

## PHYS 232

### CGS Units:

- (a) Always remember to keep your units in CGS; centimeters, grams, seconds, ergs, etc.

### Imaginary Numbers and Trigonometric Identities:

- (a) Euler's number can be defined by:

$$\cos(\theta) + i\sin(\theta) = e^{i\theta}.$$

$i$  is defined by:

$$i = \sqrt{-1}.$$

- (b) Here are some trigonometric identities that may be important for solving problems:

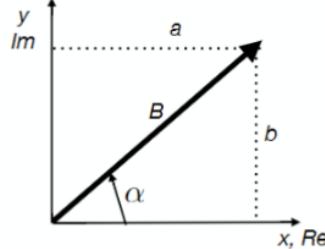
$$\begin{aligned} \text{i. } \cos(\omega t) &= \frac{e^{i\omega t} + e^{-i\omega t}}{2} \\ \text{ii. } \sin(\omega t) &= \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \end{aligned}$$

#### **Cartesian**

$$\begin{aligned} \mathbf{B} &= a\hat{x} + b\hat{y} \\ |\mathbf{B}|^2 &= a^2 + b^2 \end{aligned}$$

#### **Complex**

$$\begin{aligned} \mathbf{B} &= a + ib \\ \alpha &= \arctan(b/a) \end{aligned}$$



$$|\mathbf{B}|^2 = (a + ib)(a - ib) = \mathbf{B}\mathbf{B}^*$$

$$\mathbf{B} = \sqrt{(a^2 + b^2)}e^{i\alpha}$$

### Periodic Motion:

- (a) Given a period  $T$ , an angular frequency  $\omega$ , and a frequency  $f$ :

$$\begin{aligned} \text{i. } \omega &= \frac{2\pi}{T} \\ \text{ii. } \omega &= 2\pi f \\ \text{iii. } f &= \frac{1}{T} \end{aligned}$$

- (b) The equation of motion for a spring is:

$$mx'' = -k_1x - k_2x^2 - k_3x^3 - \dots$$

but with a small angle approximation:

$$mx'' = -kx_1$$

The position of the spring, thus, can be described by:

$$x(t) = A \sin(\omega t + \phi_0)$$

The angular frequency is defined by:

$$\omega = \sqrt{\frac{k_1}{m}}$$

$A$  is the amplitude of oscillation, and  $\phi_0$  is the initial phase. It is also important to note that when  $x = 0$ , the spring is at equilibrium position where speed is at a maximum, and the magnitude of acceleration is at a minimum.

- (c) The position of a spring can be defined in one of 4 ways:

$$x_0(t) = A \sin(\omega t + \phi_0),$$

$$x_1(t) = B \cos(\omega t + \phi_1),$$

$$x_2(t) = \operatorname{Re}[C e^{i(\omega t + \phi_2)}],$$

$$x_3(t) = \operatorname{Re}[Z e^{i\omega t}].$$

To understand the value of  $Z$ , consider this derivation:

$$x(t) = A e^{i(\omega t + \phi_0)} = A e^{i\omega t} e^{i\phi_0} = A (\cos(\phi_0) + i \sin(\phi_0)) e^{i\omega t} = Z e^{i\omega t}$$

Repeated earlier in “Imaginary Numbers and Trigonometric Identities”:

If we consider  $\cos(\phi_0) + i \sin(\phi_0)$  as being a unit vector on the real-imaginary axis, then:

$$\phi_0 = \tan^{-1}\left(\frac{\operatorname{Im}[Z]}{\operatorname{Re}[Z]}\right),$$

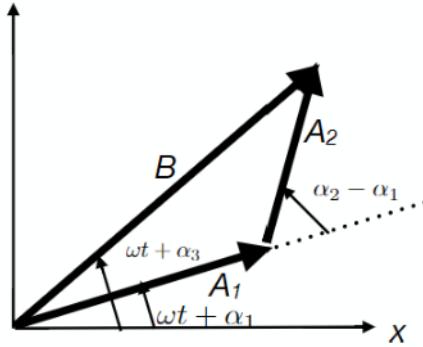
$$A = |Z| = \sqrt{ZZ^*}.$$

### Vibration Combinations:

- (a) Given 2 vibrations of equal frequency,  $x_1 = A \cos(\omega t + \alpha)$ , and  $x_2 = B \cos(\omega t + \beta)$ . We can combine these vibrations in the following way:

$$x_1 + x_2 = A \cos(\omega t + \alpha) + B \cos(\omega t + \beta) = C \cos(\omega t + \gamma).$$

The combination can be expressed with the following vector diagram:



The values of  $C$  and  $\gamma$  can be derived in the following way:

$$\begin{aligned}
 x &= A\cos(\omega t + \alpha) + B\cos(\omega t + \beta) = \operatorname{Re}[Ae^{i(\omega t + \alpha)}] + \operatorname{Re}[Be^{i(\omega t + \beta)}] = \operatorname{Re}[(Ae^{i\alpha} + Be^{i\beta})e^{i\omega t}] \\
 &= \operatorname{Re}[Ce^{i\gamma}e^{i\omega t}] = \operatorname{Re}[Ce^{i(\omega t + \gamma)}] = C\cos(\omega t + \gamma) \\
 C^2 &= (Ae^{i\alpha} + Be^{i\beta})(Ae^{-i\alpha} + Be^{-i\beta}) = A^2 + B^2 + AB e^{i(\alpha - \beta)} + AB e^{i(\beta - \alpha)} \\
 &= A^2 + B^2 + 2AB\cos(\alpha - \beta) \\
 \gamma &= \tan^{-1}\left(\frac{B\sin(\alpha) + A\sin(\beta)}{A\cos(\alpha) + B\cos(\beta)}\right) \quad \leftarrow \text{??}
 \end{aligned}$$

- (b) Given 2 vibrations of different frequency, but the same amplitude,  $x_1 = Ae^{i\omega_1 t}$ , and  $x_2 = Ae^{i\omega_2 t}$ . The sum of these vibrations can be combined in the following way:

$$x = x_1 + x_2 = Ae^{i\omega_1 t} + Ae^{i\omega_2 t} = Ae^{i\frac{\omega_1 + \omega_2}{2}t} (e^{i\frac{\omega_1 - \omega_2}{2}t} + e^{i\frac{\omega_2 - \omega_1}{2}t}) = 2A\cos\left(\frac{\omega_2 - \omega_1}{2}\right)e^{i\frac{\omega_1 + \omega_2}{2}t}.$$

This equation describes a beat frequency, where  $|\frac{\omega_2 - \omega_1}{2\pi}|$  is the beat frequency.

- (c) Given the parameterizations:

$$x = A_1\cos(\omega t + \alpha_1), \quad y = A_2\cos(\omega t + \alpha_2)$$

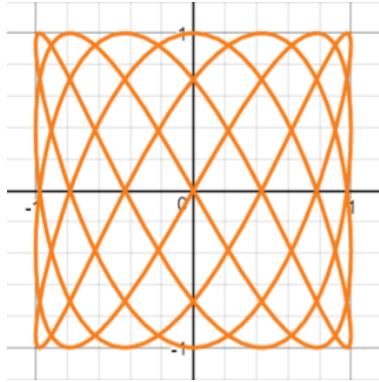
By choosing the values of  $A_1$ ,  $A_2$ ,  $\alpha_1$ , and  $\alpha_2$ , an ellipse (or a straight line) can be formed that fits within  $x \in [-A_1, A_1] \cup y \in [-A_2, A_2]$

- (d) Lissajous Curves are defined by the parameterizations:

$$x = A\sin(at), \quad y = B\sin(bt)$$

The appearance of the Lissajous Curve is based off of the least common factor ratio of  $a$  and  $b$ . The LCF ratio ( $\frac{4}{2}$  would be  $\frac{2}{1}$ ) defines its appearance, where there are LCF( $a$ ) nodes in the  $x$ -direction and LCF( $b$ ) nodes in the  $y$ -direction. Irrational values of  $a$  and  $b$  will give .

