

# FIRST-ORDER LOGIC

## CHAPTER 8

# Outline

- ◇ Why FOL?
- ◇ Syntax and semantics of FOL
- ◇ Fun with sentences
- ◇ Wumpus world in FOL

## Pros and cons of propositional logic

- 😊 Propositional logic is **declarative**: pieces of syntax correspond to facts
- 😊 Propositional logic allows partial/disjunctive/negated information (unlike most data structures and databases)
- 😊 Propositional logic is **compositional**:  
meaning of  $B_{1,1} \wedge P_{1,2}$  is derived from meaning of  $B_{1,1}$  and of  $P_{1,2}$
- 😊 Meaning in propositional logic is **context-independent**  
(unlike natural language, where meaning depends on context)
- 😞 Propositional logic has very limited expressive power  
(unlike natural language)  
E.g., cannot say “pits cause breezes in adjacent squares”  
except by writing one sentence for each square

# First-order logic

Whereas propositional logic assumes world contains **facts**,  
first-order logic (like natural language) assumes the world contains

- **Objects**: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries . . .
- **Relations**: red, round, bogus, prime, multistoried . . . ,  
brother of, bigger than, inside, part of, has color, occurred after, owns,  
comes between, . . .
- **Functions**: father of, best friend, third inning of, one more than, end of  
. . .

## Logics in general

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

## Syntax of FOL: Basic elements

Constants *KingJohn, 2, UCB, ...*

Predicates *Brother, >, ...*

Functions *Sqrt, LeftLegOf, ...*

Variables *x, y, a, b, ...*

Connectives  $\wedge \vee \neg \Rightarrow \Leftrightarrow$

Equality  $=$

Quantifiers  $\forall \exists$

## Atomic sentences

Atomic sentence =  $\text{predicate}(\text{term}_1, \dots, \text{term}_n)$   
or  $\text{term}_1 = \text{term}_2$

Term =  $\text{function}(\text{term}_1, \dots, \text{term}_n)$   
or *constant* or *variable*

E.g.,  $\text{Brother}(\text{KingJohn}, \text{RichardTheLionheart})$   
 $> (\text{Length}(\text{LeftLegOf}(\text{Richard})), \text{Length}(\text{LeftLegOf}(\text{KingJohn})))$

## Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

E.g.  $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$

$$>(1, 2) \vee \leq(1, 2)$$

$$>(1, 2) \wedge \neg >(1, 2)$$



# Truth in first-order logic

Sentences are true with respect to a **model** and an **interpretation**

Model contains  $\geq 1$  objects (**domain elements**) and relations among them

Interpretation specifies referents for

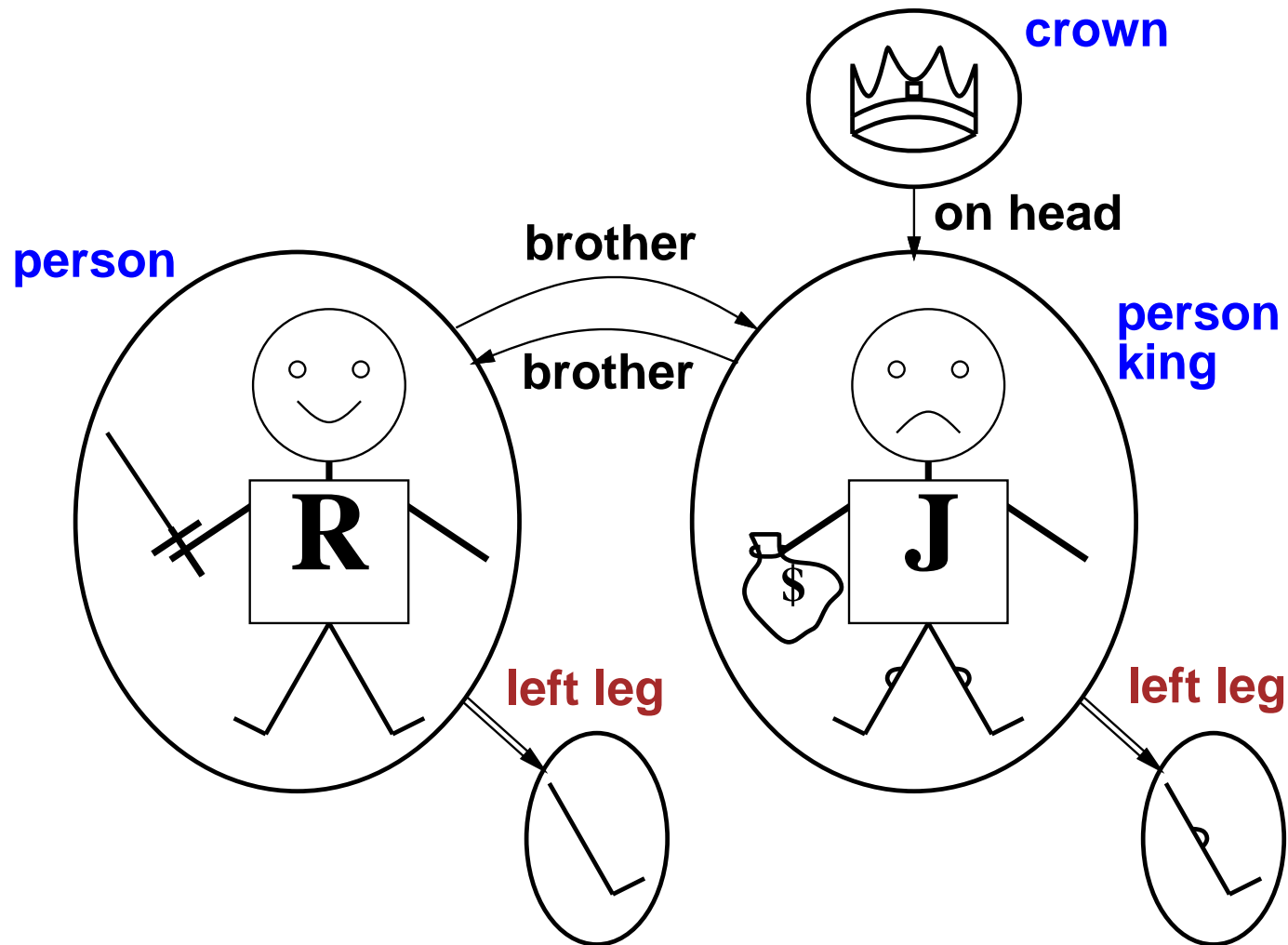
constant symbols  $\rightarrow$  objects

