PHYS 340 - Final Review

Vector Calculus:

(a) $\vec{A} \cdot \vec{B} = \sum_{i,j} A_i B_j (\hat{u}_i \cdot \hat{u}_j) = \sum_{i,j} A_i B_j \delta_{ij} = \sum_i A_i B_i = |A||B|cos(\theta)$

(b) $\vec{A} \times \vec{B} = \sum_i A_i \hat{u}_i \times \sum_j B_j \hat{u}_j = \sum_{i,j} A_i B_j (\hat{u}_i \times \hat{u}_j) = \sum_{i,j,k} A_i B_j \hat{u}_k \varepsilon_{ijk} = |A||B|sin(\theta)\hat{n}$

i. $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = -\varepsilon_{132} = -\varepsilon_{321} = -\varepsilon_{213} = 1$

ii. $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

iii. $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

iv. $\varepsilon_{ijk}\varepsilon_{mlk} = \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm}$

(c) $\vec{\nabla} \cdot \vec{v} = (\frac{d}{dx}\hat{x} + \frac{d}{dy}\hat{y} + \frac{d}{dz}\hat{z}) \cdot (v_x\hat{x} + v_y\hat{y} + v_z\hat{z}) = \frac{d}{dx}v_x + \frac{d}{dy}v_y + \frac{d}{dz}v_z = \sum_{i,j} d_i\hat{u}_i \cdot v_j\hat{u}_j = \sum_{i,j} d_i v_j (\hat{u}_i \cdot \hat{u}_j) = \sum_{i,j} d_i v_j \delta_{ij} = \sum_i d_i v_i$

(d) $\vec{\nabla} \times \vec{v} = (\frac{d}{dx}\hat{x} + \frac{d}{dy}\hat{y} + \frac{d}{dz}\hat{z}) \times (v_x\hat{x} + v_y\hat{y} + v_z\hat{z}) = \sum_{i,j,k} d_i\hat{u}_i \times v_j\hat{u}_j = \sum_{i,j,k} d_i v_j (\hat{u}_i \times \hat{u}_j) = \sum_{i,j,k} d_i v_j \hat{u}_k \varepsilon_{ijk}$

(e) Product Rules:

i. $\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$

ii. $\vec{\nabla} fg = g\vec{\nabla} f + f\vec{\nabla} g$

iii. $\vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A}$

iv. **Divergence**:

α) $\vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\vec{\nabla} f)$

β) $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

v. **Curl**:

α) $\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times (\vec{\nabla} f)$

β) $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$

(f) Second Derivatives:

i. $\vec{\nabla} \cdot (\vec{\nabla} T) = d_i\hat{u}_i \cdot (d_j\hat{u}_j T) = d_i d_j T(\hat{u}_i \cdot \hat{u}_j) = d_i d_j T\delta_{ij} = d_i d_i T = \vec{\nabla}^2 T$

ii. $\vec{\nabla} \times (\vec{\nabla} T) = 0$

iii. $\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \leftarrow$ Homework

iv. $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

v. $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{A}$

Integral Calculus:

(a) Line Integrals:

i. $\int_a^b \vec{v} \cdot d\vec{l}$

α) Normally the path taken from $a \to b$ is critical, but vector functions for which the line integrals are **independent** of path, and are determined entirely by the end points are special. A force that has this property is called a **conservative** force.

ii. Closed path: $\oint_P \vec{v} \cdot d\vec{l}$

α) A closed line is a surface.

β) A conservative force will have: $\oint \vec{F} \cdot d\vec{l} = 0$

(b) Surface Integrals:

i. $\int_S \vec{v} \cdot d\vec{a}$

α) This integral is known as the flux.

β) Normally the surface chosen is critical, but vector functions for which the surface integrals are **independent** of the surface, and are determined entirely by the boundary are special.

ii. Closed path: $\oint \vec{v} \cdot d\vec{a}$

α) A closed surface is a volume.

