Discrete Fourier Transformation (DFT)

Torsten Mayer-Gürr

Institute of Geodesy
Graz University of Technology

Time Series Analysis and Filtering

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Fourier transform

Fourier transformation $$F(\eta) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\eta t} dt$$

Fourier integral $$f(t) = \int_{-\infty}^{\infty} F(\eta) e^{i2\pi\eta t} d\eta$$

Time domain $f(t)$ $\rightleftarrows$ Frequency domain $F(\eta)$

Operator

$$F(\eta) = \mathcal{F}[f(t)] \qquad \Leftrightarrow \qquad f(t) = \mathcal{F}^{-1}[F(\eta)]$$

with $$\mathcal{F}[(\bullet)] = \int_{-\infty}^{\infty} (\bullet) e^{-i2\pi\eta t} dt$$

Fourier transform

Fourier transformation $F(\eta) = \int\limits_{-\infty}^{\infty} f(t) e^{-i2\pi\eta t} dt$

Fourier integral $f(t) = \int\limits_{-\infty}^{\infty} F(\eta) e^{i2\pi\eta t} d\eta$

Time domain $f(t)$ ⇄ Frequency domain $F(\eta)$

Fourier transform is complex

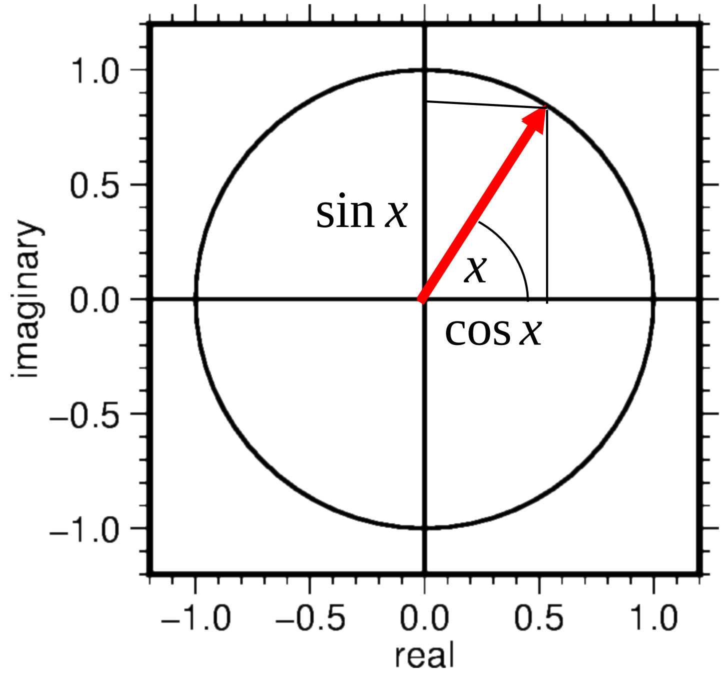
$$F(\eta) = a(\eta) + ib(\eta)$$
$$= A(\eta) e^{i\varphi(\eta)}$$

Amplitude

$$A(\eta) = |F(\eta)| = \sqrt{F(\eta)^* F(\eta)} = \sqrt{a^2(\eta) + b^2(\eta)}$$

Phase

$$\tan\varphi(\eta) = \frac{b(\eta)}{a(\eta)} \quad \varphi = \mathrm{atan2}(b, a)$$



The boxcar function and the sinc function

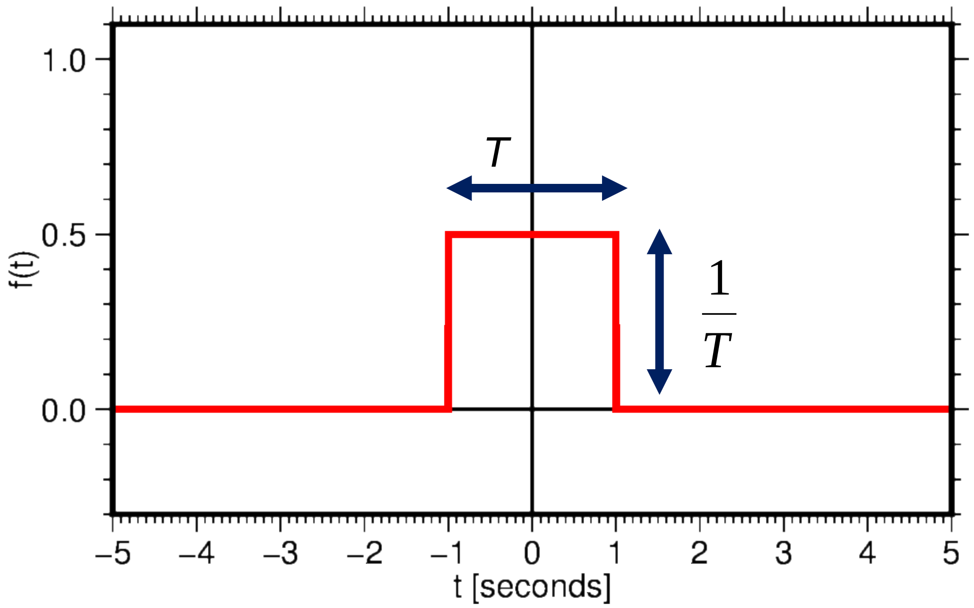
Boxcar function

Definition

$$f(t) = \frac{1}{T}\begin{cases} 1 & \text{for} \quad -T/2 < t < T/2 \\ 0 & \text{else} \end{cases}$$

Area

$$A = \int_{-\infty}^{\infty} f(t)\,dt = 1$$



Boxcar function

Fourier transform

$$F(\eta)=\int_{-\infty}^{\infty} f(t)e^{-i2\pi\eta t}dt$$

$$F(\eta)=\frac{1}{T}\int_{-T/2}^{T/2}1\cdot e^{-i2\pi\eta t}dt \quad =\frac{1}{T}\left[\frac{1}{-i2\pi\eta}e^{-i2\pi\eta t}\right]_{-T/2}^{T/2} \quad =-\frac{1}{i2\pi T\eta}\left[e^{-i\pi\eta T}-e^{i\pi\eta T}\right]$$

$$=-\frac{1}{i2\pi T\eta}\left[\cos(\pi T\eta)-i\sin(\pi T\eta)-\cos(\pi T\eta)-i\sin(\pi T\eta)\right]$$