Here is an example of  $x = \sin(4t)$ , and  $y = \sin(7t)$ :



Hooke's Law:

- (a) Hooke's law defines the oscillation of an inertial component that carries kinetic energy, and an elastic component that stores the potential energy, acting as the restoring force oppositely directed from the displacement from the equilibrium position. As stated in the "Periodic Motion" section, the equation of motion for a Hooke's Law system is:

$$-kx = mx''$$

- (b) The energy conservation law can be derived from this system by:

$$-\int kx \, mx = \int mx'' \, dx \rightarrow \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E$$

- (c) By solving for  $x$ ,  $x'$ , and  $x''$ , a general solution can be solved for  $x(t)$  where  $\omega = \pm\sqrt{\frac{k}{m}}$ :

$$x(t) = C_1 e^{i\omega t + \alpha} + C_2 e^{-i\omega t + \alpha} = Re[Z e^{i\omega t}] = Re[A e^{i(\omega t + \alpha)}]$$

- (d) Elasticity and Young's Modulus:

- i. A cylinder of mass is composed of atoms (inertial component) with molecular bonds (elastic component). A cylinder can be thought of as a line of spring that can contract by  $\Delta L$  if the whole length is  $L$ ; based off of this we call the strain of this cylinder  $Strain = \frac{\Delta L}{L}$ .
- ii. If we had instead two cylinders next to each other and tried to contract it with the same force, it would only contract half as much. We have essentially doubled the area of the cylinder, and this can be called the stress where  $Stress = \frac{F}{A}$ .
- iii. Every material is able to stretch and shrink by different amounts because the molecular bonds are stronger/weaker. Every material has its respective "Young's Modulus",  $Y$ , which can be defined by  $\frac{F}{A} = \frac{\Delta L}{L} Y$ . The natural length of the cylinder is when  $\Delta L = 0$ , similarly to a spring!, so if we instead represent  $\Delta L$  as  $x$ , then we can rearrange the equation to

$$F = -\frac{YA}{L}x$$

(e) Buoyancy:

- i. An object in a liquid displaces a certain volume. This object has a mass  $m$ , and for simplicity's sake, a cross-sectional area  $A$ . This object has a height  $y$  that it dips into the liquid. The liquid has a density  $\rho$ .
- ii. Since the object goes into the liquid, a volume of the liquid, equal to the volume of the object that goes into the liquid, is placed above the object. The liquid wants to go back into its body, so there is an upward force. This force is proportional to gravity (since the water is being pulled downwards by gravity). Thus, the buoyancy force can be defined by:

$$my'' = -g\rho Ay.$$

(f) Pendulum:

- i. Consider a pendulum with a string of length  $L$  and a mass  $m$  where the equilibrium position is where the string points in the direction of gravity, at  $\theta = 0$ . We can define the displacement from this equilibrium position as:

$$x = L\sin(\theta) \approx L\theta.$$

- ii. The force of gravity is always trying to pull the mass towards the equilibrium position, a force perpendicular to the string  $F_{\perp}$ . This restoring force can be defined as:

$$F_{\perp} = mgsin(-\theta) \approx -mg\theta.$$

- iii. We can then solve for the equation of motion of a pendulum in the following way:

$$x = L\theta \rightarrow x' = L\theta' \rightarrow x'' = L\theta''$$

$$mx'' = F_{\perp} \rightarrow m(L\theta'') = -mg\theta \rightarrow \theta'' = \frac{-g}{L}\theta.$$

(g) U-Tube:

- i. Assume a U-tube with a cross-sectional area  $A$ , length  $l$ , liquid density  $\rho$ , and where the liquid is displaced a distance  $y$  above the equilibrium (therefore 2 displacements, one for each side of the tube). The restoring force can be defined as:

$$F = -g(2y\rho A)$$

- ii. We can say that the whole mass of the liquid is defined as  $\rho Al$ . From this we can describe the equation of motion:

$$my'' = F \rightarrow \rho Aly'' = -2gA\rho \rightarrow y'' = \frac{-2g}{l}y.$$

(h) Torsional Pendulum:

- i. Imagine a mass that is connected at its center of mass by a string. If the mass is rotated, then there will be a restoring torque supplied by the string.

ii. Given a torsional constant  $k$  of the string, and a moment of inertia  $I$  of the mass:

$$\theta'' = \frac{-k}{I}\theta.$$

(i) Air Spring:

i. Given a vessel of gas with a piston that is fit flush on top of it so that the gas inside of the vessel applies a pressure  $p$  on the piston upwards. The piston has a mass  $m$  and an area  $A$ . At equilibrium, there is a length  $l$  of the vessel, below the piston, that has gas within it. If the piston were pushed down some displacement  $y$ , the pressure would push the piston towards the equilibrium position, putting the piston into motion. The force exerted is proportional to the change in pressure  $F = A\Delta p$ . The oscillatory motion can be defined depending on the type of gas inside of the vessel.

a) An **isothermal** system defines the relationship between pressure and volume as:

$$pV = nKT = \text{const.}$$

The equation of motion for an isothermal system is:

$$y'' = \frac{-Ap}{ml}y$$

$\beta)$  An **adiabatic** system defines the relationship between pressure and volume as:

$$pV^\gamma = nKT = \text{const.}$$

The equation of motion for an adiabatic system is:

$$y'' = \frac{-A\gamma p}{ml}y$$

The value of  $\gamma$  is dependent on the type of gas. A monatomic gas has  $\gamma = 1.67$ , and a diatomic gas has  $\gamma = 1.40$ .

Damped Oscillations:

(a) Damped oscillation implies that there is a damping force proportional to the velocity that impedes the movement of the inertial component (i.e. friction). The equation of motion with this new impedance is:

$$mx'' = -kx - bx' \rightarrow x'' + \gamma x' + \omega_0^2 x = 0$$

In this EOM we have that  $\gamma = \frac{b}{m}$ , and  $\omega_0^2 = \frac{k}{m}$ . We can solve for  $x(t)$  by assuming a complex solution  $z$ :

$$x = Re[z] \rightarrow z'' + \gamma z' + \omega_0^2 z = 0 \rightarrow z = Ae^{i(pt+\alpha)} \rightarrow (-p^2 + ip\gamma + \omega_0^2)Ae^{i(pt+\alpha)} = 0.$$

The solution for  $p$  must be imaginary since there is the term  $i\gamma p$ ,  $-p^2 + ip\gamma + \omega_0^2 = 0$ . We

can substitute  $p = n + is$ :

$$-n^2 + 2ins + s^2 + in\gamma - s\gamma + \omega_0^2 = 0$$

The real and imaginary terms must both be equal to zero, so we can set up two equations:

$$-n^2 + s^2 - s\gamma + \omega_0^2 = 0$$

$$-2ns + n\gamma = 0$$

We can thus solve for  $s$  and  $n$ :

$$s = \frac{\gamma}{2}$$

$$n = \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

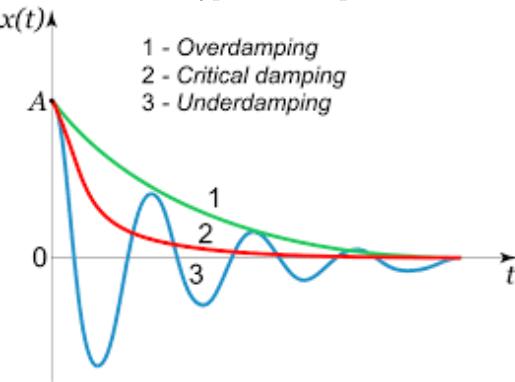
The final solution of motion is:

$$x(t) = Ae^{\frac{-b}{2m}t} \cos(nt + \alpha)$$

Where:

$$n = \pm \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

- (b) There are three types of damped oscillations, represented in the following image:



- i. An **underdamped** oscillation is defined by  $\frac{k}{m} > \frac{b^2}{4m^2}$ .
- ii. A **critically damped** oscillation is defined by  $\frac{k}{m} = \frac{b^2}{4m^2}$ .
- iii. An **overdamped** oscillation is defined by  $\frac{k}{m} < \frac{b^2}{4m^2}$ .

