9 If $A=QR$ is the full QR, what is cholesky of A^TA

$$A^TA = (QR)^T QR = R^T Q^T Q R = R^T R := LL^T$$

$L = R^T$ is lower triangular

10 (a) $A = \hat{Q}\hat{R}$ $\quad \underline{\hat{Q}^T\hat{Q} = I}$ $\qquad \boxed{} = \boxed{\hat{Q}}\,\boxed{\hat{R}}$

$$A^TA = \hat{R}^T\hat{Q}^T\hat{Q}\hat{R} = \hat{R}^T\hat{R}, \quad (A^TA)^{-1} = \hat{R}^{-1}\hat{R}^{-T}$$

$$P = A(A^TA)^{-1}A^T = \hat{Q}\hat{R}\hat{R}^{-1}\hat{R}^{-T}\hat{R}^T\hat{Q}^T = \hat{Q}\hat{Q}^T$$

(b) $A = \hat{U}\hat{\Sigma}V^T$, $A^T = V\hat{\Sigma}^T\hat{U}^T$ $\qquad \boxed{} = \boxed{\hat{U}}\,\boxed{\hat{\Sigma}}\,\boxed{V}^T$

$$A^TA = V\hat{\Sigma}^T\underline{\hat{U}^T\hat{U}}\hat{\Sigma}V^T = V\hat{\Sigma}^T\hat{\Sigma}V^T$$

$$(A^TA)^{-1} = V^{-T}\hat{\Sigma}^{-1}\hat{\Sigma}^{-T}V^T$$

$$P = A(A^TA)^{-1}A^T = \hat{U}\hat{\Sigma}\underline{V^TV^{-T}}\hat{\Sigma}^{-1}\hat{\Sigma}^{-T}\underline{V^TV}\hat{\Sigma}^T\hat{U}^T$$

$$= \hat{U}\underline{\hat{\Sigma}\hat{\Sigma}^{-1}}\,\underline{\hat{\Sigma}^{-T}\hat{\Sigma}^T}\hat{U}^T = \hat{U}\hat{U}^T$$

11
$$\underset{A}{\begin{bmatrix} 1 & 1 & 4 \\ -1 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & -2 \end{bmatrix}} = \underset{\hat{Q}}{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}} \underset{\hat{R}}{\begin{bmatrix} 2 & 1 & 4 \\ & 1 & 2 \\ & & 2 \end{bmatrix}}, \quad \text{solve LS of } Ax = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Compute
$$y = \hat{Q}^Tb = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2+1-1 \\ 2 \\ 2-1-1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Solve
$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$x_3 = 0$

$x_2 + 0 = 1 \Rightarrow x_2 = 1$

$2x_1 + 1 = 1 \Rightarrow x_1 = 0$

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

12
$$B = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & -1 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 2 & & & 0 \\ & 0.02 & & 0 \\ & & 0.01 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}^T$$

(a) $B = U\Sigma V^T$

$$B^T = V\Sigma^TU^T = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}}_{\text{reduced}} \underbrace{\begin{bmatrix} 2 & & \\ & 0.02 & \\ & & 0.01 \end{bmatrix}}_{\text{reduced}} \underset{U^T}{\frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & -1 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \end{bmatrix}^T}$$

(b)

$$B^T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 0.02 & \\ & & 0.01 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & -1 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \end{bmatrix}^T := ADC^T$$

compute $z = A^T b = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Solve $\begin{bmatrix} 2 & & \\ & 0.02 & \\ & & 0.01 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} y_1 = 0.5 \\ y_2 = 0 \\ y_3 = 100 \end{array}$

$$x = Cy = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & -1 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0.5 \\ 0 \\ 100 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{2\sqrt{2}} + \frac{100}{\sqrt{2}} \\ \frac{1}{2} - 100 \\ \frac{1}{2\sqrt{2}} + \frac{100}{\sqrt{2}} \end{bmatrix}$$