# DIAGRAM CHASES: EXAMPLES & EXERCISES

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## 1. EXAMPLES OF DIAGRAM CHASES

1.1. **Notation.** Throughout, we write $R$ for a commutative ring and $\mathbf{Mod}_R$ for the category of $R$-modules.

1.2. **Recollection.**

> ‣ Let $k$ be a field. Then $\mathbf{Mod}_k$ is the category $\mathbf{Vect}_k$ of $k$-vector spaces and $k$-linear transformations.
> ‣ The category $\mathbf{Mod}_{\mathbf{Z}}$ of modules over the ring of integers $\mathbf{Z}$ is simply the category $\mathbf{Ab}$ of abelian groups.

1.3. **Lemma** (5-lemma). *Consider a commutative diagram*

(1.4)
$$
\begin{array}{ccccccccc}
M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & M_3 & \xrightarrow{\alpha_3} & M_4 & \xrightarrow{\alpha_4} & M_5 \\
\downarrow{\gamma_1} & & \downarrow{\gamma_2} & & \downarrow{\gamma_3} & & \downarrow{\gamma_4} & & \downarrow{\gamma_5} \\
M_1' & \xrightarrow{\alpha_1'} & M_2' & \xrightarrow{\alpha_2'} & M_3' & \xrightarrow{\alpha_3'} & M_4' & \xrightarrow{\alpha_4'} & M_5'.
\end{array}
$$

*in* $\mathbf{Mod}_R$. *If the rows are exact, then the following statements hold.*

(1.3.a) *If $\gamma_2$ and $\gamma_4$ are injective and $\gamma_1$ is surjective, then $\gamma_3$ is injective.*
(1.3.b) *If $\gamma_2$ and $\gamma_4$ are surjective and $\gamma_5$ is injective, then $\gamma_3$ is surjective.*

*Proof.* First we prove (1.3.a). So suppose that $\gamma_2$ and $\gamma_4$ are injective and $\gamma_1$ is surjective. Start with $m_3 \in M_3$ with the property that $\gamma_3(m_3) = 0$. The goal is to show that $m_3 = 0$. Then since $\gamma_3(m_3) = 0$, we also know that $\alpha_3'\gamma_3(m_3) = 0$, hence by the commutativity of the diagram (2.5) we have $\gamma_4\alpha_3(m_3) = 0$. Since $\gamma_4$ is injective, this implies that $\alpha_3(m_3) = 0$, hence $m_3 \in \ker \alpha_3$. Since the top row is exact, $\ker \alpha_3 = \operatorname{im} \alpha_2$, so there exists an $m_2 \in M_2$ such that $\alpha_2(m_2) = m_3$. Then since $\gamma_3(m_3) = 0$ we see that $\gamma_3\alpha_2(m_2) = 0$. Then by the commutativity of the diagram (1.4) we see that $\alpha_2'\gamma_2(m_2) = 0$. Therefore $\gamma_2(m_2) \in \ker \alpha_2'$.

Since the bottom row is exact, $\ker(\alpha_2') = \operatorname{im}(\alpha_1')$, so there exists an $m_1' \in M_1'$ so that $\alpha_1'(m_1') = \gamma_2(m_2)$. Since $\gamma_1$ is surjective, there exists a $m_1 \in M_1$ so that $\gamma_1(m_1) = m_1'$. Then by the commutativity of the diagram (1.4), we have

$$
\gamma_2\alpha_1(m_1) = \alpha_1'\gamma_1(m_1) = \alpha_1'(m_1') = \gamma_2(m_2).
$$

Thus by the injectivity of $\gamma_2$ we have $\alpha_1(m_1) = m_2$. This shows that

$$
\alpha_2\alpha_1(m_1) = \alpha_2(m_2) = m_3.
$$

---

But by the exactness of the top row, we have $\alpha_2\alpha_1(m_1) = 0$, so $m_3 = 0$, as desired. Hence $\ker \gamma_3 = 0$, so $\gamma_3$ is injective. A summary of this chase is displayed below.

$$
\begin{array}{ccccccc}
m_1 & \longmapsto & m_2 & \longmapsto & m_3 & \longmapsto & \alpha_3(m_3) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
m_1' & \longmapsto & \gamma_2(m_2) & \longmapsto & 0 & \longmapsto & 0 \, .
\end{array}
$$

