(d-r),\mathbb{G}_{\mathrm{m}}) \end{tikzcd}$$

with the bottom arrow induced by $\underline{p} : \mathfrak{T}_n(d-r) \hookrightarrow \mathfrak{T}_{n+1}(d-r)$ of Proposition 6.7. The transition maps in the definition of $H^q(B,\mathfrak{H}^i(X,\mathbb{Q}_p/\mathbb{Z}_p(r)))$ are defined by the commutative diagram

$$\begin{tikzcd} H^q(B,\mathfrak{H}^i(X,\mathfrak{T}_n(r))) \arrow[r, dotted] \arrow[d, "(6.2)"', "\cong"] & H^q(B,\mathfrak{H}^i(X,\mathfrak{T}_{n+1}(r))) \arrow[d, "(6.2)"', "\cong"] \\ \mathrm{Ext}^q_B(R^{2(d-1)-i}\pi_!\mathfrak{T}_n(d-r),\mathbb{G}_{\mathrm{m}}) \arrow[r] & \mathrm{Ext}^q_B(R^{2(d-1)-i}\pi_!\mathfrak{T}_{n+1}(d-r),\mathbb{G}_{\mathrm{m}}) \end{tikzcd}$$

with the bottom arrow induced by $\mathscr{R}^1 : \mathfrak{T}_{n+1}(d-r) \twoheadrightarrow \mathfrak{T}_n(d-r)$ of Proposition 6.7. Taking the projective limit of the spectral sequence (6.1) with respect to $n \geqq 1$, we obtain a convergent spectral sequence of $\mathbb{Z}_p$-modules

$$E_2^{a,b} = H^a(B,\mathfrak{H}^b(X,\mathbb{Z}_p(r))) \Longrightarrow H^{a+b}(X,\mathbb{Z}_p(r)). \tag{6.3}$$

This spectral sequence yields a spectral sequence of $\mathbb{Q}_p$-vector spaces:

$$E_2^{a,b} = H^a(B,\mathfrak{H}^b(X,\mathbb{Q}_p(r))) \Longrightarrow H^{a+b}(X,\mathbb{Q}_p(r)). \tag{6.4}$$

Theorem 6.10 (1) *$H^q(X,\mathbb{Z}_p(r))$ and $H^q(B,\mathfrak{H}^i(X,\mathbb{Z}_p(r)))$ are finitely generated over $\mathbb{Z}_p$ for any $q,i\in\mathbb{Z}$ and any $r \geqq d$.*

(2) *$H^q(X,\mathbb{Q}_p/\mathbb{Z}_p(r))$ and $H^q(B,\mathfrak{H}^i(X,\mathbb{Q}_p/\mathbb{Z}_p(r)))$ are cofinitely generated over $\mathbb{Z}_p$ for any $q,i\in\mathbb{Z}$ and any $r \geqq d$.*

(3) *We have* $\mathrm{rank}_{\mathbb{Z}_p}\ H^q(B,\mathfrak{H}^i(X,\mathbb{Z}_p(r))) = \mathrm{corank}_{\mathbb{Z}_p}\ H^q(B,\mathfrak{H}^i(X,\mathbb{Q}_p/\mathbb{Z}_p(r)))$ *for any $q,i\in\mathbb{Z}$ and any $r \geqq d$.*

Proof. We explain only the case (G); the case (L) is similar and left to the reader. The assertions for $H^q(X, \mathbb{Z}_p(r))$ and $H^q(X, \mathbb{Q}_p/\mathbb{Z}_p(r))$ follow from a standard argument using Propositions 6.7 and 6.9. By the Artin-Verdier duality ([Ma], [KCT] §3), the assertions for $H^q(B, \mathfrak{H}^i(X, \mathbb{Z}_p(r)))$ and $H^q(B, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$ are reduced to those for $H^{3-q}_c(B, R^{i'}\pi_!\mathbb{Q}_p/\mathbb{Z}_p(d-r))$ and $H^{3-q}_c(B, R^{i'}\pi_!\mathbb{Z}_p(d-r))$ with $i' := 2(d-1)-i$, which are standard and omitted. □

7 Comparison over local fields

Setting 7.1 Let $\mathfrak{O}, B, K, p$ and X be as in Setting 4.1. Fix another prime number ℓ independently of p. In this section, we assume $[K : \mathbb{Q}_\ell] < \infty$ and that $\mathfrak{O}$ is the valuation ring of K. Let k be the residue field of K, and let $I_K = \mathrm{Gal}(\overline{K}/K^{\mathrm{nr}})$ be the inertia subgroup of G_K. Put $Y := (X \otimes_{\mathfrak{O}} k)_{\mathrm{red}}$ and $\overline{Y} := Y \otimes_k \overline{k}$. Note that Y defined here is different from that in Setting 4.1 unless $\ell = p$. We often write v for the closed point of B, i.e., $v = \mathrm{Spec}(k)$.

We assume that $\pi : X \to B$ is *proper* and that $X_K := X \otimes_{\mathfrak{O}} K$ is *geometrically connected over* K. We assume further that $\pi : X \to B$ is *log smooth around* Y, if $\ell = p$ and $r = d$. Unless indicated otherwise, all cohomology groups of schemes are taken over the étale topology. For each $i, q \geqq 0$ and $r \leqq 0$, we put

$$V^i := H^i(X_{\overline{K}}, \mathbb{Q}_p) \quad \text{and} \quad H^q(B, R^i\pi_*\mathbb{Q}_p(r)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n \geqq 1} H^q(B, R^i\pi_*\mathfrak{T}_n(r)).$$

For $i \geqq 0$ and $r \in \mathbb{Z}$, we put

$$H^1_{/f}(K, V^i(r)) := H^1(K, V^i(r))/H^1_f(K, V^i(r)).$$

Theorem 7.2 *For any $i \geqq 0$ and $r \geqq d := \dim X$, we have*

$$H^q(B, \mathfrak{H}^i(X, \mathbb{Q}_p(r))) = \begin{cases} H^1_f(K, V^i(r)) & (q = 1) \\ 0 & (q \neq 1). \end{cases}$$

Moreover, if $\ell \neq p$, then we have $H^1(B, \mathfrak{H}^i(X, \mathbb{Q}_p(r))) = 0$ for any $r \geqq d$ and $i \geqq 0$.

Corollary 7.3 *The spectral sequence* (6.4) *degenerates at E_2, and we have*

$$H^i(X, \mathbb{Q}_p(r)) \cong H^1_f(K, V^{i-1}(r))$$

for any $i \geqq 0$ and $r \geqq d$.

Remark 7.4 If $\ell \neq p$, then we obtain $H^i(X, \mathbb{Q}_p(r)) = 0$ for any $r \geqq d$ and $i \geqq 0$, from the proper base change theorem and a theorem of Deligne [D] Corollary 3.3.4. Theorem 7.2 refines this fact.

7.1 Reduction to dual statements

We first check that Theorem 7.2 is reduced to the following:

Theorem 7.5 *Assume that $\pi : X \to B$ is log smooth around Y, if $\ell = p$ and $s = 0$. Then for any $i \geqq 0$ and $s \leqq 0$, we have $V^i(d-s)^{G_K} = 0$ and*

$$H^q(B, R^i\pi_*\mathbb{Q}_p(s)) \cong \begin{cases} V^i(s)^{G_K} & (\textit{if } q = 0) \\ H^1_f(K, V^i(s)) & (\textit{if } q = 1) \\ 0 & (\textit{otherwise}) \end{cases}$$

In the case $s < 0$, it is easy to see that $H^q(B, R^i\pi_*\mathbb{Q}_p(s)) = 0$ for any q and i. In this case Theorem 7.5 asserts that the groups on the right hand side are zero.

"Theorem 7.5 ⇒ Theorem 7.2". Consider the localization long exact sequence with $(q', i', s) := (3 - q, 2d - 2 - i, d - r)$

$$\cdots \longrightarrow H_v^{q'}(B, R^{i'}\pi_*\mathbb{Q}_p(s)) \longrightarrow H^{q'}(B, R^{i'}\pi_*\mathbb{Q}_p(s)) \longrightarrow H^{q'}(K, V^{i'}(s)) \longrightarrow \cdots .$$

By this exact sequence and Theorem 7.5, we have

$$H_v^{q'}(B, R^{i'}\pi_*\mathbb{Q}_p(s)) \cong \begin{cases} 0 & (q' \neq 2, 3) \\ H^1_{/f}(K, V^{i'}(s)) & (\text{if } q' = 2) \\ H^2(K, V^{i'}(s)) & (\text{if } q' = 3). \end{cases}$$

The first assertion of Theorem 7.2 follows from this fact, Theorem 5.4 and the local Tate duality for cohomology of B:

$$H_v^{q'}(B, R^{i'}\pi_*\mathbb{Q}_p(s)) \times H^q(B, \mathfrak{H}^i(X, \mathbb{Q}_p(r))) \longrightarrow H_v^3(B, \mathfrak{H}^{2d-2}(X, \mathbb{Q}_p(d))) \cong \mathbb{Q}_p,$$

where the last isomorphism is obtained from the relative trace isomorphism in Example 6.4 (2) and the trace isomorphism $H_v^3(B, \mathfrak{T}_n(1)) \cong \mathbb{Z}/p^n\mathbb{Z}$. The second assertion of Theorem 7.2 follows from the vanishing $V^i(r)^{G_K} = 0$ and the equality

$$\dim_{\mathbb{Q}_p}(V^i(r)^{G_K}) = \dim_{\mathbb{Q}_p} H^1(k, V^i(r)^{I_K}).$$

Thus Theorem 7.2 is reduced to Theorem 7.5. □

Because $R^i\pi_*\mathfrak{T}_n(s)_{\overline{v}} \cong H^i(\overline{Y}, \mathfrak{T}_n(s)|_{\overline{Y}})$, we see that the second assertion of Theorem 7.5 is reduced to the computations on the cospecialization map

$$\mathrm{cosp}^{i,s} : H^i(\overline{Y}, \mathbb{Q}_p(s)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^i(\overline{Y}, \mathfrak{T}_n(s)|_{\overline{Y}}) \longrightarrow V^i(s)^{I_K}.$$

7.2 The case $\ell \neq p$

In this subsection, we give an outline of a proof of Theorem 7.5, assuming $\ell \neq p$.

Proposition 7.6 *Let $i \geqq 0$ be an integer.*

(1) *We have $V^i(r)^{G_K} = 0$ for any $r \geqq d$.*

(2) *The cospecialization map*

$$\mathrm{cosp}^{i,s} : H^i(\overline{Y}, \mathbb{Q}_p(s)) \longrightarrow V^i(s)^{I_K}$$

induces an isomorphism

$$H^q(k, H^i(\overline{Y}, \mathbb{Q}_p(s))) \cong H^q(k, V^i(s)^{I_K})$$

for $q = 0, 1$ and any $s \leqq 0$.

Outline of Proof. When X has good reduction, the assertions of Proposition 7.6 follows from the proper smooth base change theorem and a theorem of Deligne [D] 3.3.9. The general case is reduced to the case that X has strict semi-stable reduction by the alteration theorem of de Jong [dJ] and a standard norm argument using the absolute purity [FG].

To explain the outline of our proof of the strict semi-stable case, we introduce some notation. Let $\overline{\jmath}$ be the canonical map $X_{\overline{K}} \to X^{\mathrm{nr}} := X \otimes_{\mathfrak{O}} \mathfrak{O}^{\mathrm{nr}}$, and let $\overline{\iota}$ be the closed immersion $\overline{Y} \to X^{\mathrm{nr}}$. By the properness of $\pi : X \to B$, we have the following Leray spectral sequence for any $n \geqq 1$:

$$E_2^{a,b} = H^a(\overline{Y}, \overline{\iota}^* R^b\overline{\jmath}_*\mathbb{Z}/p^n\mathbb{Z}) \Longrightarrow H^{a+b}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}). \tag{7.1}$$

By Rapoport-Zink [RZ] Theorem 2.23, we have the following facts:

- *The E_2-terms of are finite.*
- *I_K acts trivially on the E_2-terms.*

We obtain a spectral sequence

$$E_2^{a,b} = H^a(\overline{Y}, \bar{\iota}^* R^b \bar{j}_* \mathbb{Q}_p) \Longrightarrow H^{a+b}(X_{\overline{K}}, \mathbb{Q}_p) = V^{a+b} \tag{7.2}$$

from (7.1) by taking the projective limit with respect to $n \geqq 1$ and the tensor product with $\mathbb{Q}_p$ over $\mathbb{Z}_p$. Note that the canonical map $E_2^{i,0} = H^i(\overline{Y}, \mathbb{Q}_p) \to E^i = V^i$ agrees with the cospecialization map $\mathrm{cosp}_X^{i,0}$.

