

Recurrence Relations

Class 13

Definitions

- you are very familiar with function definitions in math
- a function is defined with an algebraic rule

$$f(x) = x^2 - 3x + 2$$

- it can be translated directly into a C++ function

```
double f(double x)
{
    return x * x - 3 * x + 2;
}
```

Sequences

- there is another type of definition
- commonly used to define **sequences** of values
- the Fibonacci sequence can be **listed** as $\{1, 1, 2, 3, 5, \dots\}$
- and it can also be defined by a **rule**

$$f(n) = f(n-1) + f(n-2) \text{ given } f(0) = f(1) = 1$$

- this type of rule is called a **recurrence relation**
- with **initial conditions**

Recurrence Relations

- a **recurrence relation** is an equation or inequality
- defines an arbitrary element in a sequence in terms of one or more of its predecessors

Recurrence Relations

- a **recurrence relation** is an equation or inequality
- defines an arbitrary element in a sequence in terms of one or more of its predecessors
- a recursive algorithm **implements** a recurrence relation
- a recurrence relation **describes** a recursive algorithm

Recurrence to Recursion

- recurrence relations translate into code
- the initial conditions turn into **base cases**
- the code has **recursive** calls

```
unsigned fib(unsigned n)
{
    if (n == 0 || n == 1)
    {
        return 1;
    }
    return fib(n - 1) + fib(n - 2);
}
```

Solving Recurrence Relations

- to **solve** a recurrence relation means to give a formulation for an arbitrary element in a sequence in terms that does **not** use any other elements in the sequence
- a solution is also called a **closed form**
- there are many techniques for solving recurrence relations
- we will only look at one, as our interest is in using their results

Solving by Substitution

Let

$$T(n) = T(n-1) + n \text{ given } T(0) = 0$$

- off to the side, replace every occurrence of n with $n - 1$
- we can do this because n is arbitrary
- this substitution gives us

$$\begin{aligned} T(n-1) &= T(n-1-1) + (n-1) \\ &= T(n-2) + (n-1) \end{aligned}$$

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$$T(n) = T(n-2) + (n-1) + n$$

Solving by Substitution

- using the original formulation, off to the side substitute every occurrence of n by $n - 2$ to get

$$T(n - 2) = T(n - 3) + (n - 2)$$

- and use this expression for $T(n - 2)$ in the last expression of the previous slide

$$\begin{aligned} T(n) &= T(n - 2) + (n - 1) + n \\ &= T(n - 3) + (n - 2) + (n - 1) + n \end{aligned}$$

Solving by Substitution

- continuing the series, we have

$$\begin{aligned}T(n) &= T(n-1) + n \\&= T(n-2) + (n-1) + n \\&= T(n-3) + (n-2) + (n-1) + n \\&\vdots\end{aligned}$$

- how long can this process go on?

Solving by Substitution

- the series ends at the initial condition (base case) $T(0) = 0$

$$\begin{aligned}T(n) &= T(n-1) + n \\&= T(n-2) + (n-1) + n \\&= T(n-3) + (n-2) + (n-1) + n \\&\vdots \\&= T(n - (n-1)) + (n - (n-2)) + (n - (n-3)) + \cdots \\&\quad + (n-1) + n \\&= T(n-n) + (n - (n-1)) + (n - (n-2)) + (n - (n-3)) \\&\quad + \cdots + (n-1) + n \\&= 0 + 1 + 2 + \cdots + n \\&= \frac{n(n+1)}{2}\end{aligned}$$

Analysis

- thus we have the **closed form**

$$\begin{aligned} T(n) &= T(n-1) + n \text{ given } T(0) = 0 \\ &= \frac{n(n+1)}{2} \end{aligned}$$

- the **solution** of a recurrence relation is identical to the **analysis** of its matching recursive algorithm
- we analyze recursive algorithms by
 - writing the recurrence relation for the algorithm
 - solving that recurrence relation
- thus we have

$$T(n) \in \Theta(n^2)$$

Analyzing Recursive Functions

- unfortunately, many recurrence relations are hard to solve
- substitution **only** works when the terms differ in position by exactly one $n \rightarrow n - 1$
- however, most of the recurrence relations we deal with in this course do **not** have this form

The Master Theorem

- most of our recurrence relations instead have this form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

- where $f(n)$ is a polynomial of degree d e.g., $f(n) = kn^d$
- this cannot be solved by substitution
- we will use the Master Theorem for this form of recurrence relation

Example

use the Master Theorem to analyze the recursive algorithm whose behavior is expressed by

$$T(n) = 4T\left(\frac{n}{2}\right) + kn$$

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- $a = 4$
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- $d = 1$

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use the Master Theorem to analyze the recursive algorithm whose behavior is expressed by

$$T(n) = 4T\left(\frac{n}{2}\right) + kn$$

- $a = 4$
- $b = 2$
- $d = 1$

We must now ask the question:

$$a \stackrel{?}{<} b^d$$

$$4 \stackrel{?}{<} 2^1$$

$$4 \not< 2^1 \text{ No}$$

Example

- $a = 4$
- $b = 2$
- $d = 1$

Next ask:

$$a \stackrel{?}{=} b^d$$

$$4 \stackrel{?}{=} 2^1$$

$$4 \neq 2^1 \text{ No}$$

Example

- $a = 4$
- $b = 2$
- $d = 1$

Finally ask:

$$a \stackrel{?}{>} b^d$$

$$4 \stackrel{?}{>} 2^1$$

$$4 > 2^1 \text{ Yes!}$$

Therefore

$$\begin{aligned} T(n) &\in \Theta(n^{\log_2 4}) \\ &\in \Theta(n^2) \end{aligned}$$

