

**Fall 2017 Math 395 Written Homework 7 Key** 100 total. -5 for no stapling

7.1 Let  $f(x) = 3/2 - x/2, g(x) = \frac{1}{x}$ . Both functions belong to  $C[1, 2]$ . Find

(a)  $\|f - g\|_2$

(b)  $\|f - g\|_\infty$  (use derivative to find max)

**Solution**

$$\begin{aligned} \text{(a) } \|f - g\|_2^2 &= \int_1^2 \left(\frac{3}{2} - \frac{x}{2} - \frac{1}{x}\right)^2 dx = \int_1^2 \frac{9}{4} + \frac{x^2}{4} + \frac{1}{x^2} - \frac{3x}{2} - \frac{3}{x} + 1 dx = \frac{13}{4} + \frac{x^3}{12} \Big|_1^2 - \frac{1}{x} \Big|_1^2 - \frac{3x^2}{4} \Big|_1^2 - 3 \ln x \Big|_1^2 \\ &= \frac{13}{4} + \frac{8-1}{12} + \frac{1}{2} - \frac{3}{4}(4-1) - 3 \ln 2 = 25/12 - 3 \ln 2 \end{aligned}$$

$$\|f - g\|_2 = \sqrt{25/12 - 3 \ln 2} \approx 0.062$$

$$\text{(b) } h(x) = f(x) - g(x) = \frac{3}{2} - \frac{x}{2} - \frac{1}{x}.$$

$h(1) = 0, h(2) = 0$ . Since  $\frac{1}{x}$  is convex on  $(0, \infty)$ , the line  $f$  is above  $g$ , so  $|h(x)| = h(x)$

$$h'(x) = -\frac{1}{2} + x^{-2} = 0 \Rightarrow x = \sqrt{2} \quad (x = -\sqrt{2} \text{ is discarded})$$

$$\|f - g\|_\infty = \max_{x \in [1, 2]} |h(x)| = \max_{x \in [1, 2]} h(x) = h(\sqrt{2}) = 3/2 - \sqrt{2} \approx 0.086$$

7.2 Find the polynomial interpolant of  $(1, 7/4)$ ,  $(2, 3/2)$ ,  $(3.5, 0)$ , and  $(5, 3/4)$  by

(a) Vandermonde method (feel free to use python)

(b) Lagrange method (handwritten, don't simplify)

**Solution**

$$\text{(a) Solve } \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7/4 \\ 1 & 2 & 4 & 8 & 3/2 \\ 1 & 3.5 & 3.5^2 & 3.5^3 & 0 \\ 1 & 5 & 25 & 125 & 3/4 \end{array} \right], \text{ we get that the coefficients are } (0, 3.15, -1.6, 0.2)$$

$$p(x) = 3.15x - 1.6x^2 + 0.2x^3.$$

$$\text{(b) } \varphi_0(x) = \frac{(x-2)(x-3.5)(x-5)}{(1-2)(1-3.5)(1-5)}, \varphi_1(x) = \frac{(x-1)(x-3.5)(x-5)}{(2-1)(2-3.5)(2-5)}, \varphi_3(x) = \frac{(x-1)(x-2)(x-3.5)}{(5-1)(5-2)(5-3.5)}$$

$$p(x) = \frac{7}{4}\varphi_0(x) + \frac{3}{2}\varphi_1(x) + \frac{3}{4}\varphi_3(x)$$

7.3 Use python to compute the condition number of the following two Vandermonde matrices, and comment on the results briefly.

(a) generated by 0, 0.2, 0.4, 0.6, 0.8.

(b) generated by 0, 0.2, 0.22, 0.6, 0.8.

