

## Fall 2017 Math 395 Written Homework 8 Key 10 × 6 total

- 8.1 For the problem “find the point on the line  $x + y = 1$  that is closest to  $(-1, 0)$ .”, reformulate it in the form of (8.1) and determine whether it is a convex optimization problem.

**Solution**

$$\begin{array}{ll} \min & \sqrt{(x+1)^2 + y^2} \\ \text{subject to} & x + y = 1 \end{array} \iff \begin{array}{ll} \min & (x+1)^2 + y^2 \\ \text{subject to} & x + y - 1 \leq 0 \\ & -x - y + 1 \leq 0 \end{array}$$

$f_0(x, y) = (x+1)^2 + y^2$  is a convex function, because this is a quadratic with a positive definite matrix  $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ .

$f_1(x, y) = x + y - 1$ ,  $f_2(x, y) = -x - y + 1$  are both convex because they are linear.

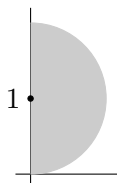
This is a convex optimization problem.

Remark 1: It is annoying to compute the Hessian of  $\sqrt{(x+1)^2 + y^2}$ , but notice that minimizing the distance is equivalent to minimizing distance squared.  $(x+1)^2 + y^2$  is a quadratic form and it is easy to determine if its convex. Same technique is used in #3.

Remark 2: Most people formulate the problem as “minimizing  $\sqrt{(x+1)^2 + (1-x)^2}$  with no restriction”. In this case, it is a univariate function so only need to take its 2nd derivative. But even better, you should reformulate the problem as “minimizing  $(x+1)^2 + (1-x)^2$  with no restriction”.

- 8.2 Find the gradient of  $f(x_1, x_2, x_3) = x_1 e^{x_2} - 2x_3 x_2 + \sin(x_1 x_3)$ .

**Solution**  $\nabla f = \begin{bmatrix} e^{x_2} + x_3 \cos(x_1 x_3) \\ x_1 e^{x_2} - 2x_3 \\ -2x_2 + x_1 \cos(x_1 x_3) \end{bmatrix}$



- 8.3 Rewrite this convex set (gray area, including boundary) as

$\{f_i(x) \leq 0, i = 1, \dots, m\}$  where  $f_i$  are convex functions.

**Solution** This gray area is bounded by two curves:  $x_1 = 0$  and half circle  $x_1^2 + (x_2 - 1)^2 = 1, x_1 > 0$ .

We can write the restrictions as  $x_1 \geq 0$  and  $x_1 \leq \sqrt{1 - (x_2 - 1)^2}$ , which is equivalent to

$$-x_1 \leq 0 \text{ and } x_1^2 + (x_2 - 1)^2 - 1 \leq 0$$

$f_1(x_1, x_2) = -x_1$  is linear, so convex.

$f_2(x_1, x_2) = x_1^2 + (x_2 - 1)^2 - 1$  is a quadratic term with a positive definite matrix  $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ , so it is convex too.

- 8.4 In Example 8.8

(a) What happens if we pick  $t^{(k)} = 1$ ?

(b) Pick  $t^{(k)} = a$ . Determine the range of  $a$  so that the sequence will converge to 0.

**Solution**  $f(x) = x^2$ .  $f'(x) = 2x$ .

(a) We start with  $x^{(0)} = 3$  and pick  $t^{(k)} = 1$  for all iterations. Then we get the iteration formula  $x^{(k+1)} = x^{(k)} - 2x^{(k)} = -x^{(k)}$ . This means the iterations are 3, -3, 3, -3, ...

$$(b) \ x^{(k+1)} = x^{(k)} - \alpha 2x^{(k)} = (1 - 2\alpha)x^{(k)}$$

This is a geometric sequence with common ratio  $1 - 2\alpha$ . To ensure it converges to 0, we need  $|1 - 2\alpha| < 1 \implies -1 < 2\alpha - 1 < 1 \implies 0 < \alpha < 1$ .

### 8.5 In Example 8.9

(a) Verify the function  $f$  is convex everywhere.

(b) With  $x^{(0)} = (0, 0)$ , what step size should you pick (different step size in each iterate) in order to have  $x^{(k+1)} = x^{(k)} + (1, 1)$ ?

**Solution**  $f(x_1, x_2) = \exp(-x_1 - x_2)$ .  $\nabla f(x) = \exp(-x_1 - x_2)(-1, -1)$ .

(a)  $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = -\exp(-x_1 - x_2)$ . We can use the Hessian matrix.

$$Hf(x) = \begin{bmatrix} \exp(-x_1 - x_2) & \exp(-x_1 - x_2) \\ \exp(-x_1 - x_2) & \exp(-x_1 - x_2) \end{bmatrix} = \exp(-x_1 - x_2) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

We can use determinants to verify that this matrix is positive semi-definite for all values of  $x_1, x_2$ , so it is convex everywhere.

(b) **Let**  $x^{(k)} = (x_1^{(k)}, x_2^{(k)})$ .

$$x^{(k+1)} = x^{(k)} - t^{(k)} \nabla f(x^{(k)}) = x^{(k)} + t^{(k)} \exp(-x_1^{(k)} - x_2^{(k)})(1, 1),$$

This means  $t^{(k)} = \exp(x_1^{(k)} + x_2^{(k)})$ . **Many people stopped here, which is kind of OK. But you can do further.**

To simplify it further,  $x^{(k+1)} = x^{(k)} + (1, 1)$  implies that  $x^{(k)} = (k, k)$ , so

$$t^{(k)} = \exp(x_1^{(k)} + x_2^{(k)}) = \exp(k + k) = \exp(2k).$$

### 8.6 Find the domain on which $f(x, y) = -e^{-x^2-y^2}$ is convex.

**Solution**  $\frac{\partial f}{\partial x} = 2xe^{-x^2-y^2}$ ,  $\frac{\partial f}{\partial y} = 2ye^{-x^2-y^2}$

$$Hf(x, y) = \begin{bmatrix} 2e^{-x^2-y^2} - 4x^2e^{-x^2-y^2} & -4xye^{-x^2-y^2} \\ -4xye^{-x^2-y^2} & 2e^{-x^2-y^2} - 4y^2e^{-x^2-y^2} \end{bmatrix} = 2e^{-x^2-y^2} \begin{bmatrix} 1 - 2x^2 & -2xy \\ -2xy & 1 - 2y^2 \end{bmatrix}$$

To ensure positive definiteness, we need  $1 - 2x^2 \geq 0$ , and  $(1 - 2x^2)(1 - 2y^2) - 4x^2y^2 \geq 0$

(a)  $1 - 2x^2 \geq 0 \implies x^2 < 1/2$

(b)  $(1 - 2x^2)(1 - 2y^2) - 4x^2y^2 \geq 0 \implies 1 - 2x^2 - 2y^2 \geq 0$ , which implies (a),

so the domain is  $x^2 + y^2 < 1/2$ .