LECTURE 3: CUBIC HYPERSURFACES II: UNIRATIONALITY

Let k be the ground field, which we will later assume to have characteristic 0. Our goal is to show that a smooth cubic projective hypersurface over k that contains a line is unirational. In particular, this is the case for all smooth cubic hypersurfaces if k is algebraically closed.

1. Projections

In this section we review some basic facts about projections. Let V be a finite-dimensional linear space over k and W a proper, nonzero linear subspace of V. The canonical map $V \to V/W$ induces a closed embedding $\Lambda = \mathbf{P}(V/W) \hookrightarrow \mathbf{P}(V) = \mathbf{P}$. The injective map $W \hookrightarrow V$ also induces a map, the *linear projection with center* Λ

$$\varphi \colon \mathbf{P} \smallsetminus \Lambda \to \mathbf{P}(W).$$

This can be described geometrically as follows. If we choose a splitting of the inclusion $W \hookrightarrow V$, we also get an embedding $\mathbf{P}(W) \hookrightarrow \mathbf{P}$ such that $\mathbf{P}(W) \cap \Lambda = \emptyset$. In this case, $\varphi(p)$ is given by intersecting the linear span of p and Λ with $\mathbf{P}(W)$.

After choosing a splitting as above, we may choose coordinates $x_0, \ldots, x_n$ on $\mathbf{P}$ such that

$$\mathbf{P}(W) = (x_{r+1} = \ldots = x_n = 0) \quad \text{and} \quad \Lambda = (x_0 = \ldots = x_r = 0).$$

In these coordinates, the map φ is given by $\varphi(x_0, \ldots, x_n) = (x_0, \ldots, x_r)$.

The rational map φ becomes a morphism on the blow-up of $\mathbf{P}$ along Λ. In order to see this and in order to describe the corresponding morphism, we begin by fixing a splitting of $W \hookrightarrow V$. Therefore we have an isomorphism $V \simeq W \oplus V/W$. Consider on $\mathbf{P}(W)$ the vector bundle

$$\mathcal{E} = \mathcal{O}_{\mathbf{P}(W)}(1) \oplus \left(V/W \otimes \mathcal{O}_{\mathbf{P}(W)} \right)$$

and let $g \colon B = \mathbf{P}_{\mathbf{P}(W)}(\mathcal{E}) \to \mathbf{P}(W)$ be the corresponding projective bundle with the tautological line bundle $\mathcal{L} = \mathcal{O}_B(1)$. Note that $\mathcal{E}$ is globally generated and

$$H^0\left(\mathbf{P}(W), \mathcal{E}\right) \simeq W \oplus V/W \simeq V.$$

Therefore we have a surjective morphism

$$V \otimes \mathcal{O}_{\mathbf{P}(W)} \to \mathcal{E}$$

which gives a closed immersion

$$j \colon B \hookrightarrow \mathbf{P}(V) \times \mathbf{P}(W),$$

that embeds B as a projective subbundle over $\mathbf{P}(W)$.

If we compose j with the first projection we obtain a morphism $h\colon B \to \mathbf{P}(V)$. This is the morphism defined by the globally generated line bundle $\mathcal{L}$ on B and the canonical map

$$H^0(B, \mathcal{L}) \otimes \mathcal{O}_B = H^0(\mathbf{P}(W), \mathcal{E}) \otimes \mathcal{O}_B = V \otimes \mathcal{O}_B \to \mathcal{L}.$$

Note that on $\mathbf{P}(W)$ we have a surjective morphism of vector bundles

$$\mathcal{E} \to V/W \otimes \mathcal{O}_{\mathbf{P}(W)}$$

that induces a closed embedding

$$E := \mathbf{P}_{\mathbf{P}(W)}(V/W \otimes \mathcal{O}_{\mathbf{P}(W)}) = \mathbf{P}(V/W) \times \mathbf{P}(W) \hookrightarrow B$$

as a projective subbundle over $\mathbf{P}(W)$. Note that

$$\mathcal{L}|_E \simeq p^* \mathcal{O}_{\mathbf{P}(V/W)}(1),$$

where $p\colon E \to \mathbf{P}(V/W)$ is the first projection, and the following diagram is commutative:

$$
\begin{array}{ccc}
E & \hookrightarrow & B \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle h} \\
\mathbf{P}(V/W) & \hookrightarrow & \mathbf{P}(V).
\end{array}
$$