Lemma 7.7 *In the spectral sequence* (7.2)*, we have $E_2^{a,b} = 0$ unless $0 \leqq a \leqq 2(d-b-1)$ and $0 \leqq b \leqq d-1$. Moreover, for a pair (a,b) with $0 \leqq a \leqq 2(d-b-1)$ and $0 \leqq b \leqq d-1$, the weights of $E_2^{a,b}$ are at least $\max\{2b, 2(a+2b+1-d)\}$ and at most $a+2b$.*

Proof. See [Sa5] Lemma 5.9. □

By this lemma, the kernel and the cokernel of the cospecialization map $\mathrm{cosp}_X^{i,s}$ have only positive weights for any $s \leqq 0$, which implies Proposition 7.6 (2). One can also derive Proposition 7.6 (1) from Lemma 7.7 easily. □

7.3 The case $\ell = p$

In this subsection, we prove Theorem 7.5, assuming $\ell = p$. Note that $H^i(\overline{Y}, \mathbb{Q}_p(s)) = 0$ if $s < 0$ by the definition of $\mathfrak{T}_n(s)$.

Proposition 7.8 *Let $i \geqq 0$ be an integer, and put $V := V^i$.*

(1) *We have $V(r)^{I_K} = 0$ unless $0 \leqq r \leqq d-1 = \dim(X_K)$.*

(2) *We have $H^1(k, V(s)^{I_K}) = H^1_f(K, V(s))$ as subspaces of $H^1(K, V(s))$ for any $s \leqq 0$. In particular, we have $H^1_f(K, V(s)) = 0$ if $s < 0$.*

(3) *If $\pi : X \to B$ is log smooth around Y, then the cospecialization map*

$$\mathrm{cosp}^{i,0} : H^i(\overline{Y}, \mathbb{Q}_p) \longrightarrow V^{I_K}$$

is bijective.

Proof. The assertions (1) and (2) are reduced to the case where X has semi-stable reduction by the alteration theorem of de Jong [dJ] and a standard norm argument. So we prove (1)–(3) in the semi-stable case and then prove (3) in the log smooth case.

(I) *Proof of* (1)–(3) *in the semi-stable reduction case.* Let $B_{\mathrm{crys}}, B_{\mathrm{st}}, B_{\mathrm{dR}}^+$ and B_{dR} be Fontaine's rings as in §5.1. Put $D := H^i_{\text{log-crys}}(Y/W(k))$. By the Fontaine-Jannsen conjecture ([Ts]), there is a canonical isomorphism

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V \cong B_{\mathrm{st}} \otimes_{W(k)} D, \tag{7.3}$$

which preserves the Frobenius operator ϕ, the monodromy operator N, the action of G_K, and the Hodge filtration $\mathrm{F}_H^\bullet$ after taking $\otimes_{B_{\mathrm{st}}} B_{\mathrm{dR}}$. By the isomorphism (7.3), we have

$$V(r) \cong \left(B_{\mathrm{st}} \otimes_{W(k)} D\right)^{N=0,\, \phi=p^r} \cap \mathrm{F}_H^r \left(B_{\mathrm{dR}} \otimes_{W(k)} D\right)$$

and

$$V(r)^{I_K} \subset (H^i_{\text{log-crys}}(\overline{Y}/W(\overline{k}))_{\mathbb{Q}_p})^{\varphi=p^r}, \tag{7.4}$$

for any $r \in \mathbb{Z}$. Here φ denotes the Frobenius operator acting on $H^i_{\text{log-crys}}(\overline{Y}/W(\overline{k}))$, and we have used the fact that $(B_{\text{st}})^{I_K} = \operatorname{Frac}(W(\overline{k}))$ ([F2] 5.1.2, 5.1.3). Proposition 7.8 (1) follows from (7.4) and the fact that

$$(H^i_{\text{log-crys}}(\overline{Y}/W(\overline{k}))_{\mathbb{Q}_p})^{\varphi=p^r} = 0 \quad \text{unless } 0 \leqq r \leqq d-1.$$

Proposition 7.8 (3) is due to Wu [W] Theorem 1. Recall that we have $H^1(k, V(s)^{I_K}) \subset H^1_f(K, V(s))$ by Proposition 5.3 (2). To prove Proposition 7.8 (2), it remains to show the following claim:

Claim. We have $\dim_{\mathbb{Q}_p} H^1(k, V(s)^{I_K}) = \dim_{\mathbb{Q}_p} H^1_f(K, V(s))$ *for any* $s \leqq 0$.

Proof. Since V is a de Rham representation, we have

$$\dim_{\mathbb{Q}_p} H^1_f(K, V(s)) = \dim_{\mathbb{Q}_p}(\mathrm{DR}(V)/\mathrm{DR}(V(s))^0) + \dim_{\mathbb{Q}_p} V(s)^{G_K}$$

by (5.3). Since $s \leqq 0$, we have

$$\mathrm{DR}(V) \cong H^i_{\mathrm{dR}}(X_K/K) = \mathrm{F}^s_H H^i_{\mathrm{dR}}(X_K/K) \cong \mathrm{DR}(V(s))^0,$$

so the claim follows from the equalities

$$\dim_{\mathbb{Q}_p} H^1_f(K, V(s)) = \dim_{\mathbb{Q}_p} V(s)^{G_K} = \dim_{\mathbb{Q}_p} H^1(k, V(s)^{I_K}).$$

This completes the proof of Proposition 7.8 in the semi-stable reduction case.

(II) *Proof of* (3) *in the log smooth reduction case.* By the alteration theorem of de Jong [dJ] Theorem 6.5, there exists a proper generically étale morphism $f : X' \to X$ such that X' is regular and flat over B and has (strict) semi-stable reduction over the normalization B' of B in X'. Let L (resp. k') be the function field of B' (resp. the residue field of L'), Y' for the special fiber of $\pi' : X' \to B'$. Let $\nu : B' \to B$ be the canonical map. We derive the bijectivity of $\mathrm{cosp}^{i,0}$ for X from that for X'. There exists a trace homomorphism

$$\mathrm{tr}_f : \nu_* R^i\pi'_*\mathbb{Z}/p^n\mathbb{Z} \to R^i\pi_*\mathbb{Z}/p^n\mathbb{Z}$$

on $B_{\text{ét}}$ for each $n \geqq 1$ by (T7) of Theorem 4.3, which yields a commutative diagram

$$\begin{tikzcd}
H^i(\overline{Y}, \mathbb{Q}_p) \arrow[r, "f^\sharp"] \arrow[d, "\mathrm{cosp}^{i,0}_X"'] \arrow[rr, bend left, "\times[L(X'):K(X)]"] & H^i(\overline{Y'}, \mathbb{Q}_p) \arrow[r, "\mathrm{tr}_f"] \arrow[d, "\mathrm{cosp}^{i,0}_{X'}"'] & H^i(\overline{Y}, \mathbb{Q}_p) \arrow[d, "\mathrm{cosp}^{i,0}_X"'] \\
V^{I_K} \arrow[r, "f^\sharp"] \arrow[rr, bend right, "\times[L(X'):K(X)]"'] & H^i(X'_{\overline{L}}, \mathbb{Q}_p)^{I_L} \arrow[r, "\mathrm{tr}_f"] & V^{I_K}.
\end{tikzcd}$$

Since $\mathrm{cosp}^{i,0}_{X'}$ is bijective by (I), we see that $\mathrm{cosp}^{i,0}_X$ is bijective as well by a diagram chase. This completes the proof of Proposition 7.8 and Theorems 7.5, 7.2. □

By Proposition 7.8 (1) and the exact sequence (5.2) for $V^i(r) = H^i(X_{\overline{K}}, \mathbb{Q}_p(r))$, we obtain the following corollary:

Corollary 7.9 *The exponential map of Bloch-Kato induces an isomorphism*

$$\exp : H^i_{\mathrm{dR}}(X_K/K) \xrightarrow{\simeq} H^1_f(K, V^i(r))$$

for any $i \geqq 0$ *and* $r \geqq d$.

8 Comparison over global fields

Setting 8.1 Let $\mathfrak{O}, B, K, p, X$ and Y be as in Setting 4.1. In this section, assume that K is a number field and that $\mathfrak{O}$ is the integer ring of K. For a place v of K, we write K_v for the completion of K at v.

We assume that $\pi : X \to B$ is *proper* and that $X_K := X \otimes_{\mathfrak{O}} K$ is *geometrically connected over* K. We assume further that $\pi : X \to B$ is *log smooth around* Y, if $r = d$. We fix a finite set S of places of K including all places which divide $p\cdot\infty$ or where X has bad reduction. We put $G_S := \mathrm{Gal}(K_S/K)$, where K_S denotes the maximal extension of K which is unramified outside of S. Unless indicated otherwise, all cohomology groups of schemes are taken over the étale topology. For each $i \geqq 0$, we put $T^i := H^i(X_{\overline{K}}, \mathbb{Z}_p)$ and $V^i := T^i \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Theorem 8.2 *Let q, i and r be integers with $r \geqq d := \dim X$ and $i \geqq 0$. Assume Conjecture 5.15 for $T^i(r)$ if $q = 2$ and $i - 2r \leqq -3$. Then we have*

$$H^q(B, \mathfrak{H}^i(X, \mathbb{Q}_p(r))) \cong \begin{cases} H^1_f(K, V^i(r)) & (\text{if } q = 1) \\ \mathbb{Q}_p & (\text{if } (q, i, r) = (3, 2d-2, d)) \\ 0 & (\text{otherwise}). \end{cases}$$

Corollary 8.3 *Assume Conjecture 5.15 for $T^{i-2}(r)$, if $(i, r) \neq (2d+1, d)$. Then we have*

$$H^i(X, \mathbb{Q}_p(r)) \cong \begin{cases} \mathbb{Q}_p & (\text{if } (i, r) = (2d+1, d)) \\ H^1_f(K, V^{i-1}(r)) & (\text{otherwise}) \end{cases}$$

for any $r \geqq d$ and any $i \geqq 0$.

8.1 Proof of Theorem 8.2

We first check the assertions other than the vanishing of $H^2(B, \mathfrak{H}^i(X, \mathbb{Q}_p(r)))$.

Proposition 8.4 *Assume $r \geqq d$. Then*:

(1) *$H^q(B, \mathfrak{H}^i(X, \mathbb{Z}_p(r)))$ is finite in each of the following cases*:

(i) $i < 0$ (ii) $i > 2d-2$ (iii) $q \leqq 0$ (iv) $q > 3$ (v) $q = 3,\ (i, r) \neq (2d-2, d)$

(2) *For any $i \geqq 0$, we have*

$$H^1(B, \mathfrak{H}^i(X, \mathbb{Q}_p(r))) = H^1_f(K, V^i(r)).$$

Proof. (1) The cases (i) and (ii) are clear by the definition of $\mathfrak{H}^i(X, \mathfrak{T}_n(r))$ (see Definition 6.3). The case (iii) with $q < 0$ follows from the fact that $\mathfrak{H}^i(X, \mathfrak{T}_n(r))$ is concentrated in degrees $\geqq 0$ (see Corollary 6.6). When $q = 0$, the restriction map

$$H^0(B, \mathfrak{H}^i(X, \mathbb{Z}_p(r))) \longrightarrow H^i(X_{\overline{K}}, \mathbb{Z}_p(r))^{G_K}$$

is injective by Proposition 6.5 (1) and the last group is finite by [D] Corollary 3.3.9. Hence $H^0(B, \mathfrak{H}^i(X, \mathbb{Z}_p(r)))$ is finite as well. See [Sa5] Proposition 6.1 for the cases (iv) and (v).

