Direct comparison between different growth models

Dominik Brandstetter
Department of Physics, Faculty of Natural Sciences, Graz University
(Dated: November 17, 2020)

As part of the exercise "Methods of Modelling and Simulation", I present a direct comparison between the following different models to simulate the population growth on an island: Linear Growth Model, Exponential Growth Model, Logistic Growth Model, Goal Seeking Growth Model, Goal Setting Growth Model.

1. Introduction

We will look at the population growth of an island possibly affected by birth & death rates, immigration, feedback loops for the growth rate or target population goals. First all five different growth models used will be outlined theoretically briefly and the results of its simulation discussed individually in the section 2 to 6. After that we will look at a comparison between them in section 7. If not stated otherwise, assume the handout for the exercise from the 10.08.2020 to be the source of information. All simulations have been performed in the Python language (v3.8.5) with the "NumPy" package (v1.19.1) and "MatPlotLib" (v3.3.1) has been used to make the plots. For access to the source code, please ask the author.

2. Linear Growth Model

2.1. Theory

The linear growth model is probably the most simple and basic type of model to simulate a population affected by immigration. At its core, it assumes a constant base population x_0 at the time t_0 being the subject to a constant stream of immigration $I(t) = I$ (the growth function) every unit of time. Thus we can state the differential form of this model as

$$\frac{dx}{dt} = I = const. \tag{1}$$

and by simple integration over dt we get the analytical solution:

$$x(t) = I * (t - t_0) + x_0 \tag{2}$$

2.2. Simulation

The simulation confirms what the analytical solution predicts. The initial population has no effect on its growth as the growth rate does not depend on the population at all. The growth rate has a direct linear effect on the growth rate.

3. Exponential Growth Model

3.1. Theory

The simplest of our models affected by a feedback loop is the exponential growth model. It is a common patterns in nature, and we will use it to model a population affected by birth b and death d rates. While the rates are independent of the current population, the absolute number of deaths and births does depend on it. Stating the functional relation

$$x(t + dt) = x(t) * [1 + b * dt - d * dt] \tag{3}$$

we can see that we can simplify this equation by grouping together both the birth and the death rate to what we will call the growth rate $r = b - d$. It's sign tells us whether the population will grow or shrink, and it's magnitude the speed by which it will happen. Rearanging (3)

$$x(t + dt) - x(t) = dx = r * x(t) * dt \tag{4}$$

of which the solution is of course

$$x(t) = x_0 e^{r(t-t_0)} \tag{5}$$

3.2. Simulation

Most people nowadays are very familiar with exponential growth and its potential harm as it quickly outgrows all boundaries. Our simulation shows exactly this. The growth is very sensitive to the growth rate r while the initial population has a comparative insignificant effect.

4. Logisic Growth Model

4.1. Theory

In the logistic growth model the population will assume its maximum size (capacity) K. For this we assume a linear negative dependence of the growth rate on the current population size

$$r(x) = 1 - \frac{x(t)}{K} \tag{6}$$

For $x(t) \to K \Rightarrow r(x) \to 0$. Thus we get

$$\frac{dx}{dt} = x(t) * r(x) = x(t) - \frac{x^2(t)}{K} \tag{7}$$

with a solution

$$x(t) = \frac{K}{1 + (\frac{K}{x_0} - 1)e^{-r(t-t_0)}} \tag{8}$$

This equation is known as Fermi-Dirac statistics in physics.

4.2. Simulation

The purpose and capability of this model is very apparent when varying th capacity K. In all simulation runs the system quickly grows close to this upper boundary and then asymptotically draws near it. Even relatively high growth rates r, which are very potent as we saw in the exponential growth model, only affect the speed by which the population approaches the boundary, but does not affect the boundary itself.

5. Goal Seeking Growth Model

5.1. Theory

In this model we depend the growth rate on the difference between current population and the goal Z. This will drive the system towards the goal population.

$$\frac{dx}{dt} = r * [Z - x(t)] \tag{9}$$

$$\Rightarrow x(t) = Z * \left[1 - e^{-r(t-t_0)} + x_0 e^{-r(t-t_0)}\right] \tag{10}$$

5.2. Simulation

As we see in varying the initial population x_0, this model will always tend towards the goal Z, regardless of whether the system starts below or above this goal.

