Fall 2017 Math 395 Written Homework 4 Key 100 total. -5 for no stapling

3.9 Compute x^TAx, where $x = (x_1, x_2, x_3)$, and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ (A is symmetric.)

$$x^TAx = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{12}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3.$$

3.10 Prove that if B is invertible, then B^TB is positive definite.

Proof. First $(B^TB)^T = B^TB$ so B^TB is symmetric.

Second we will use criteria II. For any vector x, $x^TB^TBx = (Bx)^TBx = \|Bx\|_2^2 \geq 0$. When equality holds, $Bx = 0$ which implies $x = 0$ because B is invertible. □

3.11 Check that $C = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ is positive definite using both criteria I and III.

Criteria I: $\det \begin{bmatrix} 2-r & 0 & -1 \\ 0 & 2-r & -1 \\ -1 & -1 & 3-r \end{bmatrix} = (2-r)\begin{vmatrix} 2-r & -1 \\ -1 & 3-r \end{vmatrix} - \begin{vmatrix} 0 & 2-r \\ -1 & -1 \end{vmatrix} = (2-r)[r^2 - 5r + 6 - 1] - (2-r) = (2-r)(r^2 - 5r + 5 - 1) = (2-r)(r^2 - 5r + 4) = (2-r)(r-4)(r-1)$

Eigenvalues are 2,4,1, all positive.

Criteria III: $2 > 0$, $\det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4 > 0$, $\det C = 2\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} - \begin{vmatrix} 0 & 2 \\ -1 & -1 \end{vmatrix} = 2(6-1) - 2 = 8 > 0$

3.12 Check that $D = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ is not positive definite using criteria II.

$x^TDx = x_1^2 + 4x_1x_2 + 3x_2^2 = (x_1 + 2x_2)^2 - x_2^2$.

To prove it is not PD, we just need to find a vector to make x^TDx negative. We can pick, for example $x = (2, -1)$. Then $x^TDx = -1 < 0$

4.1 How many **exact** flops are needed for the following computations?

(a) $\langle x, y \rangle$, where x, y are vectors in $\mathbb{R}^9$.

(b) ABC, where A, B, C are 5×5 matrices.

(c) Ux, where U is $n \times n$ upper triangular matrix and x is a vector in $\mathbb{R}^n$.

Solution

(a) 2*9-1=17

(b) Matrix multiplication twice. (2*5-1)*5*5*2= 450

(c) $\displaystyle\sum_{i=1}^{n}(2i-1) = n(n+1) - n = n^2$

4.2 Given $n \times n$ matrices A, B, C, how many flops are needed for the following computations? Answer in terms of the order $n, n^2, n^3, \cdots$.

(a) ABC

(b) $A^{-1}B$

(c) $A+B$

(d) LU, where L is $n\times n$ lower triangular, U is $n\times n$ upper triangular

Solution

(a) $(2n-1)n^2 * 2 = O(n^3)$

(b) This is to solve $AX=B$: $O(n^3)$

(c) n^2

(d) Suppose the columns of U are $u_1, u_2, \cdots, u_n$. Computing LU is computing Lu_i.

Lu_1: $\sum_{j=1}^{n}(2*1-1) = n$

Lu_2: $\sum_{j=1}^{2}(2j-1) + \sum_{j=3}^{n}(2*2-1)$

...

Lu_i: $\sum_{j=1}^{i}(2j-1) + \sum_{j=i+1}^{n}(2i-1) = i^2 + (n-i)(2i-1) = -i^2 + (2n+1)i - n$

Sum them up: $\sum_{i=1}^{n} -i^2+(2n+1)i-n = -\frac{n(n+1)(2n+1)}{6} + (2n+1)\frac{n(n+1)}{2} - n^2 = \frac{n(n+1)(2n+1)}{3} - n^2 = \frac{n}{3}\left[(n+1)(2n+1)-3n\right] = \frac{n}{3}[2n^2+1] = O(n^3)$

4.3 Consider the linear system $\begin{bmatrix} 2 & 2\times10^{20} & \vdots & 2\times10^{20} \\ 1 & 1 & \vdots & 2 \end{bmatrix}$. Find its 'solution' using Gaussian elimination, in a machine where numbers are in standard IEEE double-precision format.

