

Fall 2017 Math 395 Written Homework 1 Key 100 total. -5 for no stapling

1. (12 pts) Use Newton's method to approximate $\sqrt{2}$. Let $x_0 = 1$. Compute 2 iterates only.

$$f(x) = x^2 - 2$$

$$x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k} = \frac{1}{2}x_k + \frac{1}{x_k}.$$

$$x_0 = 1$$

$$x_1 = 1/2 + 1 = 3/2$$

$$x_2 = 3/4 + 2/3 = 17/12$$

$17/12 \approx 1.4167$, which is only about 0.002 away from $\sqrt{2}$, with only 2 iterations performed.

2. (12 pts) Prove that Newton's method will converge to 0 given any initial value x_0 if we are solving $x^2 = 0$.

$$f(x) = x^2, f'(x) = 2x$$

$$x_{k+1} = x_k - \frac{x_k^2}{2x_k} = \frac{1}{2}x_k.$$

This means that the iterations just keep being divided by 2. It is a geometric sequence with common ratio $1/2$!

$$\{x_k\}_{k=0}^{\infty} = \{x_0, \frac{x_0}{2}, \frac{x_0}{4}, \frac{x_0}{8}, \dots\}$$

$$\lim_{k \rightarrow \infty} x_k = \lim_{k \rightarrow \infty} \frac{x_0}{2^k} = 0$$

3. (15 pts) Write down the first three iterates of the second method for solving $x^2 - 3 = 0$, starting with $x_0 = 0$ and $x_1 = 1$.

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} = x_k - \frac{(x_k^2 - 3)(x_k - x_{k-1})}{x_k^2 - x_{k-1}^2} = x_k - \frac{x_k^2 - 3}{x_k + x_{k-1}}.$$

$$x_2 = 1 - \frac{-2}{1} = 3$$

$$x_3 = 3 - \frac{3^2 - 3}{4} = 3 - 6/4 = 3/2$$

$$x_4 = 3/2 - \frac{(3/2)^2 - 3}{3 + 3/2} = 3/2 - \frac{9 - 12}{12 + 6} = 3/2 - \frac{-3}{18} = 3/2 + 1/6 = 10/6 = 5/3.$$

$5/3 \approx 1.667$, not too far away from $\sqrt{3}$.

4. We can compute $1/3$ by solving $f(x) = 0$ with $f(x) = x^{-1} - 3$.
- (a) (12 pts) Write down the Newton iteration for this problem, and compute by hand the first 2 Newton iterates for approximating $1/3$, starting with $x_0 = 0.5$.
- (b) (6 pts) What happens if you start with $x_0 = 1$?
- (c) (+10 pts) *In the case of (b), show that the iterates $x_k \rightarrow -\infty$ as $k \rightarrow \infty$.
- (d) (12 pts) Use the theory of fixed point iteration to determine an interval about $1/3$ from which Newton's method will converge to $1/3$.

$$(a) \ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{1/x_k - 3}{-1/x_k^2} = 2x_k - 3x_k^2$$

$$x_0 = 0.5$$

$$x_1 = 1 - 3 * 0.5^2 = 1/4$$

$$x_2 = 2\frac{1}{4} - 3\left(\frac{1}{4}\right)^2 = \frac{8-3}{4^2} = \frac{5}{16}$$

$$(b) \ x_1 = 2 - 3 = -1$$

$$x_2 = -2 - 3 = -5$$

$$x_3 = -10 - 75 = -85$$

...

x_k is getting more negative and definitely does not converge to $1/3$.

(c) It suffices to show that $x_k \leq -k$. We will prove this by induction.

It is true that $x_1 \leq -1$. Assuming $x_k \leq -k$, then

$$x_{k+1} = 2x_k - 3x_k^2 \leq -2k - 3k^2 < -(k+1).$$

(d) The iteration formula is $x_{k+1} = \varphi(x_k)$ with $\varphi(x) = 2x - 3x^2$. $\varphi'(x) = 2 - 6x$

$$|2 - 6x| < 1 \implies -1 < 6x - 2 < 1 \implies 1 < 6x < 3 \implies 1/6 < x < 1/2$$

We can pick x_0 from $(1/6, 1/2)$, which is centered around $1/3$.

5. Let function $\varphi(x) = (x^2 + 4)/5$.

(a) (8 pts) Find the fixed point(s) of $\varphi(x)$.

(b) (8 pts) Would the fixed point iteration, $x_{k+1} = \varphi(x_k)$, converge to a fixed point in the interval $[0, 2]$ for all initial guesses $x_0 \in [0, 2]$?

(c) (+10 pts) *Find a function $f(x)$ such that its Newton iterations are $x_{k+1} = \varphi(x_k)$. (Hint: You need to solve a separable differential equation (Calc II material))

$$(a) \ (x^2 + 4)/5 = x \implies x^2 + 4 = 5x \implies (x - 1)(x - 4) = 0 \implies x = 1, x = 4$$

$$(b) \ \varphi'(x) = 2x/5$$

$$|\varphi'(x)| < 1 \implies -5/2 < x < 5/2.$$

The answer is yes because $[0, 2]$ is contained in the interval $(-5/2, 5/2)$.

6. (15 pts) Compare Bisection method and Newton's method. List their pros and cons. Think of a way to combine Bisection method and Newton's method to overcome the drawbacks of the Newton's method. Open question. You can find the answer in Project 1.