6. Goal Setting Growth Model

6.1. Theory

Last but not least we will look at the goal setting growth model. It can be viewed as a combination of linear growth model and the exponential growth model as it factors in both immigration and growth rate. Thus

$$\frac{dx}{dt} = I - r * x(t) \tag{11}$$

$$\Rightarrow x(t) = -\frac{I}{r} + (x_0 + \frac{I}{r}) * e^{r(t-t_0)} \tag{12}$$

6.2. Simulation

This model will, similarly to the goal seeking model, tend towards a fixed value. However, this goal is not explicitly set but is rather "set" by the ratio of immigration I to growth rate r. However, this only applies for opposite forces, i.e. the immigration and the growth rate are working against each other. In the case of positive growth rate and positive immigration the model grows unbounded. In the case of both a negative immigration rate and a negative growth rate the population reach zero in a non-asymptotically way.

7. Comparison

For this plot the exponential growth model got its own y axis, which demonstrates clearly that this model grows far far quicker than any of the other models presented here. It grows without boundary at an ever increasing speed. The only other model also growing boundlessly is the linear model, however at a slower pace. All the other models tend towards a goal/equilibrium/capacity.

Appendix A

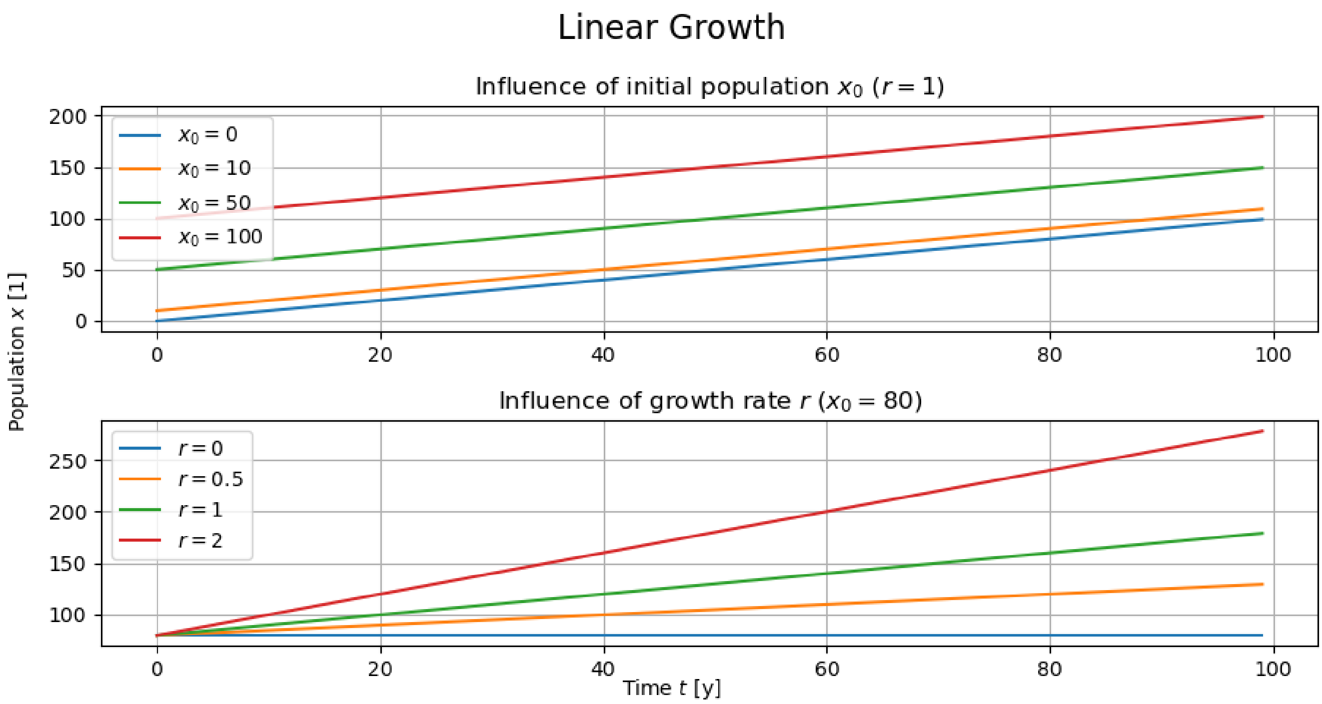

FIG. 1: Simulation of the linear growth model and the influence of different parameters (initial population x_0 in (a) and growth rate r in (b)) to its growth.

