A simple dynamic Malthusian model

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Abstract

This Lecture note, for economic history, develops the technical aspects of the basic dynamic Malthusian macro model. It complements the discussion in Clark (2007, Chapter 2-3).

1 Basic elements of the model

Consider an economy in the process of development. Time is discrete, $t = 0, 1, 2...\infty$. The economy is closed. As a result, total production equals total income. As noted in Clark (2007) the Malthusian model involves three basic model elements. As a matter of fact, two will do, as will be clear.

The first assumption is that labor is subject to diminishing returns in production. We capture this by invoking the following Codd-Douglas production function

$$Y_t = L_t^{1-\alpha} (AX)^{\alpha}, \ 0 < \alpha < 1,$$

where t denotes time, $t = 0, 1, 2..$, Y_t is output, A is the (constant) level of technology, L_t is labor input whereas X is a fixed factor of production: land. Observe that we can write output per worker

$$y_t = (AX/L_t)^{\alpha}. \tag{1}$$

Accordingly, as L increases, output per worker declines due to the imposed diminishing returns which follow from $\alpha < 1$, given AX is constant. For now we will ignore the possibility that A could be growing; technological change is thus viewed merely as a series of discrete shocks (largely upward). Of course, one might wonder if persistent *growth* in A could change the result derived below. We return to this issue as a robustness check.

The second key assumption is that the birth rate in the economy (i.e., births per capita) rises with per capita income

$$n_t = \eta y_t, \tag{2}$$

where η is a positive parameter. That is, each period every person in the economy (and the average, or "representative", individual) has n_t off spring; mere reproduction from the point of view of the "housedhold" would thus be associated with $n_t = 1$. We assume that the number of kids that is preferred is rising in per capita income.[1] With this interpretation one could further think of η as being bounded below one, implying that each parent in the economy uses a fixed fraction, η, of their income to rear children (food, clothing and so on). The remaining part is consumed by the adult. Implicitly therefore η can be thought to capture e.g. preferences for family size, cost of child rearing etc.[2]

We will assume, for simplicitly, that everyone works; i.e., no unemployment. Accordingly, L_t is the size of the labor force and total employment. This means that we can assume the size of the labor force (and thus employment) evolves according to

$$L_{t+1} = n_t L_t + (1-\mu) L_t, \;\; L_0 \text{ given.} \tag{3}$$

Hence, the size of the labor force next period (L_{t+1}) equals the new entrants into the population (period t births per person times total population, $n_t L_t$) plus the part of this periods labor force, which remains in the labor force (here: the fraction $(1-\mu)$). If people work until they drop, we can think of μ as reflecting mortality; a higher μ means greater mortality. In an effort not to maximize complexity for given level of insight we keep it as a parameter, rather than as being dependent on y as in Clark (2007). Hence, this is the element we are "doing away with" to keep things simple, yet without loss of essential insight. With this the model is complete.[3]

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[1] For the technically interested. One could easily assume people have different income levels, and thus different levels of fertility. But in the aggregate this would be of no consequence when the "demand function" (equation 2) is *linear* in income. If individuals are denoted by $i = 1, ..N$, we would have $n_{it} = \eta y_{it}$. The average level if fertility in the economy would be

$$n_t \equiv \frac{1}{N}\sum n_{it} = \eta \frac{1}{N}\sum y_{it} = \eta y_t$$

with $y_t \equiv 1/N \sum y_{it}$, exactly as in equation (2). Hence for the aggregate dynamics the distribution of income does not matter, in this case. Hence, following Clark we supress distributive issues in the remaining.

[2] For the interested reader, the Appendix offers som microfoundations for this equation. A study that derives optimal family size from first principles and proceed to examine Malthusian dynamics, as well as empirical evidence, is found in Ashraf and Galor (2011).

[3] The point is that we can, in principle, dispense with one of the three assumptions highlighted by Clark. Here I drop the negative link between mortality and income. But alternatively one could retain this assumption, and in stead drop the positive link between n and y, yet obtain the same results are those derived below. The assumption I make is not motivated by a deep felt belif that the link between n and y somehow should be regarded as nessesarily more fundamental. In some pre-historic societies it probably was; but in others it might not have been as relevant, in which case the link between mortality and y is required for the results derived below. Either way, Occam's razor would suggest we take the simplest version of the model, and I believe this is it.

