1

Complex Numbers and Function of a Complex Variable

1. 1 Introduction

In the opening unit we shall present an account of complex numbers very shortly. We shall also introduce a function of complex variable.

A complex number is an ordered pair (x, y) of real numbers x and y. It is denoted by z. Thus, $z = (x, y)$.
Here, x is the real part and y the imaginary part of z.
Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.

1.2 Sum, Product and Quotient of Two Complex Numbers.

Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y)$ be two complex numbers. Then

1. The sum of two complex numbers is a complex number, and is defined by
   $z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

2. The product of two complex numbers is also a complex number, and is defined by
   $z_1z_2 = (x_1, y_1) . (x_2, y_2) = (x_1x_2 - y_1y_2, \ \ x_1y_2 + x_2y_1)$
3. The quotient of two complex numbers is defined by
   $$\frac{z_1}{z_2} = \frac{(x_1, y_1)}{(x_2, y_2)} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right)$$

Imaginary Unit

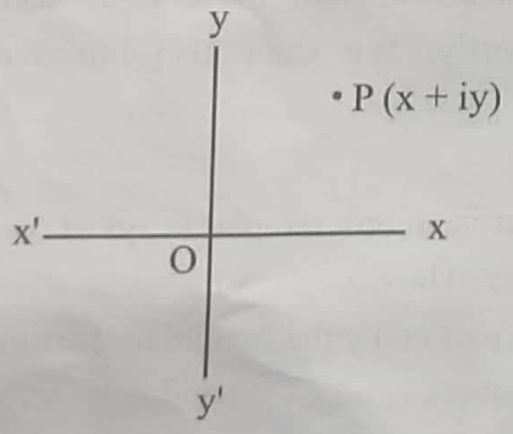
A complex number whose real part is zero and imaginary part one, is called an imaginary unit. It is denoted by i. Thus, $i = (0, 1)$

We observe that

(i) $i^2 = -1$ and (ii) $(x, y) = x + iy$.

1.3 Representation of a Complex Number

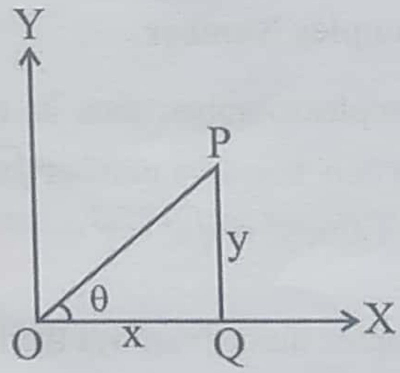
We can represent the complex number $z = x + iy$ by a point whose Cartesian coordinates are (x, y) referred to rectangular axes X'OX and YOY'.



A plane whose points are represented by complex numbers is called a complex plane, and the horizontal and vertical axes are known as Real and Imaginary axes.

1.4 Complex Number in Polar Form

Let $z = x + iy$ be a complex number. It is located at a point P.



Let OP = r, $\angle POQ = \theta$. Then $x = r\cos\theta$, and $y = r\sin\theta$

$$\therefore \quad r = \sqrt{x^2 + y^2}, \text{ and } \tan\theta = \frac{y}{x} \qquad (1)$$

Now, $x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$, where r and θ are given by (1). Thus,

$$x + iy = r(\cos\theta + i\sin\theta), \text{ which is the polar form.}$$

Here, θ is called the **amplitude or argument of z**, and r the modulus of z.

The value of θ that lies in the interval $-\pi < x \le \pi$ is called the **principal value** of z.

1.5 Conjugate Complex Numbers

Let $z = x + iy$, then the complex number $x - iy$ is called the conjugate of the complex number $z = x + iy$. The conjugate of z is denoted by $\bar{z}$. Thus $\bar{z} = x - iy$.

The complex conjugates enjoy the following properties.

1. $z + \bar{z} = 2\text{Re}z$
2. $z - \bar{z} = 2i\ \text{Im}.z$
3. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
4. $\overline{z_1 z_2} = \bar{z}_1.\bar{z}_2$.

1.6 Modulus of a Complex Number

If $z = x + iy$ is a complex number, then its modulus (or absolute value) is defined by a non–negative number $\sqrt{x^2 + y^2}$. The modulus of z is denoted by $|z|$. Thus, $|z| = \sqrt{x^2 + y^2}$.

