# Numerical Analysis

## Lecture 12: Solution of Differential Equations

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# Outline

1. Initial-Value Problems for ODEs
2. Euler's Method
3. Higher-Order Taylor Methods
4. Runge-Kutta Methods

# Introduction

## Initial-Value Problems for ODEs

Approximating the solution $y(t)$ to a problem of the form

$$\frac{dy}{dt} = f(t, y), \quad \text{for} \quad a \leq t \leq b$$

subject to an initial condition $y(a) = \alpha$.

- The methods do not produce a continuous approximation to the solution of the initial-value problem.
- Rather, approximations are found at certain specified, and often equally spaced, points.
- Some method of interpolation, is used if intermediate values are needed.

## Definition: Lipschitz Condition

A function $f(t, y)$ is said to satisfy a Lipschitz condition in the variable $y$ on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

whenever $(t, y_1)$ and $(t, y_2)$ are in $D$. The constant $L$ is called a Lipschitz constant for $f$.

# Elementary Theory of IVPs: Lipschitz Condition

## Example

Show that $f(t, y) = t|y|$ satisfies a Lipschitz condition on the interval $D = \{(t, y) | 1 \le t \le 2 \text{ and } -3 \le y \le 4\}$.

## Solution

For each pair of points $(t, y_1)$ and $(t, y_2)$ in $D$, we have

$$|f(t, y_1) - f(t, y_2)| = |t|y_1| - t|y_2|| = |t|||y_1| - |y_2|| \le 2|y_1 - y_2|.$$

Thus $f$ satisfies a Lipschitz condition on $D$ in the variable $y$ with Lipschitz constant 2. The smallest value possible for the Lipschitz constant for this problem is $L = 2$, because, for example,

$$|f(2, 1) - f(2, 0)| = |2 - 0| = 2|1 - 0|.$$

## Example

Show that $f(x) = \sqrt{x}$ does not satisfy a Lipschitz condition on the interval $D = \{x|0 \leq x \leq 1\}$.

Proof: Given $L$, $x_1 = 0, x_2 = \frac{1}{4L^2}$, $|f(x_1) - f(x_2)| > L|x_1 - x_2|$.

## Example

Show that $f(x) = \frac{1}{x}$ does not satisfy a Lipschitz condition on the interval $D = \{x|0 \leq x \leq 1\}$.

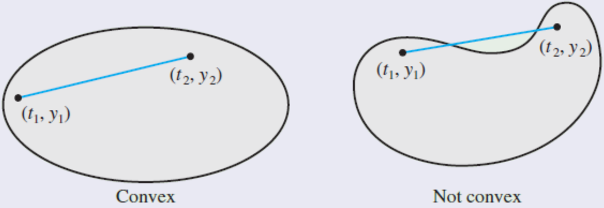
Proof: Given $L^2$, $x_1 = \frac{1}{L}, x_2 = \frac{1}{2L}$, $|f(x_1) - f(x_2)| > L^2|x_1 - x_2|$.

## Convex Set

A set $D \subset \mathbb{R}^2$ is said to be convex if whenever $(t_1, y_1)$ and $(t_2, y_2)$ belong to $D$, then

$$((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)$$

also belongs to $D$ for every $\lambda$ in [0, 1].



# Elementary Theory of IVPs: Sufficient Condition

## Theorem: Sufficient Conditions

Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left| \frac{\partial f}{\partial y}(t, y) \right| \leq L, \quad \text{for all} \quad (t, y) \in D$$

then $f$ satisfies a Lipschitz condition on $D$ in the variable $y$ with Lipschitz constant $L$.

## Theorem: Existence & Uniqueness

Suppose that $D = \{(t, y) | a \le t \le b \text{ and } -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on $D$. If $f$ satisfies a Lipschitz condition on $D$ in the variable $y$, then the initial-value problem

$$y'(t) = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

has a unique solution $y(t)$ for $a \le t \le b$.

