# PHYS 257

Errors and Uncertainties in Physical Measurements:

(a) It is essential that Physicists state the reliability of measured quantities found through experimentation.

(b) The following cannot be done without estimating experimental uncertainty:

  i. Make (dis)agreements between theoretical predictions and experimental measurements.

  ii. Make (dis)agreements about the reproducibility of measurements in experiments.

  iii. Conclude about new scientific discoveries.

(c) The **sample mean** is the best estimate, found with a sample of data, to the true mean of data. It is defined by:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

The **sample variance** is $\sigma^2$, where $\sigma$ is the **sample standard deviation**; it can be defined as the width of a data distribution and as:

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

The **standard error in the mean** is the uncertainty (see Precision below) in $\bar{x}$. The more measurements that are taken in an experiment, the smaller the standard error, and the smaller the true "width" of the data. It is defined by:

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

(d) Error is the difference between the measurement of a quantity and the true value of said quantity.

(e) The sample mean is calculated from a set of data. Since it is not independently determined from the sample data, it is a "biased value". When calculating the standard deviation of a set of $N$ data points, since the sample mean is used to calculate it, one **degree of freedom** is taken away from the calculation, meaning that $N \to N-1$ (take a look at the standard deviation). If the degree of freedom is not accounted for, the standard deviation will be systematically underestimated.

  i. Precision defines the uncertainty of data. An experiment with high precision is reproducible, but is not necessarily accurate. The quantification of uncertainty (precision) is referred to as random error; experiments always have random fluctuations in values (random error).

  ii. Accuracy is the difference between the sample mean of a data set and the "true" value of the value represented by the data. The difference between these value is referred to as the bias or system error. It is much more difficult to have accurate data because systematic error will happen with every experiment.

(f) A **parent distribution function** is a function defining the expected distribution of cetain measurements. The measurements referred to so far (mean, etc...) are normally related to the Gaussian/Normal parent function. The only other function referred to in this class is the Chi-Square function.

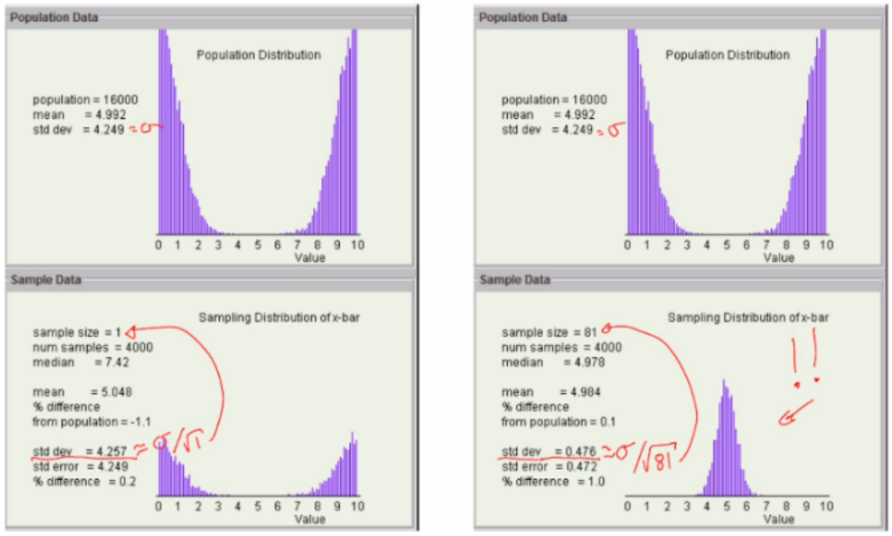
Significant figures and the five golden rules

(a) The **five golden rules** for reporting parameters are as follows:
   i. The best estimate of a parameter is the mean.
   ii. The error is the standard error in ghe mean.
   iii. Round up the error to the appropriate number of **significant figures**.
   iv. Match the number of decimal places in the mean to the standard error.
   v. Include units.

(b) All experimentally derived values have uncertainty (standard error). Any data with standard error has strict formatting:
   i. After calculating the standard error, round it off to one significant figure, e.g. $.028 \rightarrow 0.03$
   ii. Measurements should be expressed to the same to the same precision that the uncertainty is, e.g. $3.0891 \pm 0.03 \rightarrow 3.09 \pm 0.03$
   iii. It's important to note that when calculating values, the rounding of values should only occur at the end of the calculation.
   iv. Uncertainties are often estimated, given special circumstances (check error propagation).

