3

Conformal Mapping

3.1 Introduction

In real variables, the equation $y = f(x)$ represents a correspondence between points on x-axis and points on y-axis. We can exhibit the relationship by drawing a graph in the xy-plane. If $w = f(z)$ is a function of a complex variable z, it will not be possible to draw a graph since there are four variables x, y, u, and v. For complex function $w = f(z) = u + iv$ we consider two planes:

(i) z-plane for representing the variable $z = x + iy$, and

(ii) w-plane for representing the variable $w= u + iv$.

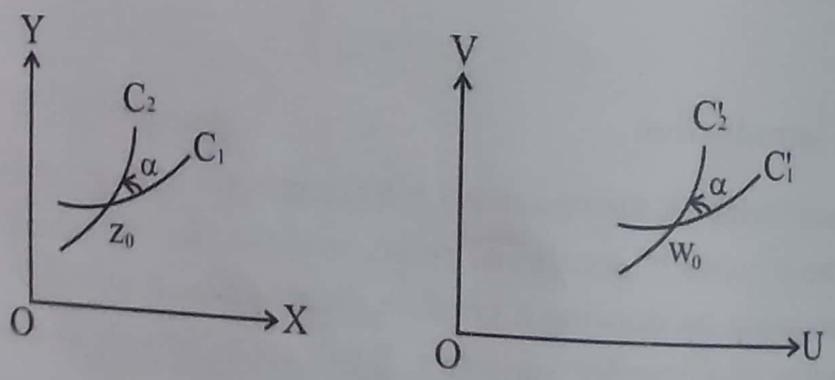
Then we can establish a correspondence between points on the z-plane and points on the w-plane. The equations that define the correspondence are

$$u = u(x, y), \qquad v = v(x, y) \qquad (1)$$

Thus, corresponding to each point (x, y) in the z-plane, there exists a point (u, v) in the w-plane. Such a correspondence defined by equation (1) is called a mapping of points in the z-plane into points of the w-plane by the function f; and we write $w = f(z)$. The corresponding sets of points in the two planes are called images o each other.

A mapping that preserves angles between the curves both in magnitude and sense is called a conformal mapping.

Suppose, two curves C_1 and C_2 intersect at a point $z = z_o$ in the z-plane making an angle α. If their corresponding curves C_1' and C_2' in the w-plane intersect at $w = w_o$ at the same angle α and if the sense of rotation is also preserved, we say that the mapping is conformal.



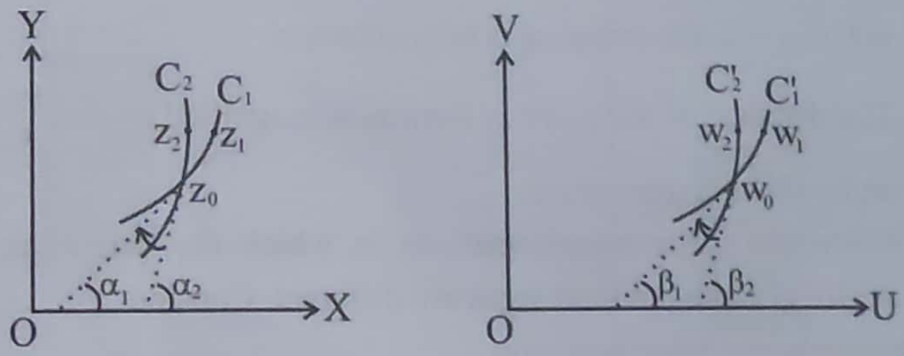
Here, we have, not only conserved the magnitude of the angle but also conserved the sign of angle.
The object of this chapter is to discuss in more detail the nature of such a mapping.

3.2 Condition for $w = f(z)$ to be Conformal

Let f(z) be an analytic function of z in a region R of the z-plane, and let z_0 be an interior point of R. Let C_1 and C_2 be two continuous curves passing through z_0. Let the tangents to those curves at z_0 make angles α_1 and α_2 with the x-axis. Now, we have to find the representation of this figure in the w-plane.

We suppose that $f'(z_0) \neq 0$.

Let z_1 and z_2 be the points of curves C_1 and C_2 such that
$|z_1 - z_0| = |z_2 - z_0| = r$. Then
$z_1 - z_0 = re^{i\theta_1}$, $z_2 - z_0 = re^{i\theta_2}$.



As z_1 and z_2 tend to zero, $1_1 \to \alpha_1$ and $1_2 \to \alpha_2$.

Let z_o be mapped into w_o in the w-plane, and let C_1, C_2 be mapped into C_1' and C_2'. Let w_1 and w_2 be the images of z_1 and z_2 respectively. Let the tangents to the curves $C_1'$and C_2'at $w = w_o$ make angles β_1 and β_2 respectively, with the u–axis.

Let $w_1 - w_o = \rho_1 e^{i\phi_1}$, $w_2 - w_o = \rho_2 e^{i\phi_2}$, then

$$f'(z_o) = \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0}$$

$$= \lim_{z_1 \to z_0} \frac{\rho_1 e^{i\phi_1}}{re^{i\theta_1}}$$

$$= \lim_{z_1 \to z_0} \frac{\rho_1}{r} e^{i(\phi_1 - \theta_1)}$$

Since $f'(z_0) \neq 0$, so we may take $f'(z_0) = Re^{i\delta}$

$$\therefore Re^{i\delta} = \lim_{z_1 \to z_0} \frac{\rho_1}{r} e^{i(\phi_1 - \theta_1)}$$

$$\Rightarrow \lim_{z_1 \to z_0} (\phi_1 - \theta_1) = \delta \quad \text{or } \beta_1 - \alpha_1 = \delta \quad \text{or } \beta_1 = \alpha_1 + \delta$$

Similarly, it can be shown that $\beta_2 = \alpha_2 + \delta$

$\therefore \beta_2 - \beta_1 = \alpha_2 - \alpha_1$

Hence the curves C_1', C_2' intersect at the same angle as C_1, C_2. Also, they have the same sense of rotation. Since the angles between the curves are preserved both in magnitude and direction, so mapping $w = f(z)$ is conformal.

