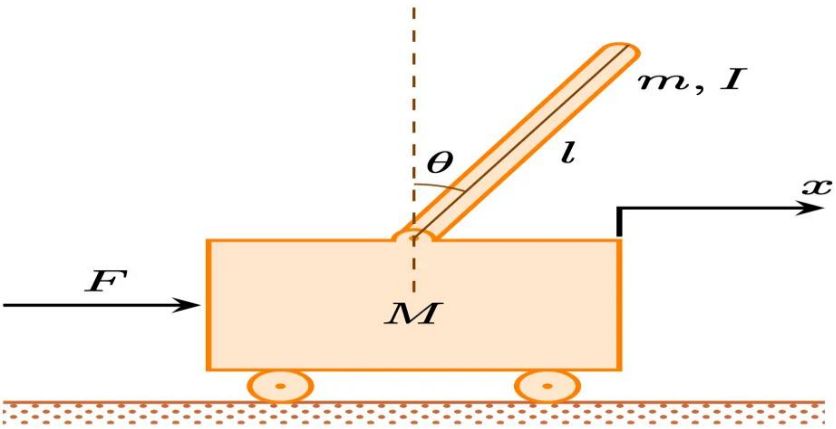
Stability Analysis of an Inverted Pendulum

EAD PROJECT G-4

2023-24



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ABSTRACT

A regular pendulum has two equilibrium positions; its bottom position and its top position. When its pivot is fixated, only the bottom equilibrium position is regarded stable. This changes when the pendulum's pivot is being vertically oscillated with appropriate amplitude and frequency. The pendulum might become stable in top equilibrium position and unstable in bottom equilibrium position. In this report, this phenomenon is investigated. First, a stability analysis is performed and a model is developed describing the pendulum's (in)stability. This model is successfully validated by conducting an experiment. The results from both the developed model and the computer simulation are then used to design and build an experimental setup in which an inverted pendulum is being stabilized. Lastly, an experiment is conducted using the created setup and the results are being compared to both the model and the simulation.

ACKNOWLEDGEMENT

The project was organized by Mechanical and Industrial Engineering Department, IIT Roorkee. We would like to thank our Professor M. M. Joglekar who provided us the golden opportunity to do this wonderful project, which helped us in doing a lot of research and learning new things.

We would like to extend our sincere gratitude to our teaching assistants Mr. Rahul Singh for guiding us at each step and explaining everything in a detailed manner that enabled us to complete the project within limited time frame.

Lastly, we would like to thank all the people who supported us in various ways throughout the project.

We had a great time exploring and learning new things. Hoping to learn more such things in future.

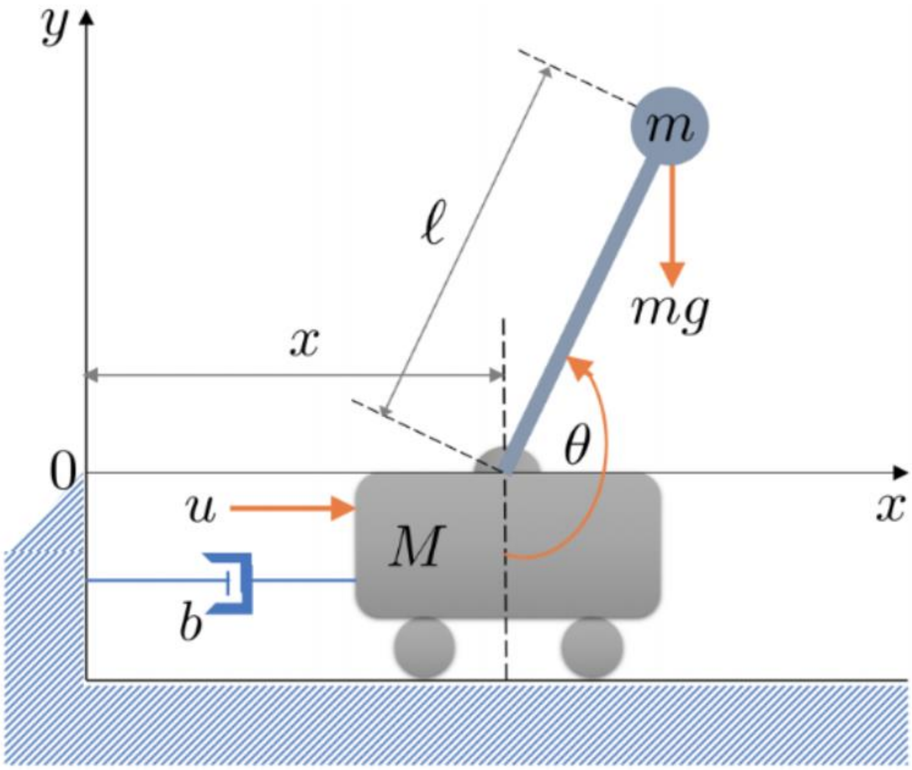
INTRODUCTION

Dynamic stability of an ordinary pendulum can be approached in an intuitive manner. The pendulum has two equilibrium positions; one at its bottom and one at its top position, where generally only one of them is stable. Under standard conditions, that is when rotating around a fixated pivot point, the bottom position is stable and the top position is unstable. However, when the pivot point is vertically oscillated with appropriate amplitude and frequency, the stability position can shift from bottom to top equilibrium position.

