Vibroacoustic Quantification of a Coastal Seabed

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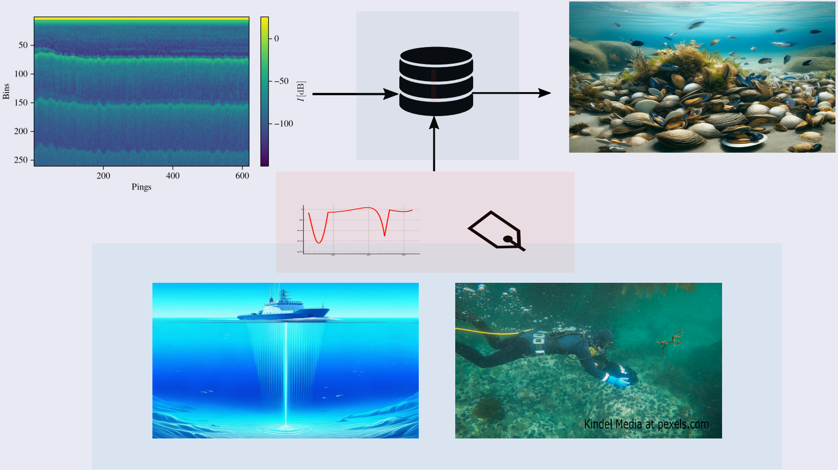
MÁSTER EN MATEMÁTICA INDUSTRIAL

UNIVERSIDADE DA CORUÑA

Introduction

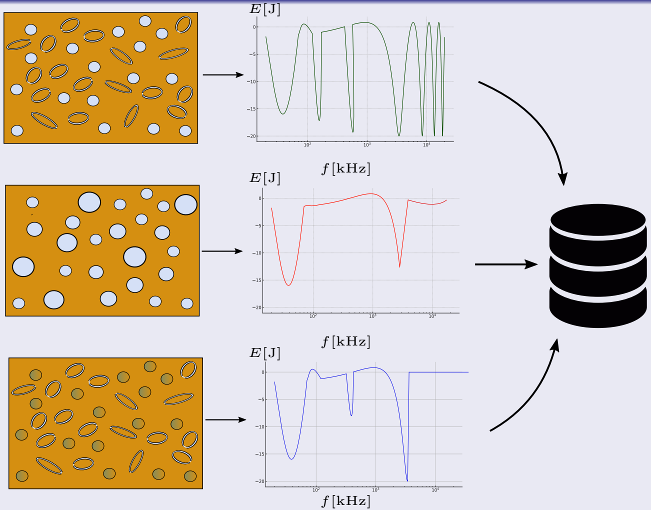
Interest in monitor the marine ecosystem



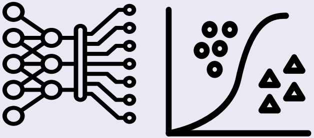
- A priori estimations.
- Database necessity.
- Traditional workflows.
- Highly time-consuming.

Objective

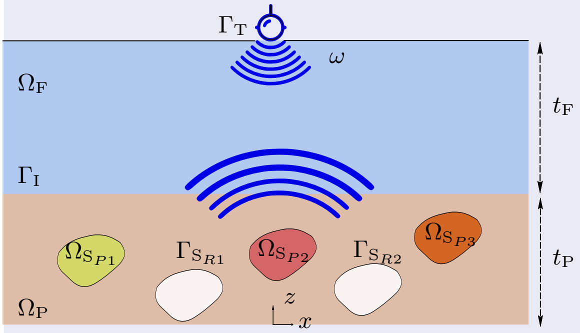
Can this database be generated virtually?



- Data base constructed
- Can we take advantage of this database and generate classifiers?
- Recognise new scenarios

