5

Zeros and Singularities

5.1 Introduction

In this section, we shall focus our attention on zeros and singularities of a function. Residue of a function and Cauchy's Residue theorem will also be considered.

5.2 Zeros of an Analytic Function

Let $f(z)$ be an analytic function. A point $z = z_0$ is called the **zero** of the function if $f(z_0) = 0$.

If $f(z_0) = 0$ and $f'(z_0) \neq 0$, then z_0 is called a **simple zero** or a **zero of the first order**.

If $f(z_0) = 0$, $f'(z_0) = 0$ but $f''(z_0) \neq 0$, then z_0 is called a **zero of order 2**. Similarly, if $f(z_0) = f'(z_0) = f''(z_0) = f'''(z_0) = \ldots = f^{(n-1)}(z_0) = 0$, but $f^{(n)}(z_0) \neq 0$, then z_0 is called a **zero of order n**.

Examples

a. $z - 1$ has a zero of order 1 at $z = 1$.
b. $(z-1)^2$ has a zero of order 2 at $z = 1$.
c. $(z-1)^3$ has a zero of order 3 at $z = 1$.
d. $z^2 + 1$ has simple zeros at $z = i$ and $z = -i$.

5.3 Singularity

A point z_0 at which a function f(z) is not analytic is called **singular point or singularity** of f(z).

For example, a function $f(z) = \frac{z}{z-3}$ is not analytic at $z = 3$; so the point $z = 3$ is the singular point of the function.

If the function f(z) is analytic in some neighbourhood of the point z_0 except at z_0 itself, then z_0 is called the **isolated singular point** or **isolated singularity** of f(z).

Examples

a. The function $f(z) = \frac{1}{z-2}$ has an isolated singularity at $z = 2$.

b. The function $f(z) = \frac{z}{(z-1)(z-3)}$ has isolated singularities at $z = 1$ and $z = 3$.

If z_0 is an isolated singularity of f(z), there is an annulus on which f(z) is analytic. Hence f(z) can be represented by Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (1)$$

The first part of f(z) i.e. $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is called the analytic part, and the second part is called the principal part of f(z).

There are now three possible cases.

Case I. If the principal part of f(z) contains a finite number terms, the isolated singular point z_0 is called the pole of f(z); and in that case it may be of the form

$$\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \frac{b_3}{(z - z_0)^3} + \ldots\ldots + \frac{b_m}{(z - z_0)^m}.$$

Then $z = z_0$ is a pole, and its order is m.

A pole of order 1 is also called a simple pole.

Examples

a. $f(z) = \frac{1}{z-3}$ has a simple pole at $z = 3$.

b. $f(z) = \frac{1}{z-3} + \frac{1}{(z-3)^2} + \frac{1}{(z-3)^4}$ has a pole of order 4 at $z = 3$.

Poles can also be determined by using the following technique:

If $\lim_{z \to z_0} (z - z_0)^m f(z)$ is a finite non-zero quantity, then $z = z_0$ is a pole of order m.

Consider the function $f(z) = \frac{1}{z} + \frac{1}{z^3}$

Here, $z = 0$ is the pole of order 3, since

$\lim_{z \to 0} z^3 f(z) = \lim_{z \to 0} z^3 . \left(\frac{1}{z} + \frac{1}{z^3}\right) = \lim_{z \to 0} (z^2 + 1) = 1$, which is a finite non-zero quantity.

Remarks:

1. If f(z) has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$.
2. If f(z) has a pole of order m, then $\frac{1}{f(z)}$ has a zero of order m and conversely, if f(z) has a zero of order m, then $\frac{1}{f(z)}$ has a pole of order m.
3. Poles are isolated. That is, if z_0 is a pole of f(z), then there exists a neighbourhood of z_0 which contains no other pole of f(z).
4. Zeros are isolated.
5. The limit point of zeros is an isolated essential singularity of f(z).
6. The limit point of a sequence of poles is a non–isolated essential singularity of f(z).

Case II. If the principal part of f(z) contains infinitely many terms, then the point z_o is called the essential singularity of f(z).

For examples,

a. $f(z) = e^{1/z}$ has an essential singularity at z = 0, since

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \dots$$

b. $f(z) = \sin\left(\frac{1}{1-z}\right)$ has an essential singularity at the point z = 1, since

$$\sin\left(\frac{1}{1-z}\right) = \frac{1}{1-z} - \frac{1}{3!}\frac{1}{(1-z)^3} + \frac{1}{5!}\frac{1}{(1-z)^5} - \dots\dots$$

Case III. If the principal of f(z) at $z = z_o$ contains no term, then z_0 is said to be a removable singularity. In such case, the singularity can be removed by defining f(z) in such a way that it becomes analytic at z_o.

Consider, for example, the function $f(z) = \frac{\sin z}{z}$.

It has a removable singularity at the point z = 0, since

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots$$

We observe that there is no term in the principal part of f(z). Here, we can remove the singularity by defining the function as follows:

$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

5.4 Singularity at Infinity

If we have to study a function at infinity, we set $z = \frac{1}{w}$. Then the behavior of the function f(z) at infinity depends on the behavior of $f\left(\frac{1}{w}\right)$ at w = 0.

We say that

f(z) is analytic, has a singularity, has a pole,

$f\left(\frac{1}{w}\right)$ has the same property at w = 0.

Examples

a. The function $f(z) = z^3$ has a pole of order

$f\left(\frac{1}{w}\right) = \frac{1}{w^3}$ has a pole of order 3 at w = 0.

b. The function $f(z) = \frac{1}{z^3}$ is analytic at ∞.

analytic at w = 0.

c. The function $f(z) = e^z$ has an essential si

since $f\left(\frac{1}{w}\right) = e^{1/w}$ has an essential singularit

Some Further Examples

1. Determine the location and order of z functions

(a) $\tan\pi z$ (b) $\cos^2\frac{Z}{2}$ (c

a. Let $f(z) = \tan\pi z$

Here, $f(z) = 0 \Rightarrow \tan\pi z = 0$ or, $\pi z =$ n

$\pm1, \pm2, \dots)$

$f'(z) = \pi\sec^2\pi z$ $\therefore f'(\pi) = \pi s$

So, f'(z) has a zero of order 1 at z = n

b. Let $f(z) = \cos^2\frac{Z}{2}$

Now, $f(z) = 0 \Rightarrow \cos^2\frac{Z}{2} = 0 \Rightarrow \cos\frac{Z}{2}$

or, $z = (2n+1)\pi$

We say that

$f(z)$ is analytic, has a singularity, has a pole, etc., at infinity if $f\left(\frac{1}{w}\right)$ has the same property at $w = 0$.

