with cohomological descent still induces a quasi-isomorphism, even though non-filtered. Gluing it to filtered cohomology $R_\sigma(\,\cdot\,,\mathbb{Q})$ via the non-filtered version, one gets $\mathbb{Z}$-Hodge complexes.

9 l-adic Cohomology

Our next aim is the definition of the étale or more precisely of the l-adic realization of a simplicial variety.

We fix an algebraic closure $\bar{k}$ of our ground field k. Let $G_k = Gal(\bar{k}/k)$ be the absolute Galois group. Very soon, we will have to assume k finitely generated over $\mathbb{Q}$. In addition, we choose a fixed prime l.

9.1 Galois Modules

Definition 9.1.1 *Let $\underline{\mathbb{Z}/l^n[G_k]}$ be the category of $\mathbb{Z}/l^n\mathbb{Z}$-modules with a continuous $\mathbb{Z}/l^n$-linear G_k-operation i.e. it factorizes over the Galois group of some finite extension of k.*

Let $\underline{\mathbb{Z}_l[G_k]}$ be the category of $\mathbb{Z}_l$-modules with a continuous operation of G_k. More precisely, we mean the category of projective systems $(A_n)_{n\in\mathbb{N}}$ where A_n is an object in $\underline{\mathbb{Z}/l^n[G_k]}$. Morphisms are systems f_n of $\mathbb{Z}/l^n[G_k]$-morphisms which are compatible with the structural morphisms of the projective systems.

Let $\underline{\mathbb{Q}_l[G_k]} = \underline{\mathbb{Z}_l[G_k]} \otimes \mathbb{Q}$. This category has the same objects as $\underline{\mathbb{Z}_l[G_k]}$, but on the groups of morphisms the tensor product with $\mathbb{Q}$ is taken. It is called the category of Galois modules.

Remark: The category $\underline{\mathbb{Z}_l[G_k]}$ is abelian, where the kernels and cokernels are computed componentwise. It has enough injectives ([38] 1.1).i The operation $\cdot \otimes \mathbb{Q}$ turns an abelian category into another abelian category, and the functor $\underline{\mathbb{Z}_l[G_k]} \longrightarrow \underline{\mathbb{Q}_l[G_k]}$ is exact and maps injectives to injectives.

A compatible system of elements of the A_n can be seen as an "element" of the object $(A_n)_{n\in\mathbb{N}}$ in $\underline{\mathbb{Z}_l[G_k]}$. An object A of $\underline{\mathbb{Q}_l[G_k]}$ is zero if and only if there is an $m \in \mathbb{N}$ such that $m \cdot A_n = 0$ for all components A_n of A. Important: this condition is stronger than being torsion object in $\underline{\mathbb{Z}_l[G_k]}$. For objects of $\underline{\mathbb{Q}_l[G_k]}$ one should not speak of "elements".

Definition 9.1.2 *An object A in $\underline{\mathbb{Z}_l[G_k]}$ is called constructible if all A_n are finite and all transition morphisms are of the form $A_{n+1} \longrightarrow A_{n+1}/l^n = A_n$. The projective limit (formed in the category of abelian groups) is automatically finitely generated as $\mathbb{Z}_l$-module.*

A Galois module is called constructible if it is isomorphic in $\underline{\mathbb{Q}_l[G_k]}$ to an object which is constructible as object of $\underline{\mathbb{Z}_l[G_k]}$.

We have recapitulated the definition of (constructible) l-adic sheaves over Spec(k) (cf. [51] Ch. V §1 p. 163 ff or [27] I 12.3). The category of constructible Galois modules is an abelian subcategory of $\underline{\mathbb{Q}_l[G_k]}$. Non-constructible Galois modules are going to used in our constructions. However, they should only be seen as background, as our real interest is in the constructible ones.