predicate symbols  $\rightarrow$  relations

function symbols  $\rightarrow$  functional relations

An atomic sentence  $\textit{predicate}(\textit{term}_1, \dots, \textit{term}_n)$  is true  
iff the **objects** referred to by  $\textit{term}_1, \dots, \textit{term}_n$   
are in the **relation** referred to by  $\textit{predicate}$

# Models for FOL: Example



## Truth example

Consider the interpretation in which

*Richard* → Richard the Lionheart

*John* → the evil King John

*Brother* → the brotherhood relation

Under this interpretation, *Brother*(*Richard*, *John*) is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

## Models for FOL: Lots!

Entailment in propositional logic can be computed by enumerating models

We **can** enumerate the FOL models for a given KB vocabulary:

For each number of domain elements  $n$  from 1 to  $\infty$

For each  $k$ -ary predicate  $P_k$  in the vocabulary

For each possible  $k$ -ary relation on  $n$  objects

For each constant symbol  $C$  in the vocabulary

For each choice of referent for  $C$  from  $n$  objects ...

Computing entailment by enumerating FOL models is not easy!

# Universal quantification

$\forall \langle variables \rangle \langle sentence \rangle$

Everyone at Berkeley is smart:

$\forall x \text{ } At(x, Berkeley) \Rightarrow Smart(x)$

$\forall x \text{ } P$  is true in a model  $m$  iff  $P$  is true with  $x$  being **each** possible object in the model

**Roughly** speaking, equivalent to the conjunction of instantiations of  $P$

$(At(KingJohn, Berkeley) \Rightarrow Smart(KingJohn))$   
 $\wedge (At(Richard, Berkeley) \Rightarrow Smart(Richard))$   
 $\wedge (At(Berkeley, Berkeley) \Rightarrow Smart(Berkeley))$   
 $\wedge \dots$

## A common mistake to avoid

Typically,  $\Rightarrow$  is the main connective with  $\forall$

Common mistake: using  $\wedge$  as the main connective with  $\forall$ :

$$\forall x \text{ } At(x, Berkeley) \wedge Smart(x)$$

means “Everyone is at Berkeley and everyone is smart”

## Existential quantification

$\exists \langle variables \rangle \langle sentence \rangle$

Someone at Stanford is smart:

$\exists x \text{ } At(x, Stanford) \wedge Smart(x)$

$\exists x \text{ } P$  is true in a model  $m$  iff  $P$  is true with  $x$  being **some** possible object in the model

**Roughly** speaking, equivalent to the **disjunction** of **instantiations** of  $P$

$$\begin{aligned} & (At(KingJohn, Stanford) \wedge Smart(KingJohn)) \\ \vee & (At(Richard, Stanford) \wedge Smart(Richard)) \\ \vee & (At(Stanford, Stanford) \wedge Smart(Stanford)) \\ \vee & \dots \end{aligned}$$

## Another common mistake to avoid

Typically,  $\wedge$  is the main connective with  $\exists$

Common mistake: using  $\Rightarrow$  as the main connective with  $\exists$ :

$$\exists x \text{ } At(x, Stanford) \Rightarrow Smart(x)$$

is true if there is anyone who is not at Stanford!