Note

1. If $f'(z_0) = 0$, the mapping is not conformal.
2. The distance of the point z_1 from z_0 is magnified by $\lim \frac{\rho_1}{r} = R$ under the transformation.
3. There are some transformations in which the magnitude of angles is same but their signs are different. Consider,

for example, the transformation

$w = \bar{z}$, where $\bar{z}$ is the complex conjugate of z.

It replaces every point by its reflection in the real axis, and so the angles are preserved but their signs are changed. Of course, it is true for every transformation of the form $w = f(\bar{z})$, where $f(z)$ is analytic.

It is a combination of two transformations:

(i) $z = \bar{z}$, (ii) $w = f(z)$.

Examples

1. Let us consider the transformation

$$\mathbf{w = z + (1 - i)} \qquad (1)$$

Now, we wish to determine a region R' in the w-plane corresponding to the rectangular region R in the z-plane bounded by the lines $x = 0$, $y = 0$, $x = 1$ and $y = 2$.

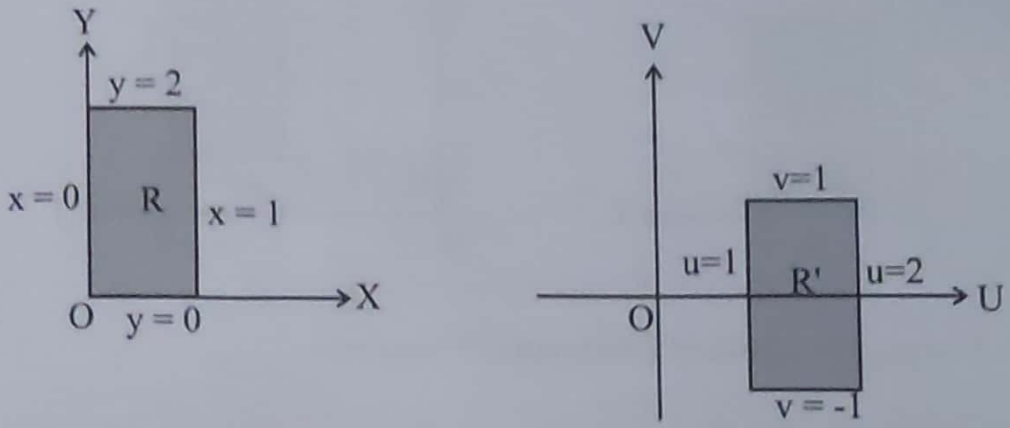
The transformation is given by $w = z + (1 - i)$.

$$u + iv = x + iy + (1 - i)$$
$$= x+1 + i(y - 1)$$

$\therefore$ $u = x + 1$, and $v = y - 1$

The lines $x = 0$, $y = 0$, $x = 1$, and $y = 2$ are mapped onto the lines

$u = 1$, $v = -1$, $u = 2$ and $v = 1$ respectively. The two regions are shown below.



2. Transform the rectangular region ABCD in z-plane bounded by $x = 1$, $x = 3$, $y = 0$ and $y = 3$ under the transformation $w = z + (2 + i)$.

Here, $w = z + (2 + i)$

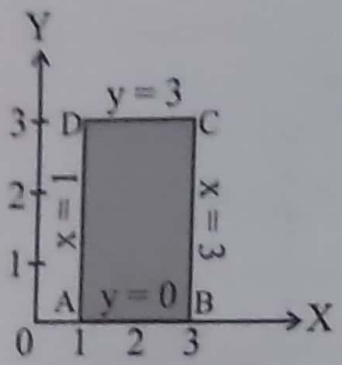
$$u + iv = x + iy + (2 + i)$$
$$= (x + 2) + i(y + 1)$$

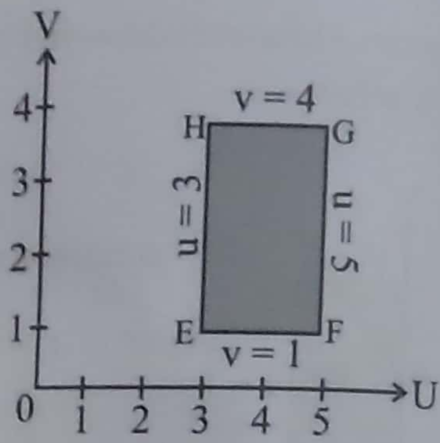
By equating real and imaginary quantities we have

$u = x + 2$ and $v = y + 1$. Then

z-plane	w-plane	z-plane	w-plane
x	$u = x + 2$	y	$v = y + 1$
1	$= 1 + 2 = 3$	0	$= 0 + 1 = 1$
3	$= 3 + 2 = 5$	3	$= 3 + 1 = 4$

Here the lines $x = 1$, $x = 3$, $y = 0$ and $y = 1$ in the z-plane are transformed onto the lines $u = 3$, $u = 5$, $v = 1$ and $v = 4$ in the w-plane. The region ABCD in z-plane is transformed into the region EFGH in w-plane.





3.3 Some Elementary Conformal Mappings

In this section we deal with some elementary conformal mappings. They are

(i) Translation
(ii) Rotation
(iii) Magnification
(iv) Rotation and Magnification
(v) Inversion.