Examples

a. The function $f(z) = z^3$ has a pole of order 3 at infinity since $f\left(\frac{1}{w}\right) = \frac{1}{w^3}$ has a pole of order 3 at $w = 0$.

b. The function $f(z) = \frac{1}{z^3}$ is analytic at ∞, since $f\left(\frac{1}{w}\right) = w^3$ is analytic at $w = 0$.

c. The function $f(z) = e^z$ has an essential singularity at infinity, since $f\left(\frac{1}{w}\right) = e^{1/w}$ has an essential singularity at $w = 0$.

Some Further Examples

1. Determine the location and order of zeros of the following functions

(a) $\tan\pi z$ (b) $\cos^2\frac{z}{2}$ (c) $\sin\left(\frac{1}{1-z}\right)$

a. Let $f(z) = \tan\pi z$

Here, $f(z) = 0 \Rightarrow \tan\pi z = 0$ or, $\pi z = n\pi$ or, $z = n$ $(n = 0, \pm1, \pm2, \ldots)$

$$f'(z) = \pi\sec^2\pi z \qquad \therefore f'(\pi) = \pi\sec^2 n\pi = \pi \neq 0$$

So, $f'(z)$ has a zero of order 1 at $z = n$ $(n = 0, \pm1, \pm2, \ldots)$

b. Let $f(z) = \cos^2\frac{z}{2}$

Now, $f(z) = 0 \Rightarrow \cos^2\frac{z}{2} = 0 \Rightarrow \cos\frac{z}{2} = 0 \Rightarrow \frac{z}{2} = (2n+1)\frac{\pi}{2}$

or, $z = (2n+1)\pi$

$$f'(z) = -\frac{1}{2}\sin z \quad \therefore f'[(2n+1)\pi] = -\frac{1}{2}\sin(2n+1)\pi = 0$$

$$f''(z) = -\frac{1}{2}\cos z \quad \therefore f''[(2n+1)\pi] = -\frac{1}{2}\cos(2n+1)\pi$$

$$= -\frac{1}{2}\times(-1) = \frac{1}{2} \neq 0.$$

$\Rightarrow \cos^2\frac{z}{2}$ has a zero of order 2 at $z = (2n+1)\pi$ $(n = 0, \pm1, \pm2, \ldots)$

c. Let $f(z) = \sin\left(\frac{1}{1-z}\right)$

Now, $f(z) = 0 \Rightarrow \sin\left(\frac{1}{1-z}\right) = 0$ or $\frac{1}{1-z} = n\pi$

or, $1 - z = \frac{1}{n\pi}$ or, $z = 1 - \frac{1}{n\pi}$

$$f'(z) = \cos\left(\frac{1}{1-z}\right).(1-z)^{-2} = \frac{\cos\left(\frac{1}{1-z}\right)}{(1-z)^2}$$

$$\therefore f'\left(1 - \frac{1}{n\pi}\right) = n^2\pi^2\cos n\pi \neq 0$$

So, f(z) has a zero of order 1 at $z = 1 - \frac{1}{n\pi}$ $(n = 0, \pm1, \pm2, \ldots)$

2. Find the poles of the functions

(a) $f(z) = \frac{1}{(z-1)(z-3)^3}$ (b) $f(z) = \frac{\sinh z}{(z-\pi i)^2}$

a. The poles are given by

$$(z-1)(z-3)^3 = 0$$

Either $z - 1 = 0$ or, $(z-3)^3 = 0$

$z - 1 = 0$ gives $z = 1$

And, $(z-3)^3 = 0$ gives $z = 3$.

Thus, $z = 1$ is a pole of order 1, and $z = 3$ is a pole of order 3.

Alternatively,

$$\lim_{z\to1}(z-1)\,f(z) = \lim_{z\to1}\frac{1}{(z-3)}$$

zero quantity.

$\therefore$ $z = 1$ is a pole of order 1.

And, the point $z = 3$ is a pole of or

$$\lim_{z\to3}(z-3)^3\,f(z) = \lim_{z\to3}\frac{1}{z-}$$

zero quantity.

b. The singular point is given by (z -

At first, it appears that $z = \pi i$ is

For,

$$\lim_{z\to\pi i}(z-\pi i)^2\,f(z) = \lim_{z\to\pi i}\sinh z = \mathrm{si}$$

$$= \frac{1}{2}(\cos\pi + i\sin\pi - \cos\pi +$$

$\Rightarrow$ $z = \pi i$ is not a pole of order 2

$$\lim_{z\to\pi i}(z-\pi i)\,f(z) = \lim_{z\to\pi i}\frac{\sinh z}{z-\pi i}$$

$$= \lim_{z\to\pi i}\frac{\cosh z}{1} = \cosh\pi i =$$

finite non-zero quantity.

3. Find the type of the singularity

(a) $f(z) = \frac{1-e^z}{z^2}$ at $z = 0$

(b) $f(z) = \cot 2z$ at $z = \infty$

(c) $f(z) = (\sin z - \cos z)^{-1}$ at

a. $f(z) = \frac{1-e^z}{z^2} = \frac{1}{z^2}\left[1 - \left(1 + z +\right.\right.$

$$= -\left[\frac{1}{z} + \frac{1}{2!} + \frac{1}{3!}z + \frac{1}{4!}z^2\right.$$

$\lim_{z\to 1}(z-1)\,f(z) = \lim_{z\to 1}\dfrac{1}{(z-3)^2} = -\dfrac{1}{8}$, which is a finite non zero quantity.

$\therefore$ $z = 1$ is a pole of order 1.

And, the point $z = 3$ is a pole of order 3, since

$\lim_{z\to 3}(z-3)^3\,f(z) = \lim_{z\to 3}\dfrac{1}{z-1} = \dfrac{1}{2}$, which is a finite non-zero quantity.

b. The singular point is given by $(z-\pi i)^2 = 0$ or, $z = \pi i$

At first, it appears that $z = \pi i$ is a pole of order 2 but it is not. For,

$$\lim_{z\to\pi i}(z-\pi i)^2 f(z) = \lim_{z\to\pi i}\sinh z = \sinh\pi i = \frac{1}{2}(e^{i\pi} - e^{-i\pi})$$

$$= \frac{1}{2}(\cos\pi + i\sin\pi - \cos\pi + i\sin\pi) = 0$$

$\Rightarrow$ $z = \pi i$ is not a pole of order 2. It is a pole of order 1. For,

$$\lim_{z\to\pi i}(z-\pi i)\,f(z) = \lim_{z\to\pi i}\frac{\sinh z}{z-\pi i} \quad \left(\frac{0}{0}\right)$$

$= \lim_{z\to\pi i}\dfrac{\cosh z}{1} = \cosh\pi i = \dfrac{1}{2}(e^{i\pi} + e^{-i\pi}) = -1$, which is a finite non-zero quantity.