$$=-\frac{1}{i2\pi T\eta}\left[-2i\sin(\pi T\eta)\right]$$

$$=\frac{\sin(\pi T\eta)}{\pi T\eta}$$

$$=:\operatorname{sinc}(T\eta)$$

Boxcar function

Sinc function

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

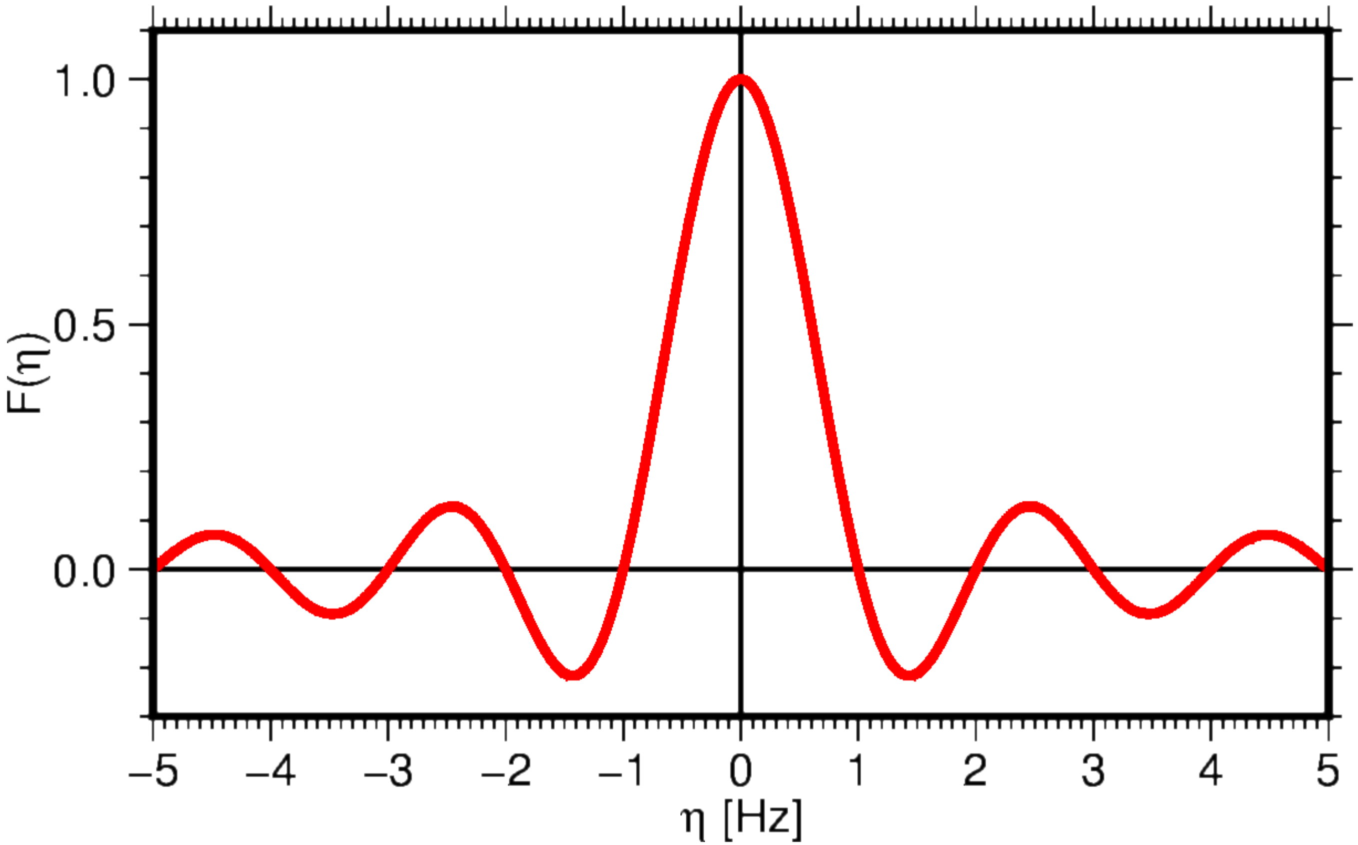
Two definitions

- Unormalized sinc function

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}$$

- Normalized sinc function

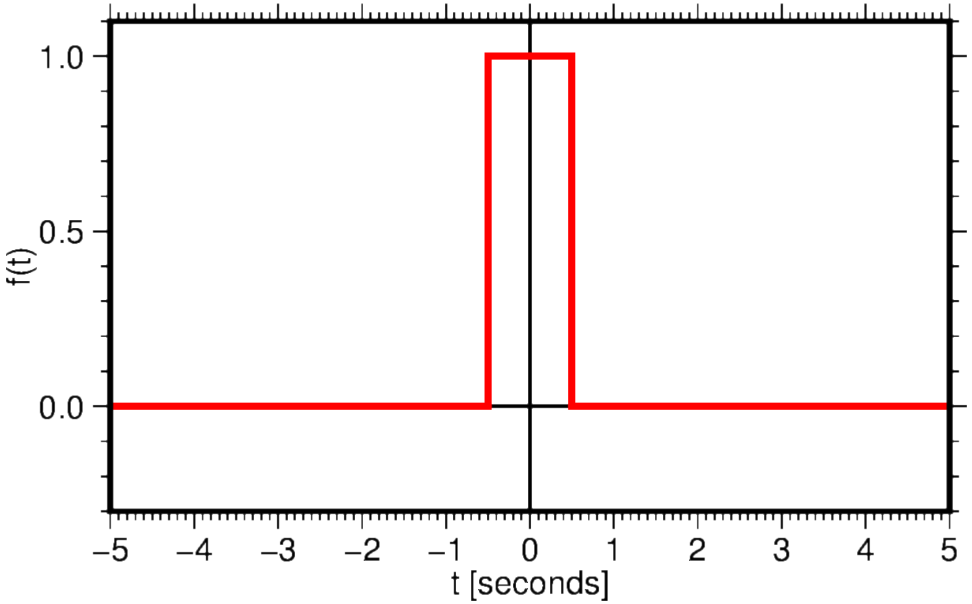
$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

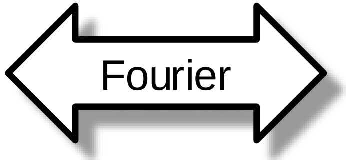


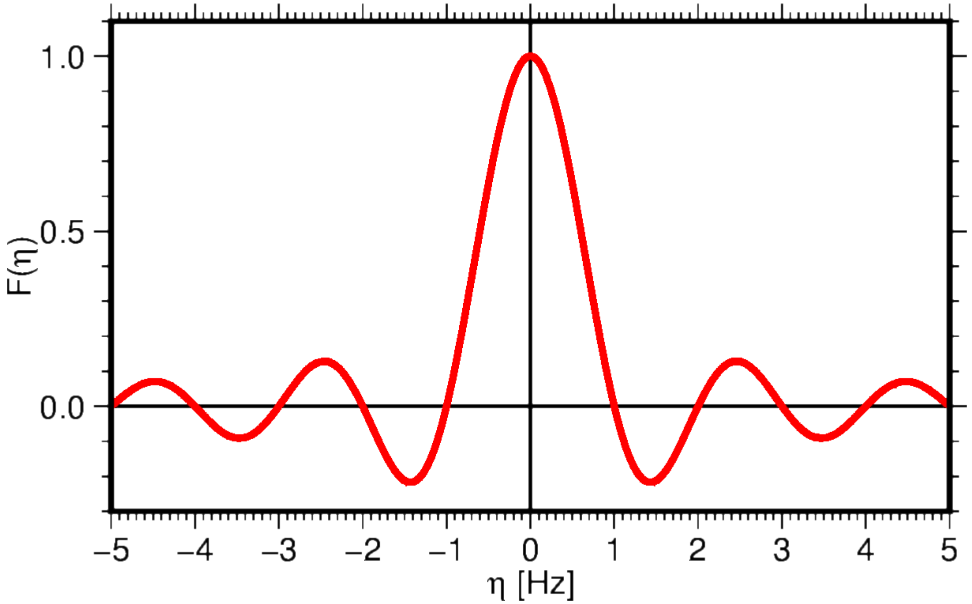
Boxcar function

$$f(t) = \frac{1}{T}\begin{cases} 1 & \text{for} \quad -T/2 < t < T/2 \\ 0 & \text{else} \end{cases}$$

$$F(\eta) = \mathrm{sinc}(T\eta)$$





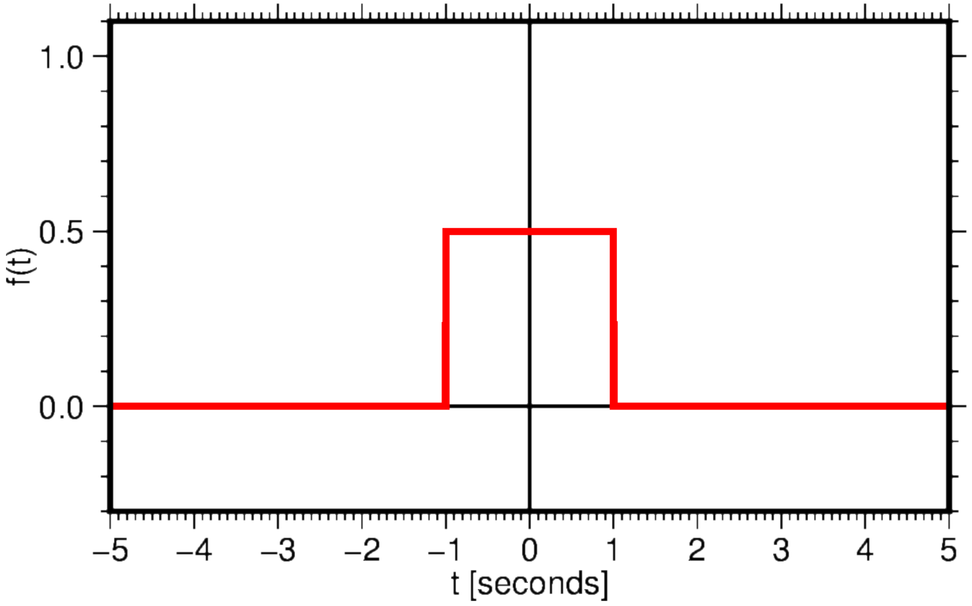


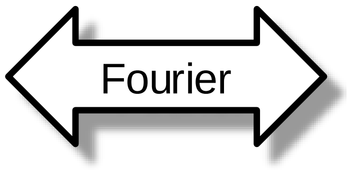
with T = 1s

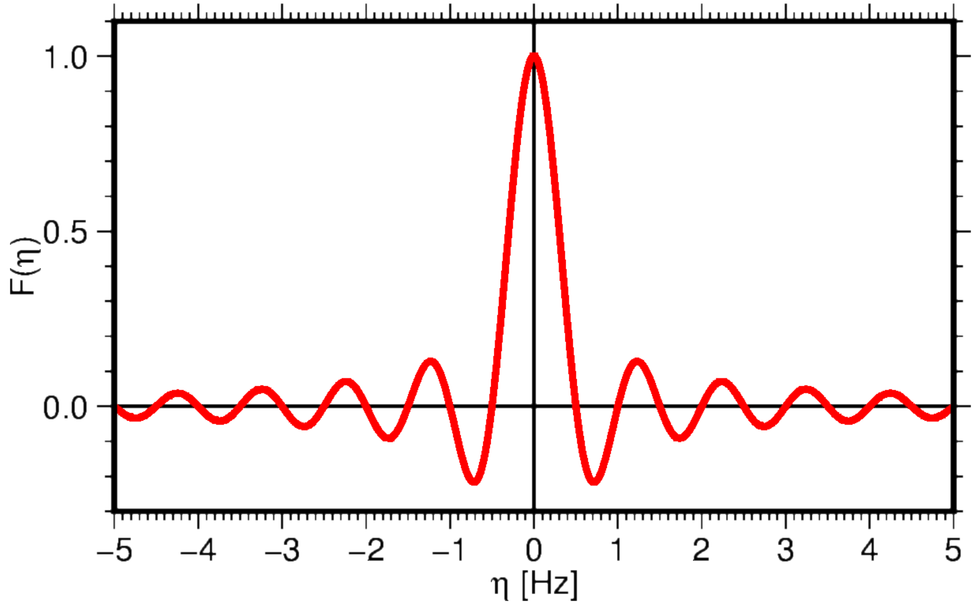
Boxcar function

$$f(t) = \frac{1}{T}\begin{cases} 1 & \text{for} \quad -T/2 < t < T/2 \\ 0 & \text{else} \end{cases}$$

