

Written exam Economics summer 2024

Introduction to Programming and Numerical Analysis

22nd-24th June 2024

This exam question consists of 7 pages in total

(An electronic version of the notebook with the exam questions is also available)

Answers only in English.

You should hand-in a single zip-file named with your group name only. The zip-file should contain:

- 1. A general README.md for your portfolio**
- 3. Your inaugural project (in the folder /inauguralproject)**
- 4. Your data analysis project (in the folder /datapoint)**
- 5. Your model analysis project (in the folder /modelproject)**
- 6. Your exam project (in the folder /examproject)**

Use of AI tools is permitted. You must explain how you have used the tools. When text is solely or mainly generated by an AI tool, the tool used must be quoted as a source.

Be careful not to cheat at exams!

Exam cheating is for example if you:

- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Reuse parts of a written paper that you have previously submitted and for which you have received a pass grade without making use of quotation marks or source references (self-plagiarism)
- Receive help from others in contrary to the rules in the Faculty of Social Science's common part of the curriculum

You can read more about the rules on exam cheating on your Study Site and in the Faculty of Social Science's common part of the curriculum.

Exam cheating is always sanctioned by a written warning and expulsion from the exam in question. In most cases, the student will also be expelled from the University for one semester.

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```
In [ ]: # Write your code here
import numpy as np
from types import SimpleNamespace
```

1. Problem 1: Production economy and CO2 taxation

Consider a production economy with two firms indexed by $j \in \{1, 2\}$. Each produce its own good. They solve

$$\begin{aligned} \max_{y_j} \pi_j &= p_j y_j - w_j \ell_j \\ \text{s.t. } y_j &= A \ell_j^\gamma. \end{aligned}$$

Optimal firm behavior is

$$\begin{aligned} \ell_j^*(w, p_j) &= \left(\frac{p_j A \gamma}{w} \right)^{\frac{1}{1-\gamma}} \\ y_j^*(w, p_j) &= A \left(\ell_j^*(w, p_j) \right)^\gamma \end{aligned}$$

The implied profits are

$$\pi_j^*(w, p_j) = \frac{1-\gamma}{\gamma} w \cdot \left(\frac{p_j A \gamma}{w} \right)^{\frac{1}{1-\gamma}}$$

A single consumer supplies labor, and consumes the goods the firms produce. She also receives the implied profits of the firm.

She solves:

$$\begin{aligned} U(p_1, p_2, w, \tau, T) &= \max_{c_1, c_2, \ell} \log(c_1^\alpha c_2^{1-\alpha}) - \nu \frac{\ell^{1+\epsilon}}{1+\epsilon} \\ \text{s.t. } p_1 c_1 + (p_2 + \tau) c_2 &= w \ell + T + \pi_1^*(w, p_1) + \pi_2^*(w, p_2) \end{aligned}$$

where τ is a tax and T is lump-sum transfer.

For a given ℓ , it can be shown that optimal behavior is

$$\begin{aligned} c_1(\ell) &= \alpha \frac{w \ell + T + \pi_1^*(w, p_1) + \pi_2^*(w, p_2)}{p_1} \\ c_2(\ell) &= (1 - \alpha) \frac{w \ell + T + \pi_1^*(w, p_1) + \pi_2^*(w, p_2)}{p_2 + \tau} \end{aligned}$$

Such that optimal behavior is:

$$\ell^* = \arg \max_{\ell} \log((c_1(\ell))^\alpha \cdot (c_2(\ell))^{1-\alpha}) - \nu \frac{\ell^{1+\epsilon}}{1+\epsilon}$$

With optimal consumption:

$$\begin{aligned} c_1^* &= c_1(\ell^*) \\ c_2^* &= c_2(\ell^*) \end{aligned}$$

The government chooses τ and balances its budget so $T = \tau c_2^*$. We initially set $\tau, T = 0$.

Market clearing requires:

1. Labor market: $\ell^* = \ell_1^* + \ell_2^*$
2. Good market 1: $c_1^* = y_1^*$
3. Good market 2: $c_2^* = y_2^*$

Question 1: Check market clearing conditions for p_1 in `linspace(0.1,2.0,10)` and p_2 in `linspace(0.1,2.0,10)`. We choose $w = 1$ as numeraire.