- (c) The damping occurs exponentially, scaling off of  $\frac{\gamma}{2}$ . For every factor it decreases, we call  $\tau$  where:

$$\frac{\gamma}{2} = \frac{1}{\tau}$$

When  $\tau = 1$ , the amplitude of oscillation has decreased to  $\frac{A}{e} \approx A \cdot 37\%$  of its original value.

- (d) We also have a number  $N$  which is defined as the amount of oscillations that occur in one  $\tau$  amount of time, specifically,

$$N = \frac{\tau}{T}$$

This can only occur for underdamped oscillations because oscillations only occur for underdamped oscillations.

- (e) The quality of a damped oscillator is defined by the amount that the oscillation will decay in one oscillation. This means a good quality oscillator has a large  $Q$ -factor.  $Q$  is defined by:

$$Q = \frac{\omega_0}{\gamma}.$$

### Forced Vibration:

- (a) A forced vibration is a driving force that exerts a constant oscillatory force. Forced oscillations tend towards resonance, in which the forced oscillation frequency and the natural frequency (are the same) create a peak amplitude (highest possible amplitude of any frequency). The equation of motion for such a system can be defined by:

$$mx'' = -kx + F_0 \cos(\omega t)$$

By assuming the solution  $x = C \cos(\omega t)$ , we can solve for the amplitude as:

$$C = \frac{\frac{F_0}{m}}{\omega_0^2 - \omega^2}$$

- (b) When the natural frequency  $\omega$  matches the frequency of the driving frequency  $\omega_0$ , the amplitude of the oscillation maxes out, tending towards infinity if they are equal.
- (c) With damping, forced vibrations become a bit tricky. The equation of motion for such a system becomes:

$$mx'' = -kx + bx' + F_0 \cos(\omega t).$$

If we set that  $x = Re[z]$ , we can define the equation of motion with  $z(t)$  as:

$$z'' = -\omega_0^2 z - \gamma z' + \frac{F_0}{m} e^{i\omega t}.$$

- (d) The solution for  $z(t)$  is of the form  $z(t) = Ae^{i(\omega t - \beta)}$ . The values of  $A$  and  $\beta$  were solved during the second assignment, the values can be defined as:

$$A = \frac{\frac{F_0}{m}}{((\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2)^{\frac{1}{2}}} = \frac{F}{k} \frac{\frac{\omega_0}{\omega}}{[(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + \frac{1}{Q^2}]^{\frac{1}{2}}}$$

$$\beta = \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right) = \frac{1}{Q} \frac{1}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}}$$

- (e) The amplitude of such a forced damped oscillator is based off of the quality  $Q$ . For such a quality there is a peak amplitude when the frequency of the forced oscillation is at resonance. The fractional width of this resonant response is defined as the frequency difference at which the amplitude is half of the peak amplitude. As the quality increases,

the fractional width gets smaller. The fractional width  $\delta\omega$  is defined as:

$$\delta\omega = \frac{\omega_0}{2Q}.$$

- (f) Resonance in an electric circuit can be defined similarly to a damped oscillator due to the impedance of its components.
- In an LC (or LRC, but never defined without a driving voltage) circuit, the flow of charge  $q(t)$  is defined as a damped oscillator relating to the inductance  $L$  and capacitance  $C$  of such a circuit:

$$q(t) = Qe^{i(\omega_0 t + \alpha)}$$

The frequency is defined by:

$$\omega_0 = \pm \frac{1}{\sqrt{LC}}$$

- In an LRC (or LC, but never defined with a driving voltage) circuit, the flow of charge  $q(t)$  can be solved for in a similar way as a damped driven oscillator; the impedance of its components along with the driving voltage cause it to act in this way:

$$q'' + \frac{R}{L}q' + \frac{1}{LC}q = \frac{V_0}{L}\sin(\omega t)$$

- (g) The power generated due to a force can be defined in the following way:

$$P = Fv = F\frac{dx}{dt} = F_0\cos(\omega t)\frac{dx}{dt}$$

- For an undamped driven oscillator we know that:

$$x(t) = \frac{\frac{F_0}{m}}{\omega_0^2 - \omega^2}\cos(\omega t) = C\cos(\omega t)$$

From this we can derive that:

$$\frac{dx}{dt} = -\omega C\sin(\omega t)$$

And finally that the generated power is:

$$P(t) = F(t)\frac{dx}{dt} = -\omega C F_0 \sin(\omega t) \cos(\omega t)$$

- For a damped driven oscillator we know that:

$$x(t) = \frac{F_0}{k} \frac{\frac{\omega_0}{\omega}}{((\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + \frac{1}{Q^2})^{1/2}} \cos(\omega t - \beta) = v_0 \cos(\omega t - \beta)$$

From this we can derive that:

$$\frac{dx}{dt} = -v_0 \omega \sin(\omega t - \beta)$$

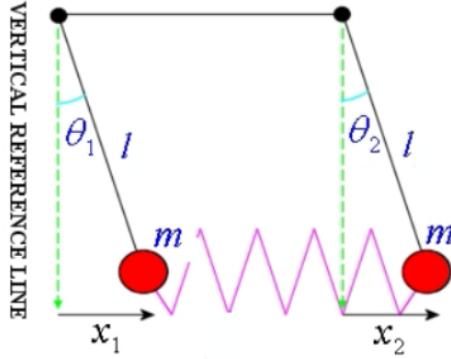
And finally that the generated power is:

$$P(t) = F(t) \frac{dx}{dt} = -v_0 \omega F_0 \sin(\omega t - \beta) \cos(\omega t)$$

$$= -v_0 \omega F_0 \cos(\beta) \sin(\omega t) \cos(\omega t) + v_0 \omega F_0 \sin(\beta) \cos^2(\omega t)$$

### Coupled Oscillators:

- (a) Many systems in nature exist with multiple resonant frequencies due to coupled (or tripled, etc..) systems. These natural resonant frequencies are called *modes*. These systems can be modeled by coupled (or tripled, etc...) oscillators with 2 or more independent inertial components which react to each others movements by elastic components.
- (b) Consider the following system:



We can begin by looking at the two modes of the system. Notice that if both pendulums swing at the same angle then the spring won't act on the system. Similarly, if the pendulums begin at opposite angles of each other, the spring will act on the pendulums in an equal and opposite fashion (the other system is also equal and opposite because of no force). Lets look at the forces acting on both masses:

$$mx''_1 = -mg \frac{x_1}{l} - k(x_1 - x_2)$$

$$mx''_2 = -mg \frac{x_2}{l} - k(x_2 - x_1)$$

These equations can be converted into:

$$x''_1 = -\omega_0^2 x_1 - \omega_k^2(x_1 - x_2)$$

$$x''_2 = -\omega_0^2 x_2 - \omega_k^2(x_2 - x_1)$$

Where  $\omega_k^2 = \frac{k}{m}$  and  $\omega_0^2 = \frac{g}{l}$ . We can reorganize these equations to setup to solve these equations with *eigenvalues*:

$$(\omega_0^2 + \omega_k^2)x_1 - \omega_k^2 x_2 + x''_1 = 0$$

$$(\omega_0^2 + \omega_k^2)x_2 - \omega_k^2 x_1 + x''_2 = 0$$

I won't be solving these equations with superposition as it is good practice to solve these equations with eigenvalues (as it becomes easier with larger numbers of variables). We can begin by changing two of our variables:  $x_1'' = -\omega^2 x_1$  and  $x_2'' = -\omega^2 x_2$ . We derive this from the equation  $x_1 = Ae^{i\omega t + \beta} \rightarrow x_1'' = i^2 \omega^2 A e^{i\omega t + \beta} \rightarrow x_1'' = i^2 \omega^2 x_1 \rightarrow x_1'' = -\omega^2 x_1$ . The same can be said for  $x_2'' = -\omega^2 x_2$ . We can then plug into our formulae and rearrange:

$$(\omega_0^2 + \omega_k^2)x_1 - \omega_k^2 x_2 - \omega^2 x_1 = 0$$

$$-\omega_k^2 x_1 + (\omega_0^2 + \omega_k^2)x_2 - \omega^2 x_2 = 0$$

Notice that we can rewrite these functions as  $A\vec{x} - \lambda\vec{x} = 0$  where:  $A = \begin{bmatrix} \omega_0^2 + \omega_k^2 & -\omega_k^2 \\ -\omega_k^2 & \omega_0^2 + \omega_k^2 \end{bmatrix}$ ,  $\lambda = \omega^2$ , and  $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ . To solve such an equation we must note that  $\det(A - \lambda I) = 0$ . We then will solve using the quadratic formula:

$$\det \begin{vmatrix} \omega_0^2 + \omega_k^2 - \omega^2 & -\omega_k^2 \\ -\omega_k^2 & \omega_0^2 + \omega_k^2 - \omega^2 \end{vmatrix} = 0 \rightarrow$$

$$(\omega_0^2 + \omega_k^2 - \omega^2)^2 - (-\omega_k^2)^2 = 0 \rightarrow \omega^2 = \omega_0^2, \omega_0^2 + 2\omega_k^2$$

- i. We can define the motion of both of the oscillators with a more general formula:

$$x_1 = \frac{1}{2}(C \cos(\omega_0 t) + D \cos(\omega_k t))$$

$$x_2 = \frac{1}{2}(C \cos(\omega_0 t) - D \cos(\omega_k t))$$

- ii. We can use the equations in part α to explore the way in which the two masses will interact. By using the functions:

$$\cos(at) + \cos(bt) = 2\cos\left(\frac{a-b}{2}t\right)\cos\left(\frac{a+b}{2}t\right)$$

$$\cos(at) - \cos(bt) = 2\sin\left(\frac{a-b}{2}t\right)\sin\left(\frac{a+b}{2}t\right)$$

We can see that the two masses interact with an oscillation in the following form:

$$\cos\left(\frac{\sqrt{\omega_0^2 + 2\omega_k^2} - \omega_0}{2}t\right)\cos\left(\frac{\sqrt{\omega_0^2 + 2\omega_k^2} + \omega_0}{2}t\right)$$

$$\sin\left(\frac{\sqrt{\omega_0^2 + 2\omega_k^2} - \omega_0}{2}t\right)\sin\left(\frac{\sqrt{\omega_0^2 + 2\omega_k^2} + \omega_0}{2}t\right)$$

- (c) We can represent the final solution of this problem by an eigenvector:

$$\vec{x} = \sum_j Z_j \begin{pmatrix} x_{j,1} \\ x_{j,2} \end{pmatrix} e^{i\omega_j t}$$

Vibrating String:

- (a) Assume we have  $N$  masses, each of mass  $m$  that are connected by a massless string. The string is connected at its ends, and the tension has a tension  $T$ ; the string is of length  $L$ . The can be treated as if it has been divided into  $N+1$  pieces where each piece has length  $l$ .
- (b) Each piece of string will oscillate with the form  $y_p = A_p \cos(\omega t)$  for all pieces labeled  $p$ . The amplitude can be solved for (difficult proof) as being  $A_p = C \sin(p\theta)$ . This is guess from a derived relationship:  $\frac{A_{p-1} + A_{p+1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$ ; in these equations,  $\omega$  is the frequency of oscillation of a string subsegment, and  $\omega_0^2 = \frac{T}{ml}$ . By the guess we made for  $A_p$  it is solved that  $\frac{A_{p-1} + A_{p+1}}{A_p} = 2\cos\theta$ . Since the string is fixed down:  $A_0 = A_{N+1} = 0$ . From out of nowhere (that I can tell) we also can assume that  $(N+1)\theta = n\pi$  where  $(n = 1, 2, 3, \dots << N)$ , where  $n$  is the harmonic number. The harmonic number is just the number of nodes (times the x-axis is crossed) - 1. From this we get that,  $\theta = \frac{n\pi}{N+1} \rightarrow A_p = C \sin\left(\frac{pn\pi}{N+1}\right)$ . This proof sucks, the result is:

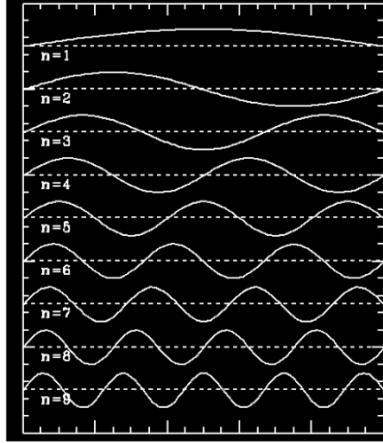
$$\omega_n = n \frac{\pi}{L} \sqrt{\frac{T}{\mu}}$$

We have that  $\mu = \frac{m}{l}$ , the mass density of the string.

- (c) The wavelength of a standing wave can be defined by the distance between 3 nodes. We can, therefore, relate the wavelength to the number of harmonics and the length of the whole string:

$$\lambda_n = \frac{2L}{n}$$

Here is a diagram of the harmonics of a sine wave for clarification:



### Fourier Transformation:

- (a) A periodic function with period  $T$  and frequency  $\omega_0 = \frac{2\pi}{T}$  can be represented with the sum of infinite trigonometric functions:

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

- (b) Fourier decomposition can also be done over a finite range. For a range  $x = [0, L]$ , the

function  $y(x)$  can be defined as either:

$$y(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

or,

$$y(x) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)$$

(c) To solve for the constants in the Fourier Transform, the following formulae can be used:

$$a_0 = \frac{1}{L} \int_0^L y(x) dx, \quad a_n = \frac{2}{L} \int_0^L \cos\left(\frac{n\pi x}{L}\right) y(x) dx$$

$$b_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) y(x) dx$$

(d) I'm skipping the derivation of the Fourier Transforms... way too much unnecessary math that we don't even know the conclusions of... similarly I will not be showing the derivations of inverse fourier transforms. It is most important to understand that Fourier transforms are about changing functions from time-space into frequency space. In fourier space  $F(\omega)$ , there is a dirac delta at each frequency for which a sine/cosine wave makes up the original equation in time-space. The height of each dirac delta function is equal to the amplitude of each of these sine/cosine waves.

### Progressive Waves:

(a) We spoke about standing waves prior. Standing waves are waves with fixed end points and modes. Standing waves are mathematically equivalent to a superposition of two *travelling waves* (see later) moving in opposite directions with the same frequency and wavelengths. Standing waves come in harmonics because of the fact that the frequency of the travelling waves are matching. Here is the equation of a standing wave:

$$y(x, t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t) = \frac{A_n}{2} [\sin\left(\frac{n\pi x}{L} - \omega_n t\right) + \sin\left(\frac{n\pi x}{L} + \omega_n t\right)]$$

i. The key trigonometric function needed to understand this former relationship is:

$$\sin\theta \cos(\phi) = \frac{1}{2} (\sin(\theta + \phi) + \sin(\theta - \phi))$$

ii. The function can more easily be deciphered by rewriting it as:

$$y(x, t) = \frac{A_n}{2} [\sin\left(\frac{n\pi}{L}(x - vt)\right) + \sin\left(\frac{n\pi}{L}(x + vt)\right)]$$

iii. For these equations we have been assuming the values of  $\omega_n = \frac{n\pi}{L} \sqrt{\frac{T}{\mu}}$  and  $v = \sqrt{\frac{T}{\mu}}$ . As usual,  $\mu = \frac{m}{L}$ , the mass density of the string per length,  $T$  is the tension of the string, and  $v$  is the *velocity* of the string.