(c) Volume Integrals:

i. $\int_V \vec{v} \cdot d\vec{\tau}$

(d) Fundamental Theorem of Calculus: $\int_a^b (\frac{df}{dx})dx = f(b) - f(a)$

i. Gradients: $\int_a^b (\vec{\nabla} T) \cdot d\vec{l} = T(b) - T(a)$

ii. Divergences: $\int_V (\vec{\nabla} \cdot \vec{v})d\vec{V} = \oint_S \vec{v} \cdot d\vec{a}$

α) ?*From Notes*? Divergence of a vector function is a scalar function, and only its surface integral matters.

iii. Curls: $\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$

α) Curls cross out for every $d\vec{a}$ because the x & y components cancel. The only parts of the curl that don't cancel are the x & y components on the surface's border.

(e) Integration by Parts: $\int_a^b g\frac{df}{dx}dx = -\int_a^b f\frac{dg}{dx}dx + (fg)|_a^b$

i. There are many combinations of "Integration by Parts" that can be derived. Check homeworks to see more examples.

(f) Spherical Coordinates:

i. Check the formula sheet, and pray you remember Advanced Calculus.

(g) Cylindrical Coordinates:

i. Check the formula sheet, and pray you remember Advanced Calculus.

Dirac Delta Function:

(a) $\delta(x) = 0$ if $x \neq 0$ & $\delta(x) = \infty$ if $x = 0$.

(b) $\int_{-\infty}^{\infty} \delta(x)dx = 1$

(c) $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$

(d) $\int_{-\infty}^{\infty} \delta(ax)dx = \frac{1}{|a|}$

(e) $\frac{d}{dx}\delta(x) = \frac{-\delta(x)}{x}$

i. The rate of change from 0^- is ∞, while from 0^+ it is $-\infty$.

(f) Given $\vec{v} = \frac{\hat{r}}{r^2}$:

i. $\oint_S \vec{v} \cdot d\vec{a} = 4\pi$

ii. $\vec{\nabla} \cdot \vec{v} = 4\pi\delta^3(\vec{r})$

    α) $\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$

    β) $\int_{allspace} \delta^3\vec{r}\,d\vec{\tau} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz = 1$

    γ) $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$

Theory of Vector Fields:

(a) Potential: $\vec{E} = -\vec{\nabla} V$

(b) Curl-less Fields:

i. $\vec{\nabla} \times \vec{E} = 0$

ii. $\vec{E} = -\vec{\nabla}\vec{V}$

iii. $\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b -(\vec{\nabla}\vec{V}) \cdot d\vec{l} = V(a) - V(b)$

iv. $\oint_P \vec{E} \cdot d\vec{l} = 0$

(c) Divergence-less Fields: (Not relevant for midterm)

i. ...

ii. ...

iii. ...

iv. ...

Electric Field:

(a) The Basics:

i. The test charge is Q, and the source charges are $\{q_1, q_2, ...\}$.

ii. The **principle of superposition** states that the interaction between any two charges is completely unaffected by the presence of others. This means that to determine the force on Q, we can first compute $\vec{F_1}$ due to q_1, and so... $\vec{F} = \sum_{i=1}^{n} \vec{F_i}$.

iii. Most problems are worked with respect to the origin. The vector that points to the test charge is $\vec{r}$, while the vector that points to the source charge is $\vec{r'}$, so the vector that seperates the source charge and test charge is $\vec{\mathscr{r}}$, where $\vec{\mathscr{r}} = \vec{r} - \vec{r'}$.

iv.

(b) Coulomb's Law:

$$\vec{F} = \frac{qQ}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}}$$

(c) Electric Field:

i. $\vec{F} = \vec{F_1} + \vec{F_2} + ... = \frac{1}{4\pi\epsilon_0}(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + ...) = \frac{Q}{4\pi\epsilon_0}(\frac{q_1}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + ...) = Q\vec{E}$

ii. $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i$

iii. For a charge distributed continuously over some region, the sum becomes an integral:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}dq$$

    α) $dq \to \lambda dl' \sim \sigma da' \sim \rho d\tau'$

β) For a line charge with $\lambda = \frac{charge}{length}$:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r'})}{\mathscr{r}^2} \hat{\mathscr{r}} dl'$$

γ) For a surface charge with $\sigma = \frac{charge}{area}$:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{r'})}{\mathscr{r}^2} \hat{\mathscr{r}} da'$$