## Properties of quantifiers

$\forall x \forall y$  is the same as  $\forall y \forall x$  (why??)

$\exists x \exists y$  is the same as  $\exists y \exists x$  (why??)

$\exists x \forall y$  is **not** the same as  $\forall y \exists x$

$\exists x \forall y \text{ Loves}(x, y)$

“There is a person who loves everyone in the world”

$\forall y \exists x \text{ Loves}(x, y)$

“Everyone in the world is loved by at least one person”

**Quantifier duality:** each can be expressed using the other

$\forall x \text{ Likes}(x, \text{IceCream}) \quad \neg \exists x \neg \text{Likes}(x, \text{IceCream})$

$\exists x \text{ Likes}(x, \text{Broccoli}) \quad \neg \forall x \neg \text{Likes}(x, \text{Broccoli})$

## Fun with sentences

Brothers are siblings

## Fun with sentences

Brothers are siblings

$\forall x, y \text{ } Brother(x, y) \Rightarrow Sibling(x, y).$

“Sibling” is symmetric

## Fun with sentences

Brothers are siblings

$$\forall x, y \text{ Brother}(x, y) \Rightarrow \text{Sibling}(x, y).$$

“Sibling” is symmetric

$$\forall x, y \text{ Sibling}(x, y) \Leftrightarrow \text{Sibling}(y, x).$$

One's mother is one's female parent

## Fun with sentences

Brothers are siblings

$$\forall x, y \text{ Brother}(x, y) \Rightarrow \text{Sibling}(x, y).$$

“Sibling” is symmetric

$$\forall x, y \text{ Sibling}(x, y) \Leftrightarrow \text{Sibling}(y, x).$$

One's mother is one's female parent

$$\forall x, y \text{ Mother}(x, y) \Leftrightarrow (\text{Female}(x) \wedge \text{Parent}(x, y)).$$

A first cousin is a child of a parent's sibling

## Fun with sentences

Brothers are siblings

$$\forall x, y \text{ } Brother(x, y) \Rightarrow Sibling(x, y).$$

“Sibling” is symmetric

$$\forall x, y \text{ } Sibling(x, y) \Leftrightarrow Sibling(y, x).$$

One's mother is one's female parent

$$\forall x, y \text{ } Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y)).$$

A first cousin is a child of a parent's sibling

$$\forall x, y \text{ } FirstCousin(x, y) \Leftrightarrow \exists p, ps \text{ } Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)$$

# Equality

$term_1 = term_2$  is true under a given interpretation  
if and only if  $term_1$  and  $term_2$  refer to the same object

E.g.,  $1 = 2$  and  $\forall x \neg (Sqrt(x), Sqrt(x)) = x$  are satisfiable  
 $2 = 2$  is valid

E.g., definition of (full) *Sibling* in terms of *Parent*:

$$\forall x, y \text{ Sibling}(x, y) \Leftrightarrow [\neg(x = y) \wedge \exists m, f \neg(m = f) \wedge \\ \text{Parent}(m, x) \wedge \text{Parent}(f, x) \wedge \text{Parent}(m, y) \wedge \text{Parent}(f, y)]$$

## Interacting with FOL KBs

Suppose a wumpus-world agent is using an FOL KB  
and perceives a smell and a breeze (but no glitter) at  $t = 5$ :

$Tell(KB, Percept([Smell, Breeze, None], 5))$

$Ask(KB, \exists a \text{ Action}(a, 5))$

I.e., does  $KB$  entail any particular actions at  $t = 5$ ?

Answer:  $Yes, \{a/Shoot\} \leftarrow \text{substitution (binding list)}$

Given a sentence  $S$  and a substitution  $\sigma$ ,

$S\sigma$  denotes the result of plugging  $\sigma$  into  $S$ ; e.g.,

$S = Smarter(x, y)$

$\sigma = \{x/Hillary, y/Bill\}$

$S\sigma = Smarter(Hillary, Bill)$

$Ask(KB, S)$  returns some/all  $\sigma$  such that  $KB \models S\sigma$



## Knowledge base for the wumpus world

“Perception”

$\forall b, g, t \text{ Percept}([Smell, b, g], t) \Rightarrow Smelt(t)$

$\forall s, b, t \text{ Percept}([s, b, Glitter], t) \Rightarrow AtGold(t)$

Reflex:  $\forall t \text{ AtGold}(t) \Rightarrow \text{Action}(Grab, t)$

Reflex with internal state: do we have the gold already?

