# Numerical Analysis

## Lecture 7: Interpolation & Polynomial Approximation

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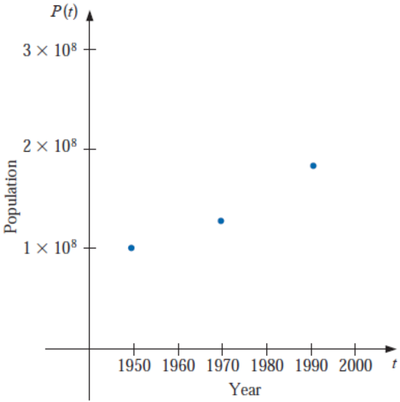
# Outline

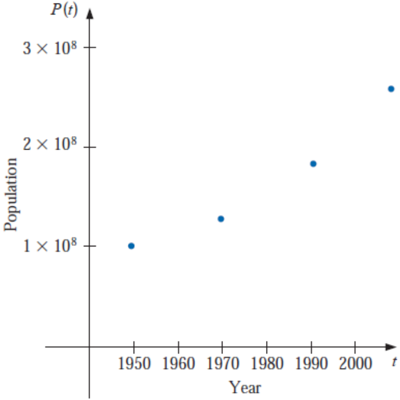
# Motivation

## Example: Population Census



# Motivation

## Example: Population Census



# Algebraic Polynomials

## Algebraic Polynomials

The algebraic polynomials are the set of functions of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $n$ is a nonnegative integer and $a_0, \cdots, a_n$ are real constants.

## Benefits of Algebraic Polynomials

- They uniformly approximate continuous functions.
- The derivative and indefinite integral of a polynomial are easy to determine and are also polynomials.

# Weierstrass Approximation Theorem

## Weierstrass Approximation Theorem

Suppose that $f$ is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial $P(x)$, with the property that

$$|f(x) - P(x)| < \epsilon, \quad \text{for all } x \text{ in } [a, b].$$

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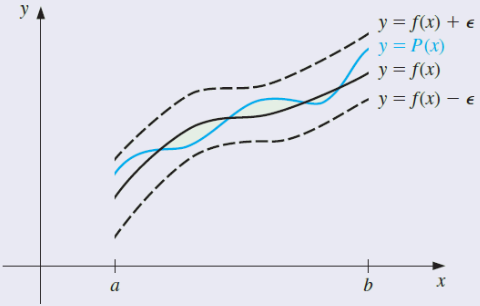


# Outline

# Taylor Polynomials

## Taylor's Theorem

Suppose $f \in C^n[a,b]$, that $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between $x_0$ and $x$ with

$$f(x) = P_n(x) + R_n(x),$$

where

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$$

Here, $P_n(x)$ is called the $n$th Taylor polynomial for $f$ about $x_0$, and $R_n(x)$ is called the remainder term (or truncation error) associated with $P_n(x)$.

# Taylor Polynomials

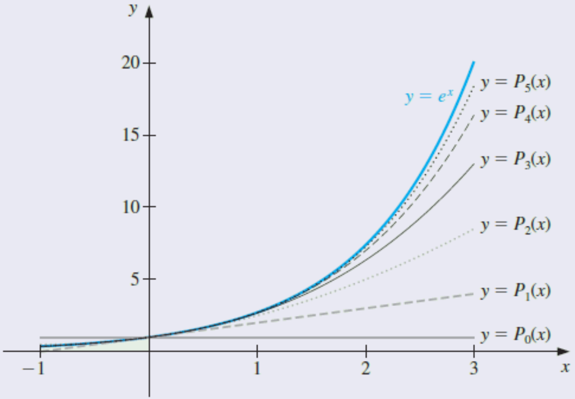
## Example1: $f(x) = e^x$

Calculate the first six Taylor polynomials about $x_0 = 0$ for $f(x) = e^x$.

$$P_0 x = 1$$
$$P_1(x) = 1 + x$$
$$P_2(x) = 1 + x + \frac{x^2}{2}$$
$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$
$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$
$$P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

# Taylor Polynomials

## Example1: $f(x) = e^x$




- Notice that even for the higher-degree polynomials, the error becomes progressively worse as we move away from zero.

# Taylor Polynomials

## Example2: $f(x) = 1/x$

Use Taylor polynomials of various degrees for $f(x) = 1/x$ expanded about $x_0 = 1$ to approximate $f(3) = 1/3$.

# Taylor Polynomials

## Example2: $f(x) = 1/x$

Use Taylor polynomials of various degrees for $f(x) = 1/x$ expanded about $x_0 = 1$ to approximate $f(3) = 1/3$.

The Taylor polynomials are

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{n} (-1)^k (x-1)^k.$$

# Taylor Polynomials

## Example2: $f(x) = 1/x$

Use Taylor polynomials of various degrees for $f(x) = 1/x$ expanded about $x_0 = 1$ to approximate $f(3) = 1/3$.

