PHYS 241

Random Basic Facts/Equations:

(a) Given a signal whose current is defined by the function $I(t)$, with peak current I_p, (or $V(t)$ with peak voltage V_p), we can derive the average current (or voltage) as:

$$I_{RMS} = \sqrt{\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} (I_p sin(\omega t))^2 dt}$$

T_2 and T_1 are times at the beginning and end of a period of oscillation. This only works with oscillating/constant functions (constant functions are themselves answers).

Voltage, Current, Power, and Resistance:

(a) Current is the change in charge over time, the *velocity* of charge, and is measured in amperes:

$$I = \frac{Q}{t}$$

Think of current as the rate of electron flow. Electrons flow out of the negative end of a battery, but current is always depicted as flowing out of the positive end of a battery.

(b) Voltage is the change in electric potential energy divided by charge and is measured in volts:

$$V = \frac{\Delta E}{q}$$

Think of voltage as the energy in each electron.

(c) If you were to consider power in terms of a person in a fight, it wouldn't only be the energy of a kick/punch but also the frequency of those kicks/punches; aka, the frequency and energy in each punch/kick. Similarly, the power disipated in a circuit is proportional to the kick and frequency (voltage & current) of each electron:

$$P = IV$$

(d) Resistance can also be thought of physically. If a fighter is wearing a weighted suit, they would need to exert much more energy per kick/punch to exert the same power (more voltage per hit), or they would need to punch/kick more frequently to be able to keep up the same power (raise current). We can relate resistance in the following way:

$$V = IR$$

Resistance is measured in Ohms. This aforementioned equation is Ohm's law, and it can be used to measure the relationship of Resistance, Voltage, and Current in different circuit components.

i. With Ohm's Law we can derive two more relationships for Power: $P = I^2R$ and $P = \frac{V^2}{R}$.

ii. Resistors in series add their resistance. Think of this as adding more weights to a fighter, more weights for one fighter will give him more resistance:

$$R_{Total} = R_1 + R_2 + \ ... \ + R_n$$

iii. Resistors in parallel give a resistance that is smaller than just one of the resistances, they add inversely. Picture we now have two fighters, the amount of power they output together against their opponent will still be greater than one of the fighters by themselves, even if they derived their energy from the same pool:

$$\frac{1}{R_{Total}} = \frac{1}{R_1} + \frac{1}{R_2} + \ ... \ + \frac{1}{R_n}$$

(e) A material can be characterized by its resistiviy which is defined by the following equation:

$$\rho = R\frac{A}{l}$$

If you can imagine a material that is part of a resistor with a certain volume with a length and consistent area (cylinder, rectangular prism, etc.), then this equation defines this material where A is the area, l is the length, R is resistance, and ρ is resistivity.

i. Typical resistance is given at room temperature (20^oC). Resistance increases as temperature goes up. If the resistance at room temperature is R_{RT}, the temperature is T, and the temperature coefficient is μ_T (generally around 100 $ppm/^oC$, then we can defined the resistance as:

$$R = R_{RT}(1 + \mu_T \cdot (T - 20^oC))$$

ii. We can redefine the equation of resistivity by plugging in our new equation for resistance to get that:

$$\rho = R_{RT}(1 + \mu_T \cdot (T - 20^oC))\frac{A}{l}$$

iii. Conductivity is measured as the inverse of resistivity: $\sigma = \frac{1}{\rho}$.

Capacitors:

(a) Capacitors are constructed of two parallel conducting plates with some area A and separation d. The charge q stored by a capacitor is proportional to the voltage across the capacitor and the capacitance of the capacitor C. The capacitance of a capacitor is proportional to the area, separation distance and dielectric permittivity of the material imbetween the plates:

$$C = \frac{\epsilon A}{d}$$

The dielectric permittivity scales off of the permittiviy of free space ϵ_0 and the dimensionless constant κ_ϵ which is a property of the dielectric material. $\epsilon_0 = 8.85 \cdot 10^{-12}\frac{F}{m}$.

(b) Using that $q = CV$ and $I = \frac{dq}{dt}$, we can relate the voltage and current:

$$I = C\frac{dV}{dt}, \quad V = \frac{1}{C}\int I \cdot dt$$

If a capacitor is initially uncharged then the initial voltage is $V_0 = 0$. Using this we can solve for the function of voltage $V(t)$.

