2

Analytic Functions

2.1 Introduction

This unit is devoted to the study of analytic function. Before describing it we shall consider some important notions that will be needed in the sequel. Harmonic function will also be studied.

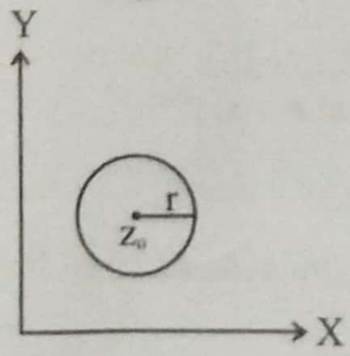
2.2 Circles, Disks and Annulus

Let z_o be a given complex number. The set of all complex numbers z such that $|z - z_o| = r$ (1)

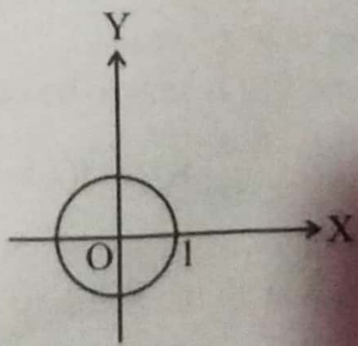
is called a circle of radius r and centre z_o. If $z_o = 0$ and $r = 1$, then (1) reduces to $|z| = 1$

This is a unit circle of radius 1 and centre 0.

Circle of Radius r having centre z_0

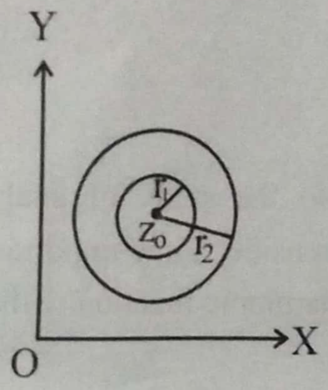


Unit circle having centre 0



Let z_o be a complex number. The set of all z such that $|z - z_o| < r$ is called an **open circular disk**; and the set of all z such that $|z - z_o| \leq r$ is called a **closed circular disk**.
An open circular disk is also called a neighbourhood of z_o. And it is the interior of the circle $|z - z_o| = r$
The exterior of the circle $|z - z_o| = r$ is $|z - z_o| > r$.

The set of all z such that $r_1 < |z - z_o| < r_2$ is called an **open annulus**. The **closed annulus** is given by $r_1 \leq |z - z_o| \leq r_2$



An open circle divides the complex plane in two open domains.

2.3 Limit and Continuity

A function f(z) is said to have a limit l if given $\varepsilon > 0$, there exists a positive number δ such that $|f(z) - l| < \varepsilon$ for all $z \neq z_o$ in the disk $|z - z_o| < \delta$
And, we write

$$\lim_{z \to z_0} f(z) = l,$$

no matter how $z \to z_0$.
A function f(z) is said to be continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_o).$$

A function is said to be continuous in a domain D if f(z) is continuous at all the points of D.

2.4 Differentiability

A function f(z) is said to be differentiable at a point z if

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

exists, no matter how $\Delta z \to 0$.

The derivative of f at a point z is denoted by f '(z). Thus,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

If the limit does not exist, we say that the function is non–differentiable at z.

Hence in order to prove non–differentiability of the function f(z), we should try different parts for Δz. Convenient paths for Δz are along real axis and imaginary axis.

Examples

1. Show $f(z) = z^2$ is differentiable for all z.

We have $f(z) = z^2$ so that $f(z+\Delta z) = (z+\Delta z)^2$

$$= z^2 + 2z\Delta z + (\Delta z)^2$$

$$\text{Now, } f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z(2z + \Delta z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

$\therefore f(z) = z^2$ is differentiable for all z.

2 . Show that the function $|z|^2$ is nowhere differentiable **except at** the origin.

