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D-Modules, Perverse Sheaves, and Representation Theory

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Preface

D-Modules, Perverse Sheaves, and Representation Theory is a greatly expanded translation of the Japanese edition entitled *D kagun to daisugun* (*D-Modules and Algebraic Groups*) which was published by Springer-Verlag Tokyo, 1995. For the new English edition, the two authors of the original book, R. Hotta and T. Tanisaki, have added K. Takeuchi as a coauthor. Significant new material along with corrections and modifications have been made to this English edition.

In the summer of 1982, a symposium was held in Kinosaki in which the subject of D-modules and their applications to representation theory was introduced. At that time the theory of regular holonomic D-modules had just been completed and the Kazhdan–Lusztig conjecture had been settled by Brylinski–Kashiwara and Beilinson–Bernstein. The articles that appeared in the published proceedings of the symposium were not well presented and of course the subject was still in its infancy. Several monographs, however, did appear later on D-modules, for example, Björk [Bj2], Borel et al. [Bor3], Kashiwara–Schapira [KS2], Mebkhout [Me5] and others, all of which were taken into account and helped us make our Japanese book more comprehensive and readable. In particular, J. Bernstein's notes [Ber1] were extremely useful to understand the subject in the algebraic case; our treatment in many aspects follows the method used in the notes. Our plan was to present the combination of D-module theory and its typical applications to representation theory as we believe that this is a nice way to understand the whole subject.

Let us briefly explain the contents of this book. Part I is devoted to D-module theory, placing special emphasis on holonomic modules and constructible sheaves. The aim here is to present a proof of the Riemann–Hilbert correspondence. Part II is devoted to representation theory. In particular, we will explain how the Kazhdan–Lusztig conjecture was solved using the theory of D-modules. To a certain extent we assume the reader's familiarity with algebraic geometry, homological algebras, and sheaf theory. Although we include in the appendices brief introductions to algebraic varieties and derived categories, which are sufficient overall for dealing with the text, the reader should occasionally refer to appropriate references mentioned in the text.

The main difference from the original Japanese edition is that we made some new chapters and sections for analytic D-modules, meromorphic connections, perverse sheaves, and so on. We thus emphasized the strong connections of D-modules with various other fields of mathematics.

We express our cordial thanks to A. D'Agnolo, C. Marastoni, Y. Matsui, P. Schapira, and J. Schürmann for reading very carefully the draft of the English version and giving us many valuable comments. Discussions with M. Kaneda, K. Kimura, S. Naito, J.-P. Schneiders, K. Vilonen, and others were also very helpful in completing the exposition. M. Nagura and Y. Sugiki greatly helped us in typing and correcting our manuscript. Thanks also go to many people for useful comments on our Japanese version, in particular to T. Ohsawa. Last but not least, we cannot exaggerate our gratitude to M. Kashiwara throughout the period since 1980 on various occasions.

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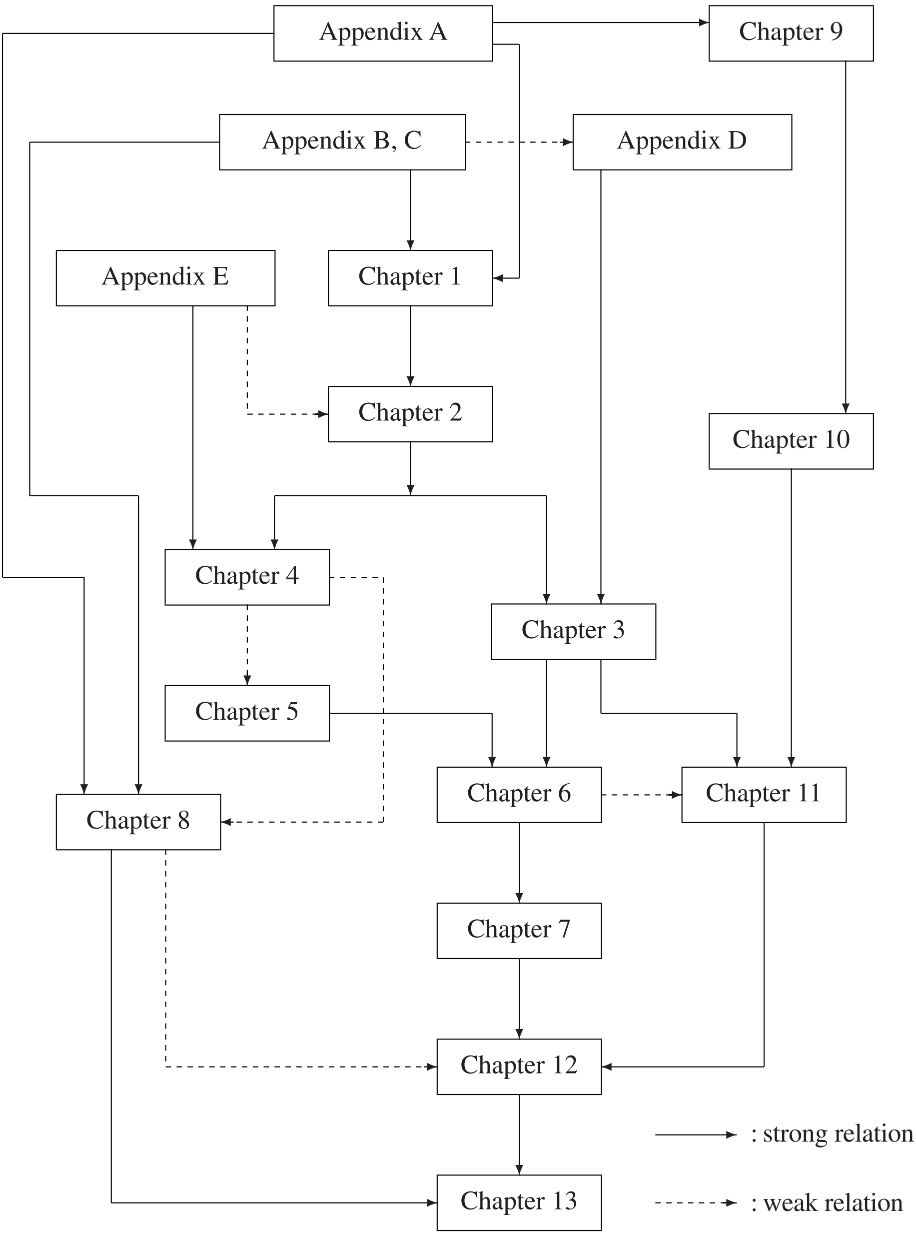


Introduction

The theory of D-modules plays a key role in algebraic analysis. For the purposes of this text, by "algebraic analysis," we mean analysis using algebraic methods, such as ring theory and homological algebra. In addition to the contributions by French mathematicians, J. Bernstein, and others, this area of research has been extensively developed since the 1960s by Japanese mathematicians, notably in the important contributions of M. Sato, T. Kawai, and M. Kashiwara of the Kyoto school.

To this day, there continue to be outstanding results and significant theories coming from the Kyoto school, including Sato's hyperfunctions, microlocal analysis, D-modules and their applications to representation theory and mathematical physics. In particular, the theory of regular holonomic D-modules and their solution complexes (e.g., the theory of the Riemann–Hilbert correspondence which gave a sophisticated answer to Hilbert's 21st problem) was a most important and influential result. Indeed, it provided the germ for the theory of perverse sheaves, which was a natural development from intersection cohomologies. Moreover, M. Saito used this result effectively to construct his theory of Hodge modules, which largely extended the scope of Hodge theory. In representation theory, this result opened totally new perspectives, such as the resolution of the Kazhdan–Lusztig conjecture.

As stated above, in addition to the strong impact on analysis which was the initial main motivation, the theory of algebraic analysis, especially that of D-modules, continues to play a central role in various fields of contemporary mathematics. In fact, D-module theory is a source for creating new research areas from which new theories emerge. This striking feature of D-module theory has stimulated mathematicians in various other fields to become interested in the subject.

Our aim is to give a comprehensive introduction to D-modules. Until recently, in order to really learn it, we had to read and become familiar with many articles, which took long time and considerable effort. However, as we mentioned in the preface, thanks to some textbooks and monographs, the theory has become much more accessible nowadays, especially for those who have some basic knowledge of complex analysis or algebraic geometry. Still, to understand and appreciate the real significance of the subject on a deep level, it would be better to learn both the theory and its typical applications.

In Part I of this book we introduce D-modules principally in the context of presenting the theory of the Riemann–Hilbert correspondence. Part II is devoted to explaining applications to representation theory, especially to the solution to the Kazhdan–Lusztig conjecture. Since we mainly treat the theory of algebraic D-modules on smooth algebraic varieties rather than the (original) analytic theory on complex manifolds, we shall follow the unpublished notes [Ber3] of Bernstein (the book [Bor3] is also written along this line). The topics treated in Part II reveal how useful D-module theory is in other branches of mathematics. Among other things, the essential usefulness of this theory contributed heavily to resolving the Kazhdan–Lusztig conjecture, which was of course a great breakthrough in representation theory.

As we started Part II by giving a brief introduction to some basic notions of Lie algebras and algebraic groups using concrete examples, we expect that researchers in other fields can also read Part II without much difficulty.

Let us give a brief overview of the topics developed in this text. First, we explain how D-modules are related to systems of linear partial differential equations. Let X be an open subset of $\mathbb{C}^n$ and denote by $\mathcal{O}$ the commutative ring of complex analytic functions globally defined on X. We denote by D the set of linear partial differential operators with coefficients in $\mathcal{O}$. Namely, the set D consists of the operators of the form

$$\sum_{i_1,i_2,\ldots,i_n}^{\infty} f_{i_1,i_2,\ldots,i_n} \left(\frac{\partial}{\partial x_1}\right)^{i_1} \left(\frac{\partial}{\partial x_2}\right)^{i_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n} \qquad (f_{i_1,i_2,\ldots,i_n} \in \mathcal{O})$$

(each sum is a finite sum), where $(x_1, x_2, \ldots, x_n)$ is a coordinate system of $\mathbb{C}^n$. Note that D is a non-commutative ring by the composition of differential operators. Since the ring D acts on $\mathcal{O}$ by differentiation, $\mathcal{O}$ is a left D-module. Now, for $P \in D$, let us consider the differential equation

$$Pu = 0 \tag{0.0.1}$$

for an unknown function u. According to Sato, we associate to this equation the left D-module $M = D/DP$. In this setting, if we consider the set $\mathrm{Hom}_D(M, \mathcal{O})$ of D-linear homomorphisms from M to $\mathcal{O}$, we get the isomorphism

$$\begin{aligned}\mathrm{Hom}_D(M, \mathcal{O}) &= \mathrm{Hom}_D(D/DP, \mathcal{O}) \\ &\simeq \{\varphi \in \mathrm{Hom}_D(D, \mathcal{O}) \mid \varphi(P) = 0\}.\end{aligned}$$

Hence we see by $\mathrm{Hom}_D(D, \mathcal{O}) \simeq \mathcal{O}$ $(\varphi \mapsto \varphi(1))$ that

$$\mathrm{Hom}_D(M, \mathcal{O}) \simeq \{f \in \mathcal{O} \mid Pf = 0\}$$

$(Pf = P\varphi(1) = \varphi(P1) = \varphi(P) = 0)$. In other words, the (additive) group of the holomorphic solutions to the equation (0.0.1) is naturally isomorphic to $\mathrm{Hom}_D(M, \mathcal{O})$. If we replace $\mathcal{O}$ with another function space $\mathcal{F}$ admitting a natural action of D (for example, the space of C^∞-functions, Schwartz distributions, Sato's hyperfunctions, etc.), then $\mathrm{Hom}_D(M, \mathcal{F})$ is the set of solutions to (0.0.1) in that function space.

More generally, a system of linear partial differential equations of l-unknown functions $u_1, u_2, \dots, u_l$ can be written in the form

$$\sum_{j=1}^{l} P_{ij} u_j = 0 \quad (i = 1, 2, \dots, k) \tag{0.0.2}$$

by using some $P_{ij} \in D$ $(1 \leq i \leq k,\ 1 \leq j \leq l)$. In this situation we have also a similar description of the space of solutions. Indeed if we define a left D-module M by the exact sequence

$$D^k \xrightarrow{\varphi} D^l \longrightarrow M \longrightarrow 0 \tag{0.0.3}$$

$$\varphi(Q_1, Q_2, \dots, Q_k) = \left(\sum_{i=1}^{k} Q_i P_{i1}, \sum_{i=1}^{k} Q_i P_{i2}, \dots, \sum_{i=1}^{k} Q_i P_{il} \right),$$

then the space of the holomorphic solutions to (0.0.2) is isomorphic to $\mathrm{Hom}_D(M, \mathcal{O})$. Therefore, systems of linear partial differential equations can be identified with the D-modules having some finite presentations like (0.0.3), and the purpose of the theory of linear PDEs is to study the solution space $\mathrm{Hom}_D(M, \mathcal{O})$. Since the space $\mathrm{Hom}_D(M, \mathcal{O})$ does not depend on the concrete descriptions (0.0.2) and (0.0.3) of M (it depends only on the D-linear isomorphism class of M), we can study these analytical problems through left D-modules admitting finite presentations. In the language of categories, the theory of linear PDEs is nothing but the investigation of the contravariant functor $\mathrm{Hom}_D(\bullet, \mathcal{O})$ from the category $M(D)$ of D-modules admitting finite presentations to the category $M(\mathbb{C})$ of $\mathbb{C}$-modules.

In order to develop this basic idea, we need to introduce sheaf theory and homological algebra. First, let us explain why sheaf theory is indispensable. It is sometimes important to consider solutions locally, rather than globally on X. For example, in the case of ordinary differential equations (or more generally, the case of integrable systems), the space of local solutions is always finite dimensional; however, it may happen that the analytic continuations (after turning around a closed path) of a solution are different from the original one. This phenomenon is called monodromy. Hence we also have to take into account how local solutions are connected to each other globally.

Sheaf theory is the most appropriate language for treating such problems. Therefore, sheafifying $\mathcal{O}$, D, let us now consider the sheaf $\mathcal{O}_X$ of holomorphic functions and the sheaf D_X (of rings) of differential operators with holomorphic coefficients. We also consider sheaves of D_X-modules (in what follows, we simply call them D_X-modules) instead of D-modules. In this setting, the main objects to be studied are left D_X-modules admitting locally finite presentations (i.e., coherent D_X-modules). Sheafifying also the solution space, we get the sheaf $\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ of the holomorphic solutions to a D_X-module M. It follows that what we should investigate is the contravariant functor $\mathcal{H}om_{D_X}(\bullet, \mathcal{O}_X)$ from the category $\mathrm{Mod}_c(D_X)$ of coherent D_X-modules to the category $\mathrm{Mod}(\mathbb{C}_X)$ of (sheaves of) $\mathbb{C}_X$-modules.

Let us next explain the need for homological algebra. Although both $\mathrm{Mod}_c(D_X)$ and $\mathrm{Mod}(\mathbb{C}_X)$ are abelian categories, $\mathcal{H}om_{D_X}(\bullet, \mathcal{O}_X)$ is not an exact functor. Indeed, for a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \tag{0.0.4}$$

in the category $\mathrm{Mod}_c(D_X)$ the sequence

$$0 \to \mathcal{H}om_{D_X}(M_3, \mathcal{O}_X) \to \mathcal{H}om_{D_X}(M_2, \mathcal{O}_X) \to \mathcal{H}om_{D_X}(M_1, \mathcal{O}_X) \tag{0.0.5}$$

associated to it is also exact; however, the final arrow $\mathcal{H}om_{D_X}(M_2, \mathcal{O}_X) \to \mathcal{H}om_{D_X}(M_1, \mathcal{O}_X)$ is not necessarily surjective. Hence we cannot recover information about the solutions of M_2 from those of M_1, M_3. A remedy for this is to consider also the "higher solutions" $\mathcal{E}xt^i_{D_X}(M, \mathcal{O}_X)$ $(i = 0, 1, 2, \dots)$ by introducing techniques in homological algebra. We have $\mathcal{E}xt^0_{D_X}(M, \mathcal{O}_X) = \mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ and the exact sequence (0.0.5) is naturally extended to the long exact sequence

$$\begin{aligned}\cdots \to \mathcal{E}xt^i_{D_X}(M_3, \mathcal{O}_X) &\to \mathcal{E}xt^i_{D_X}(M_2, \mathcal{O}_X) \to \mathcal{E}xt^i_{D_X}(M_1, \mathcal{O}_X)\\ \to \mathcal{E}xt^{i+1}_{D_X}(M_3, \mathcal{O}_X) &\to \mathcal{E}xt^{i+1}_{D_X}(M_2, \mathcal{O}_X) \to \mathcal{E}xt^{i+1}_{D_X}(M_1, \mathcal{O}_X) \to \cdots .\end{aligned}$$

Hence the theory will be developed more smoothly by considering all higher solutions together.

Furthermore, in order to apply the methods of homological algebra in full generality, it is even more effective to consider the object $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ in the derived category (it is a certain complex of sheaves of $\mathbb{C}_X$-modules whose i-th cohomology sheaf is $\mathcal{E}xt^i_{D_X}(M, \mathcal{O}_X)$) instead of treating the sheaves $\mathcal{E}xt^i_{D_X}(M, \mathcal{O}_X)$ separately for various i's. Among the many other advantages for introducing the methods of homological algebra, we point out here the fact that the sheaf of a hyperfunction solution can be obtained by taking the local cohomology of the complex $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ of holomorphic solutions. This is quite natural since hyperfunctions are determined by the boundary values (local cohomologies) of holomorphic functions.

Although we have assumed so far that X is an open subset of $\mathbb{C}^n$, we may replace it with an arbitrary complex manifold. Moreover, also in the framework of smooth algebraic varieties over algebraically closed fields k of characteristic zero, almost all arguments remain valid except when considering the solution complex $R\mathcal{H}om_{D_X}(\bullet, \mathcal{O}_X)$, in which case we need to assume again that $k = \mathbb{C}$ and return to the classical topology (not the Zariski topology) as a complex manifold. In this book we shall mainly treat D-modules on smooth algebraic varieties over $\mathbb{C}$; however, in this introduction, we will continue to explain everything on complex manifolds. Hence X denotes a complex manifold in what follows.

There were some tentative approaches to D-modules by D. Quillen, Malgrange, and others in the 1960s; however, the real intensive investigation leading to later development was started by Kashiwara in his master thesis [Kas1] (we also note that this important contribution to D-module theory was also made independently by Bernstein [Ber1],[Ber2] around the same period). After this groundbreaking work, in collaboration with Kawai, Kashiwara developed the theory of (regular) holonomic

D-modules [KK3], which is a main theme in Part I of this book. Let us discuss this subject.

It is well known that the space of the holomorphic solutions to every ordinary differential equation is finite dimensional. However, when X is higher dimensional, the dimensions of the spaces of holomorphic solutions can be infinite. This is because, in such cases, the solution contains parameters given by arbitrary functions unless the number of given equations is sufficiently large. Hence our task is to look for a suitable class of D_X-modules whose solution spaces are finite dimensional. That is, we want to find a generalization of the notion of ordinary differential equations in higher-dimensional cases.

For this purpose we consider the characteristic variety $\mathrm{Ch}(M)$ for a coherent D_X-module M, which is a closed analytic subset of the cotangent bundle T^*X of X (we sometimes call this the singular support of M and denote it by $\mathrm{SS}(M)$). We know by a fundamental theorem of algebraic analysis due to Sato–Kawai–Kashiwara [SKK] that $\mathrm{Ch}(M)$ is an involutive subvariety in T^*X with respect to the canonical symplectic structure of T^*X. In particular, we have $\dim \mathrm{Ch}(M) \geq \dim X$ for any coherent D_X-module $M \neq 0$.

Now we say that a coherent D_X-module M is holonomic (a maximally overdetermined system) if it satisfies the equality $\dim \mathrm{Ch}(M) = \dim X$. Let us give the definition of characteristic varieties only in the simple case of D_X-modules

$$M = D_X/I, \quad I = D_X P_1 + D_X P_2 + \cdots + D_X P_k$$

associated to the systems

$$P_1 u = P_2 u = \cdots = P_k u = 0 \quad (P_i \in D_X) \tag{0.0.6}$$

for a single unknown function u. In this case, the characteristic variety $\mathrm{Ch}(M)$ of M is the common zero set of the principal symbols $\sigma(Q)$ $(Q \in I)$ (recall that for $Q \in D_X$ its principal symbol $\sigma(Q)$ is a holomorphic function on T^*X). In many cases $\mathrm{Ch}(M)$ coincides with the common zero set of $\sigma(P_1), \sigma(P_2), \dots, \sigma(P_k)$, but it sometimes happens to be smaller (we also see from this observation that the abstract D_X-module M itself is more essential than its concrete expression (0.0.6)).

To make the solution space as small (finite dimensional) as possible we should consider as many equations as possible. That is, we should take the ideal $I \subset D_X$ as large as possible. This corresponds to making the ideal generated by the principal symbols $\sigma(P)$ $(P \in I)$ (in the ring of functions on T^*X) as large as possible, for which we have to take the characteristic variety $\mathrm{Ch}(M)$, i.e., the zero set of the $\sigma(P)$'s, as small as possible. On the other hand, a non-zero coherent D_X-module is holonomic if the dimension of its characteristic variety takes the smallest possible value $\dim X$. This philosophical observation suggests a possible connection between the holonomicity and the finite dimensionality of the solution spaces. Indeed such connections were established by Kashiwara as we explain below.

Let us point out here that the introduction of the notion of characteristic varieties is motivated by the ideas of microlocal analysis. In microlocal analysis, the sheaf $\mathcal{E}_X$ of microdifferential operators is employed instead of the sheaf D_X of differential operators. This is a sheaf of rings on the cotangent bundle T^*X containing $\pi^{-1}D_X$ ($\pi : T^*X \to X$) as a subring. Originally, the characteristic variety $\mathrm{Ch}(M)$ of a coherent D_X-module M was defined to be the support $\mathrm{supp}(\mathcal{E}_X \otimes_{\pi^{-1}D_X} \pi^{-1}M)$ of the corresponding coherent $\mathcal{E}_X$-module $\mathcal{E}_X \otimes_{\pi^{-1}D_X} \pi^{-1}M$. A guiding principle of Sato–Kawai–Kashiwara [SKK] was to develop the theory in the category of $\mathcal{E}_X$-modules even if one wants results for D_X-modules. In this process, they almost completely classified coherent $\mathcal{E}_X$-modules and proved the involutivity of $\mathrm{Ch}(M)$.

Let us return to holonomic D-modules. In his Ph.D. thesis [Kas3], Kashiwara proved for any holonomic D_X-module M that all of its higher solution sheaves $\mathcal{E}xt^i_{D_X}(M, \mathcal{O}_X)$ are constructible sheaves (i.e., all its stalks are finite-dimensional vector spaces and for a stratification $X = \bigsqcup X_i$ of X its restriction to each X_i is a locally constant sheaf on X_i). From this result we can conclude that the notion of holonomic D_X-module is a natural generalization of that of linear ordinary differential equations to the case of higher-dimensional complex manifolds. We note that it is also proved in [Kas3] that the solution complex $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ satisfies the conditions of perversity (in language introduced later). The theory of perverse sheaves [BBD] must have been motivated (at least partially) by this result.

In the theory of linear ordinary differential equations, we have a good class of equations called equations with regular singularities, that is, equations admitting only mild singularities. We also have a successful generalization of this class to higher dimensions, that is, to regular holonomic D_X-modules. There are roughly two methods to define this class; the first (traditional) one will be to use higher-dimensional analogues of the properties characterizing ordinary differential equations with regular singularities, and the second (rather tactical) will be to define a holonomic D_X-module to be regular if its restriction to any algebraic curve is an ordinary differential equation with regular singularities. The two methods are known to be equivalent. We adopt here the latter as the definition. Moreover, we note that there is a conceptual difference between the complex analytic case and the algebraic case for the global meaning of regularity.

Next, let us explain the Riemann–Hilbert correspondence. By the monodromy of a linear differential equation we get a representation of the fundamental group of the base space. The original 21st problem of Hilbert asks for its converse: that is, for any representation of the fundamental group, is there an ordinary differential equation (with regular singularities) whose monodromy representation coincides with the given one? (there exist several points of view in formulating this problem more precisely, but we do not discuss them here. For example, see [AB], and others).

Let us consider the generalization in higher dimensions of this problem. A satisfactory answer in the case of integrable connections with regular singularities was given by P. Deligne [De1]. In this book, we deal with the problem for regular holonomic D_X-modules. As we have already seen, for any holonomic D_X-module M, its solutions $\mathcal{E}xt^i_{D_X}(M, \mathcal{O}_X)$ are constructible sheaves. Hence, if we denote by $D^b_c(\mathbb{C}_X)$ the derived category consisting of bounded complexes of $\mathbb{C}_X$-modules whose cohomology sheaves are constructible, the holomorphic solution complex $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ is an object of $D^b_c(\mathbb{C}_X)$. Therefore, denoting by $D^b_{rh}(D_X)$ the derived category consisting of bounded complexes of D_X-modules whose cohomology sheaves are regular holonomic D_X-modules, we can define the contravariant functor

$$R\mathcal{H}om_{D_X}(\bullet, \mathcal{O}_X) : D^b_{rh}(D_X) \longrightarrow D^b_c(\mathbb{C}_X). \tag{0.0.7}$$

One of the most important results in the theory of D-modules is the (contravariant) equivalence of categories $D^b_{rh}(D_X) \simeq D^b_c(\mathbb{C}_X)$ via this functor. The crucial point of this equivalence (the Riemann–Hilbert correspondence, which we noted is the most sophisticated solution to Hilbert's 21st problem) lies in the concept of regularity and this problem was properly settled by Kashiwara–Kawai [KK3]. The correct formulation of the above equivalence of categories was already conjectured by Kashiwara in the middle 1970s and the proof was completed around 1980 (see [Kas6]). The full proof was published in [Kas10]. For this purpose, Kashiwara constructed the inverse functor of the correspondence (0.0.7). We should note that another proof of this correspondence was also obtained by Mebkhout [Me4]. For the more detailed historical comments, compare the foreword by Schapira in the English translation [Kas16] of Kashiwara's master thesis [Kas1]. As mentioned earlier we will mainly deal with algebraic D-modules in this book, and hence what we really consider is a version of the Riemann–Hilbert correspondence for algebraic D-modules. After the appearance of the theory of regular holonomic D-modules and the Riemann–Hilbert correspondence for analytic D-modules, A. Beilinson and J. Bernstein developed the corresponding theory for algebraic D-modules based on much simpler arguments. Some part of this book relies on their results.

The content of Part I is as follows. In Chapters 1–3 we develop the basic theory of algebraic D-modules. In Chapter 4 we give a survey of the theory of analytic D-modules and present some properties of the solution and the de Rham functors. Chapter 5 is concerned with results on regular meromorphic connections due to Deligne [De1]. As for the content of Chapter 5, we follow the notes of Malgrange in [Bor3], which will be a basis of the general theory of regular holonomic D-modules described in Chapters 6 and 7. In Chapter 6 we define the notion of regular holonomic algebraic D-modules and show its stability under various functors. In Chapter 7 we present a proof of an algebraic version the Riemann–Hilbert correspondence. The results in Chapters 6 and 7 are totally due to the unpublished notes of Bernstein [Ber3] explaining his work with Beilinson. In Chapter 8 we give a relatively self-contained account of the theory of intersection cohomology groups and perverse sheaves (M. Goresky–R. MacPherson [GM1], Beilinson–Bernstein–Deligne [BBD]) assuming basic facts about constructible sheaves. This part is independent of other parts of the book. We also include a brief survey of the theory of Hodge modules due to Morihiko Saito [Sa1], [Sa2] without proofs.

We finally note that the readers of this book who are only interested in algebraic D-module theory (and not in the analytic one) can skip Sections 4.4 and 4.6, and need not become involved with symplectic geometry.

In the rest of the introduction we shall give a brief account of the content of Part II which deals with applications of D-module theory to representation theory.

The history of Lie groups and Lie algebras dates back to the 19th century, the period of S. Lie and F. Klein. Fundamental results about semisimple Lie groups such as those concerning structure theorems, classification, and finite-dimensional representation theory were obtained by W. Killing, E. Cartan, H. Weyl, and others until the 1930s. Afterwards, the theory of infinite-dimensional (unitary) representations was initiated during the period of World War II by E. P. Wigner, V. Bargmann, I. M. Gelfand, M. A. Naimark, and others, and partly motivated by problems in physics. Since then and until today the subject has been intensively investigated from various points of view. Besides functional analysis, which was the main method at the first stage, various theories from differential equations, differential geometry, algebraic geometry, algebraic analysis, etc. were applied to the theory of infinite-dimensional representations. The theory of automorphic forms also exerted a significant influence. Nowadays infinite-dimensional representation theory is a place where many branches of mathematics come together. As contributors representing the development until the 1970s, we mention the names of Harish-Chandra, B. Kostant, R. P. Langlands.

On the other hand, the theory of algebraic groups was started by the fundamental works of C. Chevalley, A. Borel, and others [Ch] and became recognized widely by the textbook of Borel [Bor1]. Algebraic groups are obtained by replacing the underlying complex or real manifolds of Lie groups with algebraic varieties. Over the fields of complex or real numbers algebraic groups form only a special class of Lie groups; however, various new classes of groups are produced by taking other fields as the base field. In this book we will only be concerned with semisimple groups over the field of complex numbers, for which Lie groups and algebraic groups provide the same class of groups. We regard them as algebraic groups since we basically employ the language of algebraic geometry.

The application of algebraic analysis to representation theory was started by the resolution of the Helgason conjecture [six] due to Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, and M. Tanaka. In this book, we focus however on the resolution of the Kazhdan–Lusztig conjecture which was the first achievement in representation theory obtained by applying D-module theory.

Let us explain the problem. It is well known that all finite-dimensional irreducible representations of complex semisimple Lie algebras are highest weight modules with dominant integral highest weights. For such representations the characters are described by Weyl's character formula. Inspired by the works of Harish-Chandra on infinite-dimensional representations of semisimple Lie groups, D. N. Verma proposed in the late 1960s the problem of determining the characters of (infinite-dimensional) irreducible highest weight modules with not necessarily dominant integral highest weights. Important contributions to this problem by a purely algebraic approach were made in the 1970s by Bernstein, I. M. Gelfand, S. I. Gelfand, and J. C. Jantzen, although the original problem was not solved.

A breakthrough using totally new methods was made around 1980. D. Kazhdan and G. Lusztig introduced a family of special polynomials (the Kazhdan–Lusztig polynomials) using Hecke algebras and proposed a conjecture giving the explicit form of the characters of irreducible highest weight modules in terms of these polynomials [KL1]. They also gave a geometric meaning for Kazhdan–Lusztig polynomials using the intersection cohomology groups of Schubert varieties. Promptly responding to this, Beilinson–Bernstein [BB] and J.-L. Brylinski–Kashiwara independently solved the conjecture by establishing a correspondence between highest weight modules and the intersection cohomology complexes of Schubert varieties via D-modules on the flag manifold. This successful achievement, i.e., employing theories and methods, from other fields, was quite astonishing for the specialists who had been studying the problem using purely algebraic means. Since then D-module theory has brought numerous new developments in representation theory.

Let us explain more precisely the methods used to solve the Kazhdan–Lusztig conjecture. Let G be an algebraic group (or a Lie group), $\mathfrak{g}$ its Lie algebra and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. If X is a smooth G-variety and $\mathcal{V}$ is a G-equivariant vector bundle on X, the set $\Gamma(X, \mathcal{V})$ of global sections of $\mathcal{V}$ naturally has a $\mathfrak{g}$-module structure. The construction of the representation of $\mathfrak{g}$ (or of G) in this manner is a fundamental technique in representation theory.

Let us now try to generalize this construction. Denote by $D_X^{\mathcal{V}} \subset \mathcal{E}nd_{\mathbb{C}}(\mathcal{V})$ the sheaf of rings of differential operators acting on the sections of $\mathcal{V}$. Then $D_X^{\mathcal{V}}$ is isomorphic to $\mathcal{V} \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \mathcal{V}^*$ which coincides with the usual D_X when $\mathcal{V} = \mathcal{O}_X$. In terms of $D_X^{\mathcal{V}}$ the $\mathfrak{g}$-module structure on $\Gamma(X, \mathcal{V})$ can be described as follows. Note that we have a canonical ring homomorphism $U(\mathfrak{g}) \to \Gamma(X, D_X^{\mathcal{V}})$ induced by the G-action on $\mathcal{V}$. Since $\mathcal{V}$ is a $D_X^{\mathcal{V}}$-module, $\Gamma(X, \mathcal{V})$ is a $\Gamma(X, D_X^{\mathcal{V}})$-module, and hence a $\mathfrak{g}$-module through the ring homomorphism $U(\mathfrak{g}) \to \Gamma(X, D_X^{\mathcal{V}})$. From this observation, we see that we can replace $\mathcal{V}$ with other $D_X^{\mathcal{V}}$-modules. That is, for any $D_X^{\mathcal{V}}$-module M the $\mathbb{C}$-vector space $\Gamma(X, M)$ is endowed with a $\mathfrak{g}$-module structure.

