Fall 2017 Math 395 Written Homework 6 Key 100 total. -5 for no stapling

6.2 Find the full QR decomposition of $\begin{bmatrix} 1 & 1 & 4 \\ -1 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & -2 \end{bmatrix}$ using the reduced one (6.4).

Solution $\begin{bmatrix} 1 & 1 & 4 \\ -1 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} := \hat{Q}\hat{R}$. To find the full Q, we just need to add a unit norm column that is orthogonal to all 3 columns in $\hat{Q}$, which is to find the null space of $\hat{Q}^T$.

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -2 & -2 \end{bmatrix} \implies \begin{array}{c} x_3 = -x_4 \\ x_2 = -x_4 \\ x_1 + x_4 - x_4 - x_4 = 0 \end{array}$$

We can pick $x_4 = 1$, after normalization, we get $q_4 = \frac{1}{2}(1, -1, -1, 1)$.

$$\begin{bmatrix} 1 & 1 & 4 \\ -1 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} := QR$$

6.3 Let A be an $n \times n$ matrix. Prove that if $Ax = 0$ for every vector $x \in \mathbb{R}^d$, then A must be the zero matrix.

Proof. Let $A = [a_1, a_2, \cdots, a_n]$ where a_i are its columns. Since $Ax = 0$ for every vector x, in particular, it is true for $e_1 = (1, 0, \cdots, 0)$. $0 = Ae_1 = a_1$, meaning the first column must be a zero column.

Using the same idea, we can let x be e_j, then $0 = Ae_j = a_j$, which says that the jth column of A has to be 0. This is saying all columns of A are 0 columns, thus A is the zero matrix. □

6.4 Show that H_u in (6.6) is symmetric and orthonormal.

Solution $H_u = I - 2\frac{uu^T}{u^Tu}$.

Symmetry: Both I and uu^T are symmetric, so H_u is symmetric.

Orthonormal: H_u is square, and $H_u^T H_u = (I - 2\frac{uu^T}{u^Tu})(I - 2\frac{uu^T}{u^Tu}) = I - 4\frac{uu^T}{u^Tu} + 4\frac{uu^Tuu^T}{(u^Tu)^2} = I - 4\frac{uu^T}{u^Tu} + 4\frac{(u^Tu)uu^T}{(u^Tu)^2} = I - 4\frac{uu^T}{u^Tu} + 4\frac{uu^T}{u^Tu} = I$.

In general, $(A - B)(A - B) = A^2 + B^2 - AB - BA \neq A^2 - 2AB + B^2$.

6.5 Show that $H_{-2u} = H_u$.

$$H_{-2u} = I - 2\frac{(-2u)(-2u)^T}{(-2u)^T(-2u)} = I - 2\frac{4uu^T}{4u^Tu} = I - 2\frac{uu^T}{u^Tu} = H_u.$$

6.6 Given $x = (1, 2, -2)$,

(a) find an orthonormal matrix Q such that $Qx = 3e_1$.

(b) find an orthonormal matrix Q' such that $Q'x = -3e_2$.

(c) can you find an orthonormal matrix Q'' such that $Q''x = 2e_1$? Why?

Solution

(a) $\|x\| = 3$

$u = x - 3e_1 = (1, 2, -1) - (3, 0, 0) = (-2, 2, -2)$.

Since any scalar multiple of u will produce the same H_u, we will take $u = (-1, 1, -1)$

$$Q = H_u = I - 2\frac{uu^T}{u^Tu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}.$$

(b) $u' = x + 3e_2 = (1, 2, -1) + (0, 3, 0) = (1, 5, -1)$.

$$Q' = H_{u'} = I - 2\frac{u'u'^T}{u'^Tu'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{27}\begin{bmatrix} 1 & 5 & -1 \\ 5 & 25 & -5 \\ -1 & -5 & 1 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 25 & -10 & 2 \\ -10 & -23 & 10 \\ 2 & 10 & 25 \end{bmatrix}.$$

By the way, householder reflector is a good way to generate orthonormal matrices, if you ever need one.

(c) An orthonormal matrix preserves the length/norm. See (3.2) of the notes (page 16).

$\|Q''x\| = \|x\|$, should be 3, but it is 2 instead.

6.7 Let U be an $n\times n$ matrix. It is in the partitioned form as $U = \begin{bmatrix} I_{k\times k} & 0 \\ 0 & B \end{bmatrix}$, where B is an $(n-k)\times(n-k)$ orthonormal matrix. Show that

(a) The first k rows of UA is the same as the first k rows of A using block matrix multiplication.

(b) U is orthonormal using block matrix multiplication.

Solution

(a) Partition A as $A = \begin{bmatrix} A_k \\ D \end{bmatrix}$, where A_k is the first k rows of A.

$$UA = \begin{bmatrix} I_{k\times k} & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} A_k \\ D \end{bmatrix} = \begin{bmatrix} I_{k\times k}A_k \\ BD \end{bmatrix} = \begin{bmatrix} A_k \\ BD \end{bmatrix}.$$

(b) $U^TU = \begin{bmatrix} I_{k\times k} & 0 \\ 0 & B^T \end{bmatrix}\begin{bmatrix} I_{k\times k} & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} I_{k\times k} & 0 \\ 0 & B^TB \end{bmatrix} = \begin{bmatrix} I_{k\times k} & 0 \\ 0 & I_{(n-k)\times(n-k)} \end{bmatrix} = I_n$

6.8 Redo Example 6.12 where Step 1 is kept, but in Step 2 and Step 3, we let $u_i = a_i + \|a_i\|e_1$.

Step 1: $a_1 = (1, -1, 1, -1)$, $u_1 = a_1 - \|a_1\|e_1 = (1, -1, 1, -1) - (2, 0, 0, 0) = (-1, -1, 1, -1)$. (We could also choose $u_1 = a_1 + \|a_1\|e_1$.)

$$Q_1 = H_{u_1} = I_4 - 2\frac{u_1u_1^T}{u_1^Tu_1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}, \qquad A_1 = Q_1A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}.$$

Step 2: $a_2 = (0, 1, 0), u_2 = a_2 + \|a_2\|e_1 = (0, 1, 0) + (1, 0, 0) = (1, 1, 0)$.

$$H_{u_2} = I_3 - 2\frac{u_2u_2^T}{u_2^Tu_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & H_{u_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_2 = Q_2A_1 = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Step 3: $a_3 = (0, -2), u_3 = a_3 + \|a_3\|e_1 = (2, -2)$.

$$H_{u_3} = I_2 - 2\frac{u_3 u_3^T}{u_3^T u_3} = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right], Q_3 = \left[\begin{array}{cc} I_2 & 0 \\ 0 & H_{u_3} \end{array}\right] = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

$$A_3 = Q_3 A_2 = \left[\begin{array}{ccc} 2 & 1 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{array}\right].$$

In the end, $Q = Q_1^T Q_2^T Q_3^T, R = A_3$.