Exercises for chapter 9

8 1 If A and B are similar, then they have same eigenvalues

$$A = UBU^{-1}$$

if $AV = \lambda V$

then $UBU^{-1}V = \lambda V$

$BU^{-1}V = U^{-1}\lambda V$

$B(U^{-1}V) = \lambda(U^{-1}V) \Rightarrow \lambda$ is also an eigenvalue of B with eigenvector $U^{-1}V$

8 2 $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}^{-1}$

$\lambda_1 = 1, \quad V_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\lambda_2 = -5, \quad V_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_3 = -1, \quad V_3 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

10 3 $U\begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}U^{-1}$ has eigenvalues 1, 1, 2, and is diagonalizable by def.

We just need to pick U that is not orthonormal.

pick $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, so it's easy to compute its inverse

$$U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & -1 \\ & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & 2 \\ & & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Several students have $\begin{bmatrix} 1 & a & b \\ & 1 & c \\ & & 2 \end{bmatrix}$ as the answer with specific a, b, c values. In this case, you need to justify $\begin{bmatrix} 1 & a & b \\ & 1 & c \\ & & 2 \end{bmatrix}$ is diagonalizable, meaning find the eigenvectors 3 indep.

✓

5 4 $QQ^T = I$, $PP^T = I$

$$QP(QP)^T = QPP^TQ^T = QQ^T = I$$

15 5 $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$

See previous 2 paragraphs of Example 9.1 or Remark 6.13

$a_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $u_1 = a_1 \pm \|a_1\| e_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \sim \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$H_1 = I - 2\frac{u_1u_1^T}{u_1^Tu_1} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}\left(\begin{bmatrix} 5 & \\ & 5 \end{bmatrix} - \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{5}\begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}$$

$$Q_1 = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & -4 \\ 0 & -4 & 3 \end{bmatrix}$$

$$A_1 = Q_1AQ_1^T = \frac{1}{25}\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & -4 \\ 0 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & -4 \\ 0 & -4 & 3 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 5 & 15 & 20 \\ -25 & -3 & -4 \\ 0 & -4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & -4 \\ 0 & -4 & 3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 25 & -125 & 0 \\ -25\times5 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A_1 will be symmetric, only need to compute these entries

5 6 $Av = \lambda v$

$$\frac{v^TAv}{v^Tv} = \frac{v^T\lambda v}{v^Tv} = \lambda$$

25 7 A has eigenvalues 1, 2, 3, 4

5+5+10+5

(a) Find 4, rate $= \frac{3}{4}$

(b) $A - I$ has 0, 1, 2, 3. Find 3 (corresponding to 4), rate $= \frac{2}{3}$

$\boxed{3}$

7(c) What is the value of $A-\alpha I$ so that it finds $4-\alpha$ withe the best rate

7(c) is hard. graded generously.

$A-\alpha I$: $1-\alpha, 2-\alpha, 3-\alpha, 4-\alpha$

first need to make sure $4-\alpha$ is the biggest <u>in magnitude</u>.

if $0\le\alpha\le 1$, then $4-\alpha$ is the biggest in absolute value

rate $= \dfrac{3-\alpha}{4-\alpha}$ is minimized by $\alpha=1$ $\boxed{\frac{2}{3}}$

if $1\le\alpha\le 2$ $\max\{|1-\alpha|, |2-\alpha|, |3-\alpha|, |4-\alpha|\} = |4-\alpha|$ as well

because $\alpha-1 \le 3-\alpha < 4-\alpha$

rate $= \dfrac{3-\alpha}{4-\alpha}$ is minimized by $\alpha=2$ $\boxed{\frac{1}{2}}$

if $2<\alpha\le 3$

$\alpha-1 \overset{>}{,} \alpha-2, 3-\alpha \overset{<}{,} 4-\alpha$

need $\alpha-1 < 4-\alpha \Rightarrow \alpha < 2.5$

also $3-\alpha < \alpha-1 < 4-\alpha \Rightarrow \alpha-1$ 2nd biggest

rate $= \dfrac{\alpha-1}{4-\alpha}$ for $2<\alpha<2.5$ minimized by 2

if $\alpha>3$, then $\alpha-1 > 4-\alpha$

Overall, best rate $= 0.5$, attained by $\boxed{\alpha=2}$

7(d)

$\dfrac{1}{1-1.9}, \dfrac{1}{2-1.9}, \dfrac{1}{3-1.9}, \dfrac{1}{4-1.9}$

2nd 1st

will find $\dfrac{1}{2-1.9}$

rate $= \dfrac{\left|\frac{1}{-0.9}\right|}{\frac{1}{2-1.9}}$

$= \dfrac{0.1}{0.9} = \dfrac{1}{9}$

10) 8 $\det(A-\lambda I_4) = \begin{vmatrix} B-\lambda I_2 & C \\ 0 & D-\lambda I_2 \end{vmatrix} := \begin{vmatrix} a & b & c_1 & c_2 \\ c & d & c_3 & c_4 \\ 0 & 0 & x & y \\ 0 & 0 & z & w \end{vmatrix}$ where $B-\lambda I_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$D-\lambda I_2 = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

expand along 1st column $a\begin{vmatrix} d & c_3 & c_4 \\ 0 & x & y \\ 0 & z & w \end{vmatrix} - c\begin{vmatrix} b & c_1 & c_2 \\ 0 & x & y \\ 0 & z & w \end{vmatrix} = ad\begin{vmatrix} x & y \\ z & w \end{vmatrix} - cb\begin{vmatrix} x & y \\ z & w \end{vmatrix}$

$= (ad-bc)\det(D-\lambda I_2) = \det(B-\lambda I_2)\det(D-\lambda I_2)$

14 points free