Let us give an example. Let $G = SL_2(\mathbb{C})$. Since G acts on $X = \mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$ by the linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x) = \left(\frac{ax+b}{cx+d} \right) \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,\ (x) \in X \right),$$

it follows from the above arguments that $\Gamma(X, M)$ is a $\mathfrak{g}$-module for any D_X-module M. Let us consider the D_X-module $M = D_X\delta$ given by Dirac's delta function δ at the point $x = \infty$. In the coordinate $z = \frac{1}{x}$ in a neighborhood of $x = \infty$, the equation satisfied by Dirac's delta function δ is

$$z\delta = 0,$$

so we get

$$M = D_X / D_X z$$

in a neighborhood of $x = \infty$. Set $\delta_n = (\frac{d}{dz})^n \delta$. Then $\{\delta_n\}_{n=0}^{\infty}$ is the basis of $\Gamma(X, M)$ and we have $\frac{d}{dz}\delta_n = \delta_{n+1}$, $z\delta_n = -n\delta_{n-1}$.

Let us describe the action of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ on $\Gamma(X, M)$. For this purpose consider the following elements in $\mathfrak{g}$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(these elements h, e, f form a basis of $\mathfrak{g}$). Then the ring homomorphism $U(\mathfrak{g}) \to \Gamma(X, D_X)$ is given by

$$h \longmapsto 2z\frac{d}{dz}, \quad e \longmapsto z^2\frac{d}{dz}, \quad f \longmapsto -\frac{d}{dz}.$$

For example, since

$$\exp(-te) \cdot \begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ z/(1-tz) \end{pmatrix},$$

for $\varphi(z) \in \mathcal{O}_X$ we get

$$(e \cdot \varphi)(z) = \frac{d}{dt}\varphi\left(\frac{z}{1-tz}\right)\bigg|_{t=0} = \left(z^2\frac{d}{dz}\varphi\right)(z)$$

and $e \mapsto z^2\frac{d}{dz}$. Therefore we obtain

$$h \cdot \delta_n = -2(n+1)\delta_n, \quad e \cdot \delta_n = n(n+1)\delta_{n-1}, \quad f \cdot \delta_n = -\delta_{n+1},$$

from which we see that $\Gamma(X, M)$ is the infinite-dimensional irreducible highest weight module with highest weight -2.

For the proof of the Kazhdan–Lusztig conjecture, we need to consider the case when G is a semisimple algebraic group over the field of complex numbers and the G-variety X is the flag variety of G. For each Schubert variety Y in X we consider a D_X-module M satisfied by the delta function supported on Y. In our previous example, i.e., in the case of $G = SL_2(\mathbb{C})$, the flag variety is $X = \mathbb{P}^1$ and $Y = \{\infty\}$ is a Schubert variety. Since Schubert varieties $Y \subset X$ may have singularities for general algebraic groups G, we take the regular holonomic D_X-module M characterized by the condition of having no subquotient whose support is contained in the boundary of Y. For this choice of M, $\Gamma(X, M)$ is an irreducible highest weight $\mathfrak{g}$-module and $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ is the intersection cohomology complex of Y. A link between highest weight $\mathfrak{g}$-modules and the intersection cohomology complexes of Schubert varieties $Y \subset X$ (perverse sheaves on the flag manifold X) is given in this manner. Diagrammatically the strategy of the proof of the Kazhdan–Lusztig conjecture can be explained as follows:

$$\boxed{\mathfrak{g}\text{-modules}}$$
$$\updownarrow$$
$$\boxed{D\text{-modules on the flag manifold } X}$$
$$\updownarrow$$
$$\boxed{\text{perverse sheaves on the flag manifold } X}$$

Here the first arrow is what we have briefly explained above, and the second one is the Riemann–Hilbert correspondence, a general theory of D-modules. The first arrow is called the Beilinson–Bernstein correspondence, which asserts that the category of $U(\mathfrak{g})$-modules with the trivial central character and that of D_X-modules are equivalent. By this correspondence, for a D_X-module M on the flag manifold X, we associate to it the $U(\mathfrak{g})$-module $\Gamma(X, M)$. As a result, we can translate various problems for $\mathfrak{g}$-modules into those for regular holonomic D-modules (or through the Riemann–Hilbert correspondence, those for constructible sheaves).

The content of Part II is as follows. We review some preliminary results on algebraic groups in Chapters 9 and 10. In Chapters 11 and 12 we will explain how the Kazhdan–Lusztig conjecture was solved. Finally, in Chapter 13, a realization of Hecke algebras will be given by the theory of Hodge modules, and the relation between the intersection cohomology groups of Schubert varieties and Hecke algebras will be explained.

Let us briefly mention some developments of the theory, which could not be treated in this book. We can also formulate conjectures, similar to the Kazhdan–Lusztig conjecture, for Kac–Moody Lie algebras, i.e., natural generalizations of semisimple Lie algebras. In this case, we have to study two cases separately: (a) the case when the highest weight is conjugate to a dominant weight by the Weyl group, (b) the case when the highest weight is conjugate to an anti-dominant weight by the Weyl group. Moreover, Lusztig proposed certain Kazhdan–Lusztig type conjectures also for the following objects: (c) the representations of reductive algebraic groups in positive characteristics, (d) the representations of quantum groups in the case when the parameter q is a root of unity. The conjecture of the case (a) was solved by Kashiwara (and Tanisaki) [Kas15], [KT2] and L. Casian [Ca1]. Following the so-called Lusztig program, the other conjectures were also solved:

(A) the equivalence of (c) and (d): H. H. Andersen, J. C. Jantzen, W. Soergel [AJS].
(B) the equivalence of (b) and (d) for affine Lie algebras: Kazhdan–Lusztig [KL3].
(C) the proof of (b) for affine Lie algebras: Kashiwara–Tanisaki [KT3] and Casian [Ca2].

A

Algebraic Varieties

A.1 Basic definitions

Let $k[X_1, X_2, \ldots, X_n]$ be a polynomial algebra over an algebraically closed field k with n indeterminates $X_1, \ldots, X_n$. We sometimes abbreviate it as $k[X] = k[X_1, X_2, \ldots, X_n]$. Let us associate to each polynomial $f(X) \in k[X]$ its zero set

$$V(f) := \{x = (x_1, x_2, \ldots, x_n) \in k^n \mid f(x) = f(x_1, x_2, \ldots, x_n) = 0\}$$

in the n-fold product set k^n of k. For any subset $S \subset k[X]$ we also set $V(S) = \bigcap_{f \in S} V(f)$. Then we have the following properties:

(i) $V(1) = \emptyset$, $V(0) = k^n$.
(ii) $\bigcap_{i \in I} V(S_i) = V(\bigcup_{i \in I} S_i)$.
(iii) $V(S_1) \cup V(S_2) = V(S_1 S_2)$, where $S_1 S_2 := \{fg \mid f \in S_1,\ g \in S_2\}$.

The inclusion $\subset$ of (iii) is clear. We will prove only the inclusion $\supset$. For $x \in V(S_1 S_2) \setminus V(S_2)$ there is an element $g \in S_2$ such that $g(x) \neq 0$. On the other hand, it follows from $x \in V(S_1 S_2)$ that $f(x)g(x) = 0\ ({}^\forall f \in S_1)$. Hence $f(x) = 0\ ({}^\forall f \in S_1)$ and $x \in V(S_1)$. So the part $\supset$ was also proved.

By (i), (ii), (iii) the set k^n is endowed with the structure of a topological space by taking $\{V(S) \mid S \subset k[X]\}$ to be its closed subsets. We call this topology of k^n the *Zariski topology*. The closed subsets $V(S)$ of k^n with respect to it are called *algebraic sets* in k^n. Note that $V(S) = V(\langle S \rangle)$, where $\langle S \rangle$ denotes the ideal of $k[X]$ generated by S. Hence we may assume from the beginning that S is an ideal of $k[X]$. Conversely, for a subset $W \subset k^n$ the set

$$I(W) := \left\{ f \in k[X] \mid f(x) = 0\ ({}^\forall x \in W) \right\}$$

is an ideal of $k[X]$. When W is a (Zariski) closed subset of k^n, we have clearly $V(I(W)) = W$. Namely, in the diagram

$$\boxed{\text{ideals in } k[X]} \underset{I}{\overset{V}{\rightleftarrows}} \boxed{\text{closed subsets in } k^n}$$

we have $V \circ I = \mathrm{Id}$. However, for an ideal $J \subset k[X]$ the equality $I(V(J)) = J$ does not hold in general. We have only $I(V(J)) \supset J$. The difference will be clarified later by Hilbert's Nullstellensatz.

Let V be a Zariski closed subset of k^n (i.e., an algebraic set in k^n). We regard it as a topological space by the relative topology induced from the Zariski topology of k^n. We denote by $k[V]$ the k-algebra of k-valued functions on V obtained by restricting polynomial functions to V. It is called the *coordinate ring* of V. The restriction map $\rho_V : k[X] \to k[V]$ given by $\rho_V(f) := f|_V$ is a surjective homomorphism of k-algebras with $\mathrm{Ker}\,\rho_V = I(V)$, and hence we have $k[V] \simeq k[X]/I(V)$. For each point $x = (x_1, x_2, \ldots, x_n) \in k^n$ of V define a homomorphism $e_x : k[V] \to k$ of k-algebras by $e_x(f) = f(x)$. Then we get a map

$$e : V \longrightarrow \mathrm{Hom}_{k\text{-alg}}(k[V],\, k) \quad (x \longmapsto e_x),$$

where $\mathrm{Hom}_{k\text{-alg}}(k[V],\, k)$ denotes the set of the k-algebra homomorphisms from $k[V]$ to k. Conversely, for a k-algebra homomorphism $\phi : k[V] \to k$ define $x = (x_1, x_2, \ldots, x_n) \in k^n$ by $x_i = \phi\rho_V(X_i)$ $(1 \le i \le n)$. Then we have $x \in V$ and $e_x = \phi$. Hence we have an identification $V = \mathrm{Hom}_{k\text{-alg}}(k[V],\, k)$ as a set. Moreover, the closed subsets of V are of the form $V(\rho_V^{-1}(J)) = \{x \in V \mid e_x(J) = 0\} \subset V$ for ideals J of $k[V]$. Therefore, the topological space V is recovered from the k-algebra $k[V]$. It indicates the possibility of defining the notion of algebraic sets starting from certain k-algebras without using the embedding into k^n. Note that the coordinate ring $A = k[V]$ is finitely generated over k, and *reduced* (i.e., does not contain non-zero nilpotent elements) because $k[V]$ is a subring of the ring of functions on V with values in the field k.

In this chapter we give an account of the classical theory of "algebraic varieties" based on reduced finitely generated (commutative) algebras over algebraically closed fields (in the modern language of schemes one allows general commutative rings as "coordinate algebras").

The following two theorems are fundamental.

Theorem A.1.1 (A weak form of Hilbert' Nullstellensatz). *Any maximal ideal of the polynomial ring $k[X_1, X_2, \ldots, X_n]$ is generated by the elements $X_i - x_i$ $(1 \le i \le n)$ for a point $x = (x_1, x_2, \ldots, x_n) \in k^n$.*

Theorem A.1.2 (Hilbert's Nullstellensatz). *We have $I(V(J)) = \sqrt{J}$, where $\sqrt{J}$ is the* **radical** *$\{f \in k[X] \mid f^N \in J$ for some $N \gg 0\}$ of J.*

For the proofs, see, for example, [Mu]. □

For a finitely generated k-algebra A denote by $\mathrm{Specm}\,A$ the set of the maximal ideals of A. For an algebraic set $V \subset k^n$ we have a bijection

$$\mathrm{Hom}_{k\text{-alg}}(k[V],\, k) \simeq \mathrm{Specm}\,k[V] \quad (e \longmapsto \mathrm{Ker}\,e)$$

by Theorem A.1.1. Under this correspondence, the closed subsets in $\mathrm{Specm}\,k[V] \simeq V$ are the sets

$$V(I) = \{\mathfrak{m} \in \mathrm{Specm}\,k[V] \mid \mathfrak{m} \supset I\},$$

where I ranges through the ideals of $k[V]$. Theorem A.1.2 implies that there is a one-to-one correspondence between the closed subsets in $\operatorname{Specm} k[V]$ and the *radical ideals* I ($I = \sqrt{I}$) of $k[V]$.

A.2 Affine varieties

Motivated by the arguments in the previous section, we start from a finitely generated *reduced* commutative k-algebra A to define an algebraic variety. Namely, we set $V = \operatorname{Specm} A$ and define its topology so that the closed subsets are given by

$$\left\{V(I) = \{\mathfrak{m} \in \operatorname{Specm} A \mid I \subset \mathfrak{m}\} \mid I\text{: ideals of } A\right\}.$$

By Hilbert's Nullstellensatz (its weak form), we get the identification

$$V \simeq \operatorname{Hom}_{k\text{-alg}}(A, k).$$

We sometimes write a point $x \in V$ as $\mathfrak{m}_x \in \operatorname{Specm} A$ or $e_x \in \operatorname{Hom}_{k\text{-alg}}(A, k)$. Under this notation we have

$$f(x) = e_x(f) = (f \mod \mathfrak{m}_x) \in k$$

for $f \in A$. Here, we used the identification $k \simeq A/\mathfrak{m}_x$ obtained by the composite of the morphisms $k \hookrightarrow A \to A/\mathfrak{m}_x$. Hence the ring A is regarded as a k-algebra consisting of certain k-valued functions on V.

Recall that any open subset of V is of the form $D(I) = V \setminus V(I)$, where I is an ideal of A. Since A is a noetherian ring (finitely generated over k), the ideal I is generated by a finite subset $\{f_1, f_2, \ldots, f_r\}$ of I. Then we have

$$D(I) = V \setminus \left(\bigcap_{i=1}^{r} V(f_i)\right) = \bigcup_{i=1}^{r} D(f_i),$$

where $D(f) = \{x \in V \mid f(x) \neq 0\} = V \setminus V(f)$ for $f \in A$. We call an open subset of the form $D(f)$ for $f \in A$ a *principal open subset* of V. Principal open subsets form a basis of the open subsets of V. Note that we have the equivalence

$$D(f) \subset D(g) \iff \sqrt{(f)} \subset \sqrt{(g)} \iff f \in \sqrt{(g)}$$

by Hilbert's Nullstellensatz.

Assume that we are given an A-module M. We introduce a sheaf $\widetilde{M}$ on the topological space $V = \operatorname{Specm} A$ as follows. For a multiplicatively closed subset S of A we denote by $S^{-1}M$ the localization of M with respect to S. It consists of the equivalence classes with respect to the equivalence relation $\sim$ on the set of pairs $(s, m) = m/s$ ($s \in S, m \in M$) given by $m/s \sim m'/s' \iff t(s'm - sm') = 0$ ($\exists t \in S$). By the ordinary operation rule of fractional numbers $S^{-1}A$ is endowed with a ring structure and $S^{-1}M$ turns out to be an $S^{-1}A$-module. For $f \in A$ we set

$M_f = S_f^{-1}M$, where $S_f := \{1, f, f^2, \dots\}$. Note that for two principal open subsets $D(f) \subset D(g)$ a natural homomorphism $r_f^g : M_g \to M_f$ is defined as follows. We have $f^n = hg$ ($h \in A$, $n \in \mathbb{N}$), and then the element $m/g^l \in M_g$ is mapped to $m/g^l = h^l m/h^l g^l = h^l m/f^{nl} \in M_f$. In the case of $D(f) = D(g)$ we easily see $M_f \simeq M_g$ by considering the inverse.

Theorem A.2.1.
(i) *For an A-module M there exists a unique sheaf $\widetilde{M}$ on $V = \operatorname{Specm} A$ such that for any principal open subset $D(f)$ we have $\widetilde{M}(D(f)) = M_f$, and the restriction homomorphism $\widetilde{M}(D(f)) \to \widetilde{M}(D(g))$ for $D(f) \subset D(g)$ is given by r_f^g.*
(ii) *The sheaf $\mathcal{O}_V := \widetilde{A}$ is naturally a sheaf of k-algebras on V.*
(iii) *For an A-module M the sheaf $\widetilde{M}$ is naturally a sheaf of $\mathcal{O}_V$-module. The stalk of $\widetilde{M}$ at $x \in V$ is given by*

$$M_{\mathfrak{m}_x} = (A \setminus \mathfrak{m}_x)^{-1}M = \varinjlim_{f(x)\neq 0} M_f.$$

This is a module over the local ring $\mathcal{O}_{V,x} := A_{\mathfrak{m}_x} = (A \setminus \mathfrak{m}_x)^{-1}A$.

The key point of the proof is the fact that the functor $D(f) \mapsto M_f$ on the category of principal open subsets $\{D(f) \mid f \in A\}$ satisfies the "axioms of sheaves (for a basis of open subsets)," which is assured by the next lemma.

Lemma A.2.2. *Assume that the condition $\langle f_1, f_2, \dots, f_r\rangle \ni 1$ is satisfied in the ring A. Then for an A-module M we have an exact sequence*

$$0 \longrightarrow M \xrightarrow{\ \alpha\ } \prod_{i=1}^{r} M_{f_i} \xrightarrow{\ \beta\ } \prod_{i,j} M_{f_i f_j},$$

where the last arrow maps $(s_i)_{i=1}^r$ to $s_i - s_j \in M_{f_i f_j}$ $(1 \le i, j \le r)$.

Proof. We first show that for any $N \in \mathbb{N}$ there exists $g_1, \dots, g_r$ satisfying $\sum_i g_i f_i^N = 1$. Note that our assumption $\langle f_1, f_2, \dots, f_r\rangle \ni 1$ is equivalent to saying that for any $\mathfrak{m} \in \operatorname{Specm} A$ there exists at least one element f_i such that $f_i \notin \mathfrak{m}$ ($\operatorname{Specm} A = \bigcup_{i=1}^r D(f_i)$). If $f_i \notin \mathfrak{m}$, then we have $f_i^N \notin \mathfrak{m}$ for any N since $\mathfrak{m}$ is a prime ideal. Therefore, we get $\langle f_1^N, f_2^N, \dots, f_r^N\rangle \ni 1$, and the assertion is proved.

Let us show the injectivity of α. Assume $m \in \operatorname{Ker} \alpha$. Then there exists $N \gg 0$ such that $f_i^N m = 0$ $(1 \le i \le r)$. Combining it with the equality $\sum_i g_i f_i^N = 1$ we get $m = \sum_i g_i f_i^N m = 0$.

Next assume that $(m_l) \in \prod_l M_{f_l} \in \operatorname{Ker} \beta$. We will show that there is an element $m \in M$ such that $\alpha(m) = (m_l)$. It follows from our assumption that $m_i = m_j$ in $M_{f_i f_j}$ $(1 \le i, j \le r)$. This is equivalent to saying that $(f_i f_j)^N (m_i - m_j) = 0$ $(1 \le i, j \le r)$ for a large number N. That is, $f_j^N f_i^N m_i = f_i^N f_j^N m_j$. Now set $m = \sum_{i=1}^r g_i (f_i^N m_i)$ $(\sum g_i f_i^N = 1)$. Then for any $1 \le l \le r$ we have $f_l^N m = f_l^N \sum_i g_i (f_i^N m_i) = \sum_i g_i f_i^N (f_l^N m_i) = \sum_i g_i f_i^N f_l^N m_l = f_l^N m_l$ and $f_l^N (m - m_l) = 0$. Hence we have $\alpha(m) = (m_l)$. $\square$

For a general open subset $U = \bigcup_{i=1}^{r} D(f_i)$ of U we have

$$\Gamma(U, \widetilde{M}) = \left\{(s_i) \in M_{f_i} \mid s_i = s_j \text{ in } M_{f_i f_j} \ (1 \le i, j \le r)\right\}.$$

In particular, for $\mathcal{O}_V = \widetilde{A}$ we have

$$\Gamma(U, \mathcal{O}_V) = \{f : U \to k \mid \text{for each point of } U,$$
$$\text{there is an open neighborhood } D(g) \text{ such that } f|_{D(g)} \in A_g.\}$$

Let $(X, \mathcal{O}_X)$ be a pair of a topological space X and a sheaf $\mathcal{O}_X$ of k-algebras on X consisting of certain k-valued functions. We say that the pair $(X, \mathcal{O}_X)$ (or simply X) is an *affine variety* if $(X, \mathcal{O}_X)$ is isomorphic to some $(V, \mathcal{O}_V)$ ($V =$ Specm A, $\mathcal{O}_V = \widetilde{A}$) in the sense that there exists a homeomorphism $\phi : X \xrightarrow{\sim} V$ such that the correspondence $f \mapsto f \circ \phi$ gives an isomorphism $\Gamma(\phi(U), \mathcal{O}_V) \xrightarrow{\sim} \Gamma(U, \mathcal{O}_X)$ for any open subset U of V. In this case we have a natural isomorphism $\phi^\sharp : \phi^{-1}\mathcal{O}_V \xrightarrow{\sim} \mathcal{O}_X$ of a sheaf of k-algebras. In particular, we have an isomorphism $\phi_x^\sharp : \mathcal{O}_{V,\phi(x)} \xrightarrow{\sim} \mathcal{O}_{X,x}$ of local rings for any $x \in X$.

A.3 Algebraic varieties

Let $(X, \mathcal{O}_X)$ be a pair of a topological space X and a sheaf $\mathcal{O}_X$ of k-algebras consisting of certain k-valued functions. We say that the pair $(X, \mathcal{O}_X)$ (or simply X) is called a *prevariety* over k if it is locally an affine variety (i.e., if for any point $x \in X$ there is an open neighborhood $U \ni x$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine variety). In such cases, we call the sheaf $\mathcal{O}_X$ the *structure sheaf* of X and sections of $\mathcal{O}_X$ are called *regular functions*. A morphism $\phi : X \to Y$ between prevarieties X, Y is a continuous map so that for any open subset U of Y and any $f \in \Gamma(U, \mathcal{O}_Y)$ we have $f \circ \phi \in \Gamma(f^{-1}U, \mathcal{O}_X)$.

A prevariety X is called an *algebraic variety* if it is *quasi-compact* and *separated*. Let us explain more precisely these two conditions.

The quasi-compactness is a purely topological condition. We say that a topological space X is quasi-compact if any open covering of X admits a finite subcovering. We say "quasi" because X is not assumed to be Hausdorff. In fact, an algebraic variety is not Hausdorff unless it consists of finitely many points. In particular, an algebraic variety X is covered by finitely many affine varieties.

The condition of separatedness plays an alternative role to that of the Hausdorff condition. To define it we need the notion of products of prevarieties.

We define the product $V_1 \times V_2$ of two affine varieties V_1, V_2 to be the affine variety associated to the tensor product $A_1 \otimes_k A_2$ ($A_i = \Gamma(V_i, \mathcal{O}_{V_i})$) of k-algebras. This operation is possible thanks to the following proposition.

Proposition A.3.1. *For any pair A_1, A_2 of finitely generated reduced k-algebras the tensor product $A_1 \otimes_k A_2$ is also reduced.*

Proof. Consider the embeddings $V_i \subset \mathbb{A}^{n_i}$ $(i = 1, 2)$ and $V_1 \times V_2 \subset \mathbb{A}^{n_1+n_2}$. Next, define the ring $k[V_1 \times V_2]$ to be the restriction of the polynomial ring $k[\mathbb{A}^{n_1+n_2}]$ to $V_1 \times V_2$. Then the restriction map $\varphi : k[V_1] \otimes_k k[V_2] \to k[V_1 \times V_2]$ is bijective (the right-hand side is obviously reduced). The surjectivity is clear. We can also prove the injectivity, observing that for any linearly independent elements $\{f_i\}$ (resp. $\{g_j\}$) in $k[V_1]$ (resp. $k[V_2]$) over k the elements $\{\varphi(f_i g_j)\}$ are again linearly independent in $k[V_1 \times V_2]$. $\square$

By definition the product $V_1 \times V_2$ of affine varieties V_1, V_2 has a finer topology than the usual product topology.

Now let us give the definition of the product of two prevarieties X, Y. Let $X = \bigcup_i V_i$, $Y = \bigcup_j U_j$ be affine open coverings of X and Y, respectively. Then the product set $X \times Y$ is covered by $\{ V_i \times U_j \}_{(i,j)}$ $(X \times Y = \bigcup_{(i,j)} V_i \times U_j)$. Note that we regard the product sets $V_i \times U_j$ as affine varieties by the above arguments. Namely, the structure sheaf $\mathcal{O}_{V_i \times U_j}$ is associated to the tensor product $\Gamma(V_i, \mathcal{O}_{V_i}) \otimes_k \Gamma(U_j, \mathcal{O}_{U_j})$. Then we can glue $(V_i \times U_j, \mathcal{O}_{V_i \times U_j})$ to get a topology of $X \times Y$ and a sheaf $\mathcal{O}_{X\times Y}$ of k-algebras consisting of certain k-valued functions on $X \times Y$, for which $(X \times Y, \mathcal{O}_{X\times Y})$ is a prevariety. This prevariety is called the product of two prevarieties X and Y. It is the "fiber product" in the category of prevarieties.

Using these definitions, we say that a prevariety X is *separated* if the diagonal set $\Delta = \{(x, x) \in X \times X\}$ is closed in the self-product $X \times X$.

Let us add some remarks.

(i) If $\phi : X \to Y$ is a morphism of algebraic varieties, then its graph $\Gamma_\phi = \{ (x, \phi(x)) \in X \times Y \}$ is a closed subset of $X \times Y$.
(ii) Affine varieties are separated (hence they are algebraic varieties).

A.4 Quasi-coherent sheaves

Let $(X, \mathcal{O}_X)$ be an algebraic variety. We say that a sheaf F of $\mathcal{O}_X$-module (hereafter, we simply call F an $\mathcal{O}_X$-module) is *quasi-coherent* over $\mathcal{O}_X$ if for each point $x \in X$ there exists an affine open neighborhood $V \ni x$ and a module M_V over $A_V = \mathcal{O}_X(V)$ such that $F|_V \simeq \widetilde{M}_V$ as $\mathcal{O}_V$-modules ($\widetilde{M}_V$ is an $\mathcal{O}_V$-module on $V = \operatorname{Specm} A_V$ constructed from the A_V-module M_V by Theorem A.2.1). If, moreover, every M_V is finitely generated over A_V, we say that F is *coherent* over $\mathcal{O}_X$. The next theorem is fundamental.

Theorem A.4.1.

(i) (Chevalley) *The following conditions on an algebraic variety X are equivalent:*
  (a) *X is an affine variety.*
  (b) *For any quasi-coherent $\mathcal{O}_X$-module F we have $H^i(X, F) = 0$ $(i \geq 1)$.*
  (c) *For any quasi-coherent $\mathcal{O}_X$-module F we have $H^1(X, F) = 0$.*
(ii) *Let X be an affine variety and $A = \mathcal{O}_X(X)$ its coordinate ring. Then the functor*

$$\operatorname{Mod}(A) \ni M \longmapsto \widetilde{M} \in \operatorname{Mod}_{qc}(\mathcal{O}_X)$$

from the category $\mathrm{Mod}(A)$ *of* A*-modules to the category* $\mathrm{Mod}_{qc}(\mathcal{O}_X)$ *of quasi-coherent* $\mathcal{O}_X$*-modules induces an equivalence of categories. Namely, any quasi-coherent* $\mathcal{O}_X$*-module is isomorphic to the sheaf* $\widetilde{M}$ *constructed from an* A*-module* M*, and there exists an isomorphism*

$$\mathrm{Hom}_A(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}).$$

In particular, for a quasi-coherent $\mathcal{O}_X$*-module* F *we have*

$$F \simeq \widetilde{F(X)}.$$

By Theorem A.4.1 (ii) local properties of a quasi-coherent $\mathcal{O}_X$-module F can be deduced from those of the $\mathcal{O}_X(V)$-module $F(V)$ for an affine neighborhood V. For example, F is locally free (resp. coherent) if and only if every point $x \in X$ has an affine open neighborhood V such that $F(V)$ is free (resp. finitely generated) over $\mathcal{O}_X(V)$.

Let us give some examples.

Example A.4.2. *Tangent sheaf* Θ_X and *cotangent sheaf* Ω_X^1 (In this book, Ω_X stands for the sheaf $\Omega_X^n := \bigwedge^n \Omega_X^1$ ($n = \dim X$) of differential forms of top degree). We denote by $\mathcal{E}nd_k\mathcal{O}_X$ the sheaf of k-linear endomorphisms of $\mathcal{O}_X$. We say that a section $\theta \in (\mathcal{E}nd_k\mathcal{O}_X)(X)$ is a vector field on X if for each open subset $U \subset X$ the restriction $\theta(U) := \theta|_U \in (\mathcal{E}nd_k\mathcal{O}_X)(U)$ satisfies the condition

$$\theta(U)(fg) = \theta(U)(f)g + f\theta(U)(g) \quad (f, g \in \mathcal{O}_X(U)).$$

For an open subset U of X, denote the set of the vector fields θ ($\in (\mathcal{E}nd_k\mathcal{O}_U)(U)$) on U by $\Theta(U)$. Then $\Theta(U)$ is an $\mathcal{O}_X(U)$-module, and the presheaf $U \mapsto \Theta(U)$ turns out to be a sheaf (of $\mathcal{O}_X$-modules). We denote this sheaf by Θ_X and call it the *tangent sheaf* of X. When U is affine, we have $\Theta_U \simeq \widetilde{\mathrm{Der}_k(A)}$ for $A = \mathcal{O}_X(U)$, where the right-hand side is the $\mathcal{O}_U$-module associated to the A-module

$$\mathrm{Der}_k(A) := \{\theta \in \mathrm{End}_k A \mid \theta(fg) = \theta(f)g + f\theta(g) \ (f, g \in A)\}$$

of the derivations of A over k. It follows from this fact that Θ_X is a coherent $\mathcal{O}_X$-module. Indeed, if $A = k[X]/I$ (here $k[X] = k[X_1, X_2, \ldots, X_n]$ is a polynomial ring), then we have

$$\mathrm{Der}_k(k[X]) = \bigoplus_{i=1}^{n} k[X]\partial_i \quad \left(\partial_i := \frac{\partial}{\partial X_i}\right)$$

(free $k[X]$-module of rank n) and

$$\mathrm{Der}_k(A) \simeq \{\theta \in \mathrm{Der}_k(k[X]) \mid \theta(I) \subset I\}.$$

Hence $\mathrm{Der}_k(A)$ is finitely generated over A.

On the other hand we define the *cotangent sheaf* of X by $\Omega_X^1 := \delta^{-1}(\mathcal{J}/\mathcal{J}^2)$, where $\delta : X \to X \times X$ is the diagonal embedding, $\mathcal{J}$ is the ideal sheaf of $\delta(X)$ in $X \times X$ defined by

$$\mathcal{J}(V) = \{f \in \mathcal{O}_{X \times X}(V) \mid f(V \cap \delta(X)) = \{0\}\}$$

for any open subset V of $X \times X$, and δ^{-1} stands for the sheaf-theoretical inverse image functor. Sections of the sheaf Ω_X^1 are called differential forms. By the canonical morphism $\mathcal{O}_X \to \delta^{-1}\mathcal{O}_{X \times X}$ of sheaf of k-algebras Ω_X^1 is naturally an $\mathcal{O}_X$-module. We have a morphism $d : \mathcal{O}_X \to \Omega_X^1$ of $\mathcal{O}_X$-modules defined by $df = f \otimes 1 - 1 \otimes f \mod \delta^{-1}\mathcal{J}^2$. It satisfies $d(fg) = d(f)g + f(dg)$ for any $f, g \in \mathcal{O}_X$. For $\alpha \in \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$ we have $\alpha \circ d \in \Theta_X$, which gives an isomorphism $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) \simeq \Theta_X$ of $\mathcal{O}_X$-modules.

A.5 Smoothness, dimensions and local coordinate systems

Let x be a point of an algebraic variety X. We say that X is *smooth* (or non-singular) at $x \in X$ if the stalk $\mathcal{O}_{X,x}$ is a regular local ring. This condition is satisfied if and only if the cotangent sheaf Ω_X^1 is a free $\mathcal{O}_X$-module on an open neighborhood of x. The smooth points of X form an open subset of X. Let us denote this open subset by X_{reg}. An algebraic variety is called *smooth* (or non-singular) if all of its points are smooth. It is equivalent to saying that Ω_X^1 is a locally free $\mathcal{O}_X$-module. In this case Θ_X is also locally free of the same rank by $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) \simeq \Theta_X$. For a smooth point $x \in X$ the *dimension* of X at x is defined by

$$\dim_x X := \operatorname{rank}_{\mathcal{O}_{X,x}} \Theta_{X,x} = \operatorname{rank}_{\mathcal{O}_{X,x}} \Omega_{X,x}^1,$$

where $\Theta_{X,x}$ and $\Omega_{X,x}^1$ are the stalks of Θ_X and Ω_X^1 at x, respectively. It also coincides with the Krull dimension of the regular local ring $\mathcal{O}_{X,x}$. We define the *dimension* of X to be the locally constant function on X_{reg} defined by

$$(\dim X)(x) := \dim_x X.$$

If X is irreducible, the value $\dim_x X$ does not depend on the point $x \in X_{\mathrm{reg}}$.