Let $f(z) = |z|^2$, then $f(z + \Delta z) = |z + \Delta z|^2$

$$\text{Now, } f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\bar{z} - \Delta \bar{z}) - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z\,\bar{z} - z\,\Delta\bar{z} + \bar{z}\,\Delta z - \Delta z\,\Delta\bar{z} - z\,\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z}\,\Delta z - z\,\Delta\bar{z} - \Delta z\Delta\bar{z}}{\Delta \bar{z}}$$

$$= \lim_{\Delta z \to 0} \left(\bar{z} - z.\frac{\Delta\bar{z}}{\Delta z} - \Delta\bar{z}\right)$$

$$= \lim_{\Delta z \to 0} \left[\bar{z} - z\frac{\Delta\bar{z}}{\Delta z}\right] \quad (\because \lim_{\Delta z \to 0} \Delta\bar{z} = 0)$$

When $z = 0$, then the above limit is zero so that $f'(0) = 0$.

For $z \neq 0$, suppose first that Δz is wholly real, so that $\Delta y = 0$.

$$\therefore f'(z) = \lim_{\Delta x \to 0} \left(\bar{z} - z.\frac{\Delta x}{\Delta x}\right)$$

$$= \lim_{\Delta x \to 0} (\bar{z} - z)$$

$$= \bar{z} - z$$

$$= x - iy - x - iy$$

$$= -2iy$$

Further suppose that Δz be wholly imaginary, then $\Delta x = 0$

$$\therefore f'(z) = \lim_{\Delta y \to 0} \left(\bar{z} - z.\frac{(-i\Delta y)}{i\Delta y}\right)$$

$$= \lim_{\Delta y \to 0} (\overline{z} + z)$$

$$= \overline{z} + z$$

$$= 2x$$

Since the two limits are not equal, so the given function is not differentiable except at the origin.

3. Find the derivative of the function $f(z) = z + \dfrac{1}{z}$ $(z \neq 0)$ at $z = 1 + i$

We have, $f(z) = z + \dfrac{1}{z}$ $\therefore f(z + \Delta z) = z + \Delta z + \dfrac{1}{z + \Delta z}$

Now, $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(z + \Delta z) - f(z)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{z + \Delta z + \dfrac{1}{z + \Delta z} - z - \dfrac{1}{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\frac{\Delta z + \left(\dfrac{1}{z + \Delta z} - \dfrac{1}{z} \right)}{\Delta z} \right]$$

$$= \lim_{\Delta z \to 0} \left[1 + \frac{1}{\Delta z} \left(\frac{1}{z + \Delta z} - \frac{1}{z} \right) \right]$$

$$= \lim_{\Delta z \to 0} \left[1 + \frac{1}{\Delta z} \left(\frac{z - z - \Delta z}{z\,(z + \Delta z)} \right) \right]$$

$$= \lim_{\Delta z \to 0} \left[1 - \frac{1}{z(z + \Delta z)} \right]$$

$$= 1 - \frac{1}{z^2}$$

$$\therefore f'((1 + i) = 1 - \frac{1}{(1 + i)^2}$$

$$= 1 - \frac{1}{1+2i+i^2} = 1 - \frac{1}{2i}$$

4. Find the derivate of $f(z) = (z-i)^7$ at $z = 1+i$

Here, $f(z+\Delta z) = (z+\Delta z - i)^7$

By definition, $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z - i)^7 - (z-i)^7}{\Delta z} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{\Delta z \to 0} 7(z+\Delta z - i)^6 = 7(z-i)^6$$

$\therefore f'(1+i) = 7(1+i-i)^6 = 7$

Rules of differentiation

The property of differentiability implies that a constant function is differentiable. A positive integral power of z is differentiable. Similarly, the sum of two differentiable functions is differentiable, the product of two differential functions is also differentiable, and the quotient of two differentiable functions is differentiable provided that the denominator does not vanish, and we have

i. $\dfrac{d}{dz}[f(z) \pm g(z)] = \dfrac{d}{dz} f(z) \pm \dfrac{d}{dz} g(z)$

ii. $\dfrac{d}{dz}[cf(z)] = c\dfrac{d}{dz} f(z)$, where c is a complex constant

iii. $\dfrac{d}{dz}[f(z)\, g(z)] = f(z)\dfrac{d}{dz} g(z) + g(z)\dfrac{d}{dz} f(z)$

iv. $\dfrac{d}{dz}\left[\dfrac{f(z)}{g(z)}\right] = \dfrac{g(z)\dfrac{d}{dz} f(z) - f(z)\dfrac{d}{dz} g(z)}{[g(z)]^2}$, $g(z) \neq 0$

where f(z) and g(z) are differentiable functions.