(2) Let S be a finite set of places of K including all places with $v|p\cdot\infty$ and all finite places where X has bad reduction. To prove the assertion, it is enough to check that there is an exact sequence of $\mathbb{Q}_p$-vector spaces

$$0 \longrightarrow H^1(B, \mathfrak{H}^i(X, \mathbb{Q}_p(r))) \longrightarrow H^1(G_S, V^i(r)) \xrightarrow{\mathrm{Res}} \bigoplus_{v \in S \cap B_0} H^1_{/f}(K_v, V^i(r)),$$

where B_0 denotes the set of the closed points of B. One obtains this exact sequence by a standard localization argument on étale cohomology and the isomorphisms

$$H^q_v(B_v, \mathfrak{H}^i(X_v, \mathbb{Q}_p(r))) \cong \begin{cases} 0 & (\text{if } q \neq 2,3, \text{ by Proposition 6.5 (1)}) \\ H^1_{/f}(K_v, V^i(r)) & (\text{if } q = 2, \text{ by Theorem 7.2}) \\ H^2(K_v, V^i(r)) & (\text{if } q = 3, \text{ by Theorem 7.2}) \end{cases}$$

Here B_v denotes $\mathrm{Spec}(O_v)$ with O_v the valuation ring of K_v, and $X_v := X \times_B B_v$. $\square$

Proposition 8.5 *Let $\alpha^{i,r,+}$ be the following local-global map*:

$$\alpha^{i,r,+} : \frac{H^1(K, H^i(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r)))}{H^1_f(K, H^i(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r)))} \longrightarrow \bigoplus_{v \in P} \frac{H^1(K_v, H^i(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r)))}{H(B_v, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))}.$$

Then for any $i \geqq 0$ and $r \geqq d$, there is a canonical map

$$\mathrm{Coker}(\alpha^{i,r,+}) \longrightarrow H^2(B, \mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))),$$

which has finite kernel and cokernel. Consequently, under Conjecture 5.15 for $T^i(r)$ with $i-2r \leqq -3$, the group $H^2(B, \mathfrak{H}^i(X, \mathbb{Z}_p(r)))$ is finite and we have $H^2(B, \mathfrak{H}^i(X, \mathbb{Q}_p(r))) = 0$. See also Theorem 6.10 (3).

Proof. If $(i,r) = (2d-2, d)$, then we have

$$H^2(B, \mathfrak{H}^{2d-2}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))) \cong H^2(B, \mathbb{Q}_p/\mathbb{Z}_p(1))) \cong \mathrm{Br}(O_K)[p^\infty],$$

which is zero (if $p \neq 2$) or finite 2-torsion (if $p = 2$) by the classical Hasse principle for $\mathrm{Br}(K)$. Assume $(i,r) \neq (2d-2, d)$ in what follows and consider the following commutative diagram with exact rows, where both rows are obtained from localization sequences of étale cohomology, and the coefficients $\mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r))$ (resp. $\mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))$) in the upper row (resp. the lower row) are omitted:

$$\begin{tikzcd}
H^1(K) \arrow[r] \arrow[d, "\alpha"] & \bigoplus_{v \in B_0} H^2_v(B) \arrow[r] \arrow[d, "\delta"] & H^2(B) \arrow[r] \arrow[d, "\beta"] & H^2(K) \arrow[r] \arrow[d, "\gamma"] & \bigoplus_{v \in B_0} H^3_v(B) \arrow[d, "\delta"] \\
\bigoplus_{v \in B_0} H^1_{/B_v}(K_v) \arrow[r, "(*)"] & \bigoplus_{v \in B_0} H^2_v(B_v) \arrow[r] & \bigoplus_{v \in B_0} H^2(B_v) \arrow[r] & \bigoplus_{v \in B_0} H^2(K_v) \arrow[r] & \bigoplus_{v \in B_0} H^3_v(B_v).
\end{tikzcd}$$

Here for each $v \in B_0$, we put

$$H^1_{/B_v}(K_v) := \frac{H^1(K_v, H^i(X_{\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r)))}{H(B_v, \mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r)))}.$$

We have the following facts concerning this diagram:

- The arrows δ are bijective by étale excision (and a rigidity lemma in [Sa5] 3.9).
- The arrow γ has finite kernel and cokernel by a Hasse principle of Jannsen in [J] p. 337, Theorem 3 (c).
- The arrow $(*)$ is injective by the defition of $H^1_{/B_v}(K_v)$.
- $H^2(B_v)$ is finite for any $v \in B_0$ by Theorem 7.2, and zero for any $v \in B_0 \smallsetminus S$ by Example 6.4 (1).

Hence we see that there is a canonical map

$$\mathrm{Coker}(\alpha^{i,r,+}) \longrightarrow H^2(B, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))$$

and that this map has finite kernel and cokernel. $\square$

9 The case of arithmetic surfaces

Setting 9.1 Let the notation be as in Setting 8.1. We assume further that X is an arithmetic surface, i.e., $d = 2$. For $i \geqq 0$, put $T^i := H^i(X_{\overline{K}}, \mathbb{Z}_p)$.

For a finite place v of K, we write k_v (resp. Y_v, $Y_{\overline{v}}$) for the residue field at v (resp. $X \otimes_{O_K} k_v$, $X \otimes_{O_K} \overline{k_v}$), and B_v (resp. X_v, $X_{\overline{v}}$) for $\mathrm{Spec}(O_v)$ (resp. $X \otimes_{O_K} O_v$, $X \otimes_{O_K} O_{\overline{v}}^{\mathrm{sh}}$), where O_v (resp. $O_{\overline{v}}^{\mathrm{sh}}$) denotes the completion of O_K at v (resp. the strict henselization of O_v at its maximal ideal). We put $q_v := \# k_v$.

9.1 Integral comparison

Lemma 9.2 *We have*

$$H^1(B_v, \mathfrak{H}^i(X_v, \mathbb{Z}_p(r))) = H^1_f(K_v, T^i(r))$$

as subgroups of $H^1(K_v, T^m(r))$, for any finite place v of K, $i \geqq 0$ and $r \geqq 2$.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
H^1(K_v, \mathfrak{H}^i(X_v, \mathbb{Z}_p(r))) \arrow[rr, "d"] \arrow[d, equal] & & H^2_v(B_v, \mathfrak{H}^i(X_v, \mathbb{Z}_p(r))) \arrow[d, "b"] \\
H^1(K_v, T^i(r)) \arrow[r, "a"] & H^1_{/f}(K_v, V^i(r)) \arrow[r, hook, "d'"] & H^2_v(B_v, \mathfrak{H}^i(X_v, \mathbb{Q}_p(r))),
\end{tikzcd}$$

where the arrows d and d' are connecting maps of localization sequences of cohomology of B_v, and the existence and the injectivity of d' is a consequence of Theorem 7.2 for $q = 1$. The arrow a is the natural map, and we have $\mathrm{Ker}(a) = H^1_f(K_v, T^i(r))$ by definition. On the other hand, since $H^1_v(B_v, \mathfrak{H}^i(X_v, \mathbb{Z}_p(r))) = 0$ by Proposition 6.5 (1), we have

$$\mathrm{Ker}(d) = H^1(B_v, \mathfrak{H}^i(X_v, \mathbb{Z}_p(r))).$$

Thus it remains to check that the arrow b is injective, which follows from the facts that

$$H^2_v(B_v, \mathfrak{H}^i(X_v, \mathbb{Z}_p(r))) = 0 \qquad \text{if } v|p \text{ and } r \geqq 3$$

and that otherwise

$$H^2_v(B_v, \mathfrak{H}^i(X_v, \mathbb{Z}_p(r))) \cong H^1(k_v, H^{2-i}(Y_{\overline{v}}, \mathbb{Q}_p/\mathbb{Z}_p(2-r)))^*$$

is torsion-free, because $\dim(Y_v) = 1$ and $\mathrm{cd}(k_v) = 1$. □

As a corollary of Lemma 9.2, we obtain

Corollary 9.3 *We have*

$$H^1(B, \mathfrak{H}^i(X, \mathbb{Z}_p(r))) \cong H^1_f(K, T^i(r))$$

for any $i \geqq 0$ and $r \geqq 2$.

9.2 *p*-adic Abel-Jacobi maps

Let r be an integer with $r \geqq 2$. We define the motivic cohomology of X as

$$H^i_{\mathscr{M}}(X, \mathbb{Z}(r)) := H^i_{\mathrm{zar}}(X, \mathbb{Z}(r)),$$

and define the motivic cohomology with $\mathbb{Z}/p^n\mathbb{Z}$-coefficients as

$$H^i_{\mathscr{M}}(X, \mathbb{Z}/p^n\mathbb{Z}(r)) := H^i_{\mathrm{zar}}(X, \mathbb{Z}(r) \otimes \mathbb{Z}/p^n\mathbb{Z}) \qquad (n \geqq 1).$$

Lemma 9.4 *Assume that $r \geqq 2$, and that $p \geqq 3$ or $B(\mathbb{R}) = \emptyset$. Then the cycle class map*

$$\mathrm{cl}^{i,r}_{\mathbb{Z}/p^n\mathbb{Z}} : H^i_{\mathscr{M}}(X, \mathbb{Z}/p^n\mathbb{Z}(r)) \longrightarrow H^i(X, \mathfrak{T}_n(r))$$

is bijective for any $i \in \mathbb{Z}$ with $(i, r) \neq (5, 2)$ and any $n \geqq 1$. Consequently, there exists a short exact sequence

$$0 \longrightarrow H^i_{\mathscr{M}}(X, \mathbb{Z}(r))/p^n \longrightarrow H^i(X, \mathfrak{T}_n(r)) \longrightarrow {}_{p^n}H^{i+1}_{\mathscr{M}}(X, \mathbb{Z}(r)) \longrightarrow 0$$

for the same (i, n), where for an abelian group M, ${}_{p^n}M$ (resp. M/p^n) denotes the kernel (resp. cokernel) of the map $M \xrightarrow{\times p^n} M$.

Proof. See [Sa5] Lemma 7.1 (3). □

We define a p-adic cycle class map

$$\mathrm{cl}^{i,r}_p : H^i_{\mathscr{M}}(X, \mathbb{Z}(r)) \widehat{\otimes} \mathbb{Z}_p \longrightarrow H^i(X, \mathbb{Z}_p(r))$$

as the projective limit with respect to $n \geqq 1$ of the cycle class map

$$\mathrm{cl}^{i,r}_{/p^n} : H^i_{\mathscr{M}}(X, \mathbb{Z}(r))/p^n \longrightarrow H^i_{\mathscr{M}}(X, \mathbb{Z}/p^n\mathbb{Z}(r)) \underset{\simeq}{\xrightarrow{\mathrm{cl}^{i,r}_{\mathbb{Z}/p^n\mathbb{Z}}}} H^i(X, \mathfrak{T}_n(r)).$$

Since $X_{\overline{K}}$ is a curve, $H^i(X_{\overline{K}}, \mathbb{Z}_p(r))$ is torsion-free, and

$$H^0(B, \mathfrak{H}^i(X, \mathbb{Z}_p(r))) \subset H^i(X_{\overline{K}}, \mathbb{Z}_p(r))^{G_K} = 0$$

by Proposition 6.5 (1) and for the reason of weights. We define a p-adic Abel-Jacobi mapping

$$\mathrm{aj}^{i,r}_p : H^i_{\mathscr{M}}(X, \mathbb{Z}(r)) \widehat{\otimes} \mathbb{Z}_p \longrightarrow H^1(B, \mathfrak{H}^{i-1}(X, \mathbb{Z}_p(r))) \tag{9.1}$$

as the map induced by $\mathrm{cl}^{i,r}_p$ and an edge map of the spectral sequence

$$E^{a,b}_2 = H^a(B, \mathfrak{H}^b(X, \mathbb{Z}_p(r))) \Longrightarrow H^{a+b}(X, \mathbb{Z}_p(r)).$$

By Corollary 9.3, the map (9.1) is rewritten as follows:

$$\mathrm{aj}^{i,r}_p : H^i_{\mathscr{M}}(X, \mathbb{Z}(r)) \widehat{\otimes} \mathbb{Z}_p \longrightarrow H^1_f(K, T^{i-1}(r)). \tag{9.2}$$

The following proposition is a summary of known facts and results on this p-adic Abel-Jacobi maps, where the Voevodsky-Rost theorem [V2], [V3] plays a crucial role:

Proposition 9.5 ([Sa5] Corollary 7.7) *Assume that $r \geqq 2$, and that $p \geqq 3$ or $B(\mathbb{R}) = \emptyset$. When $r \geqq 3$, assume further Conjecture 5.15 for $T = H^1(X_{\overline{K}}, \mathbb{Z}_p(r))$ in* (3) *and* (4) *below. Then*:

(0) *$H^i_{\mathscr{M}}(X, \mathbb{Z}(r))$ is uniquely p-divisible for $i \leqq 0$ and $i \geqq 5$, and zero for $i > r + 2$.*

(1) *$\mathrm{cl}^{1,r}_p$ and $\mathrm{aj}^{1,r}_p$ are injective.*

(2) *$\mathrm{cl}^{2,r}_p$ is injective, and $\mathrm{aj}^{2,r}_p$ has finite kernel.*

(3) *$\mathrm{cl}^{3,r}_p$ is bijective, and $\mathrm{aj}^{3,r}_p$ has finite kernel and cokernel.*

(4) *$\mathrm{cl}^{4,r}_p$ is bijective, and $H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ is finite.*
Moreover, we have $H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\} \cong H^4_{\mathscr{M}}(X, \mathbb{Z}(r)) \widehat{\otimes} \mathbb{Z}_p$, and $\mathrm{aj}^{4,r}_p$ is zero.