Theorem A.5.1. *Let X be a smooth algebraic variety of dimension n. Then for each point $p \in X$, there exist an affine open neighborhood V of p, regular functions $x_i \in k[V] = \mathcal{O}_X(V)$, and vector fields $\partial_i \in \Theta_X(V)$ $(1 \le i \le n)$ satisfying the conditions*

$$\begin{cases} [\partial_i, \partial_j] = 0, \quad \partial_i(x_j) = \delta_{ij} \ (1 \le i, \ j \le n) \\ \Theta_V = \bigoplus_{i=1}^n \mathcal{O}_V \, \partial_i. \end{cases}$$

Moreover, we can choose the functions $x_1, x_2, \dots, x_n$ so that they generate the maximal ideal $\mathfrak{m}_p$ of the local ring $\mathcal{O}_{X,p}$ at p.

Proof. By the theory of regular local rings there exist $n(= \dim_x X)$ functions $x_1, \dots, x_n \in \mathfrak{m}_p$ generating the ideal $\mathfrak{m}_p$. Then $dx_1, \dots, dx_n$ is a basis of the free $\mathcal{O}_{X,p}$-module $\Omega^1_{X,p}$. Hence we can take an affine open neighborhood V of p such that $\Omega^1_X(V)$ is a free module with basis $dx_1, \dots, dx_n$ over $\mathcal{O}_X(V)$. Taking the dual basis $\partial_1, \dots, \partial_n \in \Theta_X(V) \simeq \mathrm{Hom}_{\mathcal{O}_X(V)}(\Omega^1_X(V), \mathcal{O}_X(V))$ we get $\partial_i(x_j) = \delta_{ij}$. Write $[\partial_i, \partial_j]$ as $[\partial_i, \partial_j] = \sum_{l=1}^n g^l_{ij}\partial_l$ $(g^l_{ij} \in \mathcal{O}_X(V))$. Then we have $g^l_{ij} = [\partial_i, \partial_j]x_l = \partial_i\partial_j x_l - \partial_j\partial_i x_l = 0$. Hence $[\partial_i, \partial_j] = 0$. □

Definition A.5.2. The set $\{ x_i, \partial_i \mid 1 \le i \le n \}$ defined over an affine open neighborhood of p satisfying the conditions of Theorem A.5.1 is called a *local coordinate system* at p.

It is clear that this notion is a counterpart of the local coordinate system of a complex manifold. Note that the local coordinate system $\{x_i\}$ defined on an affine open subset V of a smooth algebraic variety does not necessarily separate the points in V. We only have an *étale morphism* $V \to k^n$ given by $q \mapsto (x_1(q), \dots, x_n(q))$.

We have the following relative version of Theorem A.5.1.

Theorem A.5.3. *Let Y be a smooth subvariety of a smooth algebraic variety X. Assume that* $\dim_p Y = m$, $\dim_p X = n$ *at $p \in Y$. Then we can take an affine open neighborhood V of p in X and a local coordinate system $\{x_i, \partial_i \mid 1 \le i \le n\}$ such that $Y \cap V = \{q \in V \mid x_i(q) = 0 \ (m < i \le n)\}$ (hence $k[Y \cap V] = k[V]/\sum_{i>m} k[V]x_i$) and $\{x_i, \partial_i \mid 1 \le i \le m\}$ is a local coordinate system of $Y \cap V$. Here we regard ∂_i $(1 \le i \le m)$ as derivations on $k[Y \cap V]$ by using the relation $\partial_i x_j = 0$ $(j > m)$.*

Proof. The result follows from the fact that smooth $\Rightarrow$ locally complete intersection. □

B

Derived Categories and Derived Functors

In this appendix, we give a brief account of the theory of derived categories without proofs. The basic references are Hartshorne [Ha1], Verdier [V2], Borel et al. [Bor3, Chapter 1], Gelfand–Manin [GeM], Kashiwara–Schapira [KS2], [KS4]. We especially recommend the reader to consult Kashiwara–Schapira [KS4] for details on this subject.

B.1 Motivation

The notion of derived categories is indispensable if one wants to fully understand the theory of D-modules. Many operations of D-modules make sense only in derived categories, and the Riemann–Hilbert correspondence, which is the main subject of Part I, cannot be formulated without this notion. Derived categories were introduced by A. Grothendieck [Ha1], [V2]. We hear that M. Sato arrived at the same notion independently in his way of creating algebraic analysis. In this section we explain the motivation of the theory of derived categories and give an outline of the theory.

Let us first recall the classical definition of right derived functors. Let $\mathcal{C}$, $\mathcal{C}'$ be abelian categories and $F : \mathcal{C} \to \mathcal{C}'$ a left exact functor. Assume that the category $\mathcal{C}$ has *enough injectives*, i.e., for any object $X \in \mathrm{Ob}(\mathcal{C})$ there exists a monomorphism $X \to I$ into an injective object I. Then for any $X \in \mathrm{Ob}(\mathcal{C})$ there exists an exact sequence

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

such that I^k is an injective object for any $k \in \mathbb{Z}$. Such an exact sequence is called an *injective resolution* of X. Next consider the complex

$$I^\cdot = \left[0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow I^3 \cdots\right]$$

in $\mathcal{C}$ and apply to it the functor F. Then we obtain a complex

$$F(I^\cdot) = \left[0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow F(I^3) \cdots\right]$$

in $\mathcal{C}'$. As is well known in homological algebra the nth cohomology group of $F(I^\cdot)$:

$$H^n F(I^{\cdot}) = \mathrm{Ker}[F(I^n) \longrightarrow F(I^{n+1})]/\,\mathrm{Im}[F(I^{n-1}) \longrightarrow F(I^n)]$$

does not depend on the choice of injective resolutions, and is uniquely determined up to isomorphisms. Set $R^n F(X) = H^n F(I^{\cdot}) \in \mathrm{Ob}(\mathcal{C}')$. Then $R^n F$ defines a functor $R^n F : \mathcal{C} \to \mathcal{C}'$. We call $R^n F$ the nth *derived functor* of F. For $n < 0$ we have $R^n F = 0$ and $R^0 F = F$. Similar construction can be applied also to complexes in $\mathcal{C}$ which are bounded below. Indeed, consider a complex

$$X^{\cdot} = \left[\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^k \longrightarrow X^{k+1} \longrightarrow X^{k+2} \longrightarrow \cdots\right]$$

in $\mathcal{C}$ such that $X^i = 0$ for any $i < k$. Then there exists a complex

$$I^{\cdot} = \left[\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow I^k \longrightarrow I^{k+1} \longrightarrow I^{k+2} \longrightarrow \cdots\right]$$

of injective objects in $\mathcal{C}$ and a *quasi-isomorphism* $f : X^{\cdot} \to I^{\cdot}$, i.e., a morphism of complexes $f : X^{\cdot} \to I^{\cdot}$ which induces an isomorphism $H^i(X^{\cdot}) \simeq H^i(I^{\cdot})$ for any $i \in \mathbb{Z}$. We call $I^{\cdot}$ an injective resolution of $X^{\cdot}$. Since the injective resolution $I^{\cdot}$ and the complex $F(I^{\cdot})$ are uniquely determined up to homotopy equivalences, the nth cohomology group $H^n F(I^{\cdot}) \in \mathrm{Ob}(\mathcal{C}')$ of $F(I^{\cdot})$ is uniquely determined up to isomorphisms. Set $R^n F(X^{\cdot}) = H^n F(I^{\cdot}) \in \mathrm{Ob}(\mathcal{C}')$. If we introduce the homotopy category $K^+(\mathcal{C})$ of complexes in $\mathcal{C}$ which are bounded below (for the definition see Section B.2 below), this gives a functor $R^n F : K^+(\mathcal{C}) \to \mathcal{C}'$. Such derived functors for "complexes" are frequently used in algebraic geometry.

However, this classical construction of derived functors has some defects. Since we treat only cohomology groups $\{H^n F(I^{\cdot})\}_{n\in\mathbb{Z}}$ of $F(I^{\cdot})$, we lose various important information of the complex $F(I^{\cdot})$ itself. Moreover, the above construction of derived functors is not convenient for the composition of functors. For example, let $G : \mathcal{C}' \longrightarrow \mathcal{C}''$ be another left exact functor. Then, for $X \in \mathrm{Ob}(\mathcal{C})$ the equality $R^{i+j}(G \circ F)(X) = R^i G(R^j F(X))$ cannot be expected in general. The theory of spectral sequences was invented as a remedy for such problems, but the best way is to treat everything at the level of complexes without taking cohomology groups. Namely, we want to introduce certain categories of complexes and define a lifting RF (which will be also called a derived functor of F) of $R^n F$'s between such categories of complexes. This is the theory of derived categories. Indeed, the language of derived categories allows one to formulate complicated relations among various functors in a very beautiful and efficient way.

Now let us briefly explain the construction of derived categories. Let $C(\mathcal{C})$ be the category of complexes in $\mathcal{C}$. Since the injective resolutions $f : X^{\cdot} \to I^{\cdot}$ of $X^{\cdot}$ are just quasi-isomorphisms in $C(\mathcal{C})$, we should change the family of morphisms of $C(\mathcal{C})$ so that quasi-isomorphisms are isomorphisms in the new category. For this purpose we use a general theory of localizations of categories (see Section B.4). However, this localization cannot be applied directly to the category $C(\mathcal{C})$. So we first define the homotopy category $K(\mathcal{C})$ by making homotopy equivalences in $C(\mathcal{C})$ invertible, and then apply the localization. The derived category $D(\mathcal{C})$ thus obtained is an additive category and not an abelian category any more. Therefore, we cannot consider short exact sequences $0 \to X^{\cdot} \to Y^{\cdot} \to Z^{\cdot} \to 0$ of complexes in $D(\mathcal{C})$ as in $C(\mathcal{C})$.

Nevertheless, we can define the notion of distinguished triangles in $D(\mathcal{C})$ which is a substitute for that of short exact sequences of complexes. In other words, the derived category $D(\mathcal{C})$ is a triangulated category in the sense of Definition B.3.6. As in the case of short exact sequences in $C(\mathcal{C})$, from a distinguished triangle in $D(\mathcal{C})$ we can deduce a cohomology long exact sequence in $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor and assume that $\mathcal{C}$ has enough injectives. Denote by $D^+(\mathcal{C})$ (resp. $D^+(\mathcal{C}')$) the full subcategory of $D(\mathcal{C})$ (resp. $D(\mathcal{C})$) consisting of complexes which are bounded below. Then we can construct a (right) derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ of F, which sends distinguished triangles to distinguished triangles. If we identify an object X of $\mathcal{C}$ with a complex

$$\left[\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots\right]$$

concentrated in degree 0 and hence with an object of $D^+(\mathcal{C})$, we have an isomorphism $H^n(RF(X)) \simeq R^nF(X)$ in $\mathcal{C}'$. From this we see that the new derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ extends classical ones. Moreover, this new construction of derived functors turns out to be very useful for the compositions of various functors. Let $G : \mathcal{C}' \to \mathcal{C}''$ be another left exact functor and assume that $\mathcal{C}'$ has enough injectives. Then also the derived functors RG and $R(G \circ F)$ exist and (under a sufficiently weak hypothesis) we have a beautiful composition rule $RG \circ RF = R(G \circ F)$. Since in the theory of D-modules we frequently use the compositions of various derived functors, such a nice property is very important.

B.2 Categories of complexes

Let $\mathcal{C}$ be an abelian category, e.g., the category of R-modules over a ring R, the category $\mathrm{Sh}(T_0)$ of sheaves on a topological space T_0. Denote by $C(\mathcal{C})$ the category of complexes in $\mathcal{C}$. More precisely, an object $X^\cdot$ of $C(\mathcal{C})$ consists of a family of objects $\{X^n\}_{n\in\mathbb{Z}}$ in $\mathcal{C}$ and that of morphisms $\{d^n_{X^\cdot} : X^n \longrightarrow X^{n+1}\}_{n\in\mathbb{Z}}$ in $\mathcal{C}$ satisfying $d^{n+1}_{X^\cdot} \circ d^n_{X^\cdot} = 0$ for any $n \in \mathbb{Z}$. A morphism $f : X^\cdot \longrightarrow Y^\cdot$ in $C(\mathcal{C})$ is a family of morphisms $\{f^n : X^n \longrightarrow Y^n\}_{n\in\mathbb{Z}}$ in $\mathcal{C}$ satisfying the condition $d^n_{Y^\cdot} \circ f^n = f^{n+1} \circ d^n_{X^\cdot}$ for any $n \in \mathbb{Z}$. Namely, a morphism in $C(\mathcal{C})$ is just a chain map between two complexes in $\mathcal{C}$. For an object $X^\cdot \in \mathrm{Ob}(C(\mathcal{C}))$ of $C(\mathcal{C})$ we say that $X^\cdot$ is bounded below (resp. bounded above, resp. bounded) if it satisfies the condition $X^i = 0$ for $i \ll 0$ (resp. $i \gg 0$, resp. $|i| \gg 0$). We denote by $C^+(\mathcal{C})$ (resp. $C^-(\mathcal{C})$, resp. $C^b(\mathcal{C})$) the full subcategory of $C(\mathcal{C})$ consisting of objects which are bounded below (resp. bounded above, resp. bounded). These are naturally abelian categories. Moreover, we identify $\mathcal{C}$ with the full subcategory of $C(\mathcal{C})$ consisting of complexes concentrated in degree 0.

Definition B.2.1. We say that a morphism $f : X^\cdot \to Y^\cdot$ in $C(\mathcal{C})$ is a *quasi-isomorphism* if it induces an isomorphism $H^n(X^\cdot) \simeq H^n(Y^\cdot)$ between cohomology groups for any $n \in \mathbb{Z}$.

Definition B.2.2.

(i) For a complex $X^{\cdot} \in \mathrm{Ob}(C(\mathcal{C}))$ with differentials $d_{X^{\cdot}}^{n} : X^{n} \to X^{n+1}$ $(n \in \mathbb{Z})$ and an integer $k \in \mathbb{Z}$, we define the *shifted complex* $X^{\cdot}[k]$ by

$$\begin{cases} X^{n}[k] = X^{n+k}, \\ d_{X^{\cdot}[k]}^{n} = (-1)^{k} d_{X^{\cdot}}^{n+k} : X^{n}[k] = X^{n+k} \longrightarrow X^{n+1}[k] = X^{n+k+1}. \end{cases}$$

(ii) For a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $C(\mathcal{C})$, we define the *mapping cone* $M_{f}^{\cdot} \in \mathrm{Ob}(C(\mathcal{C}))$ by

$$\begin{cases} M_{f}^{n} = X^{n+1} \oplus Y^{n}, \\ d_{M_{f}^{\cdot}}^{n} : M_{f}^{n} = X^{n+1} \oplus Y^{n} \quad \longrightarrow \quad M_{f}^{n+1} = X^{n+2} \oplus Y^{n+1} \\ \qquad\qquad\qquad \Psi \qquad\qquad\qquad\qquad\qquad\qquad \Psi \\ \qquad\qquad (x^{n+1}, y^{n}) \quad \longmapsto \quad (-d_{X^{\cdot}}^{n+1}(x^{n+1}), f^{n+1}(x^{n+1}) + d_{Y^{\cdot}}^{n}(y^{n})). \end{cases}$$

There exists a natural short exact sequence

$$0 \longrightarrow Y^{\cdot} \xrightarrow{\alpha(f)} M_{f}^{\cdot} \xrightarrow{\beta(f)} X^{\cdot}[1] \longrightarrow 0$$

in $C(\mathcal{C})$, from which we obtain the cohomology long exact sequence

$$\cdots \longrightarrow H^{n-1}(M_{f}^{\cdot}) \longrightarrow H^{n}(X^{\cdot}) \longrightarrow H^{n}(Y^{\cdot}) \longrightarrow H^{n}(M_{f}^{\cdot}) \longrightarrow \cdots$$

in $\mathcal{C}$. Since the connecting homomorphisms $H^{n}(X^{\cdot}) \longrightarrow H^{n}(Y^{\cdot})$ in this long exact sequence coincide with $H^{n}(f) : H^{n}(X^{\cdot}) \longrightarrow H^{n}(Y^{\cdot})$ induced by $f : X^{\cdot} \to Y^{\cdot}$, we obtain the following useful result.

Lemma B.2.3. *A morphism $f : X^{\cdot} \to Y^{\cdot}$ in $C(\mathcal{C})$ is a quasi-isomorphism if and only if $H^{n}(M_{f}^{\cdot}) = 0$ for any $n \in \mathbb{Z}$.*

Definition B.2.4. For a complex $X^{\cdot} \in \mathrm{Ob}(C(\mathcal{C}))$ in $\mathcal{C}$ and an integer $k \in \mathbb{Z}$ we define the *truncated complexes* by

$$\tau^{\leqslant k} X^{\cdot} = \tau^{<k+1} X^{\cdot} := [\cdots \to X^{k-1} \to Z^{k} = \mathrm{Ker}\, d_{X^{\cdot}}^{k} \to 0 \to 0 \to \cdots],$$
$$\tau^{\geqslant k+1} X^{\cdot} = \tau^{>k} X^{\cdot} := [\cdots \to 0 \to 0 \to B^{k+1} = \mathrm{Im}\, d_{X^{\cdot}}^{k} \to X^{k+1} \to \cdots].$$

For $X^{\cdot} \in \mathrm{Ob}(C(\mathcal{C}))$ there exists a short exact sequence

$$0 \longrightarrow \tau^{\leqslant k} X^{\cdot} \longrightarrow X^{\cdot} \longrightarrow \tau^{>k} X^{\cdot} \longrightarrow 0$$

in $C(\mathcal{C})$ for each $k \in \mathbb{Z}$. Note that the complexes $\tau^{\leqslant k} X^{\cdot}$ and $\tau^{>k} X^{\cdot}$ satisfy the following conditions, which explain the reason why we call them "truncated" complexes:

$$H^{j}(\tau^{\leqslant k} X^{\cdot}) \simeq \begin{cases} H^{j}(X^{\cdot}) & j \leq k \\ 0 & j > k, \end{cases}$$

$$H^{j}(\tau^{>k} X^{\cdot}) \simeq \begin{cases} H^{j}(X^{\cdot}) & j > k \\ 0 & j \leq k. \end{cases}$$

B.3 Homotopy categories

In this section, before constructing derived categories, we define homotopy categories. Derived categories are obtained by applying a localization of categories to homotopy categories. In order to apply the localization, we need a family of morphisms called a multiplicative system (see Definition B.4.2 below). But the quasi-isomorphisms in $C(\mathcal{C})$ do not form a multiplicative system. Therefore, for the preparation of the localization, we define the *homotopy categories* $K^{\#}(\mathcal{C})$ ($\# = \emptyset, +, -, b$) of an abelian category $\mathcal{C}$ as follows. First recall that a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $C^{\#}(\mathcal{C})$ ($\# = \emptyset, +, -, b$) is *homotopic to* 0 (we write $f \sim 0$ for short) if there exists a family $\{s_n : X^n \to Y^{n-1}\}_{n\in\mathbb{Z}}$ of morphisms in $\mathcal{C}$ such that $f^n = s^{n+1} \circ d_{X^{\cdot}}^n + d_{Y^{\cdot}}^{n-1} \circ s^n$ for any $n \in \mathbb{Z}$:

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \arrow[r, "d_{X^{\cdot}}^{n-1}"] \arrow[d, "f^{n-1}"'] & X^n \arrow[r, "d_{X^{\cdot}}^{n}"] \arrow[d, "f^n"'] \arrow[dl, "s^n"'] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"'] \arrow[dl, "s^{n+1}"'] & \cdots \\
\cdots \arrow[r] & Y^{n-1} \arrow[r, "d_{Y^{\cdot}}^{n-1}"] & Y^n \arrow[r, "d_{Y^{\cdot}}^{n}"] & Y^{n+1} \arrow[r] & \cdots .
\end{tikzcd}$$

We say also that two morphisms $f, g \in \mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot})$ in $C^{\#}(\mathcal{C})$ are *homotopic* (we write $f \sim g$ for short) if the difference $f - g \in \mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot})$ is homotopic to 0.

Definition B.3.1. For $\# = \emptyset, +, -, b$ we define the homotopy category $K^{\#}(\mathcal{C})$ of $\mathcal{C}$ by

$$\begin{cases} \mathrm{Ob}(K^{\#}(\mathcal{C})) = \mathrm{Ob}(C^{\#}(\mathcal{C})), \\ \mathrm{Hom}_{K^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot}) = \mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot}) \big/ \mathrm{Ht}(X^{\cdot}, Y^{\cdot}), \end{cases}$$

where $\mathrm{Ht}(X^{\cdot}, Y^{\cdot})$ is a subgroup of $\mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot})$ defined by $\mathrm{Ht}(X^{\cdot}, Y^{\cdot}) = \big\{ f \in \mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot}) \mid f \sim 0 \big\}$.

The homotopy categories $K^{\#}(\mathcal{C})$ are not abelian, but they are still additive categories. We may regard the categories $K^{\#}(\mathcal{C})$ ($\# = +, -, b$) as full subcategories of $K(\mathcal{C})$. Moreover, $\mathcal{C}$ is naturally identified with the full subcategories of these homotopy categories consisting of complexes concentrated in degree 0. Since morphisms which are homotopic to 0 induce zero maps in cohomology groups, the additive functors $H^n : K^{\#}(\mathcal{C}) \to \mathcal{C}$ ($X^{\cdot} \longmapsto H^n(X^{\cdot})$) are well defined. We say that a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $K^{\#}(\mathcal{C})$ is a quasi-isomorphism if it induces an isomorphism $H^n(X^{\cdot}) \simeq H^n(Y^{\cdot})$ for any $n \in \mathbb{Z}$. Recall that a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $C^{\#}(\mathcal{C})$ is called a *homotopy equivalence* if there exists a morphism $g : Y^{\cdot} \to X^{\cdot}$ in $C^{\#}(\mathcal{C})$ such that $g \circ f \sim \mathrm{id}_{X^{\cdot}}$ and $f \circ g \sim \mathrm{id}_{Y^{\cdot}}$. Homotopy equivalences in $C^{\#}(\mathcal{C})$ are isomorphisms in $K^{\#}(\mathcal{C})$ and hence quasi-isomorphisms. As in the case of the categories $C^{\#}(\mathcal{C})$, we can also define truncation functors $\tau^{\geqslant k} : K(\mathcal{C}) \to K^{+}(\mathcal{C})$ and $\tau^{\leqslant k} : K(\mathcal{C}) \to K^{-}(\mathcal{C})$.

Since the homotopy category $K^{\#}(\mathcal{C})$ is not abelian, we cannot consider short exact sequences in it any more. So we introduce the notion of distinguished triangles in

$K^{\#}(\mathcal{C})$ which will be a substitute for that of short exact sequences in the derived category $D^{\#}(\mathcal{C})$.

Definition B.3.2.

(i) A sequence $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$ of morphisms in $K^{\#}(\mathcal{C})$ is called a *triangle*.

(ii) A *morphism of triangles* between two triangles $X_1^{\cdot} \longrightarrow Y_1^{\cdot} \longrightarrow Z_1^{\cdot} \longrightarrow X_1^{\cdot}[1]$ and $X_2^{\cdot} \longrightarrow Y_2^{\cdot} \longrightarrow Z_2^{\cdot} \longrightarrow X_2^{\cdot}[1]$ in $K^{\#}(\mathcal{C})$ is a commutative diagram

$$\begin{tikzcd}
X_1^{\cdot} \arrow[r] \arrow[d, "h"'] & Y_1^{\cdot} \arrow[r] \arrow[d] & Z_1^{\cdot} \arrow[r] \arrow[d] & X_1^{\cdot}[1] \arrow[d, "h[1]"'] \\
X_2^{\cdot} \arrow[r] & Y_2^{\cdot} \arrow[r] & Z_2^{\cdot} \arrow[r] & X_2^{\cdot}[1]
\end{tikzcd}$$

in $K^{\#}(\mathcal{C})$.

(iii) We say that a triangle $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$ in $K^{\#}(\mathcal{C})$ is a *distinguished triangle* if it is isomorphic to a mapping cone triangle $X_0^{\cdot} \xrightarrow{f} Y_0^{\cdot} \xrightarrow{\alpha(f)} M_f^{\cdot} \xrightarrow{\beta(f)} X_0^{\cdot}[1]$ associated to a morphism $f : X_0^{\cdot} \longrightarrow Y_0^{\cdot}$ in $C^{\#}(\mathcal{C})$, i.e., there exists a commutative diagram

$$\begin{tikzcd}
X_0^{\cdot} \arrow[r, "f"] \arrow[d, "\wr"'] & Y_0^{\cdot} \arrow[r, "\alpha(f)"] \arrow[d, "\wr"'] & M_f^{\cdot} \arrow[r, "\beta(f)"] \arrow[d, "\wr"'] & X_0^{\cdot}[1] \arrow[d, "\wr"'] \\
X^{\cdot} \arrow[r] & Y^{\cdot} \arrow[r] & Z^{\cdot} \arrow[r] & X^{\cdot}[1]
\end{tikzcd}$$

in which all vertical arrows are isomorphisms in $K^{\#}(\mathcal{C})$.

A distinguished triangle $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$ is sometimes denoted by $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \xrightarrow{+1}$ or by

$$\begin{tikzcd}
Y^{\cdot} \arrow[rr] & & Z^{\cdot} \arrow[dl, "{[+1]}"] \\
& X^{\cdot}. \arrow[ul] &
\end{tikzcd}$$

Proposition B.3.3. *The family of distinguished triangles in $\mathcal{C}_0 = K^{\#}(\mathcal{C})$ satisfies the following properties* (TR0) $\sim$ (TR5)*:*

(TR0) *A triangle which is isomorphic to a distinguished triangle is also distinguished.*

(TR1) *For any $X^{\cdot} \in \mathrm{Ob}(\mathcal{C}_0)$, $X^{\cdot} \xrightarrow{\mathrm{id}_{X^{\cdot}}} X^{\cdot} \longrightarrow 0 \longrightarrow X^{\cdot}[1]$ is a distinguished triangle.*

(TR2) *Any morphism* $f : X^{\cdot} \longrightarrow Y^{\cdot}$ *in* $\mathcal{C}_0$ *can be embedded into a distinguished triangle* $X^{\cdot} \xrightarrow{f} Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$.

(TR3) *A triangle* $X^{\cdot} \xrightarrow{f} Y^{\cdot} \xrightarrow{g} Z^{\cdot} \xrightarrow{h} X^{\cdot}[1]$ *in* $\mathcal{C}_0$ *is distinguished if and only if* $Y^{\cdot} \xrightarrow{g} Z^{\cdot} \xrightarrow{h} X^{\cdot}[1] \xrightarrow{-f[1]} Y^{\cdot}[1]$ *is distinguished.*

(TR4) *Given two distinguished triangles* $X_1^{\cdot} \xrightarrow{f_1} Y_1^{\cdot} \longrightarrow Z_1^{\cdot} \longrightarrow X_1^{\cdot}[1]$ *and* $X_2^{\cdot} \xrightarrow{f_2} Y_2^{\cdot} \longrightarrow Z_2^{\cdot} \longrightarrow X_2^{\cdot}[1]$ *and a commutative diagram*

$$\begin{tikzcd}
X_1^{\cdot} \arrow[r, "f_1"] \arrow[d, "h"'] & Y_1^{\cdot} \arrow[d] \\
X_2^{\cdot} \arrow[r, "f_2"] & Y_2^{\cdot}
\end{tikzcd}$$

in $\mathcal{C}_0$*, then we can embed them into a morphism of triangles, i.e., into a commutative diagram in* $\mathcal{C}_0$*:*

$$\begin{tikzcd}
X_1^{\cdot} \arrow[r] \arrow[d, "h"'] & Y_1^{\cdot} \arrow[r] \arrow[d] & Z_1^{\cdot} \arrow[r] \arrow[d, dashed, "\psi"'] & X_1^{\cdot}[1] \arrow[d, "h[1]"'] \\
X_2^{\cdot} \arrow[r] & Y_2^{\cdot} \arrow[r] & Z_2^{\cdot} \arrow[r] & X_2^{\cdot}[1].
\end{tikzcd}$$

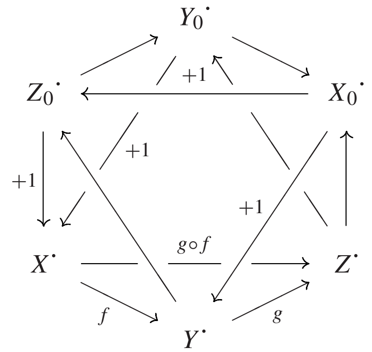
(TR5) *Let*

$$\begin{cases}
X^{\cdot} \xrightarrow{f} Y^{\cdot} \longrightarrow Z_0^{\cdot} \longrightarrow X^{\cdot}[1] \\
Y^{\cdot} \xrightarrow{g} Z^{\cdot} \longrightarrow X_0^{\cdot} \longrightarrow Y^{\cdot}[1] \\
X^{\cdot} \xrightarrow{g\circ f} Z^{\cdot} \longrightarrow Y_0^{\cdot} \longrightarrow X^{\cdot}[1],
\end{cases}$$

be three distinguished triangles. Then there exists a distinguished triangle $Z_0^{\cdot} \longrightarrow Y_0^{\cdot} \longrightarrow X_0^{\cdot} \longrightarrow Z_0^{\cdot}[1]$ *which can be embedded into the commutative diagram*

$$\begin{tikzcd}
X^{\cdot} \arrow[r, "f"] \arrow[d, equal, "\mathrm{id}"'] & Y^{\cdot} \arrow[r] \arrow[d, "g"'] & Z_0^{\cdot} \arrow[r] \arrow[d] & X^{\cdot}[1] \arrow[d, equal, "\mathrm{id}"'] \\
X^{\cdot} \arrow[r, "g\circ f"] \arrow[d, "f"'] & Z^{\cdot} \arrow[r] \arrow[d, equal, "\mathrm{id}"'] & Y_0^{\cdot} \arrow[r] \arrow[d] & X^{\cdot}[1] \arrow[d, "f[1]"'] \\
Y^{\cdot} \arrow[r, "g"] \arrow[d] & Z^{\cdot} \arrow[r] \arrow[d] & X_0^{\cdot} \arrow[r] \arrow[d, equal, "\mathrm{id}"'] & Y^{\cdot}[1] \arrow[d] \\
Z_0^{\cdot} \arrow[r] & Y_0^{\cdot} \arrow[r] & X_0^{\cdot} \arrow[r] & Z_0^{\cdot}[1].
\end{tikzcd}$$

For the proof, see [KS2, Proposition 1.4.4]. The property (TR5) is called the *octahedral axiom*, because it can be visualized by the following figure:



Corollary B.3.4. *Set* $\mathcal{C}_0 = K^{\#}(\mathcal{C})$ *and let* $X^{\bullet} \xrightarrow{f} Y^{\bullet} \xrightarrow{g} Z^{\bullet} \xrightarrow{h} X^{\bullet}[1]$ *be a distinguished triangle in* $\mathcal{C}_0$.

(i) *For any* $n \in \mathbb{Z}$ *the sequence* $H^n(X^{\bullet}) \xrightarrow{H^n(f)} H^n(Y^{\bullet}) \xrightarrow{H^n(g)} H^n(Z^{\bullet})$ *in* $\mathcal{C}$ *is exact.*
(ii) *The composite* $g \circ f$ *is zero.*
(iii) *For any* $W^{\bullet} \in \mathrm{Ob}(\mathcal{C}_0)$, *the sequences*

$$\begin{cases} \mathrm{Hom}_{\mathcal{C}_0}(W^{\bullet}, X^{\bullet}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_0}(W^{\bullet}, Y^{\bullet}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_0}(W^{\bullet}, Z^{\bullet}) \\ \mathrm{Hom}_{\mathcal{C}_0}(Z^{\bullet}, W^{\bullet}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_0}(Y^{\bullet}, W^{\bullet}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_0}(X^{\bullet}, W^{\bullet}) \end{cases}$$

associated to the distinguished triangle $X^{\bullet} \xrightarrow{f} Y^{\bullet} \xrightarrow{g} Z^{\bullet} \xrightarrow{h} X^{\bullet}[1]$ *are exact in the abelian category* $\mathcal{A}b$ *of abelian groups.*

Corollary B.3.5. *Set* $\mathcal{C}_0 = K^{\#}(\mathcal{C})$.

