Revisiting Classical Splitting Results

Joint work with. Sanath Devalapurkar

arXiv: 1912.04130 'On the James and Hilton-Milnor Splittings, & the metastable EHP Sequence'

won't talk about this today

Outline.

(1) Classical splitting results & the motivic James splitting

(2) The 'fundamental' James splitting

(3) Mather's 2^{nd} cube Lemma & our results

(4) New motivic examples

1. Classical Splittings & the motivic James Splitting

James Splitting. Let X be a pointed connected space. There is a natural splitting

$$\Sigma\underbrace{\Omega\Sigma X}_{\text{free } E_1\text{-group on } X} \simeq \bigvee_{i\geq 1} \Sigma X^{\wedge i}.$$

Analogy: $xR[x] = \bigoplus_{i\geq 1} x^i R.$

Hilton–Milnor Splitting. Let X and Y be pointed connected spaces. There is a natural equivalence

$$\underbrace{\Omega\Sigma(X\vee Y)}_{\text{free } E_1\text{-group on } X\vee Y} \simeq \Omega\Sigma X \times \Omega\Sigma Y \times \Omega\Sigma\left(\bigvee_{i,j\geq 1} X^{\wedge i}\wedge Y^{\wedge j}\right).$$

Analogy: $R[x,y] = R[x]\times R[y]\times R[xy, x^2y, xy^2, \dots, x^iy^j, \dots]$.

Quick overview of motivic Spaces. Let S be a scheme

$$\begin{tikzcd} \underbrace{H(S)}_{\text{motivic spaces}} \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \underset{\text{Nisnevich topology}}{Sh_{nis}}(\underbrace{Sm_S}_{\text{smooth } S\text{-schemes}}; Spc) \arrow[l, shift right, "L_{mot}"'] \end{tikzcd}$$

$pr_1^*: F(X) \xrightarrow{\sim} F(X\times_S \mathbb{A}^1_S)$

> ét $\geq$ nis $\geq$ zar
> K-theory satisfies Nisnevich but not étale descent
> Purity: $Th(N_{Z\subset X}) \simeq \frac{X}{X\setminus Z}$

> L_{mot} preserves finite products, but not finite limits & $H(S)$ is not an ∞-topos

$$L_{mot}\left(\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{1\} \arrow[d, hook] \\ \{0\} \arrow[r, hook] & \mathbb{A}^1 \end{tikzcd}\right) \text{ is not a pullback}$$

Really a corollary, but basically equivalent

Morel's Unstable $\mathbb{A}^1$-Connectivity Theorem. If k is a perfect field, then $L_{mot}: Sh_{nis}(Sm_k) \to H(Spec(k))$ commutes with Ω.

⚠ Ayoub showed this is not true over more general base schemes.

Theorem (Wickelgren–Williams). Let k be a perfect field. The James Splitting holds for $\underbrace{\mathbb{A}^1\text{-connected}}_{\text{connected in } Sh_{nis}(Sm_S)}$ pointed motivic spaces over k.

Proof. Not the proof they present...

(1) James Splitting in Spc $\Longrightarrow$ James Splitting in $PSh(Sm_S)$

(co) limits computed pointwise

(2) $H(S)$ is a localization of $PSh(Sm_S)$ and the left adjoint commutes with finite (smash) products and Ω:

unstable connectivity ✓ left exact ✓

$$H(S) \underset{\longrightarrow}{\overset{L_{mot}}{\underset{\perp}{\longleftarrow}}} Sh_{nis}(Sm_S) \underset{\longrightarrow}{\overset{L_{nis}}{\underset{\perp}{\longleftarrow}}} PSh(Sm_S).$$ $\square$

Proposition. Same for the Hilton–Milnor Splitting — same proof!

Question. Do these splittings hold in motivic spaces over more general bases? What generality do these results hold in?

> We'll answer this!

- First we'll generalize to remove the connectedness hypothesis.

2. The 'fundamental' James Splitting

This was known to Ganea

don't need connectedness!