```
In [ ]: par = SimpleNamespace()

# firms
par.A = 1.0
par.gamma = 0.5

# households
par.alpha = 0.3
par.nu = 1.0
par.epsilon = 2.0

# government
par.tau = 0.0
par.T = 0.0

# Question 3
par.kappa = 0.1
```

```
In [ ]: # write your answer here
```

Question 2: Find the equilibrium prices p_1 and p_2 .

Hint: you can use Walras' law to only check 2 of the market clearings

```
In [ ]: # write your answer here
```

Assume the government care about the social welfare function:

$$SWF = U - \kappa y_2^*$$

Here κ measures the social cost of carbon emitted by the production of y_2 in equilibrium.

Question 3: What values of τ and (implied) T should the government choose to maximize SWF ?

In []: `# write your answer here`

2. Problem 2: Career choice model

Consider a graduate i making a choice between entering J different career tracks. Entering career j yields utility $u_{i,j}^k$. This value is unknown to the graduate ex ante, but will ex post be:

$$u_{i,j}^k = v_j + \epsilon_{i,j}^k$$

They know that $\epsilon_{i,j}^k \sim \mathcal{N}(0, \sigma^2)$, but they do not observe $\epsilon_{i,j}^k$ before making their career choice.

Consider the concrete case of $J = 3$ with:

$$\begin{aligned} v_1 &= 1 \\ v_2 &= 2 \\ v_3 &= 3 \end{aligned}$$

If the graduates know the values of v_j and the distribution of $\epsilon_{i,j}^k$, they can calculate the expected utility of each career track using simulation:

$$\mathbb{E} \left[u_{i,j}^k | v_j \right] \approx v_j + \frac{1}{K} \sum_{k=1}^K \epsilon_{i,j}^k$$