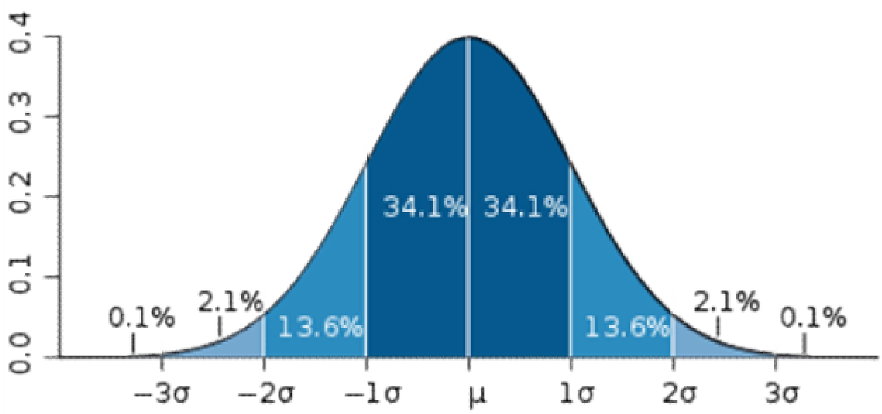
Gaussian (Normal) Distribution

(a) A **Gaussian distribution** describes a distribution around a mean value. No matter the distribution, all distributions can be "turned into" a Gaussian distribution. Take a distribution of thousands of samples. If these samples are reduced to subsamples, where each subsample refers to the mean of a subgroup of the original data, then the the new distribution of subsamples will look more similar to a Gaussian distribution.




(b) The Gaussian distribution can be defined as:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$




i. As can be seen from the diagram, 68.2% of data should fall between within 1 standard deviation, $1\sigma$, of the mean $\mu$, 95.4% for within $2\sigma$, etc.

ii. If the result of an experiment is $T = 6.2 \pm 0.2$ seconds, then a scientist would say: The experiment determined a time of 6.2 seconds. This result is considered accurate within 0.2 seconds, 68 out of 100 times.

Error Propagation

(a) With every value discovered, a good estimate of uncertainty should also be kept track of.

(b) The rule of thumb for an **analog device** (devices with an infinite set of possible values), is to round to half a division, where a division is the smallest labeled value, e.g. $3.12m \rightarrow 3.12m \pm 0.05m$. This is because a value between 3.10 and 3.20 meters was guessed by the experimenter.

(c) The rule of thumb for a **digital device** (devices with a finite set of possible values), is to round to a division, where a division is the smallest labeled value, e.g. $2.438s \rightarrow 2.438m \pm 0.001s$. This is because the value of 2.438s is shown by the digital device.

(d) Given a measured independent variable $x$, the uncertainty of a measurement function, $f(x)$ is:

$$\sigma_f = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(V(x_i) - V(\bar{x}))^2} \approx \frac{f(\bar{x}+\sigma_{\bar{x}}) - f(\bar{x}-\sigma_{\bar{x}})}{2} \approx \frac{\partial V}{\partial x}\Big|_{x=\bar{x}} * \sigma_{\bar{x}}^2$$

(e) If $f$ relies on both $x$ and $y$, then given $\Delta x, \Delta y$, and $f(x,y)$, then:

$$\Delta f^2 \approx \Delta x^2 (\frac{\partial V}{\partial x})^2_{\bar{x},\bar{y}} + \Delta y^2 (\frac{\partial V}{\partial y})^2_{\bar{x},\bar{y}}$$

- Example: $\bar{x}$=0.5±0.1, $\bar{y}$=0.9±0.3.

  What is the error on z = $x^2+y^2$ ? Use differential approach

$$\frac{\partial V}{\partial x} = 2x \ ; \ \frac{\partial V}{\partial y} = 2y$$

$$\sigma_z^2 = (\underset{\sigma_{\bar{x}}}{0.1})^2 (2x|_{x=0.5})^2 + (\underset{\sigma_{\bar{y}}}{0.3})^2 (2y|_{y=0.9})^2 = 0.3016$$

$$\bar{z} = \bar{x}^2 + \bar{y}^2 = 1.06 \ ; \ \sigma_z = \sqrt{\sigma_z^2} = 0.549$$

so $\boxed{\bar{z} = 1.1 \pm 0.5}$

Best-fit Lines

(a) Given that every data point on a plot has **error bars**, it is expected that a good linear fit will pass through 68% of them. This is because an error bar represents one standard deviation, and 68% should lie within one standard deviation of the mean, where the best fit line is the "mean-line".

i. If the best-fit line runs through > 68% error bars, and no uncertainties were guessed, then it is likely that this is just a probabilistic anomaly.

ii. If the best-fit line runs through > 68% error bars, and many uncertainties were guessed, then the guessed uncertainties are probably too large and should be lowered to better reflect the data.

iii. If the best-fit line runs through < 68% error bars, and no uncertainties were guessed, then its likely that there is unidentified constant human error. Guessing larger uncertainties will better reflect the data.

iv. If the best-fit line runs through < 68% error bars, and many uncertainties were guessed, then the guessed uncertainties are probably too small and should be increased to better reflect the data.