Now we prove (1.3.b).[†] So suppose that $\gamma_2$ and $\gamma_4$ are surjective and $\gamma_5$ is injective. Start by considering an element $m_3' \in M_3'$: the goal is to show that $m_3'$ lies in the image of $\gamma_3$. Map $m_3'$ to $\alpha_3'(m_3')$. Then by the surjectivity of $\gamma_4$ there exists an element $m_4 \in M_4$ so that $\gamma_4(m_4) = \alpha_3'(m_3')$. Now consider $\alpha_4(m_4)$. Notice that by the commutativity of the diagram (1.4), we have

$$\gamma_5\alpha_4(m_4) = \alpha_4'\gamma_4(m_4) = \alpha_4'\alpha_3'(m_3').$$

Then by the exactness of the bottom row, $\alpha_4'\alpha_3' = 0$, so $\gamma_5\alpha_4(m_4) = 0$. Therefore, by the injectivity of $\gamma_5$ we have $\alpha_4(m_4) = 0$, so $m_4 \in \ker(\alpha_4)$. By the exactness of the top row, $\ker(\alpha_4) = \mathrm{im}(\alpha_3)$, so there exists $m_3 \in M_3$ so that $\alpha_3(m_3) = m_4$.

Now consider $m_3' - \gamma_3(m_3)$. Notice that by the commutativity of the diagram (1.4),

$$\alpha_3'(m_3' - \gamma_3(m_3)) = \alpha_3'(m_3') - \alpha_3'\gamma_3(m_3) = \alpha_3'(m_3') - \gamma_4\alpha_3(m_3)$$

$$= \alpha_3'(m_3') - \gamma_4(m_4) = \alpha_3'(m_3') - \alpha_3'(m_3') = 0.$$

This shows that $m_3' - \gamma_3(m_3)$ is in the kernel of $\alpha_3'$, which is equal to $\mathrm{im}\,\alpha_2'$ by the exactness of the lower sequence. Hence there exists an $m_2'$ so that $\alpha_2'(m_2') = m_3' - \gamma_3(m_3)$. Then by the surjectivity of $\gamma_2$ there exists an $m_2 \in M_2$ so that $\gamma_2(m_2) = m_2'$. Now consider $m_3 + \alpha_2(m_2)$ and notice that by the commutativity of the diagram (1.4), we have

$$\gamma_3(m_3 + \alpha_2(m_2)) = \gamma_3(m_3) + \gamma_3\alpha_2(m_2) = \gamma_3(m_3) + \alpha_2'\gamma_2(m_3)$$

$$= \gamma_3(m_3) + (m_3' - \gamma_3(m_3)) = m_3'.$$

This shows that $\gamma_3$ is surjective, as desired. $\qquad\square$

**1.5. Remark.** The standard application of the 5-lemma in homological algebra is the following: in the diagram (1.4) the homomorphisms $\gamma_1$, $\gamma_2$, $\gamma_4$, and $\gamma_5$ are isomorphisms, from which we conclude that $\gamma_3$ is an isomorphism.

Now we prepare to prove the Snake lemma (Proposition 1.14) with a few easy lemmas. First we recall a few definitions.

**1.6. Recollection.** Let $R$ be a commutative ring and $f\colon A \to B$ a morphism of $R$-modules. Recall that the *kernel* of $f$ is an $R$-module $\ker(f)$ equipped with a morphism a morphism $i\colon \ker(f) \to A$ satisfying the following *universal property*: the composite $fi$ is the zero morphism and for any morphism $g\colon K \to A$ such that $fg = 0$, there exists a unique morphism $\bar{g}\colon K \to \ker(f)$ making the triangle

$$
\begin{array}{ccc}
 & K & \\
\bar{g}\Big\downarrow & & \searrow^{g} \\
\ker(f) & \xrightarrow{\ i\ } & A
\end{array}
$$

commute.

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[†]Actually this follows immediately by duality since $\mathbf{Mod}_R$ is an abelian category and could have been proven in the setting of abelian categories — maybe we'll talk about these at some point.
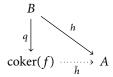
The kernel may explicitly be realized as the submodule of $A$ defined by

$$\ker(f) := \{\, a \in A \mid f(a) = 0 \,\},$$

so that the morphism $i \colon \ker(f) \to A$ in the universal property is simply the inclusion $\ker(f) \hookrightarrow A$.