(c) The energy stored by the electric field of a capacitor can be derived in the following way:

$$W = \int_0^t (power)dt = \int_0^t v \cdot Idt = \int_0^t Cv\frac{dv}{dt}dt = C\int_0^V vdv = \frac{1}{2}CV^2$$

(d) Capacitors combine in the opposite way that resistors (and inductors) do.

i. Capacitors in series add inversely:

$$\frac{1}{C_{Total}} = \frac{1}{C_1} + \frac{1}{C_2} + \ ... \ \frac{1}{C_n}$$

ii. Capacitors in parallel add directly:

$$C_{Total} = C_1 + C_2 + \ ... \ + C_n$$

(e) A DC RC Circuit in series displays the way in which capacitors charge. A capacitor initially has 0 voltage drop, but as charge increases, it goes up until eventually it acts like *infinite resistance*, absorbing all of the voltage:

$$V_0 = V_R + V_C = IR + \frac{1}{C}\int_{t_0}^t Id\tau \rightarrow 0 = RC\frac{dI}{dt} + I \rightarrow I = -RC\frac{dI}{dt}$$

$$\rightarrow \frac{dI}{I} = -\frac{dt}{RC} \rightarrow -\frac{1}{RC}\int dt = \int \frac{1}{I}dI \rightarrow \frac{-t}{RC} = ln(\frac{I}{I_0}) \rightarrow i(t) = I_0 e^{\frac{-t}{RC}} = \frac{V_0}{R}e^{\frac{-t}{\tau_0}}$$

i. Similarly for voltage:

$$V(t) = V_0(1 - e^{-\frac{t}{\tau_0}})$$

Inductors:

(a) Inductors store energy by storing magnetic fields that keep current flowing. These magnetic fields are produced by the coiling of wires. Inductance is measured in Henries. Wires are generally coiled around some sort of material, forming a loop (circle). The area of the loop (circle) A, the length of the coil l, the number of turns the coil makes n, and the magnetic permeability of the material μ define the inductance:

$$L = \frac{n^2 \mu A}{l}$$

(b) Inductors have their voltage and current defined in the following way:

$$V = L\frac{di}{dt}, \quad I = \frac{1}{L}\int V \cdot dt$$

(c) The energy stored in an inductor can be derived in the following way:

$$W = \int (Power) \cdot dt = \int_0^t Vidt = \int_0^t Vidt = \int L\frac{di}{dt}idt = L\int_0^I i \cdot di = \frac{1}{2}LI^2$$

(d) Inductance adds in the same way that resistors and the opposite of capacitors.

i. Inductors in series add directly:

$$L_{Total} = L_1 + L_2 + \; ... \; L_n$$

ii. Inductors in parallel add inversely:

$$\frac{1}{L_{Total}} = \frac{1}{L_1} + \frac{1}{L_2} + \; ... \; + \frac{1}{L_n}$$

Transformers:

(a) Transformers are toroids wrapped on both sides by wires. If a current is supplied, the wires create a magnetic field that supplies a voltage through the toroid, and this voltage then supplies a magnetic field on the wires on the other side of the toroid, and finally, these wires are supplied a current. The amount of times the wires wrap around both sides of the toroid strictly defines the ratio of voltage and current that is transfered from one side to the other. Define n_1 as the amount of turns the wire makes around side 1, and define n_2 as the same for side 2, we can define the voltage/current transfer as:

$$\frac{V_1}{V_2} = \frac{I_2}{I_1} = \frac{n_1}{n_2}$$

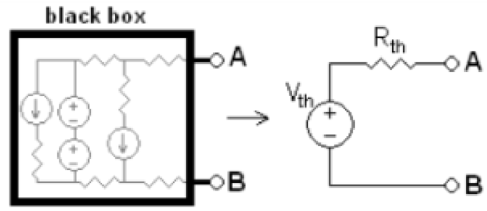
Circuit Theorems and Logic:

(a) The voltage across a capacitor can not instantaneously change, but it can for an inductor. On the other hand, the current through an inductor can not instantaneously change, but it can for a capcitor.

(b) Switching off voltage sources implies replacing all voltage sources with wires (short circuit). Switching off current sources implies replacing all current sources with nothing (open circuit).

