The homotopy-invariance of constructible sheaves of spaces

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Overview

A classical result from sheaf theory says that the functor

$$X \mapsto \mathrm{LC}(X; \mathbf{Set})$$

that sends a topological space X to the category of locally constant sheaves of sets on X is homotopy-invariant. More generally, if P is a poset then the functor

$$S \mapsto \mathrm{Cons}_P(S; \mathbf{Set})$$

that sends a P-stratified topological space S to the category of constructible sheaves of sets on S is invariant under stratified homotopy equivalences. In this note we explain how to use the material of [HA, §A.2] to generalize this result to the setting of sheaves of *spaces*. Specifically, we show that the functor that sends a P-stratified topological space S to the ∞-category of constructible *hypersheaves* of spaces on S is homotopy-invariant (Theorem 2.3).

Since constructible sheaves are functorial in with respect to the sheaf pullback, but the sheaf pullback of a hypercomplete object need not be hypercomplete, a bit of care is needed to formulate Theorem 2.3. Section 1 explains what we mean by a 'constructible hypersheaf' and the relevant functoriality. In Section 2, we state the main homotopy-invariance result (Theorem 2.3) as well as some consequences. Section 3 proves Theorem 2.3. Section 4 is an 'appendix' where include a proof of the easier variant of [HA, Proposition A.2.5] that we make use of in § 3; we include this as it is more straightforward than the proof of [HA, Proposition A.2.5].

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1 Constructible hypersheaves

In this section we explain what we mean by the term 'constructible hypersheaf' (Definition 1.8). We start with locally constant hypersheaves; the main subtlety is that sending a topological space X to the ∞-category of locally constant sheaves on X that are also hypercomplete is not functorial with with respect to hypersheaf pullback. We instead need to work with the slightly larger ∞-category of locally constant objects of the ∞-topos of hypersheaves on X.

1.1 Notation. Let X be a topological space. We write $\mathrm{Sh}(X)$ for the ∞-topos of sheaves of spaces on X. We write $\mathrm{Sh}^{\mathrm{hyp}}(X) \subset \mathrm{Sh}(X)$ for the full subcategory spanned by the *hypersheaves*, and write $(-)^{\mathrm{hyp}} \colon \mathrm{Sh}(X) \to \mathrm{Sh}^{\mathrm{hyp}}(X)$ for the left exact left adjoint to the inclusion.

The reader unfamiliar with hypercomplete objects and hypercompletion should consult [HTT, §§6.5.2–6.5.4] or [Exo, §3.11].

1.2 Recollection. Let $f \colon Y \to X$ be a map of topological spaces. Then the sheaf pushforward $f_* \colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ carries hypersheaves to hypersheaves. However, if $F \in \mathrm{Sh}(X)$ is hypercomplete, then the pullback $f^*(F)$ need not be hypercomplete. We write $f^{*,\mathrm{hyp}}$ for the composite

$$f^{*,\mathrm{hyp}} \colon \mathrm{Sh}^{\mathrm{hyp}}(X) \xrightarrow{f^*} \mathrm{Sh}(Y) \xrightarrow{(-)^{\mathrm{hyp}}} \mathrm{Sh}^{\mathrm{hyp}}(Y) .$$

Note that $f^{*,\mathrm{hyp}}$ is left adjoint to $f_* \colon \mathrm{Sh}^{\mathrm{hyp}}(Y) \to \mathrm{Sh}^{\mathrm{hyp}}(X)$.

1.3 Recollection. If the sheaf pullback $f^* \colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ admits a left adjoint, then f^* carries hypersheaves to hypersheaves [HA, Lemma A.2.6]. In particular, if $U \subset X$ is an open subset, then the restriction functor $(-)|_U \colon \mathrm{Sh}(X) \to \mathrm{Sh}(U)$ carries hypersheaves to hypersheaves.

1.4 Recollection. Let $\boldsymbol{X}$ be an ∞-topos. An object $L \in \boldsymbol{X}$ is *locally constant* if there exists an effective epimorphism $\coprod_{\alpha \in A} U_\alpha \twoheadrightarrow 1_{\boldsymbol{X}}$ such that for each $\alpha \in A$ the product $L \times U_\alpha$ is a constant object of the ∞-topos $\boldsymbol{X}_{/U_\alpha}$. We write $\mathrm{LC}(\boldsymbol{X}) \subset \boldsymbol{X}$ for the full subcategory spanned by the locally constant objects.