(i) *Let*

$$\begin{tikzcd} X_1^{\bullet} \arrow[r] \arrow[d, "f"'] & Y_1^{\bullet} \arrow[r] \arrow[d, "g"'] & Z_1^{\bullet} \arrow[r] \arrow[d, "h"'] & X_1^{\bullet}[1] \arrow[d, "f[1]"'] \\ X_2^{\bullet} \arrow[r] & Y_2^{\bullet} \arrow[r] & Z_2^{\bullet} \arrow[r] & X_2^{\bullet}[1]. \end{tikzcd}$$

be a morphism of distinguished triangles in $\mathcal{C}_0$. *Assume that* f *and* g *are isomorphisms. Then* h *is also an isomorphism.*
(ii) *Let* $X^{\bullet} \xrightarrow{u} Y^{\bullet} \longrightarrow Z^{\bullet} \longrightarrow X^{\bullet}[1]$ *and* $X^{\bullet} \xrightarrow{u} Y^{\bullet} \longrightarrow Z'^{\bullet} \longrightarrow X^{\bullet}[1]$ *be two distinguished triangles in* $\mathcal{C}_0$. *Then* $Z^{\bullet} \simeq Z'^{\bullet}$.
(iii) *Let* $X^{\bullet} \xrightarrow{u} Y^{\bullet} \longrightarrow Z^{\bullet} \longrightarrow X^{\bullet}[1]$ *be a distinguished triangle in* $\mathcal{C}_0$. *Then* u *is an isomorphism if and only if* $Z^{\bullet} \simeq 0$.

By abstracting the properties of the homotopy categories $K^{\#}(\mathcal{C})$ let us introduce the notion of triangulated categories as follows. In the case when $\mathcal{C}_0 = K^{\#}(\mathcal{C})$ for an abelian category $\mathcal{C}$, the automorphism $T : \mathcal{C}_0 \longrightarrow \mathcal{C}_0$ in the definition below is the degree shift functor $(\bullet)[1] : K^{\#}(\mathcal{C}) \to K^{\#}(\mathcal{C})$.

Definition B.3.6. Let $\mathcal{C}_0$ be an additive category and $T : \mathcal{C}_0 \to \mathcal{C}_0$ an automorphism of $\mathcal{C}_0$.

(i) A sequence of morphisms $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$ in $\mathcal{C}_0$ is called a triangle in $\mathcal{C}_0$.
(ii) Consider a family $\mathcal{T}$ of triangles in $\mathcal{C}_0$, called distinguished triangles. We say that the pair $(\mathcal{C}_0, \mathcal{T})$ is a *triangulated category* if the family $\mathcal{T}$ of distinguished triangles satisfies the axioms obtained from (TR0) $\sim$ (TR5) in Proposition B.3.3 by replacing $(\bullet)[1]$'s with $T(\bullet)$'s everywhere.

It is clear that Corollary B.3.4 (ii), (iii) and Corollary B.3.5 are true for any triangulated category $(\mathcal{C}_0, \mathcal{T})$. Derived categories that we introduce in the next section are also triangulated categories. Note also that the morphism ψ in (TR4) is not unique in general, which is the source of some difficulties in using triangulated categories.

Definition B.3.7. Let $(\mathcal{C}_0, \mathcal{T})$, $(\mathcal{C}'_0, \mathcal{T}')$ be two triangulated categories and $T : \mathcal{C}_0 \to \mathcal{C}_0$, $T' : \mathcal{C}'_0 \to \mathcal{C}'_0$ the corresponding automorphisms. Then we say that an additive functor $F : \mathcal{C}_0 \to \mathcal{C}'_0$ is a *functor of triangulated categories* (or a ∂*-functor*) if $F \circ T = T' \circ F$ and F sends distinguished triangles in $\mathcal{C}_0$ to those in $\mathcal{C}'_0$.

Definition B.3.8. An additive functor $F : \mathcal{C}_0 \longrightarrow \mathcal{A}$ from a triangulated category $(\mathcal{C}_0, \mathcal{T})$ into an abelian category $\mathcal{A}$ is called a *cohomological functor* if for any distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$, the associated sequence $F(X) \longrightarrow F(Y) \longrightarrow F(Z)$ in $\mathcal{A}$ is exact.

The assertions (i) and (iii) of Corollary B.3.4 imply that the functors $H^n : K^\#(\mathcal{C}) \longrightarrow \mathcal{C}$ and $\mathrm{Hom}_{K^\#(\mathcal{C})}(W^\cdot, \bullet) : K^\#(\mathcal{C}) \longrightarrow \mathcal{A}b$ are cohomological functors, respectively. Let $F : \mathcal{C}_0 \longrightarrow \mathcal{A}$ be a cohomological functor. Then by using the axiom (TR3) repeatedly, from a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$ in $\mathcal{C}_0$ we obtain a long exact sequence

$$\begin{aligned} \cdots \longrightarrow F(T^{-1}Z) \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow F(TX) \\ \longrightarrow F(TY) \longrightarrow \cdots \end{aligned}$$

in the abelian category $\mathcal{A}$.

B.4 Derived categories

In this section, we shall construct derived categories $D^\#(\mathcal{C})$ from homotopy categories $K^\#(\mathcal{C})$ by adding morphisms so that quasi-isomorphisms are invertible in $D^\#(\mathcal{C})$. For this purpose, we need the general theory of localizations of categories. Now let $\mathcal{C}_0$ be a category and S a family of morphisms in $\mathcal{C}_0$. In what follows, for two functors $F_1, F_2 : \mathcal{A}_1 \to \mathcal{A}_2$ $(\in Fun(\mathcal{A}_1, \mathcal{A}_2))$ we denote by $\mathrm{Hom}_{Fun(\mathcal{A}_1,\mathcal{A}_2)}(F_1, F_2)$ the set of *natural transformations* (i.e., morphisms of functors) from F_1 to F_2.

Definition B.4.1. A *localization* of the category $\mathcal{C}_0$ by S is a pair $((\mathcal{C}_0)_S, Q)$ of a category $(\mathcal{C}_0)_S$ and a functor $Q : \mathcal{C}_0 \to (\mathcal{C}_0)_S$ which satisfies the following universal properties:

(i) $Q(s)$ is an isomorphism for any $s \in S$.
(ii) For any functor $F : \mathcal{C}_0 \to \mathcal{C}_1$ such that $F(s)$ is an isomorphism for any $s \in S$, there exists a functor $F_S : (\mathcal{C}_0)_S \to \mathcal{C}_1$ and an isomorphism $F \simeq F_S \circ Q$ of functors:

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "Q"] \arrow[d, "F"'] & (\mathcal{C}_0)_S \arrow[dl, dashed, "\exists F_S"] \\ \mathcal{C}_1. & \end{tikzcd}$$

(iii) Let $G_1, G_2 : (\mathcal{C}_0)_S \to \mathcal{C}_1$ be two functors. Then the natural morphism

$$\mathrm{Hom}_{Fun((\mathcal{C}_0)_S,\mathcal{C}_1)}(G_1, G_2) \longrightarrow \mathrm{Hom}_{Fun(\mathcal{C}_0,\mathcal{C}_1)}(G_1 \circ Q, G_2 \circ Q)$$

is a bijection.

By the property (iii), F_S in (ii) is unique up to isomorphisms. Moreover, since the localization $((\mathcal{C}_0)_S, Q)$ is characterized by universal properties (if it exists) it is unique up to equivalences of categories. We call this operation "a localization of categories" because it is similar to the more familiar localization of (non-commutative) rings. As we need the so-called "Ore conditions" for the construction of localizations of rings, we have to impose some conditions on S to ensure the existence of the localization $((\mathcal{C}_0)_S, Q)$.

Definition B.4.2. Let $\mathcal{C}_0$ be a category and S a family of morphisms. We call the family S a *multiplicative system* if it satisfies the following axioms:

(M1) $\mathrm{id}_X \in S$ for any $X \in \mathrm{Ob}(\mathcal{C}_0)$.
(M2) If $f, g \in S$ and their composite $g \circ f$ exists, then $g \circ f \in S$.
(M3) Any diagram

$$\begin{tikzcd} & Y' \arrow[d, "s"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{C}_0$ with $s \in S$ fits into a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "t"'] & Y' \arrow[d, "s"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{C}_0$ with $t \in S$. We impose also the condition obtained by reversing all arrows.
(M4) For $f, g \in \mathrm{Hom}_{\mathcal{C}_0}(X, Y)$ the following two conditions are equivalent:

(i) $^{\exists}s : Y \longrightarrow Y'$, $s \in S$ such that $s \circ f = s \circ g$.
(ii) $^{\exists}t : X' \longrightarrow X$, $t \in S$ such that $f \circ t = g \circ t$.

Let $\mathcal{C}_0$ be a category and S a multiplicative system in it. Then we can define a category $(\mathcal{C}_0)_S$ by

(objects): $\mathrm{Ob}((\mathcal{C}_0)_S) = \mathrm{Ob}(\mathcal{C}_0)$.
(morphisms): For $X, Y \in \mathrm{Ob}((\mathcal{C}_0)_S) = \mathrm{Ob}(\mathcal{C}_0)$, we set

$$\mathrm{Hom}_{(\mathcal{C}_0)_S}(X, Y) = \left\{(X \xleftarrow{s} W \xrightarrow{f} Y) \mid s \in S\right\} / \sim$$

where two diagrams $(X \xleftarrow{s_1} W_1 \xrightarrow{f_1} Y)$ $(s_1 \in S)$ and $(X \xleftarrow{s_2} W_2 \xrightarrow{f_2} Y)$ $(s_2 \in S)$ are equivalent ($\sim$) if and only if they fit into a commutative diagram

$$\begin{tikzcd}
 & W_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\
X & W_3 \arrow[l, "s_3"'] \arrow[u] \arrow[d] & Y \\
 & W_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

with $s_3 \in S$. We omit the details here. Let us just explain how we compose morphisms in the category $(\mathcal{C}_0)_S$. Assume that we are given two morphisms

$$\begin{cases} \left[(X \xleftarrow{s} W_1 \xrightarrow{f} Y)\right] \in \mathrm{Hom}_{(\mathcal{C}_0)_S}(X, Y) \\ \left[(Y \xleftarrow{t} W_2 \xrightarrow{g} Z)\right] \in \mathrm{Hom}_{(\mathcal{C}_0)_S}(Y, Z) \end{cases}$$

$(s, t \in S)$ in $(\mathcal{C}_0)_S$. Then by the axiom (M3) of multiplicative systems we can construct a commutative diagram

$$\begin{tikzcd}
 & & W_3 \arrow[dl, "u"'] \arrow[dr, "h"] & & \\
 & W_1 \arrow[dl, "s"'] \arrow[dr, "f"] & & W_2 \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

with $u \in S$ and the composite of these two morphisms in $(\mathcal{C}_0)_S$ is given by

$$\left[(X \xleftarrow{s \circ u} W_3 \xrightarrow{g \circ h} Z)\right] \in \mathrm{Hom}_{(\mathcal{C}_0)_S}(X, Z).$$

Moreover, there exists a natural functor $Q : \mathcal{C}_0 \longrightarrow (\mathcal{C}_0)_S$ defined by

$$\begin{cases} Q(X) = X \quad \text{for } X \in \mathrm{Ob}(\mathcal{C}_0), \\ \mathrm{Hom}_{\mathcal{C}_0}(X, Y) \longrightarrow \mathrm{Hom}_{(\mathcal{C}_0)_S}(Q(X), Q(Y)) \\ \qquad \Psi \qquad\qquad\qquad\qquad \Psi \\ \qquad f \qquad \longmapsto \quad \left[(X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y)\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for any } X, Y \in \mathrm{Ob}(\mathcal{C}_0). \end{cases}$$

We can easily check that the pair $((\mathcal{C}_0)_S, Q)$ satisfies the conditions of the localization of $\mathcal{C}_0$ by S. Since morphisms in S are invertible in the localized category $(\mathcal{C}_0)_S$, a morphism $[(X \xleftarrow{s} W \xrightarrow{f} Y)]$ in $(\mathcal{C}_0)_S$ can be written also as $Q(f) \circ Q(s)^{-1}$. If, moreover, $\mathcal{C}_0$ is an additive category, then we can show that $(\mathcal{C}_0)_S$ is also an additive category and $Q : \mathcal{C}_0 \to (\mathcal{C}_0)_S$ is an additive functor. Since we defined the localization $(\mathcal{C}_0)_S$ of $\mathcal{C}_0$ by universal properties, also the following category $(\mathcal{C}_0)^S$ satisfies the conditions of the localization:

(objects): $\mathrm{Ob}((\mathcal{C}_0)^S) = \mathrm{Ob}(\mathcal{C}_0)$.
(morphisms): For $X, Y \in \mathrm{Ob}((\mathcal{C}_0)^S) = \mathrm{Ob}(\mathcal{C}_0)$, we set

$$\mathrm{Hom}_{(\mathcal{C}_0)^S}(X, Y) = \left\{(X \xrightarrow{f} W \xleftarrow{s} Y) \mid s \in S\right\} / \sim$$

where we define the equivalence $\sim$ of diagrams similarly. Namely, a morphism in the localization $(\mathcal{C}_0)_S$ can be written also as $Q(s)^{-1} \circ Q(f)$ for $s \in S$. The following elementary lemma will be effectively used in the next section.

Lemma B.4.3. *Let $\mathcal{C}_0$ be a category and S a multiplicative system in it. Let $\mathcal{J}_0$ be a full subcategory of $\mathcal{C}_0$ and denote by T the family of morphisms in $\mathcal{J}_0$ which belong to S. Assume, moreover, that for any $X \in \mathrm{Ob}(\mathcal{C}_0)$ there exists a morphism $s : X \to J$ in S such that $J \in \mathrm{Ob}(\mathcal{J}_0)$. Then T is a multiplicative system in $\mathcal{J}_0$, and the natural functor $(\mathcal{J}_0)_T \to (\mathcal{C}_0)_S$ gives an equivalence of categories.*

Now let us return to the original situation and consider a homotopy category $\mathcal{C}_0 = K^{\#}(\mathcal{C})$ $(\# = \emptyset, +, -, b)$ of an abelian category $\mathcal{C}$. Denote by S the family of quasi-isomorphisms in it. Then we can prove that S is a multiplicative system.

Definition B.4.4. We set $D^{\#}(\mathcal{C}) = (K^{\#}(\mathcal{C}))_S$ and call it a *derived category* of $\mathcal{C}$. The canonical functor $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$ is called the *localization functor*.

By construction, quasi-isomorphisms are isomorphisms in the derived category $D^{\#}(\mathcal{C})$. Moreover, if we define distinguished triangles in $D^{\#}(\mathcal{C})$ to be the triangles isomorphic to the images of distinguished triangles in $K^{\#}(\mathcal{C})$ by Q, then $D^{\#}(\mathcal{C})$ is a triangulated category and $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$ is a functor of triangulated categories. We can also prove that the canonical morphisms $\mathcal{C} \to D(\mathcal{C})$ and $D^{\#}(\mathcal{C}) \to D(\mathcal{C})$ $(\# = +, -, b)$ are fully faithful. Namely, the categories $\mathcal{C}$ and $D^{\#}(\mathcal{C})$ $(\# = +, -, b)$ can be identified with full subcategories of $D(\mathcal{C})$. By the results in the previous section, to a distinguished triangle $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$ in the derived category $D^{\#}(\mathcal{C})$ we can associate a cohomology long exact sequence

$$\cdots \longrightarrow H^{-1}(Z^{\cdot}) \longrightarrow H^0(X^{\cdot}) \longrightarrow H^0(Y^{\cdot}) \longrightarrow H^0(Z^{\cdot}) \longrightarrow H^1(X^{\cdot}) \longrightarrow \cdots$$

in $\mathcal{C}$. The following lemma is very useful to construct examples of distinguished triangles in $D^{\#}(\mathcal{C})$.

Lemma B.4.5. *Any short exact sequence* $0 \longrightarrow X^{\cdot} \xrightarrow{f} Y^{\cdot} \xrightarrow{g} Z^{\cdot} \longrightarrow 0$ *in* $C^{\#}(\mathcal{C})$ *can be embedded into a distinguished triangle* $X^{\cdot} \xrightarrow{f} Y^{\cdot} \xrightarrow{g} Z^{\cdot} \longrightarrow X^{\cdot}[1]$ *in* $D^{\#}(\mathcal{C})$.

Proof. Consider the short exact sequence

$$0 \longrightarrow M_{\mathrm{id}_{X^{\cdot}}}{}^{\cdot} \xrightarrow[\begin{pmatrix} \mathrm{id}_{X^{\cdot}} & 0 \\ 0 & f \end{pmatrix}]{} M_f{}^{\cdot} \xrightarrow[(0,\, g)]{\varphi} Z^{\cdot} \longrightarrow 0$$

in $C^{\#}(\mathcal{C})$. Since the mapping cone $M_{\mathrm{id}_{X^{\cdot}}}{}^{\cdot}$ is quasi-isomorphic to 0 by Lemma B.2.3, we obtain an isomorphism $\varphi : M_f{}^{\cdot} \simeq Z^{\cdot}$ in $D^{\#}(\mathcal{C})$. Hence there exists a commutative diagram

$$\begin{tikzcd}
X^{\cdot} \arrow[r, "f"] \arrow[d, equal, "\mathrm{id}"'] & Y^{\cdot} \arrow[r, "\alpha(f)"] \arrow[d, equal, "\mathrm{id}"'] & M_f{}^{\cdot} \arrow[r, "\beta(f)"] \arrow[d, "\varphi\wr"'] & X^{\cdot}[1] \arrow[d, equal, "\mathrm{id}"'] \\
X^{\cdot} \arrow[r, "f"] & Y^{\cdot} \arrow[r, "g"] & Z^{\cdot} \arrow[r, "\beta(f)\circ\varphi^{-1}"] & X^{\cdot}[1]
\end{tikzcd}$$

in $D^{\#}(\mathcal{C})$, which shows that $X^{\cdot} \xrightarrow{f} Y^{\cdot} \xrightarrow{g} Z^{\cdot} \xrightarrow{\beta(f)\circ\varphi^{-1}} X^{\cdot}[1]$ is a distinguished triangle. □

Definition B.4.6. An abelian subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called a *thick subcategory* if for any exact sequence $X_1 \to X_2 \to X_3 \to X_4 \to X_5$ in $\mathcal{C}$ with $X_i \in \mathrm{Ob}(\mathcal{C}')$ $(i = 1, 2, 4, 5)$, X_3 belongs to $\mathcal{C}'$.

Proposition B.4.7. *Let* $\mathcal{C}'$ *be a thick abelian subcategory of an abelian category* $\mathcal{C}$ *and* $D^{\#}_{\mathcal{C}'}(\mathcal{C})$ *the full subcategory of* $D^{\#}(\mathcal{C})$ *consisting of objects* $X^{\cdot}$ *such that* $H^n(X^{\cdot}) \in \mathrm{Ob}(\mathcal{C}')$ *for any* $n \in \mathbb{Z}$. *Then* $D^{\#}_{\mathcal{C}'}(\mathcal{C})$ *is a triangulated category.*

B.5 Derived functors

Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories and $F : \mathcal{C} \to \mathcal{C}'$ an additive functor. Let us consider the problem of constructing a ∂-functor $\tilde{F} : D^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$ between their derived categories which is naturally associated to $F : \mathcal{C} \to \mathcal{C}'$. This problem can be easily solved if $\# = +$ or $-$ and F is an exact functor. Indeed, let $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$, $Q' : K^{\#}(\mathcal{C}') \to D^{\#}(\mathcal{C}')$ be the localization functors and consider the functor $K^{\#}F : K^{\#}(\mathcal{C}) \to K^{\#}(\mathcal{C}')$ defined by $X^{\cdot} \mapsto F(X^{\cdot})$. Then by Lemma B.2.3 the functor $K^{\#}F$ sends quasi-isomorphisms in $K^{\#}(\mathcal{C})$ to those in $K^{\#}(\mathcal{C}')$. Hence it follows from the universal properties of the localization $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$ that there exists a unique functor $\tilde{F} : D^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$ which makes the following diagram commutative:

$$\begin{tikzcd} K^{\#}(\mathcal{C}) \arrow[r, "K^{\#}F"] \arrow[d, "Q"'] & K^{\#}(\mathcal{C}') \arrow[d, "Q'"] \\ D^{\#}(\mathcal{C}) \arrow[r, "\tilde{F}"'] & D^{\#}(\mathcal{C}'). \end{tikzcd}$$

In this situation, we call the functor $\tilde{F} : D^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$ a "localization" of $Q' \circ K^{\#}F : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$. However, many important additive functors that we encounter in sheaf theory or homological algebra are not exact. They are only left exact or right exact. So in such cases the functor $Q' \circ K^{\#}F : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$ does not factorize through $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$ in general. In other words, there is no localization of the functor $Q' \circ K^{\#}F$. As a remedy for this problem we will introduce the following notion of right (or left) localizations. In what follows, let $\mathcal{C}_0$ be a general category, S a multiplicative system in $\mathcal{C}_0$, $F : \mathcal{C}_0 \to \mathcal{C}_1$ a functor. As before we denote by $Q : \mathcal{C}_0 \to (\mathcal{C}_0)_S$ the canonical functor.

Definition B.5.1. A *right localization* of F is a pair (F_S, τ) of a functor $F_S : (\mathcal{C}_0)_S \to \mathcal{C}_1$ and a morphism of functors $\tau : F \to F_S \circ Q$ such that for any functor $G : (\mathcal{C}_0)_S \to \mathcal{C}_1$ the morphism

$$\mathrm{Hom}_{Fun((\mathcal{C}_0)_S, \mathcal{C}_1)}(F_S, G) \longrightarrow \mathrm{Hom}_{Fun(\mathcal{C}_0, \mathcal{C}_1)}(F, G \circ Q)$$

is bijective. Here the morphism above is obtained by the composition of

$$\mathrm{Hom}_{Fun((\mathcal{C}_0)_S, \mathcal{C}_1)}(F_S, G) \longrightarrow \mathrm{Hom}_{Fun(\mathcal{C}_0, \mathcal{C}_1)}(F_S \circ Q, G \circ Q)$$

$$\xrightarrow{\tau} \mathrm{Hom}_{Fun(\mathcal{C}_0, \mathcal{C}_1)}(F, G \circ Q).$$

We say that F is *right localizable* if it has a right localization.

The notion of left localizations can be defined similarly. Note that by definition the functor F_S is a representative of the functor

$$G \longrightarrow \mathrm{Hom}_{Fun(\mathcal{C}_0, \mathcal{C}_1)}(F, G \circ Q).$$

Therefore, if a right localization (F_S, τ) of F exists, it is unique up to isomorphisms. Let us give a useful criterion for the existence of the right localization of $F : \mathcal{C}_0 \to \mathcal{C}_1$.

Proposition B.5.2. *Let $\mathcal{J}_0$ be a full subcategory of $\mathcal{C}_0$ and denote by T the family of morphisms in $\mathcal{J}_0$ which belong to S. Assume the following conditions:*

(i) *For any $X \in \mathrm{Ob}(\mathcal{C}_0)$ there exists a morphism $s : X \to J$ in S such that $J \in \mathrm{Ob}(\mathcal{J}_0)$.*
(ii) *$F(t)$ is an isomorphism for any $t \in T$.*

Then F is localizable.

A very precise proof of this proposition can be found in Kashiwara–Schapira [KS4, Proposition 7.3.2]. Here we just explain how the functor $F_S : (\mathcal{C}_0)_S \to \mathcal{C}_1$ is defined. First, by Lemma B.4.3 there exists an equivalence of categories $\Phi : (\mathcal{J}_0)_T \xrightarrow{\sim} (\mathcal{C}_0)_S$. Let $\iota : \mathcal{J}_0 \to \mathcal{C}_0$ be the inclusion. Then by the condition (ii) above the functor $F \circ \iota : \mathcal{J}_0 \to \mathcal{C}_1$ factorizes through the localization functor $\mathcal{J}_0 \to (\mathcal{J}_0)_T$ and we obtain a functor $F_T : (\mathcal{J}_0)_T \to \mathcal{C}_1$. The functor $F_S : (\mathcal{C}_0)_S \to \mathcal{C}_1$ is defined by $F_S = F_T \circ \Phi^{-1}$:

$$\begin{tikzcd} (\mathcal{J}_0)_T \arrow[r, "F_T"] \arrow[d, "\Phi \wr"'] & \mathcal{C}_1 \\ (\mathcal{C}_0)_S. \arrow[ur, "F_S"] & \end{tikzcd}$$

Now let us return to the original situation and assume that $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is a left exact functor. In this situation, by Proposition B.5.2 we can give a criterion for the existence of a right localization of the functor $Q' \circ K^+F : K^+(\mathcal{C}) \to D^+(\mathcal{C}')$ as follows.

Definition B.5.3. A *right derived functor* of F is a pair (RF, τ) of a ∂-functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ and a morphism of functors $\tau : Q' \circ K^+(F) \to RF \circ Q$

$$\begin{tikzcd} K^+(\mathcal{C}) \arrow[r, "K^+(F)"] \arrow[d, "Q"'] & K^+(\mathcal{C}') \arrow[d, "Q'"] \arrow[dl, dashed, "\tau"'] \\ D^+(\mathcal{C}) \arrow[r, "RF"] & D^+(\mathcal{C}') \end{tikzcd}$$

such that for any functor $\tilde{G} : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ the morphism

$$\mathrm{Hom}_{Fun(D^+(\mathcal{C}), D^+(\mathcal{C}'))}(RF, \tilde{G}) \longrightarrow \mathrm{Hom}_{Fun(K^+(\mathcal{C}), D^+(\mathcal{C}'))}(Q' \circ K^+(F), \tilde{G} \circ Q)$$

induced by τ is an isomorphism. We say that F is *right derivable* if it admits a right derived functor.

Similarly, for right exact functors F we can define the notion of *left derived functors* $LF : D^-(\mathcal{C}) \to D^-(\mathcal{C}')$. By definition, if a right derived functor (RF, τ) of a left exact functor F exists, it is unique up to isomorphisms. Moreover, for an exact functor F the natural functor $D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ defined simply by $X^\cdot \mapsto F(X^\cdot)$ gives a right derived functor. In other words, any exact functor is right (and left) derivable.

Definition B.5.4. Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor between abelian categories. We say that a full additive subcategory $\mathcal{J}$ of $\mathcal{C}$ is *F-injective* if the following conditions are satisfied:

(i) For any $X \in \mathrm{Ob}(\mathcal{C})$, there exists an object $I \in \mathrm{Ob}(\mathcal{J})$ and an exact sequence $0 \to X \to I$.

(ii) If $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathcal{C}$ and $X', X \in \mathrm{Ob}(\mathcal{J})$, then $X'' \in \mathrm{Ob}(\mathcal{J})$.
(iii) For any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ such that $X', X, X'' \in \mathrm{Ob}(\mathcal{J})$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ in $\mathcal{C}'$ is also exact.

Similarly we define *F-projective* subcategories of $\mathcal{C}$ by reversing all arrows in the conditions above.

Example B.5.5.
(i) Assume that the abelian category $\mathcal{C}$ has enough injectives. Denote by $\mathcal{I}$ the full additive subcategory of $\mathcal{C}$ consisting of injective objects in $\mathcal{C}$. Then $\mathcal{I}$ is F-injective for any additive functor $F : \mathcal{C} \to \mathcal{C}'$ (use the fact that any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X' \in \mathrm{Ob}(\mathcal{I})$ splits).
(ii) Let T_0 be a topological space and set $\mathcal{C} = \mathrm{Sh}(T_0)$. Let $F = \Gamma(T_0, \bullet) : \mathrm{Sh}(T_0) \to \mathcal{A}b$ be the global section functor. Then $F = \Gamma(T_0, \bullet)$ is left exact and

$$\mathcal{J} = \{\text{flabby sheaves on } T_0\} \subset \mathrm{Sh}(T_0)$$

is an F-injective subcategory of $\mathcal{C}$.
(iii) Let T_0 be a topological space and $\mathcal{R}$ a sheaf of rings on T_0. Denote by $\mathrm{Mod}(\mathcal{R})$ the abelian category of sheaves of left $\mathcal{R}$-modules on T_0 and let $\mathcal{P}$ be the full subcategory of $\mathrm{Mod}(\mathcal{R})$ consisting of flat $\mathcal{R}$-modules, i.e., objects $M \in \mathrm{Mod}(\mathcal{R})$ such that the stalk M_x at x is a flat $\mathcal{R}_x$-module for any $x \in T_0$. For a right $\mathcal{R}$-module N, consider the functor $F_N = N \otimes_{\mathcal{R}} (\bullet) : \mathrm{Mod}(\mathcal{R}) \to \mathrm{Sh}(T_0)$. Then the category $\mathcal{P}$ is F_N-projective. For the details see Section C.1.

Assume that for the given left exact functor $F : \mathcal{C} \to \mathcal{C}'$ there exists an F-injective subcategory $\mathcal{J}$ of $\mathcal{C}$. Then it is well known that for any $X^{\boldsymbol{\cdot}} \in \mathrm{Ob}(K^+(\mathcal{C}))$ we can construct a quasi-isomorphism $X^{\boldsymbol{\cdot}} \to J^{\boldsymbol{\cdot}}$ into an object $J^{\boldsymbol{\cdot}}$ of $K^+(\mathcal{J})$. Such an object $J^{\boldsymbol{\cdot}}$ is called an *F-injective resolution* of $X^{\boldsymbol{\cdot}}$. Moreover, by Lemma B.2.3 we see that the functor $Q' \circ K^+F : K^+(\mathcal{J}) \to D^+(\mathcal{C}')$ sends quasi-isomorphisms in $K^+(\mathcal{J})$ to isomorphisms in $D^+(\mathcal{C}')$. Therefore, applying Proposition B.5.2 to the special case when $\mathcal{C}_0 = K^+(\mathcal{C})$, $\mathcal{C}_1 = D^+(\mathcal{C}')$, $\mathcal{J}_0 = K^+(\mathcal{J})$ and S is the family of quasi-isomorphisms in $K^+(\mathcal{C})$, we obtain the fundamental important result.

Theorem B.5.6. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a left (resp. right) exact functor and assume that there exists an F-injective (resp. F-projective) subcategory of $\mathcal{C}$. Then the right (resp. left) derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ (resp. $LF : D^-(\mathcal{C}) \to D^-(\mathcal{C}')$) of F exists.*

Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor and $\mathcal{J}$ an F-injective subcategory of $\mathcal{C}$. Then the right derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ is constructed as follows. Denote by S (resp. T) the family of quasi-isomorphisms in $K^+(\mathcal{C})$ (resp. $K^+(\mathcal{J})$). Then there exist an equivalence of categories $\Phi : K^+(\mathcal{J})_T \xrightarrow{\sim} K^+(\mathcal{C})_S = D^+(\mathcal{C})$ and a natural functor $\Psi : K^+(\mathcal{J})_T \to D^+(\mathcal{C}')$ induced by $K^+F : K^+(\mathcal{C}) \to K^+(\mathcal{C}')$ such that $RF = \Psi \circ \Phi^{-1}$. Consequently the right derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ sends $X^{\boldsymbol{\cdot}} \in \mathrm{Ob}(D^+(\mathcal{C}))$ to $F(J^{\boldsymbol{\cdot}}) \in \mathrm{Ob}(D^+(\mathcal{C}'))$, where $J^{\boldsymbol{\cdot}} \in \mathrm{Ob}(K^+(\mathcal{J}))$

is an F-injective resolution of $X^{\cdot}$. If $\mathcal{C}$ has enough injectives and denote by $\mathcal{I}$ the full subcategory of $\mathcal{C}$ consisting of injective objects in $\mathcal{C}$, then we have, moreover, an equivalence of categories $K^+(\mathcal{I}) \simeq D^+(\mathcal{C})$. This follows from the basic fact that quasi-isomorphisms in $K^+(\mathcal{I})$ are homotopy equivalences. Using this explicit description of derived functors, we obtain the following useful composition rule.

Proposition B.5.7. *Let $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ be abelian categories and $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}''$ left exact functors. Assume that $\mathcal{C}$ (resp. $\mathcal{C}'$) has an F-injective (resp. G-injective) subcategory $\mathcal{J}$ (resp. $\mathcal{J}'$) and $F(X) \in \mathrm{Ob}(\mathcal{J}')$ for any $X \in \mathrm{Ob}(\mathcal{J})$. Then $\mathcal{J}$ is $(G \circ F)$-injective and*

$$R(G \circ F) = RG \circ RF : D^+(\mathcal{C}) \longrightarrow D^+(\mathcal{C}'').$$

Definition B.5.8. Assume that a left exact functor $F : \mathcal{C} \to \mathcal{C}'$ is right derivable. Then for each $n \in \mathbb{Z}$ we set

$$R^n F = H^n \circ RF : D^+(\mathcal{C}) \to \mathcal{C}'.$$

Since $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ sends distinguished triangles to distinguished triangles, the functors $R^n F : D^+(\mathcal{C}) \to \mathcal{C}'$ defined above are cohomological functors. Now let us identify $\mathcal{C}$ with the full subcategory of $D^+(\mathcal{C})$ consisting of complexes concentrated in degree 0. Then we find that for $X \in \mathrm{Ob}(\mathcal{C})$, $R^n F(X) \in \mathrm{Ob}(\mathcal{C}')$ coincides with the nth derived functor of F in the classical literature.

