Numerical Analysis

Lecture 4: The Solution of Linear Systems

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Outline

Augmented Matrix

Representing a Linear System

The linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

can be represented by the following matrix equation

$$A\mathbf{x} = \mathbf{b}$$

and further forming a new matrix $[A, \mathbf{b}]$, which is called the augmented matrix.

3 Operations to Simplify a Linear System

3 Operations

The linear system

$$\begin{aligned} E_1 : \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ E_2 : \quad & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ E_n : \quad & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{aligned}$$

- $(\lambda E_i) \to E_i$
- $E_i + (\lambda E_j) \to E_i$
- $E_i \leftrightarrow E_j$

Examples

Example 1:

$$
\begin{array}{rrrrrr}
E_1: & x_1 & +x_2 & & +3x_4 & = 4 \\
E_2: & 2x_1 & +x_2 & -x_3 & +x_4 & = 1 \\
E_3: & 3x_1 & -x_2 & -x_3 & +2x_4 & = -3 \\
E_4: & -x_1 & +2x_2 & +3x_3 & -x_4 & = 4
\end{array}
$$

Examples

Example 1:

$$\begin{array}{rrrrrr}
E_1: & x_1 & +x_2 & & +3x_4 & = 4 \\
E_2: & 2x_1 & +x_2 & -x_3 & +x_4 & = 1 \\
E_3: & 3x_1 & -x_2 & -x_3 & +2x_4 & = -3 \\
E_4: & -x_1 & +2x_2 & +3x_3 & -x_4 & = 4
\end{array}$$

$$\Longrightarrow \begin{array}{rrrrrr}
E_1: & x_1 & +x_2 & & +3x_4 & = 4 \\
E_2: & & -x_2 & -x_3 & -5x_4 & = -7 \\
E_3: & & & 3x_3 & +13x_4 & = 13 \\
E_4: & & & & -13x_4 & = -13.
\end{array}$$

Examples

Example 1:

$$
\begin{array}{lrrrrr}
E_1: & x_1 & +x_2 & & +3x_4 & = 4 \\
E_2: & 2x_1 & +x_2 & -x_3 & +x_4 & = 1 \\
E_3: & 3x_1 & -x_2 & -x_3 & +2x_4 & = -3 \\
E_4: & -x_1 & +2x_2 & +3x_3 & -x_4 & = 4
\end{array}
$$

$$
\Longrightarrow \quad
\begin{array}{lrrrrr}
E_1: & x_1 & +x_2 & & +3x_4 & = 4 \\
E_2: & & -x_2 & -x_3 & -5x_4 & = -7 \\
E_3: & & & 3x_3 & +13x_4 & = 13 \\
E_4: & & & & -13x_4 & = -13.
\end{array}
$$

The simplified system is now in triangular form and can be solved for the unknowns by a backward-substitution process.

Gaussian Elimination

Gaussian Elimination

The general Gaussian elimination procedure applied to the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

is handled in the following manner.

Gaussian Elimination

Gaussian Elimination Procedure

- Step 1: Form the augmented matrix $\tilde{A} = [A, \mathbf{b}]$.

Gaussian Elimination

Gaussian Elimination Procedure

- Step 1: Form the augmented matrix $\tilde{A} = [A, \mathbf{b}]$.
- Step 2: Provided $a_{11} \neq 0$, we perform the operations corresponding to

$$\left(E_i - \frac{a_{i1}}{a_{11}} E_1\right) \to (E_i) \quad \text{for each } i = 2, 3, \cdots, n$$

  to eliminate the coefficient of x_1 in each of these rows.

Gaussian Elimination

Gaussian Elimination Procedure

- Step 1: Form the augmented matrix $\tilde{A} = [A, \mathbf{b}]$.
- Step 2: Provided $a_{11} \neq 0$, we perform the operations corresponding to
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to eliminate the coefficient of x_1 in each of these rows.
- Step 3: We follow a sequential procedure for $k = 2, 3, \cdots, n-1$ and perform the operation
$$\left(E_i - \frac{a_{ik}}{a_{kk}} E_k\right) \to (E_i) \quad \text{for each } i = k+1, k+2, \cdots, n$$
provided $a_{kk} \neq 0$.