**1.7. Remark.** This latter description is the description we encounter in a first course in algebra.

**1.8. Recollection.** Recall that the *cokernel* of a morphism of $R$-modules $f \colon A \to B$ is an $R$-module $\operatorname{coker}(f)$ equipped with a morphism a morphism $q \colon B \to \operatorname{coker}(f)$ satisfying the following *universal property*: the composite $qf$ is the zero morphism and for any morphism $h \colon B \to C$ such that $hf = 0$, there exists a unique morphism $\bar{h} \colon \operatorname{coker}(f) \to C$ making the triangle

$$
\begin{array}{ccc}
B & & \\
\ \ \downarrow{\scriptstyle q} & \searrow^{h} & \\
\operatorname{coker}(f) & \dashrightarrow[\bar{h}] & A
\end{array}
$$

commute.

The cokernel may explicitly be realized as the quotient of $B$ by the image of $f$:

$$\operatorname{coker}(f) := B/\operatorname{im}(f),$$

so that the morphism $q \colon B \to \operatorname{coker}(f)$ in the universal property is simply the quotient map $B \twoheadrightarrow \operatorname{coker}(f)$.

**1.9. Remark.** The cokernel is often not introduced in a first course in algebra, which is why we introduce it here. The "explicit" description of the cokernel is what is often introduced in commutative algebra courses, but it makes the duality between the kernel and cokernel less obvious than the universal property description.

**1.10. Lemma.** *Let $R$ be a commutative ring and*

$$(1.11)$$
$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\
A' & \xrightarrow[\ f'\ ]{} & B'
\end{array}
$$

*a commutative square in* $\mathbf{Mod}_R$.

(1.10.a) *The restriction $f|_{\ker(\alpha)} \colon \ker(\alpha) \to \ker(\beta)$ is the unique $R$-module homomorphism making the square*

$$
\begin{array}{ccc}
\ker(\alpha) & \xrightarrow{f|_{\ker(\alpha)}} & \ker(\beta) \\
{\scriptstyle i_\alpha}\uparrow & & \uparrow{\scriptstyle i_\beta} \\
A & \xrightarrow[\ f\ ]{} & B
\end{array}
$$

*commute, where $i_\alpha$ and $i_\beta$ denote the inclusions. Moreover, if $f$ is an injection, then so is $f|_{\ker(\alpha)}$.*

(1.10.b) *Dually, there exists a unique R-module homomorphism $\bar{f}\colon \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta)$ making the square*

$$
\begin{array}{ccc}
A' & \xrightarrow{\ f'\ } & B' \\
{\scriptstyle q_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle q_\beta} \\
\operatorname{coker}(\alpha) & \dashrightarrow[\bar{f}'] & \operatorname{coker}(\beta)
\end{array}
$$

*commute, where $q_\alpha$ and $q_\beta$ denote the quotient maps. Moreover, if $f$ is a surjection, then so is $\bar{f}'$.*

*Proof.* Since (1.10.b) is dual to (1.10.a), it suffices to prove (1.10.a). This follows easily from the universal property of the kernel. Consider the composite $\beta f i_\alpha$. By the commutativity of the square (1.11) and the fact that $i_\alpha$ is the inclusion of the kernel of $\alpha$ we have that

$$\beta f i_\alpha = f'\alpha i_\alpha = 0 \,.$$

Thus by the universal property of $\ker(\beta)$, there exists a unique $R$-module homomorphism $\ker(\alpha) \to \ker(\beta)$ making the square

$$
\begin{array}{ccc}
\ker(\alpha) & \dashrightarrow & \ker(\beta) \\
{\scriptstyle i_\alpha}\big\uparrow & & \big\downarrow{\scriptstyle i_\beta} \\
A & \xrightarrow{\ f\ } & B
\end{array}
$$

commute. The fact that the unique morphism $\ker(\alpha) \to \ker(\beta)$ is the restriction $f|_{\ker(\alpha)}$ follows immediately from the definition of the restriction.