This phenomenon has already been studied for over a decade. In 1908, Andrew Stephenson was the first to publish an article about the stabilization of an inverted pendulum. He found that if the pendulum is being oscillated, its top vertical position might stabilize when its oscillation frequency is fast enough . This article turned out to be the starting point of several more studies to the behaviour of an inverted pendulum. It then took until 1951 for a first real explanation for this phenomenon; Pyotr Kapitza was the first to develop a theory to support inverted pendulum stability.

The report will begin by setting up the pendulum's equation of motion using the Euler-Lagrange equations. This equation of motion can be linearized, after which it is simplified by applying time dependent switching to the pivot's acceleration. With these equations set up and its resulting solution, stability criteria are established and an experiment is conducted to validate the model.

Afterwards, the report will elaborate on a computer simulation of the inverted pendulum and a comparison will be made to the earlier developed model. The combination of resulting data from both the model and the computer simulation will ultimately be used to develop an experimental setup stabilizing an inverted pendulum. Finally, the report concludes with a comparison between the observed results from this setup and the theoretical results from model and simulation, after which a final conclusion will be drawn.

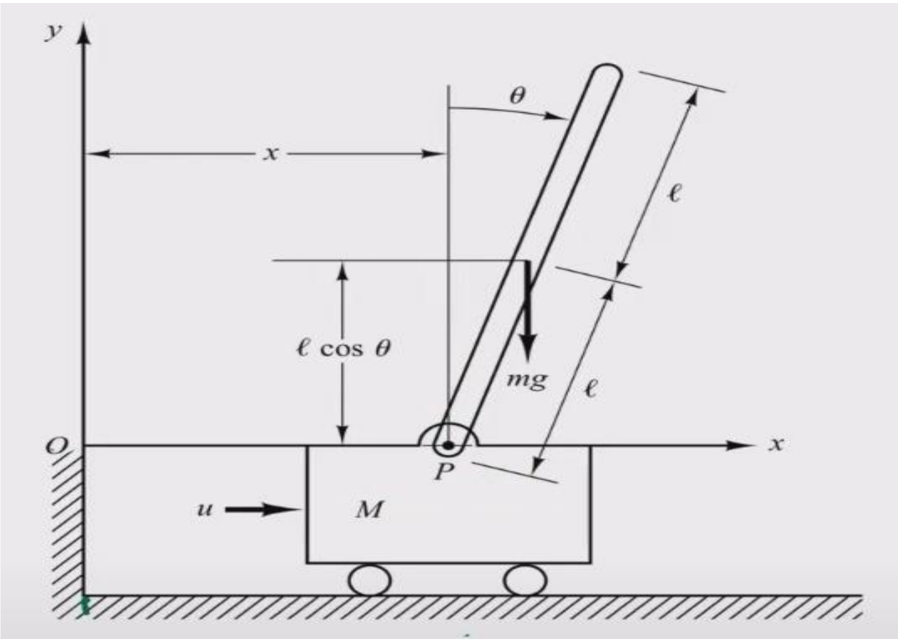


Develop inverted pendulum model:

In order to be able to understand the behaviour of a vertically oscillated inverted pendulum, a stability analysis is performed. After the situation has been introduced, the pendulum equation of motion is being set up using the Euler-Lagrange equation. Afterwards, by utilizing this equation of motion and assuming time-dependent switching for the vertical oscillation one can derive conditions for potential pendulum instability in lower position.

Situation description:

For this theoretical stability analysis, the inverted pendulum consists of a point mass M connected to a rod of length L and negligible mass. The rod rotates around a pivot point which is vertically oscillated. This oscillation behaviour is defined by the function s(t), which is a function of time. The exact shape of this function is yet undefined. The clockwise-positive angle the pendulum makes with the vertical is defined as ϕ, which is set zero at the top position. The situation is sketched in Figure 1.1.1 with corresponding 2D coordinate system.