Gaussian Elimination

Gaussian Elimination Procedure

- Step 4: The resulting matrix has the form:

$$\tilde{\tilde{A}} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{nn} & a_{n,n+1} \end{array}\right]$$

  where, except in the first row, the values of a_{ij} are not expected to agree with those in the original matrix $\tilde{A}$.

Gaussian Elimination

Gaussian Elimination Procedure

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  where, except in the first row, the values of a_{ij} are not expected to agree with those in the original matrix $\tilde{A}$.

- Step 5: Use backward substitution to solve the equations.

$$x_n = \frac{a_{n,n+1}}{a_{nn}}$$

$$x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$$

  for each $i = n-1, n-2, ..., 2, 1$.

Gaussian Elimination

A More Precise Description

Gaussian elimination procedure is described more precisely, although more intricately, by forming a sequence of augmented matrices $\tilde{A}^{(1)}, \tilde{A}^{(2)}, \cdots, \tilde{A}^{(n)}$, where $\tilde{A}^{(1)}$ is the matrix $\tilde{A}$ given earlier and $\tilde{A}^{(k)}$, for each $k = 2, 3, \cdots, n$, has entries $a_{ij}^{(k)}$, where:

$$a_{ij}^{(k)} = \begin{cases} a_{ij}^{(k-1)}, & \text{when } i = 1, 2, \cdots, k-1 \text{ and } j = 1, 2, \cdots, n+1 \\ 0, & \text{when } i = k, k+1, \cdots, n \text{ and } j = 1, 2, \cdots, k-1 \\ a_{ij}^{(k-1)} - \dfrac{a_{i,k-1}^{(k-1)}}{a_{k-1,k-1}^{(k-1)}} a_{k-1,j}^{(k-1)}, & \text{when } i = k, k+1, \cdots, n \text{ and } j = k, k+1, \cdots, n+1 \end{cases}$$

A More Precise Description

Thus,

$$\tilde{A}^{(k)} = \left[\begin{array}{ccccccc|c}
a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1,k-1}^{(1)} & a_{1,k}^{(1)} & \cdots & a_{1n}^{(1)} & a_{1,n+1}^{(1)} \\
0 & a_{22}^{(2)} & \cdots & a_{2,k-1}^{(2)} & a_{2,k}^{(2)} & \cdots & a_{2n}^{(2)} & a_{2,n+1}^{(2)} \\
\vdots & \ddots & & \vdots & \vdots & & \vdots & \vdots \\
\vdots & & & a_{k-1,k-1}^{(k-1)} & a_{k-1,k}^{(k-1)} & \cdots & a_{k-1,n}^{(k-1)} & a_{k-1,n+1}^{(k-1)} \\
\vdots & & & 0 & a_{k,k}^{(k)} & \cdots & a_{k,n}^{(k)} & a_{k,n+1}^{(k)} \\
\vdots & & & \vdots & \vdots & & \vdots & \vdots \\
\vdots & & & 0 & a_{n,k}^{(k)} & \cdots & a_{n,n}^{(k)} & a_{n,n+1}^{(k)}
\end{array}\right]$$

represents the equivalent linear system for which the variable $\mathbf{x}_{k-1}$ has just been eliminated from equations $E_k, E_{k+1}, \cdots, E_n$.

Gaussian Elimination

Remarks

The procedure will fail if one of the elements $a_{kk}^{(k)}$ is zero because

- either the following step can not be formed

$$\left(E_i - \frac{a_{i,k}}{a_{kk}^{(k)}} E_k\right) \rightarrow (E_i)$$

- or the backward substitution cannot be accomplished in the case $a_{nn}^{(n)}$.

