Exercises for chapter 9

1 If A and B are similar, then they have same eigenvalues

$$A = UBU^{-1}$$

if $AV = \lambda V$

then $UBU^{-1}V = \lambda V$

$$BU^{-1}V = U^{-1}\lambda V$$

$$B(U^{-1}V) = \lambda (U^{-1}V) \Rightarrow \lambda \text{ is also an eigenvalue of } B$$

with eigenvector $U^{-1}V$

2 $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}^{-1}$

$\lambda_1 = 1$, $V_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\lambda_2 = -5$, $V_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_3 = -1$, $V_3 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

3 $U \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} U^{-1}$ has eigenvalues 1, 1, 2, and is diagonalizable by def.

We just need to pick U that is not orthonormal.

pick $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, so it's easy to compute its inverse

$$U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & -1 \\ & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & 2 \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

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4 $QQ^T = I$, $PP^T = I$

$$QP(QP)^T = QPP^TQ^T = QQ^T = I$$

5 $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$

$a_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $u_1 = a_1 + \|a_1\| e_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \sim \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$H_1 = I - 2\frac{u_1u_1^T}{u_1^Tu_1} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}\left(\begin{bmatrix} 5 & \\ & 5 \end{bmatrix} - \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{5}\begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}$$

$$Q_1 = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & -4 \\ 0 & -4 & 3 \end{bmatrix}$$

$$A_1 = Q_1AQ_1^T = \frac{1}{25}\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & -4 \\ 0 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & -4 \\ 0 & -4 & 3 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 5 & 15 & 20 \\ -25 & -3 & -4 \\ 0 & -4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & -4 \\ 0 & -4 & 3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 25 & -125 & 0 \\ -25\times 5 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A_1 will be symmetric, only need to compute these entries

6 $Av = \lambda v$

$$\frac{v^TAv}{v^Tv} = \frac{v^T\lambda v}{v^Tv} = \lambda$$

7 A has eigenvalues 1, 2, 3, 4

(a) Find 4, rate $= \frac{3}{4}$

(b) $A - I$ has 0, 1, 2, 3. Find 3 (corresponding to 4), rate $= \frac{2}{3}$

$\boxed{3}$

7(c) What is the value of $A-\alpha I$ so that it finds $4-\alpha$ withe the best rate

$A-\alpha I$: $1-\alpha, 2-\alpha, 3-\alpha, 4-\alpha$

First need to make sure $4-\alpha$ is the biggest <u>in magnitude</u>.

if $0 \le \alpha \le 1$, then $4-\alpha$ is the biggest in absolute value

$$\text{rate} = \frac{3-\alpha}{4-\alpha} \text{ is minimized by } \alpha = 1 \qquad \boxed{\frac{2}{3}}$$

if $1 \le \alpha \le 2$ $\quad \max\{|1-\alpha|, |2-\alpha|, |3-\alpha|, |4-\alpha|\} = |4-\alpha|$ as well

because $\alpha - 1 \le 3-\alpha < 4-\alpha$

$$\text{rate} = \frac{3-\alpha}{4-\alpha} \text{ is minimized by } \alpha = 2 \qquad \boxed{\frac{1}{2}}$$

if $2 < \alpha \le 3$

$$\underline{\alpha-1} \overset{>}{,} \alpha-2, 3-\alpha \overset{<}{,} \underline{4-\alpha}$$

need $\alpha - 1 < 4-\alpha \Rightarrow \alpha < 2.5$

also $3-\alpha < \alpha-1 < 4-\alpha \Rightarrow \alpha-1$ 2nd biggest

$$\text{rate} = \frac{\alpha-1}{4-\alpha} \text{ for } 2<\alpha<2.5 \text{ minimized by } 2$$

if $\alpha > 3$, then $\alpha - 1 > 4-\alpha$

Overall, best rate $= 0.5$, attained by $\boxed{\alpha = 2}$

7(d)

$$\frac{1}{1-1.9}, \underbrace{\frac{1}{2-1.9}}_{1st}, \underbrace{\frac{1}{3-1.9}}_{2nd}, \frac{1}{4-1.9}$$

will find $\frac{1}{2-1.9}$

$$\text{rate} = \frac{\frac{1}{3-1.9}}{\frac{1}{2-1.9}} = \frac{0.1}{1.1} = \frac{1}{11}$$

8 $\det(A-\lambda I_4) = \begin{vmatrix} B-\lambda I_2 & C \\ 0 & D-\lambda I_2 \end{vmatrix} := \begin{vmatrix} a & b & c_1 & c_2 \\ c & d & c_3 & c_4 \\ 0 & 0 & x & y \\ 0 & 0 & z & w \end{vmatrix}$ where $B-\lambda I_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $D-\lambda I_2 = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

<u>expand along 1st column</u>

$$a\begin{vmatrix} d & c_3 & c_4 \\ 0 & x & y \\ 0 & z & w \end{vmatrix} - c\begin{vmatrix} b & c_1 & c_2 \\ 0 & x & y \\ 0 & z & w \end{vmatrix} = ad\begin{vmatrix} x & y \\ z & w \end{vmatrix} - cb\begin{vmatrix} x & y \\ z & w \end{vmatrix}$$

$$= (ad-bc)\det(D-\lambda I_2) = \det(B-\lambda I_2)\det(D-\lambda I_2)$$