2 Macrodynamics

Inserting equation (1) into (2), and the result into equation (3) we obtain

$$L_{t+1} = \eta L_t^{1-\alpha} (AX)^{\alpha} + (1-\mu) L_t \equiv \Phi(L_t), \; L_0 \text{ given,}$$

which is the law of motion for the labor force.

We adopt the following definition

Definition 1 *Steady state equilibrium. A steady state equilibrium of the model is a* $L_{t+1} = L_t = L^*$*, such that* $L^* = \Phi(L^*)$*.*

In order to construct the phasediagram we observe the following properties of the law of motion, Φ:

$$\begin{aligned}
\Phi(0) &= 0 \\
\Phi'(L) &= \eta(1-\alpha) L_t^{-\alpha} (AX)^{\alpha} + (1-\mu) > 0 \\
\Phi''(L) &= -\eta\alpha(1-\alpha) L_t^{-\alpha-1} (AX)^{\alpha} < 0 \\
\lim_{L\to 0} \Phi'(L) &= \infty \\
\lim_{L\to\infty} \Phi'(L) &= 1-\mu < 1
\end{aligned}$$

Figure 1 illustrates the phasediagram, plotting both $L_{t+1} = \Phi(L_t)$ and the steady state requirement $L_{t+1} = L_t$. The function Φ starts in $(0,0)$ (as $\Phi(0) = 0$), is upward sloping ($\Phi' > 0$) yet with diminishing slope ($\Phi'' < 0$). Close to the origin the slope is infinite ($\lim_{L\to 0} \Phi'(L) = \infty$) and as L rises the slope asymptotes to $1-\mu < 1$. As can be seen there is a unique intersection point between the law of motion, Φ, and the 45 degree line ($L_{t+1} = L_t$), which therefore consistutes the unique steady state equilibrium. The intersection point is ensured by the fact that $\lim_{L\to\infty} \Phi'(L) = 1-\mu$ which is stricktly smaller than the slope of the $L_{t+1} = L_t$ (or 45 degree line), which is equal to one.

We can also assess the stability properties, using the phasediagram. Starting at some arbitrary level $L_0 < L^*$ we can observe that L_1 ($= \Phi(L_0)$) evidently is greater than L_0, and similarly is $L_2 = \Phi(L_1) > L_1$ as illustrated in the figure. As we move forward in time, therefore, L gradually edges closer and closer to L^*, at which point $\Phi(L^*) = L^*$ and $L_{t+1} = L_t = L^*$. Similarly, if we begin with $L_0 > L^*$, population declines towards L^*. Accordingly, we can conclude the steady state is stable (globally, in fact, as we can choose any $L_0 > 0$, and always reach L^*).

Economically, the adjustment works as follows (here we examine the case illustrated in the figure, where $L_0 < L^*$). When L is small income per capita (y) is large (due to diminishing return), which implies fast labor force growth (i.e.,large n). In the next period the labor force is greater, which means income per worker