The Taylor polynomials are

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{n} (-1)^k (x-1)^k.$$

To approximate $f(3) = 1/3$ by $P_n(3)$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $P_n(3)$ | 1 | -1 | 3 | -5 | 11 | -21 | 43 | -85 |

# Taylor Polynomials

## Remarks

- The Taylor polynomials agree as closely as possible with a given function at a specific point, but they concentrate their accuracy near that point.
- However, a good interpolation polynomial needs to provide a relatively accurate approximation over an entire interval, and Taylor polynomials do not generally do this.
- The primary use of Taylor polynomials in numerical analysis is not for approximation purposes, but for the derivation of numerical techniques and error estimation.

# Outline

# Lagrange Interpolating Polynomials

## Polynomial Interpolation

Using a polynomial for approximation within the interval given by the endpoints is called polynomial interpolation.

## Lagrange Interpolating Polynomial

Define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

The linear Lagrange Interpolating Polynomial through $(x_0, y_0)$ and $(x_1, y_1)$ is

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1)$$

# Lagrange Interpolating Polynomial

## Example: Linear Interpolation

Determine the linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

# Lagrange Interpolating Polynomial

## Example: Linear Interpolation

Determine the linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

## Solution:

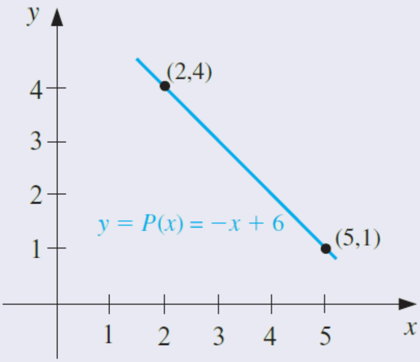
In this case we have

$$L_0(x) = \frac{x-5}{2-5} = -\frac{1}{3}(x-5) \quad \text{and} \quad L_1(x) = \frac{x-2}{5-2} = \frac{1}{3}(x-2),$$

so

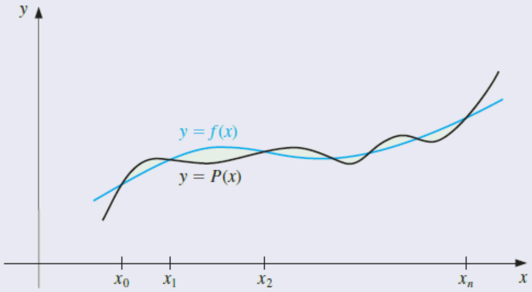
$$P(x) = -\frac{1}{3}(x-5)\cdot 4 + \frac{1}{3}(x-2)\cdot 1 = -\frac{4}{3}x + \frac{20}{3} + \frac{1}{3}x - \frac{2}{3} = -x + 6.$$

# Lagrange Interpolating Polynomial




The linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

# Lagrange Interpolating Polynomial




## Generalization

To generalize the concept of linear interpolation, consider the construction of a polynomial of degree at most $n$ that passes through the $n + 1$ points

$$(x_0, f(x_0)),\ (x_1, f(x_1)),\ \cdots,\ (x_n, f(x_n)).$$

# Lagrange Interpolating Polynomial

## Constructing the Degree $n$ Polynomial

- Construct a function $L_{n,k}(x)$ such that

$$L_{n,k}(x_i) = \begin{cases} 1 & i = k \\ 0 & otherwise \end{cases}$$

# Lagrange Interpolating Polynomial

## Constructing the Degree $n$ Polynomial

- Construct a function $L_{n,k}(x)$ such that

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- 

$$\begin{aligned} L_{n,k}(x) &= \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} \\ &= \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i} \end{aligned}$$

# Lagrange Interpolating Polynomial

## Theorem: $n$-th Lagrange Interpolating Polynomial

If $x_0, x_1, \cdots, x_n$ are $n+1$ distinct numbers and $f$ is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most $n$ exists with

$$f(x_k) = P(x_k), \quad \text{for each } k = 0, 1, \cdots, n.$$

The polynomial is given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

where, for each $k = 0, 1, \cdots, n$, $L_{n,k}(x)$ is defined as previous.

# The Lagrange Polynomial: 2nd Degree

**Example: $f(x) = 1/x$**

1. Use the number (called nodes) $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = 1/x$.
2. Use this polynomial to approximate $f(3) = 1/3$.