**Solution**

$$\text{(a) } V_1 = \left[ \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 1 & 0.2 & 0.2^2 & 0.2^3 & 0.2^4 \\ 1 & 0.4 & 0.4^2 & 0.4^3 & 0.4^4 \\ 1 & 0.6 & 0.6^2 & 0.6^3 & 0.6^4 \\ 1 & 0.8 & 0.8^2 & 0.8^3 & 0.8^4 \end{array} \right]. \quad \kappa(V_1) = 1140.27$$

You can use `numpy.linalg.cond` directly, you can also use `numpy.linalg.svd` to find all the singular values of  $V_1$  and get the ratio of biggest over smallest.

$$(b) V_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0.2 & 0.2^2 & 0.2^3 & 0.2^4 \\ 1 & 0.22 & 0.22^2 & 0.22^3 & 0.22^4 \\ 1 & 0.6 & 0.6^2 & 0.6^3 & 0.6^4 \\ 1 & 0.8 & 0.8^2 & 0.8^3 & 0.8^4 \end{bmatrix}. \quad \kappa(V_2) = 7531.24$$

Vandermonde matrices are very ill-conditioned in general. It is the most ill conditioned when there two  $x$  values are close, like 0.2 and 0.22. Ill-condition means matrix is close to being singular. In  $V_2$ , second row and third row are almost the same row because 0.2 and 0.22 are close. Having the same row means matrix is singular (condition number infinity); Having almost the same row means condition number is very big.

7.4 Given  $P_1 = (-1, 0), P_2 = (0, -1), P_3 = (1, 1), P_4 = (2, 1)$ . Find the polynomial interpolant going through

(a)  $P_2, P_3, P_4$

(b)  $P_1, P_2, P_3, P_4$

(Think about which method you should use given the connection of part (a) and part (b))

**Solution**

(a)  $x_0 = 0, x_1 = 1, x_2 = 2$

$$\text{Solve } \left[ \begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 1 & 1-0 & 0 & 1 \\ 1 & 2-0 & (2-0)(2-1) & 1 \end{array} \right], \text{ solution is } (a_0, a_1, a_2) = (-1, 2, -1)$$

$$p(x) = -1 + 2(x-0) - (x-0)(x-1) = -1 + 2x - x(x-1)$$

(b)  $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = -1$

$$\text{Solve } \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 1 & 1-0 & 0 & 0 & 1 \\ 1 & 2-0 & (2-0)(2-1) & 0 & 1 \\ 1 & -1-0 & (-1-0)(-1-1) & (-1-0)(-1-1)(-1-2) & 0 \end{array} \right]$$

$(a_0, a_1, a_2) = (-1, 2, -1)$  as computed in (a)

$$a_0 - a_1 + 2a_2 - 6a_3 = 0 \Rightarrow a_3 = 5/6$$

$$q(x) = -1 + 2(x-0) - (x-0)(x-1) = -1 + 2x - x(x-1) + \frac{5}{6}x(x-1)(x-2)$$

7.5 (a) The Chebyshev nodes defined is only for interval  $[-1,1]$ . What do you do if the interval is  $[-2,2]$ , or  $[a, b]$  in general?

(b) Write down the Chebyshev nodes for the interval  $[-1,3]$ .  $n = 6$ .

**Solution**

(a) If  $x_i$  are the chebyshev nodes for  $[a, b]$ , then  $\frac{2x_i}{b-a}$  are the chebyshev nodes for  $[\frac{2a}{b-a}, \frac{2b}{b-a}]$ . This interval has length 2. If we just shift it by the center  $\frac{a+b}{b-a}$ , we will get  $[-1,1]$ . So  $\frac{2x_i}{b-a} - \frac{a+b}{b-a}$  are the chebyshev nodes for  $[-1,1]$ .

$$\frac{2x_i}{b-a} - \frac{a+b}{b-a} = \cos \frac{\pi i}{n} \Rightarrow x_i = \frac{a+b}{2} + \frac{b-a}{2} \cos \frac{\pi i}{n}$$

(b)  $x_i = 1 + 2 \cos \frac{\pi i}{6}, i = 0, 1, \dots, 6$

7.6 \* Solve  $\min_{\{x_0, x_1\} \subset [-1,1]} \max_{x \in [-1,1]} |x - x_0| |x - x_1|$ .