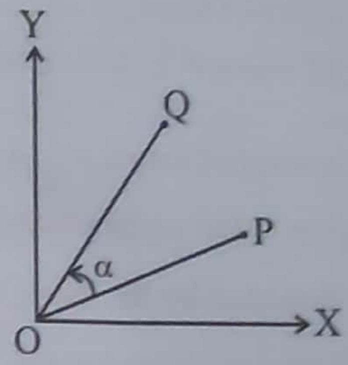
Translation : $w = z + A$, where $A = a + ib$ (a, b real constants)
This transformation displaces every point in the z-plane in the direction of A. Analytically, it can be shown as follows:
Let $z = x + iy$, and $w = u + iv$, then

$$u + iv = x + iy + a + ib$$
$$= x + a + i(y + b)$$

$\therefore\ u = x + a$ and $v = y + b$

It shows that a point (x, y) in the z-plane is mapped into a point (x +a, y + b) in the w-plane. Since it is true for every point in the region, so the image of the region is simply the translation of that region.

Rotation : $w = e^{i\alpha}z$ $(\alpha \text{ real})$



It tells us that the image Q of a point P in the z-plane is obtained by rotating the line OP through an angle α. The rotation is anticlockwise if $\alpha > 0$, and clockwise if $\alpha < 0$. Analytically, we can proceed as follows:

Let $w = pe^{i\phi}$ and $z = re^{i\theta}$, then

$$pe^{i\phi} = e^{i\alpha}\, re^{i\theta} = re^{i(\theta+\alpha)}$$

$$\therefore\ p = r,\ \ \phi = \theta + \alpha$$

Thus the modulus of w is the same as that of z, but the argument of w is increased by $\alpha(\alpha > 0)$. If $\alpha < 0$, then the argument of w is decreased by the amount α. Below is given an example.

Example

Under the mapping $w = e^{i\pi/3}z$, determine the region in the w-plane corresponding to the triangular region in the z-plane bounded by lines $x = 0$, $y = 0$, and $\sqrt{3}\,x + y = 4$.

Since $w = e^{i\pi/3}z$, we have

$$u + iv = (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})(x + iy) = \frac{1}{2}(x - \sqrt{3}\,y) + \frac{1}{2}i(\sqrt{3}\,x + y)$$

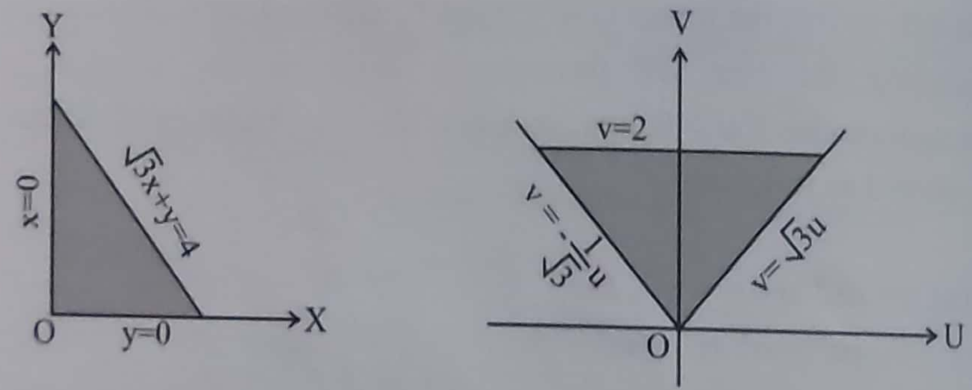
$$\therefore\ u = \frac{x}{2} - \frac{\sqrt{3}}{2}y \text{ and } v = \frac{\sqrt{3}}{2}x + \frac{1}{2}y$$

The line x = 0 is mapped into $u = -\frac{\sqrt{3}}{2} y,\ v = \frac{y}{2}$ i.e. into $v = -\frac{1}{\sqrt{3}} u$.

The line y = 0 is mapped into $u = \frac{x}{2},\ v = \frac{\sqrt{3}}{2} x$. It gives $v = \sqrt{3}\, u$

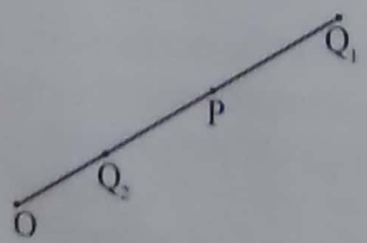
The line $\sqrt{3}\, x + y = 4$ is mapped into $v = \frac{1}{2}.4$ i.e. into v = 2

Thus the triangular region in the z-plane is mapped into a triangular region in the w-plane bounded by the lines $v = -\frac{1}{\sqrt{3}} u$, $v = \sqrt{3}\, u$, and v = 2. The two regions are shown below.



Magnification: $w = az \qquad (a > 0)$

The transformation hints that the figure in the z-plane is stretched or contracted in the direction of z according as a > 1 or 0 < a < 1.



To obtain the image of a point P, we take a point Q_2 in OP if $0<a<1$ or a point Q in OP produced such that

$OQ_1 = a.OP$ if $a > 1$

$OQ_2 = a.OP$ if $0 < a < 1$.

Let us consider the following problem.

Determine the region in the w-plane corresponding to the triangular region in the z-plane bounded by lines $x = 0$, $y = 0$ and $x + y = 3$. under the mapping $\mathbf{w = 2z}$.