3. Find the type of the singularity of the function

(a) $f(z) = \dfrac{1-e^z}{z^2}$ at $z = 0$

(b) $f(z) = \cot 2z$ at $z = \infty$

(c) $f(z) = (\sin z - \cos z)^{-1}$ at $z = \dfrac{\pi}{4}$

a. $$f(z) = \frac{1-e^z}{z^2} = \frac{1}{z^2}\left[1 - \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots\right)\right]$$

$$= -\left[\frac{1}{z} + \frac{1}{2!} + \frac{1}{3!}z + \frac{1}{4!}z^2 + \frac{1}{5!}z^3 + \ldots\right]$$

The principal part contains only one term. So, f(z) has a pole of order 1 at $z = 0$.

b. Given, $f(z) = \cot 2z = \dfrac{\cos 2z}{\sin 2z}$

Poles of f(z) are given by

$$\sin 2z = 0 \quad \text{or } 2z = n\pi, \quad z = \frac{n\pi}{2}$$

$\therefore$ Poles of f(z) are $0, \frac{\pi}{2}, 2\frac{\pi}{2}, 3\frac{\pi}{2}, \ldots\ldots$

Obviously, $z = \infty$ is limit point of these poles. Hence, f(z) has a non-isolated essential singularity at $z = \infty$.

c. $f(z) = (\sin z - \cos z)^{-1} = \dfrac{1}{\sin z - \cos z}$

Poles are given by $\sin z - \cos z = 0$

or, $\tan z = 1 \quad$ or $z = n\pi + \frac{\pi}{4} \quad (n = 0, 1, 2, \ldots)$

So, the poles are $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$ (simple)

$\therefore$ f(z) has as simple pole at $z = \frac{\pi}{4}$.

4. Find the type of singularities of the following functions at infinity.

(a) $z^2 - \frac{1}{z^2}$ (b) $\sin z$ (c) $\cosh\left(\frac{1}{z^2+4}\right)$

Here, we have to investigate the function at infinity.

Let $f(z) = z^2 - \frac{1}{z^2}$. Set $z = \frac{1}{w}$ so that $f\left(\frac{1}{w}\right) = \frac{1}{w^2} - w^2$

Clearly, $f\left(\frac{1}{w}\right)$ has a pole of order 2 at $w = 0$

$\therefore$ f(z) has a pole of order 2 at $z = \infty$.

b. Let $f(z) = \sin z$. Setting $z = \frac{1}{w}$, we have

$$f\left(\frac{1}{w}\right) = \sin\left(\frac{1}{w}\right) = \frac{1}{w} - \frac{1}{3!}\frac{1}{w^3} + \frac{1}{5!}\frac{1}{w^5} - $$

We observe that the principal part of $f\left(\frac{1}{w}\right.$

many terms. So, $f\left(\frac{1}{w}\right)$ has an essential si

Consequently, f(z) has the essential singula

c. Let $f(z) = \cosh\left(\frac{1}{z^2+4}\right)$. Set $z = \frac{1}{w}$. Then

$$f\left(\frac{1}{w}\right) = \cosh\left(\frac{w^2}{1+4w^2}\right)$$

Clearly, $f\left(\frac{1}{w}\right)$ is analytic at $w = 0$. So, f(z)

That is, there is no singularity at $z = \infty$.

5.5 Residue at Singularity

If z_0 is an isolated singularity of f(z), there is f(z) is analytic. Hence, f(z) has a Laurent serie $z = z_0$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz$ and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{-n+1}}\,dz$$

The series (1) can be written as

b. Let $f(z) = \sin z$. Setting $z = \frac{1}{w}$, we have

$$f\left(\frac{1}{w}\right) = \sin\left(\frac{1}{w}\right) = \frac{1}{w} - \frac{1}{3!}\frac{1}{w^3} + \frac{1}{5!}\frac{1}{w^5} - \dots$$

We observe that the principal part of $f\left(\frac{1}{w}\right)$ contains infinitely many terms. So, $f\left(\frac{1}{w}\right)$ has an essential singularity at $w = 0$. Consequently, $f(z)$ has the essential singularity at $z = \infty$.

c. Let $f(z) = \cosh\left(\frac{1}{z^2 + 4}\right)$. Set $z = \frac{1}{w}$. Then

$$f\left(\frac{1}{w}\right) = \cosh\left(\frac{w^2}{1 + 4w^2}\right)$$

Clearly, $f\left(\frac{1}{w}\right)$ is analytic at $w = 0$. So, $f(z)$ is analytic at $z = \infty$. That is, there is no singularity at $z = \infty$.

5.5 Residue at Singularity

If z_0 is an isolated singularity of $f(z)$, there is an annulus on which $f(z)$ is analytic. Hence, $f(z)$ has a Laurent series expansion about $z = z_0$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (1)$$

$$\text{where } a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z - z_0)^{n+1}}\, dz \quad \text{and} \qquad (2)$$

$$b_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z - z_0)^{-n+1}}\, dz \qquad (3)$$

The series (1) can be written as

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \frac{b_3}{(z - z_0)^3} + \dots \quad (4)$$

The coefficient b_1 in (4) is of particular importance, and is called the residue of f(z) at the point $z = z_0$ and is given by

$$b_1 = \underset{z=z_0}{\text{Res}}\ f(z)$$

Formula (3) tells us that

$$b_1 = \frac{1}{2\pi i} \int_c f(z)\, dz$$

It implies that

$$\int_c f(z)\, dz = 2\pi i . b_1 = 2\pi i \underset{z=z_0}{\text{Res}}\ f(z)$$

In many cases it is relatively easy to evaluate the residue at a point without the use of integration. For example, if $z = z_0$ is a simple pole, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0}$$

$$(z - z_0)\, f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+1} + b_1$$

$$\therefore \quad b_1 = \lim_{z \to z_0} (z - z_0)\, f(z)$$

This is a simple method of calculating the residue of f(z) at the simple pole $z = z_0$ (pole of order 1).