# Elementary Theory of IVPs: Existence & Uniqueness

## Example

Use the Existence & Uniqueness Theorem to show that there is a unique solution to the initial-value problem

$$y' = 1 + t\sin(ty), \qquad 0 \le t \le 2, \qquad y(0) = 0$$

## Solution (1/2)

Holding $t$ constant and applying the Mean Value Theorem to the function

$$f(t,y) = 1 + t\sin(ty)$$

we find that when $y_1 < y_2$, a number $\xi$ in $(y_1, y_2)$ exists with

$$\frac{f(t,y_2) - f(t,y_1)}{y_2 - y_1} = \frac{\partial}{\partial y} f(t,\xi) = t^2\cos(\xi t)$$

$$\frac{f(t,y_2)-f(t,y_1)}{y_2-y_1} = \frac{\partial}{\partial y} f(t,\xi) = t^2\cos(\xi t)$$

## Solution (2/2)

- Thus

  $$|f(t,y_2)-f(t,y_1)| = |y_2-y_1||t^2\cos(\xi t)| \leq 4|y_2-y_1|$$

  and $f$ satisfies a Lipschitz condition in the variable $y$ with lipschitz constant $L=4$.
- Additionally, $f(t,y)$ is continuous when $0 \leq t \leq 2$ and $-\infty < y < \infty$, so the Existence & Uniqueness Theorem implies that a unique solution exists to this initial-value problem.

### Definition: Well-Posed Problem

The initial-value problem

$$y'(t) = f(t, y), \;\; a \le t \le b, \;\; y(a) = \alpha,$$

is said to be a well-posed problem if the following 2 conditions are satisfied:

# Elementary Theory of IVPs: Well-posed Problems

## Definition: Well-Posed Problem (cont'd)

1. A unique solution, $y(t)$, to the problem exists, and
2. There exist constants $\epsilon_0 > 0$ and $k > 0$ such that for any $\epsilon$, with $\epsilon_0 > \epsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \epsilon$ for all $t$ in $[a, b]$, and when $|\delta_0| < \epsilon$, the initial-value problem

$$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \leq t \leq b, \quad z(a) = \alpha + \delta_0,$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\epsilon \quad \text{for all } t \text{ in } [a, b].$$

## Theorem: Well-Posed Problem

Suppose $D = \{(t, y) | a \leq t \leq b \text{ and } -\infty < y < \infty\}$. If $f$ is continuous and satisfies a Lipschitz condition in the variable $y$ on the set $D$, then the initial-value problem

$$y'(t) = f(t, y), \;\; a \leq t \leq b, \;\; y(a) = \alpha,$$

is well-posed.

## Example

Show that the initial-value problem

$$\frac{dy}{dt} = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

is well posed on $D = \{(t, y) | 0 \le t \le 2 \text{ and } -\infty < y < \infty\}$.

## Solution (1/3)

- Because
$$\left|\frac{\partial(y - t^2 + 1)}{\partial y}\right| = 1$$
the Lipschitz Condition Theorem implies that $f(t, y) = y - t^2 + 1$ satisfies a Lipschitz condition in $y$ on $D$ with Lipschitz constant 1.
- Since $f$ is continuous on $D$, the Theorem on Well-Posed Problems implies that the problem is well-posed.

## Solution (2/3)

- As an illustration, consider the solution to the perturbed problem

$$\frac{dz}{dt} = z - t^2 + 1 + \delta, \quad 0 \le t \le 2, \quad z(0) = 0.5 + \delta_0$$

where $\delta$ and $\delta_0$ are constants.

- The solutions to the original problem and this perturbed problem are

$$\begin{array}{rrcl} & y(t) & = & (t+1)^2 - 0.5e^t \\ \text{and} & z(t) & = & (t+1)^2 + (\delta + \delta_0 - 0.5)e^t - \delta \end{array}$$

respectively.

## Solution (3/3)

- Suppose that $\epsilon$ is a positive number. If $|\delta| < \epsilon$ and $|\delta_0| < \epsilon$, then

  $$|y(t) - z(t)| = |(\delta + \delta_0)e^t - \delta| \leq |\delta + \delta_0|e^2 + |\delta| \leq (2e^2 + 1)\epsilon$$

  for all $t$.
- This implies that the original problem is well-posed with $k(\epsilon) = 2e^2 + 1$ for all $\epsilon > 0$.

# Outline

# Euler's Method

## Objective

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

at $(N + 1)$ equally spaced numbers in the interval $[a, b]$.