$$F(\eta) = \operatorname{sinc}(T\eta)$$



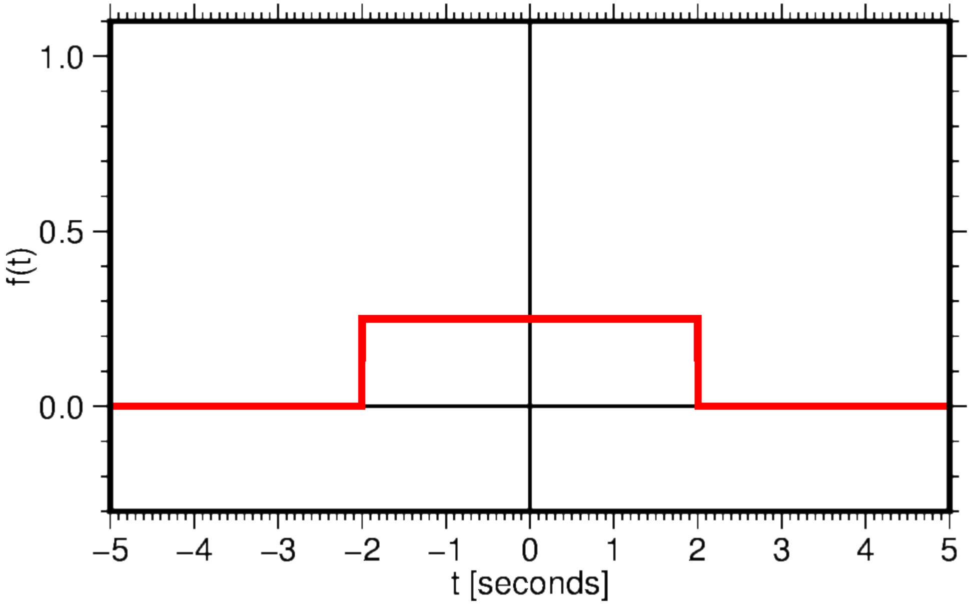




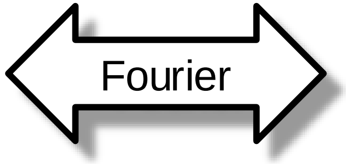
with T = 2s

Boxcar function

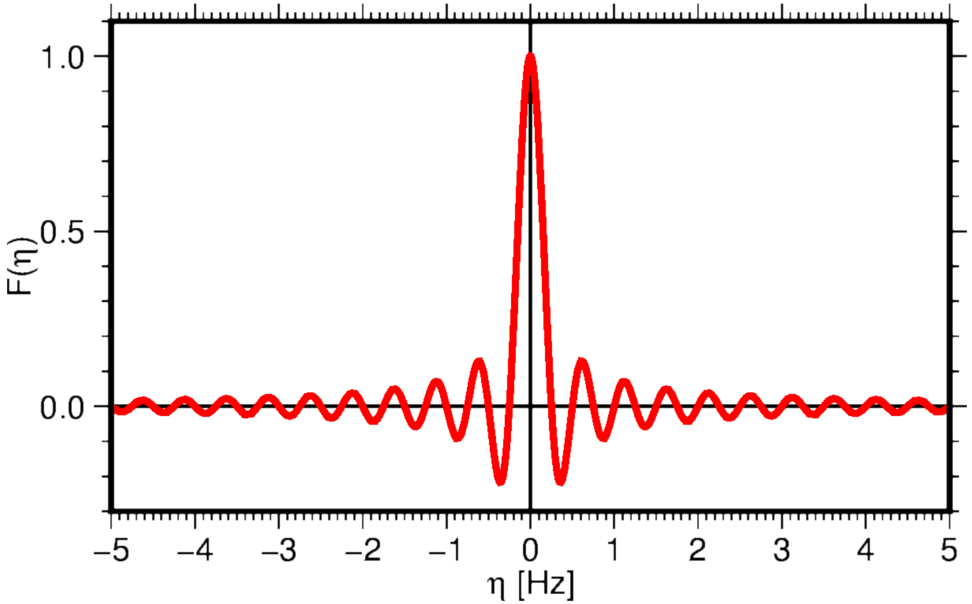
$$f(t) = \frac{1}{T}\begin{cases} 1 & \text{for} \quad -T/2 < t < T/2 \\ 0 & \text{else} \end{cases}$$



with T = 4s



$$F(\eta) = \mathrm{sinc}(T\eta)$$



Boxcar function

$$f(t) = \frac{1}{T}\begin{cases} 1 & \text{for} \quad -T/2 < t < T/2 \\ 0 & \text{else} \end{cases}$$

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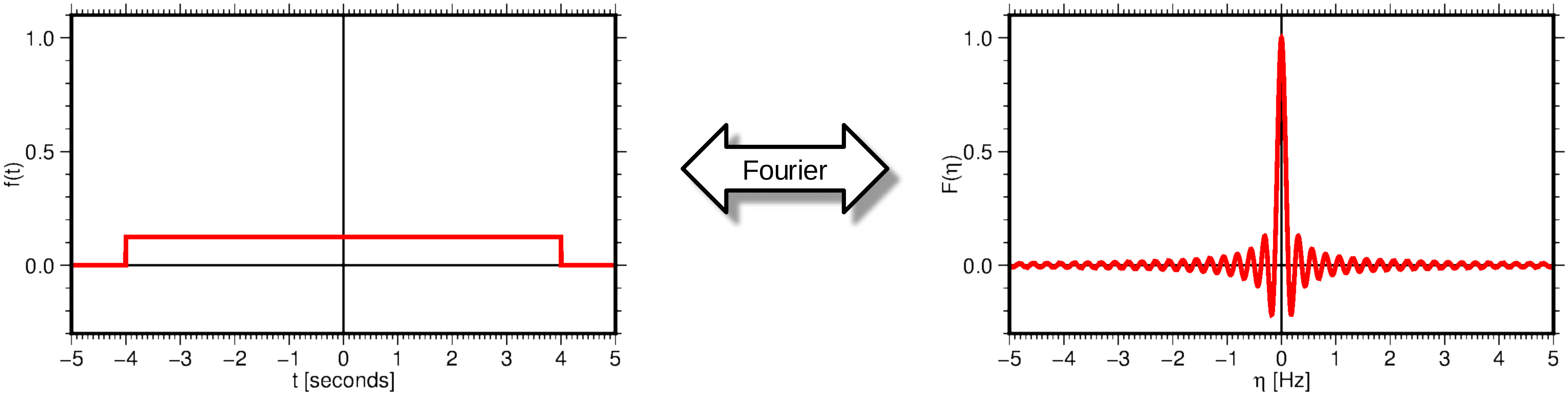


with T = 8s

Boxcar function

$$f(t)=\frac{1}{T}\begin{cases}1 & \text{for} \quad -T/2<t<T/2 \\ 0 & \text{else}\end{cases} \quad \xleftrightarrow{\text{Fourier}} \quad F(\eta)=\operatorname{sinc}(T\eta)$$

$$f(t)=\operatorname{sinc}(Tt) \quad \xleftrightarrow{\text{Fourier}} \quad F(\eta)=\frac{1}{T}\begin{cases}1 & \text{for} \quad -T/2<\eta<T/2 \\ 0 & \text{else}\end{cases}$$

calculation rules

Fourier transform

Fourier transform $$F(\eta) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\eta t} dt$$

Fourier integral $$f(t) = \int_{-\infty}^{\infty} F(\eta) e^{i2\pi\eta t} d\eta$$

Calculation rules

Linearity $$F[\alpha\, f(t)+\beta\, g(t)]=\alpha\, F[\, f(t)]+\beta\, F[g(t)]$$

Shift $$F[\, f(t-t_0)]=e^{-i2\pi\eta t_0}F[\, f(t)]$$ with $$\left|e^{-i2\pi\eta t_0}\right|=1$$