(c) Kirchoff's Laws inform us of the reality of circuitry. Since voltage is a measure of energy, voltage must be conserved just as energy is in a circuit. This statement implies that the algebraic sum of the voltages around a closed loop must be zero. Similarly, if we imagine current as just the amount of electrons flowing through circuits in a given time, then when electrons come upon a circuit junction, the same amount of electrons must pass through both junctions; in other words, the current flowing in and out of a circuit junction is the same.

(d) Thevenin's Theorem states that we can represent a section of a circuit as a black box with an equivalent voltage and resistance V_{th} and R_{th}. The black box idea is represented in the following figure:



R_{th} can be determined by eliminating all voltage sources, treating them as short circuits, and calculating the equivalent resistance from the remaining circuit. V_{th} is a bit more complicated to calculate, but it is always the voltage from A to B (in the diagram); this is most easily solved for with Kirchoff's laws.

(e) Norton's theorem is almost identical to Thevenin's Theorem; the only difference between the two theorems is that we are solving for the circuit of the black box rather than the voltage. In this case we are talking about I_{nr} and R_{nr}. $R_{nr} = R_{th}$. We can find the value of I_{nr} by short circuiting the nodes a and b (from the figure), and solving what the current through this circuit would be with the use of Kirchoff's laws.

<u>AC Circuits</u>:

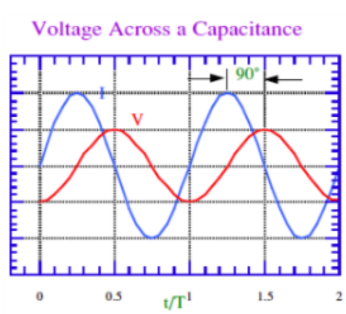
(a) Inductors and Capacitors act as resistors. This resistance is redefined as impedence Z. The impedence of the three major circuit components can be defined as:

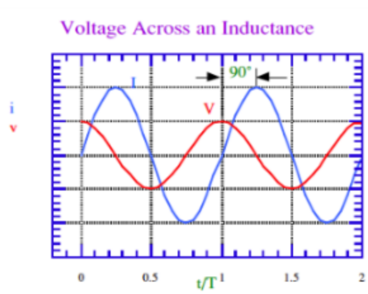
$$Z_R = R, \quad Z_C = \frac{1}{j\omega C}, \quad Z_L = j\omega L$$

i. Impedences are treated mathematically the same way as resistance, so by treating all capacitors/resistors/inductors as impedences, finding total impedence is much easier.

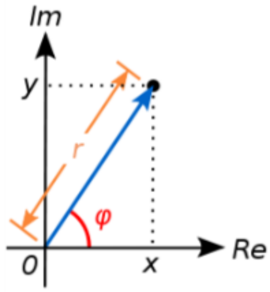
(b) We say that voltage across an inductor leads the current by 90 degrees because current can not instantaneously change in an inductor. Similarly, we say the voltage across a capacitor lags the current by 90 degrees because voltage can not instantaneously change in a capacitor. For a current $I(t) = I_p sin(\omega t)$ generated by a battery in an AC RLC-circuit, the voltage across the system can be defined as:

$$V = RI_p sin(\omega t) + L\omega I_p sin(\omega t + 90^o) + \frac{1}{\omega C} I_p sin(\omega t - 90^o)$$





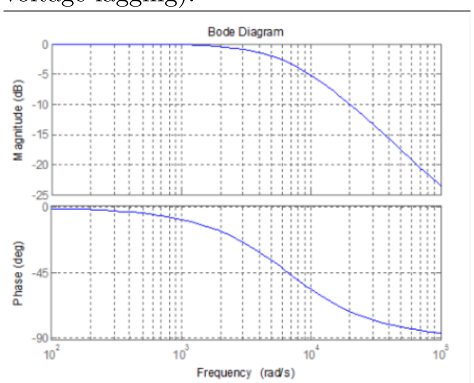
(c) Note that $re^{j\phi} = x + jy$. Voltage and current in AC circuits can be defined by sin/cos curves, both which can be defined by imaginary exponentials like the aforementioned. The impedences given by capacitors and inductors have only imaginary components, but the "resistance" that they provide to a circuit is very real. Combining imaginary and real impedences is as easy as finding the magnitude of a vector on the complex plane:



The phase difference in the vector defines the lag/lead of voltage against current in the respective circuit. This phase, ϕ can easily be solved as $\phi = tan^{-1}\frac{y}{x}$.

