PROJECT 2: MODEL ANALYSIS PROJECT

- **Vision:** *Programming is more than writing code*. The ultimate goal of the projects in this course is that you learn to structure a programming project from start to finish. Your answers should of course be correct, but your code should also be well-structured, and documented.
- **Objectives:** In your model analysis project, you should show that you can:
  1. Apply numerical model analysis methods (random numbers, numerical, optimization, root-finding, etc.)
  2. Present results in text form and in easy-to-understand figures
  3. Learn new functionalities in a known package on your own
  4. Structure a code project
  5. Document code
- **Structure:** Your model analysis project should consist of
  1. A single self-contained notebook (.ipynb) presenting the analysis
  2. Documented Python module files (.py) (if any)

  Hint: You can use the Python-files `worker.py` and `government.py` as starting point for your own code
- **Hand-in:** Upload to GitHub in your repository

  `github.com/NumEconCopenhagen/projects-2025-GROUPNAME/modelproject/`
- **Deadline:** See Calendar.
- **Feedback:** Your TA will provide feedback on your project.
- **Exam:** Your model analysis project will be a part of your exam portfolio.
  You can incorporate feedback before handing in the final version.
  This first version does not need to be perfect!

1 Labor supply

We consider workers indexed by i. Each worker has productivity $p_i > 0$ and chooses labor supply measured in hours $\ell_i \in [0, \overline{\ell}]$, where $\overline{\ell}$ is the maximum number of hours.

Post-tax income is

$$y_i = y(p_i, \ell_i) = (1-\tau)wp_i\ell_i - \zeta, \tag{1}$$

where w is the real wage, $\tau \in [0,1)$ is a proportional tax rate, and ζ is a lump-sum tax (or transfer if negative).

All workers have the same preferences, and their utility is

$$U(p_i, c_i, \ell_i) = \log c_i - \nu\frac{\ell_i^{1+\epsilon}}{1+\epsilon}. \tag{2}$$

They thus solve the following optimization problem:

$$\begin{aligned} V(p_i) = & \max_{c_i > 0, \ell_i \in [0,\overline{\ell}]} U(p_i, c_i, \ell_i) \\ & \text{s.t.} \\ & y_i = (1-\tau)wp_i\ell_i - \zeta \\ & c_i = y_i, \end{aligned} \tag{3}$$

where c_i is consumption, and $\epsilon > 0$, and $\nu > 0$.

To ensure positive consumption, the worker must choose

$$\ell_i > \underline{\ell}_i \equiv \max\left\{\frac{\zeta}{(1-\tau)\,wp_i}, 0\right\}.$$

Then the optimization problem can also be written as

$$V(p_i) = \max_{\ell_i \in (\underline{\ell}_i, \overline{\ell}]} U(p_i, (1-\tau)wp_i\ell_i - \zeta, \ell_i) \tag{4}$$

The first order condition (FOC) for ℓ_i is

$$\varphi(p_i, c_i, \ell_i) \equiv \frac{(1-\tau)\,wp_i}{c_i} - \nu\ell_i^{\epsilon} = 0. \tag{5}$$

If there is no solution to the FOC in $(\underline{\ell}_i, \overline{\ell}]$ then $\ell_i = \overline{\ell}_i$.

The calibration is $\overline{\ell} = 16$, $w = 1$, $\tau = 0.5$, $\zeta = 0.1$, $\epsilon = 1$ and $\nu = 0.015$.

1.1 Numerical optimizer vs root-finder

Assume $p_i = 1$. *Hint: Remember to use the provided* `WorkerClass` in `worker.py`.

1. Plot $U(p_i, c_i, \ell_i)$ with $c_i = y(p_i, \ell_i)$ for ℓ_i between 0.5 and $\overline{\ell}$
2. Plot $\varphi(p_i, c_i, \ell_i)$ with $c_i = y(p_i, \ell_i)$ for ℓ_i between 0.5 and $\overline{\ell}$
3. Find the optimal $\ell^\star(p_i)$ using a numerical optimizer to solve eq. 4
4. Find the optimal $\ell^\star(p_i)$ using a root-finder to solve eq. 5
5. Compare the results from the numerical optimizer and the root-finder, including which of the methods is fastest
6. Combine everything in a single figure (multiple plots are allowed)

Repeat the above for $\epsilon = 0.75$ and $\epsilon = 0.50$ writing as little as possible new code.

