Fall 2017 Math 395 Written Homework 7 Key 100 total. -5 for no stapling

7.1 7+8 Let $f(x) = 3/2 - x/2, g(x) = \frac{1}{x}$. Both functions belong to $C[1, 2]$. Find

(a) $\|f - g\|_2$

(b) $\|f - g\|_\infty$ (use derivative to find max)

Solution

(a) $\|f-g\|_2^2 = \int_1^2 (\frac{3}{2} - \frac{x}{2} - \frac{1}{x})^2 dx = \int_1^2 \frac{9}{4} + \frac{x^2}{4} + \frac{1}{x^2} - \frac{3x}{2} - \frac{3}{x} + 1 dx = \frac{13}{4} + \frac{x^3}{12}\Big|_1^2 - \frac{1}{x}\Big|_1^2 - \frac{3x^2}{4}\Big|_1^2 - 3\ln x|_1^2$
$= \frac{13}{4} + \frac{8-1}{12} + \frac{1}{2} - \frac{3}{4}(4-1) - 3\ln 2 = 25/12 - 3\ln 2$

$\|f - g\|_2 = \sqrt{25/12 - 3\ln 2} \approx 0.062$

(b) $h(x) = f(x) - g(x) = \frac{3}{2} - \frac{x}{2} - \frac{1}{x}$.

$h(1) = 0, h(2) = 0$. Since $\frac{1}{x}$ is convex on $(0, \infty)$, the line f is above g, so $|h(x)| = h(x)$

$h'(x) = -\frac{1}{2} + x^{-2} = 0 \Rightarrow x = \sqrt{2} \quad (x = -\sqrt{2}$ is discarded)

$\|f - g\|_\infty = \max_{x\in[1,2]} |h(x)| = \max_{x\in[1,2]} h(x) = h(\sqrt{2}) = 3/2 - \sqrt{2} \approx 0.086$

7.2 6+6 Find the polynomial interpolant of (1,7/4), (2,3/2), (3.5,0), and (5,3/4) by

(a) Vandermonde method (feel free to use python)

(b) Lagrange method (handwritten, don't simplify)

Solution

(a) Solve $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7/4 \\ 1 & 2 & 4 & 8 & 3/2 \\ 1 & 3.5 & 3.5^2 & 3.5^3 & 0 \\ 1 & 5 & 25 & 125 & 3/4 \end{array}\right]$, we get that the coefficients are $(0, 3.15, -1.6, 0.2)$

$p(x) = 3.15x - 1.6x^2 + 0.2x^3$.

(b) $\varphi_0(x) = \frac{(x-2)(x-3.5)(x-5)}{(1-2)(1-3.5)(1-5)}, \varphi_1(x) = \frac{(x-1)(x-3.5)(x-5)}{(2-1)(2-3.5)(2-5)}, \varphi_3(x) = \frac{(x-1)(x-2)(x-3.5)}{(5-1)(5-2)(5-3.5)}$

$p(x) = \frac{7}{4}\varphi_0(x) + \frac{3}{2}\varphi_1(x) + \frac{3}{4}\varphi_3(x)$

7.3 4+4 Use python to compute the condition number of the following two Vandermonde matrices, and comment on the results briefly.

(a) generated by $0, 0.2, 0.4, 0.6, 0.8$.

(b) generated by $0, 0.2, 0.22, 0.6, 0.8$.

Solution

(a) $V_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0.2 & 0.2^2 & 0.2^3 & 0.2^4 \\ 1 & 0.4 & 0.4^2 & 0.4^3 & 0.4^4 \\ 1 & 0.6 & 0.6^2 & 0.6^3 & 0.6^4 \\ 1 & 0.8 & 0.8^2 & 0.8^3 & 0.8^4 \end{bmatrix}$. $\kappa(V_1) = 1140.27$

You can use numpy.linalg.cond directly, you can also use numpy.linalg.svd to find all the singular values of V_1 and get the ratio of biggest over smallest.

(b) $V_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0.2 & 0.2^2 & 0.2^3 & 0.2^4 \\ 1 & 0.22 & 0.22^2 & 0.22^3 & 0.22^4 \\ 1 & 0.6 & 0.6^2 & 0.6^3 & 0.6^4 \\ 1 & 0.8 & 0.8^2 & 0.8^3 & 0.8^4 \end{bmatrix}$. $\kappa(V_2) = 7531.24$

Vandermonde matrices are very ill-conditioned in general. It is the most ill conditioned when there two x values are close, like 0.2 and 0.22. Ill-condition means matrix is close to being singular. In V_2, second row and third row are almost the same row because 0.2 and 0.22 are close. Having the same row means matrix is singular (condition number infinity); Having almost the same row means condition number is very big.