If $f^* \colon \boldsymbol{Y} \to \boldsymbol{X}$ is a left exact left adjoint functor between ∞-topoi, then f^* carries locally constant objects of $\boldsymbol{Y}$ to locally constant objects of $\boldsymbol{X}$.

Since the ∞-topoi $\mathrm{Sh}(X)$ and $\mathrm{Sh}^{\mathrm{hyp}}(X)$ are generated under colimits by the open subsets of X, locally constant objects have the following familiar reformulation.

1.5 Observation. Let X be a topological space.

(1.5.1) An object $L \in \mathrm{Sh}(X)$ is locally constant if and only if there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of X such that for each $\alpha \in A$, the sheaf $F|_{U_\alpha}$ is a constant object of $\mathrm{Sh}(U_\alpha)$.

(1.5.2) An object $L \in \mathrm{Sh}^{\mathrm{hyp}}(X)$ is locally constant if and only if there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of X such that for each $\alpha \in A$, the hypersheaf $F|_{U_\alpha}$ is a constant object of $\mathrm{Sh}^{\mathrm{hyp}}(U_\alpha)$.

1.6 Notation. Let X be a topological space. Define

$$\mathrm{LC}(X) := \mathrm{LC}(\mathrm{Sh}(X)) \qquad \text{and} \qquad \mathrm{LC}^{\mathrm{hyp}}(X) := \mathrm{LC}(\mathrm{Sh}^{\mathrm{hyp}}(X)) .$$

1.7 Warning. Notice that we have a containment

$$\mathrm{LC}(X) \cap \mathrm{Sh}^{\mathrm{hyp}}(X) \subset \mathrm{LC}^{\mathrm{hyp}}(X) .$$

However, this inclusion is not generally an equality: if $L \in \mathrm{LC}^{\mathrm{hyp}}(X)$, then L *need not* be locally constant as an object of the larger ∞-topos $\mathrm{Sh}(X)$.

Also note that if $f : Y \to X$ is a map of topological spaces, then the hypersheaf pullback $f^{*,\mathrm{hyp}}$ restricts to a functor

$$f^{*,\mathrm{hyp}} : \mathrm{LC}^{\mathrm{hyp}}(X) \to \mathrm{LC}^{\mathrm{hyp}}(Y) .$$

However, $f^{*,\mathrm{hyp}}$ *need not* carry $\mathrm{LC}(X) \cap \mathrm{Sh}^{\mathrm{hyp}}(X)$ to $\mathrm{LC}(Y) \cap \mathrm{Sh}^{\mathrm{hyp}}(Y)$. For this reason, 'locally constant hypersheaf on X' should mean a locally constant object of $\mathrm{Sh}^{\mathrm{hyp}}(X)$, rather than a locally constant object of $\mathrm{Sh}(X)$ which is a hypersheaf. More generally:

1.8 Definition. Let P be a poset and let $S \to P$ be a P-stratified topological space (see [HA, Definition A.5.1]). Let $F \in \mathrm{Sh}(S)$.

(1.8.1) We say that F is *P-constructible* if for each $p \in P$, the restriction $F|_{S_p}$ of F to the p-th stratum is a locally constant object of $\mathrm{Sh}(S_p)$.

(1.8.2) We say that F is a *P-constructible hypersheaf* if F is hypercomplete and for each $p \in P$, the hypersheaf restriction $(F|_{S_p})^{\mathrm{hyp}}$ is a locally constant object of $\mathrm{Sh}^{\mathrm{hyp}}(S_p)$.

Write $\mathrm{Cons}_P(S) \subset \mathrm{Sh}(S)$ for the full subcategory spanned by the P-constructible sheaves, and

$$\mathrm{Cons}_P^{\mathrm{hyp}}(S) \subset \mathrm{Sh}^{\mathrm{hyp}}(S)$$

for the full subcategory spanned by the P-constructible hypersheaves. Note that if $P = *$, then

$$\mathrm{Cons}_P(S) = \mathrm{LC}(S) \qquad \text{and} \qquad \mathrm{Cons}_P^{\mathrm{hyp}}(S) = \mathrm{LC}^{\mathrm{hyp}}(S) .$$

1.9 Warning. Let S be a P-stratified space. We have a containment

$$\mathrm{Cons}_P(S) \cap \mathrm{Sh}^{\mathrm{hyp}}(S) \subset \mathrm{Cons}_P^{\mathrm{hyp}}(S) ,$$

however, this inclusion need not be an equality. Also note that if F is a P-constructible sheaf, then then $F^{\mathrm{hyp}} \in \mathrm{Cons}_P^{\mathrm{hyp}}(S)$.

