Fall 2017 Math 395 Written Homework 8 Key 10 × 6 total

8.1 For the problem "find the point on the line $x + y = 1$ that is closest to (-1,0).", reformulate it in the form of (8.1) and determine whether it is a convex optimization problem.

Solution

$$\begin{array}{ll} \min & \sqrt{(x+1)^2+y^2} \\ \text{subject to} & x+y=1 \end{array} \iff \begin{array}{ll} \min & (x+1)^2+y^2 \\ \text{subject to} & x+y-1 \le 0 \\ & -x-y+1 \le 0 \end{array}$$

$f_0(x, y) = (x+1)^2 + y^2$ is a convex function, because this is a quadratic with a positive definite matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

$f_1(x, y) = x + y - 1$, $f_2(x, y) = -x - y + 1$ are both convex because they are linear.

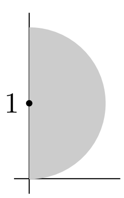
This is a convex optimization problem.

Remark 1: It is annoying to compute the Hessian of $\sqrt{(x+1)^2 + y^2}$, but notice that minimizing the distance is equivalent to minimizing distance squared. $(x + 1)^2 + y^2$ is a quadratic form and it is easy to determine if its convex. Same technique is used in #3.

Remark 2: Most people formulate the problem as "minimizing $\sqrt{(x+1)^2 + (1-x)^2}$ with no restriction". In this case, it is a univariate function so only need to take its 2nd derivative. But even better, you should reformulate the problem as "minimizing $(x + 1)^2 + (1 - x)^2$ with no restriction".

8.2 Find the gradient of $f(x_1, x_2, x_3) = x_1 e^{x_2} - 2x_3x_2 + \sin(x_1x_3)$.

Solution $\nabla f = \begin{bmatrix} e^{x_2} + x_3\cos(x_1x_3) \\ x_1e^{x_2} - 2x_3 \\ -2x_2 + x_1\cos(x_1x_3) \end{bmatrix}$



8.3 Rewrite this convex set (gray area, including boundary) as $\{f_i(x) \le 0, i = 1, \cdots, m\}$ where f_i are convex functions.

Solution This gray area is bounded by two curves: $x = 0$ and half circle $x^2 + (y - 1)^2 = 1, x > 0$. We can write the restrictions as $x_1 \ge 0$ and $x_1 \le \sqrt{1 - (x_2 - 1)^2}$, which is equivalent to

$$-x_1 \le 0 \text{ and } x_1^2 + (x_2 - 1)^2 - 1 \le 0$$

$f_1(x_1, x_2) = -x_1$ is linear, so convex.

$f_2(x_1, x_2) = x_1^2 + (x_2 - 1)^2 - 1$ is a quadratic term with a positive definite matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, so it is convex too.

8.4 In Example 8.8

(a) What happens if we pick $t^{(k)} = 1$?

(b) Pick $t^{(k)} = a$. Determine the range of a so that the sequence will converge to 0.

Solution $f(x) = x^2$. $f'(x) = 2x$.

(a) We start with $x^{(0)} = 3$ and pick $t^{(k)} = 1$ for all iterations. Then we get the iteration formula $x^{(k+1)} = x^{(k)} - 2x^{(k)} = -x^{(k)}$. This means the iterations are 3, -3, 3, -3, ...

(b) $x^{(k+1)} = x^{(k)} - \alpha 2x^{(k)} = (1-2\alpha)x^{(k)}$

This is a geometric sequence with common ratio $1-2\alpha$. To ensure it converges to 0, we need $|1-2\alpha| < 1 \implies -1 < 2\alpha - 1 < 1 \implies 0 < \alpha < 1$.

8.5 In Example 8.9

(a) Verify the function f is convex everywhere.

(b) With $x^{(0)} = (0,0)$, what step size should you pick (different step size in each iterate) in order to have $x^{(k+1)} = x^{(k)} + (1,1)$?

Solution $f(x_1, x_2) = \exp(-x_1 - x_2)$. $\nabla f(x) = \exp(-x_1 - x_2)(-1,-1)$.

(a) $\dfrac{\partial f}{\partial x_1} = \dfrac{\partial f}{\partial x_1} = -\exp(-x_1 - x_2)$. We can use the Hessian matrix.

$$Hf(x) = \begin{bmatrix} \exp(-x_1 - x_2) & \exp(-x_1 - x_2) \\ \exp(-x_1 - x_2) & \exp(-x_1 - x_2) \end{bmatrix} = \exp(-x_1 - x_2) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

We can use determinants to verify that this matrix is positive semi-definite for all values of x_1, x_2, so it is convex everywhere.

(b) Let $x^{(k)} = (x_1^{(k)}, x_2^{(k)})$.

$x^{(k+1)} = x^{(k)} - t^{(k)} \nabla f(x^{(k)}) = x^{(k)} + t^{(k)} \exp(-x_1^{(k)} - x_2^{(k)})(1,1)$,

This means $t^{(k)} = \exp(x_1^{(k)} + x_2^{(k)})$. Many people stopped here, which is kind of OK. But you can do further.

To simplify it further, $x^{(k+1)} = x^{(k)} + (1,1)$ implies that $x^{(k)} = (k,k)$, so

$t^{(k)} = \exp(x_1^{(k)} + x_2^{(k)}) = \exp(k+k) = \exp(2k)$.

8.6 Find the domain on which $f(x,y) = -e^{-x^2-y^2}$ is convex.

Solution $\dfrac{\partial f}{\partial x} = 2xe^{-x^2-y^2}, \quad \dfrac{\partial f}{\partial y} = 2ye^{-x^2-y^2}$

$$Hf(x,y) = \begin{bmatrix} 2e^{-x^2-y^2} - 4x^2e^{-x^2-y^2} & -4xye^{-x^2-y^2} \\ -4xye^{-x^2-y^2} & 2e^{-x^2-y^2} - 4y^2e^{-x^2-y^2} \end{bmatrix} = 2e^{-x^2-y^2} \begin{bmatrix} 1-2x^2 & -2xy \\ -2xy & 1-2y^2 \end{bmatrix}$$

To ensure positive definiteness, we need $1 - 2x^2 \geq 0$, and $(1-2x^2)(1-2y^2) - 4x^2y^2 \geq 0$

(a) $1 - 2x^2 \geq 0 \Rightarrow x^2 < 1/2$

(b) $(1-2x^2)(1-2y^2) - 4x^2y^2 \geq 0 \Rightarrow 1 - 2x^2 - 2y^2 \geq 0$, which implies (a),

so the domain is $x^2 + y^2 < 1/2$.