(b) For every data point, the difference between the data point and the value of the best-fit line at the same x is called the residual:

$$residual = \Delta y = y_i - y(x_i) = y_i - ax_i - b$$

All of the residual values for a set of data can get its own plot. This plot can reveal a lot of information:

i. Sometimes errorbars are quite small, and it is difficult to see whether the best-fit line crosses them. The residual plot (with a horizontal line at the x-axis) can provide a better view of this.

ii. The scattering of the residual points can provide good information about whether the function used for the best-fit line provides a good fit. If the data points have a quadratic relationship, but a linear best-fit line is used, then the center residuals would have large negative values, and the rest of the residuals would have large positive values. This distribution would paint a clear picture of what is really going on in the data.

(c) For producing the slope $a$ and vertical shift $b$ for a linear best-fit line:

4 different sums need to be evaluated to determine the best fit parameters

$$a = \frac{N(\sum x_i y_i) - (\sum x_i)(\sum y_i)}{\Delta}$$

$$b = \frac{(\sum x_i^2)(\sum y_i) - (\sum x_i)(\sum x_i y_i)}{\Delta}$$

when all data points have the same uncertainty

where $\Delta = N(\sum x_i^2) - (\sum x_i)^2$ and

Note that these expressions do not depend on the experimentally estimated uncertainty!

Errors:

$$\sigma_a^2 = \frac{N\sigma_y^2}{\Delta}$$

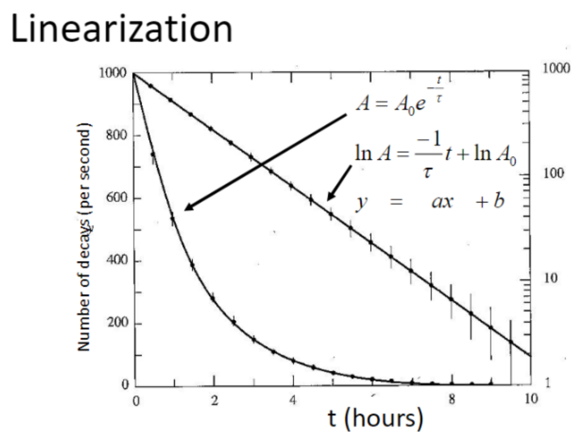
$$\sigma_b^2 = \frac{\sigma_y^2 \sum x_i^2}{\Delta}$$

where $\sigma_y^2 = \frac{1}{N-2}\sum_{i=1}^{N}(y_i - ax_i - b)^2$ ≡ Variance of the fit.

This is an estimate of the variance of the data, and should be approx. equal to the exp. estimated data variance

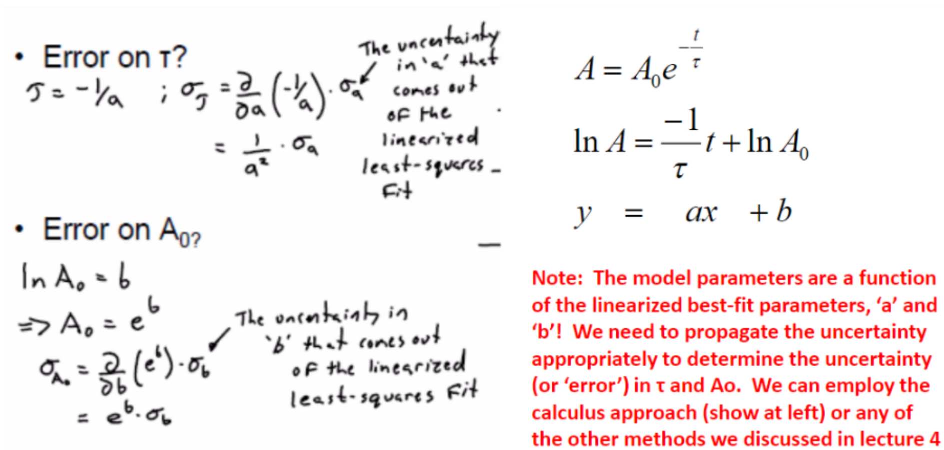
Linearization

(a) Functions can become linear and given linear fits by linearizing them. For example, an exponential function be represented linearly by having the logarithm of the function taken.