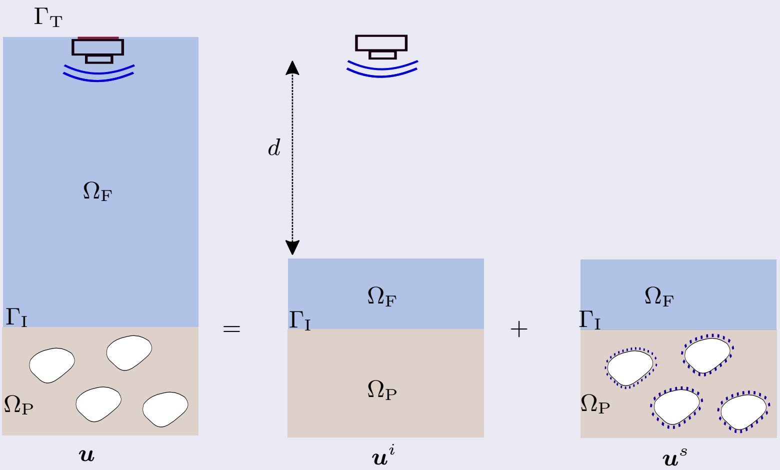


Statement of the problem: Total field



$$
\begin{cases}
-\omega^2 \rho_{\mathrm{F}} \boldsymbol{u}_{\mathrm{F}} - \nabla(K_{\mathrm{F}} \operatorname{div} \boldsymbol{u}_{\mathrm{F}}) = \mathbf{0} & \text{in } \Omega_{\mathrm{F}}, \\
-\omega^2 \rho_{\mathrm{P}}(\omega) \boldsymbol{u}_{\mathrm{P}} - \nabla(K_{\mathrm{P}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{P}}) = \mathbf{0} & \text{in } \Omega_{\mathrm{P}}, \\
-\omega^2 \rho_{\mathrm{S}}(\omega) \boldsymbol{u}_{\mathrm{S}} - \nabla(K_{\mathrm{S}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{S}}) = \mathbf{0} & \text{in } \Omega_{\mathrm{S_P}}, \\
-\rho_{\mathrm{F}} c_{\mathrm{F}}^2 \operatorname{div} \boldsymbol{u}_{\mathrm{F}} = \Pi_0 & \text{on } \Gamma_{\mathrm{T}}, \\
\rho_{\mathrm{F}} c_{\mathrm{F}}^2 \operatorname{div} \boldsymbol{u}_{\mathrm{F}} = K_{\mathrm{P}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{P}} & \text{on } \Gamma_{\mathrm{I}}, \\
\boldsymbol{u}_{\mathrm{F}} \cdot \boldsymbol{n} = \boldsymbol{u}_{\mathrm{P}} \cdot \boldsymbol{n} & \text{on } \Gamma_{\mathrm{I}}, \\
K_{\mathrm{S}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{F}} = K_{\mathrm{P}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{P}} & \text{on } \Gamma_{\mathrm{S_P}}, \\
\boldsymbol{u}_{\mathrm{P}} \cdot \boldsymbol{n} = \boldsymbol{u}_{\mathrm{S}} \cdot \boldsymbol{n} & \text{on } \Gamma_{\mathrm{S_P}}, \\
\boldsymbol{u}_{\mathrm{P}} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma_{\mathrm{S_R}}, \\
\text{radiation conditions} & \text{at } |\boldsymbol{x}| \to \infty
\end{cases}
$$

Is necessary to simulate all the water column?



Incident field known for fluid and seabed domain

$$\boxed{\boldsymbol{u} = \boldsymbol{u}^i + \boldsymbol{u}^s}$$

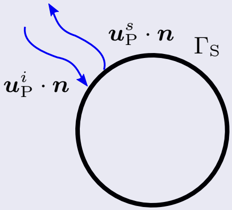
Advantages

- Computation time ✓

Inconveniences

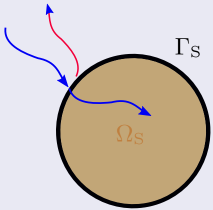
- Loss of physical domain ✗
- Analytical solution ✗

How the incident field chosen affect to the scattering objects?



Rigid $\rightarrow$ Boundary condition to the porous domain:

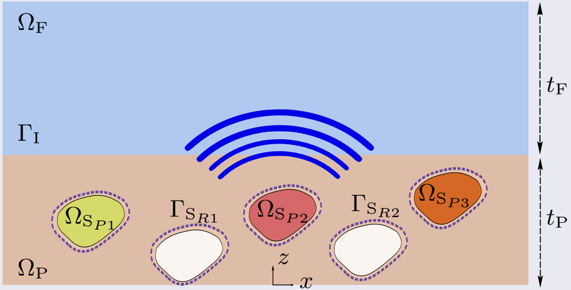
$$\boldsymbol{u}_{\mathrm{P}}^{s} \cdot \boldsymbol{n} = -\boldsymbol{u}_{\mathrm{P}}^{i} \cdot \boldsymbol{n} \quad \text{on } \Gamma_{\mathrm{S}}.$$



Porous $\rightarrow$ New medium, $\boldsymbol{u}_{\mathrm{S}}^{i}$ is considered equal to $\boldsymbol{u}_{\mathrm{P}}^{i}$:

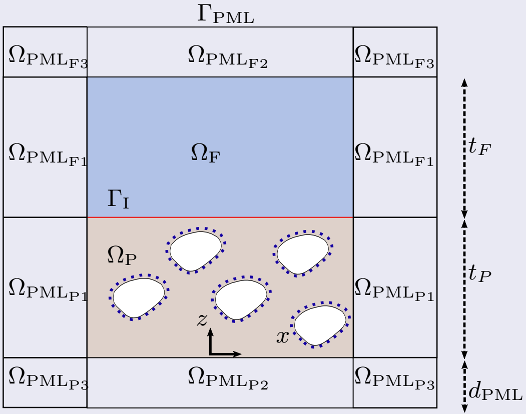
$$\begin{cases} -\rho_{\mathrm{S}}\omega^{2}\boldsymbol{u}_{\mathrm{S}}^{s} - K_{\mathrm{S}}(\omega)\nabla\left(\operatorname{div}\boldsymbol{u}_{\mathrm{S}}^{s}\right) = \left(\rho_{\mathrm{S}}\omega^{2} - k_{\mathrm{P}}^{2}K_{\mathrm{S}}(\omega)\right)\boldsymbol{u}_{\mathrm{P}}^{i} & \text{in } \Omega_{\mathrm{S}}, \\ \boldsymbol{u}_{\mathrm{S}}^{s} \cdot \boldsymbol{n} = \boldsymbol{u}_{\mathrm{P}}^{s} \cdot \boldsymbol{n} & \text{on } \Gamma_{\mathrm{S}}, \\ K_{\mathrm{P}}(\omega)\operatorname{div}\boldsymbol{u}_{\mathrm{P}}^{s} - K_{\mathrm{S}}(\omega)\operatorname{div}\boldsymbol{u}_{\mathrm{S}}^{s} = \left(1 - K_{\mathrm{S}}(\omega)/K_{\mathrm{P}}(\omega)\right)\Pi_{\mathrm{P}}^{i} & \text{on } \Gamma_{\mathrm{S}}. \end{cases}$$

Statement of the problem: Scattering field



$$
\begin{cases}
-\omega^2 \rho_{\mathrm{F}} \boldsymbol{u}_{\mathrm{F}}^s - \nabla(K_{\mathrm{F}} \operatorname{div} \boldsymbol{u}_{\mathrm{F}}^s) = \boldsymbol{0} & \text{in } \Omega_{\mathrm{F}}, \\
-\omega^2 \rho_{\mathrm{P}}(\omega) \boldsymbol{u}_{\mathrm{P}}^s - \nabla(K_{\mathrm{P}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{P}}^s) = \boldsymbol{0} & \text{in } \Omega_{\mathrm{P}}, \\
-\omega^2 \rho_{\mathrm{S}}(\omega) \boldsymbol{u}_{\mathrm{S}}^s - \nabla(K_{\mathrm{S}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{S}}^s) = \boldsymbol{f} & \text{in } \Omega_{\mathrm{S}_{\mathrm{P}}}, \\
\rho_{\mathrm{F}} c_{\mathrm{F}}^2 \operatorname{div} \boldsymbol{u}_{\mathrm{F}}^s = K_{\mathrm{P}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{P}}^s & \text{on } \Gamma_{\mathrm{I}}, \\
\boldsymbol{u}_{\mathrm{F}}^s \cdot \boldsymbol{n} = \boldsymbol{u}_{\mathrm{P}}^s \cdot \boldsymbol{n} & \text{on } \Gamma_{\mathrm{I}}, \\
K_{\mathrm{S}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{F}}^s = K_{\mathrm{P}}(\omega) \operatorname{div} \boldsymbol{u}_{\mathrm{P}}^s + g & \text{on } \Gamma_{\mathrm{S}_{\mathrm{P}}}, \\
\boldsymbol{u}_{\mathrm{P}}^s \cdot \boldsymbol{n} = \boldsymbol{u}_{\mathrm{S}}^s \cdot \boldsymbol{n} & \text{on } \Gamma_{\mathrm{S}_{\mathrm{P}}}, \\
\boldsymbol{u}_{\mathrm{P}}^s \cdot \boldsymbol{n} = -\boldsymbol{u}_{\mathrm{P}}^i \cdot \boldsymbol{n} & \text{on } \Gamma_{\mathrm{S}_{\mathrm{R}}}, \\
\text{radiation conditions} & \text{at } \|\boldsymbol{x}\| \to \infty
\end{cases}
$$

Perfectly Matched Layers



- New PML formulation has been developed
  - Conventional formulation:

$$-\rho(\omega)\omega^2 J\boldsymbol{u} - \mathrm{div}\left(K(\omega)\left(JH^{-T}\left(H^{-1} : \nabla\boldsymbol{u}\right)\right)\right) = \boldsymbol{0}$$

  - Novel formulation:

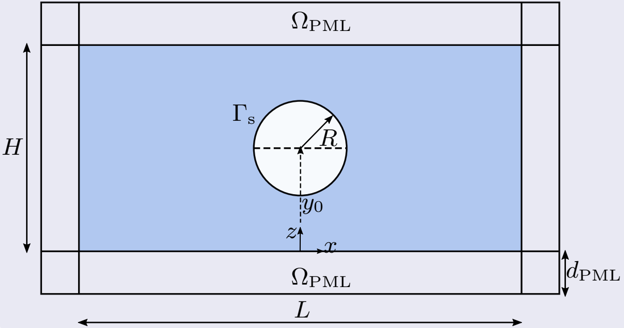
$$\boxed{-\rho(\omega)\omega^2 J^{-1}H^T H\boldsymbol{U} - \nabla\left(K(\omega)J^{-1}\mathrm{div}\boldsymbol{U}\right) = \boldsymbol{0}}$$

where $\boldsymbol{U} = JH^{-1}\boldsymbol{u}$, $H :=$ PML absorbing 2x2 tensor and $J = \det H$

- Continuity in normal displacements inside PML
- PML quadratic absorbing profile

PML verification: Conventional vs Novel PML formulation

Benchmark solved:



Analytical solution in terms of Hankel functions:

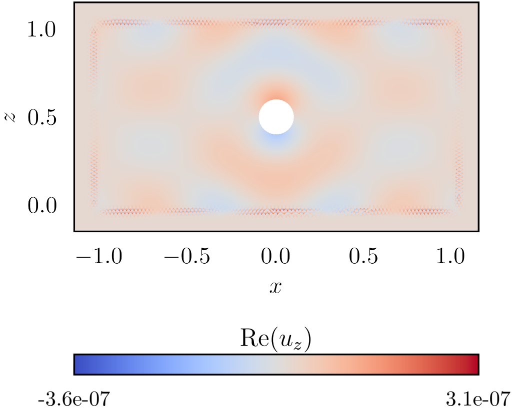
$$\Pi(x,y) = \Pi_0 \frac{H_0^{(1)}\left(k\sqrt{(x-x_0)^2+(z-z_0)^2}\right)}{H_0^{(1)}(kR)}$$

Objectives to check:

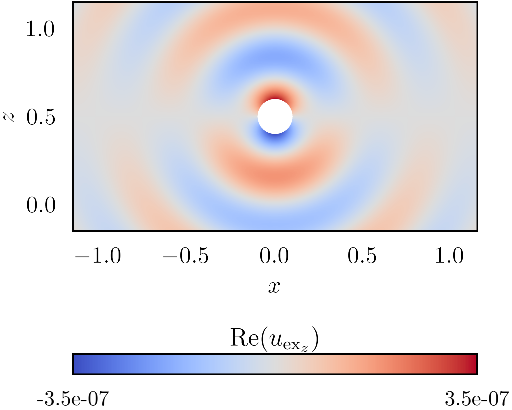
- Numerical instabilities
- New formulation ensures continuity

Benchmark results

$$-\rho\omega^2 J\boldsymbol{u} - \operatorname{div}\left(K(\omega)\left(JH^{-T}\left(H^{-1} : \nabla\boldsymbol{u}\right)\right)\right) = \mathbf{0} \quad \text{in } \Omega$$



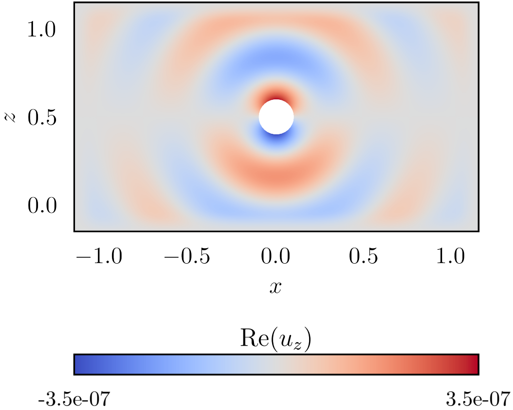
Conventional PML formulation



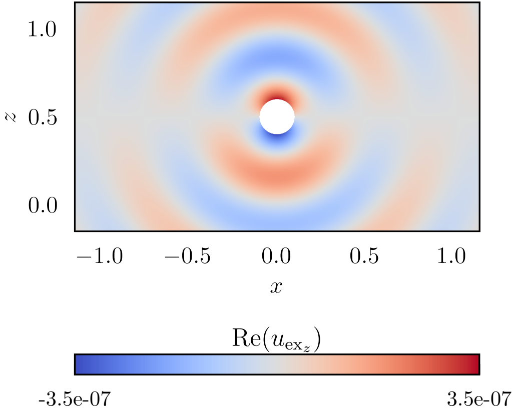
Analytical solution

Benchmark results

$$\boxed{-\rho\omega^2 J^{-1} H^T H \boldsymbol{U} - \nabla\left(K(\omega) J^{-1} \mathrm{div}\boldsymbol{U}\right) = \mathbf{0} \quad \text{in } \Omega}$$