The last thing to check is that if $f$ is an injection, then so is $f|_{\ker(\alpha)}$. To see this, notice that if $f$ is an injection, then since $i_\alpha$ is an inclusion, $f i_\alpha$ is also an injection, so the composite $i_\beta \circ f|_{\ker(\alpha)}$ is an injection, which implies that $f|_{\ker(\alpha)}$ is an injection. $\qquad\square$

**1.12. Lemma.** *Let R be a commutative ring and consider the following diagram of R-modules*

(1.13)
$$
\begin{array}{ccccc}
A & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C \\
{\scriptstyle \alpha}\big\downarrow & & {\scriptstyle \beta}\big\downarrow & & \big\downarrow{\scriptstyle \gamma} \\
A' & \xrightarrow[\ f'\ ]{} & B' & \xrightarrow[\ g'\ ]{} & C' \,.
\end{array}
$$

(1.12.a) *If the top row of (1.13) is exact and $f'$ is an injection, then the sequence*

$$\ker(\alpha) \xrightarrow{\ f|_{\ker(\alpha)}\ } \ker(\beta) \xrightarrow{\ g|_{\ker(\beta)}\ } \ker(\gamma)$$

*is exact.*

(1.12.b) *Dually, if the bottom row of (1.13) is exact and $g$ is a surjection, then the sequence*

$$\operatorname{coker}(\alpha) \xrightarrow{\ \bar{f}'\ } \operatorname{coker}(\beta) \xrightarrow{\ \bar{g}'\ } \operatorname{coker}(\gamma)$$

*is exact.*

*Proof.* As usual, since (1.12.b) is dual to (1.12.a), it suffices to prove (1.12.a). The only place where we need to show exactness is at $\ker(\beta)$. First let us show that $\operatorname{im}(f|_{\ker(\alpha)}) \subset$

$\ker(g|_{\ker(\beta)})$. So suppose that $b \in \ker(\beta)$ and $b = f(a)$ for some $a \in \ker(\alpha)$. Then since the top row of (1.13) we see that

$$g|_{\ker(\beta)}(b) = g(b) = gf(a) = 0 \,.$$

Hence $b \in \ker(g|_{\ker(\beta)})$, so we see that $\mathrm{im}(f|_{\ker(\alpha)}) \subset \ker(g|_{\ker(\beta)})$.

Now let us show that $\ker(g|_{\ker(\beta)}) \subset \mathrm{im}(f|_{\ker(\alpha)})$. Suppose that $b \in \ker(g|_{\ker(\beta)})$. Then by the exactness of the top row of (1.13) we know that $b = f(a)$ for some $a \in A$. Hence it suffices to show that $a \in \ker(\alpha)$. Since $b \in \ker(\beta)$ we know that $\beta(b) = 0$. But by the commutativity of (1.13) we see that

$$\beta(b) = \beta f(a) = f'\alpha(a) \,.$$

Since $f'$ is injective, we see that $\alpha(a) = 0$, as desired. $\qquad\qquad\qquad\qquad\qquad\square$

1.14. **Proposition** (Snake Lemma)**.** *Let R be a commutative ring and consider the following diagram of R-modules, where the rows are exact*

(1.15)
$$
\begin{array}{ccccccc}
A & \xrightarrow{\;f\;} & B & \xrightarrow{\;g\;} & C & \longrightarrow & 0 \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & A' & \rightarrowtail_{f'} & B' & \xrightarrow{\;g'\;} & C'
\end{array}
$$

*There exists a **connecting homomorphism** $\partial \colon \ker(\gamma) \to \mathrm{coker}(\alpha)$, and the sequence*

$$\ker(\alpha) \xrightarrow{f|_{\ker(\alpha)}} \ker(\beta) \xrightarrow{g|_{\ker(\beta)}} \ker(\gamma) \xrightarrow{\partial} \mathrm{coker}(\alpha) \xrightarrow{\bar{f}'} \mathrm{coker}(\beta) \xrightarrow{\bar{g}'} \mathrm{coker}(\gamma)$$

*is exact. Moreover, if $f$ is injective, then so is $f|_{\ker(\alpha)}$. Dually, if $g'$ is surjective, then so is $\bar{g}'$.*

*Proof.* There are three things to do: construct the connecting homomorphism, show that the connecting homomorphism is well-defined, and, in light of Lemma 1.12, show that the resulting sequence is exact at $\ker(\gamma)$ and $\mathrm{coker}(\alpha)$. First we construct the connecting homomorphism. We start with an element $c \in \ker(\gamma)$. Since $g$ is surjective, we can choose an element $b \in B$ such that $g(b) = c$. By the commutativity of the right-hand square of (1.15) and the fact that $c \in \ker(\gamma)$,

$$g'\beta(b) = \gamma g(b) = \gamma(c) = 0 \,,$$

so by the exactness of the lower sequence of (1.15) and the injectivity of $f'$, there exists a unique $a' \in A'$ with the property that $f'(a') = \beta(b)$. We define the connecting homomorphism $\partial \colon \ker(\gamma) \to \mathrm{coker}(\alpha)$ by the assignment

$$c \mapsto [a'] \,,$$

where $[a']$ denotes the class of $a'$ in $\mathrm{coker}(\alpha)$.