It is clear by dimension considerations that E is an irreducible divisor on B. Moreover, we have

(1) $$E = h^{-1}\mathbf{P}(V/W).$$

Indeed, $\mathbf{P}(V/W)$ is the locus in $\mathbf{P}(V)$ where the morphism of vector bundles given by the following composition

$$W \otimes \mathcal{O}_{\mathbf{P}(V)} \hookrightarrow V \otimes \mathcal{O}_{\mathbf{P}(V)} \to \mathcal{O}_{\mathbf{P}(V)}(1)$$

is zero, hence $h^{-1}\mathbf{P}(V/W)$ is the locus where the composition

$$W \otimes \mathcal{O}_B \hookrightarrow V \otimes \mathcal{O}_B \to \mathcal{L}$$

vanishes. On the other hand, it follows from definition that E is the locus where the following composition vanishes:

$$g^* \mathcal{O}_{\mathbf{P}(W)}(1) \hookrightarrow g^* \mathcal{E} \to \mathcal{L}.$$

Since we have a commutative diagram

$$
\begin{array}{ccc}
W \otimes \mathcal{O}_B & \longrightarrow & \mathcal{L} \\
\downarrow & & \downarrow \\
g^* \mathcal{O}_{\mathbf{P}(W)}(1) & \longrightarrow & \mathcal{L},
\end{array}
$$

in which the left vertical map is surjective and the right vertical map is the identity, we deduce (1).

Proposition 1.1. *With the above notation, h is the blow-up of $\mathbf{P}(V)$ along $\mathbf{P}(V/W)$, with exceptional divisor E. Moreover, the rational map $h \circ g^{-1}$ is equal to the projection map φ.*

Proof. In order to simplify the exposition, we assume that k is algebraically closed and leave the task of formalizing the argument in general as an exercise. By definition, a point $x \in B$ is given by a nonzero linear map $y \colon W \to k$ (defining the point $g(x)$) and a nonzero linear map

$$x \colon \mathcal{O}_{\mathbf{P}(W)}(1)_{(y)} \oplus V/W = W/\ker(y) \oplus V/W \to k.$$

Also, it follows from definition that $h(x)$ is given by the composition

$$V = W \oplus V/W \to W/\ker(y) \oplus V/W \xrightarrow{x} k.$$

This shows that as a subset of $\mathbf{P}(V) \times \mathbf{P}(W)$, B is equal to the graph of φ, proving the last assertion in the proposition. In particular, this shows that f is an isomorphism over $\mathbf{P}(V) \smallsetminus \mathbf{P}(V/W)$. Using (1), it follows from the universal property of the blow-up that f factors as

$$B \xrightarrow{\alpha} \widetilde{\mathbf{P}(V)} \xrightarrow{\pi} \mathbf{P}(V),$$

where π is the blowing-up along $\mathbf{P}(V/W)$. Moreover, (1) implies that the exceptional divisor of α is 0, hence α is an isomorphism. $\qquad\square$

Remark 1.2. If w is a point in $\mathbf{P}(W)$, then j induces an embedding

$$g^{-1}(w) \hookrightarrow \mathbf{P}(V) \times \{w\} \simeq \mathbf{P}(V)$$

that identifies $g^{-1}(w)$ with the linear span of $\mathbf{P}(V/W)$ and w, while $E \cap g^{-1}(w)$ lies inside as $\mathbf{P}(V/W)$.

Remark 1.3. If we have two linear subspaces $L_1, L_2 \subseteq \mathbf{P}^n$, with $L_1 \cap L_2 = \emptyset$ and $\dim(L_1) + \dim(L_2) = n - 1$, and if we consider the linear subspaces W_1, W_2 of V such that $L_1 = \mathbf{P}(V/W_1)$ and $L_2 = \mathbf{P}(V/W_2)$, then we have $W_1 \oplus W_2 = V$. Therefore we may consider the projection $\varphi \colon \mathbf{P}(V) \smallsetminus L_1 \to L_2$ and the discussion in this section applies.

2. A (UNI)RATIONALITY CRITERION FOR FIBRATIONS IN QUADRICS

The following concept is very useful when studying rationality and unirationality.