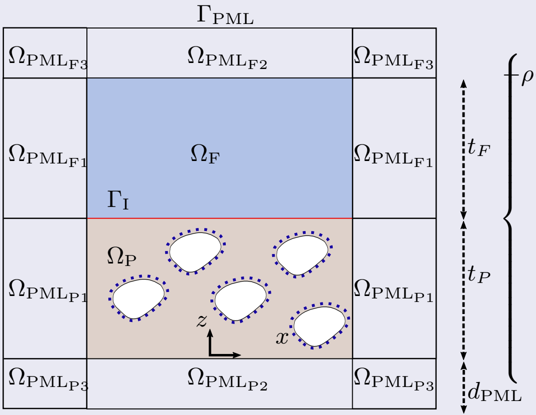


Novel PML formulation



Analytical solution

Model with PML



$$
\begin{cases}
-\rho(\omega)\omega^2 J^{-1} H^T H \boldsymbol{U}^s - \nabla\left(K(\omega) J^{-1} \mathrm{div} \boldsymbol{U}^s\right) = \boldsymbol{f} & \text{in } \Omega, \\
\rho_{\mathrm{F}} c_{\mathrm{F}}^2 \mathrm{div} \boldsymbol{U}_{\mathrm{F}}^s = K_{\mathrm{P}}(\omega) \mathrm{div} \boldsymbol{u}_{\mathrm{P}}^s & \text{on } \Gamma_{\mathrm{I}}, \\
\boldsymbol{U}_{\mathrm{F}}^s \cdot \boldsymbol{n} = \boldsymbol{U}_{\mathrm{P}}^s \cdot \boldsymbol{n} & \text{on } \Gamma_{\mathrm{I}}, \\
K_{\mathrm{S}}(\omega) \mathrm{div} \boldsymbol{U}_{\mathrm{F}}^s = K_{\mathrm{P}}(\omega) \mathrm{div} \boldsymbol{U}_{\mathrm{P}}^s + g & \text{on } \Gamma_{\mathrm{S_P}}, \\
\boldsymbol{U}_{\mathrm{P}}^s \cdot \boldsymbol{n} = \boldsymbol{U}_{\mathrm{S}}^s \cdot \boldsymbol{n} & \text{on } \Gamma_{\mathrm{S_P}}, \\
\boldsymbol{U}_{\mathrm{P}}^s \cdot \boldsymbol{n} = -\boldsymbol{U}_{\mathrm{P}}^i \cdot \boldsymbol{n} & \text{on } \Gamma_{\mathrm{S_R}}, \\
\boldsymbol{U}^s \cdot \boldsymbol{n} = 0 & \text{on } \Gamma_{\mathrm{PML}}.
\end{cases}
$$

Remark 1: $\Omega = \Omega_{\mathrm{F}} \cup \Omega_{\mathrm{P}} \cup \Omega_{\mathrm{S}} \cup \Omega_{\mathrm{PML}}$

Remark 2: In $\Omega_{\mathrm{F}} \cup \Omega_{\mathrm{P}} \cup \Omega_{\mathrm{S}}$, $H = I$ and $J = 1$ so $-\omega^2 \rho(\omega) \boldsymbol{U}^s - \nabla(K(\omega) \mathrm{div} \boldsymbol{U}^s) = \boldsymbol{0}$

What about the trial and test functions?

Trial space:

$$\boldsymbol{U}^s \in \mathrm{V} = \left\{ \boldsymbol{w} \in \left[L^2(\Omega)\right]^2 : \mathrm{div}\boldsymbol{w} \in L^2(\Omega)\ , \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_{\mathrm{PML}} \right\} \subset \mathrm{H}(\mathrm{div},\Omega).$$

Test space:

$$\boldsymbol{v} \in \mathrm{V}_0 = \left\{ \boldsymbol{w} \in \mathrm{H}\left(\mathrm{div},\Omega\right) : \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_{\mathrm{PML}} \text{ and } \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_{\mathrm{S}_R} \right\}.$$

For a given fixed angular frequency $\omega > 0$, and an incident field in the seabed domain $\boldsymbol{u}^i_{\mathrm{P}}$, find $\boldsymbol{U}^s \in \mathrm{V}$ such that $\boldsymbol{U}^s\cdot\boldsymbol{n} = \boldsymbol{u}^i_{\mathrm{P}}\cdot\boldsymbol{n}$ on Γ_{S_R} and

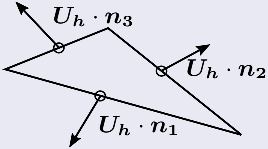
$$\underbrace{-\omega^2 \int_\Omega \rho\left(\omega\right) J^{-1}\left(H\boldsymbol{U}^s\right)\cdot\left(H\boldsymbol{v}\right)\,\mathrm{d}\Omega}_{\text{mass term}} + \underbrace{\int_\Omega K\left(\omega\right) J^{-1}\mathrm{div}\boldsymbol{U}^s\mathrm{div}\boldsymbol{v}\,\mathrm{d}\Omega}_{\text{stiffness term}} =$$

$$\underbrace{\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\Omega}_{\text{volumetric source term}} - \underbrace{\int_{\Gamma_{\mathrm{S}_P}} g\left(\boldsymbol{v}\cdot\boldsymbol{n}\right)\,\mathrm{d}\Gamma_{\mathrm{S}_P}}_{\text{boundary source term}}, \quad \forall \boldsymbol{v} \in \mathrm{V}_0.$$

What about the finite elements used to discretize the problem?