- (b) A progressive wave travelling in one direction is called a *transverse wave*. This kind of wave is no longer defined in the same physical sense; it can be thought of as a light beam instead. Given that  $\lambda$  is the wavelength of a wave,  $k$  is the wave number,  $\omega$  is the angular frequency of the wave, and  $T$  is the period of the wave, we can define that  $k = \frac{2\pi}{\lambda}$  and  $\omega = \frac{2\pi}{T}$ . We can then define the movement of a transverse wave as:

$$y(x, t) = A \sin(kx - \omega t)$$

It is key to notice that the velocity of such a wave is independent of its wavelength.

- i. A very important property of a transverse wave is its ability to define the motion of particles through a medium (such as light). If we take the second derivative of  $y(x, t)$  in terms of  $x$  as well as  $t$ , we get the relationship:

$$\frac{d^2y}{dx^2} = \frac{1}{v^2} \frac{d^2y}{dt^2}$$

This equation is known as the wave equation.

- (c) Given two transverse waves of different wavelength, and different wave numbers defined by:

$$k_{1,2} = \frac{2\pi}{\lambda_{1,2}}$$

The motion of each wave can be defined by:

$$y_{1,2} = A \sin(k_{1,2}(x - vt))$$

The superposition of these waves will remind us of the section on beat frequencies:

$$y = y_1 + y_2 = 2A[\cos(\frac{k_1 - k_2}{2}(x - vt)) \sin(\frac{k_1 + k_2}{2}(x - vt))]$$

From this equation we can derive the beat frequency as:

$$\omega_b = (k_1 - k_2)v$$

### The Ideal Gas:

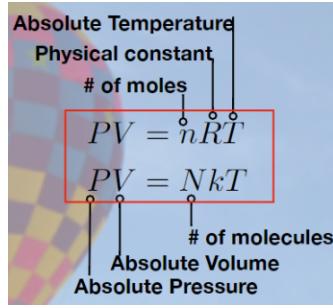
- (a) The ideal gas law is pivotal to understanding thermodynamics. The equation is dictated as either:

$$PV = nRT$$

or,

$$PV = NkT$$

- i.  $R = 8.31 \frac{J}{mol \cdot K}$  is the ideal gas constant,  $N_A = 6.02 * 10^{23} \frac{1}{mol}$  is Avogadro's number, and  $k = \frac{R}{N_A} = 1.381 * 10^{-23} \frac{J}{K}$  is Boltzmann's Constant. Refer to the following diagram for understanding the equation:



- (b) From the ideal gas law, one can derive the kinetic energy per molecule in a gas. From this we will be able to derive an equation for *temperature*. Imagine that we have a chamber with volume  $V$ , where the volume can be defined by  $V = LA$ .  $A$  is the area of one side of the chamber, this side is of the box is a piston. The length of the box is defined by  $L$  (the length from the side of the piston to the opposite side). We can thus derive the pressure created on this one wall:

$$\bar{P} = \frac{F_x}{A} = \frac{m(\bar{\Delta v}_x)}{A}$$

We can define the time it takes for the molecule to travel from one side of the chamber to the other as  $\Delta t = \frac{2L}{v_x}$ . The average change in velocity is  $\bar{\Delta v}_x = 2v_x$ . We can thus change our equation for pressure to:

$$\bar{P} = \frac{m(2v_x)}{A(\frac{2L}{v_x})} = \frac{mv_x^2}{V} \rightarrow \bar{P}V = mv_x^2$$

We know that we have  $N$  molecules, so that means we can set up an equation:

$$PV = NkT = Nmv_x^2 \rightarrow kt = mv_x^2 \rightarrow E_{k,x} = \frac{1}{2}mv_x^2 = \frac{1}{2}kT$$

- i. The energy of the system in 3 variables can be defined similarly as

$$E_k = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = 3(\frac{1}{2}kT)$$

. This would be the energy with 3 translational degrees of freedom, and only applies for a monatomic molecule. For a diatomic molecule we have two additional degrees of rotational degrees of freedom. Essentially, we have energy  $\frac{1}{2}kT$  for each degree of freedom. We can conclude that the energy of  $N$  molecules with  $f$  degrees of freedom is:

$$U_{\text{thermal}} = \frac{1}{2}NfkT$$

- ii. Additionally we can solve for the average speed of a molecule:

$$v_{rms} = \sqrt{v^2} = \sqrt{\frac{3kT}{m}}$$

## Lecture 17:

- (a) Prof says “TA taught this... SMH”

Heat and Work:

- (a) The change in energy in a system can be defined by  $\Delta U$ . This change in energy is the sum of the change in heat  $Q$  and the amount of work done by the system  $W$ . Their relation is defined by:

$$\Delta U = Q + W$$

- (b) Work is just  $W = \vec{F} \cdot \vec{dr}$ . For compression this value would be positive, and for expansion this value would be negative. The work done by isothermal compression can be defined, thus, by:

$$W = PA\Delta x = -P\Delta V$$

We can more define this equation with integrals:

$$W = - \int_{V_i}^{V_f} P(V)dV$$

- (c) The Heat Capacity  $C$  is the amount of heat  $Q$  that is needed to raise the temperature:

$$C = \frac{\Delta Q}{\Delta T} = \frac{\Delta U - W}{\Delta T}$$

This can be understood more physically by the heat capacity per unit of mass:

$$c = \frac{C}{m} = \frac{\Delta Q}{m\Delta T} = \frac{\Delta U - W}{m\Delta T}$$

- i. The heat capacity at constant volume can be defined by

$$C_V = (\frac{\Delta Q}{\Delta T})_V = (\frac{dU}{dT})_V = \frac{d}{dT}(\frac{NfkT}{2}) = f\frac{Nk}{2}$$

- ii. The heat capacity at constant pressure can be defined by:

$$C_P = (\frac{\Delta Q}{\Delta T})_P = (\frac{\Delta U - (-P\Delta V)}{\Delta T})_P = (\frac{dU}{dT})_P + P(\frac{dV}{dT})_P$$

We can define that:

$$(\frac{dV}{dT})_P = \frac{d}{dT}(\frac{NkT}{P}) = \frac{Nk}{P}$$

For an ideal gas, the derivative  $\frac{dU}{dT}$  is the same with  $P$  fixed as with  $V$  fixed, so we can derive that:

$$C_P = (\frac{dU}{dT})_P + P(\frac{dV}{dT})_P = (\frac{dU}{dT})_V + P(\frac{dV}{dT})_P = C_V + Nk$$

- (d) During phase transitions, heat can be added without changing the temperature of the material at all. During such transaction, the heat capacity is infinite, but this can't be right, we need a different equation:

$$L = \frac{\Delta Q}{m}$$

- (e) Enthalpy is the total energy which can be extracted from a system, it is defined by:

$$H = U + PV$$

This makes sense because  $U$  is the internal energy of the system which is the heat and work done by a system, but we also need the energy that it takes to get a system to its location which is defined by  $PV$ .