9.3 Comparison with local-global maps

We recall the local-global map introduced in §5.4

$$\alpha^{i,r} : \frac{H^1(K, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))}{H^1_f(K, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))} \longrightarrow \bigoplus_{v \in P} \frac{H^1(K_v, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))}{H^1_f(K_v, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))}. \tag{9.3}$$

We put $\chi(f) := \#\mathrm{Coker}(f)/\#\mathrm{Ker}(f)$ for a homomorphism $f : M \to N$ of abelian groups with finite kernel and cokernel. The following result compares the maps $\alpha^{i,r}$ for $i = 0, 1, 2$ with the p-adic Abel-Jacobi mappings $\mathrm{aj}_p^{i,r}$ for $i = 2, 3$.

Theorem 9.6 *Assume $r \geqq 2$, and that $p \geqq 3$ or $B(\mathbb{R}) = \emptyset$. Assume further Conjecture 5.15 for $T^1(r)$ and that the group $H^3_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$ is finite. Then $\mathrm{aj}_p^{2,r}$ has finite cokernel, and we have*

$$\frac{\chi(\alpha^{1,2})}{\chi(\alpha^{0,2})} = \frac{\chi(\mathrm{aj}_p^{3,2})}{\chi(\mathrm{aj}_p^{2,2})} \cdot \frac{\#\mathrm{CH}_0(X)\{p\}}{\#\mathrm{Pic}(O_K)\{p\}} \cdot \prod_{v \in S'} \frac{e_v^{2,1,2} \cdot e_v^{3,0,2}}{e_v^{2,0,2} \cdot e_v^{3,1,2}} \qquad (r = 2)$$

$$\frac{\chi(\alpha^{1,r})}{\chi(\alpha^{0,r}) \cdot \chi(\alpha^{2,r})} = \frac{\chi(\mathrm{aj}_p^{3,r})}{\chi(\mathrm{aj}_p^{2,r})} \cdot \#H^4_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\} \cdot \prod_{v \in S'} \frac{e_v^{2,1,r} \cdot e_v^{3,0,r} \cdot e_v^{3,2,r}}{e_v^{2,0,r} \cdot e_v^{2,2,r} \cdot e_v^{3,1,r}} \qquad (r \geqq 3),$$

where S' denotes the set of the places of K which divide p or where X has bad reduction; for each $v \in S'$ and $a = 2, 3$, we put

$$e_v^{a,i,r} := \#H^a(B_v, \mathfrak{H}^i(X_v, \mathbb{Z}_p(r))).$$

See Theorem 7.2 for the finiteness of $e_v^{2,i,r}$ and $e_v^{3,i,r}$.

The formulas in this theorem are based on the finiteness stated in Proposition 9.5.

Proof. For $(i, r) \neq (2, 2)$ with $r \geqq 2$, there is a commutative diagram with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\dfrac{H^1(B, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))}{H^1_f(K, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))} \arrow[r] \arrow[d] & \displaystyle\bigoplus_{v \in B_0} \dfrac{H^1(B_v, \mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r)))}{H^1_f(K, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))} \arrow[d] \\
\dfrac{H^1(K, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))}{H^1_f(K, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))} \arrow[r, "\alpha^{i,r}"] \arrow[d] & \displaystyle\bigoplus_{v \in B_0} \dfrac{H^1(K_v, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))}{H^1_f(K_v, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r))} \arrow[d] \\
\displaystyle\bigoplus_{v \in B_0} H^2_v(B, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) \arrow[r, equal, "\simeq"] \arrow[d] & \displaystyle\bigoplus_{v \in B_0} H^2_v(B, \mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))) \arrow[d] \\
H^2(B, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) \arrow[r] \arrow[d] & \displaystyle\bigoplus_{v \in B_0} H^2(B_v, \mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))) \arrow[d] \\
H^2(K, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r)) \arrow[r, "\simeq", "\text{(Jannsen)}"'] \arrow[d] & \displaystyle\bigoplus_{v \in B_0} H^2(K_v, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p(r)) \arrow[d] \\
\displaystyle\bigoplus_{v \in B_0} H^3_v(B, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) \arrow[r, equal, "\simeq"] \arrow[d] & \displaystyle\bigoplus_{v \in B_0} H^3_v(B_v, \mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))) \arrow[d] \\
0 & 0
\end{tikzcd}$$

This diagram yields an exact sequence of $\mathbb{Z}_p$-modules

$$\begin{aligned}0 \to \mathrm{Ker}(\alpha^{i,r}) \to H^1_{/f}(B, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) \to \bigoplus_{v \in B_0} H^1_{/f}(B_v, \mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r)))\\ \to \mathrm{Coker}(\alpha^{i,r}) \to H^2(B, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) \to \bigoplus_{v \in B_0} H^2(B_v, \mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))) \to 0,\end{aligned}$$

where we put

$$\begin{aligned}H^1_{/f}(B, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) &:= \frac{H^1(B, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r)))}{H^1_f(K, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p)},\\ H^1_{/f}(B_v, \mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))) &:= \frac{H^1(B_v, \mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r)))}{H^1_f(K_v, T^i \otimes \mathbb{Q}_p/\mathbb{Z}_p)}.\end{aligned}$$

By the isomorphism of finite p-groups

$$\begin{aligned}H^1_{/f}(B, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) &\cong H^2(B, \mathfrak{H}^i(X, \mathbb{Z}_p(r))),\\ H^2(B, \mathfrak{H}^i(X, \mathbb{Q}_p/\mathbb{Z}_p(r))) &\cong H^3(B, \mathfrak{H}^i(X, \mathbb{Z}_p(r))) \quad \text{under Conjcture 5.15},\\ H^1_{/f}(B_v, \mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))) &\cong H^2(B_v, \mathfrak{H}^i(X_v, \mathbb{Z}_p(r))),\\ H^2(B_v, \mathfrak{H}^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(r))) &\cong H^3(B_v, \mathfrak{H}^i(X_v, \mathbb{Z}_p(r))),\end{aligned}$$

and the above 6-term exact sequence, we obtain

$$\chi(\alpha^{i,r}) = \frac{e^{3,i,r}}{e^{2,i,r}} \times \prod_{v \in S'} \frac{e_v^{2,i,r}}{e_v^{3,i,r}} \tag{9.4}$$

for $(i, r) \neq (2, 2)$, where we put

$$e^{a,i,r} := \# H^a(B, \mathfrak{H}^i(X, \mathbb{Z}_p(r))) \quad \text{for} \quad a = 2, 3 \quad \text{with} \quad (a, i, r) \neq (3, 2, 2).$$

See also Example 6.4 (1) for the fact that $e_v^{2,i,r} = e_v^{3,i,r} = 1$ for $v \in B_0 \smallsetminus S'$.

On the other hand, $\mathrm{aj}_p^{i,r}$ for $i = 2, 3$ is identified with the natural projection

$$H^i(X, \mathbb{Z}_p(r)) \longrightarrow H^1(B, \mathfrak{H}^{i-1}(X, \mathbb{Z}_p(r))) \cong H^1_f(K, T^{i-1}(r))$$

by Proposition 9.5 (2), (3) and the finiteness assumption on $H^3_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\}$; see also the short exact sequence of Lemma 9.4. Hence we have

$$\frac{\chi(\mathrm{aj}_p^{3,r})}{\chi(\mathrm{aj}_p^{2,r})} = \frac{e^{2,0,r} \cdot e^{2,2,r} \cdot e^{3,1,r}}{e^{2,1,r} \cdot e^{3,0,r} \cdot \# H^4(X, \mathbb{Z}_p(r))}$$

for $r \geqq 2$ by the spectral sequence (6.3), and moreover

$$\begin{aligned}H^2(B, \mathfrak{H}^2(X, \mathbb{Z}_p(2))) &\cong H^2(B, \mathbb{Z}_p(1)) \cong \mathrm{Pic}(O_K) \otimes \mathbb{Z}_p \cong \mathrm{Pic}(O_K)\{p\},\\ H^3(B, \mathfrak{H}^2(X, \mathbb{Z}_p(r))) &\cong H^3(B, \mathbb{Z}_p(r-1)) \cong H^3(B[p^{-1}], \mathbb{Z}_p(r-1)) = 0 \qquad (r \geqq 3).\end{aligned}$$

The assertion follows from (9.4) and these facts. $\square$

9.4 Zeta values modulo rational numbers prime to p

Put $V^i := T^i \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. In this subsection, we relate the formula in Theorem 9.6 with zeta values assuming Conjecture 9.7 below for the motives $H^i(X_K)(r)$ with $i = 0, 1, 2$, that is, a weak version of *p-Tamagawa number conjecture* [BK2] §5. Let S' be a finite set of closed points of B containing all points of characteristic p, and all points where X has bad reduction. For $i = 0, 1, 2$ and $r \geqq 2$ with $(i, r) \neq (2, 2)$, we put

$$L_{S'}(H^i(X_K), r) := \prod_{v \in B_0 \smallsetminus S'} \det(1 - q_v^{-r} \cdot \mathrm{Fr}_v \mid V^i)^{-1}.$$

This infinite product on the right hand side converges, because $i - 2r \leqq -3$. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the prime ideal (p).

Conjecture 9.7 (Bloch-Kato) *For any $i = 0, 1, 2$ and $r \geqq 2$ with $(i, r) \neq (2, 2)$, there exists a finite-dimensional $\mathbb{Q}$-subspace $\Phi^{i,r} = \Phi_p^{i,r}$ of the $\mathbb{Q}$-vector space*

$$H^{i+1}_{\mathscr{M}}(X_K, \mathbb{Q}(r))_{\mathbb{Z}} := \mathrm{Im}\big(H^{i+1}_{\mathscr{M}}(X, \mathbb{Q}(r)) \to H^{i+1}_{\mathscr{M}}(X_K, \mathbb{Q}(r))\big)$$

which satisfies the following conditions (i) *and* (ii):

(i) *The p-adic Abel-Jacobi map*

$$H^{i+1}_{\mathscr{M}}(X_K, \mathbb{Q}(r)) \longrightarrow H^1(K, V^i(r))$$

induces an isomorphism $\Phi^{i,r} \otimes \mathbb{Q}_p \cong H^1_f(K, V^i(r))$, and Beilinson's regulator map to the real Deligne cohomology

$$H^{i+1}_{\mathscr{M}}(X_K, \mathbb{Q}(r)) \longrightarrow H^{i+1}_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{R}(r)) \tag{9.5}$$

induces an isomorphism $\Phi^{i,r} \otimes \mathbb{R} \cong H^{i+1}_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{R}(r))$.

(ii) *We define $A_p^{i,r}(K)$, the group of p-global points, as the pull-back of $\Phi^{i,r}$ under the natural map*

$$H^1_f(K, T^i(r)) \longrightarrow H^1_f(K, V^i(r)) \cong \Phi^{i,r} \otimes \mathbb{Q}_p,$$

which is a finitely generated $\mathbb{Z}_{(p)}$-module. We further fix an O_K-lattice L^i of the de Rham cohomology $H^i_{\mathrm{dR}}(X_K/K)$, and define a number $R_\Phi^{i,r} \in \mathbb{R}^\times/\mathbb{Z}_{(p)}^\times$ to be the volume of the space

$$H^{i+1}_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r))/\mathit{Image\ of}\ A_p^{i,r}(K)$$

with respect to L^i. See Definition 2.7 (*resp. Remark 9.8* (*1*) *below*) *for the definition* (*an explicit description*) *of $H^{i+1}_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r))$. On the other hand, for each $v \in B_0$ we put*

$$A_p^{i,r}(K_v) := H^1_f(K_v, T^i(r)),$$

which we call the group of p-local points at v. Then we have

$$L_{S'}(H^i(X_K), r) \equiv \frac{Ш^{(p)}(H^i(X_K)(r))}{\#\mathrm{Hom}_{G_K}(T^i, \mathbb{Q}_p/\mathbb{Z}_p(1-r))} \cdot R_\Phi^{i,r} \cdot \prod_{v \in S'} \mu_v^i(A_p^{i,r}(K_v)) \mod \mathbb{Z}_{(p)}^\times, \tag{9.6}$$

where μ_v^i for $v \nmid p$ means the cardinality, and μ_v^i for $v \mid p$ denotes the Haar measure on $A_p^{i,r}(K_v)$ constructed from that on $H^i_{\mathrm{dR}}(X_{K_v}/K_v)$ such that $\mu_v^i(L^i \otimes_{O_K} O_v) = 1$.