$\forall t \text{ AtGold}(t) \wedge \neg Holding(Gold, t) \Rightarrow \text{Action}(Grab, t)$

$Holding(Gold, t)$  cannot be observed

$\Rightarrow$  keeping track of change is essential

## Deducing hidden properties

Properties of locations:

$$\forall x, t \text{ } At(Agent, x, t) \wedge Smelt(t) \Rightarrow Smelly(x)$$

$$\forall x, t \text{ } At(Agent, x, t) \wedge Breeze(t) \Rightarrow Breezy(x)$$

Squares are breezy near a pit:

**Diagnostic** rule—infer cause from effect

$$\forall y \text{ } Breezy(y) \Rightarrow \exists x \text{ } Pit(x) \wedge Adjacent(x, y)$$

**Causal** rule—infer effect from cause

$$\forall x, y \text{ } Pit(x) \wedge Adjacent(x, y) \Rightarrow Breezy(y)$$

Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

**Definition** for the *Breezy* predicate:

$$\forall y \text{ } Breezy(y) \Leftrightarrow [\exists x \text{ } Pit(x) \wedge Adjacent(x, y)]$$

# Keeping track of change

Facts hold in **situations**, rather than eternally

E.g., *Holding(Gold, Now)* rather than just *Holding(Gold)*

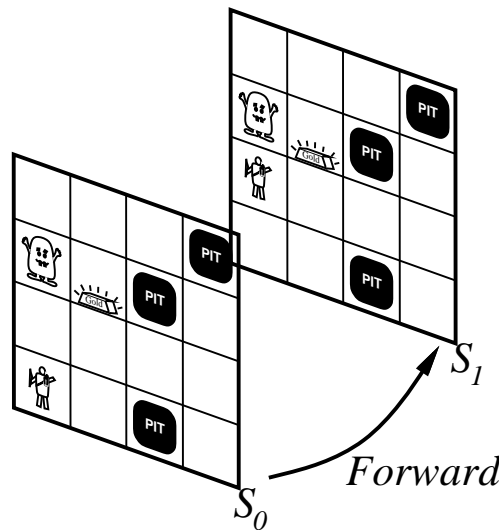
**Situation calculus** is one way to represent change in FOL:

Adds a situation argument to each non-eternal predicate

E.g., *Now* in *Holding(Gold, Now)* denotes a situation

Situations are connected by the *Result* function

*Result(a, s)* is the situation that results from doing *a* in *s*



## Describing actions I

“Effect” axiom—describe changes due to action

$$\forall s \text{ } AtGold(s) \Rightarrow Holding(Gold, Result(Grab, s))$$

“Frame” axiom—describe **non-changes** due to action

$$\forall s \text{ } HaveArrow(s) \Rightarrow HaveArrow(Result(Grab, s))$$

**Frame problem**: find an elegant way to handle non-change

- (a) representation—avoid frame axioms
- (b) inference—avoid repeated “copy-overs” to keep track of state

**Qualification problem**: true descriptions of real actions require endless caveats—what if gold is slippery or nailed down or . . .

**Ramification problem**: real actions have many secondary consequences—what about the dust on the gold, wear and tear on gloves, . . .

## Describing actions II

Successor-state axioms solve the representational frame problem

Each axiom is “about” a **predicate** (not an action per se):

$$\begin{aligned} P \text{ true afterwards} \quad \Leftrightarrow \quad & [\text{an action made } P \text{ true} \\ & \vee \quad P \text{ true already and no action made } P \text{ false}] \end{aligned}$$

For holding the gold:

$$\begin{aligned} \forall a, s \quad & Holding(Gold, Result(a, s)) \Leftrightarrow \\ & [(a = Grab \wedge AtGold(s)) \\ & \vee (Holding(Gold, s) \wedge a \neq Release)] \end{aligned}$$

## Making plans

Initial condition in KB:

$At(Agent, [1, 1], S_0)$

$At(Gold, [1, 2], S_0)$

Query:  $Ask(KB, \exists s \text{ Holding}(Gold, s))$

i.e., in what situation will I be holding the gold?

Answer:  $\{s / Result(Grab, Result(Forward, S_0))\}$

i.e., go forward and then grab the gold

This assumes that the agent is interested in plans starting at  $S_0$  and that  $S_0$  is the only situation described in the KB

## Making plans: A better way

Represent **plans** as action sequences  $[a_1, a_2, \dots, a_n]$

$PlanResult(p, s)$  is the result of executing  $p$  in  $s$

Then the query  $Ask(KB, \exists p \text{ Holding}(Gold, PlanResult(p, S_0)))$   
has the solution  $\{p/[Forward, Grab]\}$

Definition of  $PlanResult$  in terms of  $Result$ :

$$\forall s \text{ } PlanResult([], s) = s$$

$$\forall a, p, s \text{ } PlanResult([a|p], s) = PlanResult(p, Result(a, s))$$

**Planning systems** are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

# Summary

First-order logic:

- objects and relations are semantic primitives
- syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:

- conventions for describing actions and change in FOL
- can formulate planning as inference on a situation calculus KB