Some Linear Algebra review problems

1. Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & 0 & 1 & 2 \\ -2 & 1 & 2 & -1 \end{bmatrix}$, $x = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Find A^T and $A^T x$.

2. State the definition of a set of vectors $\{v_1, v_2, \cdots, v_n\}$ being linearly independent.

3. Explain why 4 vectors in $\mathbb{R}^3$ are always linearly dependent.

4. (a) Find the 3×3 matrix A such that $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_3 \end{bmatrix}$.

   (b) Find a vector u that reminas the same under A, i.e. $Au = u$.

5. Compute $\begin{vmatrix} 2 & 1 \\ -2 & 4 \end{vmatrix}$.

6. Given $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & -1 \\ 1 & 2 & -3 \end{bmatrix}$

   (a) Find the null space of A.

   (b) From your answer to (a), are the columns of A linearly independent?

7. True or False, and state reason. Suppose A, B, C are all square matrices of the same size.

   (a) If A is invertible, $(A^2)^{-1} = (A^{-1})^2$.

   (b) $A(BC) = (AB)C$.

   (c) If $AC = BC, C \neq 0$, then $A = B$.

8. Given $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$. Find A^{-1} by Gaussian elimination.

9. Without performing any computations, decide whether the following systems have no solution, one solution, or infinitely many solutions. State the reason.

   (a) $\begin{bmatrix} 0 & 1 & 5 & \vdots & 0 \\ 0 & 0 & 2 & \vdots & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 & 8 & \vdots & 9 \\ 0 & 1 & 5 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 & \vdots & 9 \\ 0 & 8 & \vdots & 2 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$

10. Given $a_1 = [1, 0, 1], a_2 = [2, 1, 1]$, $a_3 = [3, 1, 2], b_1 = [1, -1, 2], b_2 = [0, 0, 0]$

    (a) Is b_1 in span$\{a_1, a_2, a_3\}$? Find all linear combinations of a_1, a_2 that makes up b_1 if answer is yes.

    (b) Is b_2 in span$\{a_1, a_2, a_3\}$?

    (c) Find the dimension and a basis of span$\{a_1, a_2, a_3\}$

11. Given $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}$

    (a) Find rank(A). Is A invertible?

    (b) What is the column space of A (often denoted as $C(A)$)?

12. Given $u_1 = [1, -1, 1, -1]^T, u_2 = [1, 0, 1, 0]^T, u_3 = [4, 0, 2, -2]^T$. Is $\{u_1, u_2, u_3\}$ a basis of $\mathbb{R}^4$?