Spreading $$F[\, f(\alpha\, t)]=\frac{1}{|\alpha|}F(\eta/\alpha)$$

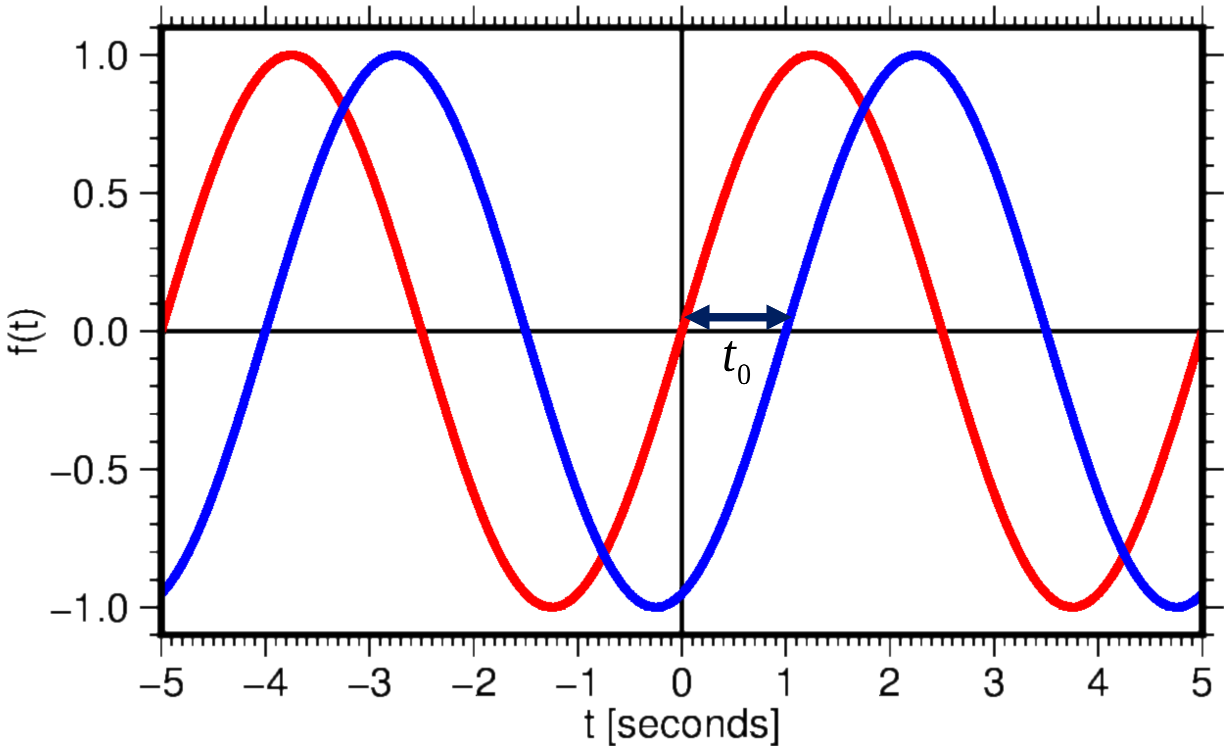
Derivation $$F\left[\frac{df}{dt}\right]=i2\pi\eta\, F[\, f(t)]$$

$$F\left[\frac{d^k f}{dt^k}\right]=(i2\pi\eta)^k\, F[\, f(t)]$$

Integration $$F\left[\int_{-\infty}^{t} f(x)dx\right]=\frac{1}{i2\pi\eta}F[\, f(t)]$$

Calculation rules

Shift $$F\left[f(t-t_0)\right]=e^{-i2\pi\eta t_0}F\left[f(t)\right]$$ with $$\left|e^{-i2\pi\eta t_0}\right|=1$$



Calculation rules

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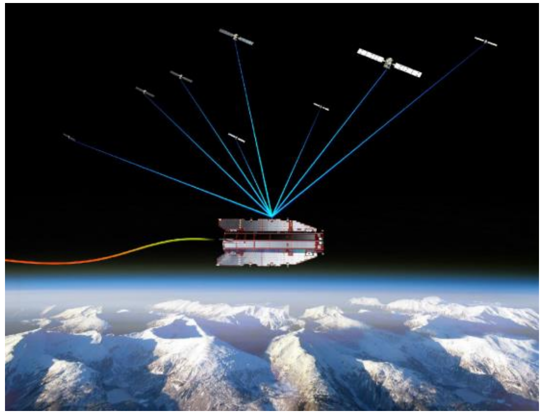
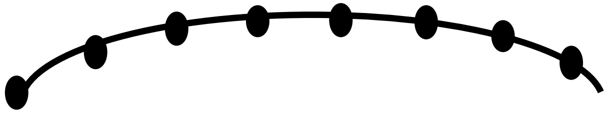
Integration $$F\left[\int_{-\infty}^{t} f(x)dx\right]=\frac{1}{i2\pi\eta}F[\, f(t)]$$

