2017 Fall Math 395 Midterm Key

1. Use Newton's method to approximate $\sqrt{2}$ by letting $f(x) = x^2 - 2$.

   (a) Pick $x_0 = 1$ to be the initial point, and compute the next two iterates.

   (b) (Easy bonus question) Draw a graph for (a) to show the first iteration and the tangent line.

   (c) Use the fixed point convergence theorem to justify that the Newton's method with $x_0 = 1$ converges to $\sqrt{2}$.

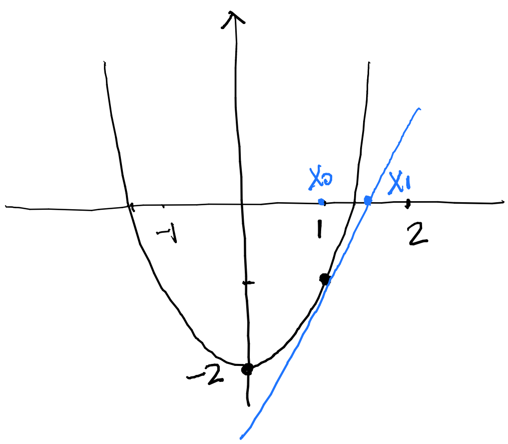
(a) $f(x) = x^2 - 2$, $f'(x) = 2x$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - 2}{2x_k} = x_k - \frac{1}{2}x_k + \frac{1}{x_k} = \frac{x_k}{2} + \frac{1}{x_k}$$

$$x_1 = \frac{1}{2} + 1 = \boxed{\frac{3}{2}}$$

$$x_2 = \frac{3}{4} + \frac{2}{3} = \frac{9+8}{12} = \boxed{\frac{17}{12}}$$

(b)



(c) $x_{k+1} = \varphi(x_k)$, $\varphi(x) = \frac{x}{2} + \frac{1}{x}$

$$\varphi'(x) = \frac{1}{2} - \frac{1}{x^2}$$

$$\left|\frac{1}{2} - \frac{1}{x^2}\right| < 1$$

$$\Rightarrow -1 < \frac{1}{2} - \frac{1}{x^2} < 1$$

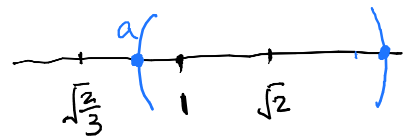
$$\Rightarrow -1 < \frac{1}{2} - \frac{1}{x^2}$$

$$\Rightarrow \frac{1}{x^2} < \frac{3}{2}$$

$$\Rightarrow x^2 > \frac{2}{3}$$

$$\Rightarrow x > \sqrt{\tfrac{2}{3}} \text{ or } \underbrace{x < -\sqrt{\tfrac{2}{3}}}$$

eliminated since we are focusing on $\sqrt{2}$



we can pick any a between $\sqrt{\frac{2}{3}}$ and 1.

Pick interval $(a, 2\sqrt{2} - a)$.

(1). On this interval $|\varphi'(x)| < 1$

(2) $1 \in (a, 2\sqrt{2} - a)$

(3) centered at $\sqrt{2}$.

So $x_0 = 1$ will give convergence.

2. How many **exact** flops are needed for the multiplication Lx, where L is a 100×100 lower triangular matrix and x is a vector in $\mathbb{R}^{100}$, but we know that the last 50 coordinates of x is 0. (Use the summation formula on the front page.)

$$\begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & * & \ddots & \\ & & & a_{100,100} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{50} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_{100} \end{bmatrix}$$

For $1 \le i \le 50$

Compute y_i needs $(2i-1)$

For $51 \le i \le 100$

compute y_i needs $2 \times 50 - 1$

$$\text{Total} = \sum_{i=1}^{50}(2i-1) + \sum_{i=51}^{100} 99 = 2 \cdot \frac{50 \times 51}{2} - 50 + 99 \times 50$$

$$= 50 \times 51 - 50 + 99 \times 50 = 50(51 - 1 + 99) = \boxed{50 \times 149}$$

3. Add in binary (No rounding involved): $0.110_2 + 1.101_2$

$$\begin{array}{r} 0.110 \\ 1.101 \\ \hline 10.011 \end{array}$$

10.011_2

4. The real number 0.1 is expressed as $1.\overline{1001}_2 \times 2^{-4}$ (therefore 0.2 is $1.\overline{1001}_2 \times 2^{-3}$ since it is twice of 0.1). With the following guidance, add 0.1 and 0.2 in a binary machine that retains 3 digits after the binary point. Use rounding to the nearest.