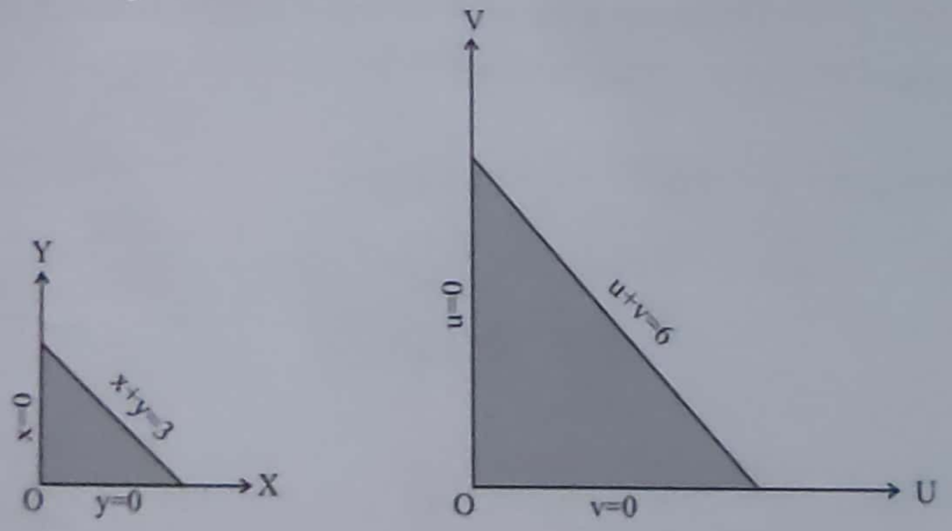
Here, the mapping is given by $w = 2z$. $(a = 2 > 1)$

It can be written as $u + iv = 2(x + iy)$

$\therefore$ $u = 2x$, and $v = 2y$

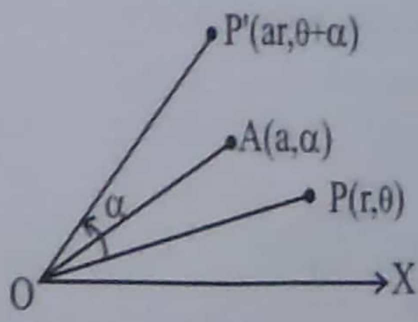
The lines $x = 0$, $y = 0$, $x + y = 3$ are mapped onto the lines $u = 0$, $v = 0$, $u + v = 6$.

The two regions are as follows.



Rotation and Magnification: $W = Az$ (A complex constant)

In polar form, let $w = \rho e^{i\phi}$, $A = ae^{i\alpha}$, $z = re^{i\theta}$. Then $\rho e^{i\phi} = ar\, e^{i(\theta+\alpha)}$



$\therefore$ $\rho = ar$ and $\phi = \theta + \alpha$

It means that a point $P(r, \theta)$ is mapped into a point $P'(ar, \theta + \alpha)$ in the w-plane. It consists of rotation of the radius vector OP about O through an angle α and then magnification of the radius vector by a.

We note that if $z = re^{i\theta}$ is transformed in $w = \rho e^{i\phi}$, then the coefficient of magnification is $\frac{\rho}{r}$, and the angle of rotation is $\phi - \theta$.

Example

Let the rectangular region R be bounded by the lines $x = 0$, $x = 1$, $y = 0$, $y = 2$. Determine a region R' of the w-plane into which R is mapped under the transformation $w = 2e^{i\pi/3}z$.

Writing $w = u + iv$ and $z = x + iy$, we have

$$u + iv = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)(x + iy)$$

$$= x - \sqrt{3}\,y + i(\sqrt{3}\,x + y)$$

$\therefore$ $u = x - \sqrt{3}\,y$ and $v = \sqrt{3}\,x + y$

The line $x = 0$ is mapped into $u = -\sqrt{3}\,y$, $v = y$ i.e. into $v = -\frac{1}{\sqrt{3}}u$

The line $x = 1$ is mapped into $u = 1 - \sqrt{3}\,y$, $v = \sqrt{3} + y$ i.e. into

$$v = -\frac{1}{\sqrt{3}}u + \frac{4}{\sqrt{3}}$$

The line $y = 0$ is mapped into $u = x$, $v = \sqrt{3}\,x$ i.e. into $v = \sqrt{3}\,u$

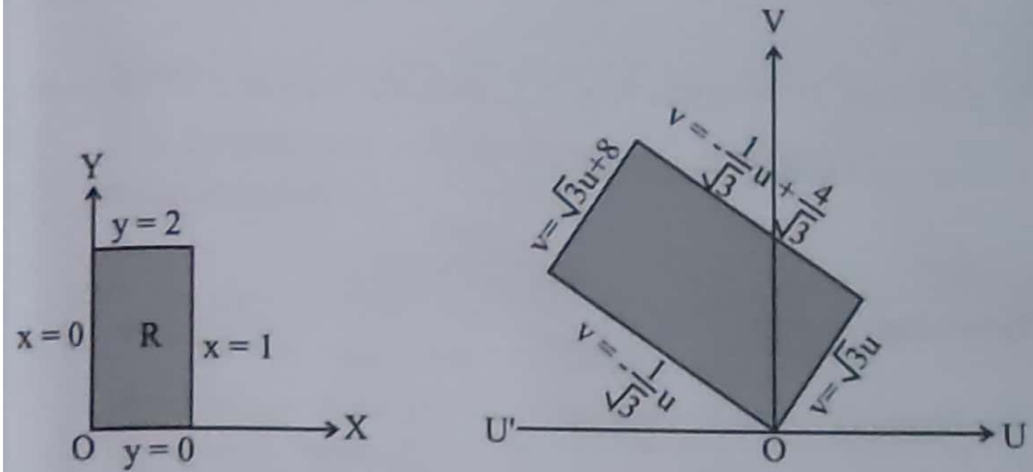
The line $y = 2$ is mapped into $u = x - 2\sqrt{3}$, $v = \sqrt{3}\,x+2$ i.e. into

$$v = \sqrt{3}\,u + 8.$$

Thus, the rectangular region in the z-plane is mapped into a rectangular region R' in the w-plane bounded by the lines

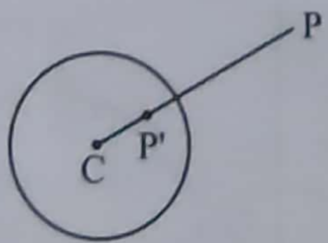
$$v = -\frac{1}{\sqrt{3}}u,\ v = -\frac{1}{\sqrt{3}}u + \frac{4}{\sqrt{3}},\ v = \sqrt{3}\,u,\ v = \sqrt{3}\,u + 8$$

The regions are shown below.