# Euler's Method

## Derivation

- Let $t_i = a + ih$, for $i = 0, 1, \cdots, N$,
  in which $h = (b-a)/N = t_{i+1} - t_i$ is called the step size.
- By Taylor's Theorem, we have
  $$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2}y''(\xi_i),$$
  for some number $\xi_i$ in $(t_i, t_{i+1})$.
- Substituting $h = t_{i+1} - t_i$ and $y' = f(t, y)$, we get
  $$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i),$$

# Euler's Method

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i),$$

## Euler's Method

Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2, \cdots, N$, by deleting the remainder term. Thus Euler's method is

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf(t_i, w_i), \quad \text{for each } \; i = 0, 1, \cdots, N-1 \end{aligned}$$

This equation is called the difference equation associated with Euler's method.

# Euler's Method: Example

## Example

Prior to introducing an algorithm for Euler's Method, we will illustrate the steps in the technique to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

at $t = 2$ using a step size of $h = 0.5$.

# Euler's Method: Example

## Solution

For this problem $f(t, y) = y - t^2 + 1$, so

$$
\begin{aligned}
w_0 &= y(0) = 0.5 \\
w_1 &= w_0 + 0.5\left(w_0 - (0.0)^2 + 1\right) = 0.5 + 0.5(1.5) = 1.25 \\
w_2 &= w_1 + 0.5\left(w_1 - (0.5)^2 + 1\right) = 1.25 + 0.5(2.0) = 2.25 \\
w_3 &= w_2 + 0.5\left(w_2 - (1.0)^2 + 1\right) = 2.25 + 0.5(2.25) = 3.375
\end{aligned}
$$

and

$$y(2) \approx w_4 = w_3 + 0.5\left(w_3 - (1.5)^2 + 1\right) = 3.375 + 0.5(2.125) = 4.4375$$

# Euler's Method: Algorithm

## Algorithm

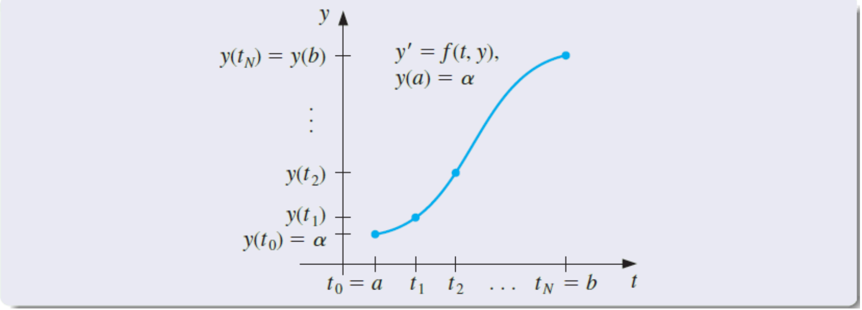
INPUT endpoints $a, b$; integer $N$; initial condition $\alpha$.
OUTPUT approximation $w$ to $y$ at the $(N+1)$ values of $t$.
Step 1 Set $h = (b-a)/N$
$t = a$
$w = \alpha$
OUTPUT $(t, w)$
Step 2 For $i = 1, 2, \ldots, N$ do Steps 3 & 4
Step 3 Set $w = w + hf(t, w)$; (*Compute* $w_i$)
$t = a + ih$. (*Compute* $t_i$)
Step 4 OUTPUT $(t, w)$
Step 5 STOP

# Euler's Method: Geometric Interpretation

To interpret Euler's method geometrically, note that when $w_i$ is a close approximation to $y(t_i)$, the assumption that the problem is well-posed implies that

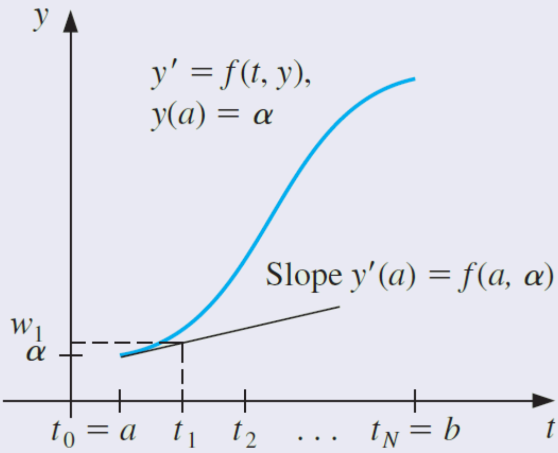
$$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$$

