Written Exam Economics summer 2023

Introduction to Programming and Numerical Analysis

25th – 27th May 2023

This exam question consists of 6 pages in total

(An electronic version of the notebook with the exam questions is also available).

Answers only in English.

You should hand-in a single zip-file named with your groupname only. The zip-file should contain:

1. A general README.md for your portfolio

3. Your inaugural project (in the folder /inauguralproject)

4. Your data analysis project (in the folder /dataproject)

5. Your model analysis project (in the folder /modelproject)

6. Your exam project (in the folder /examproject)

Be careful not to cheat at exams!

Exam cheating is for example if you:

- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Reuse parts of a written paper that you have previously submitted and for which you have received a pass grade without making use of quotation marks or source references (self-plagiarism)
- Receive help from others in contrary to the rules in the Faculty of Social Science's common part of the curriculum

You can read more about the rules on exam cheating on your Study Site and in the Faculty of Social Science's common part of the curriculum.

Exam cheating is always sanctioned by a written warning and expulsion from the exam in question. In most cases, the student will also be expelled from the University for one semester.

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1. Problem 1: Optimal taxation with government consumption

Consider a worker choosing hours of labor, $L \in [0, 24]$, to maximize utility:

$$V(w,\tau,G) = \max_{L\in[0,24]} \ln\left(C^{\alpha}G^{1-\alpha}\right) - \nu\frac{L^2}{2}$$
$$\text{s.t.}$$
$$C = \kappa + (1-\tau)wL$$

where

- C is *private* consumption with weight $\alpha \in (0, 1)$.
- $\kappa > 0$ is the *free private* consumption component.
- $C = (1-\tau)wL$ is the *costly private* consumption component.
- $w > 0$ is the real wage.
- $\tau \in (0, 1)$ is the labor-income tax rate.
- $G > 0$ is *government* consumption with weight $1 - \alpha$.
- $\nu > 0$ is the disutility of labor scaling factor

The baseline parameters are:

$$\begin{aligned}\alpha &= 0.5\\ \kappa &= 1.0\\ \nu &= \frac{1}{2\cdot 16^2}\\ w &= 1.0\\ \tau &= 0.30\end{aligned}$$

Question 1: Verify that the optimal labor supply choice is $L^{\star}(\tilde{w}) = \frac{-\kappa+\sqrt{\kappa^2+4\frac{\alpha}{\nu}\tilde{w}^2}}{2\tilde{w}}$, where $\tilde{w} = (1-\tau)w$, for $G \in \{1.0, 2.0\}$.

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Question 2: Illustrate how $L^{\star}(\tilde{w})$ depends on w.

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We now consider a government, who chooses τ and spend all of the taxes on government consumption so:

$$G = \tau w L^{\star}((1-\tau)w)$$

Question 3: Plot the implied L, G and worker utility for a grid of τ-values.

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Question 4: Find the socially optimal tax rate $\tau^{\star} \in (0,1)$ maximizing worker utility. Illustrate your result.

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A more general preference formulation for the worker is:

$$\begin{aligned}
\mathcal{V}(w,\tau,G) &= \max_{L\in[0,24]} \frac{\left[\left(\alpha C^{\frac{\sigma-1}{\sigma}}+(1-\alpha)G^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{1-\sigma}}\right]^{1-\rho}-1}{1-\rho} - \nu\frac{L^{1+\varepsilon}}{1+\varepsilon}, \quad \varepsilon,\rho,\sigma>0, \quad \rho,\sigma\neq 1 \\
&\text{s.t.} \\
&C = \kappa+(1-\tau)wL
\end{aligned}$$

Optimal labor supply is now $L^{\star}(\tilde{w},G)$.

Questions 5 and 6 must be answered with the general formulation, and for 2 different set of parameters:

- Set 1: $\sigma = 1.001$, $\rho = 1.001$ and $\varepsilon = 1.0$.
- Set 2: $\sigma = 1.5$, $\rho = 1.5$ and $\varepsilon = 1.0$.

Question 5: Find the G that solves $G = \tau w L^{\star}((1-\tau)w, G)$ using the τ found in question 4.

Hint: First write code that solves the worker problem for given values of G and τ. Then find the correct G based on this.

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Question 6: Find the socially optimal tax rate, $\tau^{\star}$, maximizing worker utility, while keeping $G = \tau w L^{\star}((1-\tau)w, G)$.

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2. Problem 2: Labor adjustment costs

You own a hair salon. You employ hairdressers, ℓ_t, to produce haircuts, $y_t = \ell_t$.

The wage for each haridresser is w.

The demand for haircuts implies that the price of haircuts you can charge is $p_t = \kappa_t y_t^{-\eta}$, where κ_t is a demand-shock and $\eta \in (0,1)$ measures the elasticity of demand.

Profits are:

$$\Pi_t = p_t y_t - w\ell_t = \kappa_t \ell_t^{1-\eta} - w\ell_t$$

Baseline parameters are:

- $\eta = 0.5$
- $w = 1.0$

Question 1: Verify numerically that $\ell_t = \left(\frac{(1-\eta)\kappa_t}{w}\right)^{\frac{1}{\eta}}$ maximises profits, for $\kappa \in \{1.0, 2.0\}$.