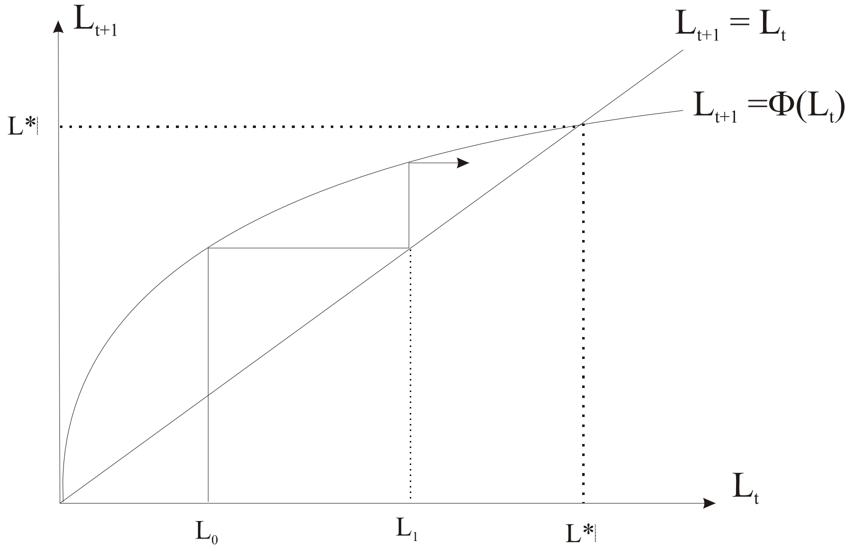

Figure 1: Phase diagram of the Malthus model.

must decline due to diminishing returns. As a result, labor force growth slows down. Gradually, therefore, income declines in transition towards the steady state, along with growth in the labor force. Eventually, L reaches a level which is consistent with an income level such that n is just high enough to keep the overall labor force constant (i.e., $n = \mu$; new entrants exactly off set those that exit). A symmetrical argument can be provided when $L_0 > L^*$; here income per worker is initially below the steady state level, which implies labor force growth below replacement in transition along with income per worker growth).

3 Steady state comparative statics

Analytically, the steady state level of L is given by:

$$\begin{aligned} L^* &= \eta \left(L^*\right)^{1-\alpha} \left(AX\right)^{\alpha} + \left(1-\mu\right) L^* \\ &\Rightarrow \; L^* = \left(\frac{\eta}{\mu}\right)^{1/\alpha} AX \end{aligned}$$

where we have used the definition $L_{t+1} = L_t = L^*$. We have the following result.

Proposition 1 *Population density (L^*/X) rises with greater levels of technology ($A \uparrow$), if more ressources are employed for child rearing ($\eta \uparrow$) or if the mortality declines ($\mu \downarrow$).*

Hence, technological advancement leads to greater population size (and density) in the long-run. In terms of the phasediagram, the Φ-function shifts up, revealing a new intersection point involving a larger labor force (the same is true if μ declines, or η goes up).

The economics are as follows, in the case of a technological innovation of a permanent nature. A permanent increase in A (a new agricultural technology, or way of organizing agriculture that increases productivity) initially increases y. As a result, labor force growth increases, and the labor force starts to rise. In the next period, however, a greater L serves to lower income per worker below the level of the previous period. As a result, growth in L slows down. This process continues until the economy is yet again in steady state. In the end, better economic opportunities admits more people to be alive simultanously, at a given amount of land. Similarly, lower mortality and more investments in child rearing will work to increase long-run population density (L/X).

Meanwhile, what about long run living standards? Inserting the solution for L^*, into equation (1) we obtain:

$$\begin{aligned} y^* &= \left[(AX/L)^*\right]^\alpha \\ &= \left[\left(\frac{\eta}{\mu}\right)^{-1/\alpha}\right]^\alpha \\ &= \frac{\mu}{\eta}. \end{aligned}$$

Proposition 2 *Living standards* (y^*) *increases if mortality rises* $(\mu\uparrow)$ *or if less resources are spend on having off-spring* $(\eta\downarrow)$. *Steady state income per capita is independent of the level of technology* (A), *the production technology more broadly (i.e.,* α*), as well as size of territory (*X*).*

Hence, we are able to reconfirm the (superficially) "counterintuitive" logic of the Malthusian model, noted by Clark (p. 27):

> Anything that raised the death rate schedule—war, disorder, disease, poor sanitary practices, or abandoning breast feeding—increased material living standards. Anything that reduced the death rate schedule—advances in medical technology, better personal hygiene, improved public sanitation, public provision for harvest failures, peace and order—reduced material living standards.

This is captured in the model by the fact that any change that serves to lower mortality (μ) will eventually lead to permanently lower income per worker, since this change induces a greater population density, L/X.

Similar remarks hold true for fertility practises that allow for higher fertility (i.e., factors captured by η). Whenever the organization of society encourages greater families (crudely captured by a greater η) population will rise, but leaving everyone poorer in the long-run.