5.6 Residue of a Function at a Pole of Order m

If f(z) has a pole of order m, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots + \frac{b_m}{(z - z_0)^m}$$

or, $f(z) = \{a_0 + a_1(z - z_0) + \dots\} + \dfrac{b_1}{z - z_0} + \dfrac{b_2}{(z - z_0)^2} + \dots + \dfrac{b_m}{(z - z_0)^m}$

Multiplying by $(z - z_0)^m$, we have

$$(z - z_0)^m f(z) = \{a_0(z - z_0)^m + a_1(z - z_0)^{m+1} + \dots\} + b_1(z - z_0)^{m-1} + b_2(z - z_0)^{m-2} + \dots + b_m$$

Differentiating w.r.t. z (m – 1) times,

$$\frac{d^{m-1}}{dz^{m-1}} \{(z - z_0)^m f(z)\} = \{a_0 m! (z - z_0) +$$

$$b_1 (m - 1)!$$

$$\therefore \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \{(z - z_0)^m f(z)\} \right] = (m - 1$$

$$\Rightarrow b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \{(z - z_0)^m f($$

(5) gives a formula for calculating resi of order m.

Thus we have the following relations.

a. If f(z) has a pole of order 1, then
$$b_1 = \lim_{z \to z_0} (z - z_0)\, f(z)$$

b. If f(z) has a pole of order 2, then
$$b_1 = \lim_{z \to z_0} \left[\frac{d}{dz} (z - z_0)^2 f(z) \right]$$

c. If f(z) has a pole of order 3, then
$$b_1 = \frac{1}{2!} \lim_{z \to z_0} \left[\frac{d^2}{dz^2} (z - z_0)^3 f(z)$$

Examples

1. Find the residue of the functions a

(a) $\dfrac{z+3}{(z-1)(z+2)^2}$

(c) $\dfrac{\sin z}{z^4}$

(e) $\cot \pi z$

a. Let $f(z) = \dfrac{z+3}{(z-1)(z+2)^2}$

The poles are given by (z – 1) (z

Clearly, z = 1 is a simple pole, an

Differentiating w.r.t. z (m – 1) times,

$$\frac{d^{m-1}}{dz^{m-1}}\{(z-z_0)^m f(z)\} = \{a_0 m!\,(z-z_0) + a_1\frac{(m+1)!}{2}(z-z_0)^2 + \dots\} + b_1(m-1)!$$

$$\therefore \lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\{(z-z_0)^m f(z)\}\right] = (m-1)!\, b_1$$

$$\Rightarrow b_1 = \frac{1}{(m-1)!}\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\{\underline{(z-z_0)^m} f(z)\}\right] \qquad (5)$$

(5) gives a formula for calculating residue of f(z) if f(z) has a pole of order m.

Thus we have the following relations.

a. If f(z) has a pole of order 1, then

$$b_1 = \lim_{z\to z_0}(z-z_0)\, f(z)$$

b. If f(z) has a pole of order 2, then

$$b_1 = \lim_{z\to z_0}\left[\frac{d}{dz}(z-z_0)^2 f(z)\right]$$

c. If f(z) has a pole of order 3, then

$$b_1 = \frac{1}{2!}\lim_{z\to z_0}\left[\frac{d^2}{dz^2}(z-z_0)^3 f(z)\right] \text{ and so on.}$$

Examples

1. Find the residue of the functions at each of its poles.

(a) $\dfrac{z+3}{(z-1)(z+2)^2}$ (b) $\dfrac{1}{z^2+a^2}$

(c) $\dfrac{\sin z}{z^4}$ (d) $\dfrac{1}{1-e^z}$

(e) $\cot\pi z$

a. Let $f(z) = \dfrac{z+3}{(z-1)(z+2)^2}$

The poles are given by $(z-1)(z+2)^2 = 0 \quad \Rightarrow z = 1$ or -2

Clearly, $z = 1$ is a simple pole, and $z = -2$ is a pole of order 2.

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}(z-1)\, f(z) = \lim_{z\to 1}\frac{z+3}{(z+3)^2} = \frac{4}{9}$$

$$\operatorname*{Res}_{z=-2} f(z) = \lim_{z\to -2}\left[\frac{d}{dz}(z+2)^2 f(z)\right] = \lim_{z\to -2}\frac{d}{dz}\left[\frac{z+3}{z-1}\right]$$

$$= \lim_{z\to -2} -\frac{4}{(z-1)^2} = -\frac{4}{9}$$

b. Let $f(z) = \dfrac{1}{z^2+a^2} = \dfrac{1}{(z+ia)(z-ia)}$

Poles are given by $(z+ia)(z-ia) = 0 \Rightarrow z = ia$ or $-ia$

Both are simple poles.

$$\operatorname*{Res}_{z=ia} f(z) = \lim_{z\to ia}(z-ia)\, f(z)$$

$$= \lim_{z\to ia}\frac{1}{z+ia} = \frac{1}{2ia}$$

$$\operatorname*{Res}_{z=-ia} f(z) = \lim_{z\to -ia}(z+ia)\, f(z)$$

$$= \lim_{z\to -ia}\frac{1}{z-ia} = -\frac{1}{2ia}$$

c. Let $f(z) = \dfrac{\sin z}{z^4} = \dfrac{1}{z^4}\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dots\dots\right)$

$$= \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}z - \dots,$$

which is a Laurent series. It has a pole of order 3 at $z = 0$.