Example

Satellite gravimetry



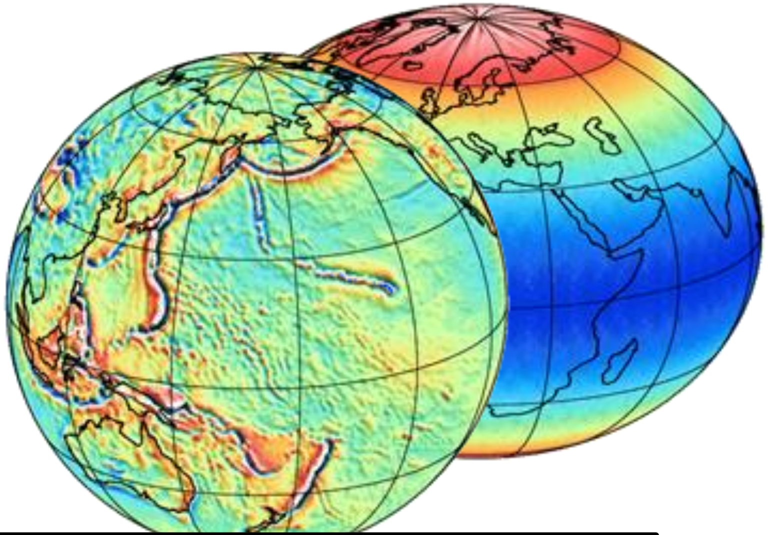


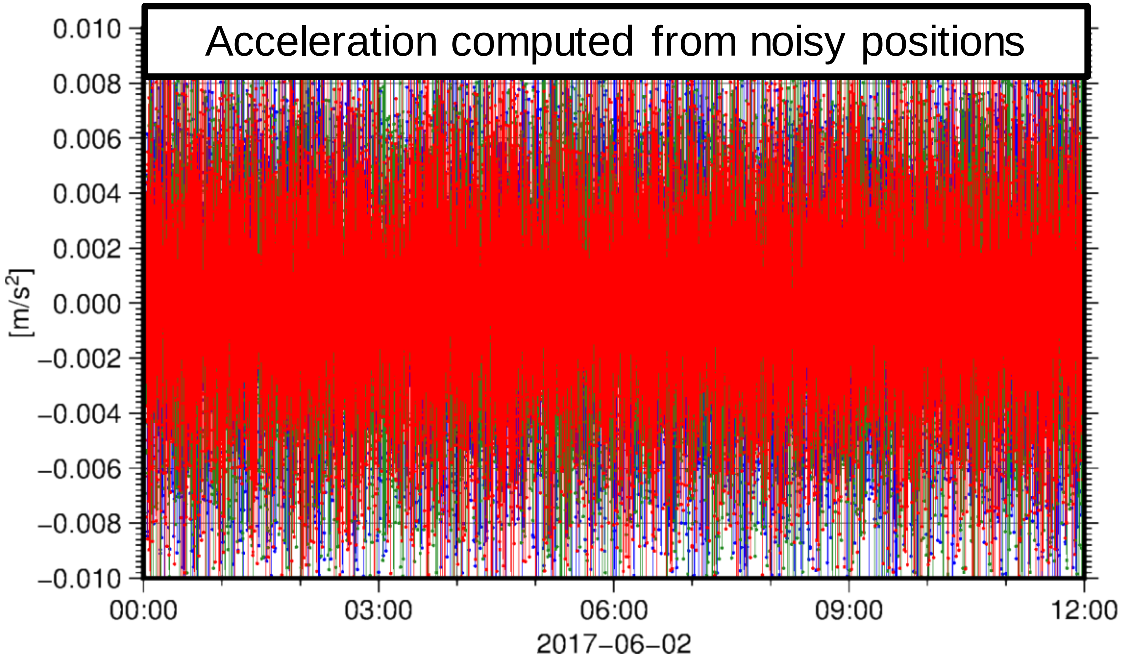
Position r_i

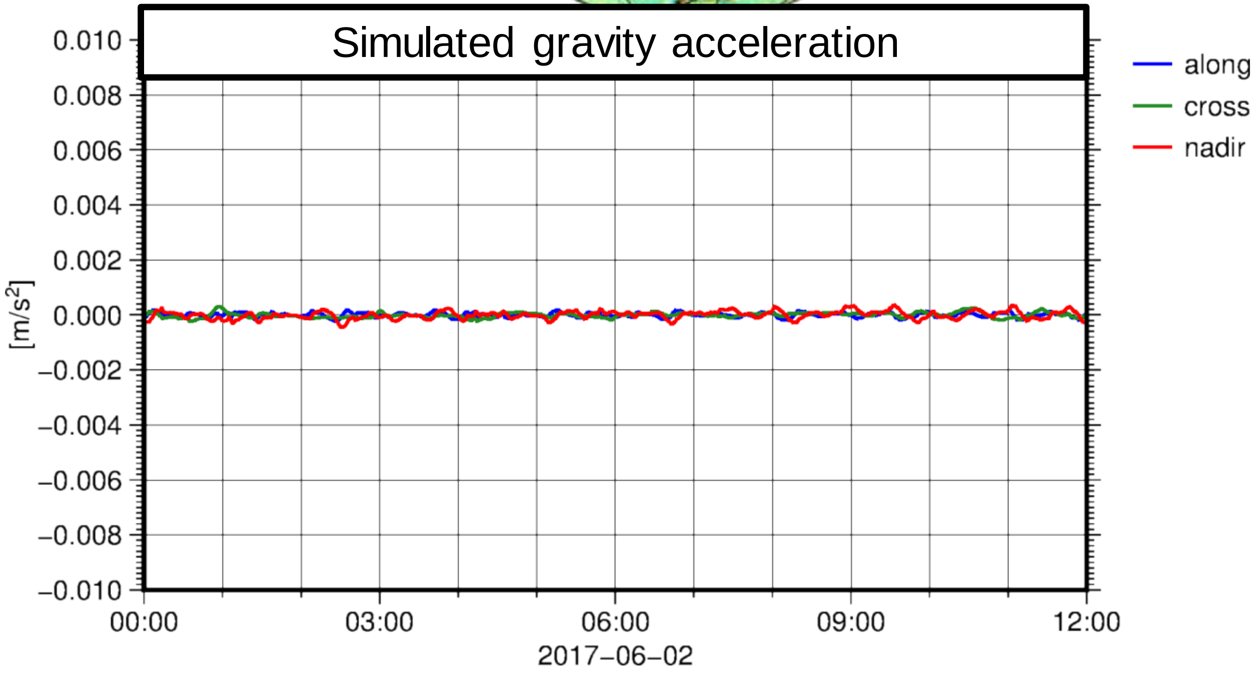
every 5s with 3 cm noise

Velocity: $\dot{r}_i = \frac{r_{i+1} - r_i}{\Delta t}$

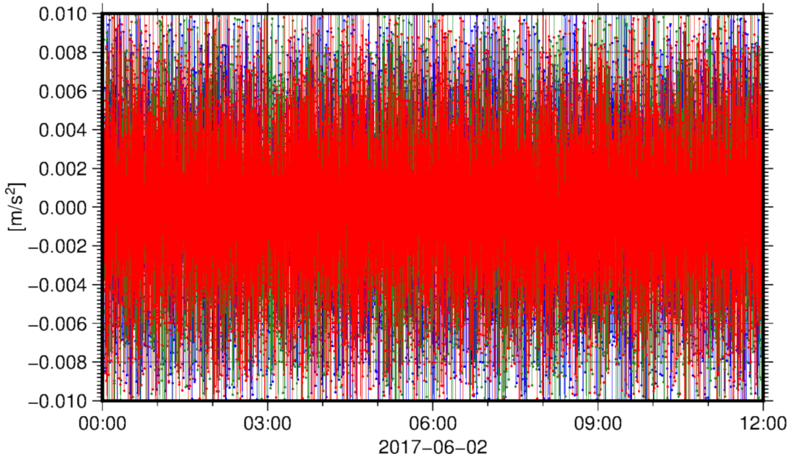
Acceleration $\ddot{r}_i = \frac{r_{i+1} - 2r_i + r_{i-1}}{2\Delta t}$