We note that it gives a rotation of R through an angle of 60° in anticlock–wise direction and a stretching of length 2.

Inverse Points



Let C be the centre and r be the radius of a circle. Two points P and P' are said to be inverse with respect to the circle if they are collinear with C and if $CP.CP' = r^2$.

Let $|z| = r$ be a circle of radius r and centre O. It can be written as $|z|^2 = r^2$ or $z\bar{z} = r^2$.

If Z' be the inverse of the point z, then the relation between z and its inverse with respect to the circle $|z| = r$ is given by $z'\bar{z} = r^2$

$\therefore z' = \dfrac{r^2}{\bar{z}}$.

Thus, the inverse of the point z with respect to the circle $|z| = r$ is

$$z' = \frac{r^2}{\bar{z}}.$$

In particular, if the circle is $|z| = 1$, then the inverse z' of the poi[nt] with respect to this circle is $z' = \frac{1}{\bar{z}}$

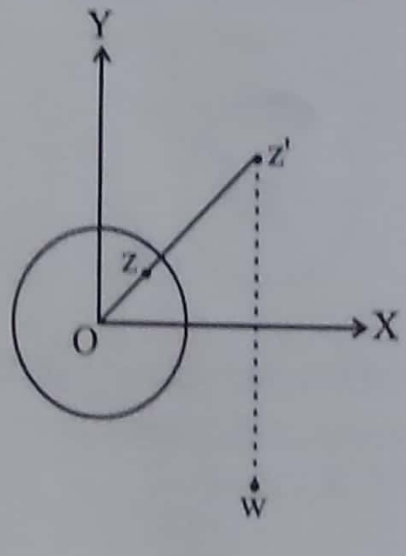
Inversion $\mathbf{w = \frac{1}{z}}$

The mapping can be written as

$$pe^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

$$\therefore p = \frac{1}{r}, \phi = -\theta$$

It is the composition of two transformations



(i) $z' = \frac{1}{\bar{z}}$, (ii) $w = \overline{z'}$, where z' is the inverse of z w.r.t. unit circle.
The first is an inversion with respect to the circle $|z| = 1$, and th[e] second is the reflection along x-axis.

Thus the points inside the unit circle are mapped outside the circl[e] and vice versa.

Examples

1. If $u = 2x^2 + y^2$ and $v = \frac{y^2}{x}$ show that the curves u = constant and v = constant cut orthogonally at all interactions but that the transformation $w = u + iv$ is not conformal.

Here, For the curves $2x^2 + y^2 = u$ (1)

$$2x^2 + y^2 = \text{constant} = k_1 \quad (\text{say})$$

Differentiating (1) we get

$$4x + 2y\frac{dy}{dx} = 0 \qquad \Rightarrow \frac{dy}{dx} = -\frac{2x}{y} \qquad (2)$$

and another curve is $\frac{y^2}{x} = v$

$$\text{Let } \frac{y^2}{x} = k_2 \qquad \text{or, } y^2 = k_2 x \qquad (3)$$

Differentiating (3) we get

$$2y\frac{dy}{dx} = k_2 \qquad \Rightarrow \frac{dy}{dx} = \frac{y}{2x} \qquad (4)$$

From (2) and (4) we see that

$$m_1 m_2 = \left(-\frac{2x}{y}\right)\left(\frac{y}{2x}\right) = -1$$

Hence two curves cut orthogonally.

But the Cauchy Riemann equations are not satisfied because $u_x \neq v_y$, $u_y \neq -v_x$.

$$\begin{bmatrix} \frac{\partial u}{\partial x} = 4x & \frac{\partial u}{\partial y} = 2y \\ \frac{\partial v}{\partial x} = -\frac{y^2}{x^2} & \frac{\partial v}{\partial y} = \frac{2y}{x} \end{bmatrix}$$

2. Under the transformation $w = \frac{1}{z}$, find the region in *w-plane* corresponding to the infinite strip $1 < y < 2$.

The transformation $w = \frac{1}{z}$ gives

$$u + iv = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$

$$\therefore\ u = \frac{x}{x^2 + y^2} \text{ and } v = -\frac{y}{x^2 + y^2} \qquad (1)$$

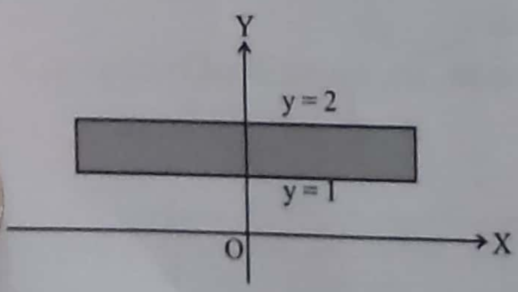
Now, $\frac{u}{v} = -\frac{x}{y}$ or $x = -\frac{uy}{v}$

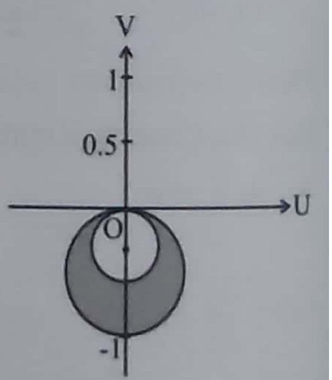
From (1), $v = -\frac{y}{\frac{u^2y^2}{v^2} + y^2} = -\frac{v^2 y}{y^2(u^2+v^2)}$

or, $y = -\frac{v}{u^2 + v^2}$

(i) If $y = 1$, then $u^2 + v^2 + v = 0$, which is a circle having centre $(0, -\frac{1}{2})$ and radius $\frac{1}{2}$.