The modulus of a complex number enjoys the following properties.

1. $|z_1 z_2| = |z_1|\,|z_2|$
2. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, provided $|z_2| \neq 0$
3. $|z| = |\bar{z}|$
4. $|z|^2 = z\,\bar{z}$.
5. $|z_1 + z_2| \le |z_1| + |z_2|$

1.7 Multiplication and Division in Polar Form

Let $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

where θ_1 is the argument of z_1 and θ_2 the argument of z_2.

Now, $z_1 z_2 = r_1 r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

or, $z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

Clearly, $\arg(z_1 z_2) = \theta_1 + \theta_2$

$\arg(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$

Also, $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left(\dfrac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\right)$

$= \dfrac{r_1}{r_2}[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)$

or $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

$\therefore \arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2$

$\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Remark: The notion of ordering (greater than or less than) does not apply to complex numbers. Thus the statements $z_1>z_2$ and $z_1<z_2$ have no meaning unless z_1 and z_2 are both real.

1.8 De Movire's Theorem

If n is a positive integer or a negative integer or a fraction, then

$$[(\cos\theta + i\sin\theta)]^n = [\cos n\theta + i\sin n\theta]$$

Proof.

Case 1. When n is a positive integer.

By actual multiplication, we have

$(\cos\theta_1 + i\sin\theta_1).(\cos\theta_2 + i\sin\theta_2)$

$$= (\cos\theta_1.\cos\theta_2 - \sin\theta_1.\sin\theta_2) + i(\sin\theta_1.\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2).$$

Similarly, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$$= \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}(\cos\theta_3 + i\sin\theta_3)$$
$$= \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$$

Proceeding in this way, we have

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots(\cos\theta_n + i\sin\theta_n)$

$$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)$$

Let $\theta_1 = \theta_2 = \theta_3 = \ldots\ldots = \theta_n = \theta$. then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Case 2. When n is a negative integer.

Let $n = -m$, where m is a positive integer. Then

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$$
$$= \frac{1}{\cos m\theta + i\sin m\theta}$$
$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos m\theta - i\sin m\theta.$$

$$= \cos(-m)\theta + i\sin(-m)\theta$$

$$= \cos n\theta + i\sin n\theta$$

Case 3. When n is a fraction, positive or negative

Let $n = \frac{p}{q}$ where q is a positive integer and p is any integer, positive or negative. Now,

$$(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})^q = (\cos q.\frac{\theta}{q} + i\sin q.\frac{\theta}{q}) = \cos\theta + i\sin\theta$$

Taking q^{th} root, we have

$$\cos\frac{\theta}{q} + i\sin\frac{\theta}{q} = (\cos\theta + i\sin\theta)^{1/q}$$

(This is one of the values of $(\cos\theta + i\sin\theta)^{1/q}$)

Raising to the p^{th} power, we get

$$(\cos\theta + i\sin\theta)^{p/q} = (\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})^p = \cos\left(\frac{p}{q}\theta\right) + i\sin\left(\frac{p}{q}\theta\right)$$

$$\therefore (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Remarks:

1. $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$
2. $(\cos\theta - i\sin\theta)^{n} = \cos n\theta - i\sin n\theta$
3. $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$

To find the q^{th} root of a complex number.

We know that $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is one of the values of $(\cos\theta+i\sin\theta)^{1/q}$

Of course, $(\cos\theta + i\sin\theta)^{1/q}$ has q different values.

We observe that

$$\cos\theta + i\sin\theta = \cos(2n\pi + \theta) + i\sin(2n\pi + \theta)$$

$$\therefore (\cos\theta + i\sin\theta)^{1/q} = [\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]^{1/q}$$

$$= \cos\frac{2n\pi+\theta}{q} + i\sin\frac{2n\pi+\theta}{q}.$$

Replacing n by 0, 1, 2, …., q – 1, we shall obtain the following q values:

i. $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ $\quad$ (n = 0)

ii. $\cos\frac{2\pi+\theta}{q} + i\sin\frac{2\pi+\theta}{q}$ $\quad$ (n=1)

iii. $\cos\frac{4\pi+\theta}{q} + i\sin\frac{4\pi+\theta}{q}$ $\quad$ (n=2)

....