'Fundamental' James Splitting. Let X be a pointed space. There is a natural equivalence

$$\Sigma\Omega\Sigma X \simeq \Sigma X \vee (X \wedge \Sigma\Omega\Sigma X)$$

Analogy: $xR[x] = xR \oplus x^2R[x]$.

in the ∞-category Spc_*. Iterating:

$$\Sigma\Omega\Sigma X \simeq \Sigma X \vee \Sigma X^{\wedge 2} \vee \cdots \vee \Sigma X^{\wedge n} \vee (X^{\wedge n} \wedge \Sigma\Omega\Sigma X).$$

Note. Taking colimits, there is a natural comparison map

$$c_X : \bigvee_{i \geq 1} \Sigma X^{\wedge i} \longrightarrow \Sigma\Omega\Sigma X.$$

⚠ c_X is not generally an equivalence!

$$c_{S^0} : \bigvee_{i \geq 1} S^1 \longrightarrow \bigvee_{i \in \mathbb{Z}} S^1$$ is the summand inclusion

Problem: S^0 is not connected

Corollary (James Splitting). If X is connected, then c_X is an equivalence.

Proof Idea.

If X is connected then $X^{\wedge n} \wedge \Sigma\Omega\Sigma X$ is n-connected, hence vanishes as $n \longrightarrow \infty$.

More precisley, this implies that c_X is $(n-1)$-connected for each $n \geq 1$. $\square$

3. Mather's 2^{nd} Cube Lemma

Definition. Let I be an ∞-category and $\mathcal{C}$ an ∞-category with pullbacks and I-shaped colimits. We say that **I-shaped colimits are universal in $\mathcal{C}$** if given

a diagram $F: I \to \mathcal{C}$ + morphisms $\operatorname{colim}_{i \in I} F(i) \longrightarrow Z \longleftarrow Y$

the natural morphism

$$\operatorname{colim}_{i \in I} \left(F(i) \times_Z Y\right) \longrightarrow \left(\operatorname{colim}_{i \in I} F(i)\right) \times_Z Y$$

is an equivalence.

Lemma. Let $\mathcal{C}$ be an ∞-category with pullbacks and pushouts. TFAE:

(1) Pushouts are universal in $\mathcal{C}$.

(2) **Mather's 2^{nd} Cube Lemma** holds in $\mathcal{C}$: given a cube

Mather originally proved for Spc.
[Kerodon, Tag 011H]

$$\begin{tikzcd}
A_0 \arrow[rr] \arrow[dr] \arrow[dd] & & A_2 \arrow[dr] \arrow[dd] & & \\
& A_1 \arrow[rr] \arrow[dd] & & A_3 \arrow[dd] \arrow[r] & Y \arrow[dd] \\
B_0 \arrow[rr] \arrow[dr] & & B_2 \arrow[dr] & & \\
& B_1 \arrow[rr] & & B_3 \arrow[r] & Z
\end{tikzcd}$$

Proof that (2) $\Rightarrow$ (1)

where the bottom square is a pushout and vertical squares are pullbacks, the top square is a pushout.

Examples

$$T \underset{\text{accessible}}{\overset{\text{left exact}}{\rightleftarrows}} PSh(C)$$

Example (∞-topoi). If T is an ∞-topos, then one of the Giraud-Lurie Axioms says all colimits are universal in T.

- $T = Sh_{nis}(Sm_S)$
- G a finite group, $T = Spc^G \overset{\text{Elmendorf}}{\simeq} Fun(Orb_G^{op}, Spc)$ G-spaces

Example (profinite spaces).

- A space X is π-finite if X is truncated, $\pi_0(X)$ is finite, and for all $x \in X$ and $i \geq 1$, the group $\pi_i(X,x)$ is finite.

Finite colimits and geometric realizations of simplicial objects are universal in $\underbrace{Pro(Spc_\pi)}_{\text{freely adjoin cofiltered limits to } Spc_\pi}$ [Lurie SAG, Theorem E.6.3.1 & Corollary E.6.3.2].

⚠ Infinite coproducts are not universal in $Pro(Spc_\pi)$. $\coprod_S * = \beta(S)$, but $\beta(S \times T) \not\simeq \beta(S) \times \beta(T)$

Example (motivic spaces). S scheme

Hoyois showed all colimits are universal in $H(S)$.

Prop. 3.15 of 'The six operations in equivariant motivic homotopy theory'

Proof of the 'fundamental' James Splitting

'Fundamental' James Splitting (Devalapurkar–H). Let $\mathcal{C}$ be an ∞-category with finite limits and pushouts satisfying Mather's 2nd Cube Lemma. For all $X \in \mathcal{C}_*$,

$$\Sigma\Omega\Sigma X \simeq \Sigma X \underbrace{\vee}_{\text{Coproduct in } \mathcal{C}_*} (X \wedge \Sigma\Omega\Sigma X).$$

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \Omega Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ * \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} X \vee Y \arrow[r, "\begin{pmatrix} id & * \\ * & id \end{pmatrix}"] \arrow[d] & X \times Y \arrow[d] \\ * \arrow[r] & X \wedge Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Lemma 1. In the above setting there is a pushout square

$$\begin{tikzcd} X \times \Omega\Sigma X \arrow[r, "pr_2"] \arrow[d, "a_X"'] & \Omega\Sigma X \arrow[d] \\ \Omega\Sigma X \arrow[r] & * \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} . \implies \mathrm{Cofib}(X \times \Omega\Sigma X \xrightarrow{pr_2} \Omega\Sigma X) \simeq \Sigma\Omega\Sigma X.$$

not usually pr_2!