δ) For a volume charge with $\rho = \frac{charge}{volume}$:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'})}{\mathscr{r}^2} \hat{\mathscr{r}} d\tau'$$

(d) Gauss' Law:

i. The flux of $\vec{E}$ can be represented by Φ where $\Phi = \int_S \vec{E} \cdot d\vec{a}$.

$$\oint \vec{E} \cdot d\vec{a} = \int \frac{1}{4\pi\epsilon_0} (\frac{Q_{enc}}{r^2} \hat{r}) \cdot (r^2 sin\theta d\theta d\phi \hat{r}) = \frac{Q_{enc}}{\epsilon_0}$$

ii. By applying the **Divergence Theorem**, $\oint \vec{E} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{E}) d\vec{\tau}$, and since $Q_{enc} = \int_V \rho d\vec{\tau}$, so Gauss' Law becomes $\int_V (\vec{\nabla} \cdot \vec{E}) d\vec{\tau} = \int_V \frac{\rho}{\epsilon_0} d\vec{\tau}$. This means that:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

(e) Work:

$$W = \int_a^b \vec{F} \cdot d\vec{l} = -Q \int_a^b \vec{E} \cdot d\vec{l} = Q[V(b) - V(a)]$$

$$V(\vec{b}) - V(\vec{a}) = \frac{\vec{W}}{Q}$$

i. $V(\vec{b}) - V(\vec{a}) = \frac{\vec{W}}{Q} \rightarrow W = Q[V(r) - V(\infty)] \rightarrow W = QV(r)$

ii. If we bring point charges together, the first point charge takes $W_1 = 0$J because it interacts with no charges. The second would take $W_2 = \frac{1}{4\pi\epsilon} q_2 (\frac{q_1}{\mathscr{r}_{12}})$. The third would take $W_3 = \frac{1}{4\pi\epsilon} q_3 (\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}})$, so for n charges it takes:

$$\frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{i>j}^{n} \frac{q_i q_j}{r_{ij}} = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_{i=1}^{n} q_i \sum_{j=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}} = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r_i})$$

Conductors:

(a) A perfect (ideal) conductor has an **infinite** amount of **free** charges.

i. Inside a conductor, $\vec{E} = 0$, everywhere, always.

ii. $\rho = 0$, $(\vec{\nabla} \cdot \vec{E}) = \frac{\rho}{\epsilon_0} = 0$

iii. $\sigma \neq 0$ if the conductor is embedded in $\vec{E}_{external} \neq 0$, this allows for $\vec{E}_{internal} = 0$

iv. The **entire** conductor is an equipotential:

$$V(\vec{b}\,) - V(\vec{a}\,) = \int_{\vec{a}}^{\vec{b}} \vec{E}\, \cdot d\vec{l} = 0$$

v. At the surface, $\vec{E}$ is **normal** to the surface.

(b) Capacitors:

i. Capacitance:

$$C = \frac{Q}{\Delta V}$$

ii. For two conductors with charge $+Q$ & $-Q$, a capacitor has properties:

$$\vec{E}_{total} = \vec{E}_{top} + \vec{E}_{bottom} = \frac{\sigma}{2\epsilon_0}(-\hat{z}) + \frac{-\sigma}{2\epsilon_0}(\hat{z}) = \frac{-\sigma}{\epsilon_0}\hat{z}$$

$$\Delta V = \int_{Bottom}^{Top} \vec{E}\, \cdot d\vec{l} = -\int \frac{-\sigma}{\epsilon_0}\hat{z} d\hat{z} = \frac{\sigma d}{\epsilon_0}$$

$$C = \frac{Q}{\Delta V} = \frac{Q}{\frac{\sigma d}{\epsilon_0}} = \frac{\epsilon_0 Q}{\sigma d} = \frac{\epsilon_0 Q}{\frac{Q}{d} d} = \frac{\epsilon_0 A}{d}$$

Technique!