Motion of equation using newton 2nd law ->

Inverted Pendulum System :

* Physical model of inverted pendulum is shown above. It is free to move in x-y plane.
* Coordinates of center of gravity w.r.t. Cartesian axis

$$\left.\begin{aligned} x_g &= x + l\sin\theta \\ y_g &= y + l\cos\theta \end{aligned}\right\} \quad (1)$$

Initial position of point P is $(x, 0)$

Balance of vertical Forces :

* Balance of forces acting on the rod in the vertical direction at the center of gravity is given by :

$$V - mg = \Sigma F = ma = m\frac{d^2}{dt^2}(l\cos\theta)$$

$$V - mg = m\frac{d^2}{dt^2}(l\cos\theta) \quad \text{— (2)}$$

$$\frac{d^2\cos\theta}{dt^2} = -\ddot{\theta}\sin\theta - \dot{\theta}\cos\theta$$

$$\Rightarrow \quad V - mg = ml[-\ddot{\theta}\sin\theta - \dot{\theta}\cos\theta]$$

$$V = mg - ml\ddot{\theta}\sin\theta - ml\dot{\theta}\cos\theta \quad \text{— (3)}$$

Balance of Horizontal Forces :

* Balance the forces acting on the rod in the horizontal direction w.r.t centre of gravity.

$$H = ma = m\frac{d^2}{dt^2}(x + l\sin\theta)$$

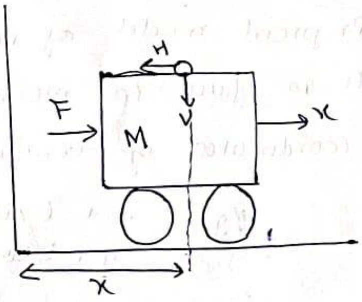
$$H = m\ddot{x} + ml\frac{d^2}{dt^2}\sin\theta \quad \text{— (4)}$$

$$\Rightarrow \quad H = m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \quad \text{— (5)}$$

Horizontal Force Balance on cart

$\Sigma F = Ma$

$F - H = M\ddot{x}$ — (6)



★ No vertical movement of the cart so net vertical forces is zero.

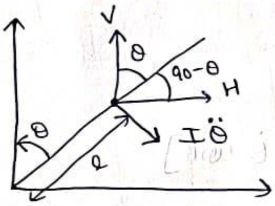
Now substituting (5) into (6),

$M\ddot{x} = F - H \Rightarrow M\ddot{x} + H = F$ — (7)

$F = M\ddot{x} + m\ddot{x} - ml\dot{\theta}^2 \sin\theta + ml\ddot{\theta}\cos\theta$

$F = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$ — (8)

Balance of forces at center of gravity



eqn of torque :

$I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta$ — (9)

Using the values of H & V :

$I\ddot{\theta} = [mg - ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta]\, l\sin\theta$

$\quad - [m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta]\, l\cos\theta$

$\Rightarrow I\ddot{\theta} = mgl\sin\theta - ml^2\ddot{\theta}\sin^2\theta - m\ddot{x}l\cos\theta - ml^2\ddot{\theta}\cos^2\theta$

$I\ddot{\theta} = mgl\sin\theta - ml^2\ddot{\theta} - m\ddot{x}l\cos\theta$

$$\boxed{(I+ml^2)\ddot{\theta} + ml\ddot{x}\cos\theta - mgl\sin\theta = 0} \quad \text{— (9)}$$

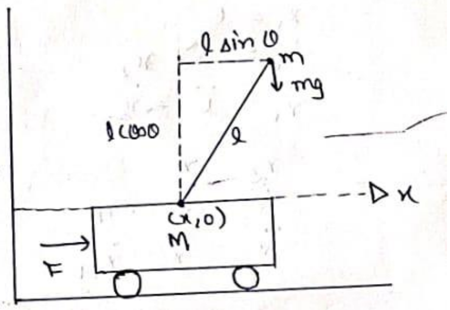
Since we must keep the inverted pendulum vertical, that means $\theta(t)$ & $\dot{\theta}(t)$ are small quantities, therefore as $\theta \to 0 \Rightarrow \sin\theta \simeq 0$ & $\cos\theta = 1$ & $\theta.\dot{\theta}^2 = 0$

$$\left.\begin{aligned} F &= (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \\ (I+ml^2)\ddot{\theta} &+ ml\ddot{x}\cos\theta - mgl\sin\theta = 0 \end{aligned}\right\}$$

Putting the values of $\sin\theta, \dot{\theta}, \cos\theta$, we get the linearize model for inverted pendulum.

$$\boxed{\begin{aligned} (M+m)\ddot{x} + ml\ddot{\theta} &= F \\ (I+ml^2)\ddot{\theta} + ml\ddot{x} &= mgl\theta \end{aligned}} \quad \text{—— (10)}$$

State space model of Inverted Pendulum



→ Consider the inverted pendulum as shown in the figure.

→ Mass is concentrated at the top of the rod, so the centre of gravity is the centre of pendulum ball.

