**Fall 2017 Math 395: Numerical Analysis** **Quiz 3**
**9/28 Thursday**

Name: Key

1. (3+4) Let $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, and $u_1, u_2, u_3$ be its 3 columns.

   (a) Find $U^{-1}$.

   (b) What linear combination of $u_1, u_2, u_3$ equals $x = (1, -1, -2)$?

   **Solution**

   (a) $U^{-1} = U^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

   (b)

   $c_1 = x^T u_1 = 1$

   $c_2 = x^T u_2 = -1/\sqrt{2} + 2/\sqrt{2} = 1/\sqrt{2}$

   $c_3 = x^T u_3 = -1/\sqrt{2} - 2/\sqrt{2} = -3/\sqrt{2}$

   $x = c_1 u_1 + c_2 u_2 + c_3 u_3$

2. (3+3) How many exact flops are needed for the following computations?

   (a) $AB$, where $A$ is $m \times n$ and $B$ is $n \times p$.

   (b) $Ux$, where $U$ is 10 by 10 upper triangular, and $x$ is a vector in $\mathbb{R}^{10}$.

   **Solution**

   (a) This is computing $mp$ inner products: $mp(2n-1)$

   (b) To compute the $i$th entry of $Ux$, it is computing an inner product with $i$ coordinates, which costs $2i - 1$ flops.

   $$\sum_{i=1}^{10}(2i-1) = 10 * 11 - 10 = 100$$

**—More on the back—**

3. (6) Find the operator norm of $\begin{bmatrix} 1/\sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & \sqrt{2} \\ 0 & 0 \end{bmatrix}$.

   We need to find the biggest singular value.

   $A = \begin{bmatrix} 1/\sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & \sqrt{2} \\ 0 & 0 \end{bmatrix}$. $A^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \end{bmatrix}$, $A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

   $\begin{vmatrix} 1-r & 2 \\ 2 & 4-r \end{vmatrix} = r^2 - 5r + 4 - 4 = r(r-5)$

   Eigenvalues of $A^T A$ are 5,0.

   Sinigular values of $A$ are $\sqrt{5}, 0$. Operator norm is $\boxed{\sqrt{5}}$.

4. (6) Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Using both criteria II (quadratic) and III (determinant) to show that it is NOT positive definite.

   Criteria II: $x^T D x = x_1^2 + 4x_1x_2 + 3x_2^2 = (x_1 + 2x_2)^2 - x_2^2$.

   To prove it is not PD, we just need to find a vector $x$ that makes $x^T D x$ negative. We can pick, for example $x = (2, -1)$. Then $x^T D x = -1 < 0$

   Criteria III: $\det A = 3 - 4 < 0$