Proposition 9.1.3 *The category of constructible Galois modules is abelian. It is equivalent to the category of finite dimensional $\mathbb{Q}_l$-vector spaces with a continuous G_k-operation. (Vector spaces are equipped with the l-adic topology, G_k with the profinite one.)*

Proof: Consider a morphism of constructible objects in $\underline{\mathbb{Z}_l[G_k]}$. Its cokernel is also constructible. The kernel is not but by the Artin-Rees lemma it is Artin-Rees equivalent to a constructible subobject ([27] I 12.2). But Artin-Rees null systems are killed by a power of l, so the kernel of our morphism is a constructible Galois module.

There is an obvious functor from the category of constructible Galois modules into the category of continuous finite dimensional G_k-modules: first we assign to an object in $\underline{\mathbb{Z}_l[G_k]}$ its inverse limit and then take the tensor product with $\mathbb{Q}$. The functor is also well defined on morphisms. It is essentially surjective because there is a G_k-invariant $\mathbb{Z}_l$-lattice in a continuous finite dimensional $\mathbb{Q}_l[G_k]$-vector space([61] Ch I 1.1 Rem. 1)). By choice of such a lattice we get a constructible object in $\underline{\mathbb{Z}_l[G_k]}$. Any morphism respects (after multiplication with the denominator of the matrix entries) these lattices, hence it is image of a morphism in $\underline{\mathbb{Q}_l[G_k]}$.

We still have to check that the functor is faithful. A constructible object in $\underline{\mathbb{Z}_l[G_k]}$ whose image in the category of $\mathbb{Q}_l$-vector spaces is zero has finite inverse limit. Multiplication with the order of the limit annihilates it in $\underline{\mathbb{Q}_l[G_k]}$. □

In the category of constructible Galois modules, it is allowed to do computations with "elements".

Definition 9.1.4 *Let k be finitely generated over $\mathbb{Q}$.*

a) A constructible Galois module M is pure of weight n if

1. *there is a smooth model U of k of finite type over* $\operatorname{Spec}\mathbb{Z}[\frac{1}{l}]$*, such that M also extends to a constructible sheaf $\tilde{M}$ on U;*
2. *and $\tilde{M}$ is pointwise pure of weight n, i.e. on the stalk of $\tilde{M}$ in every closed point x of U, the Frobenius of the residue field operates with eigenvalues of absolute value $N(x)^{\frac{m}{2}}$ (the eigenvalues are algebraic and have this absolute value with respect to all embeddings of $\bar{\mathbb{Q}}_l$ to $\mathbb{C}$).*

b) A filtration on a constructible Galois module N is called a filtration by weights if the n-th graded part is pure of weight n.

c) A constructible Galois module which admits a filtration by weights is called mixed.

Remark: In the number field case to be pure of weight n means: for all places p of k outside of a finite exceptional set, we have:

1. p is unramified, and
2. the Frobenius of p operates with eigenvalues of absolute value $N(p)^{\frac{m}{2}}$.

The properties of the category of mixed Galois modules are studied in [14] by Deligne, the definition is also his (loc. cit. 1.2.2 and 6.1.1. b)). In [39] 6.8. they are collected. Usually, the term weight filtration is used instead of filtration by weights. However, we want to stick to our definition of weight filtration as an ascending filtration on an object of an abelian category. It is the fundamental result of Deligne [14] that our weight filtration on l-adic cohomology is indeed a filtration by weights. But this will come later.

We want to assemble the most important properties:

Lemma 9.1.5 (Jannsen) *Let M and N be constructible Galois modules with the filtrations by weights.*

1. *The filtration by weights on M is uniquely determined by the mixed Galois module.*
2. *A morphism $f : M \longrightarrow N$ of mixed Galois modules is strictly compatible with the filtrations by weights.*

3. *The category of Galois modules equipped with their filtrations by weights is abelian.*

4. *If $0 \longrightarrow M \longrightarrow N \longrightarrow M' \longrightarrow 0$ is an exact sequence of filtered Galois modules (i.e.the arrows are strict) in which the filtrations on M and M' agree with the filtrations by weights then the same is true for N.*

Proof: The first three properties are [39] 6.8.1 a) and b). Because of strictness we can reduce in 4.) to the graded parts of the weight filtration. It is an elementary computation that the eigenvalues of the Frobenii have the right absolute values. □