2.5 Analytic Function

Let f(z) be a complex valued function defined on a domain D. A function f(z) is said to be analytic in D if f(z) is differentiable at all points of D. In particular, f(z) is analytic at a point z if there is an open disk about z on which f(z) is differentiable.

A function may have a derivative at a point without being analytic at the point. Consider, for example, a function $f(z) = |z|^2$.

This function have a derivative at 0 but at no other point. So the function is not analytic at 0. Remember that a function is analytic at a point if its derivative exists not only at that point but in some neighbourhood of that point.

2.6 Cauchy–Riemann Equations

The necessary conditions for f(z) to be analytic

If $f(z) = u(x, y) + iv(x, y)$ is analytic in domain D, then the partial derivatives u_x, v_x, u_y, v_y exists and satisfy $u_x = v_y$, $u_y = -v_x$.

Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D, so $f'(z)$ exists for all z, and

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

Since $f(z) = u(x, y) + iv(x, y)$, so

$$f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)$$

Therefore

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y + \Delta y) + iv(x + \Delta x, y+\Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) - u(x, y)] + i[v(x+\Delta x, y+\Delta y) - v(x, y)]}{\Delta x + i\Delta y} \quad (1)$$

First suppose that Δz is purely real, so that $\Delta y = 0$. Therefore (1) reduces to

$$f'(z) = \lim_{\Delta z \to 0} \frac{\{u(x+\Delta x, y) - u(x, y)\} + i\{v(x+\Delta x, y) - v(x, y)\}}{\Delta x}$$

$$= \lim_{\Delta z \to 0} \left[\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i\,\frac{v(x+\Delta x, y) - v(x, y)}{\Delta x}\right]$$

$$\text{or } \mathbf{f'(z) = u_x + iv_x} \quad (2)$$

Further suppose that Δz is purely imaginary so that $\Delta x = 0$. And (1) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x, y) + i[v(x, y+\Delta y) - v(x, y)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \left[\frac{1}{i}\,\frac{u(x, y+\Delta y) - u(x, y)}{\Delta y} + \frac{v(x, y+\Delta y) - v(x, y)}{\Delta y}\right]$$

$$\text{or. } \mathbf{f'(z) = v_y - iu_y} \quad (3)$$

From (2) and (3), we have $u_x = v_y$ and $u_y = -v_x$ (4)

The equations given in (4) are called **Cauchy–Riemann equations.** Thus, if a function is analytic, it must satisfy Cauchy-Riemann equations. But the converse is not necessarily true, i.e., a function may satisfy Cauchy–Riemann equation but it may not be analytic. So, the Cauchy–Riemann equations are only the necessary conditions for the function to be analytic.

Sufficient Condition

If the four partial derivatives u_x, u_y, v_x, v_y exist, are continuous and satisfy Cauchy Riemann equations, then the function f(z) is analytic in D.

Cauchy–Riemann Equations in Polar Form

Let w = f(z) be analytic in region D, and let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\therefore\ w = f(re^{i\theta})$ or $u + iv = f(re^{i\theta})$ (1)

Differentiating partially with respect to r, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = e^{i\theta}.\ f'(re^{i\theta}) \text{ or } f'(re^{i\theta}) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Again differentiating (1) partially with respect to θ,

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ire^{i\theta}\ f'(re^{i\theta}) = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Thus, we have $\dfrac{\partial u}{\partial \theta} + i\dfrac{\partial v}{\partial \theta} = -r\dfrac{\partial v}{\partial r} + ir\dfrac{\partial u}{\partial r}$

$$\Rightarrow \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \text{ and } \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}.$$

It can be written as:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

These are known as Cauchy–Riemann equations in polar from.