This is very similar to the example in the notes.

$$\begin{bmatrix} 2 & 2\times10^{20} & \vdots & 2\times10^{20} \\ 1 & 1 & \vdots & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2\times10^{20} & \vdots & 2\times10^{20} \\ 0 & -10^{20}+1 & \vdots & -10^{20}+2 \end{bmatrix} \xrightarrow{\rho2/(1-10^{20})} \begin{bmatrix} 2 & 2\times10^{20} & \vdots & 2\times10^{20} \\ 0 & 1 & \vdots & \boxed{1} \end{bmatrix}$$

$x_2 = 1$

$2x_1 + 2\times10^{20} = 2\times10^{20} \Rightarrow x_1 = 0$

4.4 Solve $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} x = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ $(Ax=b)$ by

(a) First find the LU factorization of A.

(b) Second solve $Ly=b$.

(c) Third solve $Ux=y$.

Solution

(a) $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \xrightarrow[\rho1(-2)+\rho3]{\rho1\frac{3}{2}+\rho2} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow{\rho2(3)+\rho3} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}.$

(b) $y_1 = 2$

$-1.5 * 2 + y_2 = 2 \Longrightarrow y_2 = 5$

$2 * 2 - 3 * 5 + y_3 = 3 \Longrightarrow y_3 = 14$

$\boxed{y = (2, 5, 14)}$

(c) $x_3 = 14/7 = 2$

$x_2 + 3 * 2 = 5 \Longrightarrow x_2 = -1$

$2x_1 + 6(-1) + 2 * 2 = 2 \Longrightarrow x_1 = 2$

$\boxed{x = (2, -1, 2)}$.

4.5 Given $\begin{bmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, solve the system

$$\begin{array}{llll} 3x_1 & -6x_2 & -3x_3 & = -3 \\ 2x_1 & & +6x_3 & = -22 \\ -4x_1 & +7x_2 & +4x_3 & = 3 \end{array} .$$

By the way, this is still LU factorization. L does not need to have 1's on the diagonal. It only needs to be lower triangular.

Solve $Ly = b$
$y_1 = -3/3 = -1$
$2 * (-1) + 4y_2 = -22 \Longrightarrow y_2 = -5$
$-4(-1) - 1(-5) + 2y_3 = 3 \Longrightarrow y_3 = -3$
$y = (-1, -5, -3)$

Solve $Ux = y$
$x_3 = -3$
$x_2 + 2(-3) = -5 \Longrightarrow x_2 = 1$
$x_1 - 2 - (-3) = -1 \Longrightarrow x_1 = -2$
$\boxed{x = (-2, 1, -3)}$.

4.6 Solve the system in Problem 4.5 using scaled partial pivoting strategy.

$$\begin{bmatrix} 3 & -6 & -3 & \vdots & -3 \\ 2 & 0 & 6 & \vdots & -22 \\ -4 & 7 & 4 & \vdots & 3 \end{bmatrix}$$

$s_1 = 6, s_2 = 6, s_3 = 7$

Stage 1: compare 3/6, 2/6, 4/7. We pick $\boxed{\text{row 3}}$: $\begin{bmatrix} 0 & -3/4 & 0 & \vdots & -3/4 \\ 0 & 7/2 & 8 & \vdots & -41/2 \\ -4 & 7 & 4 & \vdots & 3 \end{bmatrix}$

Stage 2: compare 3/24 (row1), 7/12 (row 2). We pick $\boxed{\text{row 2}}$: $\begin{bmatrix} 0 & 0 & 12/7 & \vdots & -36/7 \\ 0 & 7/2 & 8 & \vdots & -41/2 \\ -4 & 7 & 4 & \vdots & 3 \end{bmatrix}$

$x_3 = -3$

$7x_2/2 - 24 = -41/2 \Longrightarrow x_2 = 1$

$-4x_1 + 7 - 12 = 3 \Longrightarrow x_1 = -2$

This example, as well as the example in the notes, does not show the advantage of the scaled partial pivoting because we are using examples that's computable by hand.