1.10 Observation. For any map $f : T \to S$ of P-stratified spaces, the pullback functor

$$f^* : \mathrm{Sh}(S) \to \mathrm{Sh}(T)$$

preserves P-constructible sheaves, and the hypersheaf pullback functor

$$f^{*,\mathrm{hyp}} : \mathrm{Sh}^{\mathrm{hyp}}(S) \to \mathrm{Sh}^{\mathrm{hyp}}(T)$$

preserves P-constructible hypersheaves.

1.11 Warning. The hypersheaf pullback

$$f^{*,\mathrm{hyp}} \colon \mathrm{Sh}^{\mathrm{hyp}}(S) \to \mathrm{Sh}^{\mathrm{hyp}}(T)$$

need not carry $\mathrm{Cons}_P(S) \cap \mathrm{Sh}^{\mathrm{hyp}}(S)$ to $\mathrm{Cons}_P(T) \cap \mathrm{Sh}^{\mathrm{hyp}}(T)$. In particular, it does not appear that there is a way to extend the assignment on objects

$$S \mapsto \mathrm{Cons}_P(S) \cap \mathrm{Sh}^{\mathrm{hyp}}(S)$$

into a functor $\mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$.

2 Statement of the main result & consequences

2.1 Convention. Let P be a poset and $\sigma \colon S \to P$ be a P-stratified topological space. Write $S \times [0,1]$ for the P-stratified topological space with stratification given by the composite

$$S \times [0,1] \xrightarrow{\mathrm{pr}_S} S \xrightarrow{\sigma} P .$$

2.2 Definition. Let P be a poset. A functor $C \colon \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is *homotopy-invariant* if for each P-stratified space S, the functor

$$C(\mathrm{pr}_S) \colon C(S) \to C(S \times [0,1])$$

is an equivalence of ∞-categories.

In light of Observation 1.10, the assignment $S \mapsto \mathrm{Cons}_P^{\mathrm{hyp}}(S)$ defines a presheaf of ∞-categories on P-stratified spaces with functoriality given by hypersheaf pullback. The following is the main result of this note.

2.3 Theorem (homotopy-invariance of constructible hypersheaves)**.** *Let P be a poset. The functor*

$$\mathrm{Cons}_P^{\mathrm{hyp}} \colon \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$$

is homotopy-invariant.

After some preliminaries, we prove Theorem 2.3 at the end of § 3.

Note that in the special case that $P = *$, Theorem 2.3 says that the functor

$$\mathrm{LC}^{\mathrm{hyp}} \colon \mathbf{Top}^{\mathrm{op}} \to \mathbf{Cat}_\infty$$

that sends a topological space X to the ∞-category of locally constant hypersheaves of spaces on X is homotopy-invariant.

Theorem 2.3 implies a the following variant for truncated constructible sheaves.

2.4 Notation. Let P be a poset and $S \to P$ be a P-stratified topological space. We write

$$\mathrm{Cons}_P(S)_{<\infty} \subset \mathrm{Cons}_P(S)$$

for the full subcategory spanned by those P-constructible sheaves that are also n-truncated for some integer $n \geq 0$. Since truncated objects of an ∞-topos are hypercomplete, we see that

$$\mathrm{Cons}_P(S)_{<\infty} \subset \mathrm{Cons}_P^{\mathrm{hyp}}(S) .$$

Since left exact functors preserve truncated objects, given a map $f : T \to S$ of P-stratified spaces, the hypersheaf pullback $f^{*,\mathrm{hyp}}$ restricts to the usual sheaf pullback

$$f^* : \mathrm{Cons}_P(S)_{<\infty} \to \mathrm{Cons}_P(T)_{<\infty} .$$

The assignment $S \mapsto \mathrm{Cons}_P(S)_{<\infty}$ defines a subfunctor of $\mathrm{Cons}_P^{\mathrm{hyp}}$.