7.7 Determine an interpolant of  $(0,1)$ ,  $(1,0)$ ,  $(2,0)$   $P$  such that

$$P(x) = \begin{cases} P_2(x) (\text{degree } 2), & 0 \leq x \leq 1 \\ P_1(x) (\text{degree } 1), & 1 \leq x \leq 2 \end{cases}$$

and  $P$  is differentiable on  $[0,2]$ .

**Solution**

$P_1(x)$  is a line going through  $(1,0)$ ,  $(2,0)$ , so  $\boxed{P_1(x) = 0}$ , then  $P'_1(x) = 0$

$P'_2(1) = 0$ , we can let  $P'_2(x) = 2a(x-1)$ , so  $P_2(x) = ax^2 - 2ax + b$ .  $P_2$  goes through  $(0,1)$  and  $(1,0)$  implies  $1 = b$ ,  $0 = a - 2a + b$ .

$$\boxed{P_2(x) = x^2 - 2x + 1}$$

7.8 If  $s(x) = -x^2 + x$  when  $0 \leq x \leq 1$  and  $s(x) = P_3(x)$  on  $1 \leq x \leq 2$ . Find all possible cubic  $P_3$  that makes  $s(x)$  twice differentiable on  $[0,1]$ .

**Solution** Let  $f(x) = -x^2 + x$ ,  $f(1) = 0$ ,  $f'(1) = -1$ ,  $f''(1) = -2$

Let  $P''_3(x) = 2a(x-1) - 2$  to satisfy  $P''_3(1) = -2$

$$P'_3(x) = ax^2 - 2ax - 2x + b. \quad P'_3(1) = -1 \Rightarrow a - 2a - 2 + b = -1 \Rightarrow b = a + 1$$

$$P'_3(x) = ax^2 - 2ax - 2x + a + 1$$

$$P_3(x) = a\frac{x^3}{3} - ax^2 - x^2 + (a+1)x + c. \quad P_3(1) = 0 \Rightarrow c = -a/3$$

$$P_3(x) = a\frac{x^3}{3} - ax^2 - x^2 + (a+1)x - \frac{a}{3}$$

7.9 Finish Example 7.8 by computing the rest  $q_2, q_3, q_4$ . Plot this cubic spline.

**Solution**

$$D_2 = y_1 - z_1/6 = -\frac{5}{8}, C_2 = y_2 - y_1 + (z_1 - z_2)/6 = \frac{25}{8}, \text{ so}$$

$$q_2(x) = -\frac{z_1}{6}(x-x_2)^3 + \frac{1}{6}z_2(x-x_1)^3 + C_2(x-x_1) + D_2 = \frac{5}{8}(x-2)^3 - \frac{1}{2}(x-1)^3 + \frac{25}{8}(x-1) - \frac{5}{8}.$$

$$D_3 = y_2 - z_2/6 = \frac{3}{2}, C_3 = y_3 - y_2 + (z_2 - z_3)/6 = \frac{1}{8}, \text{ so}$$

$$q_3(x) = -\frac{z_2}{6}(x-x_3)^3 + \frac{1}{6}z_3(x-x_2)^3 + C_3(x-x_2) + D_3 = \frac{1}{2}(x-3)^3 - \frac{5}{8}(x-2)^3 + \frac{1}{8}(x-2) + \frac{3}{2}.$$

$$D_4 = y_3 - z_3/6 = \frac{21}{8}, C_4 = y_4 - y_3 + (z_3 - z_4)/6 = \frac{-29}{8}, \text{ so}$$

$$q_4(x) = -\frac{z_3}{6}(x-x_4)^3 + C_4(x-x_3) + D_4 = -\frac{5}{8}(x-4)^3 - \frac{29}{8}(x-3) + \frac{21}{8}.$$