Notice, that if we had assumed individuals face a *tax* on their income, then its impact on L^* would be isomorphic to a change in η. To see this, suppose we had specified equation (2) as $n = \theta(1-\tau)y$, where τ

is the tax rate. We could then simply redefine terms, writing $n = \eta y$, with $\eta \equiv \theta(1-\tau)$. Nothing would change in the analysis, and we could conclude that higher taxes lowers the size of the labor force in the long-run. Does this mean that taxes would increase income? The answer is "no" if we are focusing on *after tax* income. After tax income is given by $y^d = (1-\tau)\,y$, and since $y = \frac{\mu}{\theta(1-\tau)}$, it follows that steady state disposable income would be unaffected by the tax, and given by $y^d = \mu/\theta$. The only consequence of the policy would be to lower the size of the labor force. In this sense economic policy would be ineffective in affecting *livings standards* in the Malthusian world.[4]

Another way to put the same result is to say that the "deadweight loss" from taxation is zero in this economy, *when evaluated at steady state.* The deadweight loss of taxation relates to the distortion inflicted on the economy from taxation, which usually involves an income loss that then has to be weightned against the potential gains from what the tax buys.[5] But in the Malthusian economy net income will, eventually, be *the same* whether or not taxes are present. As a result, there are no such costs from taxation in this setting.[6]

We also confirm the result that technological advancement, in a Malthusian setting, does not allow for persistently higher income. This is of course only true as matter of steady state comparative statistics. Hence, this results is not inconsistent with periods during which income rises - temporarily - due to technological change; along any transition path associated with an increase in A, income per worker will be greater than before the shock. However, since population adjusts, these gains are temporary in nature. The model is therefore able to account for the apparent fact that millenia of technological change did to serve to increase living standards appreciably. Note also, that increases in the size of territory (expansion of X) has an impact on per capita income that is similar to that of increases in A: in the short run income per capita rises, but eventually population adjusts rendering income per capita unaffected.

We wil refer to y^* as *the level of susistence.* It reflects the level of income which is consistent with a constant labor force size (and population size). Hence, it should not be seen as reflecting that people are "at the point of starvation". Nor does it imply that the level of income per worker is the same everywhere; i.e., in every society. If societies differ in terms of μ and η, income per capita (in steady state) will differ as well.

[4] One might complain that the implicit assumption made is that the government (or whoever is doing the tax collection) is simply squandering the money. But the same "ineffectiveness" result applies if we were to assume that $A = (\tau Y)^{\phi}, \phi < 1$. That is, productivity is increasing in total revenue, though subject to diminishing returns. In this case the impact of τ on population size depends on the exact level of τ; but *disposable* steady state income is unaffected regardless of what level is chosen, provided $\tau > 0$. The intuition is that, as we have seen, changes in A (now implicitly affected by τ) has no impact on income per capita in this Malthusian setting.

[5] Formally, the dead weight loss (or "excess burden") of taxation is the welfare loss above and beyond the revenue collected.

[6] Traditionally these considerations involve "welfare" (or utility) comparisons rather than statements about income. Suppose people derive utility from consumption (own) and from fertility (off spring, n). Now $n = \theta(1-\tau)\,y$, and suppose consumption, $c = (1-\theta)(1-\tau)\,y$. As established above: $(1-\tau)\,y$ is left unaffected in the long-run, implying that - evaluated at the steady state - the economy cannot be suffering a welfare loss from taxation. The limitation of the argument is, of course, that there are generations that *does* suffer a loss; the ones living through the adjustment period between steady states.

4 Wait! Could A not have been growing?

At some level one might wonder if we have been "stacking the deck" in favor of stagnation by assuming a constant (parametrically given) level of technology. To see if this is the case, let us assume

$$\frac{A_{t+1}}{A_t} = g$$

where g thus is the growth factor (i.e. a positive growth *rate* in A requires $g > 1$). Stagnation in A is thus the special case where $g = 1$.

