Fall 2017 Math 395 Written Homework 1 Key 100 total. -5 for no stapling

1. (12 pts) Use Newton's method to approximate $\sqrt{2}$. Let $x_0 = 1$. Compute 2 iterates only.

   $f(x) = x^2 - 2$

   $x_{k+1} = x_k - \dfrac{x_k^2 - 2}{2x_k} = \dfrac{1}{2}x_k + \dfrac{1}{x_k}.$

   $x_0 = 1$

   $x_1 = 1/2 + 1 = 3/2$

   $x_2 = 3/4 + 2/3 = 17/12$

   $17/12 \approx 1.4167$, which is only about 0.002 away from $\sqrt{2}$, with only 2 iterations performed.

2. (12 pts) Prove that Newton's method will converge to 0 given any initial value x_0 if we are solving $x^2 = 0$.

   $f(x) = x^2, f'(x) = 2x$

   $x_{k+1} = x_k - \dfrac{x_k^2}{2x_k} = \dfrac{1}{2}x_k.$

   This means that the iterations just keep being divided by 2. It is a geometric sequence with common ratio 1/2!

   $\{x_k\}_{k=0}^{\infty} = \{x_0, \dfrac{x_0}{2}, \dfrac{x_0}{4}, \dfrac{x_0}{8}, \cdots\}$

   $\lim\limits_{k\to\infty} x_k = \lim\limits_{k\to\infty} \dfrac{x_0}{2^k} = 0$

3. (15 pts) Write down the first three iterates of the secand method for solving $x^2 - 3 = 0$, starting with $x_0 = 0$ and $x_1 = 1$.

   $x_{k+1} = x_k - \dfrac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} = x_k - \dfrac{(x_k^2 - 3)(x_k - x_{k-1})}{x_k^2 - x_{k-1}^2} = x_k - \dfrac{x_k^2 - 3}{x_k + x_{k-1}}.$

   $x_2 = 1 - \dfrac{-2}{1} = 3$

   $x_3 = 3 - \dfrac{3^3 - 3}{4} = 3 - 6/4 = 3/2$

   $x_4 = 3/2 - \dfrac{(3/2)^2 - 3}{3 + 3/2} = 3/2 - \dfrac{9 - 12}{12 + 6} = 3/2 - \dfrac{-3}{18} = 3/2 + 1/6 = 10/6 = 5/3.$

   $5/3 \approx 1.667$, not too far away from $\sqrt{3}$.

4. We can compute 1/3 by solving $f(x) = 0$ with $f(x) = x^{-1} - 3$.

   (a) (12 pts) Write down the Newton iteration for this problem, and compute by hand the first 2 Newton iterates for approximating 1/3, starting with $x_0 = 0.5$.

   (b) (6 pts) What happens if you start with $x_0 = 1$?

   (c) (+10 pts) *In the case of (b), show that the iterates $x_k \to -\infty$ as $k \to \infty$.

   (d) (12 pts) Use the theory of fixed point iteration to determine an interval about 1/3 from which Newton's method will converge to 1/3.

   (a) $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)} = x_k - \dfrac{1/x_k - 3}{-1/x_k^2} = 2x_k - 3x_k^2$

   $x_0 = 0.5$

$x_1 = 1 - 3 * 0.5^2 = 1/4$

$x_2 = 2\frac{1}{4} - 3(\frac{1}{4})^2 = \frac{8-3}{4^2} = \frac{5}{16}$

(b) $x_1 = 2 - 3 = -1$

$x_2 = -2 - 3 = -5$

$x_3 = -10 - 75 = -85$

...

x_k is getting more negative and definitely does not converge to 1/3.

(c) It suffices to show that $x_k \leq -k$. We will prove this by induction.

It is true that $x_1 \leq -1$. Assuming $x_k \leq -k$, then

$x_{k+1} = 2x_k - 3x_k^2 \leq -2k - 3k^2 < -(k+1)$.

(d) The iteration formula is $x_{k+1} = \varphi(x_k)$ with $\varphi(x) = 2x - 3x^2$. $\varphi'(x) = 2 - 6x$

$|2 - 6x| < 1 \implies -1 < 6x - 2 < 1 \implies 1 < 6x < 3 \implies 1/6 < x < 1/2$

We can pick x_0 from $(1/6, 1/2)$, which is centered around 1/3.

5. Let function $\varphi(x) = (x^2 + 4)/5$.

   (a) (8 pts) Find the fixed point(s) of $\varphi(x)$.

   (b) (8 pts) Would the fixed point iteration, $x_{k+1} = \varphi(x_k)$, converge to a fixed point in the interval $[0, 2]$ for all initial gueses $x_0 \in [0, 2]$?

   (c) (+10 pts) *Find a function $f(x)$ such that its Newton iterations are $x_{k+1} = \varphi(x_k)$. (Hint: You need to solve a separable differential equation (Calc II material))

   (a) $(x^2 + 4)/5 = x \implies x^2 + 4 = 5x \implies (x - 1)(x - 4) = 0 \implies x = 1, x = 4$

   (b) $\varphi'(x) = 2x/5$

   $|\varphi'(x)| < 1 \implies -5/2 < x < 5/2$.

   The answer is yes because:

   (1) [0,2] is contained in the interval $(-5/2, 5/2)$.

   (2) 1, as one fixed point, is the center of [0,2].

6. (15 pts) Compare Bisection method and Newton's method. List their pros and cons. Think of a way to combine Bisection method and Newton's method to overcome the drawbacks of the Newton's method.

   Open question. You can find the answer in Project 1.