2.5 Corollary. *Let P be a poset. The functor $\mathrm{Cons}_P(-)_{<\infty} : \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is homotopy-invariant.*

We end this section with some remarks on the hypercompleteness hypotheses in Theorem 2.3.

2.6 Remark. We do not know if the functor

$$\mathrm{Cons}_P : \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$$

is homotopy-invariant. We expect that for arbitrary (stratified) topological spaces, this might not be true; Theorem 2.3 is probably the best possible homotopy-invariance result for constructible sheaves of spaces. The hypercompleteness hypotheses in Theorem 2.3 should not be seen as restrictive; they are automatic in many situations in which homotopy-invariance of topological spaces is a well-behaved notion. For example, the ∞-topos of sheaves on a CW complex is hypercomplete [MO:168526].

2.7 Remark. Let X be a topological space, and write $\Pi_\infty(X) \in \mathbf{Spc}$ for the underlying ∞-groupoid of X. If X is locally of singular shape in the sense of [HA, Definition A.4.15], then there is a monodoromy equivalence

$$\mathrm{LC}(X) \simeq \mathrm{Fun}(\Pi_\infty(X), \mathbf{Spc})$$

between locally constant sheaves of spaces on X and representations of the underlying ∞-groupoid of X [HA, Theorems A.1.15 & A.4.19]. Thus when restricted to the full subcategory of topological spaces locally of singular shape, the functor $X \mapsto \mathrm{LC}(X)$ is homotopy-invariant. Notice that in this setting, locally constant sheaves are already hypercomplete [HA, Corollary A.1.17]. In particular, [HA, Theorems A.1.15 & A.4.19] do not imply a stronger homotopy-invariance result for locally constant sheaves than what we prove in this note.

With the underlying homotopy type replaced by the exit-path ∞-category, the same remark applies constructible sheaves on paracompact topological spaces locally of singular shape stratified by a poset satisfying the ascending chain condition. See [HA, Theorem A.9.3 & Remark A.9.7]

3 Proof of the homotopy-invariance of constructible hypersheaves

Throughout our proof, we make use of the fact that the sheaf pullback $\mathrm{Sh}(S) \to \mathrm{Sh}(S \times [0,1])$ is very well-behaved:

3.1 Lemma ([HA, Lemma A.2.9])**.** *Let X be a topological space. The pullback functor*

$$\mathrm{pr}_X^* : \mathrm{Sh}(X) \to \mathrm{Sh}(X \times [0,1])$$

is fully faithful and both a left and right adjoint. In particular, pr_X^* preserves hypercomplete objects (Recollection 1.3).

To prove Theorem 2.3 we need to show that the pullback functor

$$\mathrm{pr}_S^* : \mathrm{Cons}_P^{\mathrm{hyp}}(S) \hookrightarrow \mathrm{Cons}_P^{\mathrm{hyp}}(S \times [0,1])$$

is essentially surjective. We begin with the more general problem of understanding the essential image of the pullback functor pr_S^* on *all* sheaves.

3.2 Observation. Let X be a topological space and $G \in \mathrm{Sh}(X)$. The pullback $\mathrm{pr}_X^*(G)$ to $X \times [0,1]$ satisfies the following property: for each $x \in X$, the restriction $\mathrm{pr}_X^*(G)|_{\{x\}\times[0,1]}$ is constant with value the stalk of G at x.

The natural guess for the essential image of $\mathrm{pr}_X^* : \mathrm{Sh}(X) \hookrightarrow \mathrm{Sh}(X \times [0,1])$ is the subcategory of those sheaves such that the restriction to each interval $\{x\}\times[0,1]$ is constant. This guess turns out to be correct as long as we restrict to hypersheaves. For convenience we make the following variant of [HA, Definition A.2.4].

3.3 Definition. Let X be a topological space. A sheaf $F \in \mathrm{Sh}(X\times[0,1])$ is *foliated* if the following conditions are satisfied:

(3.3.1) For each point $x \in X$, the restriction $F|_{\{x\}\times[0,1]}$ is constant.

(3.3.2) The sheaf F is hypercomplete.