Definition 9.1.6 *The $\mathbb{1}$-object in $\underline{\mathbb{Z}_l[G_k]}$ is the system $(\mathbb{Z}/l^n)_{n\in\mathbb{N}}$ equipped with the trivial operation of the Galois group. The Tate-object $\mathbb{1}(1)$ is the projective system of the roots of unity μ_{l^n} in $\bar{k}$ with the natural operation of the Galois group.*

By the way, $\mathbb{1}$ and $\mathbb{1}(1)$ are isomorphic as $\mathbb{Z}_l$-modules. The category of constructible Galois modules is a rigid Tannakian category with a faithful fibre functor into the category of $\mathbb{Q}_l$-vector spaces.

In the same way as in the singular realization, we consider the derived category of $\underline{\mathbb{Q}_l[G_k]}_{\mathrm{filt}}$ and in it the subcategory of complexes with strict differentials. The category is derivably strict again, hence we get the implications of 2.1.3.

Now we have to restrict to the case of fields k which are finitely generated over $\mathbb{Q}$.

Definition 9.1.7 *Let k be finitely generated over $\mathbb{Q}$. Let $D_l(G_k)$ be the full subcategory in $D^+_{\mathrm{str}}(\underline{\mathbb{Q}_l[G_k]}_{\mathrm{filt}})$ of those complexes whose naive cohomology objects are in the category of constructible Galois modules with a filtration by weights.*

Proposition 9.1.8 *$D_l(G_k)$ is an R-category with weights.*

Proof: By 2.1.4 we get a triangulated category with t-structure. The twist is tensor product with $\mathbb{Q}_l(n)$. (As the Tate-object is flat, this operation is well behaved under the change to the derived category.) The other properties are clear. □

Remark: In [26] Ekedahl constructs "derived" categories of l-adic sheaves in a more general situation. Over a point Spec(k), he obtains a category $D_c(\mathbb{Z}_l[G_k])$. Jannsen has also given a definition [40]. At least in the case of a point, he gets the same notion. Our ad-hoc construction gives in principle the same thing again, except that we are working with $\mathbb{Q}_l$-sheaves and take care of the weight filtration.

9.2 First Step

From now on, let k be finitely generated over $\mathbb{Q}$. We start to construct the realization functor, as before at first with values in a category of complexes.

We consider étale cohomology of constant sheaves. Let $\mathcal{F}$ be an étale sheaf of abelian groups on Spec(k). $s_X^*\mathcal{F}$ is denoted $\mathcal{F}_X$ for short (s_X the structural morphism of the variety X). By adjunction a morphism of varieties $f : X \longrightarrow X'$ induces a morphism of étale sheaves $\mathcal{F}_{X'} \longrightarrow f_*\mathcal{F}_X$. Let $Gd_X(\,\cdot\,)$ be the functor which assigns to a sheaf of abelian groups on the étale site of X its Godement resolution.

Now we continue as in the definition of singular cohomology.

Definition 9.2.1 *Let $\tilde{R}_{et}(\,\cdot\,,\mathcal{F}) : \widetilde{\mathcal{V}}_0 \longrightarrow C^+(\underline{\mathbb{Z}[G_k]}_{\text{filt}})$ (where $\mathbb{Z}[G_k]$ stands for the category of abelian groups with continuous G_k-operation) be the following functor: to an object (U, X) in $\widetilde{\mathcal{V}}_0$, the complex*

$$\tilde{R}_{et}((U,X),\mathcal{F}) = s_{X*}Gd_X j_* Gd_U(\mathcal{F}_U)$$

is assigned. It is equipped with the décalage of the filtration induced by the canonical filtration $\tau_{\leq *}$ on the complex $j_*Gd_U(\mathcal{F}_U)$.

As in 6.1.2 we check that this really defines a functor. s_{X*} has values in the category of étale abelian sheaves on Spec(k). It is equivalent to the category of abelian groups with an operation of G_k.