- i. We can define the change in enthalpy by the amount of heat that is introduced or taken away from a system as well as any other work that is done on the system such as a phase transition:

$$\Delta H = \Delta Q + W_{other}$$

- (f) Note the difference between thermal and kinetic energy:

$$U_{Thermal} = \frac{f}{2}kT , U_{KE} = \frac{1}{2}mv^2$$

- (g) Thermal energy is defined by its nature to not disappear at random. It is more probable for a hot object to stay hot unless it interacts with a colder object which would probabilistically cause the object to become cold. It is not very probable a hot object to become cold again due to its random stochastic nature.
- (h) Heat is proportional to the amount of temperature that crosses a distance in a certain time:

$$Q \sim \frac{A\Delta T \Delta t}{\Delta x}$$

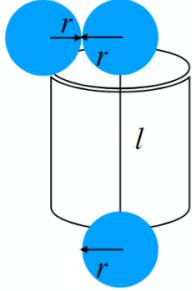
We can therefore define, with the thermal conductivity constant, which depends on the material,  $k_t$ , the change in heat over time as:

$$\frac{Q}{\Delta t} = -k_t A \frac{dT}{dx}$$

Gas Conductivity - Mean Free Path:

- (a) Imagine a gas inside a box of volume  $V$  with  $N$  molecules. The number density of the molecules is  $n = \frac{N}{V}$ . Imagine that the box has some pressure  $P$  and temperature  $T$ . If we heat up one side of the box, we can calculate the expected time for the other side of the box to heat up. Imagine we have some distance  $l$  which is the *mean free path* (the average distance) between molecules. Since the distance between two molecules (if spheres) when they hit each other is  $2r$ , we can relate this to the average volume a molecule takes up:

$$\pi(2r)^2 l \approx \frac{V}{N} \rightarrow l \approx \frac{1}{4\pi r^2 n}$$



(b) We can derive the mean free time between molecule collisions. We can relate that:

$$\frac{f}{2}kT = \frac{1}{2}mv^2 \rightarrow v = \sqrt{\frac{fkT}{m}}$$

We have the mean free distance, so we use that:

$$t = \frac{l}{v} = \frac{\frac{1}{4\pi r^2 n}}{\sqrt{\frac{fkT}{m}}} = \frac{\sqrt{m}}{\sqrt{fkT} \cdot 4\pi r^2 n}$$

### The Second Law of Thermodynamics:

(a) We must understand probability to understand temperature. We are essentially look at molecules that have a probability to hit each other. First we must understand the definition of combinations:

$$\Omega(N, n) = \frac{N!}{n!(N-n)!} = \binom{N}{n}$$

The binomial coefficient is defined by:

$$(a+b)^N = 1 \rightarrow (a+b)^N = \sum_{n=0}^N \binom{N}{n} a^n b^{N-n} = 1$$

(b) An *Einstein Solids* is a quantum mechanical model for the distribution of energy among  $N$  quantum mechanical harmonic oscillators. Each oscillator can carry a multiple  $n$  of a quantum of energy  $E_0 = hf$  where  $h = 6.626 * 10^{-34} \frac{m^2 kg}{s}$  is Planck's constant, and  $f$  is the frequency of the particle. Say that there are  $q$  quanta of energy, each with this amount of energy  $E_0$ . We can say that  $q = \sum_{i=0}^N n_i$  where  $n_i$  is the number of quanta per harmonic oscillator. Treat a quanta like a particle with energy. Each group of quanta is divided by a wall, this means we would have  $N - 1$  walls. We can thus calculate the number of possible systems (the multiplicity):

$$\Omega(N, q) = \binom{q+N-1}{q} = \frac{(q+N-1)!}{q!(N-1)!}$$

- i. If we have too many particles, the former expression cannot be solved. We use *Stirling's*

*Approximation* to estimate factorials:

$$\ln(N!) \approx N \ln N - N$$

ii. We can use this equation to solve that (for  $q \gg N$ ):

$$\Omega(N, q) \approx \left(\frac{eq}{N}\right)^N$$

- (c) If we bring two systems together, we get that  $N_{Total} = N_T = N_A + N_B$  and  $q_{Total} = q_T = q_A + q_B$ . We say that we have an equal number of oscillators to simplify the calculation ( $N_A = N_B$ ). We can also state that the number of combinations for the particles is equal to:

$$\Omega_{Total} = \Omega_T = \Omega_A \Omega_B = \left(\frac{eq_A}{N}\right)^N \left(\frac{eq_B}{N}\right)^N$$

It's easy to see from here that if  $q_A = q_B$ ,  $\Omega_T$  will have the largest possible value. This is significant because this is the same as saying that the largest subset of possibilities is when all of the energized particles in the system have the same amount of energy.

- i. Say that we have  $q_A = \frac{q_T}{2} + x$  and  $q_B = \frac{q_T}{2} - x$ . From this, our function for  $\Omega_T$  will simplify to:

$$\Omega_T = \Omega_{Max} e^{-N \frac{2x^2}{q}}$$

What we see from this equation is that,  $\Omega_T$  will decrease to  $\frac{\Omega_T}{e}$  when  $x = \pm \frac{q_T}{2\sqrt{N}}$ . This number is actually relatively large, but because  $0 < x < q$ , this is a very small fraction if  $N$  is large at all. This makes the fractional width:

$$frac. width = \frac{1}{\sqrt{N}}$$

Entropy:

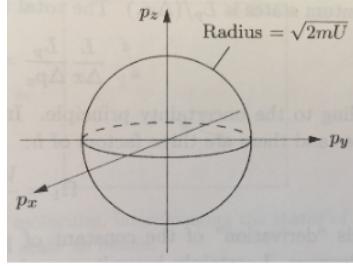
- (a) Consider a monatomic atom in a volume  $V$  with energy  $U$ . The amount of states this atom can have is proportional to the size, in other words, if we double the size of the volume the atom is in, there are double the number of location states. We can also imagine that this atom has a momentum, and in 3-dimensional space, there are three directions to have momentum; it is defined that there is also a *momentum volume*,  $V_p$  that is associated to the number of different momentum vectors. We say that the multiplicity of this atom's possible states is:

$$\Omega_1 \sim V \cdot V_p$$

The momentum vectors are called  $p_x$ ,  $p_y$ , and  $p_z$  respectively for the x, y, and z directions. The total energy  $U$  of the atom is defined as:

$$U = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

From this we can see the equation of a sphere emerge; refer to the following figure:



I'm not really sure about the following proof, so just follow along. Supposedly we need to reach into quantum mechanics to understand it:

$$\Delta x \Delta p \approx h \rightarrow \frac{L}{\Delta x} \frac{L_p}{\Delta p} = \frac{LL_p}{h} \rightarrow \Omega_1 = \frac{VV_p}{h^3}$$

Expanding this from 1 atom to N atoms:

$$\Omega_N = \frac{1}{N!} \frac{V^N}{h^{3N}} * (\text{Area of } 3N \text{ dimensional } p \text{ hypersphere})$$

The area of such a hypersphere is:

$$\text{Area} = \frac{2\pi^{\frac{d}{2}}}{(\frac{d}{2}-1)!} r^{d-1}$$

$r$  is the radius that we can see in the figure above  $r = \sqrt{2mU}$ , and  $d$  is the number of dimensions (fact-check)  $d = 3N$ . Thus, we can solve for our multiplicity for  $N$  atoms:

$$\Omega_N = \frac{1}{N!} \frac{V^N}{h^{3N}} \cdot \frac{\pi^{\frac{3N}{2}}}{(\frac{3N}{2})!} (\sqrt{2mU})^{3N} = \left( \frac{1}{N!} \frac{1}{h^{3N}} \cdot \frac{\pi^{\frac{3N}{2}}}{(\frac{3N}{2})!} (\sqrt{2m})^{3N} \right) V^N U^{\frac{3N}{2}}$$

We can see from this equation that the multiplicity is reduced by  $2^N!$  when we place all of the gas into half of the volume. Similarly, the gas wants to expand, if the volume were doubled, the multiplicity would increase by  $2^N!$ .

- (b) We define the entropy as:

$$S = k \ln \Omega$$

Multiplicities are multiplicative:  $\Omega_T = \Omega_A \Omega_B$ , so entropies are additive:  $S_T = S_A + S_B$ . If we take our previous result for the multiplicity of  $N$  atoms,  $\Omega_N$ , we end up with a result of:

$$S = Nk \left[ \ln \left( \frac{V}{N} \left( \frac{4\pi m U}{3N h^2} \right)^{\frac{3}{2}} \right) + \frac{5}{2} \right]$$

If we keep the dimensions and energy constant ( $N$  and  $U$ ), and we change the volume from  $V_i$  to  $V_f$ , then a change in entropy can be defined by:

$$\Delta S = S_f - S_i = Nk \ln \frac{V_f}{V_i}$$

We said that a volume tends to increase in volume which means that  $V_f$  will almost always want to be greater than  $V_i$ ; the change in entropy will always tend towards being positive;

a decrease in entropy is probabilistically unlikely.