The system may still have a solution, but the technique for finding it must be altered.

Gaussian Elimination

Alterations

- The kth column of $\tilde{A}^{(k-1)}$ from the kth row to the nth row is searched for the first nonzero entry.
- If $a_{pk}^{(k)} \neq 0$ for some p, with $k+1 \leq p \leq n$, then the operation $(E_k) \leftrightarrow (E_p)$ is performed to obtain $\tilde{A}^{(k-1)'}$.
- The procedure can then be continued to form $\tilde{A}^{(k)}$, and so on.
- If $a_{pk}^{(k)} = 0$ for each p, it can be shown that the linear system does not have a unique solution and the procedure stops.
- Finally, if $a_{nn}^{(n)} = 0$, the linear system does not have a unique solution, and again the procedure stops.

Gaussian Elimination Algorithm

INPUT number of unknowns and equations n; augmented matrix $A = [a_{ij}]$, where $1 \leq i \leq n$ and $1 \leq j \leq n+1$.

OUTPUT solution $x_1, x_2, \ldots, x_n$ or message that the linear system has no unique solution.

Step 1 For $i = 1, \ldots, n-1$ do Steps 2–4. (*Elimination process.*)

    Step 2 Let p be the smallest integer with $i \leq p \leq n$ and $a_{pi} \neq 0$.
    If no integer p can be found
      then OUTPUT ('no unique solution exists');
        STOP.

    Step 3 If $p \neq i$ then perform $(E_p) \leftrightarrow (E_i)$.

    Step 4 For $j = i+1, \ldots, n$ do Steps 5 and 6.

        Step 5 Set $m_{ji} = a_{ji}/a_{ii}$.

        Step 6 Perform $(E_j - m_{ji}E_i) \rightarrow (E_j)$;

Step 7 If $a_{nn} = 0$ then OUTPUT ('no unique solution exists');
        STOP.

Step 8 Set $x_n = a_{n,n+1}/a_{nn}$. (*Start backward substitution.*)

Step 9 For $i = n-1, \ldots, 1$ set $x_i = \left[a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j\right] \Big/ a_{ii}$.

Step 10 OUTPUT $(x_1, \ldots, x_n)$; (*Procedure completed successfully.*)
STOP.

Pivoting Strategies

Definition: Pivot

The number a_{kk} in the coefficient matrix A that is used to eliminate a_{pk}, where $p = k+1, k+2, \ldots, n$ is called the kth pivotal element, and the kth row is called the pivot row.

Pivoting Strategy

- Pivoting to avoid $a_{kk}^{(k)} = 0$.
- Pivoting to reduce error.

Pivoting Strategies

Pivoting to Reduce Round-off Error

- If $a_{kk}^{(k)}$ is small in magnitude compared to $a_{jk}^{(k)}$, then the magnitude of the multiplier

$$m_{jk} = \frac{a_{jk}^{(k)}}{a_{kk}^{(k)}}$$

  will be much larger than 1.
- Round-off error introduced in the computation of one of the terms $a_{kl}^{(k)}$ is multiplied by m_{jk} when computing $a_{kl}^{(k+1)}$, which compounds the original error.

Pivoting Strategies

Pivoting to Reduce Round-off Error

- Also, when performing the backward substitution for

$$x_k = \frac{a_{k,n+1}^{(k)} - \sum_{j=k+1}^{n} a_{kj}^{(k)} x_j}{a_{kk}^{(k)}}$$

with a small value of $a_{kk}^{(k)}$, any error in the numerator can be dramatically increased because of the division by $a_{kk}^{(k)}$.

Pivoting Strategies

Example

Apply Gaussian elimination to the system

$$\begin{aligned} E_1: \quad 0.003000x_1 + 59.14x_2 &= 59.17 \\ E_2: \quad 5.291x_1 - 6.130x_2 &= 46.78 \end{aligned}$$

using four-digit arithmetic with rounding, and compare the results to the exact solution $x_1 = 10.00$ and $x_2 = 1.000$.