Definition 2.1. A morphism of varieties $f \colon X \to S$ is a *quadric bundle* if it has a decomposition

$$(2) \qquad\qquad X \xhookrightarrow{\iota} S \times \mathbf{P}^n \xrightarrow{p} S,$$

with $n \geq 2$, where ι is a closed embedding and p the canonical projection, such that the fiber over each $s \in S$ is a quadric in $\mathbf{P}^{n+1}_{k(s)}$, smooth for general s. When $n = 2$, we say that f is a *conic bundle*.

A rational map $\varphi \colon X \dashrightarrow S$ is a *rational quadric bundle* if there are open subsets $U \subseteq X$ and $V \subseteq S$ such that φ corresponds to a morphism $U \longrightarrow V$, which is a quadric bundle.

Remark 2.2. Given a morphism of varieties $f \colon X \to S$ that factors as in (2), f is a rational quadric bundle if and only if the generic fiber $X_{k(\eta)}$ is a smooth quadric in $\mathbf{P}^n_{k(\eta)}$. Note that in order to check this condition, we may base-change to the algebraic closure of the ground field. We also note that the smoothness condition is automatically satisfied if X is smooth and $\mathrm{char}(k) = 0$.

This notion will be useful because of the following (uni)rationality criterion.

Proposition 2.3. *Let $\varphi\colon X \dashrightarrow S$ be a rational quadric bundle and $Y \hookrightarrow X$ a closed subvariety.*

   i) *If S is rational and φ induces a birational map $Y \dashrightarrow S$, then X is rational.*

   ii) *If Y is unirational and φ induces a dominant rational map $Y \dashrightarrow S$, then X is unirational.*

Remark 2.4. The existence of Y that satisfies condition i) above is equivalent with the fact that S is rational and φ has a rational section. When Y has the property that φ induces a dominant, generically finite rational map $Y \dashrightarrow S$, one sometimes refers to Y as a *multi-section* of φ.

Proof of Proposition 2.3. After replacing X and S by suitable open subsets, we may assume that $\varphi\colon X \to S$ is a quadric bundle. We fix a factorization as in Definition 2.1.

We first prove i). By assumption, the generic fiber is a smooth quadric X_η over the field of $k(\eta) = k(S)$ of rational functions on S. The birational map $Y \dashrightarrow S$ gives a rational section of φ, which corresponds to a rational point of X_η. Since every smooth quadric is rational as soon as it has a rational point (see Example 3.8 in Lecture 1), we conclude that X_η is birational over $k(S)$ to $\mathbf{P}^{n-1}_{k(S)}$. This implies that there is an open subset W of S such that X is birational to $W \times \mathbf{P}^{n-1}_k$, and since S is rational, we conclude that X is rational.

Suppose now that the conditions in ii) are satisfied. Since Y is unirational, we have a generically finite morphism $g\colon Z \to Y$, with Z a rational variety. Consider the Cartesian diagram

$$
\begin{array}{ccc}
Z \times_S X & \longrightarrow & X \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\
Z & \xrightarrow{\ f|_Y \circ g\ } & S.
\end{array}
$$

Note that h has the section $(1_Z, \iota \circ g)$. Moreover, by base-changing the decomposition (2), we obtain a decomposition of h as

$$Z \times_S X \hookrightarrow Z \times \mathbf{P}^n \longrightarrow Z$$

such that the generic fiber of h (which is obtained from the generic fiber of f by extending the ground field) is a smooth quadric. Note that a smooth quadric is automatically connected, since any to curves in the projective plane have nonempty intersection. We may thus replace Z by a suitable open subset so that $Z \times_S X$ a variety, in which case we see that h is a rational quadric bundle with a section. Since Z is rational, it follows from part i) that $Z \times_S X$ is rational. Using the fact that $Z \to S$ is dominant, we deduce that $Z \times_S X \to X$ is dominant, too, hence X is unirational. $\qquad\square$

Remark 2.5. It follows from the proof of Proposition 2.3 that if in ii) we assume that Y is rational and the map $Y \to S$ is generically finite, of degree d, then we have a rational, generically finite dominant map $\mathbf{P}^n \dashrightarrow X$, of degree d.

We record in the next proposition a part of the argument in the proof of Proposition 2.3 that we will apply also in other contexts.