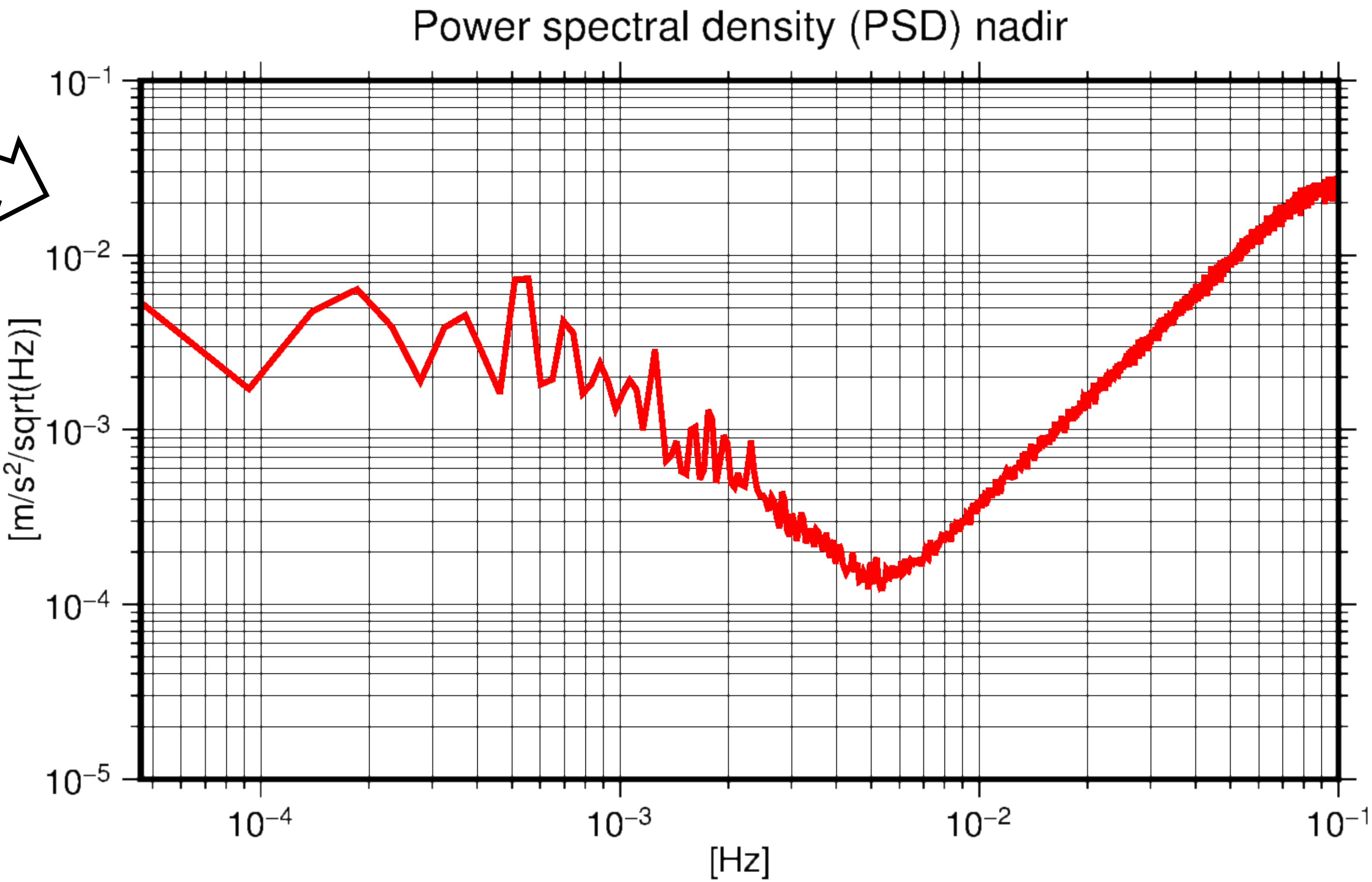






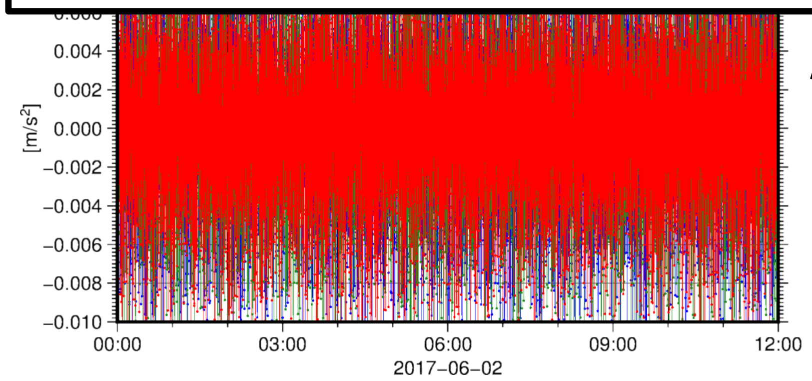
Satellite gravimetry

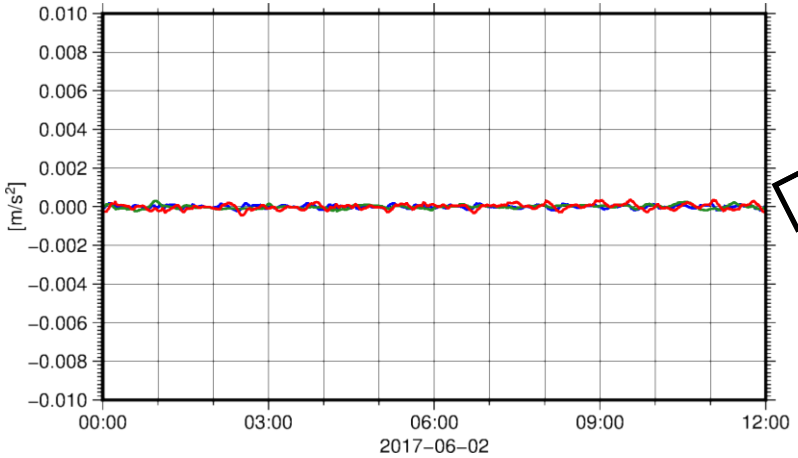


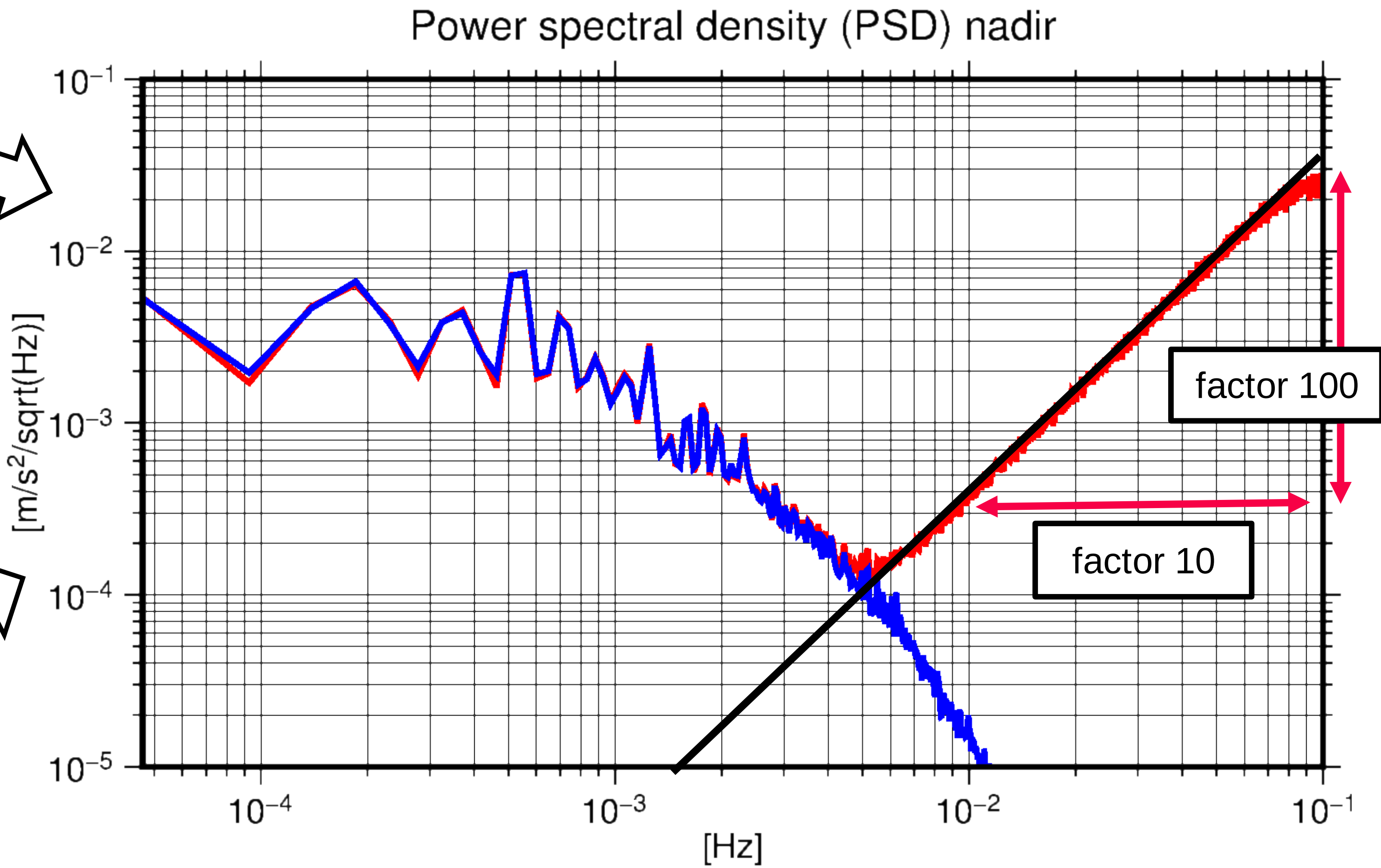


Satellite gravimetry

Acceleration (2nd derivative): $\ddot{r}_i = \frac{r_{i+1} - 2r_i + r_{i-1}}{2\Delta t}$







$$F\left[\frac{d^k f}{dt^k}\right] = (i2\pi\eta)^k F[f(t)]$$

Discrete Fourier Transform (DFT)

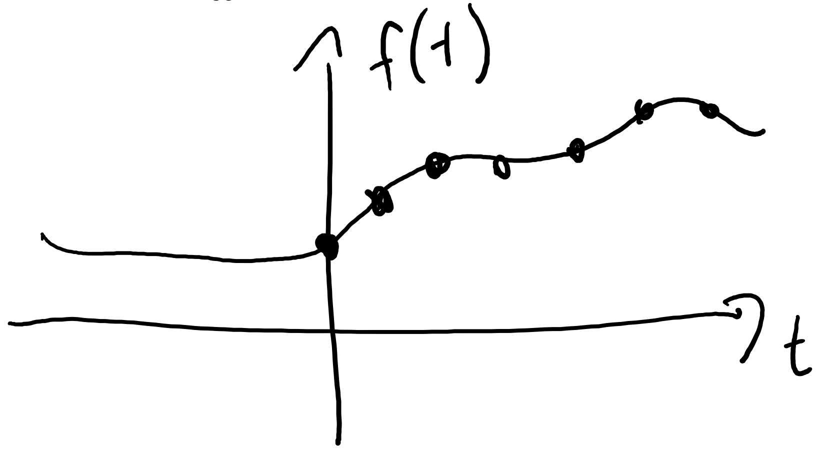
Discrete Fourier Transformation (DFT

Fourier transformation

$$F(\eta) = \cancel{\int_{-\infty}^{\infty} f(t) e^{-i2\pi\eta t} dt}$$

Fourier integral

$$f(t) = \int_{-\infty}^{\infty} F(\eta) e^{i2\pi\eta t} d\eta$$



Discrete Fourier Transformation (DFT)

$$f(t_k) = f(k\Delta t) = f_k \quad \text{with } k = 0 \dots N-1$$

C++
Python

$$F\left(\frac{n}{N\,\Delta t}\right) = \sum_{k=0}^{N-1} f(k\Delta t) e^{-\frac{i2\pi nk}{N}} \quad n = 0, \dots, N-1$$

Discrete DFT

$$F_n = F(\eta_n) = F\left(\frac{n}{N\cdot\Delta t}\right) = \sum_{k=0}^{N-1} f_k \cdot e^{-\frac{i2\pi nk}{N}}$$

$\frac{1}{\sqrt{N}}$

$$f(k\Delta t) = \frac{1}{N} \sum_{n=0}^{N-1} F\left(\frac{n}{N\,\Delta t}\right) e^{\frac{i2\pi nk}{N}}, \quad k = 0, \dots, N-1$$

Inverse DFT (iDFT)

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_N \cdot e^{+\frac{i2\pi nk}{N}}$$

$\frac{1}{\sqrt{N}}$

Discrete Fourier Transformation (DFT)

Fourier transformation $F(\eta) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi\eta t}dt$

Fourier integral $f(t) = \int_{-\infty}^{\infty} F(\eta)e^{i2\pi\eta t}d\eta$

Properties

$X_k = e^{\frac{i2\pi k}{N}}$

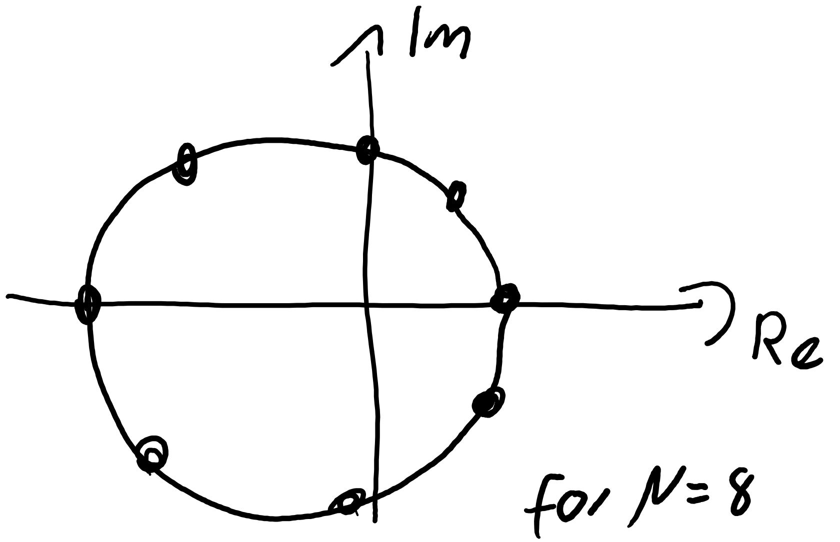
- periodic

$X_k = X_{k\pm N}$

orthogonal

$$\sum_{k=0}^{N-1} e^{\frac{i2\pi pk}{N}} \cdot e^{\frac{i2\pi qk}{N}} = \begin{cases} 0 : p \neq q \\ N : p = q \end{cases}$$

$p,q \in \mathbb{Z}$



for $N=8$

Discrete Fourier Transformation (DFT)

Forward ↔ Backward

$$f_k = \sum_{n=0}^{N-1} F_n \cdot e^{\frac{i2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} \left(\sum_{j=0}^{N-1} f_j \cdot e^{-\frac{i2\pi nj}{N}} \right) e^{\frac{i2\pi nk}{N}} = \sum_{j=0}^{N-1} f_j \cdot \underbrace{e^{-\frac{i2\pi nj}{N}} \cdot e^{\frac{i2\pi nk}{N}}}_{\substack{N \text{ for } j=k \\ 0 \text{ else}}}$$

$$= N \cdot f_k$$

Discrete Fourier Transformation (DFT) in matrix form

$$\underline{Y} = \underline{F}\,\underline{x}$$

$$\underline{Y} = \begin{bmatrix} F_0 \\ \vdots \\ F_{N-1} \end{bmatrix} = \begin{array}{c|c} & 0 \longrightarrow k \\ \hline \begin{array}{c} 0 \\ \downarrow \\ n \end{array} & \begin{bmatrix} e^{\frac{-i2\pi n k}{N}} \end{bmatrix} \end{array} \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & z^1 & z^2 & z^3 \\ 1 & z^2 & z^4 & z^6 \\ 1 & z^3 & z^6 & z^9 \end{bmatrix}$$

with $z = e^{\frac{-i2\pi}{N}}$

$\Rightarrow$ symmetric

Inverse: $F^{-1} = \frac{1}{N} F^*$

$F \cdot F^{-1} = \cdot\, \underline{I}$

sometimes defined

$\bar{F} = \frac{1}{\sqrt{N}} F$

$\bar{F}^{-1} = \bar{F}^*$

N^2 multiplications

$N = 1000 \mathrel{\hat{=}} 1\,\mathrm{ms}$

$N = 1{,}000{,}000 \mathrel{\hat{=}} 1000\,\mathrm{s}$

$\approx 16\,\mathrm{min}$

Fast Fourier Transform (FFT)