Pivoting Strategies

Example: Solution 1/4

- The first pivot element, $a_{11}^{(1)} = 0.003000$, is small, and its associated multiplier,

$$m_{21} = \frac{5.291}{0.003000} = 1763.6\bar{6}$$

  rounds to the large number 1764.

- Performing $(E_2 - m_{21}E_1) \to (E_2)$ and the appropriate rounding gives the system

$$\begin{aligned} 0.003000x_1 + 59.14x_2 &= 59.17 \\ -104300x_2 &\approx -104400 \end{aligned}$$

Pivoting Strategies

Example: Solution 2/4

We obtained

$$\begin{aligned} 0.003000x_1 + 59.14x_2 &= 59.17 \\ -104300x_2 &\approx -104400 \end{aligned}$$

instead of the exact system, which is

$$\begin{aligned} 0.003000x_1 + 59.14x_2 &= 59.17 \\ -104309.37\overline{6}x_2 &= -104309.37\overline{6} \end{aligned}$$

The disparity in the magnitudes of $m_{21}a_{13}$ and a_{23} has introduced round-off error, but the round-off error has not yet been propagated.

Pivoting Strategies

Example: Solution 3/4

Backward substitution yields

$$x_2 \approx 1.001$$

which is a close approximation to the actual value, $x_2 = 1.000$. However, because of the small pivot $a_{11} = 0.003000$,

$$x_1 \approx \frac{59.17 - (59.14)(1.001)}{0.003000} = -10.00$$

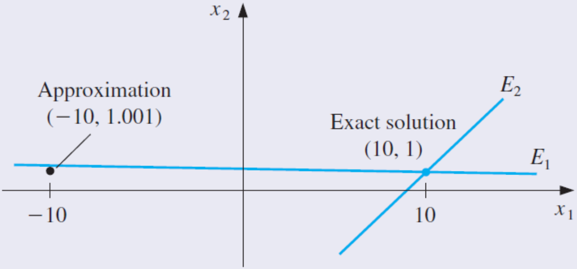
contains the small error of 0.001 multiplied by

$$\frac{59.14}{0.003000} \approx 20000$$

This ruins the approximation to the actual value $x_1 = 10.00$.

Example: Solution 4/4

This is clearly a contrived example and the graph shows why the error can so easily occur.



For larger systems it is much more difficult to predict in advance when devastating round-off error might occur.

Gaussian Elimination with Partial Pivoting

Meeting a Small Pivot Element

- The last example shows how difficulties can arise when the pivot element $a_{kk}^{(k)}$ is small relative to the entries $a_{ij}^{(k)}$, for $k \leq i \leq n$ and $k \leq j \leq n$.
- To avoid this problem, pivoting is performed by selecting an element $a_{pq}^{(k)}$ with a larger magnitude as the pivot, and interchanging the kth and pth rows.
- This can be followed by the interchange of the kth and qth columns if necessary.

Gaussian Elimination with Partial Pivoting

The Partial Pivoting Strategy

- The simplest strategy is to select an element in the same column that is below the diagonal and has the largest absolute value;
- Specifically, we determine the smallest $p \geq k$ such that

$$|a_{pk}^{(k)}| = \max_{k \leq i \leq n} |a_{ik}^{(k)}|$$

and perform $(E_k) \leftrightarrow (E_p)$.

Gaussian Elimination with Partial Pivoting

Example

Apply Gaussian elimination to the system

$$\begin{aligned} E_1 &: \quad 0.003000x_1 + 59.14x_2 = 59.17 \\ E_2 &: \quad 5.291x_1 - 6.130x_2 = 46.78 \end{aligned}$$

using four-digit arithmetic with rounding, and compare the results to the exact solution $x_1 = 10.00$ and $x_2 = 1.000$.