The Godement resolution is acyclic for higher direct images. By the Leray spectral sequence, we see that the complex $\tilde{R}_{et}((U,X),\mathcal{F})$ computes étale cohomology of the sheaf $\mathcal{F}_U$ over the algebraic closure.

Lemma 9.2.2 $\tilde{R}_{et}(\,\cdot\,,\mathcal{F})$ *is functorial in* $\mathcal{F}$.

Proof: Clear by construction. □

The important case for us is the case of the projective system of constant sheaves $\mathbb{Z}/l^n\mathbb{Z}$. Then the components of $\tilde{R}_{et}((U,X),\mathbb{Z}/l^n\mathbb{Z})$ carry a structure of $\mathbb{Z}/l^n\mathbb{Z}$-modules. The complexes form a projective system for $n \in \mathbb{N}$.

Definition 9.2.3 *Let the functor $\tilde{R}_l : \tilde{\mathcal{V}}_0 \longrightarrow C^+(\underline{\mathbb{Q}_l[G_k]}_{\mathrm{filt}})$ be given by the projective system of functors $\tilde{R}_{et}(\,\cdot\,,\mathbb{Z}/l^n\mathbb{Z})$.*

Lemma 9.2.4 *We have*

$$H^n\left(\tilde{R}_l(U,X)\right) = \left[\lim_{\leftarrow} H^n_{et}(U \times \bar{k}, \mathbb{Z}/l^n\mathbb{Z})\right] \otimes \mathbb{Q}$$

as $\mathbb{Q}_l$-vector spaces with operation of G_k. Hence $\tilde{R}_l$ computes l-adic cohomology.

Proof: As shown above, $\tilde{R}_{et}$ computes étale cohomology over the algebraic closure. The right hand side is only an explicit instruction how to view a Galois module as a vector space. This is the usual l-adic cohomology by definition ([51] V §1 p. 164). □

Proposition 9.2.5 *1. The naive cohomology objects of $\tilde{R}_l(U,X)$ are mixed. The weight filtration is the filtration by weights.*

2. The functor factors over $C^+_{(\mathrm{str})}(\underline{\mathbb{Q}_l[G_k]}_{\mathrm{filt}})$.

3. The $\mathbb{Z}$-span of morphisms of the form $\tilde{R}_\sigma(f)$ lies in $C^+_{\mathrm{str}}(\underline{\mathbb{Q}_l[G_k]}_{\mathrm{filt}})$.

Proof: The first assertion is [14] Théorème 6.1.2. The identification of the pure parts of the canonical filtration can be proved as in Hodge theory, cf. e.g. [39] 3.18 or 3.22.

Strictness of differentials will be proved by comparison with singular cohomology. The proof is postponed until later once more, cf. 10.2.4. Property 3. follows from 1. and 2. as the category of mixed Galois modules is abelian. □

9.3 Second and Third Step

Definition 9.3.1 *Let $\tilde{R}_l : \tilde{\mathcal{V}}_0^\Delta \longrightarrow C^+(\underline{\mathbb{Q}_l[G_k]}_{\mathrm{filt}})$ be the functor which is obtained from $\tilde{R}_l$ on $\tilde{\mathcal{V}}_0$ by taking the total complex. Let $\tilde{R}_l^{\mathrm{red}}$ be given by the functor* $\mathrm{Cone}(\tilde{R}_l(\mathrm{Spec}\, k) \to \tilde{R}_l(\,\cdot\,))$.

Proposition 9.3.2 *$\tilde{R}_l(U,X)$ is a strict complex with cohomology objects in the category of Galois modules with filtrations by weights. The functor maps proper hypercoverings to quasi-isomorphisms. $\tilde{R}_l^{\mathrm{red}}$ maps exact triangles to distinguished triangles.*

Proof: The assertions on strictness and on the cohomology objects follow from 9.2.5 and stability of the conditions under taking cones (see proof of 3.1.8). The assertion on exact triangles is clear as in 6.2.3. The image of a proper hypercovering is a simple quasi-isomorphism by 1.1.3 (first on the level of $\mathbb{Z}/l^n$-modules, then in general). As the complexes are strict and the cohomology groups have the form of 9.2.4 and even carry the filtration by weights, it is a filtered quasi-isomorphism. □

In the same way as in singular cohomology, we now get a functor

$$R_l : \mathcal{V}^\Delta \longrightarrow D_l(G_k)$$

by choice of a proper hypercovering of a simplicial variety.