Fast Fourier Transform (FFT) (Basic idea)

Number of elements f_k should be even $N = 2\bar{N}$

divide into even / odd elements $f'_k = f_{2k}$ $\quad k = 0 .. \bar{N}-1$

$f''_k = f_{2k+1}$

$$F_n = \sum_{k=0}^{N-1} f_k \cdot e^{-\frac{i2\pi nk}{N}} = \sum_{k=0}^{\bar{N}-1} f_{2k} \cdot e^{-\frac{i2\pi n2k}{N}} + \sum_{k=0}^{\bar{N}-1} f_{2k+1}\, e^{-\frac{i2\pi(2k+1)}{N}}$$

$$= \underbrace{\sum_{k=0}^{\bar{N}-1} f'_k \cdot e^{-\frac{i2\pi nk}{\bar{N}}}}_{\text{DFT with size } \bar{N}} + e^{-\frac{i\pi n}{\bar{N}}} \cdot \underbrace{\sum_{k=0}^{\bar{N}-1} f''_k \cdot e^{-\frac{i2\pi nk}{\bar{N}}}}_{\text{DFT with size } \bar{N} = N/2}$$

(4× faster)

Subdivision

- f_k
  - even
    - even
    - odd
  - odd
    - even
    - odd

$\rightarrow \mathcal{O}(N \cdot \log N)$ multiplication

Fast Fourier Transform (FFT)

Computational speed of the FFT

N	DFT: N^2 multiplications	FFT: $N\log_2 N$ multiplications
32	1024	160
512	262144	4608
8192	67108864	106496

Let's assume that an FFT of size N = 8192 takes 1sec. Then using the DFT would require 10min30sec.

The difference between theory and practice
is greater in practice than in theory...

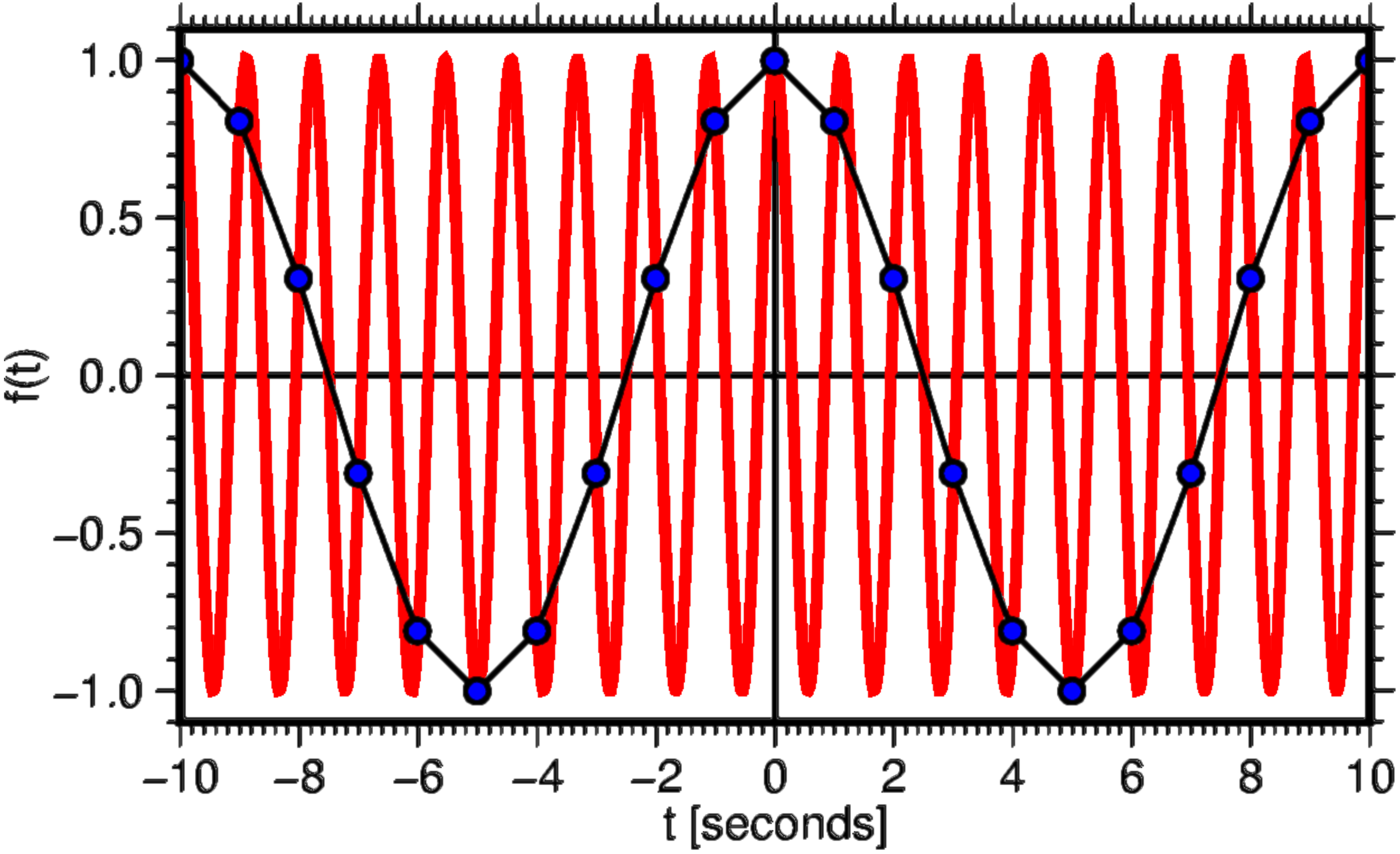
Sampling

Continuous time function

$$y = f(t)$$

Sequence of discrete, sampled values

$$y_n = f(t_n)$$



Fourier transform

Fourier transformation

$$F(\eta) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\eta t} dt$$

Fourier integral

$$f(t) = \int_{-\infty}^{\infty} F(\eta) e^{i2\pi\eta t} d\eta$$

Discrete Fourier Transformation (DFT)
Fast Fourier Transformation (FFT)

$$F\left(\frac{n}{N\,\Delta t}\right) = \sum_{k=0}^{N-1} f(k\Delta t)\, e^{-\frac{i2\pi nk}{N}}, \qquad n = 0, \dots, N-1$$

$$f(k\Delta t) = \frac{1}{N} \sum_{n=0}^{N-1} F\left(\frac{n}{N\,\Delta t}\right) e^{\frac{i2\pi nk}{N}}, \qquad k = 0, \dots, N-1$$

Units

$$\mathcal{F}(\eta) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i2\pi\eta t}\, dt$$

$\mathcal{F}(\eta)$: $\left[\frac{m}{Hz}\right]$

$f(t)$: $[m]$

dt: $[s] = \left[{}^{1}/_{Hz}\right]$

DFT as discretized integral

$$\mathcal{F}_n = \mathcal{F}(\eta_n) = \mathcal{F}\left(\frac{n}{N\cdot\Delta t}\right) \approx \sum_{k=0}^{N-1} f(t_k)\cdot e^{-i2\pi\eta_n\cdot t_k}\cdot \Delta t \qquad t_k = k\cdot\Delta t$$

$$= \Delta t \cdot \sum_{k=0}^{N-1} f(k\cdot\Delta t)\cdot e^{-i2\pi\frac{n\, k\cdot\Delta t}{N\cdot\Delta t}}$$

$$= \Delta t \cdot \underbrace{\boxed{\sum_{k=0}^{N-1} f_k \cdot e^{-\frac{i2\pi n k}{N}}}}_{\text{DFT}}$$

$\Rightarrow$ to get correct unit $\left[\frac{unit}{Hz}\right]$ you have the DFT multiply with the sampling Δt

Units

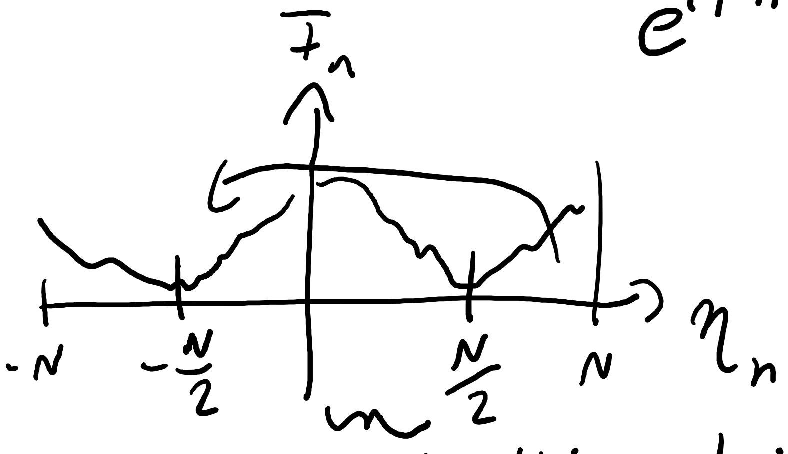
The frequencies for $n > int(N/2)$ characterize the negative components of the spectrum