Since we now have technological change, we have to work with a transformed system for a steady state to emerge. Define $l_t \equiv L_t/A_t$, and start by dividing the law of motion by A_t:

$$\begin{aligned}
\frac{L_{t+1}}{A_t} &= \eta A_t^{-1} \left(L_t\right)_t^{1-\alpha} \left(A_t X\right)^{\alpha} + (1-\mu)\frac{L_t}{A_t} \\
&\Leftrightarrow \\
\frac{L_{t+1}}{A_t} &= \eta A_t^{-(1-\alpha)} L_t^{1-\alpha} X^{\alpha} + (1-\mu)\frac{L_t}{A_t} \\
&\Leftrightarrow \\
\frac{L_{t+1}}{A_t} &= \eta \left(\frac{L_t}{A_t}\right)^{1-\alpha} X^{\alpha} + (1-\mu)\frac{L_t}{A_t}.
\end{aligned}$$

Now, L_{t+1}/A_t is obviously not l_{t+1}. But we may note that

$$\frac{L_{t+1}}{A_t} = \frac{L_{t+1}}{A_{t+1}}\frac{A_{t+1}}{A_t} = l_{t+1} g$$

If we substitute this into the law of motion above we are left with

$$l_{t+1} = \eta g^{-1} l_t^{1-\alpha} X^{\alpha} + g^{-1}(1-\mu)\, l_t \equiv \Gamma(l_t)\,, l_0 \text{ given.}$$

Definition 2 *Steady state equilibrium. A steady state equilibrium of the model is a* $l_{t+1} = l_t = l^*$, *such that* $l^* = \Gamma(l^*)$.

We can now proceed exactly as above, so as to convince ourselves that a steady state exists, is unique and stable. In the steady state we have

$$
\begin{aligned}
l^* &= \eta g^{-1} \left(l^*\right)^{1-\alpha} X^\alpha + g^{-1} \left(1-\mu\right) l^* \\
&\Leftrightarrow \\
l^* \left(1 - g^{-1} \left(1-\mu\right)\right) &= \eta g^{-1} \left(l^*\right)^{1-\alpha} X^\alpha \\
&\Leftrightarrow \\
\left(l^*\right)^\alpha &= \frac{\eta g^{-1}}{\left[1 - g^{-1} \left(1-\mu\right)\right]} X^\alpha \\
&\Leftrightarrow \\
\left(\frac{L}{AX}\right)^* &\equiv \frac{l^*}{X} = \left(\frac{\eta g^{-1}}{\left(1 - g^{-1} \left(1-\mu\right)\right)}\right)^{1/\alpha} = \left(\frac{\eta}{g+\mu-1}\right)^{1/\alpha}
\end{aligned}
$$

Hence, in the long-run the labor force is growing at the same rate as A: much like above we therefore find that innovations (technological change) serves to increase population size.[7] To examine the level of income per worker we insert the result into the production function:

$$
y^* = \left[\left(AX/L\right)^*\right]^\alpha = \frac{g+\mu-1}{\eta}
$$

We note, that *if* $g = 1$ (i.e., $A_{t+1} = A_t$ meaning no growth) we obtain the same result as above, as a special case. The bottom line is, however, that *even* with exponential growth of technology, income would stagnate though at a slightly higher level of subsistence, as $\frac{g+\mu-1}{\eta} > \frac{\mu}{\eta}$.

In practise, however, there is reason to believe that the model involving less-than-exponential growth (above taken to the extreme: stagnation) is a better description than the exponential growth version. Namely, that world population today is "only" about seven billion. To see the logic, suppose we allowed for exponential growth in A at a creeping pace of 0,1% per year; 1/20th of the rate observed today. Starting (for simplicity) with a unique human being, then 100,000 years of history should have produced a world population of about $2{,}6 \cdot 10^{37}$ rather than the observed $7 \cdot 10^9$. Hence, even with technological change 1/20th of current rates produces counter factual population numbers that are clearly out-of-this-world in size. It seems like a reasonable assumption, then, that technological change was much (much) slower than today, viewed across human history as a whole. Yet, in theory, even *if* technological change had been as high as today (at 2 pct say), income per capita would still stagnate albeit along a slightly higher level than absent growth in A.