7.4 6+6 Given $P_1 = (-1, 0), P_2 = (0, -1), P_3 = (1, 1), P_4 = (2, 1)$. Find the polynomial interpolant going through

(a) P_2, P_3, P_4

(b) P_1, P_2, P_3, P_4

(Think about which method you should use given the connection of part (a) and part (b))

Solution

(a) $x_0 = 0, x_1 = 1, x_2 = 2$

Solve $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 1 & 1-0 & 0 & 1 \\ 1 & 2-0 & (2-0)(2-1) & 1 \end{array}\right]$, solution is $(a_0, a_1, a_2) = (-1, 2, -1)$

$p(x) = -1 + 2(x-0) - (x-0)(x-1) = -1 + 2x - x(x-1)$

(b) $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = -1$

Solve $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 1 & 1-0 & 0 & 0 & 1 \\ 1 & 2-0 & (2-0)(2-1) & 0 & 1 \\ 1 & -1-0 & (-1-0)(-1-1) & (-1-0)(-1-1)(-1-2) & 0 \end{array}\right]$

$(a_0, a_1, a_2) = (-1, 2, -1)$ as computed in (a)

$a_0 - a_1 + 2a_2 - 6a_3 = 0 \Rightarrow a_3 = -5/6$

$q(x) = -1 + 2(x-0) - (x-0)(x-1) = -1 + 2x - x(x-1) - \frac{5}{6}x(x-1)(x-2)$

7.5 5+3 (a) The Chebyshev nodes defined is only for interval [-1,1]. What do you do if the interval is [-2,2], or $[a, b]$ in general?

(b) Write down the Chebyshev nodes for the interval [-1,3]. $n = 6$.

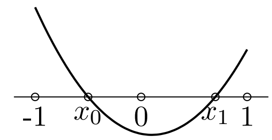
Solution

(a) If x_i are the chebyshev nodes for $[a, b]$, then $\frac{2x_i}{b-a}$ are the chebyshev nodes for $[\frac{2a}{b-a}, \frac{2b}{b-a}]$. This interval has length 2. If we just shift it by the center $\frac{a+b}{b-a}$, we will get [-1,1]. So $\frac{2x_i}{b-a} - \frac{a+b}{b-a}$ are the chebyshev nodes for [-1,1].

$$\frac{2x_i}{b-a} - \frac{a+b}{b-a} = \cos\frac{\pi i}{n} \Longrightarrow x_i = \frac{a+b}{2} + \frac{b-a}{2}\cos\frac{\pi i}{n}$$

(b) $x_i = 1 + 2\cos\frac{\pi i}{6}, i = 0, 1, \cdots, 6$

7.6 * Solve $\min_{\{x_0,x_1\}\subset[-1,1]} \max_{x\in[-1,1]} |x-x_0||x-x_1|$.



Let $f(x)=(x-x_0)(x-x_1)$. Its graph is drawn on the right

$\max_{x\in[-1,1]} |x-x_0||x-x_1| = \max_{x\in[-1,1]} |f(x)| = \max\{f(-1), -f(\frac{x_0+x_1}{2}), f(1)\}$.

If $x_0=-\sqrt{2}/2$ and $x_1=\sqrt{2}/2$, then $\max|f(x)|=0.5$. We assume that $x_0<0<x_1$ because if the are both positive, then $\max|f(x)|=f(-1)=(1+x_0)(1+x_1)>1$.

We further assume without loss of generality that x_0 is closer to 0 than x_1 is (as in the picture), then $\max_{x\in[-1,1]} |f(x)| = \max\{f(-1), -f(\frac{x_0+x_1}{2})\} = \max\{(1+x_0)(1+x_1), (x_1-x_0)^2/4\}$

Moreover, let $x_1-x_0=2h$. We can always decrease the max by shift the points to $x_0=-h, x_1=h$ because $(1+x_0)(1+x_1)\geq(1-h)(1+h)$. Now the problem becomes $\min_h \max\{(1-h^2), h^2\}$ which is optimized when $h=\sqrt{2}/2$.