Bode Plots and High/Low-Pass Filters:

(a) A bode plot strictly defines the ratio between the battery voltage and the voltage across some other part of the circuit. Bode plots are used to describe the voltage difference of AC circuits. Bode plots are often accompanied with a phase diagram which displays how the the voltage lags/leads the current. In this class Bode plots are used to describe RC, CR, LR, and RL circuits. These 4 circuit types are special as they are either high-pass or low-pass filters, as in they let high or low frequency currents pass through the second component. Here is an RC-circuit diagram of the two mentioned plots (low-pass filter, voltage lagging):



We can identify the behavior of all four circuit types based off of the way resistors, capacitors, and inductors act as impedences.

i. The impedence of a resistor: $Z_R = R$ isn't defined by the frequency of the current. High and low frequencies cause the same voltage drop across this component.

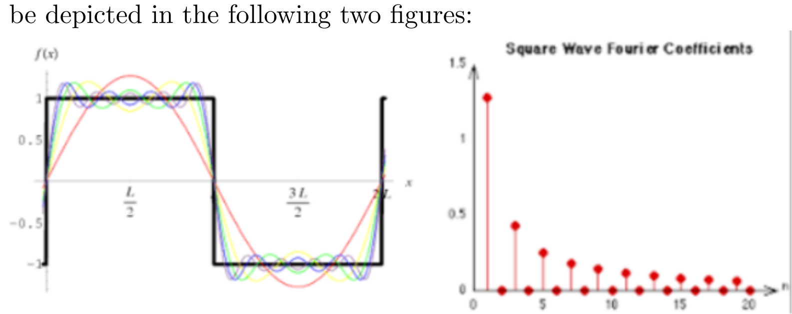
ii. The impedence of a capacitor $Z_C = \frac{1}{j\omega C}$ is defined by the frequency of the current. High frequencies don't give the capacitor enough time to charge, so they will have low impedence and act like a wire. Low frequencies give capacitors enough time to charge, so they will have high impedence and act like a *wall*.

iii. The impedence of an inductor $Z_L = j\omega L$ is defined by the frequency of the current. When the frequency is high, the inductor doesn't have enough time to generate magnetic fields and acts like a *wall* with high impedence. When the frequency is low, the inductor has enough time to generate magnetic fields and acts like a wire with low impedence.

(b) For the aforementioned reasons, an RC and LR-circuit will act as low-pass filters, but they will have opposite phase changes. Similarly, a CR and RL-circuit will act as high-pass filter with opposite phase changes. Circuits (filters) with capacitors will have negative phase changes while filters with inductors will have positive phase changes.

Fourier Transforms:

(a) Fourier transformations are based off of the understanding that all continuous functions can be generated out of the addition of sine and cosine functions. Taking a Fourier Transform means transforming a given function from its respective variable (time in this class) to a function that is in terms of frequencies. To make this understanding more clear, a sine wave, which is just one sine wave of a given frequency, would show up as a dirac delta function at the frequency of the sine wave in fourier space, with a value on the y-axis as the amplitude. Another example is the square wave; the square wave is a sum of harmonics (frequency, 1/3rd frequency, 1/5th etc...) that diminish in amplitude, this transform can be depicted in the following two figures:



(b) We want to be able to represent all functions as a sum of sine/cosine waves, essentially in the form:

$$y(x) = b_0 + \sum_{1}^{\infty} a_n sin(nx) \; + \; \sum_{1}^{\infty} b_n cos(nx)$$

This form can be derived, but it is too complicated for the course. We can, however, do a transform to get our answer in this form. To get this form we only need to solve for the values b_0, a_n and b_n. These constants can be solved with the following formulas:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} sin(nx) y(x) \cdot dx$$

$$b_n = \int_0^{2\pi} cos(nx) y(x) \cdot dx$$

$$b_0 = \frac{1}{2\pi} \int_0^{2\pi} y(x) \cdot dx$$

(c) The superposition of sine/cosine functions with close frequencies has a few significant properties (refer mostly to Heat and Waves course on beat frequencies).