3.4 Recollection. A sheaf on $[0,1]$ is locally constant if and only if it is constant [HA, Proposition A.2.1].

3.5 Example. Let P be a poset and let S be a P-stratified space. If F is a P-constructible hypersheaf on $S \times [0,1]$, then by definition for each $p \in P$ the restriction $(F|_{S_p\times[0,1]})^{\mathrm{hyp}}$ is locally constant. Hence for each $s \in S$, the restriction $F|_{\{s\}\times[0,1]}$ is constant. That is, every P-constructible hypersheaf on $S \times [0,1]$ is foliated.

The following is an easier variant of [HA, Proposition A.2.5] with $\mathbf{R}$ replaced by $[0,1]$. Since the proof is significantly more straightforward than the proof of [HA, Proposition A.2.5], we provide an exposition in §4.

3.6 Proposition (classification of foliated sheaves)**.** *Let X be a topological space. The following are equivalent for a sheaf $F \in \mathrm{Sh}(X \times [0,1])$:*

(3.6.1) *The sheaf F is foliated.*

(3.6.2) *The pushforward $\mathrm{pr}_{X,*}(F)$ is hypercomplete and the counit $\mathrm{pr}_X^* \mathrm{pr}_{X,*}(F) \to F$ is an equivalence.*

3.7 Corollary. *Let X be a topological space. A hypersheaf $F \in \mathrm{Sh}^{\mathrm{hyp}}(X \times [0,1])$ is in the essential image of the embedding*

$$\mathrm{pr}_X^* : \mathrm{Sh}^{\mathrm{hyp}}(X) \hookrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X \times [0,1])$$

if and only if F is foliated.

Thus Theorem 2.3 follows from the claim that the pushforward of a P-constructible hypersheaf is a P-constructible hypersheaf. First we prove this for locally constant hypersheaves.

3.8 Lemma. *Let X be a topological space and let G be a hypersheaf on X. Then the pullback $\mathrm{pr}_X^*(G)$ is a locally constant object of $\mathrm{Sh}^{\mathrm{hyp}}(X \times [0,1])$ if and only if G is a locally constant object of $\mathrm{Sh}^{\mathrm{hyp}}(X)$.*

Proof. For the nontrivial direction, suppose we are given a sheaf G on X such that $\mathrm{pr}_X^*(G)$ is a locally constant object of $\mathrm{Sh}^{\mathrm{hyp}}(X \times [0,1])$. By the definition of the product topology, there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of X and an open cover $\{I_\alpha\}_{\alpha \in A}$ of $[0,1]$ such that for each $\alpha \in A$, the restriction $\mathrm{pr}_X^*(G)|_{U_\alpha \times I_\alpha}$ is a constant object of $\mathrm{Sh}^{\mathrm{hyp}}(U_\alpha \times I_\alpha)$. To prove that G is locally constant, we prove that for each $\alpha \in A$, the restriction $G|_{U_\alpha}$ is a constant object of $\mathrm{Sh}^{\mathrm{hyp}}(U_\alpha)$.

To see this, fix $\alpha \in A$ and choose an element $t \in I_\alpha$. Write $i_t : U_\alpha \hookrightarrow U_\alpha \times I_\alpha$ for the inclusion $x \mapsto (x, t)$. Consider the commutative diagram

$$\begin{tikzcd}
U_\alpha \arrow[r, hook, "i_t"] \arrow[dr, equal] & U_\alpha \times [0,1] \arrow[r, hook, "j_{U_\alpha \times I_\alpha}"] \arrow[d, "\mathrm{pr}_{U_\alpha}"] & X \times [0,1] \arrow[d, "\mathrm{pr}_X"] \\
 & U_\alpha \arrow[r, hook, "j_{U_\alpha}"'] & X ,
\end{tikzcd} \tag{3.9}$$

where j_{U_α} and $j_{U_\alpha \times I_\alpha}$ are the inclusions. In light of (3.9), we see that

$$\begin{aligned}
G|_{U_\alpha} &= j_{U_\alpha}^*(G) \\
&\simeq i_t^* j_{U_\alpha \times I_\alpha}^* \mathrm{pr}_X^*(G) \\
&= i_t^*(\mathrm{pr}_X^*(G)|_{U_\alpha \times I_\alpha}) .
\end{aligned}$$

Since $G|_{U_\alpha}$ is the pullback of the constant object $\mathrm{pr}_X^*(G)|_{U_\alpha \times I_\alpha}$ of $\mathrm{Sh}^{\mathrm{hyp}}(U_\alpha \times I_\alpha)$ along i_t, we see that $G|_{U_\alpha}$ is a constant object of $\mathrm{Sh}^{\mathrm{hyp}}(U_\alpha)$, as claimed. □

The following is immediate from Lemma 3.8 and the definition of a constructible hypersheaf.