First-order Raviart-Thomas finite elements for all the domains

$$\mathrm{V}_h = \left\{ \boldsymbol{w}_h \in \mathrm{V} : \boldsymbol{w}_h|_T = \boldsymbol{a}_T + b_T(x,z)^T, \boldsymbol{a}_T \in \mathbb{C}^2, b_T \in \mathbb{C}, T \subset \Omega, T \in \mathcal{T}_h \right\}$$



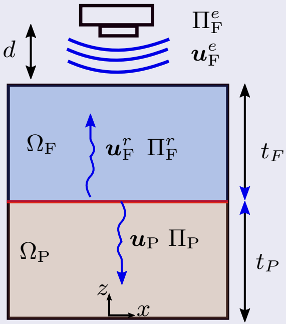
$$\left(-\omega^2 \mathcal{M} + \mathcal{K}\right) \vec{w}_h = \vec{f} + \vec{g}$$

$$[\mathcal{M}]_{ij} = \int_\Omega \rho(\omega) F J^{-1} \left(H \boldsymbol{\varphi}_i\right) \cdot \left(H \boldsymbol{\varphi}_j\right) \, \mathrm{d}\Omega \qquad [\boldsymbol{f}]_i = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}_i \, \mathrm{d}\Omega$$

$$[\mathcal{K}]_{ij} = \int_\Omega K(\omega) \, J^{-1} \mathrm{div} \boldsymbol{\varphi}_i \mathrm{div} \boldsymbol{\varphi}_j \, \mathrm{d}\Omega \qquad [\boldsymbol{g}]_i = \int_\Omega g \left(\boldsymbol{v} \cdot \boldsymbol{\varphi}_i\right) \mathrm{d}\Gamma_{\mathrm{S}_P}$$

How is the incident field computed?

Fourier Mode Decomposition of the emitted field on the interface:



$$\Pi_{\mathrm{F}}^{e}(x,t_{\mathrm{P}}) = \sum_{j=-N/2}^{N/2} \pi_{0j} e^{-ik_j x}, \quad \boldsymbol{u}_{\mathrm{F}}^{e}(x,t_{\mathrm{P}}) = \sum_{j=-N/2}^{N/2} \boldsymbol{u}_{0j} e^{-ik_j x}$$

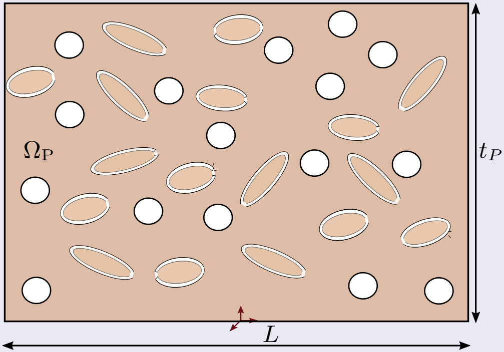
$$\Pi_{\mathrm{F}}^{r}(x,z) = \sum_j {\Pi_{\mathrm{F}}}_j^r e^{i\sqrt{k_{\mathrm{F}}^2-k_j^2}\,z} e^{-ik_j x}, \quad \Pi_{\mathrm{P}}(x,z) = \sum_j {\Pi_{\mathrm{P}}}_j e^{-i\sqrt{k_{\mathrm{P}}^2-k_j^2}\,z} e^{-ik_j x}$$

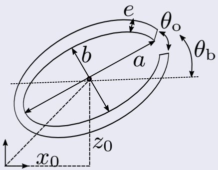
$$\begin{cases} \Delta\Pi_{\mathrm{F}}^{r} - k_{\mathrm{F}}^2 \Pi_{\mathrm{F}}^{r} = 0 & \text{in } \Omega_{\mathrm{F}}, \\ \Delta\Pi_{\mathrm{P}} - k_{\mathrm{P}}^2 \Pi_{\mathrm{P}} = 0 & \text{in } \Omega_{\mathrm{P}}, \\ \Pi_{\mathrm{F}}^{r} + {\Pi_{\mathrm{F}}}^{i} = \Pi_{\mathrm{P}} & \text{on } \Gamma_{\mathrm{I}}, \\ \dfrac{1}{\rho_{\mathrm{F}}\omega^2}\dfrac{\partial \Pi_{\mathrm{F}}^{r}}{\partial \boldsymbol{n}} + \dfrac{1}{\rho_{\mathrm{F}}\omega^2}\dfrac{\partial \Pi_{\mathrm{F}}^{e}}{\partial \boldsymbol{n}} = \dfrac{1}{\rho_{\mathrm{P}}\omega^2}\dfrac{\partial \Pi_{\mathrm{P}}}{\partial \boldsymbol{n}} & \text{on } \Gamma_{\mathrm{I}}. \end{cases}$$