B.6 Bifunctors in derived categories

In this section we introduce some important bifunctors in derived categories which will be frequently used throughout this book. First, let us explain the bifunctor $\mathrm{RHom}(\bullet, \bullet)$. Let $\mathcal{C}$ be an abelian category. For two complexes $X^{\cdot}, Y^{\cdot} \in \mathrm{Ob}(C(\mathcal{C}))$ in $\mathcal{C}$ define a new complex $\mathrm{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot}) \in \mathrm{Ob}(C(\mathcal{A}b))$ by

$$\begin{cases} \mathrm{Hom}^n(X^{\cdot}, Y^{\cdot}) = \prod\limits_{j-i=n} \mathrm{Hom}_{\mathcal{C}}(X^i, Y^j) \\ d^n = d^n_{\mathrm{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot})} : \quad \prod\limits_{j-i=n} \mathrm{Hom}_{\mathcal{C}}(X^i, Y^j) \longrightarrow \prod\limits_{j-i=n+1} \mathrm{Hom}_{\mathcal{C}}(X^i, Y^j) \\ \qquad\qquad\qquad\qquad\qquad \cup\!\!\!\!\!\!\qquad\qquad\qquad\qquad\qquad\qquad \cup\!\!\!\!\!\! \\ \qquad\qquad\qquad \{f_{ij}\} \longmapsto \quad \left\{g_{ij} = (-1)^{n+1} f_{i+1,j} \circ d^i_{X^{\cdot}} + d^{j-1}_{Y^{\cdot}} \circ f_{i,j-1}\right\}. \end{cases}$$

This is the simple complex associated to the double complex $\mathrm{Hom}(X^{\cdot}, Y^{\cdot})$, which satisfies the conditions

$$\begin{cases} \mathrm{Ker}\, d^n \simeq \mathrm{Hom}_{C(\mathcal{C})}(X^{\cdot}, Y^{\cdot}[n]) \\ \mathrm{Im}\, d^{n-1} \simeq \mathrm{Ht}(X^{\cdot}, Y^{\cdot}[n]) \\ H^n[\mathrm{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot})] \simeq \mathrm{Hom}_{K(\mathcal{C})}(X^{\cdot}, Y^{\cdot}[n]). \end{cases}$$

for any $n \in \mathbb{Z}$. We thus defined a bifunctor

$$\mathrm{Hom}^{\boldsymbol{\cdot}}(\bullet, \bullet) : C(\mathcal{C})^{\mathrm{op}} \times C(\mathcal{C}) \longrightarrow C(\mathcal{A}b),$$

where $(\bullet)^{\mathrm{op}}$ denotes the *opposite category*. It is easy to check that it induces also a bifunctor

$$\mathrm{Hom}^{\boldsymbol{\cdot}}(\bullet, \bullet) : K(\mathcal{C})^{\mathrm{op}} \times K(\mathcal{C}) \longrightarrow K(\mathcal{A}b)$$

in homotopy categories. Similarly we also obtain

$$\mathrm{Hom}^{\boldsymbol{\cdot}}(\bullet, \bullet) : K^{-}(\mathcal{C})^{\mathrm{op}} \times K^{+}(\mathcal{C}) \longrightarrow K^{+}(\mathcal{A}b),$$

taking boundedness into account. From now on, assume that the category $\mathcal{C}$ has enough injectives and denote by $\mathcal{I}$ the full subcategory of $\mathcal{C}$ consisting of injective objects. The following lemma is elementary.

Lemma B.6.1. *Let* $X^{\boldsymbol{\cdot}} \in \mathrm{Ob}(K(\mathcal{C}))$ *and* $I^{\boldsymbol{\cdot}} \in \mathrm{Ob}(K^{+}(\mathcal{I}))$. *Assume that* $X^{\boldsymbol{\cdot}}$ *or* $I^{\boldsymbol{\cdot}}$ *is quasi-isomorphic to* 0. *Then the complex* $\mathrm{Hom}^{\boldsymbol{\cdot}}(X^{\boldsymbol{\cdot}}, Y^{\boldsymbol{\cdot}}) \in \mathrm{Ob}(K(\mathcal{A}b))$ *is also quasi-isomorphic to* 0.

Let $X^{\boldsymbol{\cdot}} \in \mathrm{Ob}(K(\mathcal{C}))$ and consider the functor

$$\mathrm{Hom}^{\boldsymbol{\cdot}}(X^{\boldsymbol{\cdot}}, \bullet) : K^{+}(\mathcal{C}) \longrightarrow K(\mathcal{A}b).$$

Then by Lemmas B.2.3 and B.6.1 and Proposition B.5.2, we see that this functor induces a functor

$$R_{II}\,\mathrm{Hom}^{\boldsymbol{\cdot}}(X^{\boldsymbol{\cdot}}, \bullet) : D^{+}(\mathcal{C}) \longrightarrow D(\mathcal{A}b)$$

between derived categories. Here we write "R_{II}" to indicate that we localize the bifunctor $\mathrm{Hom}^{\boldsymbol{\cdot}}(\bullet, \bullet)$ with respect to the second factor. Since this construction is functorial with respect to $X^{\boldsymbol{\cdot}}$, we obtain a bifunctor

$$R_{II}\,\mathrm{Hom}^{\boldsymbol{\cdot}}(\bullet, \bullet) : K(\mathcal{C})^{\mathrm{op}} \times D^{+}(\mathcal{C}) \longrightarrow D(\mathcal{A}b).$$

By the universal properties of the localization $Q : K(\mathcal{C}) \to D(\mathcal{C})$, this bifunctor factorizes through Q and we obtain a bifunctor

$$R_{I}R_{II}\,\mathrm{Hom}^{\boldsymbol{\cdot}}(\bullet, \bullet) : D(\mathcal{C})^{\mathrm{op}} \times D^{+}(\mathcal{C}) \longrightarrow D(\mathcal{A}b).$$

We set $\mathrm{RHom}_{\mathcal{C}}(\bullet, \bullet) = R_{I}R_{II}\,\mathrm{Hom}^{\boldsymbol{\cdot}}(\bullet, \bullet)$ and call it the functor RHom. Similarly taking boundedness into account, we also obtain a bifunctor

$$\mathrm{RHom}_{\mathcal{C}}(\bullet, \bullet) : D^{-}(\mathcal{C})^{\mathrm{op}} \times D^{+}(\mathcal{C}) \longrightarrow D^{+}(\mathcal{A}b).$$

These are bifunctors of triangulated categories. The following proposition is very useful to construct canonical morphisms in derived categories.

Proposition B.6.2. *For* $Z^{\boldsymbol{\cdot}} \in \mathrm{Ob}(K^{+}(\mathcal{C}))$ *and* $I^{\boldsymbol{\cdot}} \in \mathrm{Ob}(K^{+}(\mathcal{I}))$ *the natural morphism*

$$Q : \mathrm{Hom}_{K^{+}(\mathcal{C})}(Z^{\boldsymbol{\cdot}}, I^{\boldsymbol{\cdot}}) \longrightarrow \mathrm{Hom}_{D^{+}(\mathcal{C})}(Z^{\boldsymbol{\cdot}}, I^{\boldsymbol{\cdot}})$$

is an isomorphism. In particular for any $X^{\boldsymbol{\cdot}}, Y^{\boldsymbol{\cdot}} \in \mathrm{Ob}(D^{+}(\mathcal{C}))$ *and* $n \in \mathbb{Z}$ *there exists a natural isomorphism*

$$H^{n}\mathrm{RHom}_{\mathcal{C}}(X^{\boldsymbol{\cdot}}, Y^{\boldsymbol{\cdot}}) = H^{n}\,\mathrm{Hom}^{\boldsymbol{\cdot}}(X^{\boldsymbol{\cdot}}, I^{\boldsymbol{\cdot}}) \xrightarrow{\sim} \mathrm{Hom}_{D^{+}(\mathcal{C})}(X^{\boldsymbol{\cdot}}, Y^{\boldsymbol{\cdot}}[n]),$$

where $I^{\boldsymbol{\cdot}}$ *is an injective resolution of* $Y^{\boldsymbol{\cdot}}$.

In the classical literature we denote $H^n\mathrm{RHom}_{\mathcal{C}}(X^\cdot, Y^\cdot)$ by $Ext^n_{\mathcal{C}}(X^\cdot, Y^\cdot)$ and call it the nth *hyperextension group* of $X^\cdot$ and $Y^\cdot$.

Next we shall explain the bifunctor $\bullet \otimes^L \bullet$. Let T_0 be a topological space and $\mathcal{R}$ a sheaf of rings on T_0. Denote by $\mathrm{Mod}(\mathcal{R})$ (resp. $\mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$) the abelian category of sheaves of left (resp. right) $\mathcal{R}$-modules on T_0. Here $\mathcal{R}^{\mathrm{op}}$ denotes the opposite ring of $\mathcal{R}$. Set $\mathcal{C}_1 = \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$ and $\mathcal{C}_2 = \mathrm{Mod}(\mathcal{R})$. Then there exists a bifunctor of tensor products

$$\bullet \otimes_{\mathcal{R}} \bullet : \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathrm{Sh}(T_0).$$

For two complexes $X^\cdot \in \mathrm{Ob}(C(\mathcal{C}_1))$ and $Y^\cdot \in \mathrm{Ob}(C(\mathcal{C}_2))$ we define a new complex $(X^\cdot \otimes_{\mathcal{R}} Y^\cdot)^\cdot \in \mathrm{Ob}(C(\mathrm{Sh}(T_0)))$ by

$$\begin{cases} (X^\cdot \otimes_{\mathcal{R}} Y^\cdot)^n = \displaystyle\prod_{i+j=n} X^i \otimes_{\mathcal{R}} Y^j \\ d^n = d^n_{(X^\cdot \otimes_{\mathcal{R}} Y^\cdot)^\cdot} : \quad \displaystyle\prod_{i+j=n} X^i \otimes_{\mathcal{R}} Y^j \quad \longrightarrow \quad \prod_{i+j=n+1} X^i \otimes_{\mathcal{R}} Y^j \\ \qquad\qquad\qquad\qquad\quad \cup\!\!\!\!\!\! \quad\qquad\qquad\qquad\qquad\qquad \cup \\ \qquad\qquad\quad \{x_i \otimes y_j\} \longmapsto \quad \left\{d^i_{X^\cdot}(x_i) \times y_j + (-1)^i x_i \otimes d^j_{Y^\cdot}(y_j)\right\}. \end{cases}$$

This is the simple complex associated to the double complex $X^\cdot \otimes_{\mathcal{R}} Y^\cdot$. We thus defined a bifunctor

$$(\bullet \otimes_{\mathcal{R}} \bullet)^\cdot : C(\mathcal{C}_1) \times C(\mathcal{C}_2) \longrightarrow C(\mathrm{Sh}(T_0))$$

which also induces a bifunctor

$$(\bullet \otimes_{\mathcal{R}} \bullet)^\cdot : K(\mathcal{C}_1) \times K(\mathcal{C}_2) \longrightarrow K(\mathrm{Sh}(T_0))$$

in homotopy categories. Similarly we also obtain

$$(\bullet \otimes_{\mathcal{R}} \bullet)^\cdot : K^-(\mathcal{C}_1) \times K^-(\mathcal{C}_2) \longrightarrow K^-(\mathrm{Sh}(T_0)),$$

taking boundedness into account. Note that in general the abelian categories $\mathcal{C}_1 = \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$ and $\mathcal{C}_2 = \mathrm{Mod}(\mathcal{R})$ do not have enough projectives (unless the topological space T_0 consists of a point). So we cannot use projective objects in these categories to derive the above bifunctor. However, it is well known that for any $Y \in \mathrm{Ob}(\mathcal{C}_2)$ there exist a flat $\mathcal{R}$-module $P \in \mathrm{Ob}(\mathcal{C}_2)$ and an epimorphism $P \to Y$. Therefore, we can use the full subcategory $\mathcal{P}$ of $\mathcal{C}_2$ consisting of flat $\mathcal{R}$-modules.

Lemma B.6.3. *Let $X^\cdot \in \mathrm{Ob}(K(\mathcal{C}_1))$ and $P^\cdot \in \mathrm{Ob}(K^-(\mathcal{P}))$. Assume that $X^\cdot$ or $P^\cdot$ is quasi-isomorphic to* 0*. Then the complex $(X^\cdot \otimes_{\mathcal{R}} P^\cdot)^\cdot \in \mathrm{Ob}(K(\mathrm{Sh}(T_0)))$ is also quasi-isomorphic to* 0.

By this lemma and previous arguments we obtain a bifunctor

$$\bullet \otimes^L_{\mathcal{R}} \bullet : D(\mathcal{C}_1) \times D^-(\mathcal{C}_2) \longrightarrow D(\mathrm{Sh}(T_0))$$

in derived categories. Taking boundedness into account, we also obtain a bifunctor

$$\bullet \otimes^L_{\mathcal{R}} \bullet : D^-(\mathcal{C}_1) \times D^-(\mathcal{C}_2) \longrightarrow D^-(\mathrm{Sh}(T_0)).$$

These are bifunctors of triangulated categories. In the classical literature we denote $H^{-n}(X^\cdot \otimes^L_{\mathcal{R}} Y^\cdot)$ by $Tor^{\mathcal{R}}_n(X^\cdot, Y^\cdot)$ and call it the nth *hypertorsion group* of $X^\cdot$ and $Y^\cdot$.

C

Sheaves and Functors in Derived Categories

In this appendix, assuming only few prerequisites for sheaf theory, we introduce basic operations of sheaves in derived categories and their main properties without proofs. For the details we refer to Hartshorne [Ha1], Iversen [Iv], Kashiwara–Schapira [KS2], [KS4]. We also give a proof of Kashiwara's non-characteristic deformation lemma.

C.1 Sheaves and functors

In this section we quickly recall basic operations in sheaf theory. For a topological space X we denote by $\mathrm{Sh}(X)$ the abelian category of sheaves on X. The abelian group of sections of $F \in \mathrm{Sh}(X)$ on an open subset $U \subset X$ is denoted by $F(U)$ or $\Gamma(U, F)$, and the subgroup of $\Gamma(U, F)$ consisting of sections with compact supports is denoted by $\Gamma_c(U, F)$. We thus obtain left exact functors $\Gamma(U, \bullet), \Gamma_c(U, \bullet) : \mathrm{Sh}(X) \to \mathcal{A}b$ for each open subset $U \subset X$, where $\mathcal{A}b$ denotes the abelian category of abelian groups. If $\mathcal{R}$ is a sheaf of rings on X, we denote by $\mathrm{Mod}(\mathcal{R})$ (resp. $\mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$) the abelian category of sheaves of left (resp. right) $\mathcal{R}$-modules on X. Here $\mathcal{R}^{\mathrm{op}}$ denotes the opposite ring of $\mathcal{R}$. For example, in the case where $\mathcal{R}$ is the constant sheaf $\mathbb{Z}_X$ with germs $\mathbb{Z}$ the category $\mathrm{Mod}(\mathcal{R})$ is $\mathrm{Sh}(X)$. For $F, G \in \mathrm{Sh}(X)$ (resp. $M, N \in \mathrm{Mod}(\mathcal{R})$) we denote by $\mathrm{Hom}(F, G)$ (resp. $\mathrm{Hom}_{\mathcal{R}}(M, N)$) the abelian group of sheaf homomorphisms (resp. sheaf homomorphisms commuting with the actions of $\mathcal{R}$) on X from F to G (resp. from M to N). We thus obtain left exact bifunctors

$$\begin{cases} \mathrm{Hom}(\bullet, \bullet) : \mathrm{Sh}(X)^{\mathrm{op}} \times \mathrm{Sh}(X) \longrightarrow \mathcal{A}b, \\ \mathrm{Hom}_{\mathcal{R}}(\bullet, \bullet) : \mathrm{Mod}(\mathcal{R})^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathcal{A}b. \end{cases}$$

For a subset $Z \subset X$, we denote by $i_Z : Z \to X$ the inclusion map.

Definition C.1.1. Let $f : X \to Y$ be a morphism of topological spaces, $F \in \mathrm{Sh}(X)$ and $G \in \mathrm{Sh}(Y)$.

(i) The *direct image* $f_*F \in \mathrm{Sh}(Y)$ of F by f is defined by $f_*F(V) = F(f^{-1}(V))$ for each open subset $V \subset Y$. This gives a left exact functor $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. If Y is the space pt consisting of one point, the functor f_* is the global section functor $\Gamma(X, \bullet) : \mathrm{Sh}(X) \to \mathcal{A}b$.

(ii) The *proper direct image* $f_!F \in \mathrm{Sh}(Y)$ of F by f is defined by $f_!F(V) = \{s \in F(f^{-1}(V)) | f|_{\mathrm{supp}\, s} : \mathrm{supp}\, s \to V \quad \text{is proper}\}$ for each open subset $V \subset Y$. This gives a left exact functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. If Y is pt, the functor $f_!$ is the global section functor with compact supports $\Gamma_c(X, \bullet) : \mathrm{Sh}(X) \to \mathcal{A}b$.

(iii) The *inverse image* $f^{-1}G \in \mathrm{Sh}(X)$ of G by f is the sheaf associated to the presheaf $(f^{-1}G)'$ defined by $(f^{-1}G)'(U) = \varinjlim G(V)$ for each open subset $U \subset X$, where V ranges through the family of open subsets of Y containing $f(U)$. Since we have an isomorphism $(f^{-1}G)_x \simeq G_{f(x)}$ for any $x \in X$, we obtain an exact functor $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$.

When we treat proper direct images $f_!$ in this book, all topological spaces are assumed to be locally compact and Hausdorff. For two morphisms $f : X \to Y$, $g : Y \to Z$ of topological spaces, we have obvious relations $g_* \circ f_* = (g \circ f)_*$, $g_! \circ f_! = (g \circ f)_!$ and $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$. For $F \in \mathrm{Sh}(X)$ and a subset $Z \subset X$ the inverse image $i_Z^{-1}F \in \mathrm{Sh}(Z)$ of F by the inclusion map $i_Z : Z \to X$ is sometimes denoted by $F|_Z$. If Z is a locally closed subset of X (i.e., a subset of X which is written as an intersection of an open subset and a closed subset), then it is well known that the functor $(i_Z)_! : \mathrm{Sh}(Z) \to \mathrm{Sh}(X)$ is exact.

Proposition C.1.2. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

be a **cartesian square** *of topological spaces, i.e., X' is homeomorphic to the fiber product $X \times_Y Y'$. Then there exists an isomorphism of functors $g^{-1} \circ f_! \simeq f'_! \circ g'^{-1}$: $\mathrm{Sh}(X) \to \mathrm{Sh}(Y')$.*

Definition C.1.3. Let X be a topological space, $Z \subset X$ a locally closed subset and $F \in \mathrm{Sh}(X)$.

(i) Set $F_Z = (i_Z)_!(i_Z)^{-1}F \in \mathrm{Sh}(X)$. Since we have $(F_Z)_x \simeq F_x$ (resp. $(F_Z)_x \simeq 0$) for any $x \in Z$ (resp. $x \in X \setminus Z$), we obtain an exact functor $(\bullet)_Z : \mathrm{Sh}(X) \to \mathrm{Sh}(X)$.

(ii) Take an open subset W of X containing Z as a closed subset of W. Since the abelian group $\mathrm{Ker}[F(W) \to F(W \setminus Z)]$ does not depend on the choice of W, we denote it by $\Gamma_Z(X, F)$. This gives a left exact functor $\Gamma_Z(X, \bullet) : \mathrm{Sh}(X) \to \mathcal{A}b$.

(iii) The subsheaf $\Gamma_Z(F)$ of F is defined by $\Gamma_Z(F)(U) = \Gamma_{Z \cap U}(U, F|_U)$ for each open subset $U \subset X$. This gives a left exact functor $\Gamma_Z(\bullet) : \mathrm{Sh}(X) \to \mathrm{Sh}(X)$. By construction we have an isomorphism of functors $\Gamma(X, \bullet) \circ \Gamma_Z(\bullet) = \Gamma_Z(X, \bullet)$.

Note that if U is an open subset of X and $j = i_U : U \to X$, then there exists an isomorphism of functors $j_* \circ j^{-1} \simeq \Gamma_U(\bullet)$.

Lemma C.1.4. *Let X be a topological space, Z a locally closed subset of X and Z' a closed subset of Z. Also let Z_1, Z_2 (resp. U_1, U_2) be closed (resp. open) subsets of X and $F \in \mathrm{Sh}(X)$.*

(i) *There exists a natural exact sequence*

$$0 \longrightarrow F_{Z\setminus Z'} \longrightarrow F_Z \longrightarrow F_{Z'} \longrightarrow 0 \tag{C.1.1}$$

in $\mathrm{Sh}(X)$.

(ii) *There exist natural exact sequences*

$$0 \longrightarrow \Gamma_{Z'}(F) \longrightarrow \Gamma_Z(F) \longrightarrow \Gamma_{Z\setminus Z'}(F), \tag{C.1.2}$$

$$0 \longrightarrow \Gamma_{Z_1\cap Z_2}(F) \longrightarrow \Gamma_{Z_1}(F) \oplus \Gamma_{Z_2}(F) \longrightarrow \Gamma_{Z_1\cup Z_2}(F), \tag{C.1.3}$$

$$0 \longrightarrow \Gamma_{U_1\cup U_2}(F) \longrightarrow \Gamma_{U_1}(F) \oplus \Gamma_{U_2}(F) \longrightarrow \Gamma_{U_1\cap U_2}(F). \tag{C.1.4}$$

in $\mathrm{Sh}(X)$.

Recall that a sheaf $F \in \mathrm{Sh}(X)$ on X is called *flabby* if the restriction morphism $F(X) \to F(U)$ is surjective for any open subset $U \subset X$.

Lemma C.1.5.

(i) *Let Z be a locally closed subset of X and $F \in \mathrm{Sh}(X)$ a flabby sheaf. Then the sheaf $\Gamma_Z(F)$ is flabby. Moreover, for any morphism $f : X \to Y$ of topological spaces the direct image f_*F is flabby.*

(ii) *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Sh}(X)$ and Z a locally closed subset of X. Assume that F' is flabby. Then the sequences $0 \to \Gamma_Z(X, F') \to \Gamma_Z(X, F) \to \Gamma_Z(X, F'') \to 0$ and $0 \to \Gamma_Z(F') \to \Gamma_Z(F) \to \Gamma_Z(F'') \to 0$ are exact.*

(iii) *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Sh}(X)$ and assume that F' and F are flabby. Then F'' is also flabby.*

(iv) *In the situation of Lemma* C.1.4, *assume, moreover, that F is flabby. Then there are natural exact sequences*

$$0 \longrightarrow \Gamma_{Z'}(F) \longrightarrow \Gamma_Z(F) \longrightarrow \Gamma_{Z\setminus Z'}(F) \longrightarrow 0, \tag{C.1.5}$$

$$0 \longrightarrow \Gamma_{Z_1\cap Z_2}(F) \longrightarrow \Gamma_{Z_1}(F) \oplus \Gamma_{Z_2}(F) \longrightarrow \Gamma_{Z_1\cup Z_2}(F) \longrightarrow 0, \tag{C.1.6}$$

$$0 \longrightarrow \Gamma_{U_1\cup U_2}(F) \longrightarrow \Gamma_{U_1}(F) \oplus \Gamma_{U_2}(F) \longrightarrow \Gamma_{U_1\cap U_2}(F) \longrightarrow 0. \tag{C.1.7}$$

in $\mathrm{Sh}(X)$.

Definition C.1.6. Let X be a topological space and $\mathcal{R}$ a sheaf of rings on X.

(i) For $M, N \in \mathrm{Mod}(\mathcal{R})$ the sheaf $\mathcal{H}om_{\mathcal{R}}(M, N) \in \mathrm{Sh}(X)$ of $\mathcal{R}$-linear homomorphisms from M to N is defined by $\mathcal{H}om_{\mathcal{R}}(M, N)(U) = \mathrm{Hom}_{\mathcal{R}|_U}(M|_U, N|_U)$ for each open subset $U \subset X$. This gives a left exact bifunctor $\mathcal{H}om_{\mathcal{R}}(\bullet, \bullet) : \mathrm{Mod}(\mathcal{R})^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{R}) \to \mathrm{Sh}(X)$. By definition we have $\Gamma(X; \mathcal{H}om_{\mathcal{R}}(M, N)) = \mathrm{Hom}_{\mathcal{R}}(M, N)$.

(ii) For $M \in \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$, $N \in \mathrm{Mod}(\mathcal{R})$ the tensor product $M \otimes_{\mathcal{R}} N \in \mathrm{Sh}(X)$ of M and N is the sheaf associated to the presheaf $(M \otimes_{\mathcal{R}} N)'$ defined by $(M \otimes_{\mathcal{R}} N)'(U) = M(U) \otimes_{\mathcal{R}(U)} N(U)$ for each open subset $U \subset X$. Since by definition we have an isomorphism $(M \otimes_{\mathcal{R}} N)_x \simeq M_x \otimes_{\mathcal{R}_x} N_x$ for any $x \in X$, we obtain a right exact bifunctor $\bullet \otimes_{\mathcal{R}} \bullet : \mathrm{Mod}(\mathcal{R}^{\mathrm{op}}) \times \mathrm{Mod}(\mathcal{R}) \to \mathrm{Sh}(X)$.

Note that for any $M \in \mathrm{Mod}(\mathcal{R})$ the sheaf $\mathcal{H}om_{\mathcal{R}}(\mathcal{R}, M)$ is a left $\mathcal{R}$-module by the right multiplication of $\mathcal{R}$ on $\mathcal{R}$ itself, and there exists an isomorphism $\mathcal{H}om_{\mathcal{R}}(\mathcal{R}, M) \simeq M$ of left $\mathcal{R}$-modules. Now recall that $M \in \mathrm{Mod}(\mathcal{R})$ is an injective (resp. a projective) object of $\mathrm{Mod}(\mathcal{R})$ if the functor $\mathrm{Hom}_{\mathcal{R}}(\bullet, M)$ (resp. $\mathrm{Hom}_{\mathcal{R}}(M, \bullet)$) is exact.

Proposition C.1.7. *Let $\mathcal{R}$ be a sheaf of rings on X. Then the abelian category* $\mathrm{Mod}(\mathcal{R})$ *has* **enough injectives**.

An injective object in $\mathrm{Mod}(\mathcal{R})$ is sometimes called an *injective sheaf* or an injective $\mathcal{R}$-module.

Definition C.1.8. We say that $M \in \mathrm{Mod}(\mathcal{R})$ is *flat* (or a flat $\mathcal{R}$-module) if the functor $\bullet \otimes_{\mathcal{R}} M : \mathrm{Mod}(\mathcal{R}^{\mathrm{op}}) \to \mathrm{Sh}(X)$ is exact.

By the definition of tensor products, $M \in \mathrm{Mod}(\mathcal{R})$ is flat if and only if the stalk M_x is a flat $\mathcal{R}_x$-module for any $x \in X$. Although in general the category $\mathrm{Mod}(\mathcal{R})$ does not have enough projectives (unless X is the space pt consisting of one point), we have the following useful result.

Proposition C.1.9. *Let $\mathcal{R}$ be a sheaf of rings on X. Then for any $M \in \mathrm{Mod}(\mathcal{R})$ there exist a flat $\mathcal{R}$-module P and an epimorphism $P \to M$.*

Lemma C.1.10. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in* $\mathrm{Mod}(\mathcal{R})$.

(i) *Assume that M' and M are injective. Then M'' is also injective.*
(ii) *Assume that M and M'' are flat. Then M' is also flat.*

Proposition C.1.11. *Let $f : Y \to X$ be a morphism of topological spaces and $\mathcal{R}$ a sheaf of rings on X.*

(i) *Let $M_1 \in \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$ and $M_2 \in \mathrm{Mod}(\mathcal{R})$. Then there exists an isomorphism*

$$f^{-1}M_1 \otimes_{f^{-1}\mathcal{R}} f^{-1}M_2 \simeq f^{-1}(M_1 \otimes_{\mathcal{R}} M_2) \tag{C.1.8}$$

in $\mathrm{Sh}(Y)$.

(ii) *Let $M \in \mathrm{Mod}(\mathcal{R})$ and $N \in \mathrm{Mod}(f^{-1}\mathcal{R})$. Then there exists an isomorphism*

$$\mathcal{H}om_{\mathcal{R}}(M, f_*N) \simeq f_*\mathcal{H}om_{f^{-1}\mathcal{R}}(f^{-1}M, N) \tag{C.1.9}$$

in $\mathrm{Sh}(X)$, *where f_*N is a left $\mathcal{R}$-module by the natural ring homomorphism $\mathcal{R} \to f_*f^{-1}\mathcal{R}$. In particular we have an isomorphism*

$$\mathrm{Hom}_{\mathcal{R}}(M, f_*N) \simeq \mathrm{Hom}_{f^{-1}\mathcal{R}}(f^{-1}M, N). \tag{C.1.10}$$

Namely, the functor f_* is a right adjoint of f^{-1}.

(iii) *Let* $M \in \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$ *and* $N \in \mathrm{Mod}(f^{-1}\mathcal{R})$. *Then there exists a natural morphism*

$$M \otimes_{\mathcal{R}} f_! N \longrightarrow f_!(f^{-1}M \otimes_{f^{-1}\mathcal{R}} N) \tag{C.1.11}$$

in $\mathrm{Sh}(X)$. *Moreover, this morphism is an isomorphism if* M *is a flat* $\mathcal{R}^{\mathrm{op}}$*-module.*

Corollary C.1.12. *Let* $f : Y \to X$ *be a morphism of topological spaces,* $\mathcal{R}$ *a sheaf of rings on* X *and* $N \in \mathrm{Mod}(f^{-1}\mathcal{R})$ *an injective* $f^{-1}\mathcal{R}$*-module. Then the direct image* f_*N *is an injective* $\mathcal{R}$*-module.*

Lemma C.1.13. *Let* Z *be a locally closed subset of* X, $\mathcal{R}$ *a sheaf of rings on* X *and* $M, N \in \mathrm{Mod}(\mathcal{R})$. *Then we have natural isomorphisms*

$$\Gamma_Z \mathcal{H}om_{\mathcal{R}}(M, N) \simeq \mathcal{H}om_{\mathcal{R}}(M, \Gamma_Z N) \simeq \mathcal{H}om_{\mathcal{R}}(M_Z, N). \tag{C.1.12}$$

Corollary C.1.14. *Let* $\mathcal{R}$ *be a sheaf of rings on* X, Z *a locally closed subset of* X *and* $M, N \in \mathrm{Mod}(\mathcal{R})$. *Assume that* N *is an injective* $\mathcal{R}$*-module. Then the sheaf* $\mathcal{H}om_{\mathcal{R}}(M, N)$ (*resp.* $\Gamma_Z N$) *is flabby* (*resp. an injective* $\mathcal{R}$*-module*). *In particular, any injective* $\mathcal{R}$*-module is flabby.*

C.2 Functors in derived categories of sheaves

Applying the results in Appendix B to functors of operations of sheaves, we can introduce various functors in derived categories of sheaves as follows.

Let X be a topological space and $\mathcal{R}$ a sheaf of rings on X. Since the category $\mathrm{Mod}(\mathcal{R})$ is abelian, we obtain a derived category $D(\mathrm{Mod}(\mathcal{R}))$ of complexes in $\mathrm{Mod}(\mathcal{R})$ and its full subcategories $D^{\#}(\mathrm{Mod}(\mathcal{R}))$ ($\# = +, -, b$). In this book for $\# = \emptyset, +, -, b$ we sometimes denote $D^{\#}(\mathrm{Mod}(\mathcal{R}))$ by $D^{\#}(\mathcal{R})$ for the sake of simplicity. For example, we set $D^+(\mathbb{Z}_X) = D^+(\mathrm{Sh}(X))$. Now let Z be a locally closed subset of X, and $f : Y \to X$ a morphism of topological spaces. Consider the following left exact functors:

$$\begin{cases} \Gamma(X, \bullet), \Gamma_c(X, \bullet), \Gamma_Z(X, \bullet) : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathcal{A}b, \\ \Gamma_Z(\bullet) : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}), \\ f_*, f_! : \mathrm{Mod}(f^{-1}\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}). \end{cases} \tag{C.2.1}$$

Since the categories $\mathrm{Mod}(\mathcal{R})$ and $\mathrm{Mod}(f^{-1}\mathcal{R})$ have enough injectives we obtain their derived functors

$$\begin{cases} \mathrm{R}\Gamma(X, \bullet), \mathrm{R}\Gamma_c(X, \bullet), \mathrm{R}\Gamma_Z(X, \bullet) : D^+(\mathcal{R}) \longrightarrow D^+(\mathcal{A}b), \\ \mathrm{R}\Gamma_Z(\bullet) : D^+(\mathcal{R}) \longrightarrow D^+(\mathcal{R}), \\ Rf_*, Rf_! : D^+(f^{-1}\mathcal{R}) \longrightarrow D^+(\mathcal{R}). \end{cases} \tag{C.2.2}$$

For example, for $M^{\cdot} \in D^+(\mathcal{R})$ the object $\mathrm{R}\Gamma(X, F^{\cdot}) \in D^+(\mathcal{A}b)$ is calculated as follows. First take a quasi-isomorphism $M^{\cdot} \xrightarrow{\sim} I^{\cdot}$ in $C^+(\mathcal{R})$ such that I^k is an injective $\mathcal{R}$-module for any $k \in \mathbb{Z}$. Then we have $\mathrm{R}\Gamma(X, F^{\cdot}) \simeq \Gamma(X, I^{\cdot})$.