[7] If the ratio L/AX is constant, it implies that the numerator and denominator are growing at the same rate; since X is constant we can conclude that L grows at the rate of A, captured by g.

5 Closing remarks

The key element that prevents technological change from translating into improvements in living standards is the positive feed back from income to population growth, equation (2). Once this feature is removed growth in income per capita becomes viable. To see this formally, suppose equation (2) is replaced with

$$n_t = \bar{n}$$

i.e., some constant (possibly) small growth factor. We have therefore killed-off the feed-back from income to population growth entirely. The law of motion for L_t becomes:

$$L_{t+1} = \bar{n}L_t + (1 - \mu) L_t$$

implying

$$L_{t+1}/L_t = 1 + \bar{n} - \mu$$

a constant rate (and positive, if births exceed deaths). What about income per capita? From the production function:

$$y_t = [(AX/L_t)]^{\alpha} \Rightarrow \frac{y_{t+1}}{y_t} = \left(\frac{A_{t+1}/A_t}{L_{t+1}/L_t}\right)^{\alpha}.$$

Hence, as long as $g > 1 + \bar{n} - \mu$ growth in income per capita ensues.

Accordingly, if A is to increase y on a more permanent basis the positive feed-back from income to population growth needs to be served. It was, eventually, in the context of *the demographic transition.* Hence, it is of first order importance to understand this fundamental transition as well, including which drivers might have been responsibel for changing the nature of the link between income and family size, which seems to have charactarized the human species (as well as basically any other species on the planet) throughout history.

As noted at the intro lecture, Clark tends to downplay the importance of the demografic transition, viewing it as *derived* from a broader- more fundamental- revolution: the Industrial revolution. Hence, at this point there is a difference in opinon between Clark (as he expresses himself in the book; it appears he has changed his mind somewhat since), and the macro literature on the subject, which we will return to later on.

A Fertility: Some microfoundations

Consider a "one person household" with the following preferences

$$u_t = \beta \log (c_t) + (1-\beta) \log (n_t), \tag{A1}$$

where $\beta \in (0,1)$. Hence, the individual derives utility from own consumption as well as from the number of off spring (n_t). Next, the budget constraint is

$$y_t \geqq \lambda n_t + c_t \tag{A2}$$

where y_t is the income of the individual, which can be spend on either child rearing (e.g., feeding and clothing the child) as well as own consumption. The relative price of off spring (in terms of consumption) is λ. We need to maximize (A1) subject to (A2).

The easiest way to proceed is to solve by substitution (and by substituting for c as we are interested in the solution for n). Hence, inserting from the budget constraint into the utility function we are left with the problem:

$$\max_{n_t} \beta \log (y_t - \lambda n_t) + (1-\beta) \log (n_t)$$

Differentiating wrt n we obtain the first order condition:

$$\begin{aligned}
\beta \frac{\lambda}{y_t - \lambda n_t} &= (1-\beta) \frac{1}{n_t} \\
&\Leftrightarrow \\
\lambda \beta n_t &= (1-\beta)(y_t - \lambda n_t) \\
&\Leftrightarrow \\
(\lambda \beta + (1-\beta) \lambda) n_t &= (1-\beta) y_t \\
&\Rightarrow \\
n_t &= \left(\frac{1-\beta}{\lambda}\right) y_t
\end{aligned}$$

Note that if we define $\eta \equiv \left(\frac{1-\beta}{\lambda}\right)$ we are left with equation (2) in the text. Inserting into the budget constraint we can derive optimal consumption, if we so desire.

As can be seen, with these microfoundations η is expected to be larger in societies where there is a particular cultural valuation of having many children (large weight on n in the utility function, $1-\beta$) as well in societies where the cost of child rearing are modest (λ).

References

Ashraf, Q. and O. Galor, 2011. Dynamics and Stagnation in the Malthusian Epoch. American Economic Review, 101 p. 2003–41

Clark, G., 2007. A Farewell to Alms. Princeton University Press.