The graph of the function highlighting $y(t_i)$ is shown below



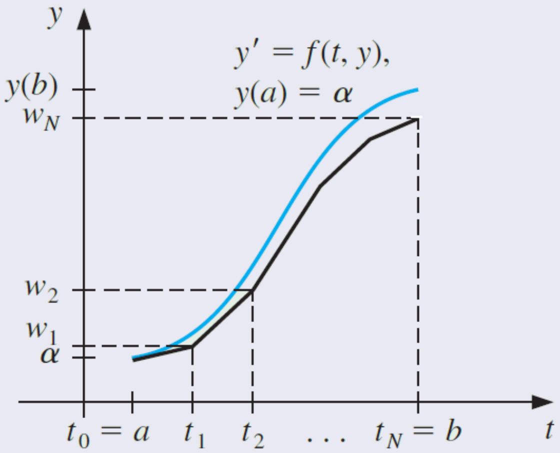
# Euler's Method: Geometric Interpretation

## One step in Euler's method



# Euler's Method: Geometric Interpretation

## A series of steps in Euler's method



# Euler's Method: Numerical Example

## Example

Use the algorithm for Euler's method with $N = 10$ to determine approximations to the solution to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

and compare these with the exact values given by

$$y(t) = (t + 1)^2 - 0.5e^t$$

# Euler's Method: Numerical Example

**Solution**

With $N = 10$, we have $h = 0.2$, $t_i = 0.2i$, $w_0 = 0.5$, so that:

$$\begin{aligned} w_{i+1} &= w_i + h(w_i - t_i^2 + 1) \\ &= w_i + 0.2[w_i - 0.04i^2 + 1] \\ &= 1.2w_i - 0.008i^2 + 0.2 \end{aligned}$$

for $i = 0, 1, \ldots, 9$. So

$$\begin{aligned} w_1 &= 1.2(0.5) - 0.008(0)^2 + 0.2 = 0.8 \\ w_2 &= 1.2(0.8) - 0.008(1)^2 + 0.2 = 1.152 \end{aligned}$$

and so on.

# Euler's Method: Numerical Example

**Results for $y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$**

| $t_i$ | $w_i$ | $y_i = y(t_i)$ | $\lvert y_i - w_i \rvert$ |
|---|---|---|---|
| 0.0 | 0.5000000 | 0.5000000 | 0.0000000 |
| 0.2 | 0.8000000 | 0.8292986 | 0.0292986 |
| 0.4 | 1.1520000 | 1.2140877 | 0.0620877 |
| 0.6 | 1.5504000 | 1.6489406 | 0.0985406 |
| 0.8 | 1.9884800 | 2.1272295 | 0.1387495 |
| 1.0 | 2.4581760 | 2.6408591 | 0.1826831 |
| 1.2 | 2.9498112 | 3.1799415 | 0.2301303 |
| 1.4 | 3.4517734 | 3.7324000 | 0.2806266 |
| 1.6 | 3.9501281 | 4.2834838 | 0.3333557 |
| 1.8 | 4.4281538 | 4.8151763 | 0.3870225 |
| 2.0 | 4.8657845 | 5.3054720 | 0.4396874 |

# Euler's Method: Numerical Example

## Comments

- Note that the error grows slightly as the value of $t$ increases.
- This controlled error growth is a consequence of the stability of Euler's method, which implies that the error is expected to grow in no worse than a linear manner.

# Error Bounds for Euler's Method–Theorem

Suppose $f$ is continuous and satisfies a Lipschitz condition on

$$D = \{(t, y) | a \leq t \leq b \text{ and } -\infty < y < \infty\}$$

and that a constant $M$ exists with

$$|y''(t)| \leq M, \quad \text{for all } t \in [a, b]$$

where $y(t)$ denotes the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

Let $w_0, w_1, \cdots, w_N$ be the approximations generated by Euler's method for some positive integer $N$. Then, for each $i = 0, 1, 2, \cdots, N$

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right]$$

# Error Bounds for Euler's Method

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \cdots, N-1 \end{aligned}$$

## Round-off Error

$$\begin{aligned} u_0 &= \alpha + \delta_0 \\ u_{i+1} &= u_i + hf(t_i, u_i) + \delta_{i+1}, \quad \text{for each } i = 0, 1, \cdots, N-1 \end{aligned}$$