Since by Lemma C.1.5 the full subcategory $\mathcal{J}$ of $\mathrm{Mod}(\mathcal{R})$ consisting of flabby sheaves is $\Gamma(X, \bullet)$-injective in the sense of Definition B.5.4, we can also take a quasi-isomorphism $M^\cdot \xrightarrow{\sim} J^\cdot$ in $C^+(\mathcal{R})$ such that $J^k \in \mathcal{J}$ for any $k \in \mathbb{Z}$ and show that $\mathrm{R}\Gamma(X, F^\cdot) \simeq \Gamma(X, J^\cdot)$. Let us apply Proposition B.5.7 to the identity $\Gamma_Z(X, \bullet) = \Gamma(X, \bullet) \circ \Gamma_Z(\bullet) : \mathrm{Mod}(\mathcal{R}) \to \mathcal{A}b$. Then by the fact that the functor $\Gamma_Z(\bullet) \simeq \mathcal{H}om_{\mathcal{R}}(\mathcal{R}_Z, \bullet)$ sends injective sheaves to injective sheaves (Corollary C.1.14), we obtain an isomorphism

$$\mathrm{R}\Gamma(X, \mathrm{R}\Gamma_Z(M^\cdot)) \simeq \mathrm{R}\Gamma_Z(X, M^\cdot) \tag{C.2.3}$$

in $D^+(\mathcal{A}b)$ for any $M^\cdot \in D^+(\mathcal{R})$. Similarly by Corollary C.1.12 we obtain an isomorphism

$$\mathrm{R}\Gamma(X, Rf_*(N^\cdot)) \simeq \mathrm{R}\Gamma(Y, N^\cdot) \tag{C.2.4}$$

in $D^+(\mathcal{A}b)$ for any $N^\cdot \in D^+(f^{-1}\mathcal{R})$ (also the similar formula $\mathrm{R}\Gamma_c(X, Rf_!(N^\cdot)) \simeq \mathrm{R}\Gamma_c(Y, N^\cdot)$ can be proved). For $M^\cdot \in D^+(\mathcal{R})$ and $i \in \mathbb{Z}$ we sometimes denote $H^i\mathrm{R}\Gamma(X, M^\cdot)$, $H^i\mathrm{R}\Gamma_Z(X, M^\cdot)$, $H^i\mathrm{R}\Gamma_Z(M^\cdot)$ simply by $H^i(X, M^\cdot)$, $H^i_Z(X, M^\cdot)$, $H^i_Z(M^\cdot)$, respectively. Now let us consider the functors

$$\begin{cases} f^{-1} : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(f^{-1}\mathcal{R}), \\ (\bullet)_Z : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}), \\ (i_Z)_! : \mathrm{Sh}(Z) \to \mathrm{Sh}(X). \end{cases} \tag{C.2.5}$$

Since these functors are exact, they extend naturally to the following functors in derived categories:

$$\begin{cases} f^{-1} : D^\#(\mathcal{R}) \longrightarrow D^\#(f^{-1}\mathcal{R}), \\ (\bullet)_Z : D^\#(\mathcal{R}) \longrightarrow D^\#(\mathcal{R}), \\ (i_Z)_! : D^\#(\mathrm{Sh}(Z)) \to D^\#(\mathrm{Sh}(X)) \end{cases} \tag{C.2.6}$$

for $\# = \emptyset, +, -, b$. Let Z' be a closed subset of Z and Z_1, Z_2 (resp. U_1, U_2) closed (resp. open) subsets of X. Then by Lemma C.1.5 and Lemma B.4.5, for $M^\cdot \in D^+(\mathcal{R})$ we obtain the following distinguished triangles in $D^+(\mathcal{R})$:

$$M_{Z \setminus Z'}{}^\cdot \longrightarrow M_Z{}^\cdot \longrightarrow M_{Z'}{}^\cdot \xrightarrow{+1}, \tag{C.2.7}$$

$$\mathrm{R}\Gamma_{Z'}(M^\cdot) \longrightarrow \mathrm{R}\Gamma_Z(M^\cdot) \longrightarrow \mathrm{R}\Gamma_{Z\setminus Z'}(M^\cdot) \xrightarrow{+1}, \tag{C.2.8}$$

$$\mathrm{R}\Gamma_{Z_1 \cap Z_2}(M^\cdot) \longrightarrow \mathrm{R}\Gamma_{Z_1}(M^\cdot) \oplus \mathrm{R}\Gamma_{Z_2}(M^\cdot) \longrightarrow \mathrm{R}\Gamma_{Z_1 \cup Z_2}(M^\cdot) \xrightarrow{+1}, \tag{C.2.9}$$

$$\mathrm{R}\Gamma_{U_1 \cup U_2}(M^\cdot) \longrightarrow \mathrm{R}\Gamma_{U_1}(M^\cdot) \oplus \mathrm{R}\Gamma_{U_2}(M^\cdot) \longrightarrow \mathrm{R}\Gamma_{U_1 \cap U_2}(M^\cdot) \xrightarrow{+1}. \tag{C.2.10}$$

The following result is also well known.

Proposition C.2.1. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

be a cartesian square of topological spaces. Then there exists an isomorphism of functors $g^{-1} \circ Rf_! \simeq Rf'_! \circ g'^{-1} : D^+(\mathrm{Sh}(X)) \to D^+(\mathrm{Sh}(Y'))$.

For the proof see [KS2, Proposition 2.6.7]. From now on, let us introduce bifunctors in derived categories of sheaves. Let $\mathcal{R}$ be a sheaf of rings on a topological space X. Then by applying the construction in Section B.6 to $\mathcal{C} = \mathrm{Mod}(\mathcal{R})$ we obtain a bifunctor

$$\mathrm{RHom}_{\mathcal{R}}(\bullet, \bullet) : D^-(\mathcal{R})^{\mathrm{op}} \times D^+(\mathcal{R}) \longrightarrow D^+(\mathcal{A}b). \tag{C.2.11}$$

Similarly we obtain a bifunctor

$$R\mathcal{H}om_{\mathcal{R}}(\bullet, \bullet) : D^-(\mathcal{R})^{\mathrm{op}} \times D^+(\mathcal{R}) \longrightarrow D^+(\mathrm{Sh}(X)). \tag{C.2.12}$$

For $M^{\boldsymbol{\cdot}} \in D^-(\mathcal{R})$ and $N^{\boldsymbol{\cdot}} \in D^+(\mathcal{R})$ the objects $\mathrm{RHom}_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, N^{\boldsymbol{\cdot}}) \in D^+(\mathcal{A}b)$ and $R\mathcal{H}om_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, N^{\boldsymbol{\cdot}}) \in D^+(\mathrm{Sh}(X))$ are more explicitly calculated as follows. Take a quasi-isomorphism $N^{\boldsymbol{\cdot}} \xrightarrow{\sim} I^{\boldsymbol{\cdot}}$ such that I^k is an injective $\mathcal{R}$-module for any $k \in \mathbb{Z}$ and consider the simple complex $\mathrm{Hom}^{\boldsymbol{\cdot}}_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, I^{\boldsymbol{\cdot}}) \in C^+(\mathcal{A}b)$ (resp. $\mathcal{H}om^{\boldsymbol{\cdot}}_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, I^{\boldsymbol{\cdot}}) \in C^+(\mathrm{Sh}(X))$) associated to the double complex $\mathrm{Hom}_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, I^{\boldsymbol{\cdot}})$ (resp. $\mathcal{H}om_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, I^{\boldsymbol{\cdot}})$) as in Section B.6. Then we have isomorphisms $\mathrm{RHom}_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, N^{\boldsymbol{\cdot}}) \simeq \mathrm{Hom}^{\boldsymbol{\cdot}}_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, I^{\boldsymbol{\cdot}})$ and $R\mathcal{H}om_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, N^{\boldsymbol{\cdot}}) \simeq \mathcal{H}om^{\boldsymbol{\cdot}}_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, I^{\boldsymbol{\cdot}})$. For $M^{\boldsymbol{\cdot}} \in D^-(\mathcal{R})$, $N^{\boldsymbol{\cdot}} \in D^+(\mathcal{R})$ and $i \in \mathbb{Z}$ we sometimes denote $H^i\mathrm{RHom}_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, N^{\boldsymbol{\cdot}})$, $H^i R\mathcal{H}om_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, N^{\boldsymbol{\cdot}})$ simply by $Ext_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, N^{\boldsymbol{\cdot}})$, $\mathcal{E}xt_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, N^{\boldsymbol{\cdot}})$, respectively. Since the full subcategory of $\mathrm{Sh}(X)$ consisting of flabby sheaves is $\Gamma(X, \bullet)$-injective, from Corollary C.1.14 and the obvious identity $\Gamma(X, \mathcal{H}om_{\mathcal{R}}(\bullet, \bullet)) = \mathrm{Hom}_{\mathcal{R}}(\bullet, \bullet)$, we obtain an isomorphism

$$\mathrm{R}\Gamma(X, R\mathcal{H}om_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, N^{\boldsymbol{\cdot}})) \simeq \mathrm{RHom}_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, N^{\boldsymbol{\cdot}}) \tag{C.2.13}$$

in $D^+(\mathcal{A}b)$ for any $M^{\boldsymbol{\cdot}} \in D^-(\mathcal{R})$ and $N^{\boldsymbol{\cdot}} \in D^+(\mathcal{R})$. Let us apply the same argument to the identities in Lemma C.1.13. Then by Lemma C.1.5 and Corollary C.1.14 we obtain the following.

Proposition C.2.2. *Let Z be a locally closed subset of X, $\mathcal{R}$ a sheaf of rings on X, $M^{\boldsymbol{\cdot}} \in D^-(\mathcal{R})$ and $N^{\boldsymbol{\cdot}} \in D^+(\mathcal{R})$. Then we have isomorphisms*

$$\mathrm{R}\Gamma_Z R\mathcal{H}om_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, N^{\boldsymbol{\cdot}}) \simeq R\mathcal{H}om_{\mathcal{R}}(M^{\boldsymbol{\cdot}}, \mathrm{R}\Gamma_Z N^{\boldsymbol{\cdot}}) \simeq R\mathcal{H}om_{\mathcal{R}}(M_Z{}^{\boldsymbol{\cdot}}, N^{\boldsymbol{\cdot}}). \tag{C.2.14}$$

Similarly, from Proposition C.1.11 (ii) we obtain the following.

Proposition C.2.3 (Adjunction formula). *Let $f : Y \to X$ be a morphism of topological spaces, $\mathcal{R}$ a sheaf of rings on X. Let $M^{\boldsymbol{\cdot}} \in D^-(\mathcal{R})$ and $N^{\boldsymbol{\cdot}} \in D^+(f^{-1}\mathcal{R})$. Then there exists an isomorphism*

$$R\mathcal{H}om_{\mathcal{R}}(M^{\cdot}, Rf_*N^{\cdot}) \simeq Rf_*R\mathcal{H}om_{f^{-1}\mathcal{R}}(f^{-1}M^{\cdot}, N^{\cdot}) \tag{C.2.15}$$

in $D^+(\mathrm{Sh}(X))$. *Moreover, we have an isomorphism*

$$\mathrm{RHom}_{\mathcal{R}}(M^{\cdot}, Rf_*N^{\cdot}) \simeq \mathrm{RHom}_{f^{-1}\mathcal{R}}(f^{-1}M^{\cdot}, N^{\cdot}) \tag{C.2.16}$$

in $D^+(\mathcal{A}b)$.

By Proposition B.6.2 and the same argument as above, we also obtain the following.

Proposition C.2.4. *In the situation of Proposition* C.2.3, *for any* $L^{\cdot} \in D^+(\mathcal{R})$ *and* $N^{\cdot} \in D^+(f^{-1}\mathcal{R})$ *there exists an isomorphism*

$$\mathrm{Hom}_{D^+(\mathcal{R})}(L^{\cdot}, Rf_*N^{\cdot}) \simeq \mathrm{Hom}_{D^+(f^{-1}\mathcal{R})}(f^{-1}L^{\cdot}, N^{\cdot}). \tag{C.2.17}$$

Namely, the functor $f^{-1} : D^+(\mathcal{R}) \to D^+(f^{-1}\mathcal{R})$ *is a left adjoint of* $Rf_* : D^+(f^{-1}\mathcal{R}) \to D^+(\mathcal{R})$.

Next we shall introduce the derived functor of the bifunctor of tensor products. Let X be a topological space and $\mathcal{R}$ a sheaf of rings on X. Then by the results in Section B.6, there exists a right exact bifunctor of tensor products

$$\bullet \otimes_{\mathcal{R}} \bullet : \mathrm{Mod}(\mathcal{R}^{\mathrm{op}}) \times \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Sh}(X), \tag{C.2.18}$$

and its derived functor

$$\bullet \otimes^L_{\mathcal{R}} \bullet : D^-(\mathcal{R}^{\mathrm{op}}) \times D^-(\mathcal{R}) \longrightarrow D^-(\mathrm{Sh}(X)). \tag{C.2.19}$$

From now on, let us assume, moreover, that $\mathcal{R}$ has *finite weak global dimension*, i.e., there exists an integer $d > 0$ such that the weak global dimension of the ring $\mathcal{R}_x$ is less than or equal to d for any $x \in X$. Then for any $M^{\cdot} \in C^+(\mathrm{Mod}(\mathcal{R}))$ (resp. $C^b(\mathrm{Mod}(\mathcal{R}))$) we can construct a quasi-isomorphism $P^{\cdot} \to M^{\cdot}$ for $P^{\cdot} \in C^+(\mathrm{Mod}(\mathcal{R}))$ (resp. $C^b(\mathrm{Mod}(\mathcal{R}))$) such that P^k is a flat $\mathcal{R}$-module for any $k \in \mathbb{Z}$. Hence we obtain also bifunctors

$$\bullet \otimes^L_{\mathcal{R}} \bullet : D^{\#}(\mathcal{R}^{\mathrm{op}}) \times D^{\#}(\mathcal{R}) \longrightarrow D^{\#}(\mathrm{Sh}(X)) \tag{C.2.20}$$

for $\# = +, b$. By definition, we immediately obtain the following.

Proposition C.2.5. *Let* $f : Y \to X$ *be a morphism of topological spaces and* $\mathcal{R}$ *a sheaf of rings on* X. *Let* $M^{\cdot} \in D^-(\mathcal{R}^{\mathrm{op}})$ *and* $N^{\cdot} \in D^-(\mathcal{R})$. *Then there exists an isomorphism*

$$f^{-1}M^{\cdot} \otimes^L_{f^{-1}\mathcal{R}} f^{-1}N^{\cdot} \simeq f^{-1}(M^{\cdot} \otimes^L_{\mathcal{R}} N^{\cdot}) \tag{C.2.21}$$

in $D^-(\mathrm{Sh}(Y))$.

The following result is also well known.

Proposition C.2.6 (Projection formula). *Let $f : Y \to X$ be a morphism of topological spaces and $\mathcal{R}$ a sheaf of rings on X. Assume that $\mathcal{R}$ has finite weak global dimension. Let $M^{\cdot} \in D^{+}(\mathcal{R}^{\mathrm{op}})$ and $N^{\cdot} \in D^{+}(f^{-1}\mathcal{R})$. Then there exists an isomorphism*

$$M^{\cdot} \otimes^{L}_{\mathcal{R}} Rf_{!}N^{\cdot} \xrightarrow{\sim} Rf_{!}(f^{-1}M^{\cdot} \otimes^{L}_{f^{-1}\mathcal{R}} N^{\cdot}) \tag{C.2.22}$$

in $D^{+}(\mathrm{Sh}(X))$.

For the proof see [KS2, Proposition 2.6.6].

Finally, let us explain the *Poincaré–Verdier duality*. Now let $f : X \to Y$ be a continuous map of locally compact and Hausdorff topological spaces. Let A be a commutative ring with finite global dimension, e.g., a field k. In what follows, we always assume the following condition for f.

Definition C.2.7. We say that the functor $f_{!} : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ has *finite cohomological dimension* if there exists an integer $d > 0$ such that for any sheaf F on X we have $H^{k}Rf_{!}(F) = 0$ for any $k > d$.

Theorem C.2.8 (Poincaré–Verdier duality theorem). *In the situation as above, there exists a functor of triangulated categories $f^{!} : D^{+}(A_Y) \to D^{+}(A_X)$ such that for any $M^{\cdot} \in D^{b}(A_X)$ and $N^{\cdot} \in D^{+}(A_Y)$ we have isomorphisms*

$$Rf_{*}R\mathcal{H}om_{A_X}(M^{\cdot}, f^{!}N^{\cdot}) \simeq R\mathcal{H}om_{A_Y}(Rf_{!}M^{\cdot}, N^{\cdot}), \tag{C.2.23}$$

$$\mathrm{RHom}_{A_X}(M^{\cdot}, f^{!}N^{\cdot}) \simeq \mathrm{RHom}_{A_Y}(Rf_{!}M^{\cdot}, N^{\cdot}) \tag{C.2.24}$$

in $D^{+}(A_Y)$ and $D^{+}(\mathrm{Mod}(A))$, respectively.

We call the functor $f^{!} : D^{+}(A_Y) \to D^{+}(A_X)$ the *twisted inverse image* functor by f. Since the construction of this functor $f^{!}(\bullet)$ is a little bit complicated, we do not explain it here. For the details see Kashiwara–Schapira [KS2, Chapter III]. Let us give basic properties of twisted inverse images. First, for a morphism $g : Y \to Z$ of topological spaces satisfying the same assumption as f, we have an isomorphism $(g \circ f)^{!} \simeq f^{!} \circ g^{!}$ of functors.

Theorem C.2.9. *Let $f : X \to Y$ be as above. Then for any $M^{\cdot} \in D^{+}(A_X)$ and $N^{\cdot} \in D^{+}(A_Y)$ we have an isomorphism*

$$\mathrm{Hom}_{D^{+}(A_X)}(M^{\cdot}, f^{!}N^{\cdot}) \simeq \mathrm{Hom}_{D^{+}(A_Y)}(Rf_{!}M^{\cdot}, N^{\cdot}).$$

Namely, the functor $f^{!} : D^{+}(A_Y) \to D^{+}(A_X)$ is a right adjoint of $Rf_{!} : D^{+}(A_X) \to D^{+}(A_Y)$.

Proposition C.2.10. *Let $f : X \to Y$ be as above. Then for any $N_1^{\cdot} \in D^{b}(A_Y)$ and $N_2^{\cdot} \in D^{+}(A_Y)$ we have an isomorphism*

$$f^{!}R\mathcal{H}om_{A_Y}(N_1^{\cdot}, N_2^{\cdot}) \simeq R\mathcal{H}om_{A_X}(f^{-1}N_1^{\cdot}, f^{!}N_2^{\cdot}).$$

Proposition C.2.11. *Assume that X is a locally closed subset of Y and let $f = i_X : X \hookrightarrow Y$ be the embedding. Then we have an isomorphism*

$$f^!(N^{\cdot}) \simeq f^{-1}(\mathrm{R}\Gamma_{f(X)}(N^{\cdot})) = (\mathrm{R}\Gamma_{f(X)}(N^{\cdot}))|_X \tag{C.2.25}$$

in $D^+(A_X)$ for any $N^{\cdot} \in D^+(A_Y)$.

Proposition C.2.12. *Assume that X and Y are real C^1-manifolds and $f : X \to Y$ is a C^1-submersion. Set $d = \dim X - \dim Y$. Then*

(i) *$H^j(f^!(A_Y)) = 0$ for any $j \neq -d$ and $H^{-d}(f^!(A_Y)) \in \mathrm{Mod}(A_X)$ is a locally constant sheaf of rank one over A_X.*
(ii) *For any $N^{\cdot} \in D^+(A_Y)$ there exists an isomorphism*

$$f^!(A_Y) \otimes^L_{A_X} f^{-1}(N^{\cdot}) \xrightarrow{\sim} f^!(N^{\cdot}). \tag{C.2.26}$$

Definition C.2.13. In the situation of Proposition C.2.12 we set $\mathrm{or}_{X/Y} = H^{-d}(f^!(A_Y)) \in \mathrm{Mod}(A_X)$ and call it the *relative orientation sheaf* of $f : X \to Y$. If, moreover, Y is the space $\{\mathrm{pt}\}$ consisting of one point, we set $\mathrm{or}_X = \mathrm{or}_{X/Y} \in \mathrm{Mod}(A_X)$ and call it the *orientation sheaf* of X.

In the situation of Proposition C.2.12 above we thus have an isomorphism $f^!(A_Y) \simeq \mathrm{or}_{X/Y}[\dim X - \dim Y]$ and for any $N^{\cdot} \in D^+(A_Y)$ there exists an isomorphism

$$f^!(N^{\cdot}) \simeq \mathrm{or}_{X/Y} \otimes_{A_X} f^{-1}(N^{\cdot})[\dim X - \dim Y]. \tag{C.2.27}$$

Note that in the above isomorphism we wrote $\otimes_{A_X}$ instead of $\otimes^L_{A_X}$ because $\mathrm{or}_{X/Y}$ is flat over A_X.

Definition C.2.14. Let $f : X \to Y$ be as above. Assume, moreover, that Y is the space $\{\mathrm{pt}\}$ consisting of one point and the morphism f is $X \to \{\mathrm{pt}\}$. Then we set $\omega_X{}^{\cdot} = f^!(A_{\{\mathrm{pt}\}}) \in D^+(A_X)$ and call it the *dualizing complex* of X. We sometimes denote $\omega_X{}^{\cdot}$ simply by ω_X.

To define the dualizing complex $\omega_X{}^{\cdot} \in D^+(A_X)$ of X, we assumed that the functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(\{\mathrm{pt}\}) = \mathcal{A}b$ for $f : X \to \{\mathrm{pt}\}$ has finite cohomological dimension. This assumption is satisfied if X is a topological manifold or a real analytic space. In what follows we assume that all topological spaces that we treat satisfy this assumption.

Definition C.2.15. For $M^{\cdot} \in D^b(A_X)$ we set

$$\mathbf{D}_X(M^{\cdot}) = R\mathcal{H}om_{A_X}(M^{\cdot}, \omega_X{}^{\cdot}) \in D^+(A_X)$$

and call it the *Verdier dual* of $M^{\cdot}$.

Since for the morphism $f : X \to Y$ of topological spaces we have $f^!\omega_Y{}^{\cdot} \simeq \omega_X{}^{\cdot}$, from Proposition C.2.10 we obtain an isomorphism

$$(f^! \circ \mathbf{D}_Y)(N^{\cdot}) \simeq (\mathbf{D}_X \circ f^{-1})(N^{\cdot}) \tag{C.2.28}$$

for any $N^{\cdot} \in D^b(A_Y)$. Similarly, from Theorem C.2.8 we obtain an isomorphism

$$(Rf_* \circ \mathbf{D}_X)(M^{\cdot}) \simeq (\mathbf{D}_Y \circ Rf_!)(M^{\cdot}) \tag{C.2.29}$$

for any $M^{\cdot} \in D^b(A_X)$.

Example C.2.16. In the situation as above, assume, moreover, that A is a field k, X is an orientable C^1-manifold of dimension n, and $Y = \{\mathrm{pt}\}$. In this case there exist isomorphisms $\omega_X{}^{\cdot} \simeq \mathrm{or}_X[n] \simeq k_X[n]$. Let $M^{\cdot} \in D^b(k_X)$ and set $\mathbf{D}'_X(M^{\cdot}) = R\mathcal{H}om_{k_X}(M^{\cdot}, k_X)$. Then by the isomorphism (C.2.29) we obtain an isomorphism $H^{n-i}(X, \mathbf{D}'_X(M^{\cdot})) \simeq [H^i_c(X, M^{\cdot})]^*$ for any $i \in \mathbb{Z}$, where we set $H^i_c(X, \bullet) = H^i\mathrm{R}\Gamma_c(X, \bullet)$. In the very special case where $M^{\cdot} = k_X$ we thus obtain the famous Poincaré duality theorem: $H^{n-i}(X, k_X) \simeq [H^i_c(X, k_X)]^*$.

C.3 Non-characteristic deformation lemma

In this section, we prove the *non-characteristic deformation lemma* (due to Kashiwara), which plays a powerful role in deriving results on global cohomology groups of complexes of sheaves from their local properties. First, we introduce some basic results on projective systems of abelian groups. Recall that a pair $M = (M_n, \rho_{n,m})$ of a family of abelian groups M_n ($n \in \mathbb{N}$) and that of group homomorphisms $\rho_{n,m} : M_m \to M_n$ ($m \geq n$) is called a *projective system* of abelian groups (indexed by $\mathbb{N}$) if it satisfies the conditions: $\rho_{n,n} = \mathrm{id}_{M_n}$ for any $n \in \mathbb{Z}$ and $\rho_{n,m} \circ \rho_{m,l} = \rho_{n,l}$ for any $n \leq m \leq l$. If $M = (M_n, \rho_{n,m})$ is a projective system of abelian groups, we denote its projective limit by $\varprojlim M$ for short. We define morphisms of projective systems of abelian groups in an obvious way. Then the category of projective systems of abelian groups is abelian. However, the functor $\varprojlim(*)$ from this category to that of abelian groups is not exact. It is only left exact. As a remedy for this problem we introduce the following notion.

Definition C.3.1. Let $M = (M_n, \rho_{n,m})$ be a projective system of abelian groups. We say that M satisfies the *Mittag-Leffler condition* (or M-L condition) if for any $n \in \mathbb{N}$ decreasing subgroups $\rho_{n,m}(M_m)$ ($m \geq n$) of M_n is stationary.

Let us state basic results on projective systems satisfying the M-L condition. Since the proofs of the following lemmas are straightforward, we leave them to the reader.

Lemma C.3.2. *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of projective systems of abelian groups.

(i) *Assume L and N satisfy the M-L condition. Then M satisfies the M-L condition.*
(ii) *Assume M satisfies the M-L condition. Then N satisfies the M-L condition.*

Lemma C.3.3. *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of projective systems of abelian groups. Assume that L satisfies the M-L condition. Then the sequence

$$0 \longrightarrow \varprojlim L \longrightarrow \varprojlim M \longrightarrow \varprojlim N \longrightarrow 0$$

is exact.

Now let X be a topological space and $F^\cdot \in D^b(\mathbb{Z}_X)$. Namely, $F^\cdot$ is a bounded complex of sheaves of abelian groups on X.

Proposition C.3.4. *Let $\{U_n\}_{n\in\mathbb{N}}$ be an increasing sequence of open subsets of X and set $U = \cup_{n\in\mathbb{N}} U_n$. Then*

(i) *The natural morphism $\phi_i : H^i(U, F^\cdot) \to \varprojlim_n H^i(U_n, F^\cdot)$ is surjective for any $i \in \mathbb{Z}$.*

(ii) *Assume that for an integer $i \in \mathbb{Z}$ the projective system $\{H^{i-1}(U_n, F^\cdot)\}_{n\in\mathbb{N}}$ satisfies the M-L condition. Then $\phi_i : H^i(U, F^\cdot) \to \varprojlim_n H^i(U_n, F^\cdot)$ is bijective.*

Proof. We may assume that each term F^i of $F^\cdot$ is a flabby sheaf. Then we have $H^i(U, F^\cdot) = H^i(F^\cdot(U)) = H^i(\varprojlim_n F^\cdot(U_n))$. Hence the morphism ϕ_i is

$$H^i(\varprojlim_n F^\cdot(U_n)) \to \varprojlim_n H^i(F^\cdot(U_n)).$$

Note that for any $i \in \mathbb{Z}$ the projective system $\{F^i(U_n)\}_{n\in\mathbb{N}}$ satisfies the M-L condition by the flabbiness of F^i. Set $Z_n^i = \mathrm{Ker}[F^i(U_n) \to F^{i+1}(U_n)]$ and $B_n^i = \mathrm{Im}[F^{i-1}(U_n) \to F^i(U_n)]$. Then we have exact sequences

$$0 \longrightarrow Z_n^i \longrightarrow F^i(U_n) \longrightarrow B_n^{i+1} \longrightarrow 0 \tag{C.3.1}$$

and by Lemma C.3.2 (ii) the projective systems $\{B_n^i\}_{n\in\mathbb{N}}$ satisfy the M-L condition. Therefore, applying Lemma C.3.3 to the exact sequences

$$0 \longrightarrow B_n^i \longrightarrow Z_n^i \longrightarrow H^i(U_n, F^\cdot) \longrightarrow 0 \tag{C.3.2}$$

we get an exact sequence

$$0 \longrightarrow \varprojlim_n B_n^i \longrightarrow \varprojlim_n Z_n^i \longrightarrow \varprojlim_n H^i(U_n, F^\cdot) \longrightarrow 0. \tag{C.3.3}$$

Since the functor $\varprojlim(*)$ is left exact, we also have isomorphisms

$$\varprojlim_n Z_n^i \simeq \mathrm{Ker}[\varprojlim_n F^i(U_n) \to \varprojlim_n F^{i+1}(U_n)] = \mathrm{Ker}[F^i(U) \to F^{i+1}(U)].$$

Now let us consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & F^{i-1}(U) & \longrightarrow & \mathrm{Ker}[F^i(U) \to F^{i+1}(U)] & \longrightarrow & H^i(U; F^{\cdot}) & \longrightarrow & 0 \\ & & \downarrow & & \wr\downarrow & & \downarrow{\scriptstyle \phi_i} & & \\ 0 & \longrightarrow & \varprojlim_n B_n^i & \longrightarrow & \varprojlim_n Z_n^i & \longrightarrow & \varprojlim_n H^i(U_n, F^{\cdot}) & \longrightarrow & 0 \end{array}$$

Then we see that ϕ_i is surjective. The assertion (i) was proved. Let us prove (ii). Assume that the projective system $\{H^{i-1}(U_n, F^{\cdot})\}_{n\in\mathbb{N}}$ satisfies the M-L condition. Then applying Lemma C.3.2 (i) to the exact sequence (C.3.2) we see that the projective system $\{Z_n^{i-1}\}_{n\in\mathbb{N}}$ satisfies the M-L condition. Hence by Lemma C.3.3 and (C.3.1) we get an exact sequence

$$0 \longrightarrow \varprojlim_n Z_n^{i-1} \longrightarrow F^{i-1}(U) \longrightarrow \varprojlim_n B_n^i \longrightarrow 0,$$

which shows that the left vertical arrow in the above diagram is surjective. Hence ϕ_i is bijective. This completes the proof. □

We also require the following.

Lemma C.3.5. *Let $\{M_t, \rho_{t,s}\}$ be a projective system of abelian groups indexed by $\mathbb{R}$. Assume that for any $t \in \mathbb{R}$ the natural morphisms*

$$\begin{cases} \alpha_t : M_t \longrightarrow \varprojlim_{s<t} M_s, \\ \beta_t : \varinjlim_{s>t} M_s \longrightarrow M_t \end{cases}$$

are injective (resp. surjective). Then for any pair $t_1 \leq t_2$ the morphism $\rho_{t_1,t_2} : M_{t_2} \to M_{t_1}$ is injective (resp. surjective).

Proof. Since the proof of injectivity is easy, we only prove surjectivity. Let $t_1 \leq t_2$ and $m_1 \in M_{t_1}$. Denote by S the set of all pairs (t, m) of $t_1 \leq t \leq t_2$ and $m \in M_t$ satisfying $\rho_{t_1,t}(m) = m_1$. Let us order this set S in the following way: $(t, m) \leq (t', m') \iff t \leq t'$ and $\rho_{t,t'}(m') = m$. Then by the surjectivity of α_s for any s we can easily prove that S is an inductively ordered set. Therefore, by *Zorn's lemma* there exists a maximal element (t, m) of S. If $t = t_2$, then $\rho_{t_1,t_2}(m) = m_1$. If $t < t_2$ then by the surjectivity of β_s's for any s, there exist t_3 with $t < t_3 \leq t_2$ and $m_3 \in M_{t_3}$ such that $\rho_{t,t_3}(m_3) = m$. This contradicts the maximality of the element (t, m). □

Now let us introduce the non-characteristic deformation lemma (due to Kashiwara). This result is very useful to derive global results from the local properties of $F^{\cdot} \in D^b(\mathbb{C}_X)$. Here we introduce only its weak form, which is enough for the applications in this book (see also [KS2, Proposition 2.7.2]).

Theorem C.3.6 (Non-characteristic deformation lemma). *Let X be a C^∞-manifold, $\{\Omega_t\}_{t\in\mathbb{R}}$ a family of relatively compact open subsets of X, and $F^{\cdot} \in D^b(\mathbb{C}_X)$. Assume the following conditions:*

(i) *For any pair* $s < t$ *of real numbers,* $\Omega_s \subset \Omega_t$.
(ii) *For any* $t \in \mathbb{R}$, $\Omega_t = \bigcup_{s<t} \Omega_s$.
(iii) *For* $^\forall t \in \mathbb{R}$, $\bigcap_{s>t}(\Omega_s \setminus \Omega_t) = \partial\Omega_t$ *and for* $^\forall x \in \partial\Omega_t$, *we have*

$$\left[\mathrm{R}\Gamma_{X\setminus\Omega_t}(F^{\cdot})\right]_x \simeq 0.$$

Then we have an isomorphism

$$\mathrm{R}\Gamma\left(\bigcup_{s\in\mathbb{R}} \Omega_s, F^{\cdot}\right) \xrightarrow{\sim} \mathrm{R}\Gamma(\Omega_t, F^{\cdot})$$

for any $t \in \mathbb{R}$.