# The Lagrange Polynomial: 2nd Degree

## Solution (1/3)

We first determine the coefficient polynomials $L_0(x)$, $L_1(x)$, and $L_2(x)$:

$$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)} = \frac{2}{3}(x-2.75)(x-4)$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x-2)(x-4)$$

$$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)} = \frac{2}{5}(x-2)(x-2.75)$$

Also, since $f(x) = 1/x$:

$$f(x_0) = f(2) = 1/2, \quad f(x_1) = f(2.75) = 4/11, \quad f(x_2) = f(4) = 1/4$$

# The Lagrange Polynomial: 2nd Degree

## Solution (2/3)

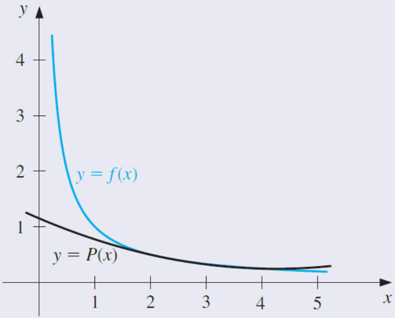
Therefore, we obtain

$$
\begin{aligned}
P(x) &= \sum_{k=0}^{2} f(x_k)L_k(x) \\
&= \frac{1}{3}(x-2.75)(x-4) - \frac{64}{165}(x-2)(x-4) + \frac{1}{10}(x-2)(x-2.75) \\
&= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}.
\end{aligned}
$$

## Solution (3/3)

An approximation to $f(3) = \frac{1}{3}$ is

$$f(3) \approx P(3) = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = \frac{29}{88} \approx 0.32955$$



# The Lagrange Polynomial

## Theorem: Theoretical Error Bound

Suppose $x_0, x_1, \cdots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each $x$ in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \cdots, x_n$, and hence in $(a, b)$, exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where $P(x)$ is the interpolating polynomial given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

# The Lagrange Polynomial

## Proof: (1/6)

Note first that if $x = x_k$, for any $k = 0, 1, \cdots, n$, then $f(x_k) = P(x_k)$, and choosing $\xi(x_k)$ arbitrarily in (a, b) yields the result:

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

if $x \neq x_k$, for all $k = 0, 1, \cdots, n$, define the function $g$ for $t$ in $[a, b]$ by

$$\begin{aligned} g(t) &= f(t) - P(t) - [f(x) - P(x)]\frac{(t - x_0)(t - x_1)\cdots(t - x_n)}{(x - x_0)(x - x_1)\cdots(x - x_n)} \\ &= f(t) - P(t) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{t - x_i}{x - x_i} \end{aligned}$$

# The Lagrange Polynomial

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$

## Proof: (2/6)

Since $f \in C^{(n+1)}[a, b]$, and $P \in C^{\infty}[a, b]$, it follows that $g \in C^{(n+1)}[a, b]$. For $t = x_k$,we have

$$\mathrm{g}(x_k) = f(x_k) - P(x_k) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{x_k - x_i}{x - x_i} = 0 - [f(x) - P(x)] \cdot 0 = 0$$

# The Lagrange Polynomial

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$

## Proof: (3/6)

We have seen that $\mathrm{g}(x_k) = 0$. Furthermore,

$$\begin{aligned} g(x) &= f(x) - P(x) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{x - x_i}{x - x_i} \\ &= f(x) - P(x) - [f(x) - P(x)] = 0 \end{aligned}$$

Thus $g \in C^{(n+1)}[a,b]$, and $g$ is zero at the $n+2$ distinct numbers $x, x_0, x_1, \cdots, x_n$.

## Proof: (4/6)

Since $g \in C^{(n+1)}[a,b]$, and $g$ is zero at the $n+2$ distinct numbers $x, x_0, x_1, \cdots, x_n$, by Generalized Rolle's Theorem, there exists a number $\xi$ in $(a,b)$ for which $g^{(n+1)}(\xi) = 0$. So

$$\begin{aligned} 0 &= g^{(n+1)}(\xi) \\ &= f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[ \prod_{i=0}^{n} \frac{t - x_i}{x - x_i} \right]_{t=\xi} \end{aligned}$$

However, $P(x)$ is a polynomial of degree at most $n$, so the $(n+1)st$ derivative, $P^{(n+1)}(x)$, is identically zero.

# The Lagrange Polynomial

## Proof: (5/6)

Also, $\prod\limits_{i=0}^{n} \frac{t-x_i}{x-x_i}$ is a polynomial of degree$(n+1)$, so

$$\prod_{i=0}^{n} \frac{t-x_i}{x-x_i} = \left[\frac{1}{\prod_{i=0}^{n}(x-x_i)}\right] t^{n+1} + (\text{lower} - \text{degree terms in t}),$$

and

$$\frac{d^{n+1}}{dt^{n+1}} \prod_{i=0}^{n} \frac{t-x_i}{x-x_i} = \frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}$$

# The Lagrange Polynomial

## Proof: (6/6)

We therefore have :

$$0 = f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[ \prod_{i=0}^{n} \frac{t - x_i}{x - x_i} \right]_{t=\xi}$$

$$= f^{(n+1)}(\xi) - 0 - [f(x) - P(x)] \frac{(n+1)!}{\prod_{i=0}^{n} (x - x_i)}$$

and upon solving for $f(x)$, we get the desired result:

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

# The Lagrange Polynomial: 2nd Degree

**Example: $f(x) = 1/x$**

1. Use the number (called nodes) $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = 1/x$.
2. Use this polynomial to approximate $f(3) = 1/3$.
3. Determine the error form for this polynomial, and the maximum error when the polynomial is used to approximate $f(x)$ for $x \in [2, 4]$.

