Fall 2017 Math 395 Written Homework 5 Key 100 total. -5 for no stapling

3.13 (5+5+3) Let A be a 3 by 3 matrix, and $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$.

(a) How are the rows or columns changed from A to ΣA?

(b) How are the rows or columns changed from A to $A\Sigma$?

(c) Use block matrix multiplication to explain either (a) or (b).

Solution

(a) Partition A as $A = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix}$.

$$\Sigma A = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} \sigma_1\rho_1 \\ \sigma_2\rho_2 \\ \sigma_3\rho_3 \end{bmatrix}.$$

Left multiplying a diagonal matrix is scaling rows.

(b) Partition A as $A = [c_1, c_2, c_3]$

$$A\Sigma = [c_1, c_2, c_3] \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = [\sigma_1 c_1, \sigma_2 c_2, \sigma_3 c_3]$$

Right multiplying a diagonal matrix is scaling columns.

(c) See (a)(b).

4.7 (6) Find the permutation matrix P so that PV is upper triangular.

$V = \begin{bmatrix} 0 & -2 & 4 & 2 \\ 0 & 0 & -2 & 7 \\ 2 & 1 & 3 & 1 \\ 0 & 0 & 0 & -43/2 \end{bmatrix}$. In order for V to be upper triangular, we need to do original $\rho_3 \to \rho_1$, original $\rho_1 \to \rho_2$, original $\rho_2 \to \rho_3$.

$$P = \text{ do the same operations to the identity matrix } = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4.8 (10) Compute the Cholesky factorization of $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

We first do LU decomposition:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{\rho_1/2+\rho_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{\rho_2\frac{2}{3}+\rho_3} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$$\overset{\text{by the first exercise part (a)}}{=} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{4/3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{4/3} \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{\text{by the first exercise part (a)(b)}}{=} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{2}/2 & \sqrt{3/2} & 0 \\ 0 & -\sqrt{2/3} & 2/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{2}/2 & \sqrt{3/2} & 0 \\ 0 & -\sqrt{2/3} & 2/\sqrt{3} \end{bmatrix}^T$$

4.9 (6) If A has full column rank (columns are independent), then A^TA is invertible. (Hint: prove A^TA is positive definite. A Positive definite matrix is always invertible because its determinant is positive.)

Proof. You can use the hint and write a proof similar to the proof of Theorem 3.5. But we will do it more directly here.

In order to prove A^TA is invertible, we set $A^TAx = 0$. (The goal is to prove $x = 0$.)

Multiply both sides by x^T, we get $x^TA^TAx = 0 \Rightarrow (Ax)^TAx = 0 \Rightarrow \|Ax\|^2 = 0 \Rightarrow Ax = 0$. Since A has full column rank, this implies that $x = 0$. □

4.10 (5+5+5+5) Let $V = \text{span}\{a_1 = (1, -1, 0), a_2 = (0, 1, -1)\}$.

(a) Find the projection matrix onto V. Call it P_1.

(b) Find an orthonormal basis $\{q_1, q_2\}$ for V. Let $Q = [q_1, q_2]$. Check that QQ^T is the same as P_1.

(c) Find $V^\perp$. (This is the same as finding all vectors that are orthogonal to a_1, a_2.)

(d) Find the projection matrix onto $V^\perp$. Call it P_2.

(e) Verify that $P_1 + P_2$ is the identity matrix.

Solution

(a) Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$. $A^TA = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $(A^TA)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

(b) $q_1 = \dfrac{1}{\sqrt{2}}(1, -1, 0)$

$$q_2 = \frac{1}{r_{22}}(a_2 - \langle a_2, q_1\rangle q_1) = \frac{1}{r_{22}}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) = \frac{1}{r_{22}}\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$QQ^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

(c) $V^\perp = \text{span}\{a_3 = (1, 1, 1)\}$.

(d) $P_2 = \dfrac{a_3a_3^T}{a_3^Ta_3} = \dfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(e) $P_1 + P_2 = \dfrac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = I$

5.1 (8) Verify the computation in (5.3) using block matrix multiplication. (U matrix is partitioned into $1 \times m$ blocks (columns), and V is partitioned into $n \times 1$ blocks (rows).)