→ Therefore $I = 0$, so using the eq^n (10) we can write:

$$\left.\begin{aligned} (M+m)\ddot{x} + ml\ddot{\theta} &= F \\ ml^2\ddot{\theta} + ml\ddot{x} &= mgl\theta \end{aligned}\right\} \text{— (11)}$$

By rearranging the above eq^n

$$\ddot{\theta} = \frac{g\theta}{l} - \frac{\ddot{x}}{l} \quad \text{—— (12)}$$

$$\ddot{x} = g\theta - l\ddot{\theta} \quad \text{—— (13)}$$

using eq^n (12)

$$(M+m)\ddot{x} + ml\left(\frac{g\theta}{l} - \frac{\ddot{x}}{l}\right) = F$$

$$(M+m)\ddot{x} + mg\theta - m\ddot{x} = F$$

$$\boxed{M\ddot{x} = F - mg\theta} \quad \text{—— (14)}$$

Using eq^n (13) in (11):

$$(M+m)\ddot{x} + ml\ddot{\theta} = F$$
$$M\ddot{x} + m\ddot{x} + ml\ddot{\theta} = F$$
$$M(g\theta - l\ddot{\theta}) + m(g\theta - l\ddot{\theta}) + ml\ddot{\theta} = F$$
$$Ml\ddot{\theta} = Mg\theta + mg\theta - F$$
$$\boxed{Ml\ddot{\theta} = (M+m)g\theta - F} \quad — (15)$$

Equation for state space model development

$$\left.\begin{aligned} M\ddot{x} &= F - mg\theta \\ Ml\ddot{\theta} &= (M+m)g\theta - F \end{aligned}\right\} (16)$$

* Let x_1, x_2, x_3 & x_4 be state variable

$$x_1 = \theta$$
$$\dot{x}_1 = \dot{\theta} = x_2$$
$$\ddot{x}_1 = \ddot{\theta} = \dot{x}_2$$
$$x_3 = x$$
$$\dot{x}_3 = \dot{x} = x_4$$
$$\ddot{x}_3 = \ddot{x} = \dot{x}_4$$

Eq^n (14) can be written as:

$$M\dot{x}_4 = F - mgx_1$$
$$\dot{x}_4 = -\frac{mg}{M}x_1 + \frac{F}{M} \quad — (17)$$

Eq^n (15) can be written as:

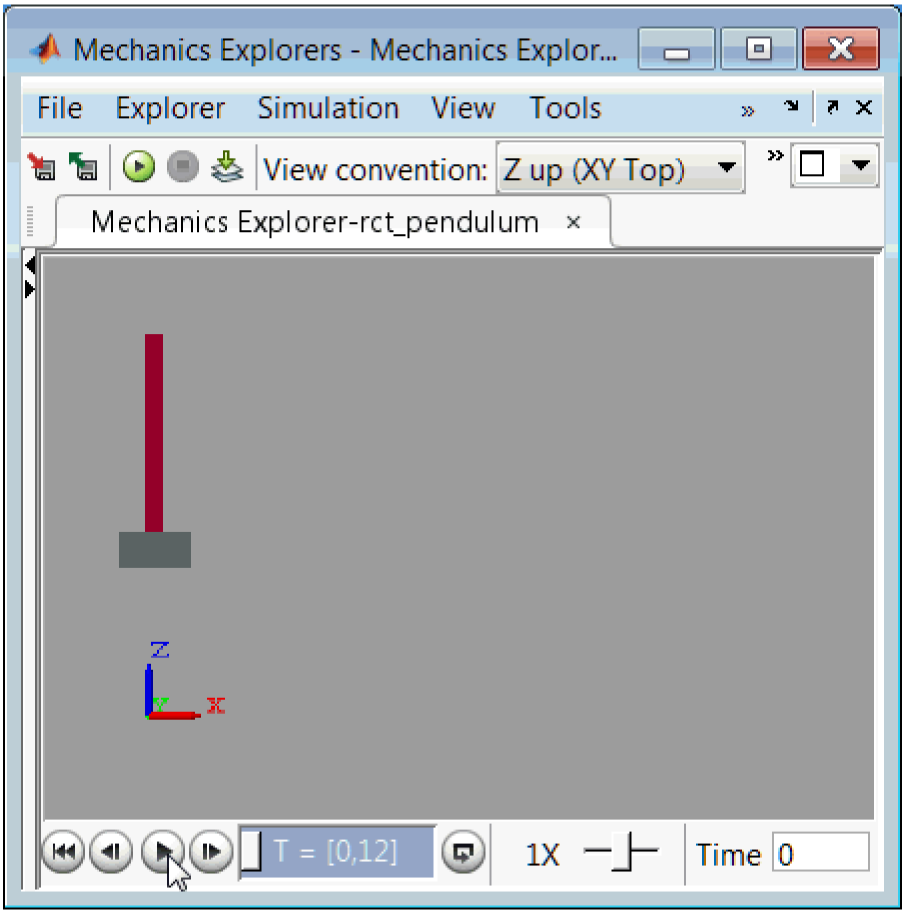
$$Ml\dot{x}_2 = (M+m)gx_1 - F$$
$$\dot{x}_2 = \frac{(M+m)g}{Ml}x_1 - \frac{F}{Ml} \quad — (18)$$