Now let us show that $\partial$ is well-defined as a set-theoretic map. Suppose that $\tilde{b} \in B$ is another element such that $g(\tilde{b}) = c$, and let $\tilde{a}' \in A'$ be the unique element with the property that $f'(\tilde{a}') = \beta(\tilde{b})$. Then since $g$ is an $R$-module homomorphism,

$$g(b - \tilde{b}) = g(b) - g(\tilde{b}) = c - c = 0 \,,$$

so by the exactness of the top row of (1.15) there exists an element $a \in A$ with the property that $f(a) = b - \tilde{b}$. By the commutativity of the left-hand square of (1.15), we see that

$$f'\alpha(a) = \beta f(a) = \beta(b - \tilde{b}) = \beta(b) - \beta(\tilde{b}) \,.$$

Since $f'(a' - \tilde{a}') = \beta(b) - \beta(\tilde{b})$ and $f'$ is injective, we see that $\alpha(a) = a' - \tilde{a}'$, that is $a' = \tilde{a}' + \alpha(a)$. Thus in $\operatorname{coker}(\alpha)$ we see that

$$[a'] = [\tilde{a}' + \alpha(a)] = [\tilde{a}'],$$

so the connecting homomorphism is actually well-defined.

Now we need to prove that $\partial$ is an $R$-module homomorphism, but this is easy since we now know that the value of $\partial(c)$ is independent of any choices we make. Suppose that we are given $r \in R$ and $c \in \ker(\gamma)$. Choose $b \in B$ so that $\beta(b) = c$. Then since $\beta$ is an $R$-module homomorphism, we have that $\beta(rb) = rc$. Let $a' \in A'$ be the unique element so that $f'(a') = \beta(b')$. Then since $f'$ and $\beta$ are $R$-module homorphisms

$$f'(ra') = rf'(a') = r\beta(b) = \beta(rb),$$

hence

$$\partial(rc) = [ra'] = r[a'] = r\partial(c).$$

Similarly, suppose that $c_1, c_2 \in \ker(\gamma)$, and let $b_1, b_2 \in B'$ be elements such that $g(b_1) = c_1$ and $g(b_2) = c_2$, and let $a_1', a_2' \in A'$ be the unique elements such that $f'(a_1') = \beta(b_1)$ and $f'(a_2') = \beta(b_2)$. Then since $\beta$ and $f'$ are $R$-module homomorphisms

$$f'(a_1' + a_2') = f'(a_1') + f'(a_2') = \beta(b_1) + \beta(b_2) = \beta(b_1 + b_2),$$

so

$$\partial(c_1 + c_2) = [a_1' + a_2'] = [a_1'] + [a_2'],$$

which completes the proof that $\partial$ is actually an $R$-module homomorphism.

Finally, let us show that the sequence

$$(1.16) \quad \ker(\alpha) \xrightarrow{f|_{\ker(\alpha)}} \ker(\beta) \xrightarrow{g|_{\ker(\beta)}} \ker(\gamma) \xrightarrow{\partial} \operatorname{coker}(\alpha) \xrightarrow{\bar{f}'} \operatorname{coker}(\beta) \xrightarrow{\bar{g}'} \operatorname{coker}(\gamma)$$

is exact. By Lemma 1.12 the sequence (1.16) is exact at $\ker(\beta)$ and $\operatorname{coker}(\beta)$, so we just need to show that (1.16) is exact at $\ker(\gamma)$ and $\operatorname{coker}(\alpha)$. First let us show that (1.16) is exact at $\ker(\gamma)$. Suppose that $c \in \operatorname{im}(g|_{\ker(\beta)})$ so that $c = g(b)$ for some $b \in \ker(\beta)$. Then $\beta(b) = 0$, so by the definition of $\partial$ we have that $\partial(c) = 0$, hence $\ker(\partial) \subset \operatorname{im}(g|_{\ker(\beta)})$.