Proposition 2.6. *Let $f\colon X \to Y$ be a dominant morphism of varieties over k and let $X_\eta \to \mathrm{Spec}\,\big(k(Y)\big)$ be the generic fiber.*

   i) *If Y is rational over k and the generic fiber X_η is rational over $k(Y)$, then X is rational over k.*

   ii) *If Y is unirational over k and the generic fiber X_η is unirational over $k(Y)$, then X is unirational over k.*

In the above statement, when we say that the generic fiber is rational or unirational, the implicit assumption is that it is irreducible (it is also reduced, as it is always the case with the generic fiber of a morphism between two varieties).

Proof of Proposition 2.6. Under the assumptions of i), we know that X_η is birational to $\mathbf{A}^n_{k(Y)}$, for some n. This implies that there is an open subset U of Y such that $f^{-1}(U)$ is birational to $U \times \mathbf{A}^n_k$. On the other hand, Y is rational, hence U is birational to some $\mathbf{A}^m$, and therefore X is birational to $\mathbf{A}^{n+m}_k$. This completes the proof of i).

Suppose now that the assumptions in ii) hold. Since Y is unirational, we have a dominant morphism $g\colon V \to Y$, with V a rational variety. Consider the following commutative diagram, with Cartesian squares:

$$
\begin{array}{ccccc}
(X_\eta)_{k(V)} & \longrightarrow & R & \overset{g'}{\longrightarrow} & X \\
{\scriptstyle f''}\big\downarrow & & {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow \\
\mathrm{Spec}\,\big(k(V)\big) & \longrightarrow & V & \overset{g}{\longrightarrow} & Y
\end{array}
$$

Since X_η is unirational, it follows that $(X_\eta)_{k(V)}$ is irreducible and (when considered with the reduced structure) unirational: see Remark 1.5 in Lecture 1. Therefore we have a rational dominant map $\mathbf{P}^n_{k(V)} \dashrightarrow (X_\eta)_{k(V)}$. Since f'' is the generic fiber of f', it follows that after possibly replacing V by an open subset, we may assume that R is irreducible and we have a dominant rational map $V \times \mathbf{P}^n_k \dashrightarrow R$. Since V is rational, $V \times \mathbf{P}^n_k$ is rational. Moreover, g is dominant, hence g' induces a dominant map $R_{\mathrm{red}} \to X$, where R_{red} is the reduced scheme structure on R. The composition

$$
V \times \mathbf{P}^n_k \dashrightarrow R_{\mathrm{red}} \to X
$$

is dominant, hence X is unirational. $\qquad\square$

3. Unirationality of cubic hypersurfaces

We now prove the main result about unirationality of cubic hypersurfaces. This is an old result, going back to Max Noether. We assume that k is a field of characteristic 0.

Theorem 3.1. *If $X \subseteq \mathbf{P}^n$, with $n \geq 3$, is a a smooth hypersurface of degree 3 that contains a line, then X is unirational.*

Corollary 3.2. *If the ground field is algebraically closed, then every smooth cubic hyper-surface $X \subseteq \mathbf{P}^n$, with $n \geq 3$, is unirational.*

Proof. We have seen in Theorem 1.1 in Lecture 2 that over an algebraically closed field, every cubic surface in $\mathbf{P}^3$ contains a line. In X is as in the statement and $\Lambda \subseteq \mathbf{P}^n$ is a 3-dimensional linear subspace that is not contained in X, then $X \cap \Lambda$ is a cubic surface in $\Lambda \simeq \mathbf{P}^3$. Therefore $X \cap \Lambda$ contains a line, hence so does X. The assertion now follows from the theorem. $\qquad\square$

Proof of Theorem 3.1. By assumption, we have a line $\Lambda \subseteq X$ and let $\mathbf{P}(W) \subseteq \mathbf{P}^n$ be a linear subspace of dimension $(n-2)$ such that $\mathbf{P}(W) \cap \Lambda = \emptyset$. Consider the projection map $\varphi\colon \mathbf{P}^n \smallsetminus \Lambda \to \mathbf{P}(W)$. If $\pi\colon \widetilde{\mathbf{P}^n} \to \mathbf{P}^n$ is the blow-up of $\mathbf{P}^n$ along Λ, with exceptional divisor E, then we have seen in §1 that the rational map $\varphi \circ \pi$ extends as a morphism $f\colon \widetilde{\mathbf{P}^n} \to \mathbf{P}(W)$. Moreover, the morphism

$$\iota = (f, \pi)\colon \widetilde{\mathbf{P}^n} \longrightarrow \mathbf{P}(W) \times \mathbf{P}^n$$

exhibits $\widetilde{\mathbf{P}^n}$ as a projective subbundle over $\mathbf{P}(W)$.

