MATHEMATICS

Structure theorem of finite rings

Asked 10 years, 9 months ago Modified 10 years, 2 months ago Viewed 3k times

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Like structure theorem for finite abelian groups or modules over PID, is there any structure theorem for finite rings? Thanks.

abstract-algebra ring-theory finite-rings

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edited Sep 12, 2013 at 13:12
azimut
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asked Apr 21, 2013 at 16:34
GA316
4,314 27 50

Finite commutative rings or finite rings in general? – user26857 Apr 21, 2013 at 16:38

finite rings in general. thanks – GA316 Apr 21, 2013 at 16:40

4 For finite commutative rings look here. In general, the question is similar to find a structure theorem for finite groups. – user26857 Apr 21, 2013 at 16:41

Related: math.stackexchange.com/questions/1825661 – Watson Dec 11, 2016 at 18:43

1 Answer

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There is no known complete classification of finite rings.

In the following, I'm assuming a ring is unitary, but not necessarily commutative.

The first step is to see that the decomposition of the additive group into groups of pairwise coprime prime power orders is also a ring-theoretic decomposition. In this way, we get a unique decomposition of any finite ring into rings of prime power order. Hence it suffices to classify rings of order p^n with p prime.

For $n = 1$, there is only the ring $\mathbb{Z}/p\mathbb{Z}$.

For $n = 2$, it's a nice exercise to show that up to isomorphism, there are the 4 rings $\mathbb{Z}/p^2\mathbb{Z}$, $\mathbb{F}_{p^2}$, $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}[X]/(X^2)$. See here and here.

The case $n = 3$ is already quite tedious. It can be found in Theorem 14 of the article R. Raghavendran, *Finite Associative Rings*, Composito Mathematica **21** (1969), 195—229. The result is that there are 11 rings of order 8 and 12 rings of order p^3 for p an odd prime. The only non-commutative ring among them is the ring of upper 2×2 triangular matrices over $\mathbb{F}_p$.

If I remember correctly, today the classification is known up to p^5 or p^6. It gets increasingly nasty, of course.

For some further classification-related properties of finite rings, see this discussion.

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edited Apr 13, 2017 at 12:21
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answered Sep 12, 2013 at 14:16
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What do you mean by "primary decomposition" here? – Martin Brandenburg Sep 12, 2013 at 14:27

And why does a decomposition of the additive group yield such a one for the ring? – Martin Brandenburg Sep 12, 2013 at 14:34

1 @MartinBrandenburg: Sorry, that was not correct, of course. What I thought of is: We decompose the additive group uniquely into groups of prime power order (one for each prime). This decomposition induces a decomposition as rings. – azimut Sep 12, 2013 at 14:37