1.2 Labor supply function

1. Find the optimal $\ell^\star(p_i)$ for p_i from 0.5 to 3.0 and plot the results
2. Find the optimal $\ell^\star(p_i)$ for p_i from 0.5 to 3.0 when $\zeta = -0.1$ and plot the results

2 Public good

We now consider an economy with N workers and a government. Productivity is log-normally distributed

$$\log p_i \sim \mathcal{N}(-0.5\sigma_p^2, \sigma_p^2), \tag{6}$$

and workers behave as described above. We now denote the optimal labor choice by $\ell^\star(p_i; \tau, \zeta)$ to indicate it depends on the government's choice of the tax system. The implied utility is similarly denoted by $U(p_i; \tau, \zeta)$.

Total tax revenue is

$$T(\tau, \zeta) = N\zeta + \sum_{i=1}^{N} \tau w p_i \ell_i^\star(p_i; \tau, \zeta). \tag{7}$$

The government must have non-negative tax revenue, $T(\tau, \zeta) \geq 0$. Let $\underline{p}$ be the smallest value of p_i in the population. The government does not impose a lump-sum tax that some individuals cannot pay. Therefore

$$\zeta < (1 - \tau) w \underline{p} \overline{\ell}. \tag{8}$$

The government uses tax revenue to finance a public good $G(\tau, \zeta) = T(\tau, \zeta)$. The government seeks to optimize a so-called Social Welfare Function (SWF) given by the sum of all utilities and a value of the public good,

$$SWF(\tau,\zeta) = \chi G(\tau,\zeta)^{\eta} + \sum_{i=1}^{N} U(p_i;\tau,\zeta). \tag{9}$$

The calibration is $N = 100$, $\sigma_p = 0.3$, $\chi = \frac{1}{2}N$ and $\eta = 0.1$.

2.1 Tax revenue and Social Welfare Function

Hint: Remember to use the provided `GovernmentClass` in `government.py`.

1. Compute $T(\tau,\zeta)$ and $SWF(\tau,\zeta)$ for $\tau = 0.50$ and $\zeta = 0.1$.
   Hint: The requires four steps
   1) Draw productivity for each worker
   2) Find the labor supply for each worker
   3) Sum over taxes paid by all workers
   4) Evaluate Social Welfare Function (SWF)
2. Plot $T(\tau,\zeta)$ and $SWF(\tau,\zeta)$ for $\zeta \in \{-0.1, 0.0, 0.1\}$ and for τ between 0.01 and 0.80
3. Indicate in your plot, where the $SWF(\tau,\zeta)$ is maximized for your current evaluations

2.2 Optimal tax system

1. Use a numerical optimizer to find optimal tax system

$$\tau^*, \zeta^* = \arg \max_{\tau\in[0,1),\zeta\in(-\infty,(1-\tau)w\underline{p}\bar{\ell})} SWF(\tau,\zeta)$$

2. Verify that your choice of starting values for the numerical optimizer do not matter for the result
3. Plot $T(\tau,\zeta)$ and $SWF(\tau,\zeta)$ for values close to (τ^*,ζ^*)
4. Plot the Lorenz-curve of consumption when $\tau = \tau^*$ and $\zeta = \zeta^*$

3 Top tax

The government considers introducing a top tax. Then post-tax income would be

$$y_i = y(p_i, \ell_i) = (1-\tau)wp_i\ell_i - \omega \max\{wp_i\ell_i - \kappa, 0\} - \zeta. \tag{10}$$

where κ is the cut-off value and $\omega \in [0, 1-\tau)$, is the additional tax rate above the cut-off. Note that for $\ell_i < \frac{\kappa}{wp_i}$ the marginal tax rate is τ, while the marginal tax is $\tau + \omega$ for $\ell_i \geq \frac{\kappa}{wp_i}$.