Remark 9.8 (1) The map $A_p^{i,r}(K) \to H_{\mathscr{D}}^{i+1}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r))$ induced by the regulator map is injective, by the condition (i) for $\Phi^{i,r}$ and [BK2] Lemma 5.10. Here

$$H_{\mathscr{D}}^{i+1}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r)) = \left(\frac{H_{\mathrm{dR}}^i(X/\mathbb{Z}) \otimes \mathbb{C}}{H_{\mathrm{sing}}^i(X \otimes_{\mathbb{Z}} \mathbb{C}, (2\pi\sqrt{-1})^r \cdot \mathbb{Z}_{(p)})} \right)^+$$

for any $i = 0, 1, 2$ and $r \geqq 2$, by definition.

(2) The product on the right hand side of (9.6) is independent of the choice of L^i.

(3) Conjecture 9.7 for $i = 0$ (resp. $i = 2$) implies that

$$\zeta_K(r) \equiv \chi(\alpha^{0,r})^{-1} \cdot R_\Phi^{0,r} \qquad (\text{resp. } \zeta_K(r-1) \equiv \chi(\alpha^{0,r-1})^{-1} \cdot R_\Phi^{0,r-1})$$

modulo $\mathbb{Z}_{(p)}^\times$ if $r \geqq 2$ (resp. $r \geqq 3$) and p is unramified in K.

(4) We have $R_\Phi^{i,r} = 1$ for any $i \geqq 3$, because $H_{\mathscr{D}}^{i+1}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r))$ is zero for such i's.

(5) For $(i, r) = (2, 2)$, we will use the classical class number formula instead of (9.6) in Proposition 9.9 below.

Assuming Conjecture 9.7, we relate the formula in Theorem 9.6 with the residue or value at $s = r$ of the zeta function $\zeta(X, s)$.

Proposition 9.9 ([Sa5] Proposition 9.3) *Assume $r \geqq 2$ and the following conditions:*

(i) *$p \geqq r + 2$.*

(ii) *For any $v \in B_0$ with $v|p$, v is absolutely unramified and X has good reduction at v.*

(iii) *Conjecture 5.15 for $T^i(r)$ and Conjecture 9.7 holds for $i = 0, 1$ (resp. $i = 0, 1, 2$) if $r = 2$ (resp. $r \geqq 3$).*

Then $H_{\mathscr{M}}^3(X, \mathbb{Z}(r))\{p\}$ is finite, and we have

$$\operatorname*{Res}_{s=2} \zeta(X, s) \equiv \operatorname*{Res}_{s=1} \zeta_K(s) \cdot \frac{\chi(\mathrm{aj}_p^{3,2}) \cdot \# \mathrm{CH}_0(X) \cdot R_\Phi^{0,2}}{\chi(\mathrm{aj}_p^{2,2}) \cdot \# \mathrm{Pic}(O_K) \cdot R_\Phi^{1,2}} \mod \mathbb{Z}_{(p)}^\times \qquad (r = 2)$$

$$\zeta(X, r) \equiv \frac{\chi(\mathrm{aj}_p^{3,r}) \cdot \# H_{\mathscr{M}}^4(X, \mathbb{Z}(r))\{p\} \cdot R_\Phi^{0,r} \cdot R_\Phi^{2,r}}{\chi(\mathrm{aj}_p^{2,r}) \cdot R_\Phi^{1,r}} \mod \mathbb{Z}_{(p)}^\times \qquad (r \geqq 3)$$

where $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at (p); $R_\Phi^{i,r} \in \mathbb{R}^\times/\mathbb{Z}_{(p)}^\times$ is defined for the lattice $L^i := H_{\mathrm{dR}}^i(X/\mathbb{Z})/H_{\mathrm{dR}}^i(X/\mathbb{Z})_{\mathrm{tors}}$. See Conjecture 9.7 for the definition of $R_\Phi^{i,r}$.

Proof. The assertion is deduced from Theorem 9.6 and the equality

$$\frac{\mu_v^1(H_f^1(K_v, T^1(r)))}{\left|\zeta(Y_v, r)(1 - q_v^{1-r})(1 - q_v^{-r})\right|_p^{-1}} = \frac{e_v^{2,1,r} \cdot e_v^{3,0,r}}{e_v^{2,0,r} \cdot e_v^{3,1,r}},$$

which holds true unconditionally (resp. under (i) and (ii)) if $v \nmid p$ (resp. if $v|p$). See [Sa5] Theorems 8.4 and 8.5 and Proof of Proposition 9.3 for details. □

Theorem 9.10 ([Sa5] Theorem 9.6) *Under the same assumptions as in Proposition 9.9, assume further that*

(iv) *$H_{\mathscr{M}}^{i+1}(X, \mathbb{Z}(r))$ is finitely generated for any $i = 0, 1, 2$ and $r \geqq 2$.*

Then the regulator map

$$\mathrm{reg}_{\mathscr{D}}^{i+1,r} : H_{\mathscr{M}}^{i+1}(X, \mathbb{Z}(r)) \longrightarrow H_{\mathscr{D}}^{i+1}(X_{/\mathbb{R}}, \mathbb{Z}(r))$$

has finite kernel for $i = 0, 1, 2, 3$, and we have

$$\zeta^*(X, r) \equiv \prod_{i=0}^{3} \left(\frac{R^{i,r}_{\mathscr{M}}}{\#\mathrm{Ker}(\mathrm{reg}^{i+1,r}_{\mathscr{D}})} \right)^{(-1)^i} \mod \mathbb{Z}^{\times}_{(p)},$$

where $\zeta^*(X, r)$ denotes the residue (resp. the value) at $s = r$ if $r = 2$ (resp. $r \geqq 3$), and $R^{i,r}_{\mathscr{M}} \in \mathbb{R}^{\times}/\mathbb{Z}^{\times}_{(p)}$ denotes the volume of the space

$$\begin{cases} H^{i+1}_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r))/\mathrm{Im}(\mathrm{reg}^{i+1,r}_{\mathscr{D}}) & (\textit{for } (i, r) \neq (2, 2)) \\ \widetilde{H}^{3}_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(2))/\mathrm{Im}(\mathrm{reg}^{3,2}_{\mathscr{D}}) & (\textit{for } (i, r) = (2, 2)) \end{cases}$$

with respect to L^i fixed in Proposition 9.9; $\widetilde{H}^{3}_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(2))$ denotes the kernel of the canonical trace map

$$\mathrm{Tr}_X : H^{3}_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(2)) \longrightarrow \mathbb{R}.$$

Proof. The map $A^{i,r}_p(K) \to H^{i+1}_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}_{(p)}(r))$ induced by the map (9.5) is injective, and the assumption (iv) implies that $H^{i+1}_{\mathscr{M}}(X, \mathbb{Z}(r)) \otimes \mathbb{Z}_p \cong H^{i+1}_{\mathscr{M}}(X, \mathbb{Z}(r)) \widehat{\otimes} \mathbb{Z}_p \cong H^{i+1}(X, \mathbb{Z}_p(r))$ for $0 \leqq i \leqq 3$. The first assertion follows from these facts and (iii). The second assertion is deduced from Proposition 9.9 and the equality

$$\frac{R^{i,r}_{\mathscr{M}}}{\#\mathrm{Ker}(c^{i,r})} = \begin{cases} R^{0,r}_{\Phi} & (i = 0) \\ \chi(\mathrm{aj}^{2,r}_p) \cdot R^{1,r}_{\Phi} & (i = 1) \\ \chi(\mathrm{aj}^{3,2}_p) \cdot \mathfrak{m}_2(\mathrm{Coker}(\varrho_K)) & ((i, r) = (2, 2)) \\ \chi(\mathrm{aj}^{3,r}_p) \cdot R^{2,r}_{\Phi} & (i = 2,\ r \geqq 3) \\ (\#\mathrm{CH}_0(X)\{p\})^{-1} & ((i, r) = (3, 2)) \\ (\# H^{4}_{\mathscr{M}}(X, \mathbb{Z}(r))\{p\})^{-1} & (i = 3,\ r \geqq 3), \end{cases}$$

where ϱ_K (resp. $\mathfrak{m}_2$) denotes the regulator map (resp. the Haar measure) considered in Proposition 2.11. See [Sa5] Proof of Theorem 9.6 for details. □

Example 9.11 Let K be an imaginary quadratic field, and let E be an elliptic curve over K with complex multiplication by the integer ring O_K of K. Let D (resp. w) be the discriminant of K (resp. the number of roots of unity contained in K). Let X be a regular model of E which is proper flat over O_K. Let p be a prime number which is prime to 6 and *good for* X in the sense that X has good reduction at each place of K lying above p. Then assuming Conjecture 5.15, we obtain a formula

$$\operatorname*{Res}_{s=2} \zeta(X, s) \equiv \frac{2\pi \cdot \chi(\mathrm{aj}^{3,2}_p) \cdot \#\mathrm{CH}_0(X) \cdot R^{0,2}_{\Phi}}{w\sqrt{-D} \cdot \chi(\mathrm{aj}^{2,2}_p) \cdot R^{1,2}_{\Phi}} \mod \mathbb{Z}^{\times}_{(p)}$$

from Proposition 9.9 and results of Kings [Ki] Theorem 1.1.5 and Huber-Kings [HK] Theorem 1.3.1. If we assume further that $H^{i+1}_{\mathscr{M}}(X, \mathbb{Z}(2))$ is a finitely generated abelian group for $i = 0, 1, 2$, then we have

$$\mathrm{rank}_{\mathbb{Z}}\, H^{1}_{\mathscr{M}}(X, \mathbb{Z}(2)) = 1, \qquad \mathrm{rank}_{\mathbb{Z}}\, H^{2}_{\mathscr{M}}(X, \mathbb{Z}(2)) = 2, \qquad \mathrm{rank}_{\mathbb{Z}}\, H^{3}_{\mathscr{M}}(X, \mathbb{Z}(2)) = 0$$

by Theorem 8.3, and obtain a stronger formula

$$\operatorname*{Res}_{s=2} \zeta(X, s) \equiv \frac{2\pi \cdot \#\mathrm{CH}_0(X) \cdot \#\mathrm{Ker}(\mathrm{reg}^{2,2}_{\mathscr{D}}) \cdot R^{0,2}_{\mathscr{M}}}{\sqrt{-D} \cdot \# H^{3}_{\mathscr{M}}(X, \mathbb{Z}(2)) \cdot R^{1,2}_{\mathscr{M}} \cdot \#\mathrm{Ker}(\mathrm{reg}^{1,2}_{\mathscr{D}})} \mod \mathbb{Z}[T^{-1}]^{\times}$$

from Theorem 9.10, where T denotes the set of all prime numbers which divide 6 or which are bad for X.

A Purity of $\mathbb{G}_{\mathrm{m}}$

Exercise 13 *Let X be a scheme, and let $i : Z \hookrightarrow X$ be a closed subscheme. Let $j : V := X \smallsetminus Z \hookrightarrow X$ be the open immersion from the open complement of Z. Let $\mathscr{F}$ be an étale sheaf on X. For each $U \in \mathrm{Ob}(\text{Ét}/X)$, we put*

$$\Gamma_{Z\times_X U}(U, \mathscr{F}) := \mathrm{Ker}(\Gamma(U, \mathscr{F}) \to \Gamma(U \times_X V, \mathscr{F}))$$

(1) *Show that the assignment*

$$\underline{\Gamma}_Z(X, \mathscr{F}) : U \in \mathrm{Ob}(\text{Ét}/X) \longmapsto \Gamma_{Z\times_X U}(U, \mathscr{F})$$

is an étale sheaf on X.

(2) *Show that the following sequence of sheaves is exact on $X_{\text{ét}}$:*

$$0 \longrightarrow \underline{\Gamma}_Z(X, \mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_* j^* \mathscr{F}.$$

(3) *Show that $\underline{\Gamma}_Z(X, \mathscr{F}) \cong i_* i^* \underline{\Gamma}_Z(X, \mathscr{F})$ for any étale sheaf on X.*

(4) *Put $i^! \mathscr{F} := i^* \underline{\Gamma}_Z(X, \mathscr{F})$ for an étale sheaf $\mathscr{F}$ on X. Then show that the functor $i^! : \mathbf{Shv}(X_{\text{ét}}) \to \mathbf{Shv}(Z_{\text{ét}})$ is right adjoint to $i_* : \mathbf{Shv}(Z_{\text{ét}}) \to \mathbf{Shv}(X_{\text{ét}})$.*

(5) *Show that the functor*

$$\Gamma_Z(X, -) : \mathbf{Shv}(X_{\text{ét}}) \longrightarrow \mathbf{Ab}, \qquad \mathscr{F} \longmapsto \Gamma_Z(X, \mathscr{F})$$

is left exact. Let $H^*_Z(X, -)$ be its right derived functor.