# Error Bounds for Euler's Method

## The Error Bound with Round-off Error

$$|y(t_i) - u_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left[e^{L(t_i-a)} - 1\right] + |\delta_0| e^{L(t_i-a)}$$

The error bound is no longer linear and it approaches infinity when $h \approx 0$. In fact, the optimal $h$ can be achieved when

$$h = \sqrt{\frac{2\delta}{M}}.$$

# Error Bounds for Euler's Method

## Comments on the Theorem

- The weakness of the error-bound theorem lies in the requirement that a bound be known for the second derivative of the solution.
- Although this condition often prohibits us from obtaining a realistic error bound, it should be noted that if $\partial f/\partial t$ and $\partial f/\partial y$ both exist, the chain rule for partial differentiation implies that

$$y''(t) = \frac{dy'}{dt}(t) = \frac{df}{dt}(t, y(t)) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t))$$

- So it is at times possible to obtain an error bound for $y''(t)$ without explicitly knowing $y(t)$

# Local Truncation Error

## Informal Definition of LTE

The local truncation error at a specified step measures the amount by which the exact solution to the differential equation fails to satisfy the difference equation being used for the approximation at that step.

## Note

- We really want to know how well the approximations generated by the methods satisfy the differential equation, not the other way around.
- However, we don't know the exact solution so we cannot generally determine this, and the local truncation will serve quite well to determine not only the local error of a method but the actual approximation error.

# Local Truncation Error

## IVP

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

## Definition of LTE

The difference method

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\phi(t_i, w_i), \quad \text{for each } i = 0, 1, \cdots, N-1, \end{aligned}$$

has local truncation error

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \cdots, N-1$, where $y_i$ and $y_{i+1}$ denote the solution at $t_i$ and $t_{i+1}$, respectively.

# Local Truncation Error

## Example

Euler's method has local truncation error at the $i$th step

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i), \quad \text{for each } i = 0, 1, \cdots, N-1$$

- This error is a local error because it measures the accuracy of the method at a specific step, assuming that the method was exact at the previous step.
- As such, it depends on the differential equation, the step size, and the particular step in the approximation.

# Local Truncation Error

## LTE in Euler's Method (Cont'd)

Earlier, we have seen that, for Euler's method:

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i),$$

so that the LTE is

$$\tau_{i+1}(h) = \frac{h}{2} y''(\xi_i), \text{ for some } \xi_i \text{ in } (t_i, t_{i+1})$$

When $y''(t)$ is known to be bounded by a constant $M$ on $[a, b]$, this implies

$$|\tau_{i+1}(h)| \leq \frac{h}{2} M$$

so the local truncation error in Euler's method is $O(h)$.

# Outline

# Higher-Order Taylor Methods

## Motivation

- One way to select difference-equation methods for solving ordinary differential equations is in such a manner that their local truncation errors are $O(h^p)$ for as large a value of $p$ as possible while keeping the number and complexity of calculations of the methods within a reasonable bound.
- Euler's method was derived by using Taylor's Theorem with $p = 1$ to approximate the solution of the differential equation.
- Can we extend this technique of derivation to larger values of $p$ in order to find methods for improving the convergence properties of difference methods?

# Higher-Order Taylor Methods

## Assumption

The solution $y(t)$ to the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

has $(n+1)$ continuous derivatives.

## Taylor Expansion about $t_i$

If we expand the solution, $y(t)$, in terms of its $n$th Taylor polynomial about $t_i$ and evaluate at $t_{i+1}$, we obtain

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

for some $\xi_i$ in $(t_i, t_{i+1})$.

# Higher-Order Taylor Methods

## Derivation (Cont'd)

Successive differentiation of the solution, $y(t)$, gives

$$y'(t) = f(t, y(t)), \quad y''(t) = f'(t, y(t)), \cdots, \quad y^{(k)}(t) = f^{(k-1)}(t, y(t))$$

Substituting these results into

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots + \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)$$

gives

$$\begin{aligned} y(t_{i+1}) = \quad & y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots \\ & + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)) \end{aligned}$$

# Higher-Order Taylor Methods

## Derivation (Cont'd)

The difference-equation method corresponding to

$$\begin{aligned} y(t_{i+1}) = \quad & y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots \\ & + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)) \end{aligned}$$

is obtained by deleting the remainder term involving $\xi_i$.