Proof. We prove the theorem by using Lemma C.3.5. First let us prove that for any $t \in \mathbb{R}$ and $i \in \mathbb{Z}$ the canonical morphism

$$\varinjlim_{s>t} H^i(\Omega_s, F^{\cdot}) \longrightarrow H^i(\Omega_t, F^{\cdot})$$

is an isomorphism. Since we have $\mathrm{R}\Gamma_{X\setminus\Omega_t}(F^{\cdot})|_{\partial\Omega_t} \simeq 0$ by the assumption (iii), we obtain

$$\mathrm{R}\Gamma(\overline{\Omega_t}, \mathrm{R}\Gamma_{X\setminus\Omega_t}(F^{\cdot})) \simeq \mathrm{R}\Gamma(\partial\Omega_t, \mathrm{R}\Gamma_{X\setminus\Omega_t}(F^{\cdot})) \simeq 0.$$

Then by the distinguished triangle

$$\mathrm{R}\Gamma_{X\setminus\Omega_t}(F^{\cdot}) \longrightarrow F^{\cdot} \longrightarrow \mathrm{R}\Gamma_{\Omega_t}(F^{\cdot}) \xrightarrow{+1}$$

we get an isomorphism

$$\mathrm{R}\Gamma(\overline{\Omega_t}, F^{\cdot}) \simeq \mathrm{R}\Gamma(\Omega_t, F^{\cdot}).$$

Taking the cohomology groups of both sides we finally obtain the desired isomorphisms

$$\varinjlim_{s>t} H^i(\Omega_s, F^{\cdot}) \simeq H^i(\Omega_t, F^{\cdot}). \tag{C.3.4}$$

Now consider the following assertions:

$$(A)_i^t : \varprojlim_{s<t} H^i(\Omega_s, F^{\cdot}) \simeq H^i(\Omega_t, F^{\cdot})$$

for $i \in \mathbb{Z}$ and $t \in \mathbb{R}$. Assume that for an integer j the assertion $(A)_i^t$ is proved for any $i < j$ and $t \in \mathbb{R}$. Then by Lemma C.3.5 we get an isomorphism $H^i(\Omega_s, F^{\cdot}) \simeq H^i(\Omega_t, F^{\cdot})$ for any $i < j$ and any pair $s > t$. This implies that for each $t \in \mathbb{R}$ the projective system $\{H^{j-1}(\Omega_{t-\frac{1}{n}}, F^{\cdot})\}_{n\in\mathbb{N}}$ satisfies the M-L condition. Hence by Proposition C.3.4 the assertion $(A)_j^t$ is proved for any $t \in \mathbb{R}$. Repeating this argument, we can finally prove $(A)_i^t$ for all $i \in \mathbb{Z}$ and all $t \in \mathbb{R}$. Together with the isomorphisms (C.3.4), we obtain by Lemma C.3.5 an isomorphism $\mathrm{R}\Gamma(\Omega_s, F^{\cdot}) \simeq \mathrm{R}\Gamma(\Omega_t, F^{\cdot})$ for any pair $s > t$. This completes the proof. □

D

Filtered Rings

D.1 Good filtration

Let A be a ring. Assume that we are given a family $F = \{F_l A\}_{l \in \mathbb{Z}}$ of additive subgroups of A satisfying

(a) $F_l A = 0$ for $l < 0$,
(b) $1 \in F_0 A$,
(c) $F_l A \subset F_{l+1} A$,
(d) $(F_l A)(F_m A) \subset F_{l+m} A$,
(e) $A = \bigcup_{l \in \mathbb{Z}} F_l A$.

Then we call (A, F) a *filtered ring*. For a filtered ring (A, F) we set

$$\operatorname{gr}^F A = \bigoplus_{l \in \mathbb{Z}} \operatorname{gr}_l^F A, \qquad \operatorname{gr}_l^F A = F_l A / F_{l-1} A.$$

The canonical map $F_l A \to \operatorname{gr}_l^F A$ is denoted by σ_l. The additive group $\operatorname{gr}^F A$ is endowed with a structure of a ring by

$$\sigma_l(a)\sigma_m(b) = \sigma_{l+m}(ab).$$

We call the ring $\operatorname{gr}^F A$ the *associated graded ring*.

Let (A, F) be a filtered ring. Let M be a (left) A-module M, and assume that we are given a family $F = \{F_p M\}_{p \in \mathbb{Z}}$ of additive subgroups of M satisfying

(a) $F_p M = 0$ for $p \ll 0$,
(b) $F_p M \subset F_{p+1} M$,
(c) $(F_l A)(F_p M) \subset F_{l+p} M$,
(d) $M = \bigcup_{p \in \mathbb{Z}} F_p M$.

Then F is called a filtration of M and (M, F) is called a filtered (left) A-module. For a filtered A-module (M, F) we set

$$\mathrm{gr}^F M = \bigoplus_{p \in \mathbb{Z}} \mathrm{gr}_p^F M, \qquad \mathrm{gr}_p^F M = F_p M / F_{p-1} M.$$

Denote the canonical map $F_p M \to \mathrm{gr}_p^F M$ by τ_p. The additive group $\mathrm{gr}^F M$ is endowed with a structure of a $\mathrm{gr}^F A$-module by

$$\sigma_l(a)\tau_p(m) = \tau_{l+p}(am).$$

We call the $\mathrm{gr}^F A$-module $\mathrm{gr}^F M$ the *associated graded module*.

We can also define the notion of a filtration of a right A-module and the associated graded module of a right filtered A-module. We will only deal with left A-modules in the following; however, parallel facts also hold for right modules.

Proposition D.1.1. *Let M be an A-module.*

(i) *Let F be a filtration of M such that $\mathrm{gr}^F M$ is finitely generated over $\mathrm{gr}^F A$. Then there exist finitely many integers p_k $(k = 1, \dots, r)$ and $m_k \in F_{p_k} M$ such that for any p we have $F_p M = \sum_{p \geq p_k} (F_{p-p_k} A) m_k$. In particular, the A-module M is generated by finitely many elements $m_1, \dots, m_k$.*

(ii) *Let M an A-module generated by finitely many elements $m_1, \dots, m_k$. For $p_k \in \mathbb{Z}$ $(k = 1, \dots, r)$ set $F_p M = \sum_{p \geq p_k} (F_{p-p_k} A) m_k$. Then F is a filtration of M such that $\mathrm{gr}^F M$ is a finitely generated $\mathrm{gr}^F A$-module.*

Proof. (i) We take integers p_k $(k = 1, \dots, r)$ and $m_k \in F_{p_k} M$ so that $\{\tau_{p_k}(m_k)\}_{1 \leq k \leq r}$ generates the $\mathrm{gr}^F A$-module $\mathrm{gr}^F M$. Then we can show $F_p M = \sum_{p \geq p_k} (F_{p-p_k} A) m_k$ by induction on p. (ii) is obvious. □

Corollary D.1.2. *The following conditions on an A-module M are equivalent:*

(i) *M is a finitely generated A-module,*

(ii) *there exists a filtration F of M such that $\mathrm{gr}^F M$ is a finitely generated $\mathrm{gr}^F A$-module.*

Let (M, F) be a filtered A-module. If $\mathrm{gr}^F M$ is a finitely generated $\mathrm{gr}^F A$-module, then F is called a *good filtration* of M, and (M, F) is called a good filtered A-module.

Proposition D.1.3. *Let M be a finitely generated A-module and let F, G be filtrations of M. If F is good, then there exist an integers a such that for any $p \in \mathbb{Z}$ we have*

$$F_p M \subset G_{p+a} M.$$

In particular, if G is also good, then for $a \gg 0$ we have

$$F_{p-a} M \subset G_p M \subset F_{p+a} M \qquad (\forall p).$$

Proof. By Proposition D.1.1 we can take elements m_k $(1 \leq k \leq r)$ of M and integers p_k $(1 \leq k \leq r)$ such that $F_p M = \sum_{p \geq p_k} (F_{p-p_k} A) m_k$. Take $q_k \in \mathbb{Z}$ such that $m_k \in G_{q_k} M$ and denote the maximal value of $q_k - p_k$ by a. Then we have

$$F_pM = \sum_{p \geq p_k} (F_{p-p_k}A)m_k \subset \sum_{p \geq p_k} (F_{p-p_k}A)G_{q_k}M$$
$$\subset \sum_{p \geq p_k} G_{p+(q_k-p_k)}M \subset G_{p+a}M.$$

The proof is complete. □

Let (M, F) be a filtered A-module, and let

$$0 \to L \to M \to N \to 0$$

be an exact sequence of A-modules. Then we have the induced filtrations of L and N defined by

$$F_pL = F_pM \cap L, \qquad F_pN = \mathrm{Im}(F_pM \to N),$$

for which we have the exact sequence

$$0 \to \mathrm{gr}^F L \to \mathrm{gr}^F M \to \mathrm{gr}^F N \to 0.$$

Hence, if (M, F) is a good filtered A-module, then so is (N, F). If, moreover, $\mathrm{gr}^F A$ is a left noetherian ring, then (L, F) is also a good filtered A-module.

Proposition D.1.4. *Let (A, F) be a filtered ring. If $\mathrm{gr}^F A$ is a left (or right) noetherian ring, then so is A.*

Proof. In order to show that A is a left noetherian ring it is sufficient to show that any left ideal I of A is finitely generated. Define a filtration F of a left A-module I by $F_pI = I \cap F_pA$. Then $\mathrm{gr}^F I$ is a left ideal of $\mathrm{gr}^F A$. Since $\mathrm{gr}^F A$ is a left noetherian ring, $\mathrm{gr}^F I$ is finitely generated over $\mathrm{gr}^F A$. Hence I is finitely generated by Corollary D.1.2. The statement for right noetherian rings is proved similarly. □

D.2 Global dimensions

Let (M, F), (N, F) be filtered A-modules. An A-homomorphism $f : M \to N$ such that $f(F_pM) \subset F_pN$ for any p is called a *filtered* **A**-*homomorphism*. In this case we write $f : (M, F) \to (N, F)$. A filtered A-homomorphism $f : (M, F) \to (N, F)$ induces a homomorphism $\mathrm{gr}\, f : \mathrm{gr}^F M \to \mathrm{gr}^F N$ of $\mathrm{gr}^F A$-modules. A filtered A-homomorphism $f : (M, F) \to (N, F)$ is called *strict* if it satisfies $f(F_pM) = \mathrm{Im}\, f \cap F_pN$. The following fact is easily proved.

Lemma D.2.1. *Let $f : (L, F) \to (M, F)$, $g : (M, F) \to (N, F)$ be strict filtered A-homomorphisms such that $L \to M \to N$ is exact. Then $\mathrm{gr}^F L \to \mathrm{gr}^F M \to \mathrm{gr}^F N$ is exact.*

Let W be a free A-module of rank $r < \infty$ with basis $\{w_k\}_{1 \leqq k \leqq r}$. For integers p_k $(1 \leqq k \leqq r)$ we can define a filtration F of W by $F^pW = \sum_k (F_{p-p_k}A)w_k$. This type of filtered A-module (W, F) is called a filtered free A-module of rank r. We can easily show the following.

Lemma D.2.2. *Assume that A is left noetherian. For a good filtered A-module (M, F) we can take filtered free A-modules (W_i, F) $(i \in \mathbb{N})$ of finite ranks and strict filtered A-homomorphisms $(W_{i+1}, F) \to (W_i, F)$ $(i \in \mathbb{N})$ and $(W_0, F) \to (M, F)$ such that*

$$\cdots \to W_1 \to W_0 \to M \to 0$$

is an exact sequence of A-modules.

For filtered A-modules (M, F), (N, F) and $p \in \mathbb{Z}$ set

$$F^p \operatorname{Hom}_A(M, N) = \{f \in \operatorname{Hom}_A(M, N) \mid f(F_q M) \subset F_{q+p} N \ (\forall q \in \mathbb{Z})\}.$$

This defines an increasing filtration of the abelian group $\operatorname{Hom}_A(M, N)$. Set

$$\begin{aligned} \operatorname{gr}_p^F \operatorname{Hom}_A(M, N) &= F_p \operatorname{Hom}_A(M, N)/F_{p-1} \operatorname{Hom}_A(M, N), \\ \operatorname{gr}^F \operatorname{Hom}_A(M, N) &= \bigoplus_p \operatorname{gr}_p^F \operatorname{Hom}_A(M, N). \end{aligned}$$

Then we have a canonical homomorphism

$$\operatorname{gr}^F \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\operatorname{gr}^F A}(\operatorname{gr}^F M, \operatorname{gr}^F N)$$

of abelian groups. The following is easily proved.

Lemma D.2.3. *Let (M, F) be a good filtered A-module and (N, F) a filtered A-module.*

(i) $\operatorname{Hom}_A(M, N) = \bigcup_{p \in \mathbb{Z}} F_p \operatorname{Hom}_A(M, N)$.
(ii) $F_p \operatorname{Hom}_A(M, N) = 0$ *for* $p \ll 0$.
(iii) *The canonical homomorphism* $\operatorname{gr}^F \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\operatorname{gr}^F A}(\operatorname{gr}^F M, \operatorname{gr}^F N)$ *is injective. Moreover, it is surjective if (M, F) is a filtered free A-module of finite rank.*

Lemma D.2.4. *Let (A, F) be a filtered ring such that $\operatorname{gr}^F A$ is left noetherian. Let (M, F) be a good filtered A-module and (N, F) a filtered A-module. Then there exists an increasing filtration F of the abelian group $\operatorname{Ext}_A^i(M, N)$ such that*

(i) $\operatorname{Ext}_A^i(M, N) = \bigcup_{p \in \mathbb{Z}} F_p \operatorname{Ext}_A^i(M, N)$,
(ii) $F_p \operatorname{Ext}_A^i(M, N) = 0$ *for* $p \ll 0$,
(iii) $\operatorname{gr}^F \operatorname{Ext}_A^i(M, N)$ *is isomorphic to a subquotient of* $\operatorname{Ext}^i_{\operatorname{gr}^F A}(\operatorname{gr}^F M, \operatorname{gr}^F N)$.

Proof. Take (W_i, F) $(i \in \mathbb{N})$, $(W_{i+1}, F) \to (W_i, F)$ $(i \in \mathbb{N})$ and $(W_0, F) \to (M, F)$ as in Lemma D.2.2. Then we have $\operatorname{Ext}_A^i(M, N) = H^i(K^\cdot)$ with $K^\cdot = \operatorname{Hom}_A(W_., N)$. Note that each term $K^i = \operatorname{Hom}_A(W_i, N)$ of $K^\cdot$ is equipped with increasing filtration F satisfying $d^i(F_p K^i) \subset F_p K^{i+1}$, where $d^i : K^i \to K^{i+1}$ denotes the boundary homomorphism.

Consider the complex $\operatorname{gr}^F K^\cdot$ with ith term $\operatorname{gr}^F K^i$. We have $\operatorname{gr}^F K^\cdot \simeq \operatorname{Hom}_{\operatorname{gr}^F A}(\operatorname{gr}^F W_., \operatorname{gr}^F N)$ by Lemma D.2.3. Hence by the exact sequence

$$\cdots \to \mathrm{gr}^F W_1 \to \mathrm{gr}^F W_0 \to \mathrm{gr}^F M \to 0,$$

(see Lemma D.2.1) we obtain

$$\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F N) = H^i(\mathrm{Hom}_{\mathrm{gr}^F A}(\mathrm{gr}^F W_\cdot, \mathrm{gr}^F N) \simeq H^i(\mathrm{gr}^F K^\cdot).$$

Now define an increasing filtration of $H^i(K^\cdot) = \mathrm{Ext}^i_A(M, N)$ by

$$F_p H^i(K^\cdot) = \mathrm{Im}(H^i(F^p K^\cdot) \to H^i(K^\cdot)).$$

For each $i \in \mathbb{N}$ we have

$$K^i = \bigcup_p F_p K^i, \qquad F_p K^i = 0 \ (p \ll 0).$$

by Lemma D.2.3. From this we easily see that

$$H^i(K^\cdot) = \bigcup_p F_p H^i(K^\cdot), \qquad F_p H^i(K^\cdot) = 0 \ (p \ll 0).$$

It remains to show that $\mathrm{gr}^F H^i(K^\cdot)$ is a subquotient of $H^i(\mathrm{gr}^F K^\cdot)$. By definition we have

$$\mathrm{gr}^F_p H^i(K^\cdot) = (F_p K^i \cap \mathrm{Ker}\, d^i + \mathrm{Im}\, d^{i-1})/(F_{p-1} K^i \cap \mathrm{Ker}\, d^i + \mathrm{Im}\, d^{i-1}),$$
$$H^i(\mathrm{gr}^F_p K^\cdot) = \mathrm{Ker}(F_p K^i \to \mathrm{gr}^F_p K^{i+1})/(F_{p-1} K^i + d^{i-1}(F_p K^{i-1})).$$

Set $L = F_p K^i \cap \mathrm{Ker}\, d^i/(F_{p-1} K^i \cap \mathrm{Ker}\, d^i + d^{i-1}(F_p K^{i-1}))$. Then we can easily check that L is isomorphic to a submodule of $H^i(\mathrm{gr}^F_p K^\cdot)$ and that $\mathrm{gr}^F_p H^i(K^\cdot)$ is a quotient of L. □

Let us consider the situation where $N = A$ (with canonical filtration F) in Lemma D.2.3 and Lemma D.2.4. Let (A, F) be a filtered ring and let (M, F) be a good filtered A-module. We easily see that the filtration F of the right A-module $\mathrm{Hom}_A(M, A)$ is a good filtration and the canonical homomorphism $\mathrm{gr}^F \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F N)$ preserves the $\mathrm{gr}^F A$-modules structure. Hence (the proof of) Lemma D.2.4 implies the following.

Lemma D.2.5. *Let (A, F) be a filtered ring such that $\mathrm{gr}^F A$ is left noetherian, and let (M, F) be a good filtered A-module. Then there exists a good filtration F of the right A-module $\mathrm{Ext}^i_A(M, A)$ such that $\mathrm{gr}^F \mathrm{Ext}^i_A(M, A)$ is isomorphic to a subquotient of $\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)$ as a right $\mathrm{gr}^F A$-module.*

Theorem D.2.6. *Let (A, F) be a filtered ring such that $\mathrm{gr}^F A$ is left (resp. right) noetherian. Then the left (resp. right) global dimension of the ring A is smaller than or equal to that of $\mathrm{gr}^F A$.*

Proof. We will only show the statement for left global dimensions. Denote the left global dimension of $\mathrm{gr}^F A$ by n. If $n = \infty$, there is nothing to prove. Assume that $n < \infty$. We need to show $\mathrm{Ext}^i_A(M, N) = 0$ $(i > n)$ for arbitrary A-modules M, N. Since A is left noetherian, we may assume that M is finitely generated. Choose a good filtration F of M and a filtration F of N. Then we have $\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F N) = 0$ $(i > n)$. Hence the assertion follows from Lemma D.2.4. □

D.3 Singular supports

Let R be a commutative noetherian ring and let M be a finitely generated R-module. We denote by $\mathrm{supp}(M)$ the set of prime ideals $\mathfrak{p}$ of R satisfying $M_{\mathfrak{p}} \neq 0$, and by $\mathrm{supp}_0(M)$ the set of minimal elements of $\mathrm{supp}(M)$. We have $\mathfrak{p} \in \mathrm{supp}(M)$ if and only if $\mathfrak{p}$ contains the annihilating ideal

$$\mathrm{Ann}_R(M) = \{r \in R \mid rM = 0\}.$$

In fact, we have

$$\sqrt{\mathrm{Ann}_R(M)} = \bigcap_{\mathfrak{p} \in \mathrm{supp}(M)} \mathfrak{p}.$$

For $\mathfrak{p} \in \mathrm{supp}_0(M)$ we denote the *length* of the artinian $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ by $\ell_{\mathfrak{p}}(M)$. We set $\ell_{\mathfrak{q}}(M) = 0$ for a prime ideal $\mathfrak{q} \notin \mathrm{supp}(M)$. For an exact sequence

$$0 \to L \to M \to N \to 0$$

of finitely generated R-modules we have

$$\mathrm{supp}(M) = \mathrm{supp}(L) \cup \mathrm{supp}(N).$$

Moreover, for $\mathfrak{p} \in \mathrm{supp}_0(M)$ we have

$$\ell_{\mathfrak{p}}(M) = \ell_{\mathfrak{p}}(L) + \ell_{\mathfrak{p}}(N).$$

In the rest of this section (A, F) denotes a filtered ring such that $\mathrm{gr}^F A$ is a commutative noetherian ring. Let M be a finitely generated A-module. By choosing a good filtration F we can consider $\mathrm{supp}(\mathrm{gr}^F M)$ and $\ell_{\mathfrak{p}}(\mathrm{gr}^F M)$ for $\mathfrak{p} \in \mathrm{supp}_0(M)$.

Lemma D.3.1. *$\mathrm{supp}(\mathrm{gr}^F M)$ and $\ell_{\mathfrak{p}}(\mathrm{gr}^F M)$ for $\mathfrak{p} \in \mathrm{supp}_0(M)$ do not depend on the choice of a good filtration F.*

Proof. We say two good filtrations F and G are "*adjacent*" if they satisfy the condition

$$F_i M \subset G_i M \subset F_{i+1} M \quad ({}^\forall i \in \mathbb{Z}).$$

We first show the assertion in this case. Consider the natural homomorphism $\varphi_i : F_i M / F_{i-1} M \to G_i M / G_{i-1} M$. Then we have $\mathrm{Ker}\,\varphi_i \simeq G_{i-1} M / F_{i-1} M \simeq \mathrm{Coker}\,\varphi_{i-1}$. Therefore, the morphism $\varphi : \mathrm{gr}^F M \to \mathrm{gr}^G M$ entails an isomorphism $\mathrm{Ker}\,\varphi \simeq \mathrm{Coker}\,\varphi$. Consider the exact sequence

$$0 \to \mathrm{Ker}\,\varphi \to \mathrm{gr}^F M \xrightarrow{\varphi} \mathrm{gr}^G M \to \mathrm{Coker}\,\varphi \to 0$$

of finitely generated $\mathrm{gr}^F A$-modules. From this we obtain

$$\mathrm{supp}(\mathrm{gr}^F M) = \mathrm{supp}(\mathrm{Ker}\,\varphi) \cup \mathrm{supp}(\mathrm{Im}\,\varphi),$$
$$\mathrm{supp}(\mathrm{gr}^G M) = \mathrm{supp}(\mathrm{Im}\,\varphi) \cup \mathrm{supp}(\mathrm{Coker}\,\varphi).$$

Hence $\operatorname{Ker}\varphi \simeq \operatorname{Coker}\varphi$ implies $\operatorname{supp}(\operatorname{gr}^F M) = \operatorname{supp}(\operatorname{gr}^G M)$. Moreover, for $\mathfrak{p} \in \operatorname{supp}_0(\operatorname{gr}^F M) = \operatorname{supp}_0(\operatorname{gr}^G M)$ we have

$$\ell_{\mathfrak{p}}(\operatorname{gr}^F M) = \ell_{\mathfrak{p}}(\operatorname{Ker}\varphi) + \ell_{\mathfrak{p}}(\operatorname{Im}\varphi) = \ell_{\mathfrak{p}}(\operatorname{gr}^G M).$$

The assertion is proved for adjacent good filtrations.

Let us consider the general case. Namely, assume that F and G are arbitrary good filtrations of M. For $k \in \mathbb{Z}$ set

$$F_i^{(k)} M = F_i M + G_{i+k} M \quad (i \in \mathbb{Z}).$$

By Proposition D.1.3 $F^{(k)}$ is a good filtration of M satisfying the conditions

$$\begin{cases} F^{(k)} = F \quad (k \ll 0), \\ F^{(k)} = G[k] \quad (k \gg 0), \\ F^{(k)} \text{ and } F^{(k+1)} \text{ are adjacent,} \end{cases}$$

where $G[k]$ is a filtration obtained form G by the degree shift $[k]$. Therefore, our assertion follows form the adjacent case. $\square$

Definition D.3.2. For a finitely generated A-module M we set

$$\begin{aligned} \operatorname{SS}(M) &= \operatorname{supp}(\operatorname{gr}^F M), \\ \operatorname{SS}_0(M) &= \operatorname{supp}_0(\operatorname{gr}^F M), \\ J_M &= \sqrt{\operatorname{Ann}_{\operatorname{gr}^F A}(\operatorname{gr}^F M)} = \bigcap_{\mathfrak{p} \in \operatorname{SS}_0(M)} \mathfrak{p}, \\ d(M) &= \text{Krull dim}\left(\operatorname{gr}^F A / J_M\right), \\ m_{\mathfrak{p}}(M) &= \ell_{\mathfrak{p}}(\operatorname{gr}^F M) \qquad (\mathfrak{p} \in \operatorname{SS}_0(M) \text{ or } \mathfrak{p} \notin \operatorname{SS}(M)), \end{aligned}$$

where F is a good filtration of M. $\operatorname{SS}(M)$ and J_M are called the *singular support* and the *characteristic ideal* of M, respectively.

Lemma D.3.3. *For an exact sequence*

$$0 \to L \to M \to N \to 0$$

of finitely generated A-modules we have

$$\begin{aligned} \operatorname{SS}(M) &= \operatorname{SS}(L) \cup \operatorname{SS}(N), \\ d(M) &= \max\{d(L), d(N)\}, \\ m_{\mathfrak{p}}(M) &= m_{\mathfrak{p}}(L) + m_{\mathfrak{p}}(N) \qquad (\mathfrak{p} \in \operatorname{SS}_0(M)). \end{aligned}$$

Proof. Take a good filtration F of M. With respect to the induced filtrations of L and N we have a short exact sequence

$$0 \longrightarrow \operatorname{gr}^F M \longrightarrow \operatorname{gr}^F N \longrightarrow \operatorname{gr}^F L \longrightarrow 0.$$

Hence the assertions for SS and $\ell_{\mathfrak{p}}$ are obvious. The assertion for d follows from the one for SS. $\square$

Since $\mathrm{gr}^F A$ is commutative, we have $[F_pA, F_qA] \subset F_{p+q-1}A$. Here, for $a, b \in A$ we set $[a, b] = ab - ba$. Hence we obtain a bi-additive product

$$\{\ ,\ \} : \mathrm{gr}^F_p A \times \mathrm{gr}^F_q A \to \mathrm{gr}^F_{p+q-1} A, \quad (\{\sigma_p(a), \sigma_q(b)\} = \sigma_{p+q-1}([a, b])).$$

Its bi-additive extension

$$\{\ ,\ \} : \mathrm{gr}^F A \times \mathrm{gr}^F A \to \mathrm{gr}^F A$$

is called the *Poisson bracket*. It satisfies the following properties:

(i) $\{a, b\} + \{b, a\} = 0$,
(ii) $\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0$,
(iii) $\{a, bc\} = \{a, b\}c + b\{a, c\}$.

We say that an ideal I of $\mathrm{gr}^F A$ is *involutive* if it satisfies $\{I, I\} \subset I$.

We state the following deep result of Gabber [Ga] without proof.

Theorem D.3.4. *Assume that (A, F) is a filtered ring such that the center of A contains a subring isomorphic to $\mathbb{Q}$ and that $\mathrm{gr}^F A$ is a commutative noetherian ring. Let M be a finitely generated A-module. Then any $\mathfrak{p} \in \mathrm{SS}_0(M)$ is involutive. In particular, J_M is involutive.*

D.4 Duality

In this section (A, F) is a filtered ring such that $\mathrm{gr}^F A$ is a regular commutative ring of pure dimension m (a commutative ring R is called a regular ring of pure dimension m if its localization at any maximal ideal is a regular local ring of dimension m). In particular, $\mathrm{gr}^F A$ is a noetherian ring whose global dimension and Krull dimension are equal to m. Hence A is a left and right noetherian ring with global dimension $\leqq m$ by Proposition D.1.4 and Theorem D.2.6. We will consider properties of the Ext-groups $\mathrm{Ext}^i_A(M, A)$ for finitely generated A-modules M.

Note first that for any (left) A-module M the Ext-groups $\mathrm{Ext}^i_A(M, A)$ are endowed with a right A-module structure (i.e., a left A^{op}-module structure, where A^{op} denotes the opposite ring) by the right multiplication of A on A. Since A has global dimension $\leqq m$, we have $\mathrm{Ext}^i_A(M, A) = 0$ for $i > m$. Moreover, if M is finitely generated, then $\mathrm{Ext}^i_A(M, A)$ are also finitely generated since A is left noetherian.

Let us give a formulation in terms of the derived category. Let $\mathrm{Mod}(A)$ and $\mathrm{Mod}_f(A)$ denote the category of (left) A-modules and its full subcategory consisting of finitely generated A-modules, respectively. Denote by $D^b(A)$ and $D^b_f(A)$ the bounded derived category of $\mathrm{Mod}(A)$ and its full subcategory consisting of complexes whose cohomology groups belong to $\mathrm{Mod}_f(A)$. Our objectives are the functors

$$\mathbb{D} = R\,\mathrm{Hom}_A(\bullet, A) : D^b_f(A) \to D^b_f(A^{\mathrm{op}})^{\mathrm{op}},$$
$$\mathbb{D}' = R\,\mathrm{Hom}_{A^{\mathrm{op}}}(\bullet, A^{\mathrm{op}}) : D^b_f(A^{\mathrm{op}}) \to D^b_f(A)^{\mathrm{op}},$$

where $D^b_f(A^{\mathrm{op}})$ is defined similarly.

Proposition D.4.1. *We have* $\mathbb{D}' \circ \mathbb{D} \simeq \mathrm{Id}$ *and* $\mathbb{D} \circ \mathbb{D}' \simeq \mathrm{Id}$.

Proof. By symmetry we have only to show $\mathbb{D}' \circ \mathbb{D} \simeq \mathrm{Id}$. We first construct a canonical morphism $M^\cdot \to \mathbb{D}'\mathbb{D}M^\cdot$ for $M^\cdot \in D^b_f(A)$. Set $H^\cdot = \mathbb{D}M^\cdot = R\operatorname{Hom}_A(M^\cdot, A)$. By

$$R\operatorname{Hom}_{A\otimes_{\mathbb{Z}}A^{\mathrm{op}}}(M^\cdot \otimes_{\mathbb{Z}} H^\cdot, A) \simeq R\operatorname{Hom}_A(M^\cdot, R\operatorname{Hom}_{A^{\mathrm{op}}}(H^\cdot, A^{\mathrm{op}}))$$

we have

$$\operatorname{Hom}_{A\otimes_{\mathbb{Z}}A^{\mathrm{op}}}(M^\cdot \otimes_{\mathbb{Z}} H^\cdot, A) \simeq \operatorname{Hom}_A(M^\cdot, R\operatorname{Hom}_{A^{\mathrm{op}}}(H^\cdot, A^{\mathrm{op}}))$$

Hence the canonical morphism $M^\cdot \otimes_{\mathbb{Z}} H^\cdot (= M^\cdot \otimes_{\mathbb{Z}} R\operatorname{Hom}_A(M^\cdot, A)) \to A$ in $D^b(A \otimes_{\mathbb{Z}} A^{\mathrm{op}})$ gives rise to a canonical morphism

$$M^\cdot \to R\operatorname{Hom}_{A^{\mathrm{op}}}(H^\cdot, A^{\mathrm{op}})(= \mathbb{D}'\mathbb{D}M)$$

in $D^b(A)$. It remains to show that $M^\cdot \to \mathbb{D}'\mathbb{D}M^\cdot$ is an isomorphism. By taking a free resolution of $M^\cdot$ we may replace $M^\cdot$ with A. In this case the assertion is clear. □

For $M^\cdot \in D^b_f(A)$ (or $D^b_f(A^{\mathrm{op}})$) we set

$$\mathrm{SS}(M^\cdot) = \bigcup_i \mathrm{SS}(H^i(M^\cdot)).$$

We easily see by Lemma D.3.3 that for a distinguished triangle

$$L^\cdot \longrightarrow M^\cdot \longrightarrow N^\cdot \xrightarrow{+1}$$

we have $\mathrm{SS}(M^\cdot) \subset \mathrm{SS}(L^\cdot) \cup \mathrm{SS}(N^\cdot)$.

Proposition D.4.2. *For* $M^\cdot \in D^b_f(A)$ (*resp.* $D^b_f(A^{\mathrm{op}})$) *we have* $\mathrm{SS}(\mathbb{D}M^\cdot) = \mathrm{SS}(M^\cdot)$ (*resp.* $\mathrm{SS}(\mathbb{D}'M^\cdot) = \mathrm{SS}(M^\cdot)$).