Gaussian Elimination with Partial Pivoting

Example: Solution 1/2

The partial-pivoting procedure first requires finding

$$max\left\{|a_{11}^{(1)}|, |a_{21}^{(1)}|\right\} = max\,\{|0.003000|, |5.291|\} = |5.291| = |a_{21}^{(1)}|$$

This requires that the operation $(E_2) \to (E_1)$ be performed to produce the equivalent system

$$\begin{aligned} E_1 : \quad & 5.291x_1 - 6.130x_2 = 46.78, \\ E_2 : \quad & 0.003000x_1 + 59.14x_2 = 59.17 \end{aligned}$$

Gaussian Elimination with Partial Pivoting

Example: Solution 2/2

The multiplier for this system is

$$m_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} = 0.0005670$$

and the operation $(E_2 - m_{21}E_1) \to (E_2)$ reduces the system to

$$\begin{aligned} 5.291x_1 - 6.130x_2 &= 46.78, \\ 59.14x_2 &\approx 59.14 \end{aligned}$$

The 4-digit answers resulting from the backward substitution are the correct values.

$$x_1 = 10.00 \quad \text{and} \quad x_2 = 1.000$$

Gaussian Elimination with Partial Pivoting

Can Partial Pivoting Fail?

- Each multiplier m_{ji} in the partial pivoting algorithm has magnitude less than or equal to 1.
- Although this strategy is sufficient for many linear systems, situations do arise when it is inadequate.

Gaussian Elimination with Partial Pivoting

Example

The linear system

$$\begin{aligned} E_1: &\quad 30.00x_1 + 591400x_2 = 591700 \\ E_2: &\quad 5.291x_1 - 6.130x_2 = 46.78 \end{aligned}$$

is the same as that in the two previous examples except that all the entries in the first equation have been multiplied by 10^4.

The partial pivoting procedure described in the algorithm with 4-digit arithmetic leads to the same incorrect results as obtained in the first example (Gaussian elimination without pivoting).

Gaussian Elimination with Partial Pivoting

Example: Solution

The maximal value in the first column is 30.00, and the multiplier,

$$m_{21} = \frac{5.291}{30.00} = 0.1764$$

leads to the system

$$\begin{aligned} 30.00x_1 + 591400x_2 &= 591700 \\ -104300x_2 &\approx -104400 \end{aligned}$$

which has the same inaccurate solutions as in the first example: $x_2 \approx 1.001$ and $x_1 \approx -10.00$.

Pivoting Strategies

Scaled Pivoting Strategies

- The effect of scaling is to ensure that the largest element in each row has a relative magnitude of 1 before the comparison for row interchange is performed.

Matrix Inversion

A Method to Compute the Matrix Inverse

$$AB = I \Rightarrow [A\ I]$$

Determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Gaussian Elimination Algorithm

INPUT number of unknowns and equations n; augmented matrix $A = [a_{ij}]$, where $1 \le i \le n$ and $1 \le j \le n+1$.

OUTPUT solution $x_1, x_2, \ldots, x_n$ or message that the linear system has no unique solution.

Step 1 For $i = 1, \ldots, n-1$ do Steps 2–4. (*Elimination process.*) O(n)

Step 2 Let p be the smallest integer with $i \le p \le n$ and $a_{pi} \ne 0$.
If no integer p can be found
then OUTPUT ('no unique solution exists');
STOP.

Step 3 If $p \ne i$ then perform $(E_p) \leftrightarrow (E_i)$.

Step 4 For $j = i+1, \ldots, n$ do Steps 5 and 6. O(n)

Step 5 Set $m_{ji} = a_{ji}/a_{ii}$.

Step 6 Perform $(E_j - m_{ji}E_i) \to (E_j)$; O(n)

$O(n^3)$

Step 7 If $a_{nn} = 0$ then OUTPUT ('no unique solution exists');
STOP.