```
In [ ]: par = SimpleNamespace()
par.J = 3
par.N = 10
par.K = 10000

par.F = np.arange(1, par.N+1)
par.sigma = 2

par.v = np.array([1, 2, 3])
par.c = 1
```

Question 1: Simulate and calculate expected utility and the average realised utility for $K = 10000$ draws, for each career choice j .

In []: `# write your answer here`

Now consider a new scenario: Imagine that the graduate does not know v_j . The *only* prior information they have on the value of each job, comes from their F_i friends that work in

each career j . After talking with them, they know the average utility of their friends (which includes their friends' noise term), giving them the prior expectation:

$$\tilde{u}_{i,j}^k(F_i) = \frac{1}{F_i} \sum_{f=1}^{F_i} (v_j + \epsilon_{f,j}^k), \quad \epsilon_{f,j}^k \sim \mathcal{N}(0, \sigma^2)$$

For ease of notation consider that each graduate have $F_i = i$ friends in each career.

For K times do the following:

1. For each person i draw $J \cdot F_i$ values of $\epsilon_{f,j}^k$, and calculate the prior expected utility of each career track, $\tilde{u}_{i,j}^k(F_i)$.
Also draw their own J noise terms, $\epsilon_{i,j}^k$
2. Each person i chooses the career track with the highest expected utility:

$$j_i^{k*} = \arg \max_{j \in \{1, 2, \dots, J\}} \left\{ \tilde{u}_{i,j}^k(F_i) \right\}$$

3. Store the chosen careers: j_i^{k*} , the prior expectation of the value of their chosen career: $\tilde{u}_{i,j_i^{k*}}^k(F_i)$, and the realized value of their chosen career track:
 $u_{i,j_i^{k*}}^k = v_{j_i^{k*}} + \epsilon_{i,j_i^{k*}}^k$.

Chosen values will be:

$$i \in \{1, 2, \dots, N\}, N = 10$$

$$F_i = i$$

So there are 10 graduates. The first has 1 friend in each career, the second has 2 friends, ... the tenth has 10 friends.

Question 2: Simulate and visualize: For each type of graduate, i , the share of graduates choosing each career, the average subjective expected utility of the graduates, and the average ex post realized utility given their choice.

That is, calculate and visualize:

$$\frac{1}{K} \sum_{k=1}^K \mathbb{I} \{j = j_i^{k*}\} \quad \forall j \in \{1, 2, \dots, J\}$$

$$\frac{1}{K} \sum_{k=1}^K \tilde{u}_{i,j_i^{k*}}^k(F_i)$$

And

$$\frac{1}{K} \sum_{k=1}^K u_{i,j_i^{k*}}^k$$

For each graduate i .

In []: # Write your answer here

After a year of working in their career, the graduates learn u_{ij}^k for their chosen job j_i^{k*} perfectly.

The can switch to one of the two remaining careers, for which they have the same prior as before, but it will now include a switching cost of c which is known. Their new priors can be written as:

$$\tilde{u}_{ij}^{k,2}(F_i) = \begin{cases} \tilde{u}_{ij}^k(F_i) - c & \text{if } j \neq j_i^{k*} \\ u_{ij=j_i^{k*}} & \text{if } j = j_i^{k*} \end{cases}$$

We will set $c = 1$.

Their realized utility will be:

$$u_{ij}^{k,2} = \begin{cases} u_{ij}^k - c & \text{if } j \neq j_i^{k*} \\ u_{ij=j_i^{k*}} & \text{if } j = j_i^{k*} \end{cases}$$

Question 3: Following the same approach as in question 2, find the new optimal career choice for each i, k . Then for each i , calculate the average subjective expected utility from their new optimal career choice, and the ex post realized utility of that career. Also, for each i , calculate the share of graduates that chooses to switch careers, conditional on which career they chose in the first year.

In []: # write your answer here

3. Problem 3: Barycentric interpolation

Problem: We have a set of random points in the unit square,

$$\mathcal{X} = \{(x_1, x_2) \mid x_1 \sim \mathcal{U}(0, 1), x_2 \sim \mathcal{U}(0, 1)\}.$$

For these points, we know the value of some function $f(x_1, x_2)$,

$$\mathcal{F} = \{f(x_1, x_2) \mid (x_1, x_2) \in \mathcal{X}\}.$$

Now we want to approximate the value $f(y_1, y_2)$ for some $y = (y_1, y_2)$, where $y_1 \sim \mathcal{U}(0, 1)$ and $y_2 \sim \mathcal{U}(0, 1)$.

Building block I

For an arbitrary triangle ABC and a point y , define the so-called barycentric coordinates as:

$$r_1^{ABC} = \frac{(B_2 - C_2)(y_1 - C_1) + (C_1 - B_1)(y_2 - C_2)}{(B_2 - C_2)(A_1 - C_1) + (C_1 - B_1)(A_2 - C_2)}$$

$$r_2^{ABC} = \frac{(C_2 - A_2)(y_1 - C_1) + (A_1 - C_1)(y_2 - C_2)}{(B_2 - C_2)(A_1 - C_1) + (C_1 - B_1)(A_2 - C_2)}$$

$$r_3^{ABC} = 1 - r_1 - r_2.$$

If $r_1^{ABC} \in [0, 1]$, $r_2^{ABC} \in [0, 1]$, and $r_3^{ABC} \in [0, 1]$, then the point is inside the triangle.

We always have $y = r_1^{ABC} A + r_2^{ABC} B + r_3^{ABC} C$.

Building block II

Define the following points:

$$A = \arg \min_{(x_1, x_2) \in \mathcal{X}} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \text{ s.t. } x_1 > y_1 \text{ and } x_2 > y_2$$

$$B = \arg \min_{(x_1, x_2) \in \mathcal{X}} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \text{ s.t. } x_1 > y_1 \text{ and } x_2 < y_2$$

$$C = \arg \min_{(x_1, x_2) \in \mathcal{X}} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \text{ s.t. } x_1 < y_1 \text{ and } x_2 < y_2$$

$$D = \arg \min_{(x_1, x_2) \in \mathcal{X}} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \text{ s.t. } x_1 < y_1 \text{ and } x_2 > y_2.$$

Algorithm:

1. Compute A , B , C , and D . If not possible return `NaN`.
2. If y is inside the triangle ABC return $r_1^{ABC} f(A) + r_2^{ABC} f(B) + r_3^{ABC} f(C)$.
3. If y is inside the triangle CDA return $r_1^{CDA} f(C) + r_2^{CDA} f(D) + r_3^{CDA} f(A)$.
4. Return `NaN`.

Sample:

```
In [ ]: rng = np.random.default_rng(2024)

X = rng.uniform(size=(50,2))
y = rng.uniform(size=(2,))
```

Questions 1: Find A , B , C and D . Illustrate these together with X , y and the triangles ABC and CDA .

```
In [ ]: # write your answer here
```

Question 2: Compute the barycentric coordinates of the point y with respect to the triangles ABC and CDA . Which triangle is y located inside?

```
In [ ]: # write your answer here
```

Now consider the function:

$$f(x_1, x_2) = x_1 \cdot x_2$$

```
In [ ]: f = lambda x: x[0]*x[1]  
F = np.array([f(x) for x in X])
```

Question 3: Compute the approximation of $f(y)$ using the full algorithm. Compare with the true value.

```
In [ ]: # write your answer here
```

Question 4: Repeat question 3 for all points in the set Y .

```
In [ ]: Y = [(0.2,0.2),(0.8,0.2),(0.8,0.8),(0.8,0.2),(0.5,0.5)]
```

```
In [ ]: # write your answer here
```