Proposition 9.3.3 *R_l is an exact realization functor.*

Proof: This is shown as in all other realizations. □

Remark: The restriction to finitely generated field (instead of fields embeddable into $\mathbb{C}$) is not important. The real problem is quite formal: The given procedure gives in general a functor with values in the derived category $D^+(\underline{\mathbb{Q}_l[G_k]}_{\mathrm{filt}})$, which is not an R-category. But all other properties are as before. The complexes have strict cohomology, and the morphisms of varieties induce strict morphisms on cohomology. This final property has to be proved by comparison with singular cohomology. Another possibility is to use the fact that all varieties, morphism and sheaves involved in the construction of $R_l(Y_\cdot)$ are already defined over some finitely generated field.

9.4 Calculation of Cohomology

As in the Hodge realization, calculation of the geometric cohomology is rather easy in the l-adic realization, too.

Proposition 9.4.1 *The geometric cohomology object*

$$\underline{\mathrm{H}}_l^i(Y_\cdot, j) = H^i\left(R_l(Y_\cdot)(j)\right)$$

agrees with the Galois representation $\mathrm{H}^i_{l-ad.}(Y_\cdot \times \mathrm{Spec}(\bar{k}), \mathbb{Q}_l) \otimes \mathbb{Q}_l(j)$*. The filtration is the filtration by weights.*

Proof: This is clear by construction and what was said before. □

The absolute cohomology groups do not have a similar identification with well-known groups. It is possible to define an unfiltered l-adic realization with values in the derived category of the abelian category $\underline{\mathbb{Q}_l[G_k]}$ by application of the (exact) forgetful functor. (Or construct it directly as the filtered l-adic realization itself.) We denote it by $R_l^{\text{unfilt}}(\,\cdot\,)$. For these we have:

Proposition 9.4.2 *The absolute cohomology groups of the unfiltered l-adic realization*

$$\mathrm{H}^n_{l,\text{unfilt}}(Y.,j) = \mathrm{Hom}_{D_l(G_k)^{\text{unfilt}}}(\mathbb{1}, R_l^{\text{unfilt}}(Y.)(j)[n])$$

agree after taking the tensor product with $\mathbb{Q}$ *with continuous étale cohomology by Jannsen in [38]. ($\mathbb{Z}_l$-coefficients are used there.)*

Proof: In the unfiltered case same functor is derived as one checks in Jannsen's construction. □

Remark: By the remark after 3.2.2 the absolute cohomology of the l-adic realization

$$\mathrm{H}^n_l(Y.,j) = \mathrm{Hom}_{D_l(G_k)}(R_l(Y.)(j)[n])$$

is related to its unfiltered version. Note that W_0 does not commute with taking cohomology of the complex which computes $\mathrm{H}^n_{l,\text{unfilt}}(Y.,j)$!

Remarks: If k' is an algebraic extension of k, then R_l is functorial. On the geometric cohomology objects the transition from G_k-modules to $G_{k'}$-modules only consists in the restriction of the operation of groups. The functor is exact.

If $k \longrightarrow k'$ is an inclusion of algebraically closed fields (of characteristic 0, as usual), then the induced transformation of the functors R_l is an isomorphism.

Fixing an inclusion $\bar{\sigma} : \bar{k} \longrightarrow \mathbb{C}$, the composition of the transformations of functors yields a comparison of l-adic cohomology over k with l-adic cohomology over $\mathbb{C}$.

10 Comparison Functors: l-adic versus Singular Realization

The aim is this time the comparison between l-adic cohomology and singular cohomology. In contrast to the considerations in chapter 8.2, the construc-