(ii) If $y = 2$, then $u^2 + v^2 = -\frac{v}{2}$ or, $u^2 + v^2 + \frac{v}{2} = 0$, which is the equation of a circle whose centre is $(0, -\frac{1}{4})$ and radius $\frac{1}{4}$.





The infinite strip $1 < y < 2$ is transformed into the region common to the circle $u^2 + v^2 + v = 0$ and $u^2 + v^2 + \frac{v}{2} = 0$ in w-plane.

3.4 Linear Transformation

A function of the form $w = az + b$, where a and b are complex constants is called a linear function. It is a composition of the above type (magnification, rotation and translation) of transformations.

3.5 Bilinear Transformation

If a, b, c, d are four complex constants such that $ad - bc \neq 0$, then the transformation defined by

$$w = \frac{az+b}{cz+d} \qquad (cz + d \neq 0) \qquad (1)$$

is called a bilinear transformation. It is also known as Mobius transformation.

The above transformation can be written as

$$cwz + dw - az - b = 0,$$

which is linear in both z and w. That is why it is called a bilinear transformation.

Solving (1) for z, we have

$$z = \frac{-dw+b}{cw-a}$$ which is the inverse transformation.

Differentiating (1) with respect to z, we have

$$\frac{dw}{dz} = \frac{(cz+d).a-(az+b).c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

If $ad - bc \neq 0$, then the bilinear transformation is conformal. However, if $ad - bc = 0$, the bilinear transformation is not conformal.

Bilinear Transformation of a Circle

Let p and q be two fixed points in the z-plane then the equation of a circle is

$$\left|\frac{z-p}{z-q}\right| = k \quad (k \neq 1) \qquad (1)$$

It can be written as

$$\left|\frac{\frac{b-dw}{cw-a}-p}{\frac{b-dw}{cw-a}-q}\right| = k$$

$$\text{or,}\quad \left|\frac{b-dw-pcw+ap}{b-dw-qcw+aq}\right| = k$$

$$\text{or,}\quad \left|\frac{b+ap-w(d+pc)}{b+aq-w(d+qc)}\right| = k$$

$$\text{or}\quad \left|\frac{d+pc}{d+qc}\right| \left|\frac{\frac{b+ap}{d+pc}-w}{\frac{b+aq}{d+qc}-w}\right| = k$$

$$\text{or,}\quad \left|\frac{w-\frac{b+ap}{d+pc}}{w-\frac{b+aq}{d+qc}}\right| = k\left|\frac{d+qc}{d+pc}\right| = k',$$

$$\text{or,}\quad \left|\frac{w-\frac{b+ap}{d+pc}}{w-\frac{b+aq}{d+qc}}\right| = k' \qquad (2)$$

which is of the form (1), and hence represents the circle.

$\therefore$ Bilinear transformation transforms circles into circles (where st. lines are considered as special cases of circles)

Example

Given the bilinear transformation $w = \dfrac{5-4z}{4z-2}$. Find the mapping of the circle $|z| = 1$, in the w-plane.

The bilinear mapping is $w = \dfrac{5-4z}{4z-2}$

Solving for z, we have $4zw - 2w = 5 - 4z$ or, $4z(w + 1) = 2w + 5$

$$\therefore z = \frac{2w+5}{4w+4}$$

Taking modulus,

$$|z| = \left|\frac{2w+5}{4w+4}\right| \quad \text{or,} \quad \left|\frac{2w+5}{4w+4}\right| = 1 \qquad (1)$$

Let $w = u + iv$, then (1) reduces to

$|(2u + 5) + i2v| = |(4u + 4) + i4v|$

or, $\sqrt{(2u+5)^2 + 4v^2} = \sqrt{(4u+4)^2 + 16v^2}$

On simplification, $12u^2 + 12v^2 + 12u = 9$

or, $u^2 + v^2 + u = \frac{3}{4}$ which is the equation of the circle whose centre is $(-\frac{1}{2}, 0)$ and radius 1.

Invariant Point (or Fixed Point)

A point $z = z_0$ is called a fixed point of the bilinear mapping $w = \frac{az+b}{cz+d}$ if $w(z_0) = z_0$.

By definition, $z = \frac{az+b}{cz+d} \Rightarrow cz^2 - (a-d)z - b = 0$

It is quadratic in z, so it has two roots. Hence, we have at the most two fixed points.

3.6 Cross-Ratio

The cross ratio of the four points z_1, z_2, z_3, z_4 (taken in order) is defined by $\frac{(z_1 - z_2)}{(z_2 - z_3)} \Big/ \frac{(z_4 - z_1)}{(z_3 - z_4)}$ i.e. $\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$

[Starting with z_1, the four elements are $z_1-z_2, z_2-z_3, z_3-z_4, z_4-z_1$: the first is put in numerator, the second in denominator, third in numerator, the fourth in denominator]

3.7 Determining Bilinear Transformation

If the points z_1, z_2, z_3 map into the points w_1, w_2, w_3 in w-pla[ne] respectively, then the bilinear mapping is determined by [the] transformation $\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

Example 1. Find the bilinear transformation which maps $z_1=0$, [z_2] $=1$, $z_3=\infty$ into $w_1=i$, $w_2=-1$, $w_3=-i$

The bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(z-z_1).z_3(\frac{z_2}{z_3}-1)}{(z_1-z_2).z_3(1-\frac{z}{z_3})}$$

$$\text{or, } \frac{(w-i)(-1+i)}{(i+1)(-i-w)} = \frac{(z-0)(0-1)}{(0-1)(1-0)} = z$$

$$\text{or, } \frac{(w-i)(-1+i)}{(w+i)(-1-i)} = z \text{ or, } \frac{(w-i)}{(w+i)}.(-i) = z$$

$$\text{or, } \frac{w-i}{w+i} = \frac{z}{-i} = iz$$

Solving for w, we have $w=\dfrac{-z+i}{1-iz}$.