Proof.

$$\begin{tikzcd} X \times \Omega\Sigma X \arrow[rr, "pr_2"] \arrow[dr] \arrow[dd] & & \Omega\Sigma X \arrow[dr] \arrow[dd] & \\ & \Omega\Sigma X \arrow[rr] \arrow[dd] & & * \arrow[dd, "i_2"] \\ X \arrow[rr] \arrow[dr] & & * \arrow[dr, "i_2"] & \\ & * \arrow[rr, "i_1"] & & \Sigma X \end{tikzcd}$$

$\square$

Lemma 2. Let $\mathcal{C}$ be an ∞-category with finite limits and pushouts, and $X, Y \in \mathcal{C}_*$. There is a pushout square

$$\begin{tikzcd}
X \times Y \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & Y \arrow[d, "\text{null}"] \\
X \arrow[r, "\text{null}"'] & \Sigma(X \wedge Y) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}.$$

Proof.

$$\begin{tikzcd}
* \arrow[r, equal] & * \arrow[r, equal] & * & & * \arrow[d, equal] \\
X \arrow[u] \arrow[d, equal] & X \vee Y \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \arrow[u] \arrow[d, equal] & & * \arrow[d] \arrow[r, squiggly, "\text{pushout}"] & P \\
X & X \times Y \arrow[l] \arrow[r] & Y & & P \\
* & X \wedge Y \arrow[l] \arrow[r] & * \arrow[rr, squiggly, "\text{pushout}"'] & & \Sigma(X \wedge Y)
\end{tikzcd}$$

pushout $\rightsquigarrow$

pushout $\}$

$\square$

Lemma 3. Let $\mathcal{C}$ be an ∞-category with finite limits and pushouts, and $X, Y \in \mathcal{C}_*$. Then $\mathrm{Cofib}(X \times Y \xrightarrow{pr_2} \underbrace{Y}_{\text{think } \Omega\Sigma X}) \simeq \Sigma X \vee \Sigma(X \wedge Y)$.

Proof.

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r, "\text{null}"'] \arrow[d] & \Sigma(X \wedge Y) \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] \\
* \arrow[r] & \Sigma X \vee \Sigma(X \wedge Y) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}.$$

Lemma 2

$\square$

Proof of 'Fundamental' James Splitting.

$$\Sigma\Omega\Sigma X \simeq \mathrm{Cofib}\left(X \times \Omega\Sigma X \xrightarrow{pr_2} \Omega\Sigma X\right)$$

Lemma 1

$$\simeq \Sigma X \vee \underbrace{\Sigma(X \wedge \Omega\Sigma X)}_{\text{Mather: } \simeq X \wedge \Sigma\Omega\Sigma X}.$$

Lemma 3

$\square$

proof uses similar techniques

'Fundamental' Hilton-Milnor Splitting (Devalapurkar-H.). Let $\mathcal{C}$ be an ∞-cat. with finite limits and pushouts satisfying Mather's 2^{nd} Cube Lemma.

For all $X, Y \in \mathcal{C}_*$,

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \wedge \Omega Y).$$

4. New Motivic Examples

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, hook] \arrow[d, hook] & \mathbb{P}^1 \smallsetminus \{0\} \cong \mathbb{A}^1 \arrow[d, hook] \\ \mathbb{A}^1 \cong \mathbb{P}^1 \smallsetminus \{\infty\} \arrow[r, hook] & \mathbb{P}^1 \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Example 1. S scheme. The James Splitting and equivalence $\Sigma \mathbb{G}_m \simeq \mathbb{P}^1_S$ imply that

$$\Sigma \Omega \mathbb{P}^1_S \simeq \Big(\bigvee_{i=1}^{n} \underbrace{S^{i+1,i}}_{S^{a,b} := \mathbb{G}_m^{\wedge b} \wedge (S^1)^{\wedge(a-b)}} \Big) \vee \left(S^{n+1,n} \wedge \Omega \mathbb{P}^1_S \right)$$