Step 8 Set $x_n = a_{n,n+1}/a_{nn}$. (*Start backward substitution.*)

Step 9 For $i = n-1, \ldots, 1$ set $x_i = \left[a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j\right] \Big/ a_{ii}$. O(n) O(n)

$O(n^2)$

Step 10 OUTPUT $(x_1, \ldots, x_n)$; (*Procedure completed successfully.*)

Outline

LU Factorization

LU Factorization

- $A\mathbf{x} = \mathbf{b}$
- $A = LU$
- $L\mathbf{y} = \mathbf{b}$
- $U\mathbf{x} = \mathbf{y}$

LU Factorization

LU Factorization

- $A\mathbf{x} = \mathbf{b}$
- $A = LU$
- $L\mathbf{y} = \mathbf{b}$
- $U\mathbf{x} = \mathbf{y}$

Solving a linear system $A\mathbf{x} = \mathbf{b}$ in factored form means that the number of operations needed is reduced from $O(n^3)$ to $O(n^2)$.

LU Factorization

Constructing L & U

- First, suppose that Gaussian elimination can be performed on the system $A\mathbf{x} = \mathbf{b}$ without row interchanges.
- With the notation used earlier, this is equivalent to having nonzero pivot elements $a_{ii}^{(i)}$, for each $i = 1, 2, \cdots, n$.
- The first step in the Gaussian elimination process consists of performing, for each $j = 2, 3, \cdots, n$, the operations

$$(E_j - m_{j,1}E_1) \to (E_j), \quad \text{where} \quad m_{j,1} = \frac{a_{j1}^{(1)}}{a_{11}^{(1)}}.$$

- These operations transform the system into one in which all the entries in the first column below the diagonal are zero.

LU Factorization

$$(E_j - m_{j,1}E_1) \rightarrow (E_j), \quad \text{where} \quad m_{j,1} = \frac{a_{j1}^{(1)}}{a_{11}^{(1)}}.$$

Constructing L & U (cont'd)

It is simultaneously accomplished by multiplying the original matrix A on the left by the matrix

$$M^{(1)} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -m_{21} & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -m_{n1} & 0 & \cdots & 0 & 1 \end{bmatrix}$$

This is called the first Gaussian transformation matrix.

LU Factorization

Constructing L & U (cont'd)

- $$A^{(2)}\mathbf{x} = M^{(1)}A\mathbf{x} = M^{(1)}\mathbf{b} = \mathbf{b}^{(2)}$$

LU Factorization

Constructing L & U (cont'd)

- $$A^{(2)}\mathbf{x} = M^{(1)}A\mathbf{x} = M^{(1)}\mathbf{b} = \mathbf{b}^{(2)}$$

- $$\begin{aligned} A^{(k+1)}\mathbf{x} &= M^{(k)}A^{(k)}\mathbf{x} \\ &= M^{(k)}\cdots M^{(1)}A\mathbf{x} \\ &= M^{(k)}\mathbf{b}^{(k)} \\ &= \mathbf{b}^{(k+1)} \\ &= M^{(k)}\cdots M^{(1)}\mathbf{b} \end{aligned}$$

LU Factorization

Constructing L & U (cont'd)

The process ends with the formation of $A^{(n)}\mathbf{x} = \mathbf{b}^{(n)}$, where $A^{(n)}$ is the upper triangular matrix

$$A^{(n)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\ 0 & \cdots & \cdots & 0 & a_{n,n}^{(n)} \end{bmatrix}$$

given by

$$A^{(n)} = M^{(n-1)}M^{(n-2)} \cdots M^{(1)}A = U$$

LU Factorization

Constructing L & U (cont'd)