i. The *fundamental* is referred to as the middle frequency in an odd number of sine/cosine functions. We call its frequency f_0, and hence its period is $T = \frac{1}{f_0}$.

ii. Since the frequencies are quite similar, the addition of trigonometric functions produce a *modulation envelope* (similar to beat frequency). This modulation is a sine/cosine itself and it has a few properties. The *packet width* is referred to as the distance between two different peaks, and it is defined as $W_p = \frac{1}{\Delta f}$ for the summation of 3 sine/cosine waves. The modulation period is the full length of the sine/cosine that is formed by the addition of the waves, and it is defined as $T_m = \frac{2}{\Delta f} = 2W_p$ for 3 sine/cosine waves.

iii. The general formula for the period of the modulation envelope for N tones is:

$$T_{MOD} = \frac{2\pi \cdot N}{\Delta\omega} = \frac{N-1}{\Delta f}$$

It should also be noted that the general formula for the packet width is $W_p = \frac{1}{\Delta f}$.

(d) This knowledge helps us understand the Bandwidth Theorem. Lets assume now that we have some superposition of sine functions that all lie within 48Hz and 52Hz. This means that $\Delta f = 4$Hz. Now lets say that we have an infinite amount of sine waves between these two frequencies. From these two assumptions we can derive a modulation period $T_{MOD} = \frac{N-1}{\Delta f} \to \infty$. A modulation period of close to infinite time is the same as a one-time pulse of length (packet width) $\Delta t = \frac{1}{\Delta f} = 0.25$s that has a frequency of 50Hz.

(e) Bandwidth theoerm states that:

$$\Delta f \Delta t \simeq 1 \;\&\; \Delta\omega\Delta t \simeq 2\pi$$

Diodes:

(a) Diodes only allow current to flow in one direction with very little voltage drop over them (typically 0.6V). This means diodes are non-linear circuit components.

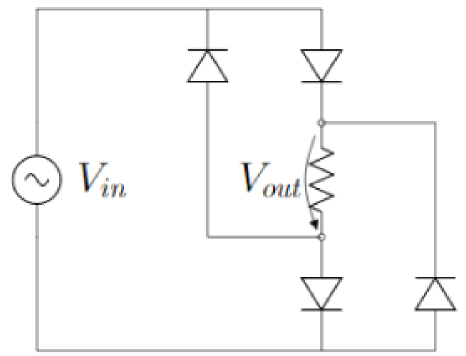
(b) There are a couple things to note for circuits that contain diodes.

i. If the voltage has not met the forward voltage of a diode then treat it like an open wire.

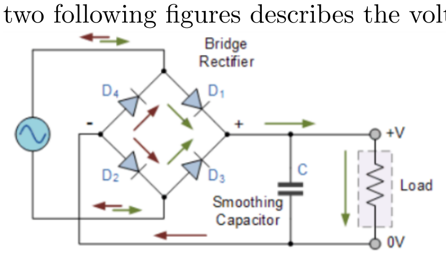
ii. If the voltage has met the forward voltage then it can be treated like a battery that absorbs a voltage equal to its forward voltage. If we consider the diode to be idea (0V forward voltage) then the diode is essentially a shorted wire in the direction of flow and an open wire in the direction opposite of flow.

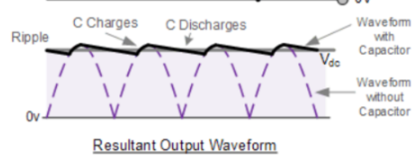
(c) In the context of AC circuits, a diode works to oppose the negative flow of the current. A half-wave rectifier is a circuit that uses a diode to not allow negative voltage to pass. This type of current isn't super useful as we lose half of the current. We can, however, use a full-wave rectifier, which supplies current as if it were the absolute value of the current using four diodes and a capacitor. Refer to the following diagrams:

Full Wave Rectifier Circuit:



This circuit is commonly drawn like the first of the following figures. The second of the two following figures describes the voltage over time with(out) a capacitor included:





Laplace Transforms:

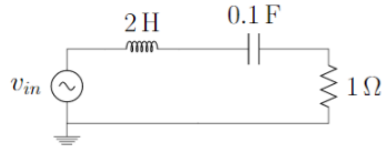
(a) Laplace transformations are a more general case of fourier transformations. With Fourier transformations we are looking at the addition of sine/cosine waves. Sine/Cosine waves can be represented as complex exponentials, so we are limiting our case to adding exponentials that are solely on the imaginary axis. In fact, we can represent all functions with exponentials on the real and imaginary axes.