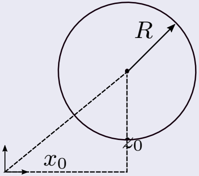
Remark: $\overset{u^i}{\boldsymbol{u}} = \frac{1}{\rho\omega^2}\nabla\Pi$

How are the scattering objects generated?

In Ω_{P} a combination of the following objects could be generated:







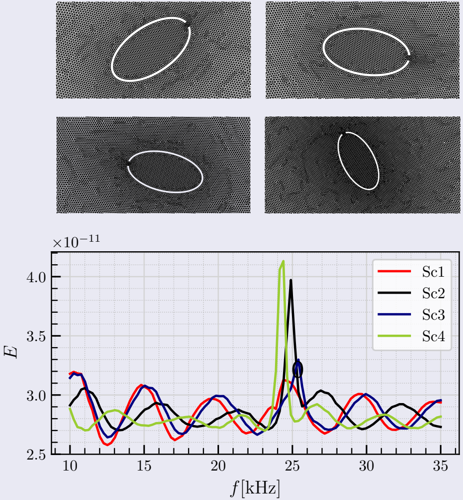
Ellipses:

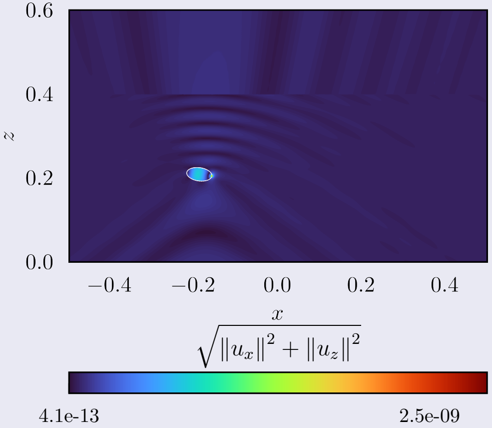
- $a \sim \mathcal{N}(a_0,\, \sigma_a^2)$
- $b \sim \mathcal{N}(b_0,\, \sigma_b^2)$
- $\theta_o \sim \mathcal{N}(\theta_0,\, \sigma_{\theta_0}^2)$
- $\theta_b \sim \mathcal{N}(\theta_b,\, \sigma_{\theta_b}^2)$
- $e \sim \mathcal{N}(a_0,\, \sigma_e^2)$
- $x_c \sim \mathcal{U}(-L/2, L/2)$
- $y_c \sim \mathcal{U}(0, t_{\mathrm{P}})$

Circles:

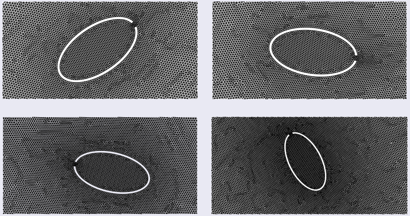
- $R \sim \mathcal{N}(R_0,\, \sigma_r^2)$
- $x_c \sim \mathcal{U}(-L/2, L/2)$
- $y_c \sim \mathcal{U}(0, t_{\mathrm{P}})$