(6) *For each $q \geqq 0$ and each sheaf $\mathscr{F} \in \mathbf{Shv}(X_{\text{ét}})$, let $\underline{H}^q_Z(X, \mathscr{F})$ be the sheaf on $X_{\text{ét}}$ associated with the presheaf*

$$U \in \mathrm{Ob}(\text{Ét}/X) \longmapsto H^q_{Z\times_X U}(U, \mathscr{F}).$$

Then show that the functor

$$i^* \underline{H}^q_Z(X, -) : \mathbf{Shv}(X_{\text{ét}}) \longrightarrow \mathbf{Shv}(Z_{\text{ét}}), \qquad \mathscr{F} \longmapsto i^* \underline{H}^q_Z(X, \mathscr{F})$$

agrees with the q-th right derived functor $R^q i^!$ of $i^!$.

Theorem A.1 (Purity of $\mathbb{G}_{\mathrm{m}}$) *Let X be a locally noetherian scheme, and let $i : Z \hookrightarrow X$ be a closed subscheme.*

(0) *If X is reduced and $\mathrm{codim}_X(Z) \geqq 1$, then we have $i^! \mathbb{G}_{\mathrm{m}} = 0$.*

(1) *If X is normal and $\mathrm{codim}_X(Z) \geqq 2$, then we have $R^1 i^! \mathbb{G}_{\mathrm{m}} = 0$.*

(1^+) *If X is regular and $\mathrm{codim}_X(Z) \geqq 1$, then we have*

$$R^1 i^! \mathbb{G}_{\mathrm{m}} \cong \bigoplus_{x \in Z \cap X^1} i_{x*} \mathbb{Z}_x,$$

where x on the right hand side runs through all points of Z which has codimension 1 in X, and i_x denotes the natural map $x \hookrightarrow Z$.

(2) *If X is regular and $\mathrm{codim}_X(Z) \geqq 1$, then we have $R^2 i^! \mathbb{G}_{\mathrm{m}} = 0$.*

(3) *If X is regular and $\mathrm{codim}_X(Z) \geqq 2$, then we have $R^3 i^! \mathbb{G}_{\mathrm{m}} = 0$.*

Remark A.2 The vanishing of (3) is called the *purity of Brauer groups* [G], [FG], [Če]. We do not explain any more about (3) in what follows.

Proof. The following fact is useful [GD2] Propositions 17.5.7, 17.5.8:

- *Let $U \to X$ be a smooth morphism of schemes. If X is reduced (resp. normal, regular), then U is also reduced (resp. normal, regular).*

Since the problems are local, we suppose that X is an affine scheme with affine ring A.

(0) Assume that A is a noetherian ring whose nilpotent radical is 0. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal prime ideals of A and put $S := A \smallsetminus (\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r)$. Then S agrees with the set of all non-zero-divisors in A [GD2] Remark 20.2.13 (ii). The assertion follows from this fact.

(1) If A is a noetherian integrally closed domain, then we have

$$A = \bigcap_{\mathfrak{p}\,:\,\mathrm{height}\,1} A_{\mathfrak{p}}, \tag{A.1}$$

where $\mathfrak{p}$ on the right hand side runs through all prime ideals of A of height 1, and the intersection is taken in the fraction field of A. The assertion follows from this fact.

(1^+) Assume that A is a regular local ring, and let $Z' \subsetneqq \mathrm{Spec}(A) = X$ be a proper closed subset. Our task is to show that there is a short exact sequence

$$0 \longrightarrow A^\times \longrightarrow \Gamma(\mathrm{Spec}(A) \smallsetminus Z', \mathbb{G}_\mathrm{m}) \xrightarrow{(\mathrm{ord}_x)_x} \bigoplus_{x \in Z' \cap \mathrm{Spec}(A)^1} \mathbb{Z} \longrightarrow 0.$$

Indeed, we may replace Z' with the union of its irreducible components $Z_1, Z_2, \ldots, Z_r$ which have codimension 1 in X, by (0) and (1). Since A is regular local, each Z_j endowed with the reduced structure is principal and defined by some prime $p_j \in A$, and the above sequence is identified with the exact sequence

$$0 \longrightarrow A^\times \longrightarrow A[p_1^{-1}, \ldots, p_r^{-1}]^\times \xrightarrow{(\mathrm{ord}_{p_j})_j} \bigoplus_{j=1}^{r} \mathbb{Z} \longrightarrow 0,$$

whose exactness at the middle follows from (A.1). Thus we obtain the assertion.

(2) Assume that A is a strict henselian regular local ring, and let $Z \subsetneqq X := \mathrm{Spec}(A)$ be a proper closed subset. Put $U := X \smallsetminus Z$. Then there is an exact sequence

$$H^1(X, \mathbb{G}_\mathrm{m}) \xrightarrow{(\star)} H^1(U, \mathbb{G}_\mathrm{m}) \longrightarrow H^2_Z(X, \mathbb{G}_\mathrm{m}) \longrightarrow 0.$$

Since we have

$$H^1(X, \mathbb{G}_\mathrm{m}) \overset{\text{(H)}}{\cong} \mathrm{Pic}(X) \cong \mathrm{CH}^1(X) \quad \text{and} \quad H^1(U, \mathbb{G}_\mathrm{m}) \overset{\text{(H)}}{\cong} \mathrm{Pic}(U) \cong \mathrm{CH}^1(U),$$

the map $(\star)$ is surjective (and in fact, these group are all zero because $H^1(X, \mathbb{G}_\mathrm{m}) = 0$), where the isomorphisms (H) follow from Hilbert's Theorem 90. Hence $H^2_Z(X, \mathbb{G}_\mathrm{m}) = 0$. □

Exercise 14 *Let X be a proper smooth geometrically integral curve over a field k.*

(1) *Assume that k is algebraically closed. Then show that $H^i(X, \mathbb{G}_\mathrm{m}) = 0$ for any $i \geqq 2$, using Tsen's theorem: the function field $k(X)$ is a C_1-field.*

(2) *Assume that k is a finite field. Then show that*

$$H^i(X, \mathbb{G}_\mathrm{m}) \cong \begin{cases} \mathbb{Q}/\mathbb{Z} & (\textit{if } i = 3) \\ 0 & (\textit{if } i = 2 \textit{ or } i \geqq 4). \end{cases}$$

B Higher cycle class map for regular schemes

Let Reg be the category of all noetherian regular schemes and all morphisms of schemes. Let $\mathscr{C}$ be a full subcategory of Reg satisfying the following conditions:

- If $f : X \to Y$ is a smooth morphism with $Y \in \mathrm{Ob}(\mathscr{C})$, then $X \in \mathrm{Ob}(\mathscr{C})$.

Let Λ be a commutative ring with unity, and let $\{\Lambda(r)\}_{r\in\mathbb{Z}}$ be a family of complexes of sheaves of Λ-modules on $\mathscr{C}_{\mathrm{zar}}$. We are concerned with the following data (D1)–(D5) and the axioms (B0)–(B9) below:

(D1) **(First Chern class)** A morphism $\varrho : \mathscr{O}^{\times}[-1] \longrightarrow \Lambda(1)$ in $D(\mathbf{Shv}(\mathscr{C}_{\mathrm{zar}}))$ is given, where $\mathscr{O}^{\times}$ denotes the sheaf $U \in \mathrm{Ob}(\mathscr{C}) \to \Gamma(U, \mathscr{O}_U)^{\times}$ on $\mathscr{C}_{\mathrm{zar}}$.

(D2) **(Product structure)** For each pair of integers $r, r' \in \mathbb{Z}$, a morphism

$$\Lambda(r) \otimes^{\mathbb{L}} \Lambda(r') \longrightarrow \Lambda(r + r') \quad \text{in} \quad D(\mathbf{Shv}(\mathscr{C}_{\mathrm{zar}}), \Lambda)$$

(D3) **(Push-forward along regular divisors)** For each closed immersion $i : D \hookrightarrow X$ of pure codimension 1 in $\mathscr{C}$ and each $r \geqq 0$, a morphism

$$i_* : i_*\Lambda(r-1)_D[-2] \to \Lambda(r)_X \quad \text{in} \quad D(\mathbf{Shv}(X_{\mathrm{zar}}), \Lambda)$$

is given, where $\Lambda(r-1)_D$ (resp. $\Lambda(r)_X$) denotes the restriction of $\Lambda(r-1)$ to D_{zar} (resp. $\Lambda(r)$ to X_{zar}).

(D4) **(Cycle classes)** For each $X \in \mathrm{Ob}(\mathscr{C})$ and each irreducible closed subset $V \subset X$ of codimension c, a cycle class

$$\mathrm{cyc}_X(V) \in H^{2c}_V(X_{\mathrm{zar}}, \Lambda(c))$$

(D5) **($\Lambda(r)$ with log poles)** For each $n, r \geqq 1$, any $X \in \mathrm{Ob}(\mathscr{C})$ and each relative hyperplane $H \subset P := X \times \mathbb{P}^n$ over X, a complex $\Lambda(r)_{(P,H)}$ of sheaves on P_{zar} is given and contravariantly functorial in (P, H). Here a morphism $f : (Y, E) \to (Y', E')$ of pairs is a morphism $f : Y \to Y'$ of schemes satisfying $f(Y \smallsetminus E) \subset Y' \smallsetminus E'$.

(B0) The 0-th cohomology sheaf $\mathscr{H}^0(\Lambda(0))$ is a sheaf of commutative rings with unity.

(B1) The product structure of (D2) is commutative, associative and compatible with the product structure on $\mathscr{H}^0(\Lambda(0))$ mentioned in (B0).

(B2) **(Fundamental class)** For any integral $X \in \mathrm{Ob}(\mathscr{C})$, the unity of $H^0(X_{\mathrm{zar}}, \Lambda(0))$ agrees with $\mathrm{cyc}_X(X)$.

(B3) **(Compatibility of D1 and D4: first Chern class)** For any $X \in \mathrm{Ob}(\mathscr{C})$ and any prime divisor D on X, $\mathrm{cyc}_X(D)$ agrees with the first Chern class $c_1^X(D)$, i.e., the value of the class of $\mathscr{O}_X(D)$ under the map

$$\varrho : H^1_D(X_{\mathrm{zar}}, \mathscr{O}_X^{\times}) \longrightarrow H^2_D(X_{\mathrm{zar}}, \Lambda(1))$$

(B4) **(Compatibility of D2 and D4: intersection formula)** For any $X \in \mathrm{Ob}(\mathscr{C})$, any prime divisor D on X and any irreducible closed subset $V \subset X$ of codimension r, we have

$$\mathrm{cyc}_X(D \cdot V) = \mathrm{cyc}_X(D) \cup \mathrm{cyc}_X(V) \quad \text{in} \quad H^{2r+2}_{D\cap V}(X_{\mathrm{zar}}, \Lambda(r+1)).$$

(B5) **(Compatibility of D3 and D4)** For any $i : D \hookrightarrow X$ as in (D3) and any $r \geqq 0$, the following diagram commutes in $D(\mathbf{Shv}(X_{\mathrm{zar}}), \Lambda)$:

$$\begin{tikzcd} i_*\Lambda_D \otimes^{\mathbb{L}} \Lambda(r)_X \arrow[r, "\mathrm{id}\otimes i^*"] \arrow[d, "\mathrm{cyc}_X(D)^{\mathrm{ad}}\otimes\mathrm{id}"'] & i_*\Lambda_D \otimes^{\mathbb{L}} i_*\Lambda(r)_D \arrow[r, "\mathrm{prod}"] & i_*\Lambda(r)_D \arrow[d, "i_*"] \\ \Lambda(1)_X[2] \otimes^{\mathbb{L}} \Lambda(r)_X \arrow[r, equal, "\sim"] & \Lambda(1)_X \otimes^{\mathbb{L}} \Lambda(r)_X[2] \arrow[r, "\mathrm{prod}"'] & \Lambda(r+1)_X[2], \end{tikzcd}$$

where $\mathrm{cyc}_X(D)^{\mathrm{ad}}$ in the left downarrow denotes the morphism $i_*\Lambda_D \to \Lambda(1)_X[2]$ corresponding to $\mathrm{cyc}_X(D) \in H^2_D(X_{\mathrm{zar}}, \Lambda(1)) \cong \mathrm{Hom}_{D(\mathbf{Shv}(X_{\mathrm{zar}}),\Lambda)}(i_*\Lambda_D, \Lambda(1)_X[2])$.