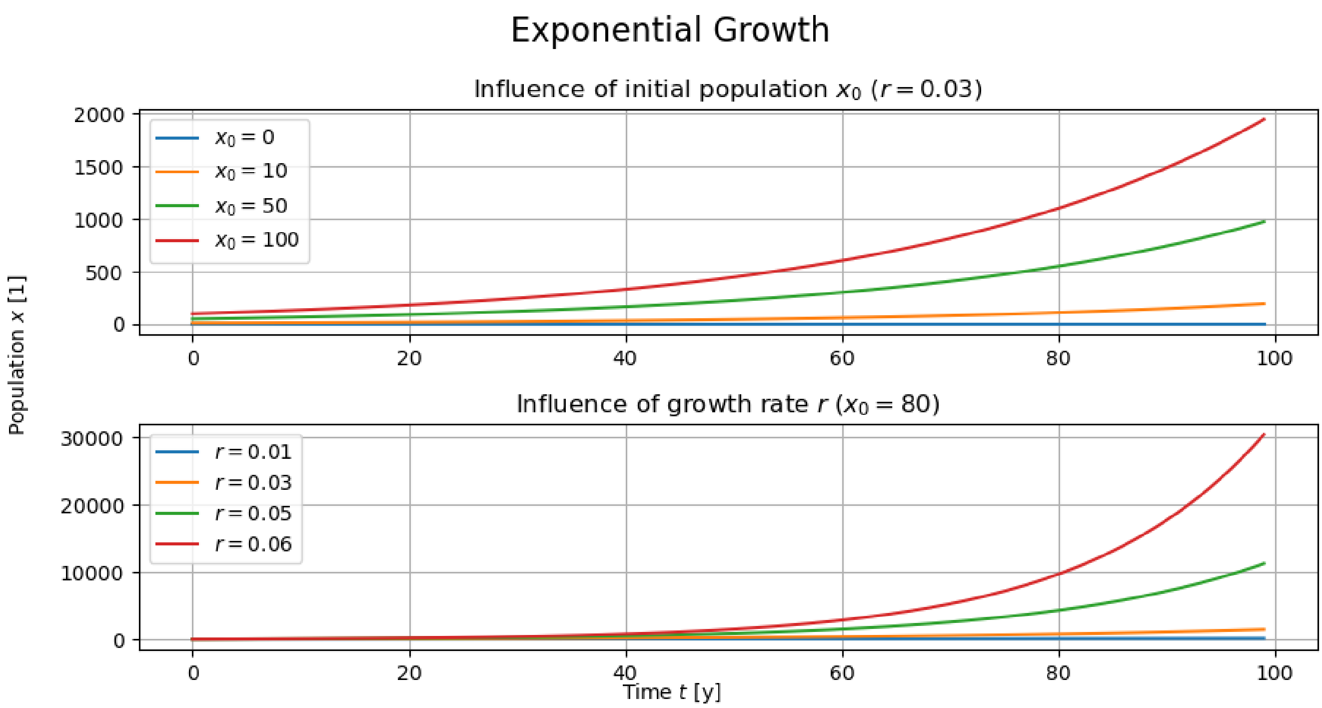

FIG. 2: Simulation of the exponential growth model and the influence of different parameters (initial population x_0 in (a) and growth rate r in (b)) to its growth.