So the minimizer is obtained when $x_0=-\sqrt{2}/2, x_1=\sqrt{2}/2$.

7.7 10 Determine an interpolant of (0,1), (1,0), (2,0) P such that

$$P(x)=\begin{cases} P_2(x)(\text{degree } 2), & 0\leq x\leq 1 \\ P_1(x)(\text{degree } 1), & 1\leq x\leq 2 \end{cases}$$

and P is differentiable on [0,2].

Solution

$P_1(x)$ is a line going through (1,0), (2,0), so $\boxed{P_1(x)=0}$, then $P_1'(x)=0$

$P_2'(1)=0$, we can let $P_2'(x)=2a(x-1)$, so $P_2(x)=ax^2-2ax+b$. P_2 goes through (0,1) and (1,0) implies $1=b, 0=a-2a+b$.

$\boxed{P_2(x)=x^2-2x+1}$

7.8 10 If $s(x)=-x^2+x$ when $0\leq x\leq 1$ and $s(x)=P_3(x)$ on $1\leq x\leq 2$. Find all possible cubic P_3 that makes $s(x)$ twice differentiable on [0,1].

Solution Let $f(x)=-x^2+x, f(1)=0, f'(1)=-1, f''(1)=-2$

Let $P_3''(x)=2a(x-1)-2$ to satisfy $P_3''(1)=-2$

$P_3'(x)=a(x-1)^2-2(x-1)+b$. $P_3'(1)=-1\Rightarrow P_3'(x)=a(x-1)^2-2(x-1)-1$

$P_3(x)=\frac{a}{3}(x-1)^3-(x-1)^2-(x-1)+c$. $P_3(1)=0\Rightarrow$

$P_3(x)=\frac{a}{3}(x-1)^3-(x-1)^2-(x-1)$

7.9 8 Finish Example 7.8 by computing the rest q_2, q_3, q_4. Plot this cubic spline.

Solution $y_0=0, y_1=0, y_2=2, y_3=2, y_4=-1$

$z_1=15/4, z_2=-3, z_3=-15/4$

$D_2=y_1-z_1/6=-\frac{5}{8}, C_2=y_2-y_1+(z_1-z_2)/6=\frac{25}{8}$, so

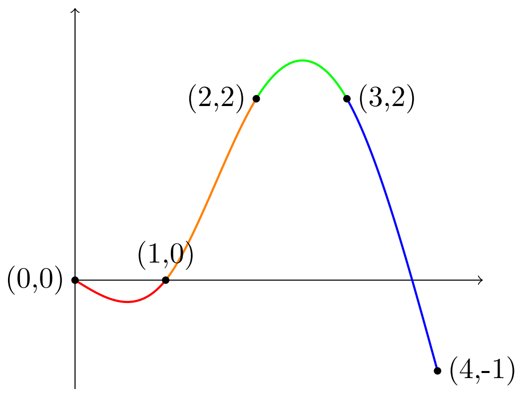
$q_2(x)=-\frac{z_1}{6}(x-x_2)^3+\frac{1}{6}z_2(x-x_1)^3+C_2(x-x_1)+D_2=-\frac{5}{8}(x-2)^3-\frac{1}{2}(x-1)^3+\frac{25}{8}(x-1)-\frac{5}{8}$.

$D_3 = y_2 - z_2/6 = \dfrac{5}{2}, C_3 = y_3 - y_2 + (z_2 - z_3)/6 = \dfrac{1}{8}$, so

$q_3(x) = -\dfrac{z_2}{6}(x-x_3)^3 + \dfrac{1}{6}z_3(x-x_2)^3 + C_3(x-x_2) + D_3 = \dfrac{1}{2}(x-3)^3 - \dfrac{5}{8}(x-2)^3 + \dfrac{1}{8}(x-2) + \dfrac{5}{2}.$

$D_4 = y_3 - z_3/6 = \dfrac{21}{8}, C_4 = y_4 - y_3 + (z_3 - z_4)/6 = \dfrac{-29}{8}$, so

$q_4(x) = -\dfrac{z_3}{6}(x-x_4)^3 + C_4(x-x_3) + D_4 = \dfrac{5}{8}(x-4)^3 - \dfrac{29}{8}(x-3) + \dfrac{21}{8}.$



17 points for free