(B6) **(Projective space)** For any $X \in \mathrm{Ob}(\mathscr{C})$ and any $n, r \geqq 0$, the morphism

$$\bigoplus_{i=0}^{n} \Lambda(r-i)_X[-2i] \longrightarrow R\pi_*\Lambda(r)_P, \quad (\alpha_i)_{i=0}^n \mapsto \sum_{i=0}^{n} \pi^*(\alpha) \cup \xi^i$$

is an isomorphism in $D(\mathbf{Shv}(X_{\mathrm{zar}}), \Lambda)$. Here π denotes the natural projection $P := X \times \mathbb{P}^n \to X$, and $\xi \in H^2(P_{\mathrm{zar}}, \Lambda(1))$ denotes the first Chern class of the tautological line bundle over P.

(B7) **(Weak purity)** For any $X \in \mathrm{Ob}(\mathscr{C})$, any integer $c \geqq 0$ and any closed subscheme $i : W \hookrightarrow X$ of codimension $\geqq c$, we have $\tau_{\leqq 2c-1} Ri^! \Lambda(c)_X = 0$.

(B8) **(Compatibility of D3 and D5)** For any pair (P, H) as in (D5) and any $r \geqq 1$, the complex $\Lambda(r)_{(P,H)}$ fits into a distinguished triangle in $D(\mathbf{Shv}(P_{\mathrm{zar}}), \Lambda)$

$$\Lambda(r-1)_H[-2] \xrightarrow{\ i_*\ } \Lambda(r)_P \longrightarrow \Lambda(r)_{(P,H)} \longrightarrow \Lambda(r-1)_H[-1]\ .$$

(B9) **(Purity along log poles)** For any pair (P, H) as in (D5) and any closed subscheme $i : W \hookrightarrow P$ of codimension $\geqq c$ with $W \subset H$, we have $\tau_{\leqq 2c} Ri^! \Lambda(c)_{(P,H)} = 0$.

Remark B.1 The axiom (B5) for $r = 0$ implies that $i_* : \Gamma(D, \Lambda) \to H^2_D(X_{\mathrm{zar}}, \Lambda(1))$ sends 1 to $\mathrm{cyc}_X(D)$. Using this fact, one can further deduce the following projection formula from (B5) for any $r \geqq 0$ and $j \in \mathbb{Z}$:

$$i_*(\alpha) \cup \beta = i_*(\alpha \cup i^*\beta) \qquad ({}^\forall\alpha \in \Gamma(D, \Lambda),\ {}^\forall\beta \in H^j(X_{\mathrm{zar}}, \Lambda(r))).$$

Example B.2 Let n be a positive integer, and let $\mathscr{C} \subset \mathrm{Reg}$ be the full-subcategory consisting of all regular noetherian schemes over $\mathbb{Z}[n^{-1}]$. Put $\Lambda := \mathbb{Z}/n\mathbb{Z}$ and

$$\Lambda(r) := \begin{cases} R\epsilon_*\mu_n^{\otimes r} & (r > 0) \\ R\epsilon_*\Lambda & (r = 0) \\ R\epsilon_*\mathscr{H}om_{\mathbf{Shv}(\mathscr{C}_{\text{ét}}),\Lambda}\big(\mu_n^{\otimes(-r)}, \Lambda\big) & (r < 0), \end{cases}$$

where ϵ denotes the continuous map $\mathscr{C}_{\text{ét}} \to \mathscr{C}_{\mathrm{zar}}$ of big sites. We define the morphism $\varrho : \mathscr{O}^\times[-1] \to \Lambda(1)$ of (D1) as the connecting morphism associated with the Kummer exact sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_{\mathrm{m}} \xrightarrow{n\text{-th power}} \mathbb{G}_{\mathrm{m}} \longrightarrow 1$$

on $\mathscr{C}_{\text{ét}}$. We define the product structure (D2) on $\{\Lambda(r)\}_{r\in\mathbb{Z}}$ as the natural one, and define the cycle class (D4) as Gabber's cycle class

$$\mathrm{cyc}_X(V) \in H^{2r}_V(X_{\text{ét}}, \mu_n^{\otimes r}) \cong H^{2r}_V(X_{\mathrm{zar}}, \Lambda(r)) \qquad (\text{for } r = \mathrm{codim}_X(V)).$$

We define the push-forward morphism (D3) by the cup product with $\mathrm{cyc}_X(D) = c_1^X(D)$. Then these data satisfy the axioms (B0)–(B3) and (B5) obviously. See [SS1] Theorem 12.5.1 (resp. loc. cit. Proposition 12.2.1) for (B4) (resp. (B6)). For a pair (P, H) as in (D5) and $r \geqq 1$, we define

$$\Lambda(r)_{(P,H)} := \varepsilon_* j_* J_{P \smallsetminus H},$$

where j (resp. ε) denotes the natural open immersion $P \smallsetminus H \to P$ (resp. the continuous map $P_{\text{ét}} \to P_{\text{zar}}$ of small sites), and $J_{P \smallsetminus H}$ denotes the Godement resolution of $\mu_n^{\otimes r}$ on $(P \smallsetminus H)_{\text{ét}}$. One can check (B7)–(B9) by the absolute purity [FG] ([SS1] Theorem 12.2.15).

Setting B.3 In the rest of this appendix, we are given a family $\{\Lambda(r)\}_{r \in \mathbb{Z}}$ of complexes of sheaves of Λ-modules on $\mathscr{C}_{\text{zar}}$ with data (D1)–(D5) satisfying (B0)–(B9). See also Setting B.6 below.

Lemma B.4 *Let (P, H) be a pair as in* (D5)*, and let W be a closed subset of P of pure codimension c. Then we have*

$$H^q_W(P_{\text{zar}}, \Lambda(c)_{(P,H)}) \cong \begin{cases} 0 & (q < 2c) \\ H^{2c}_{W \smallsetminus H}((P \smallsetminus H)_{\text{zar}}, \Lambda(c)) & (q = 2c). \end{cases}$$

Consequently, for each irreducible component V of $W \smallsetminus H$, we define a cycle class

$$\mathrm{cyc}_P(V) \in H^{2c}_W(P_{\text{zar}}, \Lambda(c)_{(P,H)})$$

as the element corresponding to $\mathrm{cyc}_{P \smallsetminus H}(V) \in H^{2c}_{Z \smallsetminus H}((P \smallsetminus H)_{\text{zar}}, \Lambda(c))$.

Proof. The assertion follows mainly from (B7), (B8), (B9). The details are left to the reader as a report exercise. □

Lemma B.5 *For any $X \in \mathrm{Ob}(\mathscr{C})$ and any pair (P, H) over X as in* (D5)*, the composite*

$$\Lambda(r)_X \xrightarrow{\pi^*} R\pi_* \Lambda(r)_P \longrightarrow R\pi_* \Lambda(r)_{(P,H)}$$

is an isomorphism in $D(\mathbf{Shv}(X_{\text{zar}}), \Lambda)$, where π denotes the projection $P \to X$.

Proof. The assertion follows mainly from (B3), (B5), (B6), (B8). The details are left to the reader as a report exercise. □

Exercise 15 *Deduce Lemma B.5 from the axioms* (B0)–(B9) *of* $\{\Lambda(r)\}_{r \in \mathbb{Z}}$.

Setting B.6 We fix a projective completion $\overline{\Delta^q}$ of Δ^q (cf. Definition 3.3) as follows:

$$\overline{\Delta^q} := \mathrm{Proj}(\mathbb{Z}[T_0, T_1, \ldots, T_q, T_\infty]/(T_0 + T_1 + \cdots + T_q = T_\infty)).$$

Let $H^q \subset \overline{\Delta^q}$ be the hyperplane at infinity, i.e., $H^q = \{T_\infty = 0\}$.

The following proposition will be useful in our construction of a cycle class morphism.

Proposition B.7 *Let q and r be integers with $q, r \geqq 0$, and let $X \in \mathrm{Ob}(\mathscr{C})$. Let U be a scheme which is étale of finite type over X. Let $\Sigma^{r,q}$ be the set of all closed subsets on $U \times \Delta^q$ of pure codimension r which meet all faces of $U \times \Delta^q$ properly* (*cf. Definition 3.3*)*. For $W \in \Sigma^{r,q}$, let $\overline{W}$ be the closure of W in $U \times \overline{\Delta^q}$. Then:*

(1) *There is a canonical Λ-homomorphism*

$$\mathrm{cyc}^{r,q} : z^r(U, q) \otimes \Lambda \longrightarrow \varinjlim_{W \in \Sigma^{r,q}} H^{2r}_{\overline{W}}\left(U \times \overline{\Delta^q}_{\text{zar}}, \Lambda(r)_{(U \times \overline{\Delta^q}, U \times H^q)}\right)$$

sending a cycle $C \in z^r(U, q)$ to the cycle class $\mathrm{cyc}_{U \times \overline{\Delta^q}}(C)$, the linear extension of the cycle class of Lemma B.4.

(2) *For each $W \in \Sigma^{r,q}$, the natural morphism*

$$\tau_{\leqq 2r}\, R\Gamma_{\overline{W}}\left(U \times \overline{\Delta^q}_{\mathrm{zar}}, \Lambda(r)_{(U\times\overline{\Delta^q}, U\times H^q)}\right) \longrightarrow H^{2r}_{\overline{W}}\left(U \times \overline{\Delta^q}_{\mathrm{zar}}, \Lambda(r)_{(U\times\overline{\Delta^q}, U\times H^q)}\right)[-2r]$$

is an isomorphism in the derived category of Λ-modules.

(3) *Let C be a cycle which belongs to $z^r(U,q)$, and let $W \in \Sigma^{r,q}$ be the support of C. Let $\bar{i} : U \times \overline{\Delta^{q-1}} \hookrightarrow U \times \overline{\Delta^q}$ be the closure of a face map $i : U \times \Delta^{q-1} \hookrightarrow U \times \Delta^q$. Then the pull-back map*

$$\bar{i}^* : H^{2r}_{\overline{W}}\left(U \times \overline{\Delta^q}_{\mathrm{zar}}, \Lambda(r)_{(U\times\overline{\Delta^q}, U\times H^q)}\right) \longrightarrow H^{2r}_{\bar{i}^{-1}(\overline{W})}\left(U \times \overline{\Delta^{q-1}}_{\mathrm{zar}}, \Lambda(r)_{(U\times\overline{\Delta^{q-1}}, U\times H^{q-1})}\right)$$

sends the cycle class $\mathrm{cyc}_{U\times\overline{\Delta^q}}(C)$ to $\mathrm{cyc}_{U\times\overline{\Delta^{q-1}}}(i^*C)$, where i^*C denotes the pull-back of the cycle C along i.

Proof. (1) and (2) follow from Lemma B.4. The assertion (3) follows from (B4). □

Theorem B.8 *For any $X \in \mathrm{Ob}(\mathscr{C})$ and $r \geqq 0$, there exists a canonical morphism*

$$\mathrm{cyc}^r : \mathbb{Z}(r)_X \otimes \Lambda \longrightarrow \Lambda(r)_X \quad \text{in} \quad D(X_{\mathrm{zar}}, \Lambda),$$

where $\mathbb{Z}(r)_X$ denotes Bloch's $\mathbb{Z}(r)$ considered on X_{zar} (see Definition 3.3).