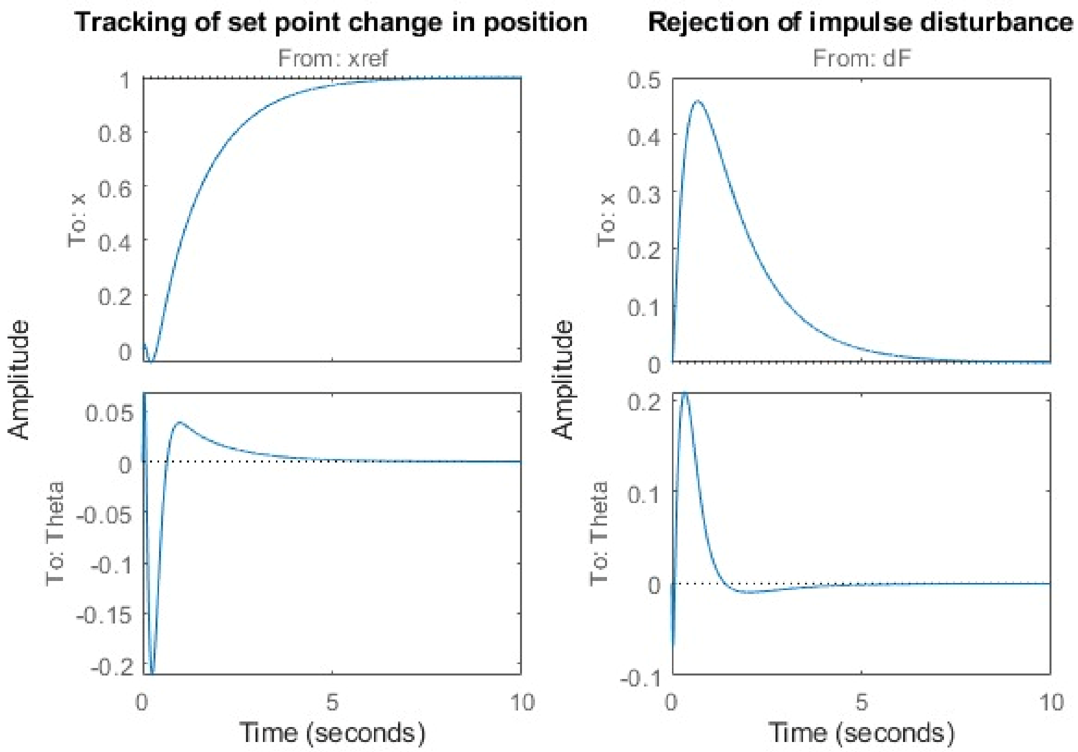
State Space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix} [F]$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0][F]$$

$$\boxed{\begin{aligned} \dot{X} &= AX + BU \\ Y &= CX + DU \end{aligned}}$$


Mechanics Explorers - Mechanics Explor...
File Explorer Simulation View Tools
View convention: Z up (XY Top)
Mechanics Explorer-rct_pendulum
T = [0,12]
1X
Time 0


Tracking of set point change in position
From: xref
Rejection of impulse disturbance
From: dF
Amplitude
To: x
To: Theta
Time (seconds)

USING EULER-LANGRANGES APPROACH->

Equation of motion:->

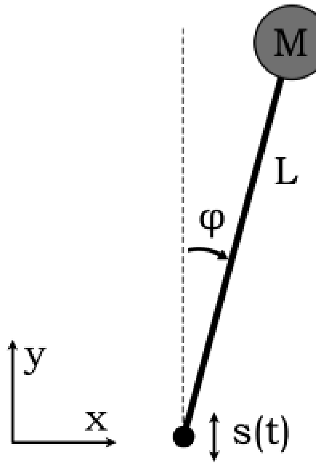


Figure 1.1.1: Situation sketch vertically oscillated inverted pendulum

Continuing from Figure 1.1.1, the pendulum's equations of motion can be derived. This is done in several sub steps, which will all be elaborated in this section.

Point mass position and its time derivative :

To start off the stability analysis, the x and y position of the point mass is expressed in known quantities. To do so, first an origin is defined; this is the pivot point in initial position. From here it can be derived that

$$x = \sin(\phi)L \qquad (1.1)$$

$$y = \cos(\phi)L + s(t) \qquad (1.2)$$

from which it directly follows that the time derivative of this position is defined as:

$$\dot{x} = \dot{\varphi}\cos(\varphi)L \quad (1.3)$$
$$\dot{y} = -\dot{\varphi}\sin(\varphi)L + \dot{s}(t) \quad (1.4)$$

Kinetic and potential energy:

Now the pendulum's x and y positions and their time derivatives have been expressed in known quantities, the pendulum's kinetic and potential energy can be derived. This is done using the regular equations for kinetic and potential energy.