Example 2. S qcqs scheme. Wickelgren showed that

$$\Sigma(\mathbb{P}^1_S \smallsetminus \{0,1,\infty\}) \simeq \Sigma(\mathbb{G}_m \vee \mathbb{G}_m).$$

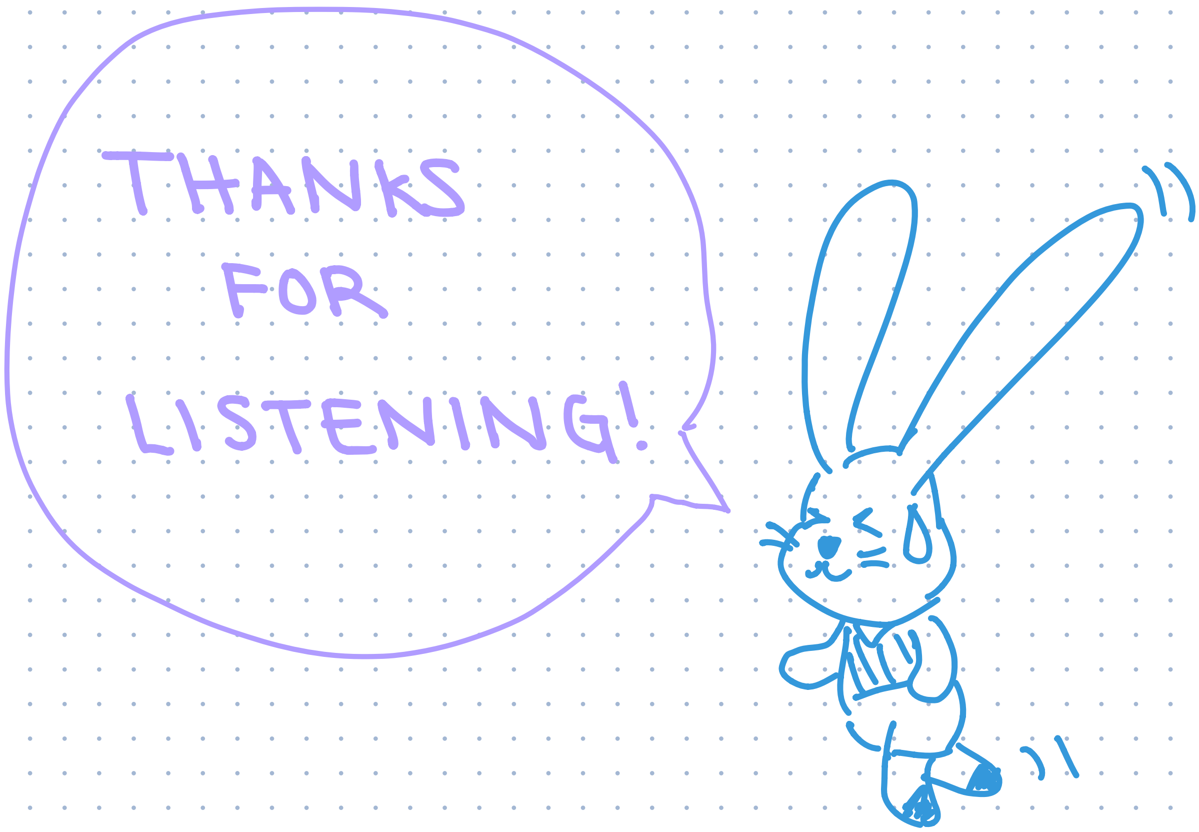
⚠ Not true before suspension

The Fundamental Hilton-Milnor Splitting shows that

$$\Omega\Sigma(\mathbb{P}^1_S \smallsetminus \{0,1,\infty\}) \simeq \Omega\mathbb{P}^1_S \times \Omega\mathbb{P}^1_S \times \Omega\Sigma(\underbrace{\Omega\mathbb{P}^1_S}_{\text{Can plug in Example 1 if desired}} \wedge \Omega\mathbb{P}^1_S).$$

Example 3. k perfect field.

$$\Omega\Sigma^2(\mathbb{P}^1_k \smallsetminus \{0,1,\infty\}) \simeq \Omega S^{3,1} \times \Omega S^{3,1} \times \Omega\Big(\bigvee_{i,j \geq 1} S^{2(i+j)+1,\, i+j} \Big).$$


THANKS FOR LISTENING!