- (c) Imagine a volume which is suddenly increased, the gase occupies a volume  $V$  and then occupies a volume  $V + \Delta V$ . The volume is thermally insulated. We can see that  $\Delta Q = 0$  and that  $W = \int PdV = 0$  which means that  $\Delta U = \Delta Q + W = 0$ . Imagine a similar scenario, but we are relating the change in entropy; imagine a volume that is divided in two with an equal amount of energetic particles ( $A$  and  $B$  particle) on each side:

$$\Delta S_A = Nk \ln \frac{V_f}{V_i} = Nk \ln 2 = \Delta S_B \rightarrow \Delta S_T = \Delta S_A + \Delta S_B = 2Nk \ln 2$$

#### Temperature:

- (a) The 2nd Law of Thermodynamics states that when two objects are in *thermal equilibrium*, then their total entropy has reached a maximum. More regularly, we say that thermal equilibrium is achieved when the temperature of two objects become equal. It is implied that entropy and temperature are related.
- (b) We say that the thermal energy of an energized particle is proportional to the number of energized particles:

$$U_A = q_A \epsilon \quad U_B = q_B \epsilon$$

Remember that as multiplicity increases, entropy increases. This means that if we increase the number of energized particle  $q_A$ , then  $S_A$  would increase, but  $S_B$  would decrease if  $q_T = q_A + q_B$ . We have derived that the total entropy of the two systems together is maximized when  $q_A = q_B$ . This means that:

$$\frac{dS_T}{dU_A} \sim \frac{dS_T}{dq_A} = 0 = \frac{dS_T}{dq_B} \sim \frac{dS_T}{dU_B}$$

We can thus prove that:

$$\frac{dS_A}{dU_A} + \frac{dS_B}{dU_A} = 0 \quad \& \quad dU_A = -dU_B \rightarrow \frac{dS_A}{dU_A} = \frac{dS_B}{dU_B}$$

If we look at the units of what we have just derived we can notice that we have  $K^{-1}$ . We can define temperature in the following way:

$$T = \left( \frac{dU}{dS} \right)_{N,V}$$

#### More Random Thermo Stuff:

- (a) The heat capacity at constant volume of a monotonic gas can be derived by:

$$C_V = \left( \frac{dU}{dT} \right)_{N,V} = \left( \frac{d(\frac{3}{2}NkT)}{dT} \right)_{N,V} = \frac{3}{2}Nk$$

(b) The partial derivative of Entropy is defined per the definition of Temperature:

$$T = \frac{\partial U}{\partial S} \rightarrow dU = dU = TdS + SdT \rightarrow dS = \frac{dU}{T} = \frac{Q}{T}$$

(c) We can use the previous definition to see how Entropy changes:

$$dS = \frac{C_V dT}{T} \rightarrow \int_{S_i}^{S_f} dS = \int_{T_i}^{T_f} \frac{C_V dT}{T}$$

$$\Delta S = S_f - S_i = \int_{T_i}^{T_f} \frac{C_V dT}{T}$$

We can recall that  $S = k \ln \Omega$  and solve:

$$\Delta S = k \ln \left( \frac{\Omega_f}{\Omega_i} \right) = \int_{T_i}^{T_f} \frac{C_V dT}{T}$$

(d) Lets see what happens when temperature reaches absolute zero,  $T = 0$ :

$$\Delta S = k \ln \left( \frac{\Omega_f}{1} \right) = \int_0^{T_f} \frac{C_V dT}{T}$$

It seems that the rightside of the equation breaks because we have  $\Delta S = C_V \cdot \ln(\frac{T_f}{0})$ . I'm not sure what to say about this :(

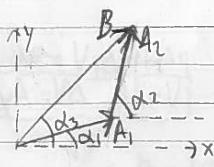
(e) I'm tired... here are the last two equations I could find:

$$P = T \left( \frac{\partial S_B}{\partial V_B} \right)_{U,N}$$

and,

$$dU = TdS - PdV$$

Midterm 1 Memorization:



$$B^2 = (A e^{i\alpha_1} e^{i\alpha_2})(A e^{-i\alpha_1} + A e^{-i\alpha_2}) \\ = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\alpha_1 - \alpha_2)$$

$$\alpha_3 = \tan^{-1} \left( \frac{A_1 \sin(\alpha_1) + A_2 \sin(\alpha_2)}{A_1 \cos(\alpha_1) + A_2 \cos(\alpha_2)} \right)$$

$$X_1 = A e^{i\omega_1 t}, X_2 = A e^{i\omega_2 t} \rightarrow X = X_1 + X_2 = 2A \cos\left(\frac{\omega_2 - \omega_1}{2}\right) e^{i\frac{\omega_1 + \omega_2}{2}t}$$

$$\text{Beat Frequency} = \left| \frac{\omega_2 - \omega_1}{2\pi} \right|, \text{ Beat angular frequency} = |\omega_2 - \omega_1|$$

Young's Modulus & Elasticity:

$$\text{Stress} = \frac{F}{A}, \text{ Strain} = \frac{\Delta L}{L} = \frac{x}{L} \Rightarrow mx'' = -\frac{YA}{L}x$$

Buoyancy:  $my'' = -g \rho_A V_{\text{mass}}$   $\Rightarrow y'' = -\frac{2\rho_A}{m} y$

Pendulum:

$$x = L \sin \theta \approx L \theta \quad (\text{small angle approximation})$$

$$F_L = mg \sin(-\theta) \approx -mg\theta \quad (\text{perp. gravity force})$$

$$x' = L\theta' \rightarrow x'' = L\theta''$$

$$\Rightarrow mx'' = F_L \Rightarrow \theta'' = -\frac{F}{m} \theta$$

V-Tube:  $my'' = F \Rightarrow \rho A y'' = -g 2 A p$

$$\Rightarrow y'' = -\frac{2g}{\rho} y$$

Torsional Pendulum:  $\theta'' = -\frac{k}{I} \theta$

Air Spring: Isothermal:  $pV = \text{const.} \Rightarrow p\Delta V + \Delta pV = 0$   
 $\Leftrightarrow \dots \quad y'' = -\frac{\Delta p}{m}y \quad \leftarrow \boxed{V = Al, \Delta V = Ay}$

Adiabatic:  $pV^\gamma = \text{const.} \Rightarrow \Delta pV^\gamma + \gamma p\Delta V = 0$   
 $\boxed{y'' = -\frac{\Delta p}{m}y}$

Damped Oscillator:  $x(t) = Ae^{\frac{-b}{2m}t} \cos(\omega t + \phi)$ ,  $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

Overdamped:  $\frac{k}{m} < \frac{b^2}{4m^2}$   
 Critically Damped:  $\frac{k}{m} = \frac{b^2}{4m^2}$   
 Underdamped:  $\frac{k}{m} > \frac{b^2}{4m^2}$

$\frac{Y}{2} = \frac{b}{2m} = \frac{1}{T}$ ,  $T = \frac{2\pi}{\omega}$ , the time to decrease to  $\frac{1}{e}$  of amplitude

$N = \frac{Q}{T} = \# \text{ of oscillations for } \frac{1}{e} \text{ amplitude drop}$

$(Q = \frac{\omega_0}{\zeta})$ , large-Q represents less damping per oscillation.