Examples

1. Show that $f(z) = z^3$ is analytic for all z.

We have, $f(z) = z^3 = (x + iy)^3 = x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3$

or, $u + iv = x^3 - 3xy^2 + i(3x^2y - y^3)$

Equating real and imaginary parts,

$u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

Here, $u_x = 3x^2 - 3y^2 \qquad v_x = 6xy$

$u_y = -6xy \qquad v_y = 3x^2 - 3y^2$

Clearly, $u_x = v_y$ and $u_y = -v_x$

Moreover, the above four partial derivatives are continuous.

$\therefore$ The given function is analytic for all z.

2. Show that $z\bar{z}$ is not analytic.

Let $f(z) = z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$

or, $u + iv = x^2 + y^2$

$\therefore u = x^2 + y^2$ and $v = 0$

Here, $u_x = 2x \qquad v_x = 0$

$u_y = 2y \qquad v_y = 0$

$\therefore u_x \neq v_y$ and $u_y \neq v_x$ (except at the origin)

Hence $z\bar{z}$ is not analytic.

3. Test for analyticity of the function $f(z) = \dfrac{\text{Re}z}{\text{Im}z}$.

We have $f(z) = \dfrac{\text{Re}z}{\text{Im}z} = \dfrac{x}{y}$

or $u + iv = \dfrac{x}{y}$

$\therefore u = \dfrac{x}{y}$ and $v = 0$

Here, $u_x = \dfrac{1}{y} \qquad v_x = 0$

$u_y = -\dfrac{x}{y^2} \qquad v_y = 0$

Since $u_x \neq v_y$ and $u_y \neq -v_x$, so the function is not analytic.

4. Check for analyticity of the function f(z) = Arg z

$$f(z) = \text{Arg } z = \theta = \tan^{-1}\left(\frac{y}{x}\right) \text{ or } u + iv = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore u = \tan^{-1}\frac{y}{x} \text{ and } v = 0$$

$$\text{Here, } u_x = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$u_y = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}. \text{ Also, } v_x = 0,\ v_y = 0$$

Clearly, $u_x \neq v_y$ and $u_y \neq -v_x$ (except at the origin)

$\therefore$ The given function is not analytic.

5. Check the analyticity of the function f(z) = ln z.

Since $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$, so

or, $u + iv = \ln r + i\theta$ $\therefore$ $u = \ln r$ and $v = \theta$

$$\text{Here, } \frac{\partial u}{\partial r} = \frac{1}{r}, \frac{\partial u}{\partial \theta} = 0, \frac{\partial v}{\partial \theta} = 1, \frac{\partial v}{\partial r} = 0$$

$$\text{Clearly, } \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad (r \neq 0)$$

So, the function is analytic except at the origin.

2.7 Harmonic Function, Laplace Equation

Before we define harmonic function, it will be convenient to introduce Laplace equation.

An equation of the form $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called Laplace's equation. This equation has frequent occurrence in gravitation, electrostatics, heat conduction etc.

Theorem.

If $f(z) = u + iv$ is analytic in a domain D, then u and v satisfies the Laplace's equation.

Proof.

Since f(z) is analytic in D, then it satisfies the Cauchy–Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (1) \text{ and } \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (2)$$

Furthermore, the partial derivatives of u and v of all orders exist and are continuous, because f(z) is analytic in D.

Differentiating (1) partially with respect to x, and (2) with respect to y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad (3)$$

$$\text{and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad (4)$$

Adding (3) and (4), $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (5) \quad [\because \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}]$

Similarly, differentiating (1) w.r.t. y, and (2) w.r.t. x, and then subtracting we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad (6)$$

The equations (5) and (6) show that u and v satisfy the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Any function f(x, y) which has continuous partial derivatives of the first and second order and satisfies Laplace's equation is called a **Harmonic Function**.