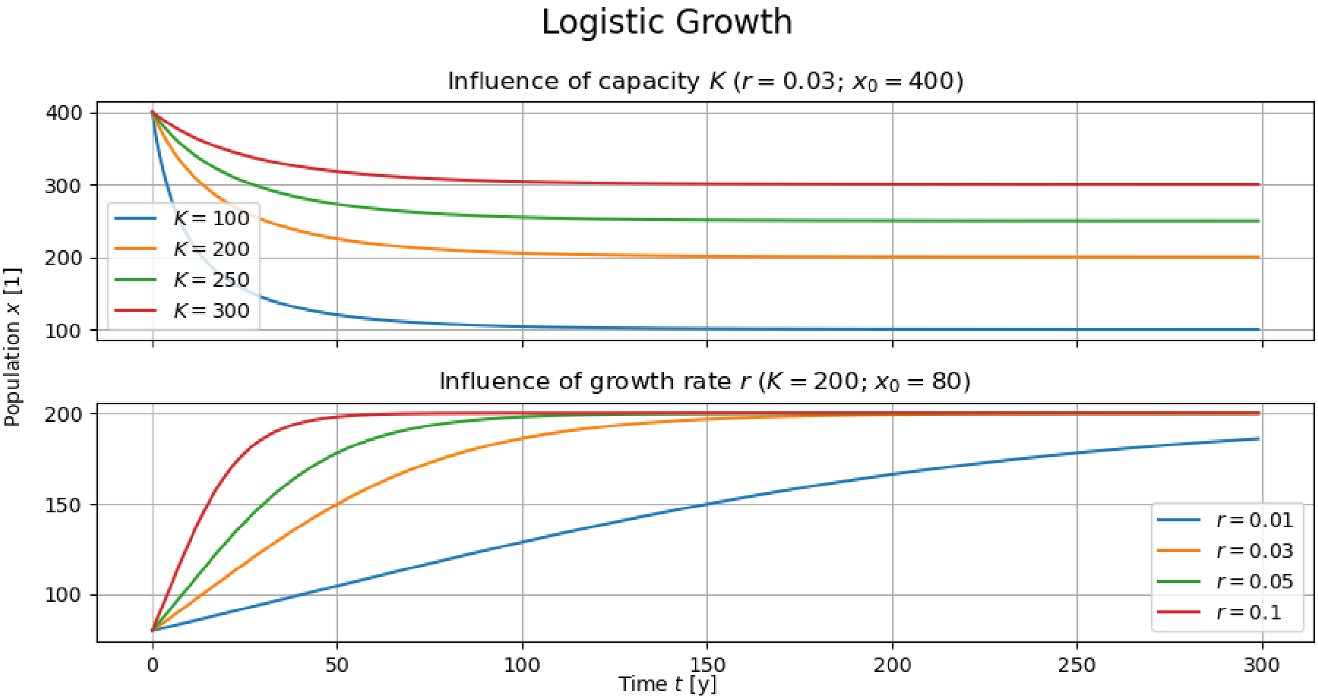

FIG. 3: Simulation of the logistic growth model and the influence of different parameters (capacity K in (a) and growth rate r in (b)) to its growth.