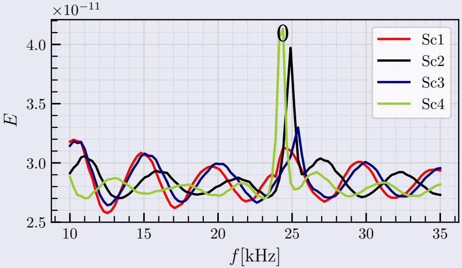
Case of study 1: One open rigid ellipse

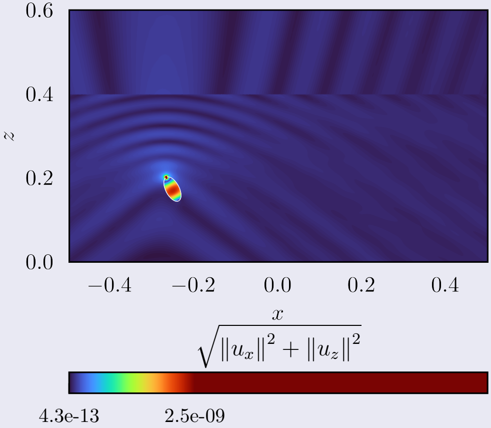




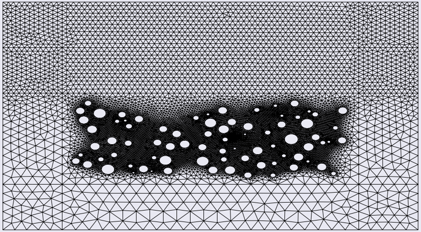
Case of study 1: One open rigid ellipse

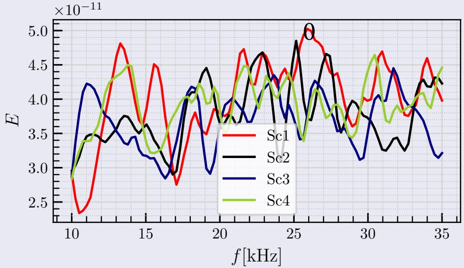


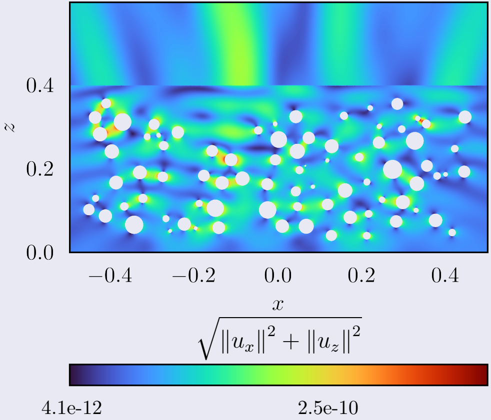




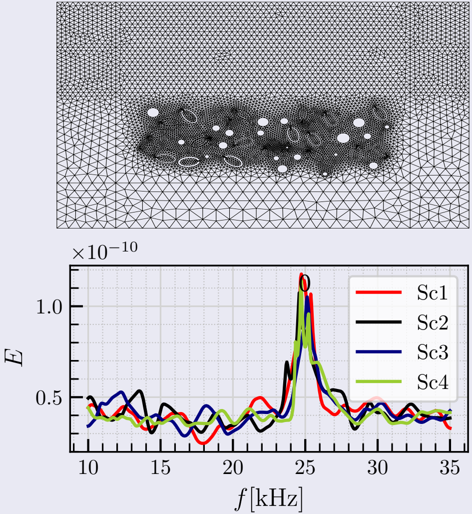
Case of study 2: Eighty rigid circles

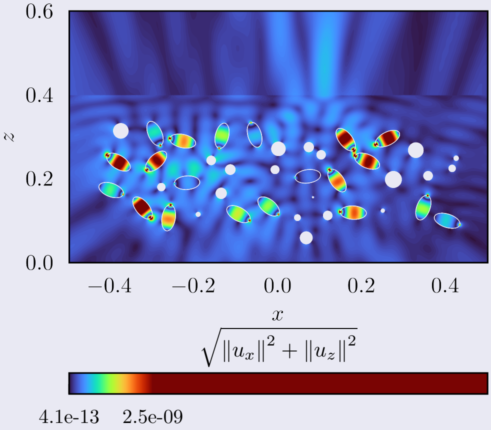




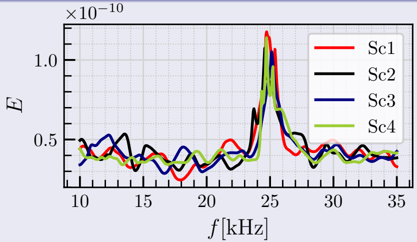


Case of study 3: Mixture of rigid circles and open rigid ellipses

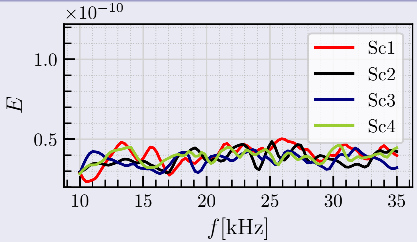




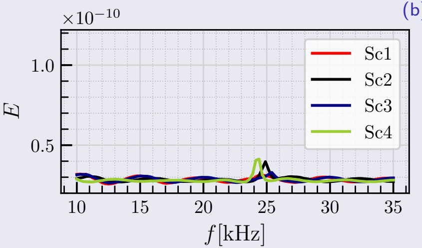
Comparison results of the three cases of study



(a) Mixture.



(b) Eighty rigid circles.



(c) Single open rigid ellipse.

Conclusions and future work

Conclusions

- A tool for generating scenarios was constructed.
- Frequency response functions that allow identification of different obstacles were obtained.

Future work

- Uncertainty quantification for a variety of scenarios.
- Mathematical modelling of the seabed using the Biot-Stoll model.

The results presented in this work have been presented at the 9th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2024) held 05-06-2024 in Lisbon, Portugal, under the title: *Numerical Quantification of the Biological Presence Buried in Granular Sediments in Coastal Environments.*

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