   (a) Compute the floating point number of 0.1 and 0.2 in this machine. Call them $fl(0.1)$ and $fl(0.2)$ respectively. (Use the midpoint method.)

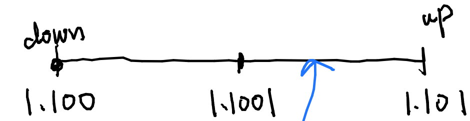
   (b) Add up $fl(0.1)$ and $fl(0.2)$ in this machine. Remember that you have to first make both number have the same power (rounding involved), second do a binary addition, and last round again.

(a) $0.1 = 1.\overline{1001}_2 \times 2^{-4}$

$fl(0.1) = 1.101_2 \times 2^{-4}$

$fl(0.2) = 1.101_2 \times 2^{-3}$

For $1.\overline{1001}_2$



(b) $1.101_2 \times 2^{-4} = 0.1101_2 \times 2^{-3} \xrightarrow{\text{Round}} 0.110_2 \times 2^{-3}$

$$\begin{array}{r} 0.110 \\ 1.101 \\ \hline 10.011 \end{array}$$

$10.011_2 \times 2^{-3} = 1.0011_2 \times 2^{-2} \xrightarrow{\text{Round}} \boxed{1.010_2 \times 2^{-2}}$

5. Given $x = (1, -2)$, find

   (a) $\|x\|_2$, $\|x\|_1$, and $\|x\|_\infty$.

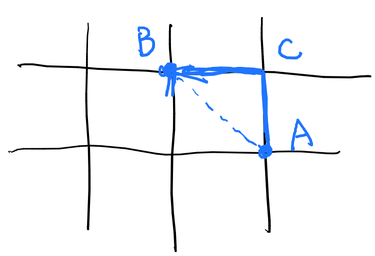
   (b) Find a vector whose Euclidean norm is 1, and is parallel to x.

   (c) (Bonus question) In a typical downtown neighborhood where streets are in grids, if one wants to compute the driving distance from one point to another, which norm from (a) do you think suits the best? Why?

(a) $\|x\|_2 = \sqrt{5}$ $\quad \|x\|_1 = 3$ $\quad \|x\|_\infty = 2$

(b) $\dfrac{x}{\|x\|_2} = \dfrac{1}{\sqrt{5}}(1, -2)$

(c) $\|x\|_1$



Driving distance

$= |AC| + |BC|$

$= \|\vec{AB}\|_1$

6. $E = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ symmetric to begin with

(a) Using two methods to check whether E is positive definite. Extra credit for using a third method.

(b) Find the operator norm and condition number of E.

(a) Method 1: $3>0$, $\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9-4 = 5 > 0$ $\boxed{\text{Yes}}$

Method 2: $\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 2^2 = (\lambda-3-2)(\lambda-3+2) = (\lambda-5)(\lambda-1)$

$\lambda = 5, 1$, both positive

Method 3: $x^T E x = 3x_1^2 + 4x_1x_2 + 3x_2^2 = \left(\sqrt{3}x_1 + \frac{2}{\sqrt{3}}x_2\right)^2 + \frac{5}{3}x_2^2 \geq 0$

Equality holds only when $x_1 = x_2 = 0$

(b) $\|E\|_2$ = biggest singular value = biggest eigenvalue if A is PD = 5

$\kappa(E) = \frac{5}{1} = 5$

7. Given two (column) vectors a_1, a_2. Write down the formula to produce orthonormal vectors q_1, q_2 such that $\text{span}\{q_1, q_2\} = \text{span}\{a_1, a_2\}$.

$$q_1 = \frac{a_1}{\|a_1\|_2}$$

$$q_2 = \frac{a_2 - \langle a_1, q_1 \rangle q_1}{\|a_2 - \langle a_1, q_1 \rangle q_1\|_2}$$