....

$\cos\frac{2(q-1)\pi}{q} + i\sin\frac{2(q-1)\pi}{q}$ $\quad$ (n = q – 1)

and each of the above q quantities is equal to one of the values of $(\cos\theta + i\sin\theta)^{1/q}$.

Examples

1. Show that $\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta$.

Here, $\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \frac{(\cos\theta + i\sin\theta)^4}{[\cos(\pi/2-\theta) + i\sin(\pi/2-\theta)]^4}$

$$= \frac{\cos 4\theta + i\sin 4\theta}{\cos(2\pi - 4\theta) + i\sin(2\pi - 4\theta)}$$

$$= \frac{\cos 4\theta + i\sin 4\theta}{\cos 4\theta - i\sin 4\theta}$$

$$= \frac{(\cos 4\theta + i\sin 4\theta)(\cos 4\theta + i\sin 4\theta)}{(\cos 4\theta - i\sin 4\theta)(\cos 4\theta + i\sin 4\theta)}$$

$= (\cos 4\theta + i\sin 4\theta)^2$

$= \cos 8\theta + i\sin 8\theta.$

2. Prove that $\left(\dfrac{1+\sin\phi+i\cos\phi}{1+\sin\phi - i\cos\phi}\right)^n = \cos(\frac{n\pi}{2} - n\phi) + i\sin(\frac{n\pi}{2} - n\phi)$

Let $1+\sin\phi + i\cos\phi = r(\cos\theta + i\sin\theta)$. Then

$r\cos\theta = 1 + \sin\phi$, and $r\sin\theta = \cos\phi$

$1 + \sin\phi - i\cos\phi = r(\cos\theta - i\sin\theta)$, and

$$\tan\theta = \frac{\cos\phi}{1 + \sin\phi}$$

$$= \frac{\sin(90\text{-}\phi)}{1 + \cos(90\text{-}\phi)}$$

$$= \frac{2\sin\left(\frac{90-\phi}{2}\right)\cos\left(\frac{90-\phi}{2}\right)}{2\cos^2\left(\frac{90-\phi}{2}\right)}$$

$$= \tan\left(\frac{90-\phi}{2}\right)$$

$$\therefore\ \theta = \left(\frac{90-\phi}{2}\right)$$

$$\text{LHS} = \frac{[r(\cos\theta + i\sin\theta)]^n}{[r(\cos\theta - i\sin\theta)]^n} = \frac{\cos n\theta + \sin n\theta}{\cos n\theta - i\sin n\theta}$$

$$= (\cos n\theta + i\sin n\theta)(\cos n\theta + i\sin n\theta)$$

$$= \cos 2n\theta + i\sin 2n\theta$$

$$= \cos 2n\left(\frac{90-\phi}{2}\right) + i\sin 2n\left(\frac{90-\phi}{2}\right)$$

$$= \cos(n\pi/2 - n\phi) + i\sin(n\pi/2 - n\phi).$$

3. Finds all values of $(1+i)^{1/3}$

Let $1 + i = r(\cos\theta + i\sin\theta)$., Then $r = \sqrt{2}$, $\theta = \pi/4$.

$\therefore (1 + i) = \sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$$(1+i)^{1/3} = [\sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})]^{1/3}$$

$$= [\sqrt{2}\,\{\cos(2n\pi+\frac{\pi}{4}) + i\sin(2n\pi+\frac{\pi}{4})\}]^{1/3}$$

$$= 2^{1/6}\,[\cos\frac{(8n+1)\pi}{12} + i\sin\frac{(8n+1)\pi}{12}]$$

Replacing n by 0, 1, 2 we obtain the required values:

$2^{1/6}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$, $2^{1/6}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$, $2^{1/6}(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12})$.

1.9 Function of a Complex Variable

Let S be a set of complex numbers. A function f defined on S is a rule which assigns to every $z(=x + iy)$ in S, a complex number $w(=u+ iv)$. We say that w is the function of z, and we write

$w = f(z)$.

Here, z is called a **complex variable**. And the set S is the domain of the function. The set of all the outputs is called the range of the function.