(a) Primary Approach:

i. Use Coulomb's Law:

$$\vec{E}\,(\vec{r}\,) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,)(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} d\vec{\tau}$$

(b) Secondary Approach:

i. Use Gauss' Law:

$$\oint \vec{E}\, \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = |E|(Area)$$

(c) Tertiary Approach:

i. Use Electric Potential:

$$\vec{V}\,(\vec{r}\,) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d\vec{\tau}$$

$$\vec{E}\,(\vec{r}\,) = -\vec{\nabla}\, V(\vec{r}\,)$$

(d) Quaternary Approach:

i. Use Poisson's or Laplace's Equations:

$$\nabla^2 V(\vec{r}\,) = \frac{-\rho(\vec{r}\,)}{\epsilon_0}$$

$$\rho(\vec{r}\,) = 0 \Rightarrow \nabla^2 V(\vec{r}\,) = 0$$

Laplace's Equation:

(a) Electric Field:

$$\vec{E}\left(\vec{r}\right) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\vec{r}')d\tau'$$

(b) Electric Potential:

$$V\left(\vec{r}\right) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\vec{r}')d\tau'$$

(c) Laplace's Equation:

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

When there is no charge, $\rho = 0$, and Laplace's equation becomes:

$$\nabla^2 V = 0$$

This is synonymous to:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

i. Laplace's equation does not allow for any local minima or maxima in the Electric Potential. All minima and maxima are extreme values (occur on the boundaries of the function).

ii. Due to the previous point, all occurences of Electric Potential in a space are averaged values of the Electric Potential adjacent (this is easiest to imagine in one dimension).

iii. The method of relaxation is used to solve for Electric Potential using Laplace's equation (we do not need to know this).

Uniqueness Theorems:

(a) Solving Laplaces equation in the 2nd and 3rd dimension is very difficult because we are confronted by partial differential equations. Sufficient boundary conditions must be given, and the uniqueness theorem tells us that Electric Potential is uniquely determined by the boundary values.

(b) "**First uniqueness theorem**: The solution to Laplace's equation in some volume V is uniquely determined if V is specified on the boundary surface S."

(c) "**Second uniqueness theorem**: In a volume V surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given..."

Method of Images:

(a) The first uniqueness theorem states that the following situations are equivalent:

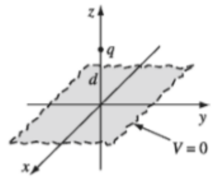


FIGURE 3.10

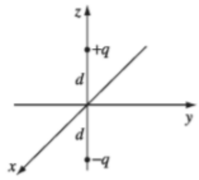


FIGURE 3.11

The infinite grounded conducting plane has 0 electric potential. The boundary conditions from the uniqueness theorem state that there is only one solution to this problem. The solution is defined by the "method of images" which states that the solution is identical to a negative charge opposite the plane.

(b) Since the potential is known, the surface charge σ induced on the conductor can be deduced:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} \rightarrow \sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{\frac{3}{2}}}$$

This means the total induced charge is:

$$Q = \int \sigma da = \int_0^{2\pi} \int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{\frac{3}{2}}} r dr d\phi = -q$$

(c) The charge is attracted to the plane because of its negative charge:

$$\vec{F} = -\frac{q^2}{4\pi\epsilon_0 (2d)^2 \hat{z}}$$

While the force can be found by using the second figure, the work done cannot. In the second figure the electric field is nonzero everywhere, but in the first figure only $z > 0$ has a nonzero field, therefore:

$$\vec{W} = \int_{\infty}^{d} \vec{F} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0} \int_{\infty}^{d} \frac{q^2}{4z^2} dz = -\frac{q^2}{4\pi\epsilon_0 4d}$$

Multipole Expansion:

(a) We have only looked at electric **monopole**s up to this point, but **multipoles** that consist of equal and opposite charges seperated by equal distances are equally important. The total charge is 0, but the electric potential is not. The **electric dipole** consists of two equal and opposite charges separated by a distance d:

$$V(\vec{r}) \cong \frac{qdcos\theta}{4\pi\epsilon_0 r^2}$$

A monopole scales with $\frac{1}{r}$, a dipole with $\frac{1}{r^2}$, a quadrapole with $\frac{1}{r^3}$ and so on. The general solution for a multipole is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{r} \rho(\vec{r}') d\tau' = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(cos\alpha) \rho(\vec{r}') d\tau'$$