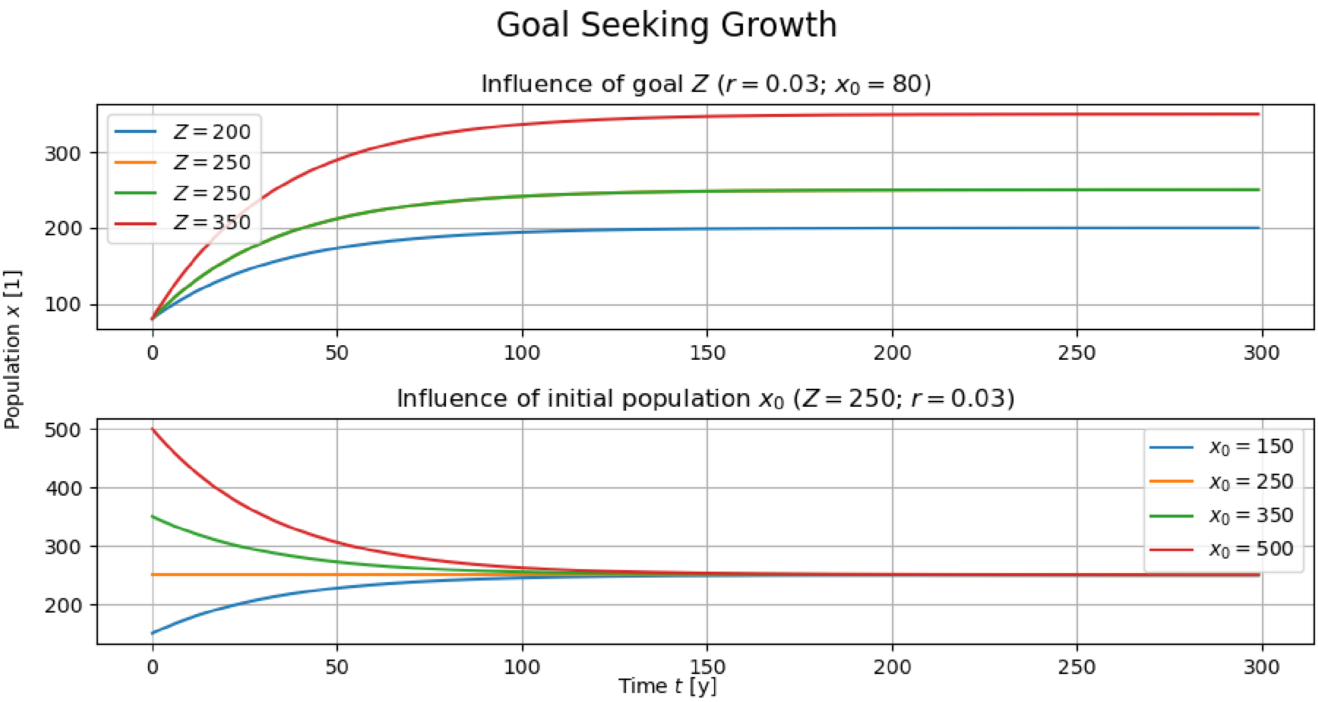

FIG. 4: Simulation of the goal seeking growth model and the influence of different parameters (goal Z in (a) and initial population x_0 in (b) and initial population x_0) to its growth.

