Fall 2017 Math 395 Written Homework 10 Key

10.1 6 If we want to write $f(x) = f(x_1, x_2) = x_1^2 + 2x_2^2 + 2x_1x_2 - 2x_1 + 5x_2 + 3$ as the standard quadratic form as $f(x) = \frac{1}{2}x^T Ax - b^T x + c$, what is A, b and c?

Solution $A = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, c = 3$

10.2 7 Given $n \times 1$ vectors x, y and $n \times n$ matrix A, show that $x^T A^T y = y^T Ax$.

Solution Method 1: LHS $= (Ax)^T y = \langle Ax, y\rangle$. RHS $= y^T(Ax) = \langle y, Ax\rangle$. So they are equal.

Method 2: $x^T A^T y$ is a 1 by 1 matrix, so it is always equal to its transpose.

$x^T A^T y = (x^T A^T y)^T = y^T (A^T)^T (x^T)^T = y^T Ax$.

10.3 10 Given a quadratic form $f(x) = \frac{1}{2}x^T Ax - b^T x + c$. Let x_0 be such that $Ax_0 = b$. Show that $f(y) - f(x_0) = \frac{1}{2}(x_0 - y)^T A(x_0 - y)$. Remember that A is symmetric in a quadratic form. You may find Exercise 2 useful.

Solution:

LHS $= f(y) - f(x_0) = \frac{1}{2}y^T Ay - b^T y + c - (\frac{1}{2}x_0^T Ax_0 - b^T x_0 + c) = \frac{1}{2}y^T Ay - b^T y - \frac{1}{2}x_0^T Ax_0 + b^T x_0 = \frac{1}{2}y^T Ay - b^T y - \frac{1}{2}x_0^T b + b^T x_0 = \frac{1}{2}y^T Ay - b^T y + \frac{1}{2}x_0^T b$.

RHS $= \frac{1}{2}(x_0 - y)^T A(x_0 - y) = \frac{1}{2}(x_0^T - y^T)(Ax_0 - Ay) = \frac{1}{2}(x_0^T Ax_0 - x_0^T Ay - y^T Ax_0 + y^T Ay) = \frac{1}{2}(x_0^T Ax_0 - 2y^T Ax_0 + y^T Ay) = \frac{1}{2}(x_0^T b - 2y^T b + y^T Ay)$.

So RHS = LHS.

10.4 5+5+5+10 Let $A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$

(a) Verify that A is positive definite.

(b) Compute the A-norm of $x = (1, 1)$.

(c) Find two vectors that are A-orthogonal.

(d) Draw the ellipse $x^T Ax = 1$. Indicate the length of major/minor axes.

Solution

(a) $1 > 0$, $\det(A) = 1 - 1/4 > 0$

(b) $\|x\|_A^2 = x^T Ax = x_1^2 + x_2^2 + x_1x_2 = 3$.

$\|x\|_A = \sqrt{3}$.

(c) If $x = (x_1, x_2), y = (y_1, y_2$, then $x^T Ay = x_1y_1 + x_2y_2 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1$

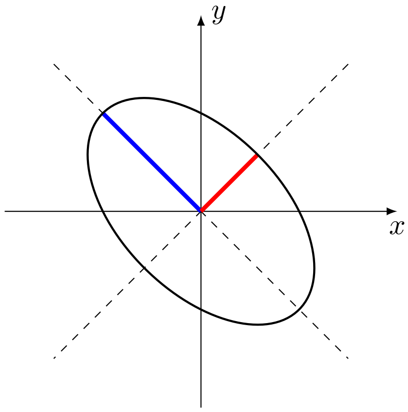
Need to find two vectors such that $x_1y_1 + x_2y_2 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1 = 0$.

We can let $x = (2, 2)$ first, then equation becomes $2y_1 + 2y_2 + y_2 + y_1 = 0$. $y = (1, -1)$ will do.

(d) $\det \begin{bmatrix} 1-r & 1/2 \\ 1/2 & 1-r \end{bmatrix} = (1-r)^2 - 1/4 = (1 - r + 1/2)(1 - r - 1/2) = (3/2 - r)(1/2 - r)$

$r_1 = 3/2, \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} x = 0, \qquad u_1 = [1, 1]^T$ ——minor axis direction

$r_2 = 1/2,\ \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} x = 0, \qquad u_2 = [1,-1]^T$ ——major axis direction



length of blue line is $\sqrt{2}$ and length of red line is $\sqrt{2/3}$.
length of major axis is $2\sqrt{2}$ and length of minor axis is $2\sqrt{2/3}$.