(b) The multipole expansion is dominated by the monopole term $V_{mon}(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r}$ at large r values. Since the total charge is 0, this term vanishes, and the dipole term becomes dominant:

$$V_{dip}(\vec{r}) = \frac{\hat{r}}{4\pi\epsilon_0 r^2} \cdot \int \vec{r}\,'\rho(\vec{r}\,')d\tau'$$

The integrated term is called the **dipole moment**:

$$\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')d\tau' \rightarrow V_{dip}(\vec{r}) = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}$$

The dipole moment of a collection of point charges is:

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}\,'$$

(c) For a perfect dipole:

$$V_{dip}(r,\theta) = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r^2} = \frac{p cos\theta}{4\pi\epsilon_0 r^2}$$

The electric field is (first equation more useful):

$$\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2cos\theta\hat{r} + sin\theta\hat{\theta}) = \frac{1}{4\pi\epsilon_0 r^3}[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}]$$

Magnetic Forces:

(a) The **Lorentz force law** defines the forces of eletric and magnetic fields:

$$\vec{F} = Q[\vec{E} + (\vec{v} \times \vec{B})] \rightarrow \vec{F}_{mag} = Q(\vec{v} \times \vec{B})$$

(b) **Cyclotron motion** involves a charged particle in a magnetic field, traveling in a circular motion. Since magnetic force is perpendicular to particle motion we can define the particle's motion:

$$QvB = \frac{mv^2}{r} \rightarrow p = QBR$$

The second equation above is the cyclotron formula that defines the momentum p of a charged particle.

(c) Magnetic forces do no work:

$$dW_{mag} = \vec{F}_{mag} \cdot d\vec{l} = Q(\vec{v} \times \vec{B}) \cdot \vec{v}\,dt = 0$$

Current:

(a) The **current** is the amount of charge per unit time passign a given point (measured in amperes). Given a line charge density $\lambda(\frac{C}{m})$, the current can be defined as:

$$\vec{I} = \lambda\vec{v}$$

With this information we can redefine the magnetic force:

$$\vec{F}_{mag} = Q(\vec{v} \times \vec{B}) = \int (\vec{v} \times \vec{B}) dq = \int (\vec{v} \times \vec{B}) \lambda dl = \int (\vec{I} \times \vec{B}) dl = I \int (d\vec{l} \times \vec{B})$$

(b) If the current instead flows over a surface, then there is a surface current density $\vec{K}$ which can be defined as $\vec{K} = \sigma \vec{v}$, and therefore:

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B}) dq = \int (\vec{v} \times \vec{B}) \sigma da = \int (\vec{K} \times \vec{B}) da$$

(c) Similarly, given a volume current density $\vec{J}$ where $\vec{J} = \rho \vec{v}$, and therefore:

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B}) dq = \int (\vec{v} \times \vec{B}) \rho d\tau = \int (\vec{J} \times \vec{B}) d\tau$$

(d) The **continuity equation** defines charge flow. Because charge is conserved, whatever flows out through a surface must come at the expense of what remains inside:

$$\oint_S \vec{J} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{J}) d\tau$$

The continuity equation states that:

$$\vec{\nabla} \cdot \vec{J} = \frac{\partial \rho}{\partial t}$$

Steady Currents:

(a) A **steady current** produces a magnetic field that is constant with time where $\frac{\partial \rho}{\partial t} = 0$ and $\frac{\partial \vec{J}}{\partial t}$.

(b) A steady current implies that:

$$\vec{\nabla} \cdot \vec{J} = 0$$

(c) A steady current has a magnetic field which can be defined by the **Biot-Savart law**:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \vec{\mathscr{r}}}{\mathscr{r}^2}$$

For surface and volume current:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}') \times \vec{\mathscr{r}}}{\mathscr{r}^2} da'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times \vec{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

(d) The magnetic field of an infinite straight wire can be defined by:

$$\oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I$$

Since the answer doesn't depend on the radius s, any loop that encloses the wire would give the same answer.
The same can be done with any amount of inclosed current:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

If the current is defined by the volume current density:

$$I_{enc} = \int \vec{J} \cdot d\vec{a}$$

We can relate:

$$\mu_0 I_{enc} = \mu_0 \int \vec{J} \cdot d\vec{a} = \oint \vec{B} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a}$$

because of Stoke's Theorem, and therefore:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

This is the curl of the magnetic field. It is called **Ampere's law**. All of these equations are important to know! Coulomb's law is to Gauss' Law as the Biot-Savart law is to Ampere's law. It is also important to note the divergence of the magnetic field which is:

$$\vec{\nabla} \cdot \vec{B} = 0$$

(e) A current carrying wire wrapped vertically along a rod is called a **Solenoid**. If we consider that the Solenoid has n winds of wire per unit length on a cylinder with steady current I, then:

$$\vec{B} = \mu_0 n I \hat{z}$$

inside the solenoid, and:

$$\vec{B} = 0$$

outside the solenoid.