(b) If we are given an exponential current function like $i(t) = Ie^{st}$, then in our previous case we had that $s = j\omega$. By allowing this we can treat our impedences as R, sL and $\frac{1}{sC}$ for resistors, inductors, and capacitors respectively. The most general form is that $s = \sigma + j\omega$. Using this we can defined how current and voltage acts in a circuit when there are all forms of real and imaginary exponential currents/voltages.

   i. A non-ideal circuit would have voltage decay over time, so this new form for our exponentials would apply for exponentially decaying AC circuitry. If we were given an input voltage such as $V_{in} = 20e^{-2t}e^{4jt}$. In this case we would have that $s = (-2+4j)$. We can then represent, say, an LCR-circuit in the following way:

Consider the following circuit with an exponentially decaying input voltage

$$v_{in} = 20e^{-2t}e^{4jt}$$



the complex s-impedances are

$$Z_L = sL = (-2 + j4) \cdot 2H = (-4 + j8)\ \Omega$$

$$Z_C = \frac{1}{sC} = \frac{10}{(-2 + j4)}\ \Omega$$

$$Z_R = 1\Omega$$

and so the current is

$$\vec{I} = \frac{\vec{V}}{R + sL + \frac{1}{sC}} = \frac{\vec{V}}{1 + (-4 + j8) + \frac{10}{-2+j4}} = \vec{V}\left(\frac{1 - j2}{8 + j14}\right)$$

and the current magnitude is

$$|I| = V\frac{\sqrt{1^2 + 2^2}}{\sqrt{8^2 + 14^2}} = 20\frac{\sqrt{5}}{\sqrt{260}} = 2.8A$$

$$\phi = \arctan(-2) - \arctan(14/8) = -123.5°$$

(in this example, the current lags the voltage by $-123.5°$).
The final solution for the current is

$\frac{-2}{1}$

$$i = \vec{I}e^{st} = 2.8e^{-j(123.5°)}e^{-2t}e^{j4t}$$

(c) In Laplace space there is generally a value for s where the transfer function goes to infinity. This location is called a pole. A zero is when the voltage acts like DC voltage and hence there is no output voltage because an inductor/capacitor is blocking the voltage.

(d) Poles and zeros can be used to characterize circuits. By plotting the poles and zeros of a circuit onto a complex coordinate system as well as the signal of interest, the transfer function ratio and phase change can be solved for.

  i. The ratio of the zero-vector length over the pole-vector length is the ratio of the output voltage over the input voltage.

  ii. The phase can be solved for by taking the difference between the phase of the numerator and the phase of the denominator. Recall that the phase of a complex number would be $tan^{-1}(\frac{Complex}{Real})$.

(e) The Laplace Transform is very similar to the Fourier Transform, it can be seen in the following equation:

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)e^{-st}dt$$

There is also an inverse Laplace Transform to go back into time space from Laplace space:

$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi j}\lim_{T\to\infty}\int_{\gamma-jT}^{\gamma+jT} e^{st}F(s)ds$$

This is often simplified for $\gamma = 0$ as:

$$f(t) = \frac{1}{2\pi j}\int_{j\omega=-\infty}^{j\omega=\infty} e^{st}F(\omega)d\omega$$

The Wave Equation:

(a) Given some wave that travels in the positive x-direction at the speed of light ($c = 3*10^8 \frac{m}{s}$) defined by the function $f(x)$. An electron moving along with this wave would describe events using his own x-coordinate $x' = x - ct$, or $x' = x + ct$ for a wave moving to the left. This means that for an electron, events would be moving as a function (with respect to time instead of distance):

$$y = f(t - \frac{x}{c}) \quad \& \quad y = f(t + \frac{x}{c})$$

(b) I will be referring to a rightward moving sine wave for all of the following mathematics, but the derivations are the same for leftward moving sine waves. Recall that a sine wave is described by the function $y = Asin\omega t'$. We can plug this into our new electron-movement-defined time axis to get ($k = \frac{\omega}{c}$):