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We now consider a *dynamic* version of the model.

- The demand-shock is a so-called AR(1) in logs,

$$\log \kappa_t = \rho \log \kappa_{t-1} + \epsilon_t, \quad \epsilon_{t+1} \sim \mathcal{N}(-0.5\sigma_\epsilon^2, \sigma_\epsilon)$$

- Any hiring or firing implies a fixed adjustment cost, $\iota > 0$.
- Future profits are discounted with a monthly factor of $R \in (0,1)$.

The initial demand shock is $\kappa_{-1} = 1$ and the planning horizon is 10 years, i.e. 120 months so $t \in \{0, 1, 2, \ldots, 119\}$. Initially you don't have any employees, $\ell_{-1} = 0$

The *ex post* value of the salon is *conditional* on the shock series is:

$$h(\epsilon_0, \epsilon_1, \ldots, \epsilon_{119}) = \left[\sum_{t=0}^{119} R^{-t}\left[\kappa_t \ell_t^{1-\eta} - w\ell_t - \boldsymbol{1}_{\ell_t \neq \ell_{t-1}}\iota\right]\right]$$

The *ex ante* expected value of the salon can be approximated by

$$H = \mathbb{E}[h(\epsilon_0, \epsilon_1, \ldots, \epsilon_{119})] \approx \frac{1}{K}\sum_{k=0}^{K} h(\epsilon_0^k, \epsilon_1^k, \ldots, \epsilon_{119}^k)$$

where each $k \in \{0, 1, \ldots, K-1\}$ is a random shock series. Maximizing profitability means maximizing H.

Baseline parameters are:

- $\rho = 0.90$
- $\iota = 0.01$
- $\sigma_\epsilon = 0.10$
- $R = (1 + 0.01)^{1/12}$

Question 2: Calculate H if the policy $\ell_t = \left(\frac{(1-\eta)\kappa_t}{w}\right)^{\frac{1}{\eta}}$ from question 1 is followed. Choose K so the approximation is good enough to not affect your results substantially.

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Next, we consider policies on the form:

$$\ell_t = \begin{cases} \ell_t^* & \text{if } |\ell_{t-1} - \ell_t^*| > \Delta \\ \ell_{t-1} & \text{else} \end{cases}$$
$$\text{where } \ell_t^* = \left(\frac{(1-\eta)\kappa_t}{w}\right)^{\frac{1}{\eta}}$$

With $\Delta \geq 0$ and $\Delta = 0$ being the previous policy.

Question 3: Calculate H if the policy above was followed with $\Delta = 0.05$. Does it improve profitability?

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Question 4: Find the optimal Δ maximizing H. Illustrate your result.

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Question 5: Suggest an alternative policy you believe might improve profitability. Implement and test your policy.

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3. Problem 3: Global optimizer with refined multi-start

We consider the Griewank function:

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{x_i^2}{4000} - \prod_{i=1}^{n} \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1$$

The **global minimum** of this function is $f(0,0) = 0$ (remember: $\cos(0) = 1$).
But the function also have a lot of **local minima**.

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def griewank(x):
    return griewank_(x[0],x[1])

def griewank_(x1,x2):
    A = x1**2/4000 + x2**2/4000
    B = np.cos(x1/np.sqrt(1))*np.cos(x2/np.sqrt(2))
    return A-B+1
```

A **refined global optimizer with multi-start** is:

1. Choose *bounds* for $\mathbf{x}$ and *tolerance* $\tau > 0$.
2. Choose number of *warm-up iterations*, $\underline{K} > 0$ and *maximum number of iterations*, $K > \underline{K}$.
3. In each iteration for $k \in \{0, 1, \ldots, K-1\}$:

   A. Draw random $\mathbf{x}^k$ uniformly within chosen bounds.

   B. If $k < \underline{K}$ go to step E.

   C. Calculate $\chi^k = 0.50 \cdot \frac{2}{1+\exp((k-\underline{K})/100)}$

   D. Set $\mathbf{x}^{k0} = \chi^k \mathbf{x}^k + (1-\chi^k)\mathbf{x}^*$

   E. Run optimizer with $\mathbf{x}^{k0}$ as initial guess and $\mathbf{x}^{k*}$ as result.

   F. Set $\mathbf{x}^* = \mathbf{x}^{k*}$ if $k = 0$ or $f(\mathbf{x}^{k*}) < f(\mathbf{x}^*)$

   G. If $f(\mathbf{x}^*) < \tau$ go to step 4.

4. Return the result $\mathbf{x}^*$.

As settings we choose:

- $x_1, x_2 \in [-600, 600]$
- $\tau = 10^{-8}$
- $\underline{K} = 10$
- $K = 1000$

The optimizer in Step 3.E is `BFGS` with a tolerance of τ.

Question 1: Implement the refined global optimizer with multi-start. Illustrate how the effective initial guesses $\mathbf{x}^{k0}$ vary with the iteration counter k.

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Question 2: Is it a better idea to set $\underline{K} = 100$? Is the convergence faster?

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