# The Lagrange Polynomial: 2nd Degree

## Solution (1/3)

Because $f(x) = x^{-1}$, we have

$$f'(x) = -\frac{1}{x^2}, \;\; f''(x) = \frac{2}{x^3}, \;\; and \;\; f'''(x) = -\frac{6}{x^4}$$

As a consequence, the second Lagrange polynomial has the error form

$$\frac{f'''(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2) = -\frac{1}{\xi(x)^4}(x-2)(x-2.75)(x-4)$$

for $\xi(x)$ in$(2,4)$. The maximum value of $\frac{1}{\xi(x)^4}$ on the interval is $\frac{1}{2^4} = \frac{1}{16}$

# The Lagrange Polynomial: 2nd Degree

## Solution (2/3)

We now need to determine the maximum value on $[2, 4]$ of the absolute value of the polynomial

$$g(x) = (x-2)(x-2.75)(x-4) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22$$

Because

$$g'(x) = 3x^2 - \frac{35}{2}x + \frac{49}{2} = \frac{1}{2}(3x-7)(2x-7)$$

the critical points occur at

$$x = \frac{7}{3} \ \textit{with} \ g(\frac{7}{3}) = \frac{25}{108} \ \textit{and} \ x = \frac{7}{2} \ \textit{with} \ g(\frac{7}{2}) = -\frac{9}{16}$$

# The Lagrange Polynomial: 2nd Degree

## Solution (3/3)

Hence, the maximum error is

$$
\begin{aligned}
&\max_{[2,4]} \left| \frac{f'''(\xi(x))}{3!} \right| \cdot \max_{[2,4]} \left| (x - x_0)(x - x_1)(x - x_2) \right| \\
&\leq \frac{1}{16} \cdot \frac{9}{16} \\
&= \frac{9}{256} \\
&\approx 0.035156
\end{aligned}
$$

# The Lagrange Polynomial: 2nd Degree

## Example: Tabulated Data

1. Suppose that a table is to be prepared for the function $f(x) = e^x$, for $x \in [0, 1]$.
2. Assume that the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent $x$-values, the step size, is $h$.
3. What step size $h$ will ensure that linear interpolation gives an absolute error of at most $10^{-6}$ for all $x \in [0, 1]$?

# The Lagrange Polynomial: 2nd Degree

## Solution (1/3)

The step size is $h$, so $x_j = jh, x_{j+1} = (j+1)h$, and

$$|f(x) - P(x)| \leq \frac{\left|f^{(2)}(\xi)\right|}{2!} \left|(x - jh)(x - (j+1)h\right|$$

Hence

$$\begin{aligned} |f(x) - P(x)| &\leq \frac{\max_{\xi \in [0,1]} e^{\xi}}{2} \max_{x_j \leq x \leq x_{j+1}} \left|(x - jh)(x - (j+1)h)\right| \\ &\leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} \left|(x - jh)(x - (j+1)h)\right| \end{aligned}$$

# The Lagrange Polynomial: 2nd Degree

## Solution (2/3)

Consider the function $g(x) = (x - jh)(x - (j+1)h)$, for $jh \leq x \leq (j+1)h$. Because

$$g'(x) = (x - (j+1)h) + (x - jh) = 2(x - jh - \frac{h}{2})$$

The only critical point for $g$ is at $x = jh + \frac{h}{2}$, with

$$g(jh + \frac{h}{2}) = (\frac{h}{2})^2 = \frac{h^2}{4}$$

Since $g(jh) = 0$ and $g((j+1)h) = 0$, the maximum value of $|g(x)|$ in $[jh, (j+1)h]$ must occur at the critical point.

# The Lagrange Polynomial: 2nd Degree

## Solution (3/3)

This implies that

$$|f(x) - P(x)| \leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |g(x)| \leq \frac{e}{2} \cdot \frac{h^2}{4} = \frac{eh^2}{8}$$

Consequently, to ensure that the error in linear interpolation is bounded by $10^{-6}$, it is sufficient for $h$ to be chosen so that
$\frac{eh^2}{8} \leq 10^{-6}$. This implies that $h < 1.72 \times 10^{-3}$.
Because $n = \frac{(1-0)}{h}$ must be an integer, a reasonable choice for the step size is $h = 0.001$.

# Example

## Example: Tabulated Data

The following table

| x | 1.0 | 1.3 | 1.6 | 1.9 | 2.2 |
|---|---|---|---|---|---|
| f(x) | 0.7651977 | 0.6200860 | 0.4554022 | 0.2818186 | 0.1103623 |

lists values of a function $f$ at various points. The approximations to $f(1.5)$ obtained by various Lagrange polynomials that use this data will be compared to try and determine the accuracy of the approximation.