# Higher-Order Taylor Methods

## Taylor's Method of order $n$

$$
\begin{aligned}
w_0 &= \alpha \\
w_{i+1} &= w_i + hT^{(n)}(t_i, w_i), \quad \text{for each } \; i = 0, 1, \cdots, N-1
\end{aligned}
$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$$

# Higher-Order Taylor Methods

## Example

Apply Taylor's method of orders

- 2 and
- 4

with $N = 10$ to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

# Higher-Order Taylor Methods

## Order 2 Method: Summary of Numerical Results

| $t_i$ | Taylor Order 2 $w_i$ | Error $\lvert y(t_i) - w_i \rvert$ |
|---|---|---|
| 0.0 | 0.500000 | 0 |
| 0.2 | 0.830000 | 0.000701 |
| 0.4 | 1.215800 | 0.001712 |
| 0.6 | 1.652076 | 0.003135 |
| 0.8 | 2.132333 | 0.005103 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| 1.6 | 4.306146 | 0.022663 |
| 1.8 | 4.846299 | 0.031122 |
| 2.0 | 5.347684 | 0.042212 |

# Higher-Order Taylor Methods

## Order 4 Method: Summary of Numerical Results

| $t_i$ | Taylor Order 4 $w_i$ | Error $\lvert y(t_i) - w_i \rvert$ |
|---|---|---|
| 0.0 | 0.500000 | 0 |
| 0.2 | 0.829300 | 0.000001 |
| 0.4 | 1.214091 | 0.000003 |
| 0.6 | 1.648947 | 0.000006 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| 1.4 | 3.732432 | 0.000032 |
| 1.6 | 4.283529 | 0.000045 |
| 1.8 | 4.815238 | 0.000062 |
| 2.0 | 5.305555 | 0.000083 |

# Higher-Order Taylor Methods

**Theorem**

If Taylor's method of order $n$ is used to approximate the solution to

$$y'(t) = f(t, y(t)), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

with step size $h$ and if $y \in C^{n+1}[a, b]$, then the local truncation error is $O(h^n)$.

# Higher-Order Taylor Methods

## Proof (1/2)

When deriving Taylor Method, we obtained the expression

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

and this can be rewritten in the form

$$\begin{aligned} y_{i+1} \quad & -y_i - hf(t_i, y_i) - \frac{h^2}{2}f'(t_i, y_i) - \cdots - \frac{h^n}{n!}f^{(n-1)}(t_i, y_i) \\ &= \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i)) \end{aligned}$$

for some $\xi_i$ in $(t_i, t_{i+1})$.

# Higher-Order Taylor Methods

$$
\begin{aligned}
y_{i+1} \quad & -y_i - hf(t_i, y_i) - \frac{h^2}{2} f'(t_i, y_i) - \cdots - \frac{h^n}{n!} f^{(n-1)}(t_i, y_i) \\
& = \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))
\end{aligned}
$$

## Proof (2/2)

So the local truncation error is

$$
\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))
$$

for each $i = 0, 1, \cdots, N-1$. Since $y \in C^{n+1}[a, b]$, we have $y^{(n+1)}(t) = f^{(n)}(t, y(t))$ bounded on $[a, b]$ and $\tau_i(h) = O(h^n)$, for each $i = 1, \cdots, N$.

# Outline

# Runge-Kutta Methods

## Taylor Methods vs. Runge-Kutta Methods

- Taylor methods have the desirable property of high-order local truncation error.
- But the disadvantage of requiring the computation and evaluation of the derivatives of $f(t,y)$.
- This is a complicated and time-consuming procedure for most problems, so the Taylor methods are seldom used in practice.
- Runge-Kutta methods have the high-order local truncation error of the Taylor methods but eliminate the need to compute and evaluate the derivatives of $f(t,y)$.