<u>Magnetic Vector Potential</u>:

(a) In electrostatics, the fact that $\vec{\nabla} \times \vec{E} = 0$ allowed us to introduce that $\vec{E} = \vec{\nabla} V$. Similarly, since we know that $\vec{\nabla} \cdot \vec{B} = 0$ allows the vector potential $\vec{A}$ to be introduced:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

The divergence of A:

$$\vec{\nabla} \cdot \vec{A} = 0$$

Given Ampere's law the following can be proven about $\vec{A}$:

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

This relation is similar to Poisson's equation. Using this we can create a function for $\vec{A}$:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{\mathscr{r}} d\tau'$$

Similarily for line and surface currents:

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}}{\mathscr{r}} dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathscr{r}} d\vec{l}'$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{\mathscr{r}} da'$$

(b) Multipole Expansion of the Vector Potential:

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(cos\alpha) d\vec{l}'$$

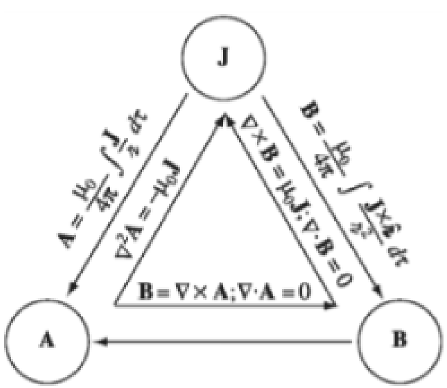
(c) The vector potential of a dipole moment is defined by:

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \hat{r}}{r^2}$$

where $\vec{m}$ is the **magnetic dipole moment**:

$$\vec{m} = I\int d\vec{a} = I\vec{a}$$

Everything to know about Magnetism in one triangle:



Polarization:

(a) **Insulators**, also known as **dielectrics**, are one of two large classes of matter that respond to electric fields; the other class is **conductors**. Dielectrics consist of nuclei with tightly bound electrons. Electric fields cause changes in the orientation of the electron.

(b) If an atom/molecule in a dielectric is subject to an Electric field, then the atom becomes **polarized**. A polarized atom has a **dipole moment**, $\vec{p}$, defined by:

$$\vec{p} = \alpha\vec{E}$$

Where α is the **atomic polarizability**.

i. If we treat any dipole like a stick with opposite charges on each end, then we can define the torque on each end, $\vec{N}$, by:

$$\vec{N} = \vec{p} \times \vec{E}$$

ii. If there is a nonuniform electric field, then the force on both ends of a dipole are different, and the force on the dipole can be defined as:

$$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E}$$

iii. The torque equation for a perfect dipole in a nonuniform electric field:

$$\vec{N} = (\vec{p} \times \vec{E}) + (\vec{r} \times \vec{F}) = (\vec{p} \times \vec{E}) + (\vec{r} \times ((\vec{p} \cdot \vec{\nabla})\vec{E}))$$

(c) When subject to an electric field, a dielectric material becomes **polarized**. The **polarization** of a dielectric is defined by $\vec{P}$ which is the dipole moment per unit volume.