Data with an exponential relationship between 'A' and 't' plotted on a linear scale (left-hand y-axis) and a logarithmic scale (right-hand y-axis). ***Note that the graph is a straight line on the logarithmic scale; a "linearized" or semi-log plot/fit. Note also that <u>if the error bars in 'A' are constant in t, then the error bars in 'lnA' are NOT constant in t</u>***

(b) Since the parameters from the non-linearized function might have uncertainties, the parameters of the linearized function may need to be calculated:




<u>Goodness of a Fit</u>

(a) After creating a function of best-fit, the goodness of fit can be determined quite easily. $\chi^2$ is the value that represents how good the fit is. It is defined as:

$$\chi^2 = \sum_{i=1}^{N}[\frac{y_i - f(x_i;a_i)}{\sigma_i}]^2$$

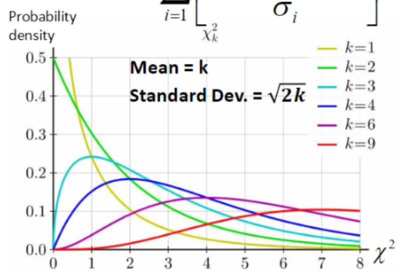
(b) The value of $\chi^2$ varies based off of the amount of the amount of degrees of freedom in the evalution of it. By default, the mean and standard deviation are used to calculate it, so its value should be $N-2$, where $N$ represents the number of data points. In fact, $\chi^2$ maps out a function that has different possible shapes based off of the number of degrees of freedom.

## Chi-Square Distribution

$$\chi^2 = \sum_{i=1}^{N} \left[ \frac{y_i - f(x_i; a_i)}{\sigma_i} \right]^2$$

**See Ch. 8 of your text**

- The **chi-squared distribution** (also **chi-square** or **$\chi^2$ - distribution**) with *k* degrees of freedom is the probability distribution that describes the sum of squares of *k* independent standard gaussian random variables.




The *standard Gaussian* distribution *has* μ=0 and σ=1. Note again that we expect the terms in the sum, $\lfloor y_i - f(x_i; a_i)/\sigma_i \rfloor$, to be Gaussian distributed with μ=0 and σ=1 if the fit is good.
**The number of "degrees of freedom", k, is the number of data points minus the number of parameters in your model**

(c) To find how likely a best-fit line is, the $\chi^2$ distribution table is used. To use the table, find the closest value to $\chi^2$ (that you would have calculated) that lines up with the number of degrees of freedom ($N-2$). The probability of this value (how probable your best-fit is to actually agree with your data-set) can be found at the top of column. If the percentage were, say, 70%, then this means that 70% of experiments would yield higher $\chi^2$ values, and 30% would yield smaller $\chi^2$ values.

(d) To determine how good a fit is, it is easiest to look at how close $\chi^2$ is to the mean ($\mu = k$), and by how many standard deviations ($\sigma = \sqrt{2k}$):

i. The model is acceptable if: $\chi^2$ is within $\pm 2\sigma$ of $k$.

ii. The model is questionable if: $\chi^2$ is within $\pm 3\sigma$ of $k$.

iii. The model is unacceptable if: $\chi^2$ is within $\pm 4\sigma$ of $k$.

(e) The value of $\chi^2$ is determined by four mathematical factors:

i. The gaussian distributed measurement values $y_i$. These values are set in stone after they are measured, so if poor steps were taken to keep measurements consistent, then a bad $\chi^2$ value will result.

ii. The values given to the uncertainty $\sigma$ of each measurement value $y_i$. If the uncertainties are poorly chosen, the a bad $\chi^2$ value will result. The estimate of these values can be changed after being designated as an implication that the original estimate was poorly chosen.

iii. The best-fit function $y(x)$. The residual function should imply a poorly chosen function, but if not noticed, the function will heavily affect the $\chi^2$ value.

iv. The parameters of $y(x)$. These values shouldn't be incorrect, but if the calculations to find the parameters are incorrect, then a bad $\chi^2$ value will result.

(f) The actual standard deviation of a plot about a function is:

$$\sqrt{\frac{1}{N-2}\sum_{i=1}^{N}(y_i - y(x_i))^2} = \sqrt{\frac{1}{N-2}\sum_{i=1}^{N}(y_i - ax_i - b)^2 (For.A.Linear.Function)}$$

The guessed uncertainty should be the same or very close to this value.

Communicating Science in Writing

(a) Experimental data is only useful if it is communicated effictively. Information must be communicated as broadly and accessibly as possible. Information may be communicated through oral presentations as well as written reports. Communicating a message is about capturing an audience's attention and

(b)

Good Luck :)