$$E_k = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}M(\dot{\varphi}^2L^2 - 2\dot{\varphi}\sin(\varphi)L\dot{s} + \dot{s}^2) \quad (1.5)$$
$$E_p = Mgy = Mg(\cos(\varphi)L + s) \quad (1.6)$$

Euler-Lagrange equation :

The Euler-Lagrange equation is-

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) - \frac{\partial\mathcal{L}}{\partial q} = F_{nc} \quad (1.7)$$

Where the Lagrangian (L) is defined as the difference between kinetic and potential energy-

$$\mathcal{L} = E_k - E_p = \frac{1}{2}M(\dot{\varphi}^2L^2 - 2\dot{\varphi}\sin(\varphi)L\dot{s} + \dot{s}^2) - Mg(\cos(\varphi)L + s) \quad (1.8)$$

In The Euler-Lagrange equation relevant generalized coordinate is ϕ. Also, the non conservative force factor Fnc can be reduced to only a damping factor d, which is linearly dependent on and works in opposite direction of angular velocity $\dot{\phi}$.

Now the Euler-Lagrange equation is :

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = -d\dot{\varphi} \quad (A.1)$$

First, the Lagrangian is differentiated with respect to $\dot{\phi}$:

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = M(L^2\dot{\varphi} - \sin(\varphi)L\dot{s}) \qquad \textbf{(A.3)}$$

this derivative is differentiated with respect to time :

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) = M(L^2\ddot{\varphi} - \sin(\varphi)L\ddot{s} - \dot{\varphi}\cos(\varphi)L\dot{s}) \qquad \textbf{(A.4)}$$

In order to get the second term of Equation , the Lagrangian is differentiated with respect to ϕ:

$$\frac{\partial \mathcal{L}}{\partial \varphi} = M(-\dot{\varphi}\cos(\varphi)L\dot{s}) - Mg(-\sin(\varphi)L) = M(-\dot{\varphi}\cos(\varphi)L\dot{s} + g\sin(\varphi)L) \qquad \textbf{(A.5)}$$

Now put in equation (A.1) to get :

$$M(L^2\ddot{\varphi} - \sin(\varphi)L\ddot{s} - \dot{\varphi}\cos(\varphi)L\dot{s}) - M(-\dot{\varphi}\cos(\varphi)L\dot{s} + g\sin(\varphi)L) = -d\dot{\varphi}$$

Which is equal to :

$$\ddot{\varphi} + \lambda\dot{\varphi} - \frac{1}{L}(g + \ddot{s})\sin(\varphi) = 0 \qquad (1.9)$$

Where -

$$\lambda = \frac{d}{ML^2}$$

Equilibrium positions and linearization:->

The nonlinear differential equation displayed in Equation 1.9 is challenging to solve analytically , but it can be linearized about equilibrium point , which is more straight-forward to solve. For that we have to fined equilibrium positions . As we know, potential energy is constant at equilibrium positions.

So , at equilibrium positions:

$$\frac{dE_p}{d\varphi} = 0 \tag{1.10}$$

$$\frac{dE_p}{d\varphi} = -Mg\sin(\varphi) = 0$$

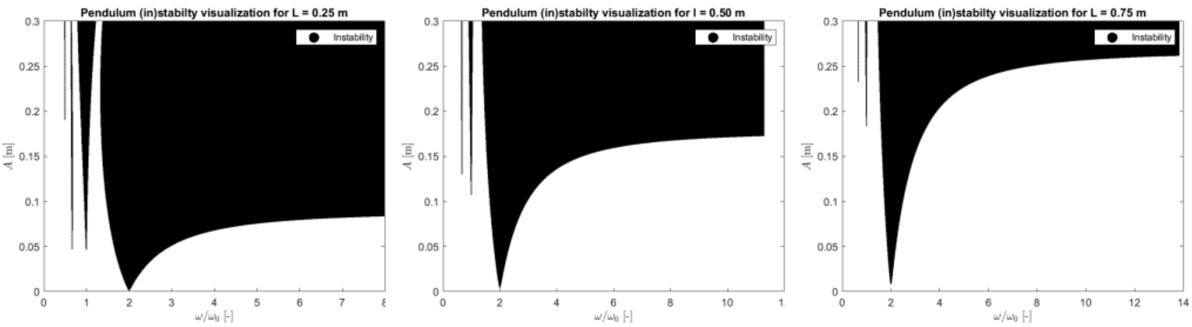
the two equilibrium positions logically are found to be $\phi = 0 + k \cdot 2\pi$ and $\phi = \pi + k \cdot 2\pi$, corresponding to the top and bottom position of the pendulum respectively.

