Fall 2017 Math 395 Written Homework 3 Key 100 total. -5 for no stapling
Chapter 3

1. Given $x = (1, -2, 3), y = (-2, 0, 2)$, find

   (a) Find $x^T y, \|x\|_2, \|x\|_1, \|x\|_\infty$

   (b) Find a vector whose Euclidean norm is 1, and is parallel to y.

   (c) Find a vector that is orthogonal to both x and y. This is the same as finding the null space of what matrix?

   Solution:

   (a) $x^T y = -2 + 6 = 4$

   $\|x\|_2 = \sqrt{1+4+9} = \sqrt{14}$

   $\|x\|_1 = 1 + 2 + 3 = 6$

   $\|x\|_\infty = 3$

   (b) $y/\|y\|_2 = \dfrac{1}{\sqrt{4+4}}(-2, 0, 2) = \dfrac{1}{\sqrt{2}}(-1, 0, 1)$

   (c) Let $z = (z_1, z_2, z_3)$ be the vector. We need to solve $x^T z = 0, y^T z = 0$, which becomes $\begin{array}{r} z_1 - 2z_2 + 3z_3 = 0 \\ -2z_1 + 2z_3 = 0 \end{array}$.

   $$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & -4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

   $z_1 - z_3 = 0, z_2 - 2z_3 = 0$. z_3 is free, we can pick $z_3 = 1$, then $z_2 = 2, z_1 = 1$.

   A possible solution is $\boxed{(1,2,1)}$.

   The same as finding the null space of matrix A.

2. Find a 3 by 3 orthonormal matrix, then find its inverse. (Don't pick the canonical basis.)

   $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is an example. It is easy to verify that all columns are orthogonal to each other and every column has unit norm.

   $$U^{-1} = U^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

3. If $Au = ru$, simplify $(3A^3 - 4A + 2I)u$ so that A disappears.

   $(3A^3 - 4A + 2I)u = 3AAAu - 4Au + 2u = 3rAAu - 4ru + 2u = 3r^2Au - 4ru + 2u = 3r^3u - 4ru + 2u = (3r^3 - 4r + 2)u$

4. $B = \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}$.

   (a) Find the eigenvalues and corresponding eigenvectors of B.

   (b) Is this matrix positive semidefinite?

   (c) Find its operator norm.

   Solution:

   (a) $\det \begin{bmatrix} 1-\lambda & 4 \\ 4 & 7-\lambda \end{bmatrix} = \lambda^2 - 8\lambda + 7 - 16 = \lambda^2 - 8\lambda - 9 = (\lambda - 9)(\lambda + 1)$.

$\lambda_1 = 9$, solve $\begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} x = 0$, $u_1 = [1, 2]^T$.

$\lambda_2 = -1$, solve $\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} x = 0, u_2 = [-2, 1]^T$.

(b) No. one eigenvalue is less than 0.

(c) Singular values are 9,1. Operator norm is 9.

5. Find the eigenvalues and corresponding eigenvectors of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$. Determine if it is diagonalizable.

$$\det \begin{bmatrix} 1-r & 0 & 0 \\ 0 & 2-r & 3 \\ 1 & 0 & 2-r \end{bmatrix} = (1-r) \begin{vmatrix} 2-r & 3 \\ 0 & 2-r \end{vmatrix} = (1-r)(2-r)^2$$

$r_1 = 1$, Solve $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} x = 0$, pick $x_3 = 1$

$u_1 = [-1, -3, 1]^T$.

$r_2 = 2$, Solve $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix} x = 0$, $x_1 = x_3 = 0$

$u_2 = [0, 1, 0]^T$

Only 2 INDEPENDENT eigenvectors. This is not diagonalizable.

6. Let u_1, u_2, u_3 be the 3 columns of the orthonormal matrix you found in Exercise 2, and let $x = (1, 2, 3)$. Find the coefficients c_1, c_2, c_3 in $x = c_1 u_1 + c_2 u_2 + c_3 u_3$.

$c_1 = x^T u_1 = 1$

$c_2 = x^T u_2 = 2/\sqrt{2} - 3/\sqrt{2} = -1/\sqrt{2}$

$c_3 = x^T u_3 = 2/\sqrt{2} + 3/\sqrt{2} = 5/\sqrt{2}$

7. If A is a 2 by 2 matrix that rotates a vector by $\pi/5$ counterclockwise, show that $\|A\|_2 = 1$ directly from its definition.

Rotation does not change Euclidean norm, i.e. $\|Ax\|_2 = \|x\|_2$

$$\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{x \neq 0} \frac{\|x\|_2}{\|x\|_2} = \sup_{x \neq 0} 1 = 1$$

8. Find the operator norm of $\begin{bmatrix} 1/\sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & \sqrt{2} \\ 0 & 0 \end{bmatrix}$.

We need to find the biggest singular value.

$$A = \begin{bmatrix} 1/\sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & \sqrt{2} \\ 0 & 0 \end{bmatrix}.\ A^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \end{bmatrix},\ A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 1-r & 2 \\ 2 & 4-r \end{vmatrix} = r^2 - 5r + 4 - 4 = r(r-5)$$

Eigenvalues of $A^T A$ are 4,0.

Sinigular values of A are $2, 0$. Operator norm is $\boxed{2}$.