# Example

## Solution (1/6)

The most appropriate linear polynomial uses $x_1 = 1.3$ and $x_2 = 1.6$ because 1.5 is between 1.3 and 1.6. The value of the interpolating polynomial at 1.5 is

$$
\begin{aligned}
P_1(1.5) &= \frac{(1.5-1.6)}{(1.3-1.6)} f(1.3) + \frac{(1.5-1.3)}{(1.6-1.3)} f(1.6) \\
&= \frac{(1.5-1.6)}{(1.3-1.6)}(0.6200860) + \frac{(1.5-1.3)}{(1.6-1.3)}(0.4554022) \\
&= 0.5102968
\end{aligned}
$$

# Example

## Solution (2/6)

Two polynomials of degree 2 can reasonably be used, one with $x_1 = 1.3$, $x_2 = 1.6$ and $x_3 = 1.9$, which gives

$$P_2(1.5) = \frac{(1.5-1.6)(1.5-1.9)}{(1.3-1.6)(1.3-1.9)}(0.6200860)$$
$$+\frac{(1.5-1.3)(1.5-1.9)}{(1.6-1.3)(1.6-1.9)}(0.4554022)$$
$$+\frac{(1.5-1.3)(1.5-1.6)}{(1.9-1.3)(1.9-1.6)}(0.2818186) = 0.5112857$$

and one with $x_0 = 1.0$, $x_1 = 1.3$, and $x_2 = 1.6$, which gives $\hat{P}_2(1.5) = 0.5124715$.

# Example

## Solution (3/6)

- In the third-degree case, there are also two reasonable choices for the polynomial. One with $x_1 = 1.3$, $x_2 = 1.6$, $x_3 = 1.9$ and $x_4 = 2.2$, which gives $P_3(1.5) = 0.5118302$.
- The second third-degree approximation is obtained with $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$ and $x_3 = 1.9$, which gives $\hat{P}_3(1.5) = 0.5118127$.
- The fourth-degree Lagrange polynomial uses all the entries in the table.
- With $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, $x_3 = 1.9$ and $x_4 = 2.2$, the approximation is$P_4(1.5) = 0.5118200$.

# Example

## Solution (4/6)

- Because $P_3(1.5)$, $\hat{P}_3(1.5)$, and $P_4(1.5)$ all agree to within $2 \times 10^{-5}$ units, we expect this degree of accuracy for these approximations.
- We also expect $P_4(1.5)$ to be the most accurate approximation, since it uses more of the given data.
- The function we are approximating is actually the Bessel function of the first kind of order zero, whose value at 1.5 is known to be 0.5118277.

# Example

## Solution (5/6)

Therefore, the true accuracies of the approximations are as follows:

$$
\begin{aligned}
|P_1(1.5) - f(1.5)| &\approx 1.53 \times 10^{-3} \\
|P_2(1.5) - f(1.5)| &\approx 5.42 \times 10^{-4} \\
\left|\hat{P}_2(1.5) - f(1.5)\right| &\approx 6.44 \times 10^{-4} \\
|P_3(1.5) - f(1.5)| &\approx 2.5 \times 10^{-6} \\
\left|\hat{P}_3(1.5) - f(1.5)\right| &\approx 1.50 \times 10^{-5} \\
|P_4(1.5) - f(1.5)| &\approx 7.7 \times 10^{-6}
\end{aligned}
$$

# Example

## Solution (6/6)

- Although$P_3(1.5)$ is the most accurate approximation, if we had no knowledge of the actual value of $f(1.5)$, we would accept $P_4(1.5)$ as the best approximation since it includes the most data about the function.
- The theoretical Lagrange error term cannot be applied here because we have no knowledge of the fourth derivative of $f$.
- Unfortunately, this is generally the case.

# Outline

# Introduction to Neville's Method

## Definition: Lagrange Polynomial $P_{m_1,m_2,\cdots,m_k}(x)$

- Let $f$ be a function defined at $x_0, x_1, x_2, \cdots, x_n$, and suppose that $m_1, m_2, \cdots, m_k$ are $k$ distinct integers, with $0 \leq m_i \leq n$ for each $i$.
- The Lagrange polynomial that agrees with $f(x)$ at the $k$ points $x_{m_1}, x_{m_2}, \cdots, x_{m_k}$ is denoted by

$$P_{m_1,m_2,\cdots,m_k}(x)$$

# Introduction to Neville's Method

## Example: $P_{1,2,4}(x)$

- Suppose that $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4, x_4 = 6$, and $f(x) = e^x$.
- Determine the interpolating polynomial denoted $P_{1,2,4}(x)$, and use this polynomial to approximate $f(5)$.

# Introduction to Neville's Method

## Example: $P_{1,2,4}(x)$

- Suppose that $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4, x_4 = 6$, and $f(x) = e^x$.
- Determine the interpolating polynomial denoted $P_{1,2,4}(x)$, and use this polynomial to approximate $f(5)$.

