Fall 2017 Math 395 Written Homework 4 Key 100 total. -5 for no stapling

3.9 (5) Compute x^TAx, where $x = (x_1, x_2, x_3)$, and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ (A is symmetric.)

$$x^TAx = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{12}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3.$$

3.10 (6) Prove that if B is invertible, then B^TB is positive definite.

Proof. First $(B^TB)^T = B^TB$ so B^TB is symmetric.

Second we will use criteria II. For any vector x, $x^TB^TBx = (Bx)^TBx = \|Bx\|_2^2 \geq 0$. When equality holds, $Bx = 0$ which implies $x = 0$ because B is invertible. □

3.11 (10) Check that $C = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ is positive definite using both criteria I and III.

Criteria I: $\det \begin{bmatrix} 2-r & 0 & -1 \\ 0 & 2-r & -1 \\ -1 & -1 & 3-r \end{bmatrix} = (2-r)\begin{vmatrix} 2-r & -1 \\ -1 & 3-r \end{vmatrix} - \begin{vmatrix} 0 & 2-r \\ -1 & -1 \end{vmatrix} = (2-r)[r^2 - 5r + 6 - 1] - (2-r) = (2-r)(r^2 - 5r + 5 - 1) = (2-r)(r^2 - 5r + 4) = (2-r)(r-4)(r-1)$

Eigenvalues are 2,4,1, all positive.

Criteria III: $2 > 0$, $\det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4 > 0$, $\det C = 2\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} - \begin{vmatrix} 0 & 2 \\ -1 & -1 \end{vmatrix} = 2(6-1) - 2 = 8 > 0$

3.12 (6) Check that $D = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ is not positive definite using criteria II.

$x^TDx = x_1^2 + 4x_1x_2 + 3x_2^2 = (x_1 + 2x_2)^2 - x_2^2$.

To prove it is not PD, we just need to find a vector x that makes x^TDx negative. We can pick, for example $x = (2, -1)$. Then $x^TDx = -1 < 0$

4.1 (3+4+5) How many **exact** flops are needed for the following computations?

(a) $\langle x, y\rangle$, where x, y are vectors in $\mathbb{R}^9$.

(b) ABC, where A, B, C are 5×5 matrices.

(c) Ux, where U is $n \times n$ upper triangular matrix and x is a vector in $\mathbb{R}^n$.

Solution

(a) 2*9-1=17

(b) Matrix multiplication twice. (2*5-1)*5*5*2= 450

(c) $\sum_{i=1}^{n}(2i-1) = 2\sum_{i=1}^{n} i - n = n(n+1) - n = n^2$

Only two students got this problem correctly. First of all, this is not asking for the operation counts of solving $Ux = y$. This is asking how many operations it takes to do matrix-vector product with a special U. To compute the ith coordinate of Ux, it is the same as doing inner product of two vectors of size i, which is $2i - 1$. To compute all coordinates, it is $\sum_{i=1}^{n}(2i-1)$.

4.2 (4+4+5) Given $n \times n$ matrices A, B, C, how many flops are needed for the following computations? Answer in terms of the order $n, n^2, n^3, \cdots$.

(a) ABC

(b) $A^{-1}B$

(c) $A + B$

(d) LU, where L is $n \times n$ lower triangular, U is $n \times n$ upper triangular

Solution

(a) $(2n-1)n^2 * 2 = O(n^3)$

(b) This is to solve $AX = B$: $O(n^3)$

(c) n^2

(d) This ended up being an extra credit problem.

Suppose the columns of U are $u_1, u_2, \cdots, u_n$. Computing LU is computing Lu_i.

Lu_1: $\sum_{j=1}^{n}(2*1-1) = n$

Lu_2: $\sum_{j=1}^{2}(2j-1) + \sum_{j=3}^{n}(2*2-1)$

$\cdots$

Lu_i: $\sum_{j=1}^{i}(2j-1) + \sum_{j=i+1}^{n}(2i-1) = i^2 + (n-i)(2i-1) = -i^2 + (2n+1)i - n$

Sum them up: $\sum_{i=1}^{n} -i^2+(2n+1)i-n = -\frac{n(n+1)(2n+1)}{6} + (2n+1)\frac{n(n+1)}{2} - n^2 = \frac{n(n+1)(2n+1)}{3} - n^2 = \frac{n}{3}\left[(n+1)(2n+1) - 3n\right] = \frac{n}{3}[2n^2+1] = O(n^3)$

4.3 (6) Consider the linear system $\left[\begin{array}{cc:c} 2 & 2\times 10^{20} & 2\times 10^{20} \\ 1 & 1 & 2 \end{array}\right]$. Find its 'solution' using Gaussian elimination, in a machine where numbers are in standard IEEE double-precision format.