Proof. Let U be étale of finite type over X. For each $q \geqq 0$, let $G(r)^{\bullet}_{(U\times\overline{\Delta^q}, U\times H^q)}$ be the Godement resolution on $(U\times\overline{\Delta^q})_{\mathrm{zar}}$ of the complex $\Lambda(r)_{(U\times\overline{\Delta^q}, U\times H^q)} = \Lambda(r)^{\bullet}_{(U\times\overline{\Delta^q}, U\times H^q)}$. There is a diagram of cochain complexes concerning $\bullet$:

$$\begin{aligned} z^r(U,q)\otimes\Lambda[-2r] &\xrightarrow{\mathrm{cyc}^{r,q}} \varinjlim_{W\in\Sigma^{r,q}} H^{2r}_{\overline{W}}\left(U \times \overline{\Delta^q}_{\mathrm{zar}}, \Lambda(r)_{(U\times\overline{\Delta^q}, U\times H^q)}\right)[-2r] \\ &\xleftarrow{\alpha^{r,q}} \varinjlim_{W\in\Sigma^{r,q}} \tau_{\leqq 2r}\,\Gamma_{\overline{W}}\left(U\times\overline{\Delta^q}, G(r)^{\bullet}_{(U\times\overline{\Delta^q}, U\times H^q)}\right) \\ &\xrightarrow{\beta^{r,q}} \Gamma\left(U\times\overline{\Delta^q}, G(r)^{\bullet}_{(U\times\overline{\Delta^q}, U\times H^q)}\right). \end{aligned}$$

Here $\alpha^{r,q}$ and $\beta^{r,q}$ are natural maps of complexes, which are contravariant for the face maps $U \times \Delta^{q-1} \hookrightarrow U \times \Delta^q$. The arrow $\mathrm{cyc}^{r,q}$ is contravariant for these face maps by Proposition B.7 (3). Hence we get homomorphisms of double complexes concerning $(\star,\bullet)$

$$\begin{aligned} z^r(U,\star)\otimes\Lambda[-2r] &\xrightarrow{\mathrm{cyc}^{r,\star}} \varinjlim_{W\in\Sigma^{r,\star}} H^{2r}_{\overline{W}}\left(U \times \overline{\Delta^\star}_{\mathrm{zar}}, \Lambda(r)_{(U\times\overline{\Delta^\star}, U\times H^\star)}\right)[-2r] \\ &\xleftarrow{\alpha^{r,\star}} \varinjlim_{W\in\Sigma^{r,\star}} \tau_{\leqq 2r}\,\Gamma_{\overline{W}}\left(U\times\overline{\Delta^\star}, G(r)^{\bullet}_{(U\times\overline{\Delta^\star}, U\times H^\star)}\right) \\ &\xrightarrow{\beta^{r,\star}} \Gamma\left(U\times\overline{\Delta^\star}, G(r)^{\bullet}_{(U\times\overline{\Delta^\star}, U\times H^\star)}\right) \\ &\longleftarrow \Gamma(U, G(r)^{\bullet}_U), \end{aligned}$$

where the differentials in the $\star$-direction are alternating sums of pull-back maps along the faces of codimension 1, and the last arrow is the inclusion to the factor of $\star = 0$. The arrow $\alpha^{r,\star}$ (resp. the last arrow) is a quasi-isomorphism on the associated total complexes by Proposition B.7 (2) (resp. Lemma B.5). We thus obtain the cycle class morphism in $D(X_{\mathrm{zar}}, \Lambda)$ by sheafifying the diagram of total complexes. □

C Report exercises

Exercise 1 (§2) *Let X be a scheme of finite type over $\mathbb{F}_q$. Then show that*

$$\zeta(X,s) = Z(X/\mathbb{F}_q, q^{-s})$$

for $\mathrm{Re}(s) > \dim X$.

Exercise 2 (§2) *Check that the value on the right hand side of* (2.3) *is independent of the choice of a bounded open subset $Z \neq \emptyset$, and that $\mathfrak{m}_1$ is a Haar measure on $\widetilde{H}^1_{\mathscr{D}}(X_{/\mathbb{R}}, \mathbb{Z}(1))$.*

Exercise 3 (§2) *Using classical facts on number fields, show that*

$$H^i_{\text{ét}}(\mathrm{Spec}(O_K), \mathbb{G}_{\mathrm{m}}) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r'} & (i=2) \\ \mathbb{Q}/\mathbb{Z} & (i=3) \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1} & (i \geqq 4,\ \textit{even}) \\ 0 & (i \geqq 5,\ \textit{odd}), \end{cases}$$

where r_1 denotes the number of real places of K, and $r' := \max\{r_1 - 1, 0\}$.

Exercise 4 (§3) *Show that Bloch's $\mathbb{Z}(r)$ satisfies* (L0) *for any regular noetherian scheme.*

Exercise 5 (§4) *Let $\mathscr{A}$ be an abelian category with enough injective objects, and let $\mathscr{N}_1 \xrightarrow{f} \mathscr{N}_2 \xrightarrow{g} \mathscr{N}_3 \xrightarrow{h} \mathscr{N}_1[1]$ be a distinguished triangle in $D^-(\mathscr{A})$. Show the following*:

(1) *Let $i : \mathscr{K} \to \mathscr{N}_2$ be a morphism with $g \circ i = 0$, and assume* $\mathrm{Hom}_{D^-(\mathscr{A})}(\mathscr{K}, \mathscr{N}_3[-1]) = 0$. *Then there exists a unique morphism $i' : \mathscr{K} \to \mathscr{N}_1$ that i factors through.*

(2) *Let $p : \mathscr{N}_2 \to \mathscr{K}$ be a morphism with $p \circ f = 0$ and assume* $\mathrm{Hom}_{D^-(\mathscr{A})}(\mathscr{N}_1[1], \mathscr{K}) = 0$. *Then there exists a unique morphism $p' : \mathscr{N}_3 \to \mathscr{K}$ that p factors through.*

(3) *Assume* $\mathrm{Hom}_{D^-(\mathscr{A})}(\mathscr{N}_2, \mathscr{N}_1) = 0$. *Then relatively to a morphism $h : \mathscr{N}_3 \to \mathscr{N}_1[1]$, the triple $(\mathscr{N}_2, f, g)$ is unique up to a unique isomorphism, and f is determined by the pair $(\mathscr{N}_2, g)$.*

Exercise 6 (§4) *Show that* cyc^r *of Theorem 4.11 is an isomorphism for $r = 0, 1$.*

Exercise 7 (§5) *Let $\xi \in H^1(K, V)$ correspond to an extension of ℓ-adic representations of G_K*

$$0 \longrightarrow V \longrightarrow E \longrightarrow \mathbb{Q}_\ell \longrightarrow 0.$$

Then show the following:

(1) *If $\ell \neq p$, then ξ belongs to $H^1_f(K, V)$ if and only if the induced sequence*

$$0 \longrightarrow V^{I_K} \longrightarrow E^{I_K} \longrightarrow \mathbb{Q}_\ell^{I_K} (= \mathbb{Q}_\ell) \longrightarrow 0$$

is exact.

(2) *If $\ell = p$, then ξ belongs to $H^1_f(K, V)$ if and only if the induced sequence*

$$0 \longrightarrow \mathrm{Crys}(V) \longrightarrow \mathrm{Crys}(E) \longrightarrow \mathrm{Crys}(\mathbb{Q}_p)(= K_0) \longrightarrow 0$$

is exact.

Exercise 8 (§5) *Let G be a profinite group and let N be a closed normal subgroup of G. Let M be a topological G-module, and put*

$$Z^1(N,M) := \big\{\varphi : N \to M \text{ continuous map} \,\big|\, {}^\forall x, {}^\forall y \in N, \varphi(xy) = \varphi(x) + x\cdot\varphi(y)\big\},$$
$$B^1(N,M) := \big\{\varphi : N \to M \text{ continuous map} \,\big|\, {}^\exists a \in M, {}^\forall x \in N, \varphi(x) = x\cdot a - a\big\}.$$

Then show the following:

(1) *For $\varphi \in Z^1(N,M)$ and $g \in G$, define a map $g\cdot\varphi : N \to M$ by*

$$(g\cdot\varphi)(x) := g\cdot(\varphi(g^{-1}xg)).$$

Then $g\cdot\varphi$ belongs to $Z^1(N,M)$, and the map

$$\gamma : G \times Z^1(N,M) \to Z^1(N,M), \quad (g,\varphi) \mapsto g\cdot\varphi$$

defines a left G-action on $Z^1(N,M)$.

(2) *$B^1(N,M)$ is a left G-submodule of $Z^1(N,M)$.*

(3) *N acts trivially on $H^1(N,M)$ via γ, i.e., $H^1(N,M)$ is a left G/N-module.*

Exercise 9 (§5) *Let G be a profinite group and let N be a closed normal subgroup of G. Put $\varGamma := G/N$. Let M be a topological G-module. Then show that there is an inflation-restriction exact sequence*

$$0 \longrightarrow H^1(\varGamma, M^N) \xrightarrow{\text{Inf}} H^1(G,M) \xrightarrow{\text{Res}} H^1(N,M)^{\varGamma}.$$

Exercise 10 (§5) *Show the equality $\star$ in the last display of* §5.2.

Exercise 11 (§5) *Show that $H^1_f(K,V)$ is independent of the choice of S as in Condition 5.6.*

Exercise 12 (§5) *Let E be an elliptic curve over K, and let $\mathrm{Sel}(E/K)^{(p)}$ be the p-primary Selmer group:*

$$\mathrm{Sel}(E/K)^{(p)} := \mathrm{Ker}\left(H^1(K, E\{p\}) \longrightarrow \prod_{v\in P} \frac{H^1(K_v, E\{p\})}{E(K_v)\otimes \mathbb{Q}_p/\mathbb{Z}_p}\right),$$

where $E\{p\}$ denotes the p-primary torsion part of $E(\overline{K})$. Put

$$V_E := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{n\geqq 1} {}_{p^n}E,$$

where ${}_{p^n}E$ denotes the p^n-torsion part of $E(\overline{K})$. Is there a natural map

$$H^1_f(K, V_E) \longrightarrow \mathrm{Sel}(E/K)^{(p)}\,?$$

If so, is the cokernel finite?

Exercise 13 (§A) *Let X be a scheme, and let $i : Z \hookrightarrow X$ be a closed subscheme. Let $j : V := X \smallsetminus Z \hookrightarrow X$ be the open immersion from the open complement of Z. Let $\mathscr{F}$ be an étale sheaf on X. For each $U \in \mathrm{Ob}(\text{Ét}/X)$, we put*

$$\Gamma_{Z \times_X U}(U, \mathscr{F}) := \mathrm{Ker}(\Gamma(U, \mathscr{F}) \to \Gamma(U \times_X V, \mathscr{F}))$$

(1) *Show that the assignment*

$$\underline{\Gamma}_Z(X, \mathscr{F}) : U \in \mathrm{Ob}(\text{Ét}/X) \longmapsto \Gamma_{Z \times_X U}(U, \mathscr{F})$$

is an étale sheaf on X.

(2) *Show that the following sequence of sheaves is exact on $X_{\text{ét}}$:*

$$0 \longrightarrow \underline{\Gamma}_Z(X, \mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_* j^* \mathscr{F}.$$

(3) *Show that $\underline{\Gamma}_Z(X, \mathscr{F}) \cong i_* i^* \underline{\Gamma}_Z(X, \mathscr{F})$ for any étale sheaf on X.*

(4) *Put $i^! \mathscr{F} := i^* \underline{\Gamma}_Z(X, \mathscr{F})$ for an étale sheaf $\mathscr{F}$ on X. Then show that the functor $i^! : \mathbf{Shv}(X_{\text{ét}}) \to \mathbf{Shv}(Z_{\text{ét}})$ is right adjoint to $i_* : \mathbf{Shv}(Z_{\text{ét}}) \to \mathbf{Shv}(X_{\text{ét}})$.*

(5) *Show that the functor*

$$\Gamma_Z(X, -) : \mathbf{Shv}(X_{\text{ét}}) \longrightarrow \mathbf{Ab}, \qquad \mathscr{F} \longmapsto \Gamma_Z(X, \mathscr{F})$$

is left exact. Let $H^*_Z(X, -)$ be its right derived functor.

(6) *For each $q \geqq 0$ and each sheaf $\mathscr{F} \in \mathbf{Shv}(X_{\text{ét}})$, let $\underline{H}^q_Z(X, \mathscr{F})$ be the sheaf on $X_{\text{ét}}$ associated with the presheaf*

$$U \in \mathrm{Ob}(\text{Ét}/X) \longmapsto H^q_{Z \times_X U}(U, \mathscr{F}).$$

Then show that the functor

$$i^* \underline{H}^q_Z(X, -) : \mathbf{Shv}(X_{\text{ét}}) \longrightarrow \mathbf{Shv}(Z_{\text{ét}}), \qquad \mathscr{F} \longmapsto i^* \underline{H}^q_Z(X, \mathscr{F})$$

agrees with the q-th right derived functor $R^q i^!$ of $i^!$.

Exercise 14 (§A) *Let X be a proper smooth geometrically integral curve over a field k.*

(1) *Assume that k is algebraically closed. Then show that $H^i(X, \mathbb{G}_{\mathrm{m}}) = 0$ for any $i \geqq 2$, using Tsen's theorem: the function field $k(X)$ is a C_1-field.*

(2) *Assume that k is a finite field. Then show that*

$$H^i(X, \mathbb{G}_{\mathrm{m}}) \cong \begin{cases} \mathbb{Q}/\mathbb{Z} & (\text{if } i = 3) \\ 0 & (\text{if } i = 2 \text{ or } i \geqq 4). \end{cases}$$

Exercise 15 (§B) *Deduce Lemma B.5 from the axioms* (B0)–(B9) *of $\{\Lambda(r)\}_{r \in \mathbb{Z}}$.*

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