<u>The Field of a Polarized Object</u>:

(a) Since $\vec{P}$ is a measure of $\frac{\vec{p}}{volume}$, we can define the electric potential of a material as being:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\nu}\frac{\vec{P}(\vec{r}\,')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

By investigating the equality: $\vec{\nabla}'(\frac{1}{\mathscr{r}}) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$, our Electric Potential can be transformed:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\nu}\frac{\vec{P}(\vec{r}\,')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' = \frac{1}{4\pi\epsilon_0}\int_{\nu}\vec{P}\cdot\vec{\nabla}'(\frac{1}{\mathscr{r}})d\tau' = \frac{1}{4\pi\epsilon_0}[\int_{\nu}\vec{\nabla}'\cdot(\frac{\vec{P}}{\mathscr{r}})d\tau' - \int_{\nu}\frac{1}{\mathscr{r}}(\vec{\nabla}'\cdot\vec{P})d\tau']$$

$$= \frac{1}{4\pi\epsilon_0}\oint_S\frac{1}{\mathscr{r}}\vec{P}\cdot d\vec{a}\,' - \frac{1}{4\pi\epsilon_0}\int_{\nu}\frac{1}{\mathscr{r}}(\vec{\nabla}'\cdot\vec{P})d\tau' = \frac{1}{4\pi\epsilon_0}\oint_S\frac{\sigma_b}{\mathscr{r}}da' + \frac{1}{4\pi\epsilon_0}\int_{\nu}\frac{\rho_b}{\mathscr{r}}d\tau'$$

Where the **bound surface charge** is defined by:

$$\sigma_b = \vec{P}\cdot\hat{n}$$

And the **bound volume charge** is defined by:

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

The Electric Displacement:

(a) The total charge density within a dielectric can be defined by: $\rho = \rho_b + \rho_f$, where ρ_f is the "**free charge**". From Gauss's Law we know that $\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho$. This means that:

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho = \rho_b + \rho_f = -\vec{\nabla} \cdot \vec{P} + \rho_f$$

From this, free charge can be defined by:

$$\vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f$$

If we define:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$\vec{D}$ is known as the **electric displacement**, and with it we can define Gauss's Law by:

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

and

$$\oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}}$$

(b) The electric displacement is deceptively similar to the electric potential. To see the difference, the cross product of $\vec{D}$ is:

$$\vec{\nabla} \times \vec{D} = \epsilon_0 (\vec{\nabla} \times \vec{E}) + (\vec{\nabla} \times \vec{P}) = \vec{\nabla} \times \vec{P}$$

Linear Dielectrics:

(a) We still have not defined polarization. With our new found knowledge we can begin to describe it:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

i. χ_e is the **electric susceptibility** of the dielectric medium.

ii. The linear relationship between $\vec{P}$ and $\vec{E}$ makes the dielectric a **linear dielectrics**.

(b) We can now define the electric displacement, $\vec{D}$, by:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1 + \chi_e) \vec{E}$$

(c) We can define electric displacement as:

$$\vec{D} = \epsilon \vec{E}$$

This definition is derived from the **permittivity** of a material:

$$\epsilon = \epsilon_0 (1 + \chi_e)$$

We can define the **relative permittivity**, also known as the **dielectric constant**, by:

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

(d) The electric displacement in a vaccuum is defined by:

$$\vec{D} = \epsilon_0 \vec{E}_{vac}$$

From this and previous equations we can define the following relation:

$$\vec{E} = \frac{1}{\epsilon}\vec{D} = \frac{1}{\epsilon_r}\vec{E}_{vac}$$

Miscellaneous Equations in Dielectrics:

(a) The electric field produced by a free charge q in a large (infinite) dielectric can be defined by:

$$\vec{E} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{r}$$

(b) A parallel-plate capacitor filled with an insulating material of dielectric constant ϵ_r has a capacitance defined by:

$$C = \epsilon_r C_{vac}$$

(c) The bound charge density ρ_b can be defined proportionally to free charge density ρ_f:

$$\rho_b = -(\frac{\chi_e}{1+\chi_e})\rho_f$$

(d) The amount of work it takes to charge up a capacitor is defined by:

$$\frac{1}{2}\int \vec{D} \cdot \vec{E}\, d\tau$$

(e)

$$F = -\frac{\epsilon_0 \chi_e w}{2d} V^2$$

This fringing force is defined where a dielectric is not completely inside of a capacitor. The width of that capacitor is defined by w.

(f) If a battery is hooked up to this capacitor (the battery does work to keep an even potential (no loss of charge)), then the fringing force can be defined by:

$$F = \frac{1}{2}V^2\frac{dC}{dx}$$

Good Luck on the Final!