

**Fall 2017 Math 395 Written Homework 3 Key 100 total. -5 for no stapling**  
Chapter 3

1. (8+4+8) Given  $x = (1, -2, 3), y = (-2, 0, 2)$ , find
- (a) Find  $x^T y, \|x\|_2, \|x\|_1, \|x\|_\infty$
  - (b) Find a vector whose Euclidean norm is 1, and is parallel to  $y$ .
  - (c) Find a vector that is orthogonal to both  $x$  and  $y$ . This is the same as finding the null space of what matrix?

**Solution:**

(a)  $x^T y = -2 + 6 = 4$  Both  $x, y$  are column vectors if in ( )

$$\|x\|_2 = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$\|x\|_1 = 1 + 2 + 3 = 6$$

$$\|x\|_\infty = 3$$

(b)  $y/\|y\|_2 = \frac{1}{\sqrt{4+4}}(-2, 0, 2) = \frac{1}{\sqrt{2}}(-1, 0, 1)$

(c) Let  $z = (z_1, z_2, z_3)$  be the vector. We need to solve  $x^T z = 0, y^T z = 0$ , which becomes

$$\begin{matrix} z_1 - 2z_2 + 3z_3 = 0 \\ -2z_1 + 2z_3 = 0 \end{matrix}.$$

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & -4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$z_1 - z_3 = 0, z_2 - 2z_3 = 0. \quad z_3 \text{ is free, we can pick } z_3 = 1, \text{ then } z_2 = 2, z_1 = 1.$$

A possible solution is  $\boxed{(1, 2, 1)}$ .

The same as finding the null space of matrix  $A$ .

2. (8) Find a 3 by 3 orthonormal matrix, then find its inverse. (Don't pick the canonical basis.)

$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$  is an example. It is easy to verify that all columns are orthogonal to each other and every column has unit norm.

$$U^{-1} = U^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

3. (6) If  $Au = ru$ , simplify  $(3A^3 - 4A + 2I)u$  so that  $A$  disappears.

$$(3A^3 - 4A + 2I)u = 3AAAu - 4Au + 2u = 3rAAu - 4ru + 2u = 3r^2Au - 4ru + 2u = 3r^3u - 4ru + 2u = (3r^3 - 4r + 2)u$$

4. (8+3+3)  $B = \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}$ .

- (a) Find the eigenvalues and corresponding eigenvectors of  $B$ .
- (b) Is this matrix positive semidefinite?
- (c) Find its operator norm.

**Solution:**

(a)  $\det \begin{bmatrix} 1 - \lambda & 4 \\ 4 & 7 - \lambda \end{bmatrix} = \lambda^2 - 8\lambda + 7 - 16 = \lambda^2 - 8\lambda - 9 = (\lambda - 9)(\lambda + 1).$

$$\lambda_1 = 9, \text{ solve } \begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} x = 0, u_1 = [1, 2]^T.$$

$$\lambda_2 = -1, \text{ solve } \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} x = 0, u_2 = [-2, 1]^T.$$

(b) No. one eigenvalue is less than 0.

(c) Singular values are 9,1. Operator norm is 9.

5. (15) Find the eigenvalues and corresponding eigenvectors of  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ . Determine if it is diagonalizable.

$$\det \begin{bmatrix} 1-r & 0 & 0 \\ 0 & 2-r & 3 \\ 1 & 0 & 2-r \end{bmatrix} = (1-r) \begin{vmatrix} 2-r & 3 \\ 0 & 2-r \end{vmatrix} = (1-r)(2-r)^2$$

$$r_1 = 1, \text{ Solve } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} x = 0, \text{ pick } x_3 = 1$$

$$u_1 = [-1, -3, 1]^T.$$

$$r_2 = 2, \text{ Solve } \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix} x = 0, x_1 = x_3 = 0$$

$u_2 = [0, 1, 0]^T$  Many students think the solution is the zero vector!! But in fact  $x_2$  is completely free and can be any value.

Only 2 INDEPENDENT eigenvectors. This is not diagonalizable.

6. (9) Let  $u_1, u_2, u_3$  be the 3 columns of the orthonormal matrix you found in Exercise 2, and let  $x = (1, 2, 3)$ . Find the coefficients  $c_1, c_2, c_3$  in  $x = c_1 u_1 + c_2 u_2 + c_3 u_3$ .

$$c_1 = x^T u_1 = 1$$

$$c_2 = x^T u_2 = 2/\sqrt{2} - 3/\sqrt{2} = -1/\sqrt{2}$$

$$c_3 = x^T u_3 = 2/\sqrt{2} + 3/\sqrt{2} = 5/\sqrt{2}$$

Please read this problem very carefully because most students got it wrong.

7. (6) If  $A$  is a 2 by 2 matrix that rotates a vector by  $\pi/5$  counterclockwise, show that  $\|A\|_2 = 1$  directly from its definition.

Rotation does not change Euclidean norm, i.e.  $\|Ax\|_2 = \|x\|_2$

$$\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{x \neq 0} \frac{\|x\|_2}{\|x\|_2} = \sup_{x \neq 0} 1 = 1$$

8. (12) Find the operator norm of  $\begin{bmatrix} 1/\sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & \sqrt{2} \\ 0 & 0 \end{bmatrix}$ .

We need to find the biggest singular value.

$$A = \begin{bmatrix} 1/\sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & \sqrt{2} \\ 0 & 0 \end{bmatrix}. A^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \end{bmatrix}, A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 1-r & 2 \\ 2 & 4-r \end{vmatrix} = r^2 - 5r + 4 - 4 = r(r-5)$$

Eigenvalues of  $A^T A$  are 5,0.

Sinigular values of  $A$  are  $\sqrt{5}, 0$ . Operator norm is  $\boxed{\sqrt{5}}$ .