If f(z) = u + iv is analytic, then both u and v are harmonic functions. In such a case, we say that v is a conjugate harmonic of u, and u the conjugate harmonic of v.

Now, we mention a theorem without proof.

Theorem.

If the harmonic functions u and v satisfy Cauchy–Riemann equations, then u + iv is an analytic function.

2.8 Finding Conjugate of Harmonic Function and Corresponding Analytic Function

To illustrate some functions are given below. At first, we verify that they are harmonic. We then find harmonic conjugate and corresponding analytic functions.

1. Let $u = y^3 - 3x^2y$

Here, $\frac{\partial u}{\partial x} = -6xy$, $\frac{\partial^2 u}{\partial x^2} = -6y$, $\frac{\partial u}{\partial y} = 3y^2 - 3x^2$, $\frac{\partial^2 u}{\partial y^2} = 6y$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ so that u is harmonic.

By Cauchy–Riemann equations, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Since $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$, so $\frac{\partial v}{\partial y} = -6xy$

Integrating w.r.t. y, $v = -3xy^2 + h(x)$ (1)

Differentiating partially w.r.t. x, $\frac{\partial v}{\partial x} = -3y^2 + h'(x)$

Or, $\quad -\frac{\partial u}{\partial y} = -3y^2 + h'(x)$

or, $3x^2 - 3y^2 = -3y^2 + h'(x)$

or $h'(x) = 3x^2$

On integration, $h(x) = x^3 + c$

Inserting it into (1), we have

$v = -3xy^2 + x^3 + c$, which is the harmonic conjugate of u.

The corresponding analytic function is given by

$f(z) = u + iv = y^3 - 3x^2y + i(x^3 - 3xy^2 + c)$

To express f(z) in term of z, we put y = 0 so that z = x.

$\therefore\ f(z) = i(z^3 + c).$

2. $\quad u = \cos x \cosh y$

Here, $\frac{\partial u}{\partial x} = -\sin x \cosh y, \qquad \frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$

$\frac{\partial u}{\partial y} = \cos x \sinh y, \qquad \frac{\partial^2 u}{\partial y^2} = \cos x \cosh y$

$\therefore\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ so that u is harmonic.

For harmonic conjugate

By Cauchy–Riemann equations, we have

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \qquad (1)$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (2)$

Since $\frac{\partial u}{\partial x} = -\sin x \cosh y$, so (1) becomes

$\frac{\partial v}{\partial y} = -\sin x \cosh y$

Integrating w.r.t.y, $v = -\sin x \sinh y + h(x)$ (3)

On differentiation, $\dfrac{\partial v}{\partial x} = -\cos x \sinh y + h'(x)$

$$\text{or, } -\frac{\partial u}{\partial y} = -\cos x \sinh y + h'(x)$$

or, $-\cos x \sinh y = -\cos x \sinh y + h'(x)$

or $h'(x) = 0$

On integration, $h(x) = c$

From (3), $v = -\sin x \sinh y + c$ which is the harmonic conjugate of u.

The corresponding analytic function is given by

$f(z) = u + iv = \cos x \cosh y + i(-\sin x \sinh y + c)$

Put $y = 0$ so that $z = x$.

$\therefore f(z) = \cos z + ic.$

3. $u = \ln|z| = \ln\sqrt{x^2+y^2} = \dfrac{1}{2}\ln(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$\therefore \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, which verifies that u is harmonic.