This is the Lagrange polynomial that agrees with $f(x)$ at $x_1 = 2, x_2 = 3$, and $x_4 = 6$. Hence

$$P_{1,2,4}(x) = \frac{(x-3)(x-6)}{(2-3)(2-6)}e^2 + \frac{(x-2)(x-6)}{(3-2)(3-6)}e^3 + \frac{(x-2)(x-3)}{(6-2)(6-3)}e^6$$

$$f(5) \approx P(5) = \frac{(5-3)(5-6)}{(2-3)(2-6)}e^2 + \frac{(5-2)(5-6)}{(3-2)(3-6)}e^3 + \frac{(5-2)(5-3)}{(6-2)(6-3)}e^6$$

$$= -\frac{1}{2}e^2 + e^3 + \frac{1}{2}e^6 \approx 218.105$$

# Recursive Lagrange Polynomial Approximations

**Theorem**

Let $f$ be defined at $x_0, x_1, \cdots, x_k$, and let $x_j$ and $x_i$ be two distinct numbers in this set. Then

$$P(x) = \frac{(x - x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x) - (x - x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{(x_i - x_j)}$$

is the $k$th Lagrange polynomial that interpolates $f$ at the $k + 1$ points $x_0, x_1, \cdots, x_k$.

# Recursive Lagrange Polynomial Approximations

For ease of notation, let

$$Q \equiv P_{0,1,\ldots,i-1,i+1,\ldots,k} \quad \text{and} \quad \hat{Q} \equiv P_{0,1,\ldots,j-1,j+1,\ldots,k}$$

Since $Q(x)$ and $\hat{Q}(x)$ are polynomials of degree $k-1$ or less, $P(x)$ is of degree at most $k$.

## Proof (1/2)

First note that $\hat{Q}(x_i) = f(x_i)$, implies that

$$P(x_i) = \frac{(x_i - x_j)\hat{Q}(x_i) - (x_i - x_i)Q(x_i)}{x_i - x_j} = \frac{(x_i - x_j)}{(x_i - x_j)} f(x_i) = f(x_i)$$

Similarly, since $Q(x_j) = f(x_j)$, we have $P(x_j) = f(x_j)$.

# Recursive Lagrange Polynomial Approximations

## Proof (2/2)

In addition, if $0 \leq r \leq k$ and $r$ is neither $i$ nor $j$, then $Q(x_r) = \hat{Q}(x_r) = f(x_r)$. So

$$P(x_r) = \frac{(x_r - x_j)\hat{Q}(x_r) - (x_r - x_i)Q(x_r)}{x_i - x_j} = \frac{(x_i - x_j)}{(x_i - x_j)} f(x_r) = f(x_r)$$

However, by definition, $P_{0,1,\ldots,k}(x)$ is the unique polynomial of degree at most $k$ that agrees with $f$ at $x_0, x_1, \ldots, x_k$. Thus, $P \equiv P_{0,1,\ldots,k}$.

# Recursive Lagrange Polynomial Approximations

## Comments

- This result implies that the interpolating polynomials can be generated recursively.
- For example we have

$$
\begin{aligned}
P_{0,1} &= \frac{1}{x_1 - x_0}\left[(x - x_0)P_1 - (x - x_1)P_0\right] \\
P_{1,2} &= \frac{1}{x_2 - x_1}\left[(x - x_1)P_2 - (x - x_2)P_1\right] \\
P_{0,1,2} &= \frac{1}{x_2 - x_0}\left[(x - x_0)P_{1,2} - (x - x_2)P_{0,1}\right]
\end{aligned}
$$

and so on.

# Neville's Method: Recursive Generation

The following table illustrate how the interpolating polynomials can be generated recursively, where each row is completed before the succeeding rows are begun.

| | | | | | |
|---|---|---|---|---|---|
| $x_0$ | $P_0$ | | | | |
| $x_1$ | $P_1$ | $P_{0,1}$ | | | |
| $x_2$ | $P_2$ | $P_{1,2}$ | $P_{0,1,2}$ | | |
| $x_3$ | $P_3$ | $P_{2,3}$ | $P_{1,2,3}$ | $P_{0,1,2,3}$ | |
| $x_4$ | $P_4$ | $P_{3,4}$ | $P_{2,3,4}$ | $P_{1,2,3,4}$ | $P_{0,1,2,3,4}$ |

The procedure that uses the results of the theorem to recursively generate interpolating polynomial approximations is called Neville's method.
Hierarchical!!!