Forced Vibrations:  $x(t) = C \cos(\omega t)$  for  $F_0 \cos(\omega t)$   
 Resonance frequency:  $\omega_r = \pm \sqrt{\frac{k}{m}}$   $C = \frac{(F_0/m)}{\omega_r^2 - \omega^2}$

With Damping:  $z(t) = Ae^{i(\omega t - \beta)}$

$$A = \frac{(F_0/m)}{\sqrt{(\omega_r^2 - \omega^2)^2 + (\gamma\omega)^2}} = K\omega \sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{4\gamma^2}{\omega^2}}$$

$$\beta = \tan^{-1} \left( \frac{\omega\gamma}{\omega_0^2 - \omega^2} \right) = \frac{1}{\alpha} \left( \frac{1}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}} \right)$$

Fractional Width =  $\delta\omega = \frac{\omega_0}{2Q}$

Crib Sheet Midterm 2 PHYS 232

$$m\ddot{x}_1 = -mg \frac{x_1}{L} - K(x_1 - x_2), \quad m\ddot{x}_2 = -mg \frac{x_2}{L} - K(x_2 - x_1)$$

$$\omega_0 = \sqrt{\frac{K}{m}}, \quad \omega_k = \sqrt{\frac{K}{m}}$$

$$\Rightarrow \ddot{x}_1 + (\omega_0^2 + \omega_k^2)x_1 - \omega_k^2 x_2 = 0 \Rightarrow \omega_1^2 = \omega_0^2 + \omega_k^2$$

$$\ddot{x}_2 + (\omega_0^2 + \omega_k^2)x_2 - \omega_k^2 x_1 = 0$$

Assume  $x_1 = z_1 e^{i\omega t}, x_2 = z_2 e^{i\omega t} \Rightarrow \ddot{x} = -\omega^2 \ddot{x}$

$$\& \begin{pmatrix} \omega_1^2 - \omega_k^2 \\ -\omega_k^2 \end{pmatrix} \ddot{x} + (-\omega^2) \ddot{x} = 0 \Rightarrow (A - \lambda I) \ddot{x} = 0$$

where  $A = \begin{pmatrix} \omega_1^2 - \omega_k^2 \\ -\omega_k^2 & \omega_1^2 \end{pmatrix} \& \lambda = \omega^2 \Rightarrow \text{Solve } \det |A - \lambda I|$

$q_1 = C \cos(\omega_0 t), q_2 = D \cos(\omega_k t)$ ,

$$x_1 = \frac{1}{2}(C \cos(\omega_0 t) + D \cos(\omega_k t))$$

$$x_2 = \frac{1}{2}(C \cos(\omega_0 t) - D \cos(\omega_k t))$$

$$\begin{aligned} \cos(\alpha t) + \cos(\beta t) &= \\ 2 \cos\left(\frac{\alpha-\beta}{2}t\right) \cos\left(\frac{\alpha+\beta}{2}t\right) & \end{aligned}$$

$$\cos(\alpha t) - \cos(\beta t) =$$

$$\begin{aligned} \cos(\alpha t) - \cos(\beta t) &= \\ 2 \sin\left(\frac{\alpha-\beta}{2}t\right) \sin\left(\frac{\alpha+\beta}{2}t\right) & \end{aligned}$$

To find ratio of the amplitudes of oscillations / find eigenvectors:

Take:  $(A - \lambda I) \ddot{x} = 0, \text{ we will have solved for } \omega^2 = \omega_1^2, \omega^2 = \omega_k^2$

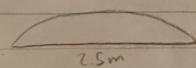
where  $\lambda_1 = \omega_1^2, \lambda_2 = \omega_k^2 \Rightarrow (A - \omega_1^2 I) \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0 \& (A - \omega_k^2 I) \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 0$

$\lambda_1$  defines one mode,  $\lambda_2$  defines the other mode

The ratio of the first mode would be  $\frac{x_{11}}{x_{12}}$  or  $\frac{x_{21}}{x_{11}}$

& the ratio of the other mode would be  $\frac{x_{11}}{x_{22}}$  or  $\frac{x_{21}}{x_{22}}$

String pluck:



$5m = \text{wavelength}$

$$\lambda = 2L, \Rightarrow f = \frac{V}{\lambda} = \frac{V}{2L}$$

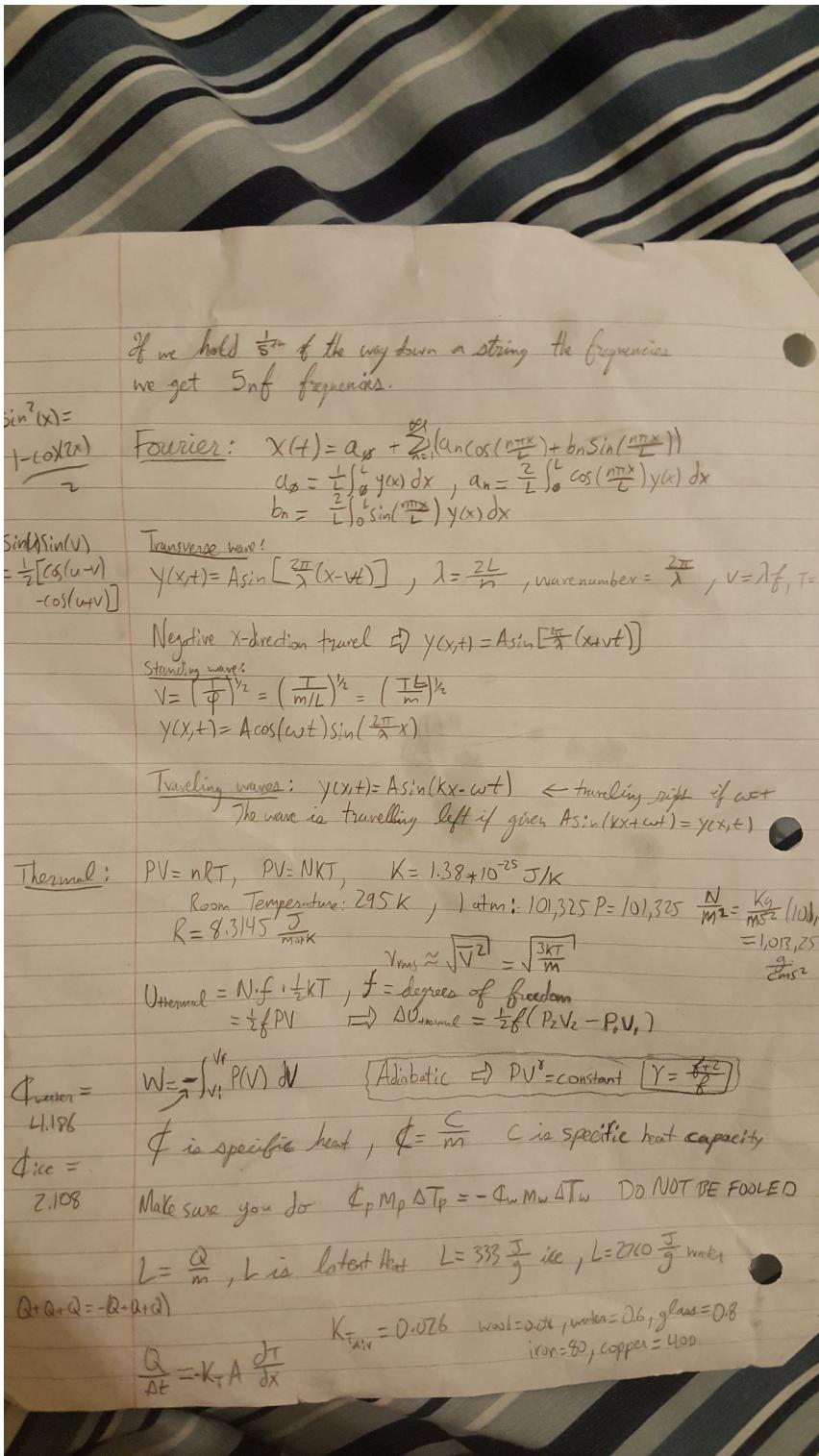
$$\mu = \text{mass density} = \frac{M}{L} \& V = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{m/L}} = \sqrt{\frac{TL}{m}}$$

$$\Rightarrow f = \frac{1}{2} \sqrt{\frac{T}{m}}$$

Fundamental Mode implies  $n=1 \Rightarrow f = \frac{1}{2} \sqrt{\frac{T}{m}}$

All frequencies (any mode) could be  $n \frac{1}{2} \sqrt{\frac{T}{m}}$

When playing a note we filter frequencies out



Good Luck on the Final!