Solution This is related to chapter 3 #13.

$$[u_1, u_2, \cdots, u_m] \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \sigma_r & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} [v_1, v_2 \cdots, v_n]^T$$

$$= [u_1, u_2, \cdots, u_m] \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \sigma_r & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$= [\sigma_1 u_1, \cdots, \sigma_r u_r, 0, \cdots, 0] \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$

5.2 (6) Let U be $m \times m$, V be $n \times n$, and $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \sigma_r & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{m \times n}$. Let $\hat{U}$ be the $m \times r$ matrix consisting of the first r columns of U, and $\hat{V}$ be the $r \times n$ matrix consisting of first r rows of V. Show that $U\Sigma V^T = \hat{U} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_r \end{bmatrix} \hat{V}^T$ using block matrix multiplication.

Solution We do the following partitions: $U = [\hat{U}, *], \Sigma = \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} \hat{V} \\ ** \end{bmatrix}$, where $\hat{\Sigma}$ is the upper left $r \times r$ submatrix.

$V^T = [\hat{V}^T, **^T]$

$U\Sigma V^T = [\hat{U}, *] \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} [\hat{V}^T, **^T] = \begin{bmatrix} \hat{U}\hat{\Sigma} \\ 0 \end{bmatrix} [\hat{V}^T, **^T] = \hat{U}\hat{\Sigma}\hat{V}^T$

5.3 (9) Compute the singular values of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -2 & 2 \end{bmatrix}$.

Solution $A^T = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \end{bmatrix}$, $A^T A = \begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix}$

$$\begin{vmatrix} 6-r & -2 \\ -2 & 6-r \end{vmatrix} = (r-6)^2 - 4 = (r-6-2)(r-6+2) = (r-8)(r-4)$$

Sinigular values of A are $\sqrt{8}, \sqrt{4}$.

5.4 (3+3+4+4) The singular value decomposition of B is

$$B = \frac{1}{\sqrt{2}} \begin{bmatrix} \sin\frac{\pi}{4} & \sin\frac{2\pi}{4} & \sin\frac{3\pi}{4} \\ \sin\frac{2\pi}{4} & \sin\frac{4\pi}{4} & \sin\frac{6\pi}{4} \\ -\sin\frac{3\pi}{4} & \sin\frac{6\pi}{4} & \sin\frac{9\pi}{4} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0.02 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}^T$$

(a) What is the V matrix in svd of B?

(b) What is the rank of B

(c) Find a basis of $C(B)$.

(d) Find a basis of $N(B)$?

(e) *On which line do the columns of B approximately lie? On which line do the rows of B approximately lie? (Easy problem!)

Solution

(a) $V = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$

(b) 3

(c) $C(B) = \text{span}\{u_1, u_2, u_3\}$ (the three columns of the first matrix).

$C(B)$ is a subspace in $\mathbb{R}^3$, since it has the full dimension (3), $C(B)$ is in fact the whole $\mathbb{R}^3$.

(d) $N(B) = \text{span}\{v_4\} = \text{span}\{[-1, 1, 1, -1]^T\}$.

(e) "line" means that we want a rank-1 approximation of B.

All columns of B will approximately belong to $\text{span}\{u_1\} = \text{span}\{[\sin\frac{\pi}{4}, \sin\frac{2\pi}{4}, -\sin\frac{3\pi}{4}]^T\}$.

All rows of B will approximately belong to $\text{span}\{v_1\} = \text{span}\{[1, 1, 1, 1]^T\}$.

6.1 (8) $A = \begin{bmatrix} 101 & 99 \\ 99 & 101 \end{bmatrix}$ has condition number 100, but it doesn't mean that solution is always sensitive to perturbation, for example, let $b = (2, -2)$ and $\hat{b} = (2.01, -2)$. Let $x, \hat{x}$ be the solutions as $Ax = b$ and $A\hat{x} = \hat{b}$. Compute the number C such that $\frac{\|x - \hat{x}\|_2}{\|x\|_2} = C \frac{\|b - \hat{b}\|_2}{\|b\|_2}$, using a calculator or computer.

Solution

$x = (1, -1)$

$\hat{x} = (1.002525, -1.002475)$

$\frac{\|x - \hat{x}\|_2}{\|x\|_2} = \frac{0.0035357}{\sqrt{2}}, \frac{\|b - \hat{b}\|_2}{\|b\|_2} = \frac{0.01}{2\sqrt{2}}$.

$C = \frac{0.0035357}{\sqrt{2}} \cdot \frac{2\sqrt{2}}{0.01} = 0.70714$. This is a small number.