Since $z = x + iy$, so f(z) will be of the form $u + iv$ where u and v are functions of two real variables x and y. We may then write

$$w = u(x, y) + iv(x, y)$$

Examples

1. Find u and v if $f(z) = z^3$. Also find the value of f at $z = 1 + i$

Let $w = f(z) = z^3$

$$= (x + iy)^3$$

$$= x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3$$
$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

or, $u + iv = x^3 - 3xy^2 + i(3x^2y - y^3)$

Equating real and imaginary part, we get

$$u = x^3 - 3xy^2$$
$$v = 3x^2y - y^3$$

The value of f at $z = 1 + i$ is

$$f(1 + i) = (1 + i)^3$$
$$= 1 + 3i + 3i^2 + i^3$$
$$= 1 + 3i - 3 - i$$
$$= -2 + 2\,i$$

2. If $f(z) = \dfrac{1}{z}$, find u and v. Also find the value of f at $z = 1 - i$.

Let $w = f(z) = \dfrac{1}{x + iy}$

$$= \frac{1}{x + iy} \times \frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2}$$

or, $u + iv = \dfrac{x}{x^2 + y^2} - i\dfrac{y}{x^2 + y^2}$

Equating real and imaginary parts,

$u = \dfrac{x}{x^2 + y^2}$ and $v = -\dfrac{y}{x^2 + y^2}$

Also, $f(1 - i) = \dfrac{1}{1 - i} = \dfrac{1 + i}{(1 - i)(1 + i)} = \dfrac{1 + i}{2} = \dfrac{1}{2}(1 + i)$

Exercise 1.1

1. Express the following in polar form

   (i) $1 + i$ (ii) $-1 + i$ (iii) $1 - i$ (iv) $3\,i$

2. Determine the principal value of the following arguments.

(i) $1 - i$ (ii) $1 + i$ (iii) $\sqrt{3} + i$

(iv) $1 + \sqrt{3}\, i$ (v) $1 - \sqrt{3}\, i$

3. Show that $\frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5} = \sin 9\theta - i\cos 9\theta$

4. Show that

$$\frac{(\cos 3\theta + i\sin\theta)^4 (\cos\theta - i\sin\theta)^3}{(\cos 5 + i\sin 5\theta)7\,(\cos 2\theta + i\sin 2\theta)^5} = \sin 13\theta - i\sin 13\theta$$

5. Find all the values

a. $1^{1/3}$ b. $(1+i)^{1/7}$ c. $(-1)^{1/6}$

6. If $(x + \frac{1}{x}) = 2\cos\theta$, show that $x^n + \frac{1}{x^n} = 2\cos n\theta$

7. Find the values of the following functions at the indicated points

a. $f(z) = z^2 - 2z,\ z = 1 + i$

b. $f(z) = z + \overline{z},\ z = 3 + i$

8. If $w = f(z) = z^2$, find Re f and Im f.

9. If $w = f(z) = z^2 + 3z$, find u and v.

10. If $w = f(z) = 3iz + 5\overline{z}$, find u and v. Also, find the value of f at $z = \frac{1}{2} + i$.

11. If $w = f(z) = z^2$, find u and v. Also, find the value of f at $z = i$.

Answers

1. (i) $\sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ (ii) $\sqrt{2}\,(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$

(iii) $\sqrt{2}\,(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})$ (iv) $3(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

2. (i) $-\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{\pi}{6}$ (iv) $\frac{\pi}{3}$ (v) $-\frac{\pi}{3}$

5. a. $1, \frac{1}{2}(-1+\sqrt{-3}), \frac{1}{2}(-1-\sqrt{-3})$

   b. $2^{1/14}\left[\cos\frac{1}{7}\left(2n\pi+\frac{\pi}{4}\right)+i\sin\frac{1}{7}\left(2n\pi+\frac{\pi}{4}\right)\right]$, n =0, 1, 2, 3, 4, 5

   c. $\cos\frac{2n\pi+\pi}{6}+i\sin\frac{2n\pi+\pi}{6}$, n = 0, 1, 2, 3, 4, 5

7. a. –2 b. 6
8. $x^2-y^2, 2xy$
9. $x^2+3x-y^2, (2x+3)y$
10. $5x-3y, 3x-5y, -\frac{1}{2}(1+7i)$
11. $x^2-y^2, 2xy, -1$