- To determine the complementary lower triangular matrix L, first recall the multiplication of $A^{(k)}\mathbf{x} = \mathbf{b}^{(k)}$ by the Gaussian transformation of $M^{(k)}$ used to obtain:

$$A^{(k+1)}\mathbf{x} = M^{(k)}A^{(k)}\mathbf{x} = M^{(k)}\mathbf{b}^{(k)} = \mathbf{b}^{(k+1)}$$

where $M^{(k)}$ generates the row operations

$$(E_j - m_{j,k}E_k) \to (E_j), \quad \text{for } j = k+1, \cdots, n.$$

LU Factorization

Constructing L & U (cont'd)

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$$A^{(k+1)}\mathbf{x} = M^{(k)}A^{(k)}\mathbf{x} = M^{(k)}\mathbf{b}^{(k)} = \mathbf{b}^{(k+1)}$$

where $M^{(k)}$ generates the row operations

$$(E_j - m_{j,k}E_k) \to (E_j), \quad \text{for } j = k+1, \cdots, n.$$

Reverse

$$\left[M^{(k)}\right]^{-1} A^{(k+1)}\mathbf{x} = A^{(k)}\mathbf{x},$$

which performs the operations $(E_j + m_{j,k}E_k) \to (E_j)$.

LU Factorization

Constructing L & U (cont'd)

$$L^{(k)} = \left[M^{(k)}\right]^{-1} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & 0 & \ddots & \ddots & & & \vdots \\ \vdots & & \vdots & m_{k+1,k} & \ddots & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 0 & m_{n,k} & 0 & \cdots & 0 & 1 \end{bmatrix}$$

LU Factorization

Constructing L & U (cont'd)

The lower-triangular matrix L in the factorization of A, then, is the product of the matrices $L^{(k)}$:

$$L = L^{(1)}L^{(2)}\cdots L^{(n-1)} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ m_{21} & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ m_{n1} & \cdots & \cdots & m_{n,n-1} & 1 \end{bmatrix}$$

since the product of L with the upper-triangular matrix $U = M^{(n-1)}\cdots M^{(2)}M^{(1)}A$ gives

LU Factorization

Constructing L & U (cont'd)

$$
\begin{aligned}
LU &= L^{(1)}L^{(2)}\cdots L^{(n-2)}L^{(n-1)} \\
&\qquad M^{(n-1)}M^{(n-2)}\cdots M^{(2)}M^{(1)}A \\
&= \left[M^{(1)}\right]^{-1}\left[M^{(2)}\right]^{-1}\cdots\left[M^{(n-2)}\right]^{-1}\left[M^{(n-1)}\right]^{-1} \\
&\qquad M^{(n-1)}M^{(n-2)}\cdots M^{(2)}M^{(1)}A \\
&= A
\end{aligned}
$$

LU Factorization

Theorem

If Gaussian elimination can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ without row interchanges, then the matrix A can be factored into the product of a lower-triangular matrix L and an upper-triangular matrix U, that is, $A = LU$, where $m_{ji} = a_{ji}^{(i)}/a_{ii}^{(i)}$,

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\ 0 & \cdots & 0 & a_{nn}^{(n)} \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{bmatrix}$$

Example

Example 2 **(a)** Determine the LU factorization for matrix A in the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}.$$

(b) Then use the factorization to solve the system

$$\begin{aligned} x_1 + x_2 + 3x_4 &= 8, \\ 2x_1 + x_2 - x_3 + x_4 &= 7, \\ 3x_1 - x_2 - x_3 + 2x_4 &= 14, \\ -x_1 + 2x_2 + 3x_3 - x_4 &= -7. \end{aligned}$$

LU factorization

Example

Solution **(a)** The original system was considered in Section 6.1, where we saw that the sequence of operations $(E_2 - 2E_1) \to (E_2)$, $(E_3 - 3E_1) \to (E_3)$, $(E_4 - (-1)E_1) \to (E_4)$, $(E_3 - 4E_2) \to (E_3)$, $(E_4 - (-3)E_2) \to (E_4)$ converts the system to the triangular system

$$\begin{aligned} x_1 + x_2 \qquad\quad + \;\; 3x_4 &= \;\;\; 4, \\ -x_2 - \;\; x_3 - \;\; 5x_4 &= -7, \\ 3x_3 + 13x_4 &= \;\; 13, \\ -13x_4 &= -13. \end{aligned}$$