This stability analysis will focus on the instability of the pendulum's bottom position because this matches with the experimental setup which will be used later to validate the model. Therefore, Equation 1.9 will be linearized around this bottom equilibrium position to get the linear equation of motion. This results in the following differential equation, describing the linearized pendulum's equation of motion around the bottom position:

$$\ddot{\theta} + \lambda\dot{\theta} + \frac{1}{L}(g + \ddot{s})\theta = 0 \tag{1.11}$$

Analysis results

The theory described above is implemented in Matlab to visualize pendulum stability for varying conditions. For this analysis, several assumptions are made. The pendulum length L is set to 0.25 m, 0.50 m and 0.75 m for three different simulations. The damping coefficient λ is set to 0.1. These values have been chosen arbitrary. As mentioned before, this analysis solely focuses on the instability of an oscillated pendulum in bottom position (i.e. $\phi 0 \approx \pi$). The result for this analysis can be seen in Figure 1.3.1, where the dark area represents an unstable behaviour of the pendulum. On the x-axis, the normalized angular frequency is plotted. This implies division of the excited angular frequency by the pendulum's eigenfrequency .On the y-axis the excitation amplitude is graphed.



(a) L = 0.25 m (b) L = 0.50 m (c) L = 0.75 m

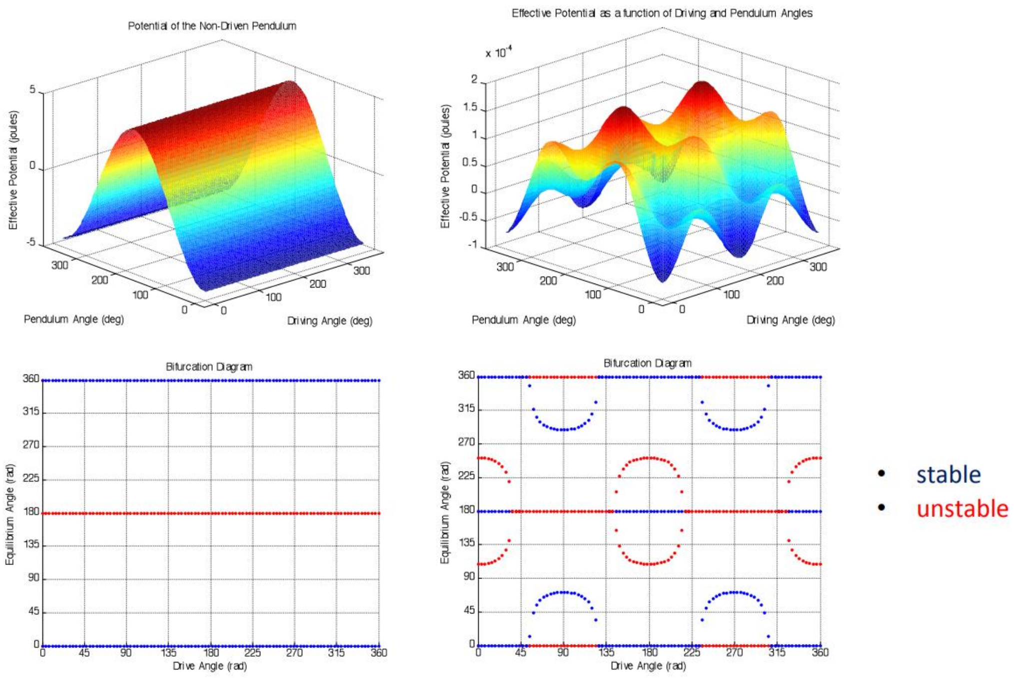
Figure 1.3.1: Pendulum (in)stability for varying oscillation amplitude and frequency

As can be observed in Figure 1.3.1, the described pendulum can both be stable and unstable for various combinations of length, oscillation frequency and amplitude. In general, the shorter the pendulum length the lower the required excitation amplitude in order to become unstable at relatively higher frequencies. Nevertheless, all three different pendulum lengths show similar behaviour in terms of parametric resonance. The largest peak in resonance is invariably located at $\omega\ \omega 0 \approx 2$, with several smaller resonance peaks to its left at $\omega\ /\omega 0 \approx 1$ and $\omega\ /\omega 0 \approx 0.67$. The right side of this largest peak does not show any resonance. This observation is in line with existing articles about this subject [3]. The results of this theoretical approach on the (in)stability of a vertically oscillated pendulum will be validated by performing an experiment, which is elaborated in chapter 2.

BIFRUCATION DIAGRAM->

Bifurcation analysis is a mathematical and computational technique used in dynamical system theory to study how to behavior of a system changes as one or more parameters are varied . it is particularly useful in understanding the qualitative changes in the system's dynamics, including the appearance of a new stable or unstable solution , periodic orbits , chaos , or other complex behaviors.

Here we will use three different variables effective potential ,pendulum angle and driving angle .
Using these three variables we will make our bifurcating diagram and stability diagrams.