Example 2. Find the bilinear transformation which maps $z=1$, [...] 1 onto the points $w=i, 0, -i$; and hence find the image of $|z|<$ [...] Also find the invariant points of the transformation.

The bilinear transformation which maps the points z_1, z_2, z_3 o[nto] the points w_1, w_2, w_3 in w-plane is given by

$$\frac{(w-w_1)}{(w_2-w_3)}.\frac{(w_1-w_2)}{(w_3-w)} = \frac{(z-z_1)}{(z_2-z_3)}.\frac{(z_1-z_2)}{(z_3-z)}$$

$$\text{or, } \frac{(w-i)}{(i-0)}.\frac{(0+i)}{(-i-w)} = \frac{(z-1)}{(1-i)}.\frac{(i+1)}{-1-z}$$

$$\text{or, } \frac{w-i}{w+i} = \frac{(z-1)}{(z+1)} \cdot \frac{(1+i)}{1-i} = \frac{(z-1)}{(z+1)} \cdot \frac{(1+i)^2}{2} = \frac{(z-1)}{z+1} i = \frac{iz-i}{z+1}$$

Applying componendo and dividendo,

$$\frac{2w}{-2i} = \frac{iz-i+z+1}{iz-i-z-1} \quad \text{or, } w = \frac{z-1-iz-i}{iz-i-z-1} = \frac{z(1-i)-(1+i)}{-z(1-i)-(1+i)}$$

$$\text{or } w = \frac{(1-i)\left[z - \frac{1+i}{1-i}\right]}{(1-i)\left[-z - \frac{1+i}{1-i}\right]} = \frac{z-i}{-z-i}$$

$\therefore$ The bilinear transformation is $w = \dfrac{z-i}{-z-i}$ (1)

Now (1) can be written as

$$z = \frac{(1-w)}{1+w} i \qquad \text{(solving for z)}$$

$|z| < | \Rightarrow \left| \dfrac{(1-w)}{(1+w)} i \right| < 1$ or $|1-w| < |1+w|$ (2)

Since $w = u + iv$, so (2) reduces to

$$|1-u-iv| < |1+u+iv|$$

$$\text{or, } \sqrt{(1-u)^2+v^2} < \sqrt{(1+u)^2+v^2}$$

$$\text{or } (1-u)^2 + v^2 < (1+u)^2 + v^2$$

$$\text{or, } -2u < 2u \text{ or } 0 < 4u \text{ or } u > 0$$

i.e. $\text{Re}(w) > 0$.

For invariant points, $w = z$.

From (1) $z = \dfrac{z-i}{-z-i}$ or, $z^2 + (1+i)z - i = 0$

$$\therefore z = \frac{-(1+i) \pm \sqrt{(1+i)^2 + 4i}}{2} = \frac{-(1+i) \pm \sqrt{3}\sqrt{2i}}{2}$$

$$= \frac{-(1+i) \pm \sqrt{3}(1+i)}{2} = \frac{(1+i)}{2}(-1 \pm \sqrt{3})$$

3.8 Some Special Transformations

In this section we shall consider some special transformations which are not linear.

1. The transformation $w = z^2$

Determine the region of the w-plane into which the region $\frac{1}{2} \leq x \leq 1$ and $\frac{1}{2} \leq y \leq 1$ is mapped by the transformation $w = z^2$.

The mapping is $w = z^2$. It can be written as
$u + iv = (x + iy)^2 = x^2 - y^2 + i2xy$
$\therefore\ u = x^2 - y^2$ and $v = 2xy$.

The line $x = \frac{1}{2}$ is mapped into $u = \frac{1}{4} - y^2, v = y$

Eliminating y, $v^2 = -(u - \frac{1}{4})$ which is the equation of the parabola.

It is open to the left, and its vertex is $(\frac{1}{4}, 0)$

The line $x = 1$ is mapped into $u = 1 - y^2, v = 2y$
They together give $v^2 = -4(u - 1)$, which is the equation of a parabola. It is also open to the left, and its vertex is (1, 0).

Thus, the region $\frac{1}{2} \leq x \leq 1$ is mapped into the region between the above two parabolas.

The line $y = \frac{1}{2}$ is mapped into $u = x^2 - \frac{1}{4}, v = x$

These relations give $v^2 = u + \frac{1}{4}$ (A)

Which is a parabola.

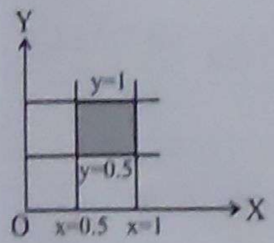
It is open to the right, and its vertex is $(-\frac{1}{4}, 0)$.