On the other hand suppose that $c \in \ker(\partial)$. Let $b \in B$ be such that $g(b) = c$ and $a' \in A'$ be the unique element so that $f'(a') = \beta(b)$. Then by the definition of $\partial$ and the fact that $\partial(c) = 0$, we have that $[a'] = 0$, that is $a' \in \operatorname{im}(\alpha)$, so there exists some element $a \in A$ so that $\alpha(a) = a'$. Thus

$$f'(a') = f'\alpha(a) = \beta f(a).$$

But $\beta(b) = f'(a')$, so we have that $\beta(b) = \beta f(a)$. Thus $b - f(a) \in \ker(\beta)$. Since $c = g(b)$ we have that

$$c = g(b - f(a) + f(a)) = g(b - f(a)) + gf(a) = g(b - f(a)).$$

Since $b - f(a) \in \ker(\beta)$, this shows that $c \in \operatorname{im}(g|_{\ker(\beta)})$, as desired. Hence the sequence (1.16) is exact at $\ker(\gamma)$.

Now let us show that the sequence (1.16) is exact at $\operatorname{coker}(\gamma)$. Suppose that $[a'] \in \operatorname{im}(\partial)$. Then $[a'] = \partial(c)$ for some $c \in \ker(\gamma)$. We know that

$$\bar{f}'[a'] = [f'(a')].$$

By the definition of $\partial$, we have that $f'(a') = \beta(b)$, where $g(b) = c$. Hence, in $\operatorname{coker}(\beta)$

$$\bar{f}'[a'] = [f'(a')] = [\beta(b)] = 0,$$

so $[a'] \in \ker(\bar{f}')$, as desired.

On the other hand, suppose that $a' \in A'$ and $\bar{f}'[a'] = 0$. This means that $f'(a') \in \text{im}(\beta)$, so $f'(a') = \beta(b)$ for some $b \in B$. Then

$$\gamma g(b) = g'\beta(b) = g'f'(a) = 0\,,$$

by the commutativity of (1.15) and the exactness of the bottom row. Thus $g(b) \in \ker(\gamma)$, and by the definition of $\partial$ we have that $\partial g(b) = [a']$, so $[a'] \in \text{im}(\partial)$, as desired.

The last statement, that if $f$ is injective, then so is $f|_{\ker(\alpha)}$, and if $g'$ is surjective, then so is $\bar{g}'$ is just an application of Lemma 1.10. $\qquad\square$

## 2. Exercises in diagram chasing

The following lemmas are good exercises in diagram chasing and are both standard results from homological algebra. As an exercise, try to prove these results.

2.1. **Exercise.** Prove (1.10.b) directly (i.e., not appealing to duality).

2.2. **Lemma** (Splitting Lemma)**.** *Let $R$ be a commutative ring and*

$$0 \longrightarrow X \overset{i}{\rightarrowtail} Z \overset{p}{\twoheadrightarrow} Y \longrightarrow 0$$

*be a short exact sequence of $R$-modules. The following are equivalent.*

(2.2.a) *The image of $i$ is a direct summand of $Z$.*
(2.2.b) *There exists a retraction $r$ of $i$.*
(2.2.c) *There exists a section $s$ of $p$.*

*In this case, we say that the sequence **splits**, and $Z \cong X \oplus Y$ in such a way that $i$ is the inclusion and $p$ is the projection.*

2.3. **Lemma.** *Let $R$ be a commutative ring. Then:*

(2.3.a) *Given an exact sequence of $R$-modules*

$$0 \longrightarrow X \overset{i}{\rightarrowtail} Z \overset{p}{\longrightarrow} Y\,,$$

*we have $\ker(p) \cong X$.*
(2.3.b) *Given an exact sequence of $R$-modules*

$$X \overset{i}{\longrightarrow} Z \overset{p}{\twoheadrightarrow} Y \longrightarrow 0\,,$$
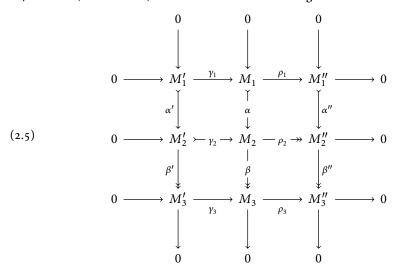
*we have $\text{coker}(i) \cong Y$.*

*(Moreover, these isomorphisms are natural in morphisms of exact sequences.)*

The Snake lemma might come in handy to prove the next lemma.