We furthermore assume that the government restricts attention to a lump-sum transfer, i.e. $\zeta \leq 0$. This implies that it is always feasible for the worker to choose $\ell_i = 0$.

The problem can be solved with a numerical optimizer as before, but the kink in the tax system implies that a first order condition approach must be amended, and it can be done in the following four steps:

Step 1: Assume $\ell_i \in [0, \frac{\kappa}{wp_i})$. Find the solution (if any) to the FOC:

$$\underline{\varphi}(p_i, c_i, \ell_i) \equiv \frac{(1-\tau)\,wp_i}{c_i} - \nu\ell_i^{\epsilon} = 0. \tag{11}$$

Denote the solution ℓ_i^b and denote the implied utility by U_i^b. If no solution $U_i^b = -\infty$.

Step 2: Denote $\ell_i^k = \frac{\kappa}{wp_i}$ and denote the implied utility by U_i^k.

Step 3: Assume $\ell_i(\frac{\kappa}{wp_i}, \overline{\ell}]$. Find the solution (if any) to the FOC:

$$\overline{\varphi}(p_i, c_i, \ell_i) \equiv \frac{(1-\tau-\omega)\,wp_i}{c_i} - \nu\ell_i^{\epsilon} = 0. \tag{12}$$

Denote the solution ℓ_i^a and denote the implied utility by U_i^a. If no solution $U_i^a = -\infty$.

Step 4: Find the optimal choice as

$$\ell^{\star}(p_i) = \begin{cases} \ell_i^b & \text{if } U_i^b > \max\{U_i^k, U_i^a\} \\ \ell_i^k & \text{if } U_i^k > \max\{U_i^b, U_i^a\} \\ \ell_i^a & \text{if } U_i^a > \max\{U_i^b, U_i^k\} \end{cases} \tag{13}$$

The government keep $\tau = \tau^*$and $\zeta = \zeta^*$ and initially tries $\kappa = 9$ and $\omega = 0.2$.

3.1 Labor supply

1. Plot $U(p_i, c_i, \ell_i)$ with $c_i = y(p_i, \ell_i)$ for ℓ_i between 0.5 and $\overline{\ell}$
2. Plot $\underline{\varphi}(p_i, c_i \ell_i)$ with $c_i = y(p_i, \ell_i)$ for ℓ_i between 0.5 and $\frac{\kappa}{w p_i}$
3. Plot $\overline{\varphi}(p_i, c_i \ell_i)$ with $c_i = y(p_i, \ell_i)$ for ℓ_i between $\frac{\kappa}{w p_i}$ and $\overline{\ell}$
4. Find the optimal $\ell^\star(p_i)$ using a numerical optimizer
5. Find the optimal $\ell^\star(p_i)$ using the four step first order condition approach described above[1]
6. Compare the results from the numerical optimizer and the root-finder, including which of the methods is fastest
7. Combine everything in a single figure (multiple plots are allowed)

Repeat the above for $p_i = 1.175$ and $p_i = 1.5$ writing as little as possible new code.

3.2 Labor supply function

1. Plot both $\ell^\star(p_i)$ and the implied $c_i = y(p_i, \ell_i)$ for p_i between 0.5 and 3.0
2. What proportions of workers has $\ell^\star(p_i) = \ell_i^b$, $\ell^\star(p_i) = \ell_i^k$ and $\ell^\star(p_i) = \ell_i^a$?

3.3 Public good

1. How does the value of the SWF change when introducing the top tax rate?
2. How does the Lorenz-curve for consumption change?
3. Can you find a ω and κ which implies an improvement in the SWF?

4 Extension

Extend the model above in one direction of your choice. E.g. you could:

1. Change the utility function
2. Make preferences heterogeneous
3. Introduce a new tax function

Solve your extended model and explain what new economic insight it brings.

[1] This is the hardest question in the project. If you can *not* solve it, all the questions in 3.2 and 3.3 can still be answered using the numerical optimizer results.