Clearly, $\operatorname*{Res}_{z=0} f(z) =$ coefficient of $\dfrac{1}{z} = -\dfrac{1}{3!} = -\dfrac{1}{6}$

Alternatively,

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{2!}\lim_{z\to 0}\left[\frac{d^2}{dz^2}\{z^3 f(z)\}\right]$$

$$= \frac{1}{2!}\lim_{z\to 0}\left[\frac{d^2}{dz^2}\frac{\sin z}{z}\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\left[\frac{d^2}{dz^2}\left(\frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots}{z}\right)\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\left[\dots\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\left[\dots\right]$$

$$= -\frac{1}{3!}$$

$$= -\frac{1}{6}$$

d. Poles of the function are given b

$1 - e^z = 0$ or, $e^z = 1 = e^{2n\pi i}$ or

Now, $\operatorname*{Res}_{z=2n\pi i} f(z) = \lim_{z\to 2n\pi i}(z - 2n\pi$

$$= \lim_{z\to 2n\pi i}\frac{z-2n\pi}{1-e^z}$$

$$= \lim_{z\to 2n\pi i}\frac{1}{-e^z}$$

$$= -\frac{1}{e^{2n\pi i}}$$

$$= -1$$

e. Let $f(z) = \cot\pi z = \dfrac{\cos\pi z}{\sin\pi z}$

Poles are given by $\sin\pi z = 0$

$$\operatorname*{Res}_{z=n} f(z) = \lim_{z\to n}(z-n)\, f(z) = \lim$$

$$= (-1)^n \lim_{z\to n}\frac{z-n}{\sin\pi z}$$

$$= (-1)^n \lim_{z\to n}\frac{1}{\pi\cos\pi z}$$

$$= (-1)^n \frac{1}{\pi\cos n\pi}$$

$$= \frac{1}{2}\lim_{z\to 0}\left[\frac{d^2}{dz^2}\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\ldots\right)\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\left[-\frac{2}{3!}+\frac{12}{5!}z^2-\ldots\right]$$

$$= -\frac{1}{3!}$$

$$= -\frac{1}{6}$$

d. Poles of the function are given by

$1-e^z=0$ or, $e^z=1=e^{2n\pi i}$ or, $z=2n\pi i$ $(n=0,\pm1,\pm2,\ldots)$

(simple poles)

Now, $\operatorname*{Res}_{z=2n\pi i} f(z) = \lim_{z\to 2n\pi i}(z-2n\pi i)\, f(z)$

$$= \lim_{z\to 2n\pi i}\frac{z-2n\pi i}{1-e^z} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{z\to 2n\pi i}\frac{1}{-e^z}$$

$$= -\frac{1}{e^{2n\pi i}}$$

$$= -1$$

e. Let $f(z)=\cot\pi z=\dfrac{\cos\pi z}{\sin\pi z}$

Poles are given by $\sin\pi z=0$ or $\pi z=n\pi$ $(n=0,\pm1,\pm2,\ldots)$

$z=n$ (simple pole)

$$\operatorname*{Res}_{z=n} f(z)=\lim_{z\to n}(z-n)\, f(z)=\lim_{z\to n}(z-n)\frac{\cos\pi z}{\sin\pi z} \qquad \left(\frac{0}{0}\right)$$

$$= (-1)^n\lim_{z\to n}\frac{z-n}{\sin\pi z}$$

$$= (-1)^n\lim_{z\to n}\frac{1}{\pi\cos\pi z}$$

$$= (-1)^n\frac{1}{\pi\cos n\pi}$$

$= \frac{1}{\pi}$

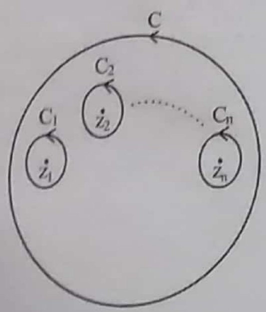
5.7 Cauchy Residue Theorem

Let f(z) be analytic inside and on a simple closed curve C, except at a finite number of singularities $z_1, z_2, \ldots, z_n$ inside C. Let the residues of f(z) at these points be $R_1, R_2, \ldots, R_n$. Then

$$\int_C f(z)\, dz = 2\pi i\,(R_1 + R_2 + \ldots + R_n)$$

[Integral of f(z) around C is equal to $2\pi i$ times the sum of residues at the singular points]

Proof.



Let $z_1, z_2, \ldots, z_n$ be the singular points which are inside C. Around each of the singular points draw small non–intersecting circles $C_1, C_2, \ldots, C_n$ as shown in the figure. Then f(z) is analytic in the region between C and the circles $C_1, C_2, \ldots, C_n$. By Cauchy's theorem,

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)dz + \ldots + \int_{C_n} f(z)\, dz$$

$$= 2\pi i\, R_1 + 2\pi i\, R_2 + \ldots + 2\pi i\, R_n$$

$$= 2\pi i\,(R_1 + R_2 + \ldots + R_n)$$

$= 2\pi i$ times sum of residues of f(z) at $z_1, z_2, \ldots, z_n$

Examples

1. Evaluate the following integrals (coun

a. $\int_C \frac{z+3}{(z-2)(z+1)^2}\, dz,$

b. $\int_C \frac{e^z}{\cos z}\, dz,$

c. $\int_C \frac{z\cosh \pi z}{z^4 + 13z^2 + 36}\, dz,$

d. $\int_C \left(\frac{ze^{\pi z}}{z^4 - 16} + ze^{\pi/z}\right) dz,$

e. $\int_C \tan\pi z\, dz,$

a. Let $f(z) = \frac{z+3}{(z-2)(z+1)^2}$

Clearly, z = 2 is a simple pole, and

Both the poles lie within the circle

$$\operatorname{Res}_{z=2} f(z) = \lim_{z\to 2} (z-2)\ \text{f}$$

$$= \lim_{z\to 2} \frac{z+3}{(z+1)^2} =$$

$$\operatorname{Res}_{z=-1} f(z) = \lim_{z\to -1} \left[\frac{d}{dz}(z\right.$$

$$= \lim_{z\to -1} \left[\frac{d}{dz}\left(\frac{z+}{z-}\right.\right.$$

$$= \lim_{z\to -1} \frac{(z-2}{}$$

$$= \lim_{z\to -1} \frac{-5}{(z-2)}$$

$$= -\frac{5}{9}$$

Examples

1. Evaluate the following integrals (counter clockwise)

a. $\displaystyle\int_C \frac{z+3}{(z-2)(z+1)^2}\,dz,$ $\qquad C : |z| = 3$

b. $\displaystyle\int_C \frac{e^z}{\cos z}\,dz,$ $\qquad C : |z| = 3$

c. $\displaystyle\int_C \frac{z\cosh \pi z}{z^4+13z^2+36}\,dz,$ $\qquad C : |z| = \pi$

d. $\displaystyle\int_C \left(\frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\right)dz,$ $\qquad C :$ ellipse $9x^2 + y^2 = 9$

e. $\displaystyle\int_C \tan\pi z\,dz,$ $\qquad C : |z| = 1$

a. Let $f(z) = \dfrac{z+3}{(z-2)(z+1)^2}$

Clearly, $z = 2$ is a simple pole, and $z = -1$ is a pole of order 2. Both the poles lie within the circle $|z| = 3$.