The multipliers m_{ij} and the upper triangular matrix produce the factorization

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} = LU.$$

Example

(b) To solve

$$A\mathbf{x} = LU\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix},$$

we first introduce the substitution $\mathbf{y} = U\mathbf{x}$. Then $\mathbf{b} = L(U\mathbf{x}) = L\mathbf{y}$. That is,

$$L\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix}.$$

This system is solved for $\mathbf{y}$ by a simple forward-substitution process:

$$\begin{aligned} y_1 &= 8; & & \\ 2y_1 + y_2 &= 7, & \text{so } y_2 &= 7 - 2y_1 = -9; \\ 3y_1 + 4y_2 + y_3 &= 14, & \text{so } y_3 &= 14 - 3y_1 - 4y_2 = 26; \\ -y_1 - 3y_2 + y_4 &= -7, & \text{so } y_4 &= -7 + y_1 + 3y_2 = -26. \end{aligned}$$

LU factorization

Example

We then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, the solution of the original system; that is,

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 26 \\ -26 \end{bmatrix}.$$

Using backward substitution we obtain $x_4 = 2$, $x_3 = 0$, $x_2 = -1$, $x_1 = 3$.

Permutation Matrices

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An $n \times n$ permutation matrix $P = [p_{ij}]$ is a matrix obtained by rearranging the rows of I_n, the identity matrix.

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Example

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Diagonally Dominant Matrices

Definition (Diagonally Dominant Matrices)

The $n \times n$ matrix A is said to be diagonally dominant when

$$|a_{ii}| \geq \sum_{j=1, j \neq i}^{n} |a_{ij}|$$

holds for each $i = 1, 2, \cdots, n$.

Definition (Strictly Diagonally Dominant)

A diagonally dominant matrix is said to be strictly diagonally dominant when

$$|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}|$$

holds for each $i = 1, 2, \cdots, n$.

Diagonally Dominant Matrices

Theorem

A strictly diagonally dominant matrix A is nonsingular. Moreover, in this case, Gaussian elimination can be performed on any linear system of the form $A\mathbf{x} = \mathbf{b}$ *to obtain its unique solution without row or column interchanges, and the computations will be stable with respect to the growth of round-off errors.*

Positive Definite Matrices

Definition (Positive Definite)

A matrix A is positive definite if it is symmetric and if $\mathbf{x}^\top A\mathbf{x} > 0$ for every n-dimensional vector $\mathbf{x} \neq 0$.

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Theorem

If A is an $n \times n$ positive definite matrix, then

1. *A has an inverse;*
2. *$a_{ii} > 0$, for each $i = 1, 2, \cdots, n$;*
3. $\max_{1\leq k,j\leq n} |a_{kj}| \leq \max_{1\leq i\leq n} |a_{ii}|$
4. *$(a_{ij})^2 < a_{ii}a_{jj}$, for each $i \neq j$.*

Positive Definite Matrices

Theorem

The symmetric matrix A is positive definite iff Gaussian elimination without row interchanges can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ *with all pivot elements positive. Moreover, in this case, the computations are stable with respect to the growth of round-off errors.*

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Corollary

The matrix A is positive definite iff A can be factored in the form $LDL^\top$*, where L is lower triangular with* 1*s on its diagonal, and D is a diagonal matrix with positive diagonal entries.*

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The matrix A is positive definite iff A can be factored in the form $LDL^\top$, *where L is lower triangular with 1s on its diagonal, and D is a diagonal matrix with positive diagonal entries.*

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The matrix A is positive definite iff A can be factored in the form $LL^\top$, *where L is lower triangular with nonzero diagonal entries.*

Assignments

- Reading Assignment: Chap 6