APPLICATIONS OF INVERTED PENDULUM

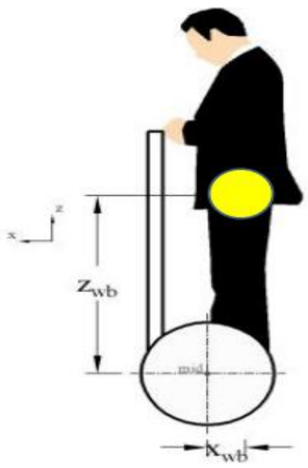
1. Rockets :

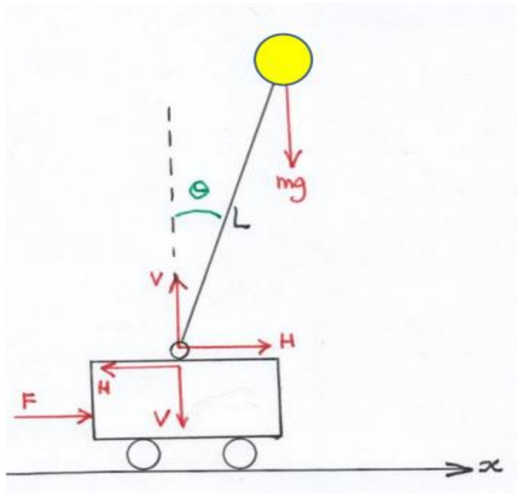
It is the belief that rockets are passively stable if the engines are at the tip and the rocket "hangs" from them. The fallacy lies in the expectation that gravity pulls the body of the rocket downwards, while the engines pull it upwards.

The problem of the inverted pendulum is to control an upside-down rigid pendulum by moving the base or applying a torque to it, with gravity exerting a tilting force. For example, by balancing a broomstick or a pencil on your hand. Rockets are not really inverted pendulums, because the disturbing torque from a misaligned thrust is independent of the orientation of the vehicle and gravity, but their response to such a misaligned thrust or outside disturbance is similar, and balancing an inverted pendulum is sometimes used as an analogy to rocket control.

2. Segway :

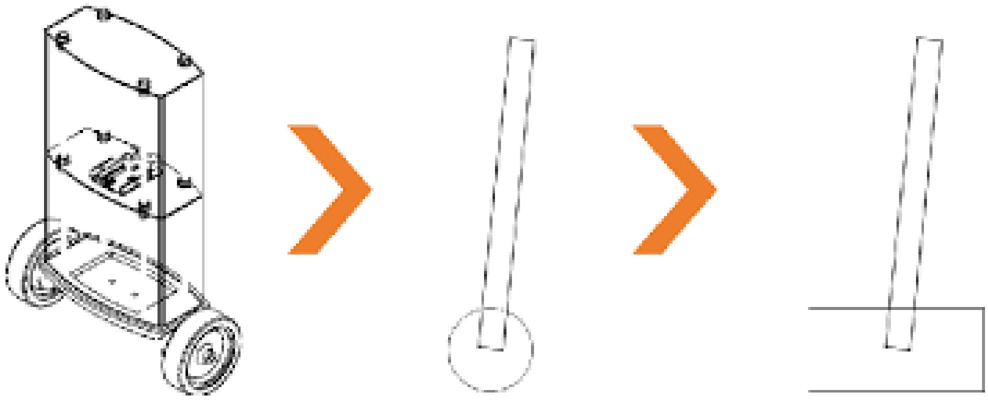
On the left is a picture of an inverted pendulum on a cart and on the right is a guy on a Segway. These two systems are identical where the yellow pendulum bob corresponds to the yellow centre of mass of the guy on the right. Such systems are unstable unless they are controlled, they will fall over; the pendulum will rotate clockwise and settle with the bob underneath the cart. The guy will rotate either forwards or backwards and will end up on the ground if he is daft





3. Self-balancing Robot :

A robot that uses the inverted pendulum is usually a tower-shaped structure that normally stands on two wheels and controls the motors autonomously so that it can keep itself upright while also moving under the control of user input It consists of both a hardware and a software implementation. The mechanical model is based on the state space design of the cart, the pendulum system. To find the stable inverted position, I used a generic feedback controller (i.e. PID controller). Depending on the situation, we need to control both the angle of the pendulum and the position of the carriage cart.



CONTRIBUTIONS

- Nikhlesh Singh
    - Derivation of Equation of motion
    - Real world analysis
    - Documentation

- Omprakash Balara
    - Euler-Lagrange equation
    - Analysis result and bifurcation analysis
    - Documentation and report

- Pooja Nunna
    - Research
    - Documentation
    - Bifurcation analysis

- Nishant Singh
    - Real life uses analysis
    - Documentation
    - Research

REFERENCES

- https://research.tue.nl/files/212921269/1459090_Stabilization_inverted_pendulum.pdf

- https://en.wikipedia.org/wiki/Inverted_pendulum

s

- Andrew Stephenson. XX. On induced stability. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 15(86):233–236, 1908.

- https://www.math.arizona.edu/~gabitov/teaching/131/math_485_585/Midterm_Reports/pendulum.pdf

- https://youtu.be/1Gf5urTHQsE?feature=shared