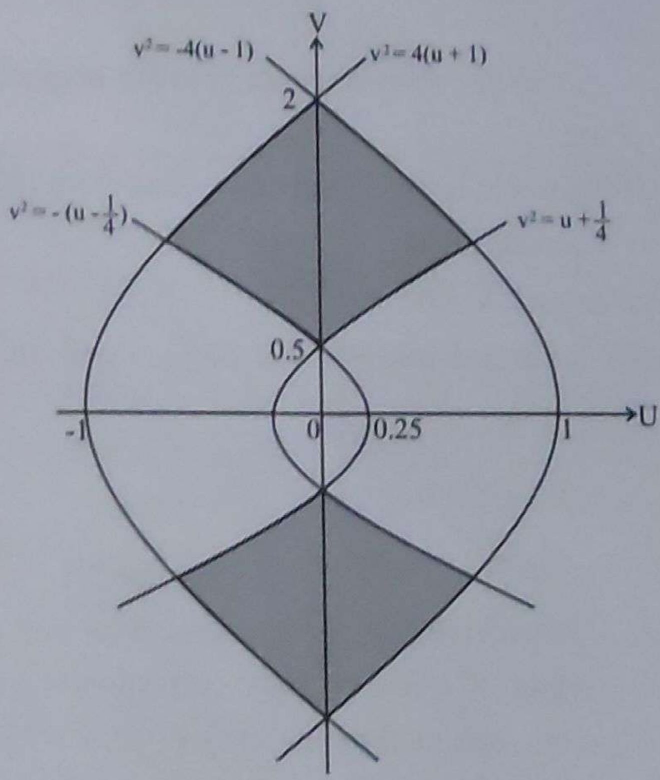
The line $y = 1$ is mapped into $u = x^2 - 1, y = 2x$

i.e. into the parabola $v^2 = 4(u + 1)$ (B). It is also open to the right, and its vertex is $(-1, 0)$.

Thus, the region $\frac{1}{2} \le y \le 1$ is mapped into the region between the parabolas (A) and (B).

The regions are shown below.





2. The Transformation $w = z + \frac{1}{z}$.

Here, $\frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2 - 1}{z^2}$

For $z = \pm 1$, $\frac{dw}{dz} = 0$. It implies that the mapping is conformal everywhere except at $z = \pm 1$.

Let $z = re^{i\theta}$, and $w = u + iv$. Then

$$u + iv = re^{i\theta} + \frac{1}{r} e^{-i\theta} = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$=(r+\frac{1}{r})\cos\theta + i(r-\frac{1}{r})\sin\theta$

$\therefore u = (r+\frac{1}{r})\cos\theta$, and $v = (r-\frac{1}{r})\sin\theta$ (1)

Eliminating θ, we have

$$\frac{u^2}{(r+1/r)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = 1$$

$\therefore$ circles $|z| = r =$ constant ($\neq 1$) are mapped onto ellipses in the w-plane.

If the circle is unit circle $r = 1$, then from (1) we have

$u = 2\cos\theta,\ v = 0$

$\Rightarrow |u| \leq 2,\ v = 0$

So, the unit circle $r = 1$ is mapped onto the segment $-2 \leq u \leq 2$ of the u-axis.

Exercise 3.1

1. Consider the transformation $w = z + (3 + i)$. Determine the region R' of the w-plane corresponding to the region R in the z-plane bounded by the lines $x = 0$, $x = 2$, $y = 0$, $y = 2$.
2. What is the region of the w-plane into which the rectangular region in the z-plane bounded by the lines $x = 0$, $y = 0$, $x = 1$, and $y = 2$ is mapped under the transformation $w = z + (2-i)$?
3. Find the region in the w-plane corresponding to the triangular region on the z-plane bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ under the mapping $w = ze^{i\pi/4}$.
4. Find the region in the w-plane corresponding to the region in the z-plane bounded by the lines $x = 1$, $y = 1$, $x + y = 4$ under the mapping $w = 3z$.
5. Let the rectangular region R in the z-plane be bounded by lines $x = 0$, $y = 0$, $x = 2$, $y = 3$. Find the region R' of the w-plane into which R is mapped under the transformation $w = \sqrt{2}\, w^{i\pi/4} z$..

6. Find the image of the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

7. Find the bilinear transformation which maps the points $z_1=2$, $z_2=i$ and $z_3 = -2$ into the points $w_1= 1$, $w_2= i$, and $w_3 = -1$.

8. Given the bilinear transformation $w = \frac{3-z}{2z+1}$, find the mapping of the circle $|z| = 1$ in the w-plane.

9. Given the bilinear transformation $w = \frac{2z+3}{z-4}$, find the mapping of the circle $x^2 + y^2 - 4x = 0$ in w-plane.

10. Find the images of the lines $x = 1$ and $x = 2$ in the z-plane under the mapping $w = z^2$.

11. Let $w = z^2$ define a transformation from z – plane to w-plane. Consider a triangle in the z-plane with vertices at A(2, 1), B(4, 1), C(4, 3). Show that the image of this triangle is a curvilinear triangle in the w-plane.

12. Show that the transformation $w = z + \frac{1}{z}$ maps the circle $|z|=2$ into the ellipse in the w-plane.

Answers

1. Region bounded by $u = 3$, $u = 5$, $v = 1$, $v = 3$
2. Region in w-plane bounded by $u = 2$, $u = 3$, $v = -1$, $v = 1$
3. Region in the w-plane bounded by lines $v = -u$, $v = u$ and $v = \frac{1}{\sqrt{2}}$
4. Region bounded by lines $u = 3$, $v = 3$, $u + v = 12$
5. Region bounded by $v = u$, $v = -u$, $v + u = 4$, $v - u = 6$
6. Region common to the circles $u^2 + (v + 2)^2 = 4$, $u^2+(v+1)^2=1$
7. $w = \dfrac{3z + 2i}{iz + 6}$
8. $3u^2 + 3v^2 + 10u - 8 = 0$ (circle)
9. $4u + 3 = 0$

11. Triangle formed by the parabolas $v^2 = 4(u+1)$, $v^2 = -64(u-16)$ and $u^2 = 2(v + 1/2)$