# Runge-Kutta Methods

## Taylor Theorem in 2 Variables (1/2)

Suppose that $f(t, y)$ and all its partial derivatives of order less than or equal to $n+1$ are continuous on $D = \{(t, y) | a \leq t \leq b, c \leq y \leq d\}$, and let $(t_0, y_0) \in D$. For every $(t, y) \in D$, there exists $\xi$ between $t$ and $t_0$ and $\mu$ between $y$ and $y_0$ with

$$f(t, y) = P_n(t, y) + R_n(t, y)$$

The function $P_n(t, y)$ is called the $n$th Taylor polynomial in two variables for the function $f$ about $(t_0, y_0)$, and $R_n(t, y)$ is the remainder term associated with $P_n(t, y)$.

# Runge-Kutta Methods

## Taylor Theorem in 2 Variables (2/2)

$$P_n(t,y) = f(t_0,y_0) + \left[(t-t_0)\frac{\partial f}{\partial t}(t_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)\right]$$

$$+\left[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0)+(t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t\partial y}(t_0,y_0)+\frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right]$$

$$+\cdots+\left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^j\frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0,y_0)\right]$$

$$R_n(t,y) = \frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t-t_0)^{n+1-j}(y-y_0)^j\frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(\xi,\mu)$$

# Runge-Kutta Methods of Order Two

## Basic Structure of RK2 Methods

Our starting point is to assume that the numerical method has the following structure:

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h \cdot a_1 f(t_i + \alpha_1, w_i + \beta_1) \end{aligned}$$

for $i = 0, 1, \cdots, N-1$, where $a_1$, $\alpha_1$, and $\beta_1$ are parameters to be determined to ensure a local truncation error of $O(h^2)$.

# Runge-Kutta Methods of Order Two

## Method of Derivation

The first step is to determine values for $a_1$, $\alpha_1$, and $\beta_1$ with the property that

$$a_1 f(t+\alpha_1, y+\beta_1) \approx T^{(2)}(t,y) = f(t,y) + \frac{h}{2} f'(t,y),$$

with error no greater than $O(h^2)$, which is the same as the order of the truncation error for the Taylor method of order two. Since

$$f'(t,y) = \frac{df}{dt}(t,y) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y) \cdot y'(t) \quad \text{and } y'(t) = f(t,y),$$

we have

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}(t,y) \cdot f(t,y)$$

# Runge-Kutta Methods of Order Two

## Method of Derivation (Cont'd)

Expanding $f(t+\alpha_1, y+\beta_1)$ in its Taylor polynomial of degree one about $(t, y)$ gives

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y)$$

$$+a_1\beta_1 \frac{\partial f}{\partial y}(t,y) + a_1 R_1(t+\alpha_1, y+\beta_1)$$

where

$$R_1(t+\alpha_1, y+\beta_1) = \frac{\alpha_1^2}{2}\frac{\partial^2 f}{\partial t^2}(\xi,\mu) + \alpha_1\beta_1 \frac{\partial^2 f}{\partial t \partial y}(\xi,\mu) + \frac{\beta_1^2}{2}\frac{\partial^2 f}{\partial y^2}(\xi,\mu)$$

for some $\xi$ between $t$ and $t+\alpha_1$ and $\mu$ between $y$ and $y+\beta_1$.

# Runge-Kutta Methods of Order Two

## Method of Derivation (Cont'd)

Matching the coefficients of $f$ and its derivatives in

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) + a_1\beta_1 \frac{\partial f}{\partial y}(t,y) + a_1 R_1(t+\alpha_1, y+\beta_1)$$

and

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}(t,y)\cdot f(t,y)$$

gives the three equations

$$a_1 = 1 \qquad a_1\alpha_1 = \frac{h}{2} \qquad a_1\beta_1 = \frac{h}{2} f(t,y)$$

# Runge-Kutta Methods of Order Two

$$a_1 = 1 \qquad a_1\alpha_1 = \frac{h}{2} \qquad a_1\beta_1 = \frac{h}{2}f(t,y)$$

## Method of Derivation (Cont'd)

The parameters $a_1$, $\alpha_1$, and $\beta_1$ are therefore

$$a_1 = 1 \qquad \alpha_1 = \frac{h}{2} \qquad \beta_1 = \frac{h}{2}f(t,y)$$

so that

$$T^{(2)}(t,y) = f\left(t + \frac{h}{2}, y + \frac{h}{2}f(t,y)\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2}f(t,y)\right)$$