This is very similar to the example in the notes.

$$\left[\begin{array}{cc:c} 2 & 2\times 10^{20} & 2\times 10^{20} \\ 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc:c} 2 & 2\times 10^{20} & 2\times 10^{20} \\ 0 & -10^{20}+1 & -10^{20}+2 \end{array}\right] \xrightarrow{\rho 2/(1-10^{20})} \left[\begin{array}{cc:c} 2 & 2\times 10^{20} & 2\times 10^{20} \\ 0 & 1 & \boxed{1} \end{array}\right]$$

$x_2 = 1$

$2x_1 + 2\times 10^{20} = 2\times 10^{20} \Rightarrow x_1 = 0$

4.4 (7+5+5) Solve $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} x = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ $(Ax = b)$ by

(a) First find the LU factorization of A.

(b) Second solve $Ly = b$.

(c) Third solve $Ux = y$.

Solution

(a) $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \xrightarrow[\rho1(-2)+\rho3]{\rho1\frac{3}{2}+\rho2} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow{\rho2(3)+\rho3} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{3}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}.$

(b) $y_1 = 2$
$-1.5 * 2 + y_2 = 2 \Longrightarrow y_2 = 5$
$2 * 2 - 3 * 5 + y_3 = 3 \Longrightarrow y_3 = 14$
$\boxed{y = (2, 5, 14)}$

(c) $x_3 = 14/7 = 2$
$x_2 + 3 * 2 = 5 \Longrightarrow x_2 = -1$
$2x_1 + 6(-1) + 2 * 2 = 2 \Longrightarrow x_1 = 2$
$\boxed{x = (2, -1, 2)}$.

4.5 (10) Given $\begin{bmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, solve the system

$$\begin{array}{llll} 3x_1 & -6x_2 & -3x_3 & = -3 \\ 2x_1 & & +6x_3 & = -22 \\ -4x_1 & +7x_2 & +4x_3 & = 3 \end{array} .$$

By the way, this is still LU factorization. L does not need to have 1's on the diagonal. It only needs to be lower triangular.

Solve $Ly = b$
$y_1 = -3/3 = -1$
$2 * (-1) + 4y_2 = -22 \Longrightarrow y_2 = -5$
$-4(-1) - 1(-5) + 2y_3 = 3 \Longrightarrow y_3 = -3$
$y = (-1, -5, -3)$

Solve $Ux = y$
$x_3 = -3$
$x_2 + 2(-3) = -5 \Longrightarrow x_2 = 1$
$x_1 - 2 - (-3) = -1 \Longrightarrow x_1 = -2$
$\boxed{x = (-2, 1, -3)}$.

4.6 (15) Solve the system in Problem 4.5 using scaled partial pivoting strategy.

$$\left[\begin{array}{ccccc} 3 & -6 & -3 & \vdots & -3 \\ 2 & 0 & 6 & \vdots & -22 \\ -4 & 7 & 4 & \vdots & 3 \end{array}\right] \qquad s_1 = 6, s_2 = 6, s_3 = 7$$

Stage 1: compare 3/6, 2/6, 4/7. We pick $\boxed{\text{row 3}}$: $\left[\begin{array}{ccccc} 0 & -3/4 & 0 & \vdots & -3/4 \\ 0 & 7/2 & 8 & \vdots & -41/2 \\ -4 & 7 & 4 & \vdots & 3 \end{array}\right]$

Several students stopped here. Yes, I know it is very tempting to say $x_2 = 1$ from first row already, but we still need to do stage 2 according to scaled partial pivoting. It is annoying by hand, but computers don't mind. It is true that this example, as well as the example in the notes, does not show the advantage of the scaled partial pivoting.

Stage 2: compare 3/24 (row1), 7/12 (row 2). We pick $\boxed{\text{row 2}}$: $\left[\begin{array}{ccccc} 0 & 0 & 12/7 & \vdots & -36/7 \\ 0 & 7/2 & 8 & \vdots & -41/2 \\ -4 & 7 & 4 & \vdots & 3 \end{array}\right]$

$x_3 = -3$

$7x_2/2 - 24 = -41/2 \Longrightarrow x_2 = 1$

$-4x_1 + 7 - 12 = 3 \Longrightarrow x_1 = -2$