2.4. **Lemma** ($3 \times 3$ lemma). *Consider a commutative diagram*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1' & \xrightarrow{\gamma_1} & M_1 & \xrightarrow{\rho_1} & M_1'' & \longrightarrow & 0 \\
 & & \alpha' \downarrow & & \alpha \downarrow & & \alpha'' \downarrow & & \\
0 & \longrightarrow & M_2' & \rightarrowtail_{\gamma_2} & M_2 & \twoheadrightarrow_{\rho_2} & M_2'' & \longrightarrow & 0 \\
 & & \beta' \downarrow & & \beta \downarrow & & \beta'' \downarrow & & \\
0 & \longrightarrow & M_3' & \xrightarrow{\gamma_3} & M_3 & \xrightarrow{\rho_3} & M_3'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

(2.5)

*in* $\mathbf{Mod}_R$, *where all of the columns are exact and the middle row is exact. Then the first row of* (2.5) *is exact if and only if the third is.*

*Proof.* First, assume that the first row of (2.5) is exact. Then by the Snake Lemma, the first two rows induce an exact sequence

$$
0 \longrightarrow \ker \alpha' \xrightarrow{\phi} \ker \alpha \xrightarrow{\psi} \ker \alpha'' \xrightarrow{\partial} \operatorname{coker} \alpha' \xrightarrow{\phi'} \operatorname{coker} \alpha \xrightarrow{\psi'} \operatorname{coker} \alpha'' \longrightarrow 0 \, .
$$

Since the columns of (2.5) are exact, $\alpha$, $\alpha'$, and $\alpha''$ are injective, so the kernels of each of these morphisms are zero. Again since the columns of (2.5) are exact, by Lemma 2.3 we have $M_3' = \operatorname{coker} \alpha'$, $M_3 = \operatorname{coker} \alpha$, and $M_3'' = \operatorname{coker} \alpha''$. Hence the sequence

$$
0 \longrightarrow M_3' \xrightarrow{\phi'} M_3 \xrightarrow{\psi'} M_3'' \longrightarrow 0
$$

is exact. By the definitions of $\phi'$ and $\psi'$ in the Snake Lemma and the commutativity of the diagram (2.5), we see that $\phi' \beta' = \gamma_3 \beta'$ and $\psi' \beta = \rho_3 \beta$. Since $\beta$ and $\beta'$ are epimorphisms in $\mathbf{Mod}_R$, this implies that $\phi' = \gamma_3$ and $\psi' = \rho_3$ so the last row of (2.5) is exact.

Now, assume that the last row of (2.5) is exact. The fact that the first row is exact follows by duality — we provide a seperate proof anyhow. Then by the Snake Lemma, the last two rows induce an exact sequence

$$
0 \longrightarrow \ker \beta' \xrightarrow{\phi} \ker \beta \xrightarrow{\psi} \ker \beta'' \xrightarrow{\partial} \operatorname{coker} \beta' \xrightarrow{\phi'} \operatorname{coker} \beta \xrightarrow{\psi'} \operatorname{coker} \beta'' \longrightarrow 0 \, .
$$

Since the columns of (2.5) are exact, $\beta$, $\beta'$, and $\beta''$ are surjective, so the cokernels of each of these morphisms are zero. Again since the columns of (2.5) are exact, by Lemma 2.3 we have $M_1' = \ker \beta'$, $M_1 = \ker \beta$, and $M_1'' = \ker \beta''$. Hence the sequence

$$
0 \longrightarrow M_1' \xrightarrow{\phi} M_1 \xrightarrow{\psi} M_1'' \longrightarrow 0
$$

is exact. By the definitions of $\phi$ and $\psi$ in the Snake Lemma and the commutativity of the diagram (2.5), we see that $\alpha\phi = \alpha\gamma_1$ and $\alpha''\psi = \alpha''\rho_1$. Since $\alpha$ and $\alpha''$ are monomorphisms in $\mathbf{Mod}_R$, this implies that $\phi = \gamma_1$ and $\psi = \rho_1$ so the first row of (2.5) is exact. $\square$