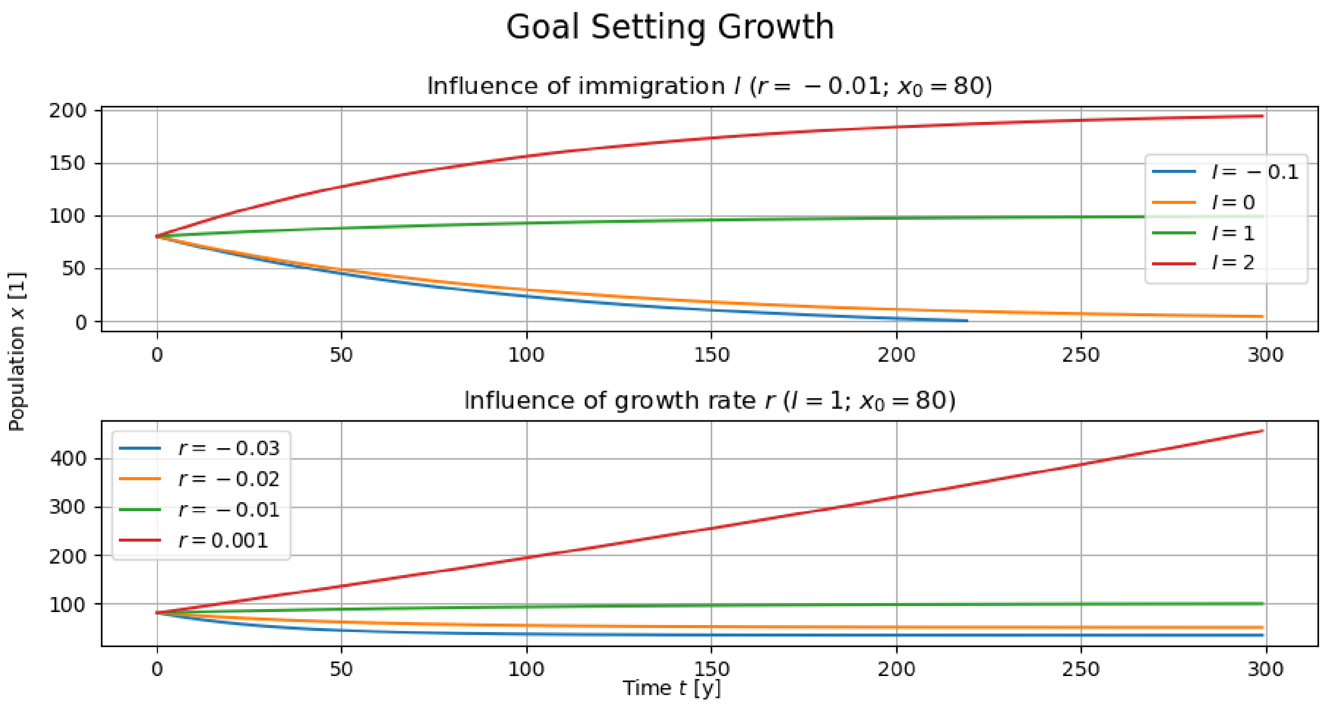

FIG. 5: Simulation of the goal setting growth model and the influence of different parameters (immigration I in (a) and growth rate r in (b)) to its growth.

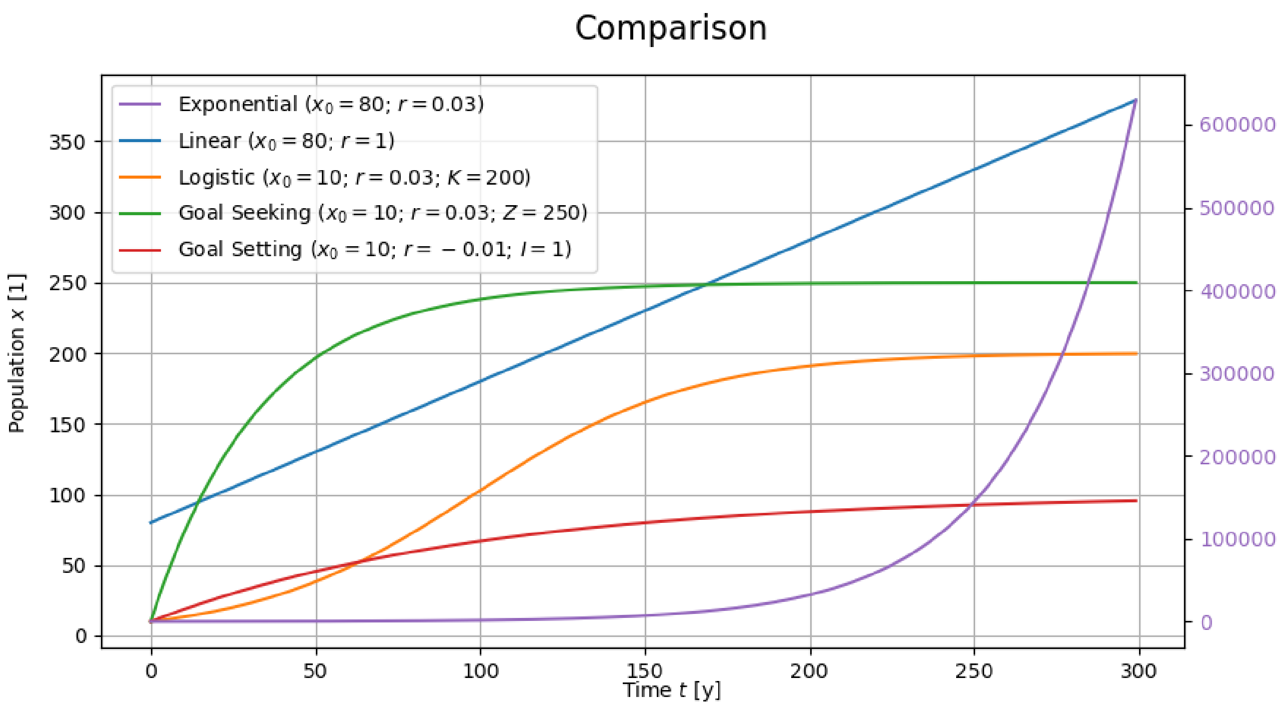

FIG. 6: Comparison between the different models.