For harmonic conjugate

By Cauchy–Riemann equations, we have

$\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$ (1) and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ (2)

Now (1) gives $\dfrac{\partial v}{\partial y} = \dfrac{x}{x^2+y^2}$

Integrating w.r.t. y, $v = x\displaystyle\int \frac{1}{y^2+x^2}dy + h(x)$ $x^2\left(\frac{y^2}{x^2}+1\right)$

$$= x \cdot \frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) + h(x)$$

$$v = \tan^{-1}\left(\frac{y}{x}\right) + h(x) \qquad (3)$$

On differentiation, $\frac{\partial v}{\partial x} = \frac{1}{(1 + y^2/x^2)} \cdot \left(-\frac{y}{x^2}\right) + h'(x)$

$$\text{or, } -\frac{y}{x^2 + y^2} = -\frac{y}{x^2 + y^2} + h'(x)$$

$$\text{or, } h'(x) = 0$$

On integration, $h(x) = c$

Inserting it into (3), $v = \tan^{-1}(y/x) + c$

$$\text{or, } v = \arg z + c$$

which is the harmonic conjugate of u.

The corresponding analytic function is

$$f(z) = u + iv = \ln|z| + i(\arg z + c)$$
$$= \ln|z| + i \arg z + ic.$$

Exercise 2.1

1. Show that the function $f(z) = z^3$ is differentiable for all z.
2. Show that the function $f(z) = \bar{z}$ is not differentiable.
3. Show that
   a. $f(z) = 1 + z$ is differentiable for all z.
   b. $f(z) = z^n$ is differentiable for all z.
4. Find the value of the derivative of functions
   a. $f(z) = \frac{z - i}{z + i}$ at $z = i$
   b. $f(z) = z^2 + \frac{1}{z^2}$ at $z = 1 + i$
   c. $f(z) = (z - 4i)^8$ at $z = 5 + 4i$.
5. a. Is z^2 analytic for all z?

b. Is z^4 analytic for all z?

6. Is $f(z) = z^2 + 5z$ analytic for all z?

7. Check for analyticity.

   a. $f(z) = z^4$ b. $f(z) = z + \dfrac{1}{z}$ $(z \neq 0)$ c. $f(z) = \bar{z}$

   d. $f(z) = e^x(\cos y + i\sin y)$ e. $f(z) = \operatorname{Re} z^3$

   f. $f(z) = \ln|z| + i \arg z$.

8. Verify that $u = x^2 - y^2$ is harmonic, and find the harmonic conjugate of u, and then find the corresponding analytic function.

9. Show that $u = x^2 + y^2$ is not harmonic.

10. Verify that $u = x^2 - y^2 - y$ is harmonic. Find the harmonic conjugate of u and then the corresponding analytic function.

11. Determine whether the following functions are harmonic. If harmonic, find the corresponding analytic functions.

    a. $u = \sin x \cosh y$ b. $u = -e^{-x} \sin y$

    c. $v = \arg z$ d. $v = x^3 - 3x^2$

    e. $u = \dfrac{x}{x^2 + y^2}$ f. $u = x^2 - y^2 - y$

    g. $v = 2xy - \dfrac{y}{x^2 + y^2}$

12. Determine the constants a and b such that $u = ax^3 + by^3$ is harmonic. Find the conjugate harmonic of u.

Answers

4. a. $-\frac{i}{2}$ b. $\frac{5}{2}(1+i)$ c. 8.5^7

5. Analytic

6. Analytic

7. a. Analytic b. Analytic c. Not analytic
   d. Analytic e. Not analytic f. Analytic

8. $2xy + c,\ z^2 + ic$

9. $2xy + x + c,\ z^2 + i(z + c)$

10. a. $\cos x \sinh y + c,\ \sin z + ic$
    b. $-e^{-x} \sin y + c,\ e^{-z} + ic$
    c. $\frac{1}{2}\ln(x^2 + y^2) + c,\ \text{Arg}z + i[\ln + |z| + c]$
    d. Not harmonic
    e. $-\frac{y}{x^2 + y^2} + c,\ \frac{1}{z} + ic$
    f. harmonic, $2xy + x + c,\ z^2 + i(z + c)$
    g. harmonic, $x^2 - y^2 + \frac{x}{x^2 + y^2} + c,\ i(z^2 + \frac{1}{2} + c)$

11. $a = 0,\ b = 0,\ v = c$