# Runge-Kutta Methods of Order Two

## Method of Derivation (Cont'd)

Earlier, we saw that

$$R_1(t+\alpha_1, y+\beta_1) = \frac{\alpha_1^2}{2}\frac{\partial^2 f}{\partial t^2}(\xi,\mu) + \alpha_1\beta_1\frac{\partial^2 f}{\partial t \partial y}(\xi,\mu) + \frac{\beta_1^2}{2}\frac{\partial^2 f}{\partial y^2}(\xi,\mu)$$

which leads to

$$R_1\left(t+\frac{h}{2}, y+\frac{h}{2}f(t,y)\right) = \frac{h^2}{8}\frac{\partial^2 f}{\partial t^2}(\xi,\mu) + \frac{h^2}{4}f(t,y)\frac{\partial^2 f}{\partial t \partial y}(\xi,\mu) + \frac{h^2}{8}(f(t,y))^2\frac{\partial^2 f}{\partial y^2}(\xi,\mu).$$

which is $O(h^2)$ if all the second-order partial derivatives of $f$ are bounded.

# Runge-Kutta Methods of Order Two

The difference-equation method resulting from replacing $T^{(2)}(t, y)$ in Taylor's method of order two by $f(t + (h/2), y + (h/2)f(t, y))$ is specific Runge-Kutta method known as the Midpoint Method.

## The Midpoint Method

$$
\begin{aligned}
w_0 &= \alpha \\
w_{i+1} &= w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right)
\end{aligned}
$$

for each $i = 0, 1, \cdots, N - 1$.

# Higher Order Runge-Kutta Methods

## The Heun Method of Order 3

The term $T^{(3)}(t,y)$ can be approximated with error $O(h^3)$ by an expression of the form

$$f(t+\alpha_1, y+\delta_1 f(t+\alpha_2, y+\delta_2 f(t,y)))$$

involving 4 parameters, but the algebra involved in the determination of $\alpha_1, \delta_1, \alpha_2$, and $\delta_2$ is quite involved. The most common $O(h^3)$ method is that of Huen, given by

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + \end{aligned}$$

$$\frac{h}{4}\left(f(t_i,w_i)+3f\left(t_i+\frac{2h}{3}, w_i+\frac{2h}{3}f\left(t_i+\frac{h}{3}, w_i+\frac{h}{3}f(t_i,w_i)\right)\right)\right)$$

# Higher Order Runge-Kutta Methods

## Runge-Kutta Order 4 Method

$$
\begin{aligned}
w_0 &= \alpha \\
k_1 &= hf(t_i, w_i) \\
k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right) \\
k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right) \\
k_4 &= hf(t_{i+1}, w_i + k_3) \\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

for each $i = 0, 1, \cdots, N-1$. This method has local truncation error $O(h^4)$, provided the solution $y(t)$ has five continuous derivatives.

# Higher Order Runge-Kutta Methods

## Example

- For the problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

  Euler's method with $h = 0.025$, the Midpoint method with $h = 0.05$, and the Runge-Kutta 4th-order method with $h = 0.1$ are compared at the common mesh points of these method 0.1, 0.2, 0.3, 0.4, and 0.5.
- Each of these techniques requires 20 function evaluations to determine the values (listed in the following table) to approximate $y(0.5)$.

# Higher Order Runge-Kutta Methods

| $t_i$ | Exact | Euler $h = 0.025$ | Modified Euler $h = 0.05$ | Runge-Kutta Order Four $h = 0.1$ |
|---|---|---|---|---|
| 0.0 | 0.5000000 | 0.5000000 | 0.5000000 | 0.5000000 |
| 0.1 | 0.6574145 | 0.6554982 | 0.6573085 | 0.6574144 |
| 0.2 | 0.8292986 | 0.8253385 | 0.8290778 | 0.8292983 |
| 0.3 | 1.0150706 | 1.0089334 | 1.0147254 | 1.0150701 |
| 0.4 | 1.2140877 | 1.2056345 | 1.2136079 | 1.2140869 |
| 0.5 | 1.4256394 | 1.4147264 | 1.4250141 | 1.4256384 |

In this example, the fourth-order method is clearly superior.

# Assignment

- Reading assignment: Chap. 5.1-5.4