Proof. By Proposition D.4.1 and symmetry we have only to show $\mathrm{SS}(\mathbb{D}M^\cdot) \subset \mathrm{SS}(M^\cdot)$ for $M^\cdot \in D^b_f(A)$. We use induction on the cohomological length of $M^\cdot$. We first consider the case where $M^\cdot = M \in \mathrm{Mod}_f(A)$. Take a good filtration F of M and consider a good filtration F of $\operatorname{Ext}^i_A(M, A)$ as in Lemma D.2.5. By Lemma D.2.5 we have

$$\begin{aligned}\mathrm{SS}(\operatorname{Ext}^i_A(M, A)) &= \operatorname{supp}(\operatorname{gr}^F \operatorname{Ext}^i_A(M, A)) \subset \operatorname{supp}(\operatorname{Ext}^i_{\operatorname{gr}^F A}(\operatorname{gr}^F M, \operatorname{gr}^F A)) \\ &\subset \operatorname{supp}(\operatorname{gr}^F M) = \mathrm{SS}(M).\end{aligned}$$

The assertion is proved in the case where $M^\cdot = M \in \mathrm{Mod}_f(A)$. Now we consider the general case. Set $k = \min\{i \mid H^i(M^\cdot) \neq 0\}$. Then we have a distinguished triangle

$$H^k(M^\cdot)[-k] \longrightarrow M^\cdot \longrightarrow N^\cdot \xrightarrow{+1},$$

where $N^\cdot = \tau^{\geqslant k+1}M^\cdot$. By

$$H^i(N^\cdot) = \begin{cases} H^i(M^\cdot) & (i \neq k) \\ 0 & (i = k) \end{cases}$$

we obtain $\mathrm{SS}(M^\cdot) = \mathrm{SS}(N^\cdot) \cup \mathrm{SS}(H^k(M^\cdot))$. Moreover, by the hypothesis of induction we have $\mathrm{SS}(\mathbb{D}N^\cdot) \subset \mathrm{SS}(N^\cdot)$ and $\mathrm{SS}(\mathbb{D}H^k(M^\cdot)) \subset \mathrm{SS}(H^k(M^\cdot))$. Hence by the distinguished triangle

$$\mathbb{D}N^\cdot \longrightarrow \mathbb{D}M^\cdot \longrightarrow (\mathbb{D}H^k(M^\cdot))[k] \xrightarrow{+1}$$

we obtain

$$\mathrm{SS}(\mathbb{D}M^\cdot) \subset \mathrm{SS}(\mathbb{D}H^k(M^\cdot)) \cup \mathrm{SS}(\mathbb{D}N^\cdot) \subset \mathrm{SS}(H^k(M^\cdot)) \cup \mathrm{SS}(N^\cdot) = \mathrm{SS}(M^\cdot).$$

The proof is complete. □

For a finitely generated A-module M set

$$j(M) := \min\{i \mid \mathrm{Ext}^i_A(M,\ A) \neq 0\}.$$

Theorem D.4.3. *Let M be a finitely generated A-module.*

(i) $j(M) + d(M) = m$,
(ii) $d(\mathrm{Ext}^i_A(M,\ A)) \leq m - i \quad (i \in \mathbb{Z})$,
(iii) $d(\mathrm{Ext}^{j(M)}_A(M,\ A)) = d(M)$.

(Recall that m denotes the global dimension of $\mathrm{gr}^F A$.)

The following corresponding fact for regular commutative rings is well known (see [Ser2], [Bj1]).

Theorem D.4.4. *Let R be a regular commutative ring of dimension m'. For a finitely generated R-module N we set $d(N) :=$ Krull $\dim(R/\operatorname{Ann}_R N)$ and $j(N) :=$* $\min\{\, i \mid \mathrm{Ext}^i_R(N,\ R) \neq 0\}$.

(i) $d(N) + j(N) = m'$,
(ii) $d(\mathrm{Ext}^i_R(N,\ R)) \leq m' - i \quad (i \in \mathbb{Z})$,
(iii) $d(\mathrm{Ext}^{j(N)}_R(N,\ R)) = d(N)$.

Proof of Theorem D.4.3. We apply Theorem D.4.4 to the case $R = \mathrm{gr}^F A$. Fix a good filtration F of M. By Lemma D.2.4 (iii) we have $\mathrm{SS}(\mathrm{Ext}^i_A(M, A)) \subset \mathrm{supp}(\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A))$ and hence

$$d(\mathrm{Ext}^i_A(M,\ A)) \leqq d(\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)).$$

Thus (ii) follows from the corresponding fact for $\mathrm{gr}^F A$. Moreover, we have $\mathrm{Ext}^i_A(M, A) = 0$ for $i < j(\mathrm{gr}^F M)$. Hence in order to show (i) and (iii) it is sufficient to verify

$$d(\operatorname{Ext}_A^{j(\operatorname{gr}^F M)}(M, A)) = d(M).$$

By Proposition D.4.2 we have

$$d(M) = \max_{i \geqq j(\operatorname{gr}^F M)} d(\operatorname{Ext}_A^i(M, A)).$$

For $i > j(\operatorname{gr}^F M)$ we have

$$d(\operatorname{Ext}_A^i(M, A)) \leqq m - i < m - j(\operatorname{gr}^F M) = d(\operatorname{gr}^F M) = d(M),$$

and hence we must have $d(\operatorname{Ext}_A^{j(\operatorname{gr}^F M)}(M, A)) = d(M)$. □

Corollary D.4.5. *For an exact sequence*

$$0 \to L \to M \to N \to 0$$

of finitely generated A-modules we have

$$j(M) = \min\{j(L), j(N)\}.$$

D.5 Codimension filtration

In this section (A, F) is a filtered ring such that $\operatorname{gr}^F A$ is a regular commutative ring of pure dimension m. For a finitely generated A-module M and $s \geq 0$ we denote by $C^s(M)$ the sum of all submodules N of M satisfying $j(N) \geq s$. Since $C^s(M)$ is finitely generated, we have $j(C^s(M)) \geq s$ by Corollary D.4.5 and hence $C^s(M)$ is the largest submodule N of M satisfying $j(N) \geq s$. By definition we have a decreasing filtration

$$0 = C^{m+1}(M) \subset C^m(M) \subset \cdots \subset C^1(M) \subset C^0(M) = M.$$

We say that a finitely generated A-module M is purely s-codimensional if $C^s(M) = M$ and $C^{s+1}(M) = 0$.

Lemma D.5.1. *For any finitely generated A-module M $C^s(M)/C^{s+1}(M)$ is purely s-codimensional.*

Proof. Set $N = C^s(M)/C^{s+1}(M)$. Then we have $j(N) \geq j(C^s(M)) \geq s$ and hence $C^s(N) = N$. Set $K = \operatorname{Ker}(C^s(M) \to N/C^{s+1}(N))$. Then by the exact sequence

$$0 \to C^{s+1}(M) \to K \to C^{s+1}(N) \to 0$$

we have $j(K) = \min\{j(C^{s+1}(M)), j(C^{s+1}(N))\} \geq s + 1$. By the maximality of $C^{s+1}(M)$ we obtain $K = C^{s+1}(M)$, i.e., $C^{s+1}(N) = 0$. □

We will give a cohomological interpretation of this filtration.

For a finitely generated A-module M and $s \geq 0$ we set

$$T^s(M) = \mathrm{Ext}^0_{A^{\mathrm{op}}}(\tau^{\geqslant s} R\,\mathrm{Hom}_A(M, A), A^{\mathrm{op}}).$$

By Proposition D.4.1 we have $T^0(M) = M$. By $j(\mathrm{Ext}^s(M, A)) \geq s$ we have $\mathrm{Ext}^{s-1}_{A^{\mathrm{op}}}(\mathrm{Ext}^s(M, A), A^{\mathrm{op}}) = 0$. Hence by applying $R\,\mathrm{Hom}_{A^{\mathrm{op}}}(\bullet, A^{\mathrm{op}})$ to the distinguished triangle

$$\mathrm{Ext}^s_A(M, A)[-s] \longrightarrow \tau^{\geqslant s} R\,\mathrm{Hom}_A(M, A) \longrightarrow \tau^{\geqslant s+1} R\,\mathrm{Hom}_A(M, A) \xrightarrow{+1}$$

and taking the cohomology groups we obtain an exact sequence

$$0 \to T^{s+1}(M) \to T^s(M) \to \mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M, A), A^{\mathrm{op}}).$$

Hence we obtain a decreasing filtration

$$0 = T^{m+1}(M) \subset T^m(M) \subset \cdots \subset T^1(M) \subset T^0(M) = M.$$

Proposition D.5.2. *For any s we have $C^s(M) = T^s(M)$.*

Proof. By $j(\mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s(M, A), A^{\mathrm{op}})) \geq s$ we see from the exact sequence

$$0 \to T^{s+1}(M) \to T^s(M) \to \mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M, A), A^{\mathrm{op}})$$

using the backward induction on s that $j(T^s(M)) \geq s$. Hence $T^s(M) \subset C^s(M)$. It remains to show the opposite inclusion. Set $N = C^s(M)$. By $j(N) \geq s$ we have

$$\tau^{\geqslant s} R\,\mathrm{Hom}_A(N, A) = R\,\mathrm{Hom}_A(N, A),$$

and hence $N = T^s(N)$. By the functoriality of T^s we have a commutative diagram

$$\begin{array}{ccc} T^s(N) & = & N \\ \downarrow & & \downarrow \\ T^s(M) & \longrightarrow & M, \end{array}$$

which implies $N \subset T^s(M)$. □

Theorem D.5.3. *Let M be a finitely generated A-module which is purely s-codimensional. Then for any $\mathfrak{p} \in \mathrm{SS}_0(M)$ we have*

$$\text{Krull dim}((\mathrm{gr}^F A)/\mathfrak{p}) = m - s.$$

Proof. The assertion being trivial for $M = 0$ we assume that $M \neq 0$. In this case we have $j(M) = s$. Let F be a good filtration of M. Then there exists a good filtration F of $N = \mathrm{Ext}^s_A(M, A)$ such that $\mathrm{gr}^F N$ is a subquotient of $\mathrm{Ext}^s_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)$. Hence

$$\begin{aligned}\mathrm{SS}(N) = \mathrm{supp}(\mathrm{gr}^F N) &\subset \mathrm{supp}(\mathrm{Ext}^s_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)) \\ &\subset \mathrm{supp}(\mathrm{gr}^F M) = \mathrm{SS}(M).\end{aligned}$$

On the other hand since M is purely s-codimensional, we have

$$M = T^s(M)/T^{s+1}(M) \subset \mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M, A), A^{\mathrm{op}}) = \mathrm{Ext}^s_{A^{\mathrm{op}}}(N, A^{\mathrm{op}})$$

and hence $\mathrm{SS}(M) \subset \mathrm{SS}(\mathrm{Ext}^s_{A^{\mathrm{op}}}(N, A^{\mathrm{op}})) \subset \mathrm{SS}(N)$. Therefore, we have

$$\mathrm{SS}(M) = \mathrm{supp}(\mathrm{gr}^F M) = \mathrm{supp}(\mathrm{Ext}^s_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)).$$

By $j(\mathrm{gr}^F M) = j(M) = s$ we have $j((\mathrm{gr}^F A)/\mathfrak{p}) \geq s$ for any $\mathfrak{p} \in \mathrm{supp}_0(\mathrm{gr}^F M)$. Set $\Lambda = \{\mathfrak{p} \in \mathrm{supp}_0(\mathrm{gr}^F M) \mid j((\mathrm{gr}^F A)/\mathfrak{p}) = s)\}$. By a well-known fact in commutative algebra there exists a submodule L of $\mathrm{gr}^F M$ such that $j(L) > s$ and $\mathrm{supp}_0(\mathrm{gr}^F M/L) = \Lambda$. We need to show $\mathrm{supp}(\mathrm{gr}^F M) = \mathrm{supp}(\mathrm{gr}^F M/L)$. We have obviously $\mathrm{supp}(\mathrm{gr}^F M) \supset \mathrm{supp}(\mathrm{gr}^F M/L)$. On the other hand by $\mathrm{Ext}^s(L, \mathrm{gr}^F A) = 0$ we have an injection $\mathrm{Ext}^s(\mathrm{gr}^F M, \mathrm{gr}^F A) \to \mathrm{Ext}^s(\mathrm{gr}^F M/L, \mathrm{gr}^F A)$, and hence

$$\begin{aligned}\mathrm{supp}(\mathrm{gr}^F M) = \mathrm{supp}(\mathrm{Ext}^s(\mathrm{gr}^F M, \mathrm{gr}^F A)) &\subset \mathrm{supp}(\mathrm{Ext}^s(\mathrm{gr}^F M/L, \mathrm{gr}^F A)) \\ &\subset \mathrm{supp}(\mathrm{gr}^F M/L).\end{aligned}$$

The proof is complete. $\square$

E

Symplectic Geometry

In this chapter we first present basic results in symplectic geometry laying special emphasis on cotangent bundles of complex manifolds. Most of the results are well known and we refer the reader to Abraham–Mardsen [AM] and Duistermaat [Dui] for details. Next we will precisely study conic Lagrangian analytic subsets in the cotangent bundles of complex manifolds. We prove that such a Lagrangian subset is contained in the union of the conormal bundles of strata in a Whitney stratification of the base manifold (Kashiwara's theorem in [Kas3], [Kas8]).

E.1 Symplectic vector spaces

Let V be a finite-dimensional vector space over a field k. A *symplectic form* σ on V is a non-degenerate anti-symmetric bilinear form on V. If a vector space V is endowed with a symplectic form σ, we call the pair (V, σ) a *symplectic vector space*. The dimension of a symplectic vector space is even. Let (V, σ) be a symplectic vector space. Denote by V^* the dual of V. Then for any $\theta \in V^*$ there exists a unique $H_\theta \in V$ such that

$$< \theta, v >= \sigma(v, H_\theta) \quad (v \in V)$$

by the non-degeneracy of σ. The correspondence $\theta \mapsto H_\theta$ defines the *Hamiltonian isomorphism* $H : V^* \simeq V$. For a linear subspace W of V consider its orthogonal complement $W^\perp = \{v \in V | \sigma(v, W) = 0\}$ with respect to σ. Then again by the non-degeneracy of σ we obtain $\dim W + \dim W^\perp = \dim V$. Now let us introduce the following important linear subspaces of V.

Definition E.1.1. A linear subspace W of V is called *isotropic* (resp. *Lagrangian*, resp. *involutive*) if it satisfies $W \subset W^\perp$ (resp. $W = W^\perp$, resp. $W \supset W^\perp$).

Note that if a linear subspace $W \subset V$ is isotropic (resp. Lagrangian, resp. involutive) then $\dim W \leq \frac{1}{2} \dim V$ (resp. $\dim W = \frac{1}{2} \dim V$, resp. $\dim W \geq \frac{1}{2} \dim V$). Moreover, a one-dimensional subspace (resp. a hyperplane) of V is always isotropic (resp. involutive).

Example E.1.2. Let W be a finite-dimensional vector space and W^* its dual. Set $V = W \oplus W^*$ and define a bilinear form σ on V by

$$\sigma((x,\xi),(x',\xi')) = < x', \xi > - < x, \xi' > \qquad ((x,\xi),(x',\xi') \in V = W \oplus W^*).$$

Then (V,σ) is a symplectic vector space. Moreover, W and W^* are Lagrangian subspaces of V.

E.2 Symplectic structures on cotangent bundles

A complex manifold X is called a (holomorphic) *symplectic manifold* if there exists a holomorphic 2-form σ globally defined on X which induces a symplectic form on the tangent space T_xX of X at each $x \in X$. The dimension of a symplectic manifold is necessarily even. As one of the most important examples of symplectic manifolds, we treat here cotangent bundles of complex manifolds.

Now let X be a complex manifold and TX (resp. T^*X) its tangent (resp. cotangent) bundle. We denote by $\pi : T^*X \to X$ the canonical projection. By differentiating π we obtain the tangent map $\pi' : T(T^*X) \to (T^*X) \times_X (TX)$ and its dual $\rho_\pi : (T^*X) \times_X (T^*X) \to T^*(T^*X)$. If we restrict ρ_π to the diagonal T^*X of $(T^*X) \times_X (T^*X)$ then we get a map $T^*X \to T^*(T^*X)$. Since this map is a holomorphic section of the bundle $T^*(T^*X) \to T^*X$, it corresponds to a (globally defined) holomorphic 1-form α_X on T^*X. We call α_X the *canonical* **1**-*form*. If we take a local coordinate $(x_1, x_2, \dots, x_n)$ of X on an open subset $U \subset X$, then any point p of $T^*U \subset T^*X$ can be written uniquely as $p = (x_1, x_2, \dots, x_n; \xi_1 dx_1 + \xi_2 dx_2 + \cdot + \xi_n dx_n)$ where $\xi_i \in \mathbb{C}$. We call $(x_1, x_2, \dots, x_n; \xi_1, \xi_2, \dots, \xi_n)$ the local coordinate system of T^*X associated to $(x_1, x_2, \dots, x_n)$. In this local coordinate of T^*X the canonical 1-form α_X is written as $\alpha_X = \sum_{i=1}^n \xi_i dx_i$. Set $\sigma_X = d\alpha_X = \sum_{i=1}^n d\xi_i \wedge dx_i$. Then we see that the holomorphic 2-form σ_X defines a symplectic structure on $T_p(T^*X)$ at each point $p \in T^*X$. Namely, the cotangent bundle T^*X is endowed with a structure of a symplectic manifold by σ_X. We call σ_X the (canonical) *symplectic form* of T^*X. Since there exists the Hamiltonian isomorphism $H : T_p^*(T^*X) \simeq T_p(T^*X)$ at each $p \in T^*X$, we obtain the global isomorphism $H : T^*(T^*X) \simeq T(T^*X)$. For a holomorphic function f on T^*X we define a holomorphic vector field H_f to be the image of the 1-form df by $H : T^*(T^*X) \simeq T(T^*X)$. The vector field H_f is called the *Hamiltonian vector field* of f. Define the *Poisson bracket* of two holomorphic functions f, g on T^*X by $\{f, g\} = H_f(g) = \sigma_X(H_f, H_g)$. In the local coordinate $(x_1, x_2, \dots, x_n; \xi_1, \xi_2, \dots, \xi_n)$ of T^*X we have the explicit formula

$$H_f = \sum_{i=1}^n \left(\frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial}{\partial \xi_i} \right).$$

We can also easily verify the following:

$$\begin{cases} \{f, g\} = -\{g, f\}, \\ \{f, hg\} = h\{f, g\} + g\{f, h\}, \\ \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0. \end{cases}$$

Moreover, we have $[H_f, H_g] = H_{\{f,g\}}$, where $[H_f, H_g]$ is the Lie bracket of H_f and H_g.

Definition E.2.1. An analytic subset V of T^*X is called isotropic (resp. Lagrangian, resp. involutive) if for any smooth point $p \in V_{\mathrm{reg}}$ of V the tangent space T_pV at p is a isotropic (resp. a Lagrangian, resp. an involutive) subspace in $T_p(T^*X)$.

By definition the dimension of a Lagrangian analytic subset of T^*X is equal to $\dim X$.

Example E.2.2.

(i) Let $Y \subset X$ be a complex submanifold of X. Then the *conormal bundle* T^*_YX of Y in X is a Lagrangian submanifold of T^*X.

(ii) Let f be a holomorphic function on X. Set $\Lambda_f = \{(x, \operatorname{grad} f(x) | x \in X\}$. Then Λ_f is a Lagrangian submanifold of T^*X.

For an analytic subset V of T^*X denote by $\mathcal{I}_V$ the subsheaf of $\mathcal{O}_{T^*X}$ consisting of holomorphic functions vanishing on V.

Lemma E.2.3. *For an analytic subset V of T^*X the following conditions are equivalent:*

(i) *V is involutive.*

(ii) $\{\mathcal{I}_V, \mathcal{I}_V\} \subset \mathcal{I}_V$.

Proof. By the definition of Hamiltonian isomorphisms, for each smooth point $p \in V_{\mathrm{reg}}$ of V the orthogonal complement $(T_pV)^\perp$ of T_pV in the symplectic vector space $T_p(T^*X)$ is spanned by the Hamiltonian vector fields H_f of $f \in \mathcal{I}_V$. Assume that V is involutive. If $f, g \in \mathcal{I}_V$ then the Hamiltonian vector field H_f is tangent to V_{reg} and hence $\{f, g\} = H_f(g) = 0$ on V_{reg}. Since $\{f, g\}$ is holomorphic and V_{reg} is dense in V, $\{f, g\} = 0$ on the whole V, i.e., $\{f, g\} \in \mathcal{I}_V$. The part (i) $\Longrightarrow$ (ii) was proved. The converse can be proved more easily. □

E.3 Lagrangian subsets of cotangent bundles

Let X be a complex manifold of dimension n. Since the fibers of the cotangent bundle T^*X are complex vector spaces, there exists a natural action of the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ on T^*X. We say that an analytic subset V of T^*X is *conic* if V is stable by this action of $\mathbb{C}^\times$. In this subsection we focus our attention on conic Lagrangian analytic subsets of T^*X.

First let us examine the image $H(\alpha_X)$ of the canonical 1-form α_X by the Hamiltonian isomorphism $H : T^*(T^*X) \simeq T(T^*X)$. In a local coordinate $(x_1, x_2, \dots, x_n; \xi_1, \xi_2, \dots, \xi_n)$ of T^*X the holomorphic vector field $H(\alpha_X)$ thus obtained has the form

$$H(\alpha_X) = -\sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i}.$$

It follows that $-H(\alpha_X)$ is the infinitesimal generator of the action of $\mathbb{C}^\times$ on T^*X. We call this vector field the *Euler vector field*.

Lemma E.3.1. *Let V be a conic complex submanifold of T^*X. Then V is isotropic if and only if the pull-back $\alpha_X|_V$ of α_X to V is identically zero.*

Proof. Assume that $\alpha_X|_V$ is identically zero on V. Then also the pull-back of the symplectic 2-form $\sigma_X = d\alpha_X$ to V vanishes. Hence V is isotropic. Let us prove the converse. Assume that V is isotropic. Then for any local section δ of the tangent bundle $TV \to V$ we have

$$\langle \alpha_X, \delta \rangle = \sigma_X(\delta, H(\alpha_X)) = 0$$

on V because the Euler vector field $-H(\alpha_X)$ is tangent to V by the conicness of V. This means that $\alpha_X|_V$ is identically zero on V. □

Corollary E.3.2. *Let Λ be a conic Lagrangian analytic subset of T^*X. Then the pull-back of α_X to the regular part Λ_{reg} of Λ is identically zero.*

Let Z be an analytic subset of the base space X and denote by $\overline{T^*_{Z_{\mathrm{reg}}}X}$ the closure of the conormal bundle $T^*_{Z_{\mathrm{reg}}}X$ of Z_{reg} in T^*X. Since the closure is taken with respect to the classical topology of T^*X, it is not clear if $\overline{T^*_{Z_{\mathrm{reg}}}X}$ is an analytic subset of T^*X or not. In Proposition E.3.5 below, we will prove the analyticity of $\overline{T^*_{Z_{\mathrm{reg}}}X}$. For this purpose, recall the following definitions.

Definition E.3.3. Let S be an analytic space. A locally finite partition $S = \bigsqcup_{\alpha \in A} S_\alpha$ of S by locally closed complex manifolds S_α's is called a *stratification* of S if for each S_α the closure $\overline{S_\alpha}$ and the boundary $\partial S_\alpha = \overline{S_\alpha} \setminus S_\alpha$ are analytic and unions of S_β's. A complex manifold S_α in it is called a *stratum* of the stratification $S = \bigsqcup_{\alpha \in A} S_\alpha$.

Definition E.3.4. Let S be an analytic space. Then we say that a subset S' of S is *constructible* if there exists stratification $S = \bigsqcup_{\alpha \in A} S_\alpha$ of S such that S' is a union of some strata in it.

For an analytic space S the family of constructible subsets of S is closed under various set-theoretical operations. Note also that by definition the closure of a constructible subset is analytic. Moreover, if $f : S \to S'$ is a morphism (resp. proper morphism) of analytic spaces, then the inverse (resp. direct) image of a constructible subset of S' (resp. S) by f is again constructible. Now we are ready to prove the following.

Proposition E.3.5. *$\overline{T^*_{Z_{\mathrm{reg}}}X}$ is an analytic subset of T^*X.*

Proof. We may assume that Z is irreducible. By Hironaka's theorem there exists a proper holomorphic map $f : Y \to X$ from a complex manifold Y and an analytic subset $Z' \neq Z$ of Z such that $f(Y) = Z$, $Z_0 := Z \setminus Z'$ is smooth, and the restriction of f to $Y_0 := f^{-1}(Z_0)$ induces a biholomorphic map $f|_{Y_0} : Y_0 \simeq Z_0$. From $f : Y \to X$ we obtain the canonical morphisms

$$T^*Y \xleftarrow{\rho_f} Y \times_X T^*X \xrightarrow{\varpi_f} T^*X.$$

We easily see that $T^*_{Z_0}X = \varpi_f \rho_f^{-1}(T^*_{Y_0}Y_0)$, where $T^*_{Y_0}Y_0 \simeq Y_0$ is the zero-section of $T^*Y_0 \subset T^*Y$. Since $T^*_{Y_0}Y_0$ is a constructible subset of T^*Y and ϖ_f is proper, $T^*_{Z_0}X$ is a constructible subset of T^*X. Hence the closure $\overline{T^*_{Z_0}X} = \overline{T^*_{Z_{\mathrm{reg}}}X}$ is an analytic subset of T^*X. □

By Example E.2.2 and Proposition E.3.5, for an irreducible analytic subset Z of X we conclude that the closure $\overline{T^*_{Z_{\mathrm{reg}}}X}$ is an irreducible conic Lagrangian analytic subset of T^*X. The following result, which was first proved by Kashiwara [Kas3], [Kas8], shows that any irreducible conic Lagrangian analytic subset of T^*X is obtained in this way.

Theorem E.3.6 (Kashiwara [Kas3], [Kas8]). *Let Λ be a conic Lagrangian analytic subset of T^*X. Assume that Λ is irreducible. Then $Z = \pi(\Lambda)$ is an irreducible analytic subset of X and $\Lambda = \overline{T^*_{Z_{\mathrm{reg}}}X}$.*

Proof. Since Λ is conic, $Z = \pi(\Lambda) = (T^*_XX) \cap \Lambda$ is an analytic subset of X. Moreover, by definition we easily see that Z is irreducible. Denote by Λ_0 the open subset of $\pi^{-1}(Z_{\mathrm{reg}}) \cap \Lambda_{\mathrm{reg}}$ consisting of points where the map $\pi|_{\Lambda_{\mathrm{reg}}}$ has the maximal rank. Then Λ_0 is open dense in $\pi^{-1}(Z_{\mathrm{reg}}) \cap \Lambda$ and the maximal rank is equal to $\dim Z$. Now let p be a point in Λ_0. Taking a local coordinate $(x_1, x_2, \dots, x_n)$ of X around the point $\pi(p) \in Z_{\mathrm{reg}}$, we may assume that $Z = \{x_1 = x_2 = \cdots = x_d = 0\}$ where $d = n - \dim Z$. Let us choose a local section $s : Z_{\mathrm{reg}} \hookrightarrow \Lambda_{\mathrm{reg}}$ of $\pi|_{\Lambda_{\mathrm{reg}}}$ such that $s(\pi(p)) = p$. Let $i_{\Lambda_{\mathrm{reg}}} : \Lambda_{\mathrm{reg}} \hookrightarrow T^*X$ be the embedding. Then by Corollary E.3.2 the pull-back of the canonical 1-form α_X to Z_{reg} by $i_{\Lambda_{\mathrm{reg}}} \circ s$ is zero. On the other hand, this 1-form on Z_{reg} has the form $\xi_{d+1}(x')dx_{d+1} + \cdots + \xi_n(x')dx_n$, where we set $x' = (x_{d+1}, \dots, x_n)$. Therefore, the point p should be contained in $\{\xi_{d+1} = \cdots = \xi_n = 0\}$. We proved that $\Lambda_0 \subset T^*_{Z_{\mathrm{reg}}}X$. Since $\dim \Lambda_0 = \dim T^*_{Z_{\mathrm{reg}}}X = n$ we obtain $\overline{T^*_{Z_{\mathrm{reg}}}X} = \overline{\Lambda_0} \subset \Lambda$. Then the result follows from the irreducibility of Λ. □

To treat general conic Lagrangian analytic subsets of T^*X let us briefly explain Whitney stratifications.

Definition E.3.7. Let S be an analytic subset of a complex manifold M. A stratification $S = \bigsqcup_{\alpha \in A} S_\alpha$ of S is called a *Whitney stratification* if it satisfies the following Whitney conditions (a) and (b):

(a) Assume that a sequence $x_i \in S_\alpha$ of points converges to a point $y \in S_\beta$ $(\alpha \neq \beta)$ and the limit T of the tangent spaces $T_{x_i}S_\alpha$ exists. Then we have $T_yS_\beta \subset T$.
(b) Let $x_i \in S_\alpha$ and $y_i \in S_\beta$ be two sequences of points which converge to the same point $y \in S_\beta$ $(\alpha \neq \beta)$. Assume further that the limit l (resp. T) of the lines l_i jointing x_i and y_i (resp. of the tangent spaces $T_{x_i}S_\alpha$) exists. Then we have $l \subset T$.

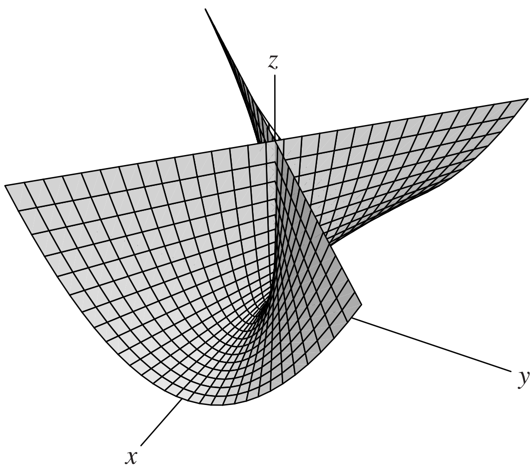
It is well known that any stratification of an analytic set can be refined to satisfy the Whitney conditions. Intuitively, the Whitney conditions means that the geometrical normal structure of the stratification $S = \bigsqcup_{\alpha \in A} S_\alpha$ is locally constant along each stratum S_α as is illustrated in the example below.

Example E.3.8 (Whitney's umbrella). Consider the analytic set $S = \{(x, y, z) \in \mathbb{C}^3 \mid y^2 = zx^2\}$ in $\mathbb{C}^3$ and the following two stratifications $S = \bigsqcup_{i=1}^2 S_i'$ and $S = \bigsqcup_{i=1}^3 S_i$ of S:

$$\begin{cases} S_1' = \{(x, y, z) \in \mathbb{C}^3 \mid x = y = 0\} \\ S_2' = S \setminus S_1' \end{cases}$$

$$\begin{cases} S_1 = \{0\} \\ S_2 = \{(x, y, z) \in \mathbb{C}^3 \mid x = y = 0\} \setminus S_1 \\ S_3 = S \setminus (S_1 \sqcup S_2) \end{cases}$$

Then the stratification $S = \bigsqcup_{i=1}^3 S_i$ satisfies the Whitney conditions (a) and (b), but the stratification $S = \bigsqcup_{i=1}^2 S_i'$ does not.



We see that along each stratum S_i ($i = 1, 2, 3$), the geometrical normal structure of $S = \bigsqcup_{i=1}^3 S_i$ is constant.

Now consider a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of a complex manifold X. Then it is a good exercise to prove that for each point $x \in X$ there exists a sufficiently small sphere centered at x which is transversal to all the strata X_α's. This result follows easily from the Whitney condition (b). For the details see [Kas8], [Schu]. Moreover, by the Whitney conditions we can prove easily that the union $\bigsqcup_{\alpha \in A} T^*_{X_\alpha} X$ of the conormal bundles $T^*_{X_\alpha} X$ is a closed analytic subset of T^*X (the analyticity follows from Proposition E.3.5). The following theorem was proved by Kashiwara [Kas3], [Kas8] and plays a crucial role in proving the constructibility of the solutions to holonomic D-modules.

Theorem E.3.9. *Let X be a complex manifold and Λ a conic Lagrangian analytic subset of T^*X. Then there exists a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X such that*

$$\Lambda \subset \bigsqcup_{\alpha \in A} T^*_{X_\alpha} X.$$

Proof. Let $\Lambda = \cup_{i \in I} \Lambda_i$ be the irreducible decomposition of Λ and set $Z_i = \pi(\Lambda_i)$. Then we can take a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X such that Z_i is a union of strata in it for any $i \in I$. Note that for each $i \in I$ there exists a (unique) stratum $X_{\alpha_i} \subset (Z_i)_{\mathrm{reg}}$ which is open dense in Z_i. Hence we have $\Lambda_i = \overline{T^*_{(Z_i)_{\mathrm{reg}}} X} = \overline{T^*_{X_{\alpha_i}} X}$ by Theorem E.3.6 and $\Lambda = \cup_{i \in I} \Lambda_i \subset \sqcup_{\alpha \in A} T^*_{X_\alpha} X$. $\square$