3.10 Corollary. *Let P be a poset and let $S \to P$ be a P-stratified space. Let G be a hypersheaf on S. Then the pullback $\mathrm{pr}_S^*(G) \in \mathrm{Sh}^{\mathrm{hyp}}(S \times [0,1])$ is a P-constructible hypersheaf if and only if G is a P-constructible hypersheaf.*

We now deduce Theorem 2.3 from the classification of foliated sheaves and Corollary 3.10.

Proof of Theorem 2.3. We need to show that for each $S \in \mathbf{Top}_{/P}$, the fully faithful functor

$$\mathrm{pr}_S^* \colon \mathrm{Cons}_P^{\mathrm{hyp}}(S) \hookrightarrow \mathrm{Cons}_P^{\mathrm{hyp}}(S \times [0,1])$$

is essentially surjective. Let $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(S \times [0,1])$. Since F is foliated (Example 3.5), by Proposition 3.6 it suffices to show that $\mathrm{pr}_{S,*}(F)$ is a P-constructible hypersheaf. This follows from the assumption that F is a P-constructible hypersheaf and Corollary 3.10. □

3.11 Remark. Combining Lemma 3.1 with a slight variant of Corollary 3.10 shows that pr_S^* restricts to an equivalence

$$\mathrm{pr}_S^* \colon \mathrm{Cons}_P(S) \cap \mathrm{Sh}^{\mathrm{hyp}}(S) \xrightarrow{\sim} \mathrm{Cons}_P(S \times [0,1]) \cap \mathrm{Sh}^{\mathrm{hyp}}(S \times [0,1]) .$$

4 Addendum: proof of the classification of foliated sheaves

We now provide a proof of Lurie's classification of foliated sheaves (Proposition 3.6). Our proof uses the following consequence of Lurie's work on the proper basechange theorem in topology [HTT, §7.3].

4.1 Lemma. *Let $f \colon Y \to X$ be a map of topological spaces and let K be a compact Hausdorff space. Then the square of ∞-topoi*

$$\begin{tikzcd} \mathrm{Sh}(Y \times K) \arrow[r, "(f \times \mathrm{id}_K)_*"] \arrow[d, "\mathrm{pr}_{Y,*}"'] & \mathrm{Sh}(X \times K) \arrow[d, "\mathrm{pr}_{X,*}"] \\ \mathrm{Sh}(Y) \arrow[r, "f_*"'] & \mathrm{Sh}(X) \end{tikzcd}$$

satisfies the left basechange condition. That is, the basechange morphism

$$\mathrm{BC} \colon f^* \, \mathrm{pr}_{X,*} \to \mathrm{pr}_{Y,*} (f \times \mathrm{id}_K)^*$$

is an equivalence.

4.2 Remark. The reader unfamiliar with basechange conditions should consult [HTT, §7.3.1], [HA, Definition 4.7.4.13], or [Exo, §7.1].

Proof. Consider the commutative diagram of ∞-topoi

$$\begin{tikzcd} \mathrm{Sh}(Y \times K) \arrow[r, "(f \times \mathrm{id}_K)_*"] \arrow[d, "\mathrm{pr}_{Y,*}"'] & \mathrm{Sh}(X \times K) \arrow[r, "\mathrm{pr}_{K,*}"] \arrow[d, "\mathrm{pr}_{X,*}"] & \mathrm{Sh}(K) \arrow[d] \\ \mathrm{Sh}(Y) \arrow[r, "f_*"'] & \mathrm{Sh}(X) \arrow[r] & \mathrm{Sh}(*) . \end{tikzcd}$$

Since K is locally compact, [HTT, Proposition 7.3.1.11] shows that the right-hand square and outer square are pullback squares of ∞-topoi. Hence the left-hand square is also a pullback square of ∞-topoi. To complete the proof, note that since K is compact Hausdorff, the global sections geometric morphism $\mathrm{Sh}(K) \to \mathrm{Sh}(*)$ is *proper* in the sense of [HTT, Definition 7.3.1.4]; see [HTT, Corollary 7.3.4.11]. $\square$

Now we prove Proposition 3.6. To prove the implication (3.6.1) $\Rightarrow$ (3.6.2), we check that the counit $\mathrm{pr}_X^* \mathrm{pr}_{X,*}(F) \to F$ is an equivalence on stalks by applying Lemma 4.1 in the case where $K = [0,1]$ and f is the inclusion of a point of X.