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2}(z-2)\,f(z)$$

$$= \lim_{z\to 2}\frac{z+3}{(z+1)^2} = \frac{5}{9}$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1}\left[\frac{d}{dz}(z+1)^2 f(z)\right]$$

$$= \lim_{z\to -1}\left[\frac{d}{dz}\left(\frac{z+3}{z-2}\right)\right]$$

$$= \lim_{z\to -1}\frac{(z-2).1-(z+3).1}{(z-2)^2}$$

$$= \lim_{z\to -1}\frac{-5}{(z-2)^2}$$

$$= -\frac{5}{9}$$

Sum of the residues $= \frac{5}{9} - \frac{5}{9} = 0$

By Residue Theorem,

$$\int_C \frac{z+3}{(z-2)(z+1)^2}\,dz = 2\pi i \times (\text{sum of the residues}) = 2\pi i \times 0 = 0$$

b. Let $f(z) = \frac{e^z}{\cos z}$

The singular points are given by

$$\cos z = 0 \qquad \text{or} \quad z = (2n+1)\frac{\pi}{2} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

The singular points are

$$\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$$

Of these singular points, only $z = \pm\frac{\pi}{2}$ lie inside the circle $z = |3|$.

The other singular points are of no interest. Also, we note that the singular points are simple poles. Now, we shall find the residues of the function at $z = \pm\frac{\pi}{2}$.

$$\operatorname*{Res}_{z=\pi/2} f(z) = \lim_{z\to\pi/2}\left(z-\frac{\pi}{2}\right) f(z) = \lim_{z\to\pi/2} \frac{\left(z-\frac{\pi}{2}\right)e^z}{\cos z} \qquad \left(\frac{0}{0}\right)$$

$$= e^{\pi/2} \lim_{z\to\pi/2} \frac{z-\frac{\pi}{2}}{\cos z} \qquad \left(\frac{0}{0}\right)$$

$$= e^{\pi/2} \lim_{z\to\pi/2} \frac{1}{-\sin z}$$

$$= e^{\pi/2}\left(\frac{1}{-1}\right) = -e^{\pi/2}$$

$$\operatorname*{Res}_{z=-\pi/2} f(z) = \lim_{z\to-\pi/2}\left(z+\frac{\pi}{2}\right) f(z)$$

$$= \lim_{z\to-\pi/2} \frac{\left(z+\frac{\pi}{2}\right)e^z}{\cos z}$$

$$= e^{-\pi/2} \lim_{z\to-\pi/2} \frac{z+\frac{\pi}{2}}{\cos z}$$

$$= e^{-\pi/2} \lim_{z\to-\pi/2} \frac{1}{-\sin z}$$

$$= e^{-\pi/2}$$

Sum of residues $= e^{-\pi/2} - e^{\pi/2}$

By Residue Theorem

$$\int_C \frac{e^z}{\cos z}\,dz = 2\pi i \times \text{sum of residues} = 2$$

$$= -4\pi i \sinh\frac{\pi}{2}$$

c. Let $f(z) = \frac{z\cosh \pi z}{z^4 + 13z^2 + 36}$

The singular points are given by

$z^4 + 13z^2 + 36 = 0$ or $(z^2 +$

or $z = \pm 2i$ or $\pm 3i$. Moreover, they

All these poles lie inside the circle |

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z\to 2i} (z-2i) f(z)$$

$$= \lim_{z\to 2i} (z-2i) \frac{}{(z^2+9}$$

$$= \lim_{z\to 2i} \frac{z\cosh \pi z}{(z+2i)(z^2+}$$

$$\operatorname*{Res}_{z=-2i} f(z) = \lim_{z\to-2i} (z+2i) f(z)$$

$$= \lim_{z\to-2i} \frac{z\cosh \pi z}{(z-2i)(z^2+}$$

$$= \frac{1}{10}\cosh 2\pi i$$

$$= \lim_{z\to-\pi/2} \frac{\left(z+\frac{\pi}{2}\right)e^z}{\cos z}$$

$$= e^{-\pi/2} \lim_{z\to-\pi/2} \frac{z+\frac{\pi}{2}}{\cos z}$$

$$= e^{-\pi/2} \lim_{z\to-\pi/2} \frac{1}{-\sin z}$$

$$= e^{-\pi/2}$$

Sum of residues $= e^{-\pi/2} - e^{\pi/2}$

By Residue Theorem

$$\int_C \frac{e^z}{\cos z}\, dz = 2\pi i \times \text{sum of residues} = 2\pi i\,(e^{-\pi/2} - e^{\pi/2})$$

$$= -4\pi i \sinh\frac{\pi}{2}$$

c. Let $f(z) = \dfrac{z\cosh \pi z}{z^4 + 13z^2 + 36}$

The singular points are given by

$$z^4 + 13z^2 + 36 = 0 \quad \text{or} \quad (z^2+4)(z^2+9) = 0$$

or $z = \pm 2i$ or $\pm 3i$. Moreover, they are simple poles.

All these poles lie inside the circle $|z| = \pi$

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z\to 2i} (z-2i)f(z)$$

$$= \lim_{z\to 2i} (z-2i)\frac{z\cosh \pi z}{(z^2+9)(z+2i)(z-2i)}$$

$$= \lim_{z\to 2i} \frac{z\cosh \pi z}{(z+2i)(z^2+9)} = \frac{1}{10}\cosh 2\pi i$$

$$\operatorname*{Res}_{z=-2i} f(z) = \lim_{z\to -2i} (z+2i)f(z)$$

$$= \lim_{z\to -2i} \frac{z\cosh \pi z}{(z-2i)(z^2+9)}$$

$$= \frac{1}{10}\cosh 2\pi i$$

$$\underset{z=3i}{\text{Res}}\ f(z) = \lim_{z\to 3i}(z-3i)\, f(z)$$

$$= \lim_{z\to 3i} \frac{z\cosh \pi z}{(z+3i)(z^2+4)}$$

$$= -\frac{1}{10}\cosh 3\pi i$$

Similarly, $\underset{z=-3i}{\text{Res}}\ f(z) = -\frac{1}{10}\cosh 3\pi i$

Sum of residues $= \frac{1}{5}\cosh 2\pi i - \frac{1}{5}\cosh 3\pi i$

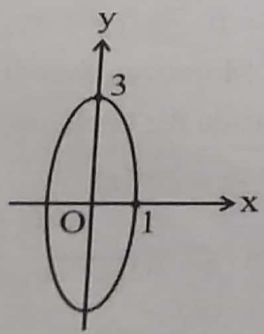
$$= \frac{1}{5}(\cosh 2\pi i - \cosh 3\pi i)$$

$$= \frac{1}{5}\left[\frac{1}{2}(e^{i2\pi} + e^{-i2\pi}) - \frac{1}{2}(e^{i3\pi} + e^{-i3\pi})\right] = \frac{2}{5}$$

By Residue Theorem,

$$\int_C \frac{z\cosh \pi z}{z^4+13z^2+36}\, dz = 2\pi i \times \text{sum of residues} = 2\pi i \times \frac{2}{5} = \frac{4\pi i}{5}$$

d. Given: $9x^2 + y^2 = 9$ or, $\frac{x^2}{1} + \frac{y^2}{9} = 1$.