8. Compute the LU factorization of $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$. Can you further do Cholesky factorization of it? 1 3 2

$$\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \xrightarrow[\rho_1(-2)+\rho_3]{\rho_1 \frac{3}{2} + \rho_2} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 6 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow{\rho_2 \cdot 3 + \rho_3} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 16 \end{bmatrix}$$

$$\Uparrow \qquad\qquad \Uparrow$$

$$\begin{bmatrix} 1 & & \\ \frac{3}{2} & 1 & \\ -2 & & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & & \\ & 1 & \\ & 2 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & \\ -\frac{3}{2} & 1 & \\ 2 & -2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 16 \end{bmatrix}$$

9. Given $\begin{bmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{bmatrix} = \underset{L}{\begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$, solve the system $\begin{array}{llll} 3x_1 & -6x_2 & -3x_3 & = -3 \\ 2x_1 & & +6x_3 & = -22 \\ -4x_1 & +7x_2 & +4x_3 & = 3 \end{array}$.

$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{bmatrix} y = \begin{bmatrix} -3 \\ -22 \\ 3 \end{bmatrix} \Rightarrow \begin{array}{l} y_1 = -1 \\ -2+4y_2 = -22 \Rightarrow y_2 = -5 \\ 4+5+2y_3 = 3 \Rightarrow y_3 = -3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} -1 \\ -5 \\ -3 \end{bmatrix} \Rightarrow \begin{array}{l} x_3 = -3 \\ x_2 - 6 = -5 \Rightarrow x_2 = 1 \\ x_1 - 2 + 3 = -1 \Rightarrow x_1 = -2 \end{array}$$

$$x = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

10. To find the orthogonal complement of span$\{(1, 2, 3), (1, 2, -3)\}$, it is the same as finding the null space of what matrix? No need to compute the null space.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & -3 \end{bmatrix}$$

11. Given $x = (3, 4)$, find an orthonormal matrix Q such that $Qx = 5e_1$.

$$u = x - 5e_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$Q = H_u = I - 2\frac{uu^T}{u^Tu}$$ we can use $u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$= I - 2\frac{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\begin{bmatrix} -1 & 2 \end{bmatrix}}{5}$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{5}\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

12. Let U be an $n \times n$ matrix. It is in the partitioned form as $U = \begin{bmatrix} I_k & 0 \\ 0 & B \end{bmatrix}$, where I_k is the $k \times k$ identity matrix. Show that the first k rows of UA are the same as the first k rows of A using block matrix multiplication.

$A = \begin{bmatrix} A_k \\ \hline C \end{bmatrix}$ $\quad A_k$ = first k rows of A

$$UA = \begin{bmatrix} I_k & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} A_k \\ C \end{bmatrix} = \begin{bmatrix} I_k A_k + 0 \cdot C \\ 0 \cdot A_k + BC \end{bmatrix} = \begin{bmatrix} A_k \\ BC \end{bmatrix}$$

13. The singular value decomposition of B is

$$B = \frac{1}{3} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \end{bmatrix} * \overset{\frac{1}{2}}{\frac{1}{4}} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}^T$$

(a) There is a mistake above, fix it. (one of them is not an orthonormal matrix.)

(b) Find the reduced SVD of B.

(c) What is the rank of B?

(d) Find a basis of $N(B)$?

(e) Compute the rank-2 approximation of B.

(a) $\frac{1}{4} \to \frac{1}{2}$

(b) $B = \frac{1}{3} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 1 & \\ & & 0.01 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}^T$

(c) 3

(d) $\{v_4\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

(e) $B_2 = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}^T$

$= \frac{1}{6} \begin{bmatrix} 2 & 2 \\ 4 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 4 & 4 & 0 & 0 \\ 5 & 5 & 3 & 3 \\ -2 & -2 & -6 & -6 \end{bmatrix}$

14. Given the reduced QR of A: $A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

(a) Find the projection matrix onto the column space of A.

(b) Let $b = (2, -1, 0, 1)$, find the least square solution of $Ax = b$ using the algorithm on the front cover.

(c) (Easy bonus question) Find the projection of b onto the column space of A using two different methods. (One uses (a), the other uses(b).)

(d) Find the full QR decomposition of A using the reduced one. (Hint: you can get your answer fast from Problem 13.)

(a) $A = QR$

$$P = QQ^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

(b) $$Q^Tb = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2+1-1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solve $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} x_2 = 1 \\ 2x_1 + 1 = 1 \end{matrix} \Rightarrow x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) Method 1: $$p = Pb = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Method 2: $$p = Ax = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(d) In the SVD of B in #13, there is already the 2 other columns that need to be filled in for Q

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$