$$y = Asin\omega(t - \frac{x}{c}) = Asin(\omega t - kx)$$

This can also be represented as:

$$y = Ae^{j(\omega t - kx)}$$

(c) If we take our function as $y = f(x - ct)$, we can find the second derivate in terms of both x and t to find the following relationships:

$$\frac{dy}{dx} = f'(x - ct) \quad \frac{d^2y}{dx^2} = f''(x - ct)$$

$$\frac{dy}{dt} = -cf'(x - ct) \quad \frac{d^2y}{dt^2} = c^2f''(x - ct)$$

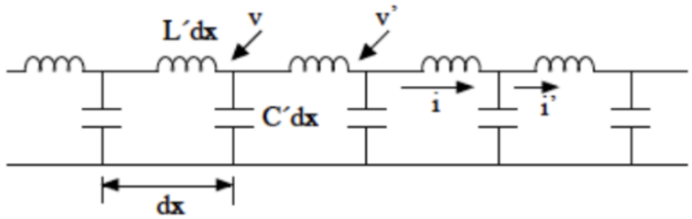
By combining these two equations we have derived the wave equation which defines the way a wave travels down the x-axis with the speed c:

$$\frac{d^2y}{dx^2} = \frac{1}{c^2}\frac{d^2y}{dt^2}$$

Transmission Lines:

(a) One assumption has been made throughout the entirety of this discussion. This assumption is that wires are perfect, but they are not. A typical wire (transmission line) transmits electricity at about $\frac{3}{4}$ of the speed of light.

(b) A wire is quite complicated, but in essesnce, each distance dx has a capacitance across it $C'dx$ and an inductance along it $L'dx$. The resistance per unit length is miniscule in comparison, and hence does not add to the total impedence. The following is a diagram of such a wire:



(c) We can apply the same derivation that we had to derive the wave equation to this circuit. In the end we can derive an equation for voltage with respect to time as well as distance:

$$\frac{d^2V}{dx^2} = L'C'\frac{d^2V}{dt^2}$$

Note that L' and C' are values of inductance and capacitance per distance. We have essentially derived a wave equation for wires where $c = \frac{1}{\sqrt{L'C'}}$.

(d) Midway through the derivation of the *wire wave equation* we would have solved a relationship between voltage per distance and current per time:

$$\frac{dV}{dx} = -L'\frac{di}{dt}$$

With this we can calculate the impedence Z_0 of the transmission line. Using that $V = f(x - ct)$, we can solve for the current. After this derivation we would get that:

$$i = \frac{f(x-ct)}{cL'} + const. = \frac{V}{cL'} + const.$$

The impedence is simply:

$$\frac{V}{i} = cL' = \frac{L'}{\sqrt{L'C'}} = \sqrt{\frac{L'}{C'}} = Z_0$$

(e) There is an issue with the model we are currently using. As signals propagate down a wire, inductors and capacitors charge and discharge causing static. This issue can mostly be solved with Fourier Transforms (fact-check**). The more startling issue is that when the signal gets to the end of the wire, it has nowhere to go, and will return back the way it came (a reflection). To solve this issue, a resistor can be placed at the other end of the wire (not the end with the voltage), which will absorb all excess power.

i. The left-going wave can be defined as:

$$V_+ + V_- = V_{Left}\frac{R_0}{R_L}$$

This has R_0 as the resistance of the wire $\frac{V}{I}$, and it has V_+ as the voltage drop when going rightwards and V_- being the voltage drop when going leftwards. The resistance we must put on the opposite side of the wire must be equal to R_0 so that $V_+ - V_- = V_L$.

ii. We call the reflection coefficent (for voltage):

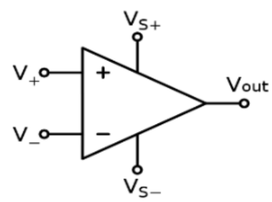
$$\rho_V = \frac{V_-}{V_+} = \frac{R_L - R_0}{R_L + R_0}$$

Op-Amps:

(a) Op-amp stands for *operational amplifier*, and as the name suggests, they amplify voltage. View the following image as reference:

$V_{out} = Ga * (V^+ - V^-)$

Vs+ and Vs- represent the power supplies to the amplifier



Good Luck on the Final!