Proof of Proposition 3.6. The implication (3.6.2) $\Rightarrow$ (3.6.1) is immediate from Observation 3.2 and the fact that pr_X^* preserves hypercomplete objects (Lemma 3.1).

Now we prove that (3.6.1) $\Rightarrow$ (3.6.2). Assume that $F \in \mathrm{Sh}(X \times [0,1])$ is foliated. Since F is hypercomplete, the pushforward $\mathrm{pr}_{X,*}(F)$ is hypercomplete. Since pr_X^* preserves hypercomplete objects, the pullback $\mathrm{pr}_X^* \mathrm{pr}_{X,*}(F)$ is also hypercomplete. Since the ∞-topos $\mathrm{Sh}^{\mathrm{hyp}}(X \times [0,1])$ has enough points [HA, Lemma A.3.9], to show that the counit $c_F \colon \mathrm{pr}_X^* \mathrm{pr}_{X,*}(F) \to F$ is an equivalence, it suffices to show that c_F becomes an equivalence after taking stalks.

Fix $(x,t) \in X \times [0,1]$. To show that the stalk of c_F at (x,t) is an equivalence, consider the commutative diagram of topological spaces

$$
\begin{tikzcd}
\{(x,t)\} \arrow[r, hook, "i_{(x,t)}"] \arrow[dr, "\sim"'] & \{x\} \times [0,1] \arrow[d, "\mathrm{pr}_{\{x\}}"] & X \times [0,1] \arrow[l, hook', "i_x"'] \arrow[d, "\mathrm{pr}_X"] \\
& \{x\} \arrow[r, hook, "x"'] & X ,
\end{tikzcd}
\tag{4.3}
$$

where the horizontal morphisms are the obvious inclusions. From the commutativity of (4.3) we see that

$$
\begin{aligned}
(\mathrm{pr}_X^* \mathrm{pr}_{X,*}(F))_{(x,t)} &\simeq i_{(x,t)}^* i_x^* \mathrm{pr}_X^* \mathrm{pr}_{X,*}(F) \\
&\simeq i_{(x,t)}^* \mathrm{pr}_{\{x\}}^* x^* \mathrm{pr}_{X,*}(F) .
\end{aligned}
$$

Lemma 4.1 applied to the square in (4.3) shows that the basechange morphism

$$
\mathrm{BC} \colon x^* \mathrm{pr}_{X,*} \to \mathrm{pr}_{\{x\},*}\, i_x^*
$$

is an equivalence. Since F is foliated, the sheaf $i_x^*(F)$ on $\{x\} \times [0,1]$ is constant. Since

$$
\mathrm{pr}_{\{x\},*} \colon \mathrm{Sh}(\{x\} \times [0,1]) \to \mathrm{Sh}(\{x\}) \simeq \mathbf{Spc}
$$

restricts to an equivalence on constant sheaves [HA, Proposition A.2.1], the counit

$$
\mathrm{pr}_{\{x\}}^* \mathrm{pr}_{\{x\},*}\, i_x^*(F) \to i_x^*(F)
$$

is an equivalence. Composing the basechange morphism with the counit $\mathrm{pr}_{\{x\}}^* \mathrm{pr}_{\{x\},*} \to \mathrm{id}$ thus provides an equivalence

$$
(\mathrm{pr}_X^* \mathrm{pr}_{X,*}(F))_{(x,t)} \simeq i_{(x,t)}^* \mathrm{pr}_{\{x\}}^* x^* \mathrm{pr}_{X,*}(F) \xrightarrow{\sim} i_{(x,t)}^* \mathrm{pr}_{\{x\}}^* \mathrm{pr}_{\{x\},*}\, i_x^*(F) \xrightarrow{\sim} F_{(x,t)} .
\tag{4.4}
$$

It follows from the definitions that the composite morphism (4.4) is the stalk of the counit $c_F : \mathrm{pr}_X^* \mathrm{pr}_{X,*}(F) \to F$ at (x,t), completing the proof. $\square$

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