# Neville's Method: Recursive Generation

To avoid the multiple subscripts, we let $Q_{i,j}(x)$, for $0 \le j \le i$, denote the interpolating polynomial of degree $j$ on the $(j+1)$ numbers $x_{i-j}, x_{i-j+1}, \cdots, x_{i-1}, x_i$; that is,

$$Q_{i,j} = P_{i-j,i-j+1,\cdots,i-1,i}.$$

Using this notation provides the following $Q$ notation array

| | | | | | |
|---|---|---|---|---|---|
| $x_0$ | $P_0 = Q_{0,0}$ | | | | |
| $x_1$ | $P_1 = Q_{1,0}$ | $P_{0,1} = Q_{1,1}$ | | | |
| $x_2$ | $P_2 = Q_{2,0}$ | $P_{1,2} = Q_{2,1}$ | $P_{0,1,2} = Q_{2,2}$ | | |
| $x_3$ | $P_3 = Q_{3,0}$ | $P_{2,3} = Q_{3,1}$ | $P_{1,2,3} = Q_{3,2}$ | $P_{0,1,2,3} = Q_{3,3}$ | |
| $x_4$ | $P_4 = Q_{4,0}$ | $P_{3,4} = Q_{4,1}$ | $P_{2,3,4} = Q_{4,2}$ | $P_{1,2,3,4} = Q_{4,3}$ | $P_{0,\ldots,4} = Q_{4,4}$ |

# Recursive Lagrange Polynomial Approximations

## Example

Values of various interpolating polynomials at $x = 1.5$ were obtained in an earlier example using the following data:

| $x$ | 1.0 | 1.3 | 1.6 | 1.9 | 2.2 |
|---|---|---|---|---|---|
| $f(x)$ | 0.7651977 | 0.6200860 | 0.4554022 | 0.2818186 | 0.1103623 |

Apply Neville's method to the data by constructing a recursive table in the Q-notation array format.

# Recursive Lagrange Polynomial Approximations

## Solution (1/6)

Let $x_0 = 1.0, x_1 = 1.3, x_2 = 1.6, x_3 = 1.9$, and $x_4 = 2.2$, then
$Q_{0,0} = f(1.0), Q_{1,0} = f(1.3), Q_{2,0} = f(1.6), Q_{3,0} = f(1.9)$, and $Q_{4,0} = f(2.2)$.
These are the five polynomials of degree zero (constants) that approximate $f(1.5)$, and are the same as data given in the example table.

# Recursive Lagrange Polynomial Approximations

## Solution (2/6)

Calculating the first-degree approximation $Q_{1,1}(1.5)$ gives

$$\begin{aligned}
Q_{1,1}(1.5) &= \frac{(x - x_0)Q_{1,0} - (x - x_1)Q_{0,0}}{x_1 - x_0} \\
&= \frac{(1.5 - 1.0)Q_{1,0} - (1.5 - 1.3)Q_{0,0}}{1.3 - 1.0} \\
&= \frac{0.5(0.6200860) - 0.2(0.7651977)}{0.3} \\
&= 0.5233449
\end{aligned}$$

# Recursive Lagrange Polynomial Approximations

## Solution (3/6)

Similarly

$$
\begin{aligned}
Q_{2,1}(1.5) &= \frac{(1.5-1.3)(0.4554022)-(1.5-1.6)(0.6200860)}{1.6-1.3} \\
&= 0.5102968 \\
Q_{3,1}(1.5) &= 0.5132634 \quad \text{and} \quad Q_{4,1}(1.5) = 0.5104270
\end{aligned}
$$

The best linear approximation is expected to be $Q_{2,1}$ because 1.5 is between $x_1 = 1.3$ and $x_2 = 1.6$.

# Recursive Lagrange Polynomial Approximations

## Solution (4/6)

In a similar manner, approximations using higher-degree polynomials are given by

$$Q_{2,2}(1.5) = \frac{(1.5 - 1.0)(0.5102968) - (1.5 - 1.6)(0.5233449)}{1.6 - 1.0}$$
$$= 0.5124715$$

$$Q_{3,2}(1.5) = 0.5112857$$
$$Q_{4,2}(1.5) = 0.5137361$$

# Recursive Lagrange Polynomial Approximations

## Solution (5/6)

| | | | | | |
|---|---|---|---|---|---|
| 1.0 | 0.7651977 | | | | |
| 1.3 | 0.6200860 | 0.5233449 | | | |
| 1.6 | 0.4554022 | 0.5102968 | 0.5124715 | | |
| 1.9 | 0.2818186 | 0.5132634 | 0.5112857 | 0.5118127 | |
| 2.2 | 0.1103623 | 0.5104270 | 0.5137361 | 0.5118302 | 0.5118200 |

# Recursive Lagrange Polynomial Approximations

## Solution (6/6)

- If the latest approximation, $Q_{4,4}$, was not sufficiently accurate, another node, $x_5$, could be selected, and another row added:

$$x_5 \quad Q_{5,0} \quad Q_{5,1} \quad Q_{5,2} \quad Q_{5,3} \quad Q_{5,4} \quad Q_{5,5}.$$

  Then $Q_{4,4}$, $Q_{5,4}$, and $Q_{5,5}$ could be compared to determine further accuracy.
- The function in this example is the Bessel function of the first kind of order zero, whose value at 2.5 is $-0.0483838$, and the next row of approximations to $f(1.5)$ is

$$2.5 \quad -0.0483838 \quad 0.4807699 \quad 0.5301984 \quad 0.5119070 \quad 0.5118430 \quad 0.5118277$$

  The final new entry, 0.5118277, is correct to all 7 decimal places.