Let $\widetilde{X}$ denote the strict transform of X and $Y = E \cap \widetilde{X}$. Therefore $\widetilde{X}$ is the blow-up of the smooth variety X along L, and Y is the exceptional divisor. This implies that both $\widetilde{X}$ and Y are smooth and irreducible. Moreover, since Y is a projective bundle over Λ, we deduce that it is rational.

Consider now the restriction $g\colon \widetilde{X} \to \mathbf{P}(W)$ of f. We show that this is a rational conic bundle. In order to check this, we may base-change to the algebraic closure of the ground field and thus assume that we work over an algebraically closed field. Recall that for every $Q \in \mathbf{P}(W)$, the map ι embeds the fiber $f^{-1}(Q)$ in $\mathbf{P}^n$ as the linear span $\langle \Lambda, Q \rangle$ of Λ and Q. Moreover, $f^{-1}(Q) \cap E$ corresponds to Λ. Note now that we get an induced isomorphism

$$g^{-1}(Q) \smallsetminus Y = (f^{-1}(Q) \cap \widetilde{X}) \smallsetminus E \simeq (\langle \Lambda, Q \rangle \cap X) \smallsetminus \Lambda.$$

For Q general, the intersection $\langle \Lambda, Q \rangle \cap X$ is not contained in Λ. Therefore it is a hypersurface of degree 3 in $\langle \Lambda, Q \rangle \simeq \mathbf{P}^2$. It contains Λ and we will shortly show that for Q general, it contains it with multiplicity 1. On the other hand, since $\widetilde{X}$ is smooth and $\mathrm{char}(k) = 0$, it follows from generic smoothness that for Q general, the fiber $g^{-1}(Q)$ is smooth (hence it is also connected, being a plane curve). We thus conclude that for Q general,

$$g^{-1}(Q) = \overline{g^{-1}(Q) \smallsetminus Y} \hookrightarrow \langle \Lambda, Q \rangle$$

is a smooth conic. Therefore g is a rational conic bundle.

We now show that for $Q \in \mathbf{P}(W)$ general, Λ appears with multiplicity 1 in $\langle \Lambda, Q \rangle \cap X$. We may choose coordinates on $\mathbf{P}^n$ such that

$$\Lambda = (x_2 = \ldots = x_n = 0) \quad \text{and} \quad \mathbf{P}(W) = (x_0 = x_1 = 0).$$

If $f \in k[x_0, \ldots, x_n]$ is an equation of X, then $f \in (x_2, \ldots, x_n)$, hence we can write $f = \sum_{i=2}^n x_i f_i$ for some homogeneous polynomials $f_2, \ldots, f_n$ of degree 2. Let $g_i(x_0, x_1) = f_i(x_0, x_1, 0, \ldots, 0)$. Note that there is i such that $g_i \neq 0$ (otherwise Λ is contained in

the singular locus of X, a contradiction). Given $Q = (\lambda_2, \ldots, \lambda_n) \in \mathbf{P}(W)$, we have coordinates a, b, c on $\langle \Lambda, Q \rangle$ such that $\Lambda = (c = 0)$ and $X \cap \langle \Lambda, Q \rangle$ is defined by

$$f(a, b, c\lambda_2, \ldots, c\lambda_n) = c \cdot \sum_{i=2}^{n} \lambda_i f_i(a, b, c\lambda_1, \ldots, c\lambda_n).$$

If Q is general, then $\sum_{i=2}^{n} \lambda_i g_i \neq 0$, which implies that Λ appears with multiplicity 1 in $\langle \Lambda, Q \rangle \cap X$.

Finally, Y is a multi-section of g (in fact, it is a $2:1$ cover of $\mathbf{P}(W)$). Indeed, it is clear that for Q general, $g^{-1}(Q) \cap E$ is the intersection of the plane conic $g^{-1}(Q)$ with the line Λ, hence it consists of 2 points, by generic smoothness. Since Y is rational, hence unirational, we can thus apply Proposition 2.3 to conclude that $\widetilde{X}$ is unirational, hence X is unirational. $\qquad\square$

Remark 3.3. It follows from the proof of Theorem 3.1 and from Remark 2.5 that under the assumptions of the theorem, we have a dominant, generically finite rational map $\mathbf{P}^{n-1} \dashrightarrow X$, of degree 2.