Analysis

- note the Master Theorem does not actually **solve** the recurrence relation
- we did **not** arrive at a closed form (because we did not specify a base case)
- with the Master Theorem, we go directly from recurrence relation to analysis, without solving for a closed form
- therefore, we do not need to specify a base case

Binary Search

- binary search is a classic recursive algorithm
- a recursive algorithm consists of
 1. one or more checks for base case(s)
 2. some amount of local work
 3. one or more recursive calls
- basic operations
 - recursive calls themselves are not counted
 - return statements themselves are not counted
 - **generating** arguments for recursive calls is counted
 - **generating** a value to return is counted
 - **local work** is counted

Binary Search

2	3	5	11	17	23	29
---	---	---	----	----	----	----

1. if the range of elements is **empty**, return not-found sentinel

Binary Search

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 - 2.1 the very **middle** element

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4. else if the searched-for value is **smaller** than the middle element, repeat step 1 on the **left half**

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5. else repeat step 1 on the **right half**

Binary Search

look at code

Binary Search Analysis

base case determination

line 5: 1 operation

line 6: 3 operations

local work

line 8: 3 operations

lines 9 and 13: 2 operations

line 11 or 15: 1 operation (lines mutually exclusive)

total: 10 operations

Binary Search Analysis

- how many recursive calls?
- how big is the input for the recursive call?
- can the algorithm end early?

Binary Search Analysis

- **how many** recursive calls?
 - either line 9 or line 13, but never both
 - therefore **one** recursive call
- **how big** is the input for the recursive call?
- can the algorithm **end early**?

Binary Search Analysis

- **how many** recursive calls?
 - either line 9 or line 13, but never both
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- **how big** is the input for the recursive call?
 - the size of the range is half the size of the original
- can the algorithm **end early**?

Binary Search Analysis

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 - either line 9 or line 13, but never both
 - therefore **one** recursive call
- **how big** is the input for the recursive call?
 - the size of the range is half the size of the original
- can the algorithm **end early**?
 - yes, because of line 19 `return mid;`
 - **not** because of line 22
(special case, equivalent to zero input size)

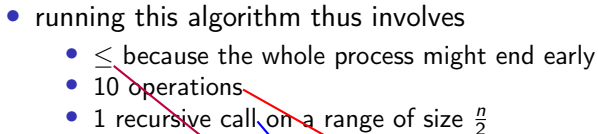
Binary Search Analysis

- running this algorithm thus involves
 - \leq because the whole process might end early
 - 10 operations
 - 1 recursive call on a range of size $\frac{n}{2}$

$$T(n) \leq aT\left(\frac{n}{b}\right) + kn^d$$

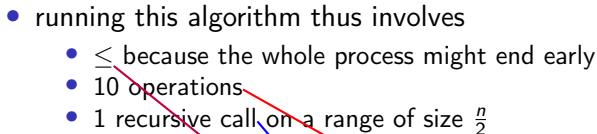
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$$T(n) \leq aT\left(\frac{n}{b}\right) + kn^d$$

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$$\begin{aligned}T(n) &\leq aT\left(\frac{n}{b}\right) + kn^d \\ &\leq 1T\left(\frac{n}{2}\right) + 10n^0\end{aligned}$$

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$$T(n) \in O(\lg n)$$

Lower Bound

- the previous slide resulted in a big-Oh result
- this is an **upper bound**
- now we need to consider the lower bound
- direct observation of the code shows that a **single** run of the algorithm **could** find the requested element and end the algorithm immediately
- thus the minimum number of basic operations is 9 (nothing to do in line 19)

$$T(n) \geq 9$$
$$\in \Omega(1)$$

- this is typical for search algorithms

Complete Analysis

- putting it all together, for recursive binary search, we have:
 - the input size is the size of the range of array elements
 - the algorithm can terminate early, so there are distinct best and worst cases
 - there are a constant 10 basic operations each time
 - the analysis is:

$$\begin{aligned} T(n) &\in O(\lg n) \\ &\in \Omega(1) \end{aligned}$$

Binary Search Considerations

- the analysis assumes the data are already in the vector, in order
- the analysis **only** treats the search function itself, not the entire program
- just to put n items into a vector is $\in \Theta(n)$
- sorting the n -item vector is $\in \Theta(n \lg n)$
- so the overall program:
 1. populate a vector with n elements
 2. sort the vector
 3. perform one binary search on the array
 4. report results
- is $\in O(n \lg n)$
- but once the vector is built, successive binary searches on that array would each be $\in O(\lg n)$

A Key Takeaway

- from a slide in Class 1:
- for our purposes, the simplest definition of a logarithm is
How many times can you divide a number n by 2
(using integer division)
before the result is 1 or 0?
- binary search works by dividing the size of the vector by two for each recursion
- this division is what causes the algorithm to have logarithmic analysis