# Recursive Lagrange Polynomial Approximations

## Example: 4-Digit Values of *lnx*

The following table lists the values of $f(x) = \ln x$ accurate to the places given.

| $i$ | $x_i$ | $\ln x_i$ |
|---|---|---|
| 0 | 2.0 | 0.6931 |
| 1 | 2.2 | 0.7885 |
| 2 | 2.3 | 0.8329 |

Use Neville's method and 4 - digit rounding arithmetic to approximate $f(2.1) = \ln 2.1$ by completing the Neville table.

# Recursive Lagrange Polynomial Approximations

## Solution: (1/2)

Because $x - x_0 = 0.1$, $x - x_1 = -0.1$, $x - x_2 = -0.2$, and we are given $Q_{0,0} = 0.6931$, $Q_{1,0} = 0.7885$, and $Q_{2,0} = 0.8329$, we have

$$Q_{1,1} = \frac{1}{0.2}[(0.1)0.7885 - (-0.1)0.6931] = \frac{0.1482}{0.2} = 0.7410$$

and

$$Q_{2,1} = \frac{1}{0.1}[(-0.1)0.8329 - (-0.2)0.7885] = \frac{0.07441}{0.1} = 0.7441.$$

The final approximation we can obtain from this data is

$$Q_{2,2} = \frac{1}{0.3}[(0.1)0.7441 - (-0.2)0.7410] = \frac{0.2276}{0.3} = 0.7420.$$

# Recursive Lagrange Polynomial Approximations

## Solution: (2/2)

The calculations are summarized in the following table:

| $i$ | $x_i$ | $x - x_i$ | $Q_{i0}$ | $Q_{i1}$ | $Q_{i2}$ |
|---|---|---|---|---|---|
| 0 | 2.0 | 0.1 | 0.6931 | | |
| 1 | 2.2 | $-0.1$ | 0.7885 | 0.7410 | |
| 2 | 2.3 | $-0.2$ | 0.8329 | 0.7441 | 0.7420 |

# Accuracy of 4-Digit Approximations

## Absolute Error vs. Error Bound (1/2)

In the preceding example, we have $f(2.1) = \ln 2.1 = 0.7419$ to four decimal places, so the absolute error is

$$|f(2.1) - P_2(2.1)| = |0.7419 - 0.7420| = 10^{-4}$$

However $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, and $f'''(x) = \frac{2}{x^3}$, so the Lagrange error formula gives the error bound

$$\begin{aligned}
|f(2.1) - P_2(2.1)| &= \left|\frac{f'''(\xi(2.1))}{3!}(x - x_0)(x - x_1)(x - x_2)\right| \\
&= \left|\frac{1}{3(\xi(2.1))^3}(0.1)(-0.1)(-0.2)\right| \\
&\le \frac{0.002}{3(2)^3} = 8.3 \times 10^{-5}
\end{aligned}$$

# Accuracy of 4-Digit Approximations

## Absolute Error vs. Error Bound (2/2)

- Notice that the actual error, $10^{-4}$ , exceeds the error bound, $8.3 \times 10^{-5}$
- This apparent contradiction is a consequence of finite-digit computations
- We used four-digit rounding arithmetic, whereas the Lagrange error formula assumes infinite-digit arithmetic
- This caused our actual errors to exceed the theoretical error estimate.

## Neville's Iterated Interpolation Algorithm

INPUT numbers $x, x_0, x_1, \ldots, x_n$; values $f(x_0), f(x_1), \ldots, f(x_n)$ as the first column $Q_{0,0}, Q_{1,0}, \ldots, Q_{n,0}$ of $Q$.

OUTPUT the table $Q$ with $P(x) = Q_{n,n}$.

*Step 1* For $i = 1, 2, \ldots, n$
for $j = 1, 2, \ldots, i$

$$\text{set } Q_{i,j} = \frac{(x - x_{i-j})Q_{i,j-1} - (x - x_i)Q_{i-1,j-1}}{x_i - x_{i-j}}.$$

*Step 2* OUTPUT $(Q)$;
STOP.

# Neville's Iterated Interpolation Algorithm

## Additional Nodes & Stopping Criteria

- The algorithm can be modified to allow for the addition of new interpolating nodes. For example, the inequality.

$$|Q_{i,j} - Q_{i-1,j-1}| < \varepsilon$$

  can be used as a stopping criterion, where $\varepsilon$ is a prescribed error tolerance
- If the inequality is ture, $Q_{i,j}$ is a reasonable approximation to $f(x)$.
- If the inequality is false, a new interpolation point, $x_{i+1}$ is added.

# Assignment

- Reading Assignment: Chap 3.1-3.2