10.5 8 Let $A = \begin{bmatrix} 10 & 3 \\ 1 & 20 \end{bmatrix}$, as in Example 10.1. Compute the spectral radius of $I - D^{-1}A$ to justify the fast convergence in that Example.

$$I - D^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/10 & 0 \\ 0 & 1/20 \end{bmatrix} \begin{bmatrix} 10 & 3 \\ 1 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3/10 \\ 1/20 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.3 \\ -0.05 & 0 \end{bmatrix}$$

Characteristic polynomial: $\lambda^2 - \dfrac{3}{200}$. Eigenvalue: $\pm\dfrac{\sqrt{6}}{20}$

spectral radius $= \dfrac{\sqrt{6}}{20} \approx 0.1225$. small

10.6 8 Let $B = \begin{bmatrix} 10 & 1 \\ 10 & 10 \end{bmatrix}$, as in Example 10.2. Compute the spectral radius of $I - L^{-1}B$ to justify the fast convergence in that Example.

$$L^{-1} = \begin{bmatrix} 10 & 0 \\ 10 & 10 \end{bmatrix}^{-1} = \frac{1}{100} \begin{bmatrix} 10 & 0 \\ -10 & 10 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$I - L^{-1}B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{10} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 10 & 1 \\ 10 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0.1 \\ 0 & 0.9 \end{bmatrix} = \begin{bmatrix} 0 & -0.1 \\ 0 & 0.1 \end{bmatrix}$$

Characteristic polynomial: $\lambda(\lambda - 0.1)$. Eigenvalue: $0, 0.1$

spectral radius $= 0.1$. small

10.7 10 Compute the first two iterations $x^{(1)}, x^{(2)}$ of Steepest Gradient Descent applied to the problem $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} x = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$ with the initial value $x^{(0)} = (-2,-2)$. (probably with a calculator)

Solution

$x^{(0)} = (-2,-2)$

$r^{(0)} = b - Ax^{(0)} = (12, 8)$

$t^{(0)} = \dfrac{(r^{(0)})^T r^{(0)}}{(r^{(0)})^T A r^{(0)}} = \dfrac{12^2 + 8^2}{312^2 + 68^2 + 4 * 12 * 8} \approx 0.1733$

$x^{(1)} = x^{(0)} + t^{(0)} r^{(0)} = (0.08, -0.6133)$

$r^{(1)} = b - Ax^{(1)} = (2.9867, -4.48)$

$t^{(1)} = \dfrac{(r^{(1)})^T r^{(1)}}{(r^{(1)})^T A r^{(1)}} = 0.3095$

$x^{(2)} = x^{(1)} + t^{(1)} r^{(1)} = (1.0044, -2)$

10.8 10 Compute the first two iterations $x^{(1)}, x^{(2)}$ of Conjugatet Gradient Descent applied to the problem $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} x = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$ with the initial value $x^{(0)} = (-2,-2)$.

Solution $r^{(0)} = b - Ax^{(0)} = (12, 8), p^{(0)} = r^{(0)} = (12, 8)$

(i) $Ap^{(0)} = (52, 72)$

(ii) $t^{(0)} = \dfrac{(p^{(0)})^T r^{(0)}}{(p^{(0)})^T Ap^{(0)}} = 0.1733$

(iii) $x^{(1)} = x^{(0)} + t^{(0)}p^{(0)} = (0.08, -0.6133)$

(iv) $r^{(1)} = r^{(0)} - t^{(0)}Ap^{(0)} = (2.9867, -4.48)$

(v) $\alpha^{(0)} = -\dfrac{(r^{(1)})^T Ap^{(0)}}{(p^{(0)})^T Ap^{(0)}} = 0.1394$

(vi) $p^{(1)} = r^{(1)} + \alpha^{(0)}p^{(0)} = (4.6592, -3.3650)$

(i) $Ap^{(1)} = (7.2476, -10.8715)$

(ii) $t^{(1)} = \dfrac{(p^{(1)})^T r^{(1)}}{(p^{(1)})^T Ap^{(1)}} = 0.4121$

(iii) $x^{(2)} = x^{(1)} + t^{(1)}p^{(1)} = (2, -2)$

10.9 6 Prove that $P^{(1)}$ is A-orthogonal to $p^{(0)}$ in the conjugate gradient descent algorithm.

We need to show that $(p^{(1)})^T Ap^{(0)} = 0$

$(p^{(1)})^T Ap^{(0)} = (r^{(1)} + \alpha^{(0)}p^{(0)})^T Ap^{(0)} = (r^{(1)})^T Ap^{(0)} + \alpha^{(0)}(p^{(0)})^T Ap^{(0)}$

$= (r^{(1)})^T Ap^{(0)} - \dfrac{(r^{(1)})^T Ap^{(0)}}{(p^{(0)})^T Ap^{(0)}}(p^{(0)})^T Ap^{(0)} = 0$

10.10 10 Compute the exact number of flops needed in one iteration of the conjugate gradient descent algorithm.

(i) $n(2n-1) = 2n^2 - n$

(ii) $2n - 1 + 2n - 1 + n = 5n - 2$

(iii) $2n$

(iv) $2n$

(v) $5n - 2$ as well

(vi) $2n$

$2n^2 - n + 5n - 2 + 2n + 5n - 2 + 2n = 2n^2 + 15n - 4$