At first, we consider the first term of the integrand.

The poles are given by $z^4 - 16 = 0$ or $(z^2+4)(z^2-4) = 0$

$\Rightarrow z = \pm 2i,\ z = \pm 2$

Of these poles, only $z = \pm 2i$ lie inside the ellipse. The other poles lie outside the ellipse, and they are of no interest.

$$\underset{z=2i}{\text{Res}}\ f(z) = \lim_{z\to 2i}(z-2i)\, f(z)$$

$$= \lim_{z\to 2i}(z-2i)\frac{ze^{\pi z}}{z^4-16}$$

$$= \lim_{z\to 2i}\frac{ze^{\pi z}}{(z+2i)(z^2-4)}$$

$$= \frac{2i\, e^{2i\pi}}{4i(-8)} = -\frac{1}{16} \quad [\because$$

$$\underset{z=-2i}{\text{Res}}\ f(z) = \lim_{z\to -2i}(z+2i) f(z)$$

$$= \lim_{z\to -2i}\frac{ze^{\pi z}}{(z-2i)(z^2-4}$$

$$= \frac{-2i\, e^{-2\pi i}}{-4i.(-8)}$$

$$= -\frac{1}{16}$$

Now, consider the 2^{nd} term of the i

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{1}{2!}\frac{\pi^2}{z^2} + \frac{1}{3!}\frac{\pi^3}{z^3} + \frac{1}{4!}\right.$$

$$= z + \pi + \frac{\pi^2}{2!}\frac{1}{z} + \frac{\pi^3}{3!}\frac{1}{z^2} + \frac{\pi^4}{4!}$$

series.

Clearly, $z = 0$ is the essential singu

Residue $= \frac{\pi^2}{2}$

$$\therefore \int_C \left(\frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\right) dz = 2\pi i \times ($$

e. The singular points are given by

$\cos \pi z = 0$ or $\pi z = (2n+1)\frac{\pi}{2}$ or

Thus, the singular points are $\pm\frac{1}{2}$,

Of these singular points only $\pm\frac{1}{2}$

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z\to 2i} (z-2i)\, f(z)$$

$$= \lim_{z\to 2i} (z-2i) \frac{ze^{\pi z}}{z^4-16}$$

$$= \lim_{z\to 2i} \frac{ze^{\pi z}}{(z+2i)(z^2-4)}$$

$$= \frac{2i\, e^{2i\pi}}{4i(-8)} = -\frac{1}{16} \quad [\because e^{2i\pi} = 1]$$

$$\operatorname*{Res}_{z=-2i} f(z) = \lim_{z\to -2i} (z+2i) f(z)$$

$$= \lim_{z\to -2i} \frac{ze^{\pi z}}{(z-2i)(z^2-4)}$$

$$= \frac{-2i\, e^{-2\pi i}}{-4i.(-8)}$$

$$= -\frac{1}{16}$$

Now, consider the 2nd term of the integrand:

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{1}{2!}\frac{\pi^2}{z^2} + \frac{1}{3!}\frac{\pi^3}{z^3} + \frac{1}{4!}\frac{\pi^4}{z^4} + \dots\right)$$

$$= z + \pi + \frac{\pi^2}{2!}\frac{1}{z} + \frac{\pi^3}{3!}\frac{1}{z^2} + \frac{\pi^4}{4!}\frac{1}{z^3} + \dots,$$ which is a Laurent series.

Clearly, $z = 0$ is the essential singularity of $f(z)$.

$$\text{Residue} = \frac{\pi^2}{2}$$

$$\therefore \int_C \left(\frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\right) dz = 2\pi i \times \left(-\frac{1}{16} - \frac{1}{16} + \frac{\pi^2}{2}\right) = \pi\left(\pi^2 - \frac{1}{4}\right) i$$

e. The singular points are given by

$$\cos\pi z = 0 \quad \text{or } \pi z = (2n+1)\frac{\pi}{2} \quad \text{or } z = \left(n + \frac{1}{2}\right) \quad (n = 0, \pm 1, \pm 2, \dots)$$

Thus, the singular points are $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2} \dots\dots$

Of these singular points only $\pm\frac{1}{2}$ lies inside the circle $|z| = 1$.

$$\operatorname*{Res}_{z=1/2} f(z) = \lim_{z\to 1/2}\left(z-\frac{1}{2}\right) f(z)$$

$$= \lim_{z\to 1/2}\left(z-\frac{1}{2}\right)\frac{\sin\pi z}{\cos\pi z} \qquad \left(\frac{0}{0}\right)$$

$$= \sin\frac{\pi}{2}\lim_{z\to 1/2}\frac{z-\frac{1}{2}}{\cos\pi z} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{z\to 1/2}\frac{1}{-\pi\sin\pi z} = -\frac{1}{\pi}$$

Similarly, $\operatorname*{Res}_{z=-1/2} f(z) = \lim_{z\to -1/2}\left(z+\frac{1}{2}\right)\frac{\sin\pi z}{\cos\pi z} = -\frac{1}{\pi}$

Sum of residues $= -\frac{1}{\pi}-\frac{1}{\pi} = -\frac{2}{\pi}$

$$\therefore \int_C \tan\pi z \, dz = 2\pi i \times \left(\frac{-2}{\pi}\right) = -4i$$

Exercise 5.1

1. Find out the zeros of the following function and specify their order.
   (i) $f(z) = (z^4-1)^3$ (ii) $f(z) = \dfrac{z^2+4}{e^z}$
   (iii) $f(z) = (z^4-z^2-6)^3$ (iv) $f(z) = (\sin z - 1)^5$
2. Determine the location and type of the singularities. If the singularities are poles, state their order.
   (i) $f(z) = \dfrac{1}{(z-1)(z-3)}$ (ii) $\dfrac{z-\sin z}{z^3}$
   (iii) $f(z) = \sin\dfrac{1}{z}$ (iv) $f(z) = \dfrac{1}{z^2+a^2}$
3. *Find the type of the singularity of the following function at infinity.*
   (i) e^z (ii) z^3 (iii) $\dfrac{\sinh z}{(z-\pi i)^2}$ (iv) $(z^2+a^2)^{-2}$
4. *Determine the poles of the function* $f(z) = \dfrac{1}{z^4+1}$

5. For the following functions, find poles (i) $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ (ii) f
6. Evaluate the following integrals theoem
   (i) $\int_c \dfrac{1+z}{z(2-z)}\,dz$ c: $|z| = 1$
   (ii) $\int_c \dfrac{4-3z}{z(z-1)(z-2)}\,dz$ c: $|z| = 3/$
   (iii) $\int_c \dfrac{12z-7}{(z-1)^2(2z+3)}\,dz$ c: $|z| = 2$
   (iv) $\int_c \dfrac{z^2}{(z-1)^2(z+2)}\,dz$ c: $|z| = 3$
   (v) $\int_c \dfrac{z-1}{(z+1)^2(z-2)}\,dz$ c: $|z-i| =$
   (vi) $\int_c \dfrac{1-2z}{z(z-1)(z-2)}\,dz$ c: $|z| = 3/$
   (vii) $\int_c \dfrac{2z^2+z}{z^2-1}\,dz$ c: $|z-1| =$
   (viii) $\int_c \dfrac{4z^2-4z+1}{(z-2)(z^2+4)}\,dz$ c: $|z| = 1$
   (ix) $\int_c \dfrac{\sin hz}{(2z-i)}\,dz$ c: $|z-i| =$
   (x) $\int_c \left(\dfrac{e^z+z}{z^3-z}\right)dz$ c: $|z| = \dfrac{\pi}{2}$
   (xi) $\int_c \left(\dfrac{z^2\sin z}{4z^2-1}\right)dz$ c: $|z| = 2$
   (xii) $\int_c \dfrac{e^z}{\cos z}\,dz$ c: $|z| = 3$
   (xiii) $\int_c \dfrac{e^{z}}{\sin 4z}\,dz$ c: $|z| = 1$
7. Evaluate $\int_c \dfrac{\tan z}{z^2-1}\,dz$ where c: $|z|$ direction.
8. Evaluate $\int_c \dfrac{4-3z}{z^2-z}\,dz$ where c counterclockwise simple closed pa

5. For the following functions, find the residues at each of its poles (i) $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ (ii) $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

6. Evaluate the following integrals by using Cauchy residue theocem

(i) $\int_c \frac{1+z}{z(2-z)} dz$ c: $|z| = 1$

(ii) $\int_c \frac{4-3z}{z(z-1)(z-2)} dz$ c: $|z| = 3/2$

(iii) $\int_c \frac{12z-7}{(z-1)^2(2z+3)} dz$ c: $|z| = 2$

(iv) $\int_c \frac{z^2}{(z-1)^2(z+2)} dz$ c: $|z| = 3$

(v) $\int_c \frac{z-1}{(z+1)^2(z-2)} dz$ c: $|z-i| = 2$

(vi) $\int_c \frac{1-2z}{z(z-1)(z-2)} dz$ c: $|z| = 3/2$

(vii) $\int_c \frac{2z^2+z}{z^2-1} dz$ c: $|z-1| = 1$

(viii) $\int_c \frac{4z^2-4z+1}{(z-2)(z^2+4)} dz$ c: $|z| = 1$

(ix) $\int_c \frac{\sin hz}{(2z-i)} dz$ c: $|z-i| = 1$

(x) $\int_c \left(\frac{e^z+z}{z^3-z}\right) dz$ c: $|z| = \frac{\pi}{2}$

(xi) $\int_c \left(\frac{z^2 \sin z}{4z^2-1}\right) dz$ c: $|z| = 2$

(xii) $\int_c \frac{e^z}{\cos z} dz$ c: $|z| = 3$

(xiii) $\int_c \frac{e^{-z^2}}{\sin 4z} dz$ c: $|z| = 1$

7. Evaluate $\int_c \frac{\tan z}{z^2-1} dz$ where c: $|z| = 3/2$ in counterclockwise direction.

8. Evaluate $\int_c \frac{4-3z}{z^2-z} dz$ where c is the curve which is counterclockwise simple closed path such that

a. 0 and 1 are inside c
b. 0 is inside, 1 is outside c
c. 1 is inside, 0 is outside c
d. 0 and 1 are outside c.

9. Evaluate $\int_c \frac{z-23}{z^2-4z-5}dz$ c: $|z-2|=4$

Answers

1. (i) 1, −1, i, −1 of order 3
(ii) $\pm 2i$, (simple)
(iii) $\pm\sqrt{3}, \pm i\sqrt{2}$ of order 3
(iv) $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots\ldots$, order 5
2. (i) 1 , 3 (simple)
(ii) 0, removable singularity
(iii) 0; essential singularity
(iv) $\pm ia$, (simple poles)
3. (i) essential singularity (ii) pole of order 3
(iii) essential singularity (iv) no singularity
4. $(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}})$
5. (i) −2 (simple) and 1 (order 2), and residues are $\frac{4}{9}$ and $\frac{5}{9}$ respectively.
(ii) simple poles are 0, 1, and 2 and residues are respectively 1/2, 1, −3/2
6. (i) πi (ii) $2\pi i$ (iii) 0 (iv) $2\pi i$
(v) 0 (vi) $3\pi i$ (vii) $4\pi i$ (viii) 0
(ix) $\pi i(\sinh\frac{i}{2})$ (x) $2\pi i(\frac{e}{2}+\frac{1}{2e}-1)$
(xi) $\frac{\pi i}{4}\sin\frac{1}{2}$ (xii) $-4\pi i \sinh\frac{\pi}{2}$ (xiii) $\frac{\pi i}{2}(1-2e^{-\pi^2/16})$
7. $2\pi i \tan 1$
8. (a) $-6\pi i$ (b) $-8\pi i$ (c) $2\pi i$ (d) 0
9. $2\pi i$