<u>example</u> for even N

$$F_{\frac{N}{2}+1} = \sum_{k=0}^{N-1} f_k \cdot \underbrace{e^{-\frac{i2\pi k}{N}\left(\frac{N}{2}+1\right)}}_{}$$

$$e^{-i\pi k} \cdot e^{-\frac{i2\pi k}{N}} = e^{+i\left(+\frac{2\pi}{N}\left(\frac{N}{2}-1\right)\right)k}$$

$$e^{+i\pi k} \cdot e^{-i\frac{2\pi}{N}k} = e^{\frac{i2\pi k}{N}\left(\frac{N}{2}-1\right)}$$



only this part is plotted usually

Frequencies are defined for

$$\eta_n = \frac{n}{N\cdot\Delta t} \qquad n \begin{cases} 0 .. N/2 & N \text{ is even} \\ 0 .. (N-1)/2 & N \text{ is odd} \end{cases}$$

Fourier transform

Fourier transformation

$$F(\eta) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\eta t} dt$$

Fourier integral

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Discrete Fourier Transformation (DFT)
Fast Fourier Transformation (FFT)

$$F\left(\frac{n}{N\,\Delta t}\right) = \sum_{k=0}^{N-1} f(k\Delta t) e^{-\frac{i2\pi nk}{N}}, \qquad n = 0, \dots, N-1$$

$$f(k\Delta t) = \frac{1}{N} \sum_{n=0}^{N-1} F\left(\frac{n}{N\,\Delta t}\right) e^{\frac{i2\pi nk}{N}}, \qquad k = 0, \dots, N-1$$

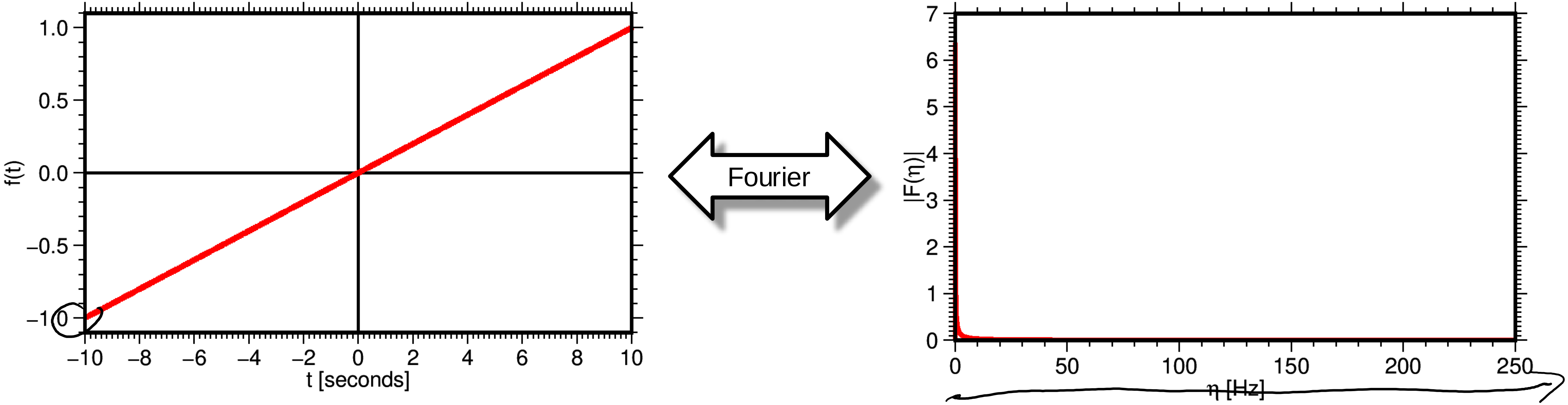
Multiply F with sampling dt to get proper units

Valid frequencies

$$\eta_n = \frac{n}{N\,\Delta t} \qquad n = \begin{cases} 0 \dots N/2 & \text{for } N \text{ even} \\ 0 \dots (N-1)/2 & \text{for } N \text{ odd} \end{cases}$$

Phase shift $F[f(t-t_0)] = e^{-i2\pi\eta t_0} F[f(t)]$ with $\left|e^{-i2\pi\eta t_0}\right| = 1$

Example: Trend



N = 10001

$$\Delta t = \frac{20s}{10000}$$

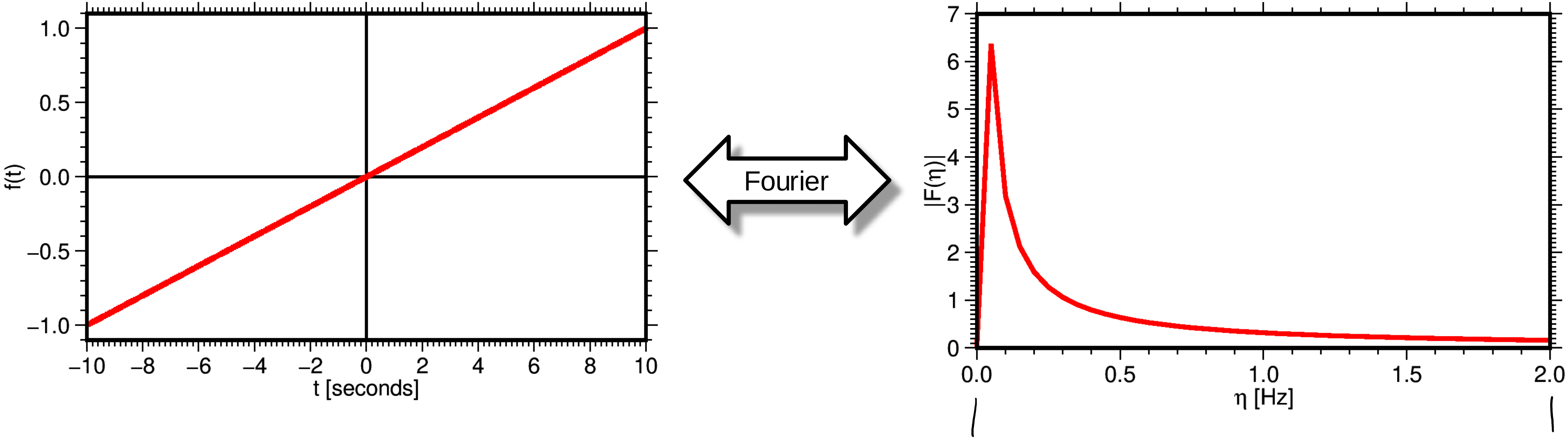
$$= \frac{1}{500} s$$

Nyquist $\eta = \frac{1}{2\Delta t} = 250\,\text{Hz}$

Spacing $\Delta\eta = \frac{1}{T} = \frac{1}{20s} = 0.05$ Hz

Torsten Mayer-Gürr

Example: Trend



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$$\Delta t = \frac{20s}{10000}$$

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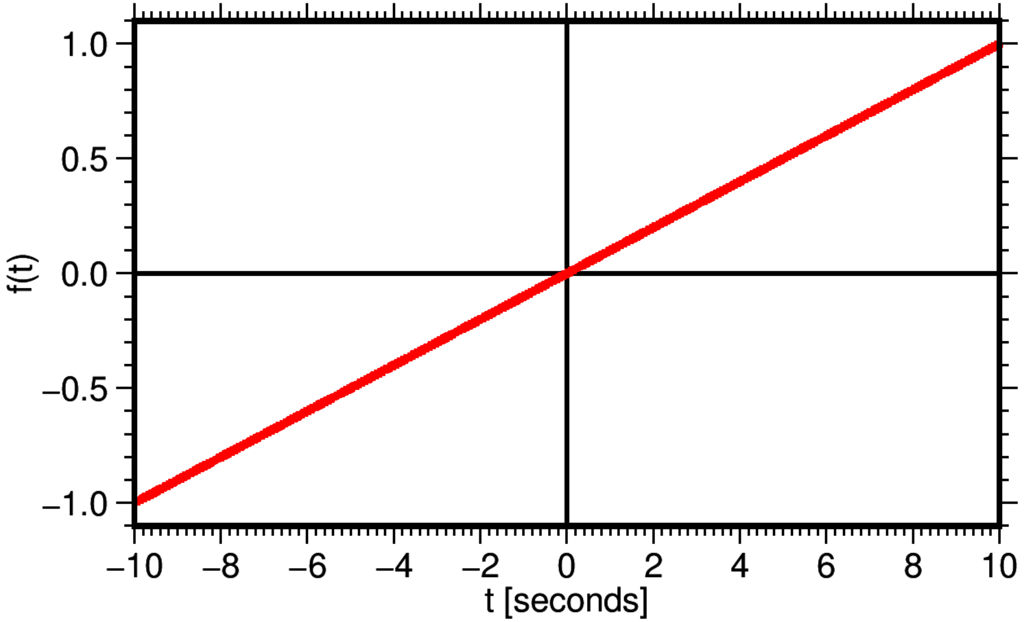
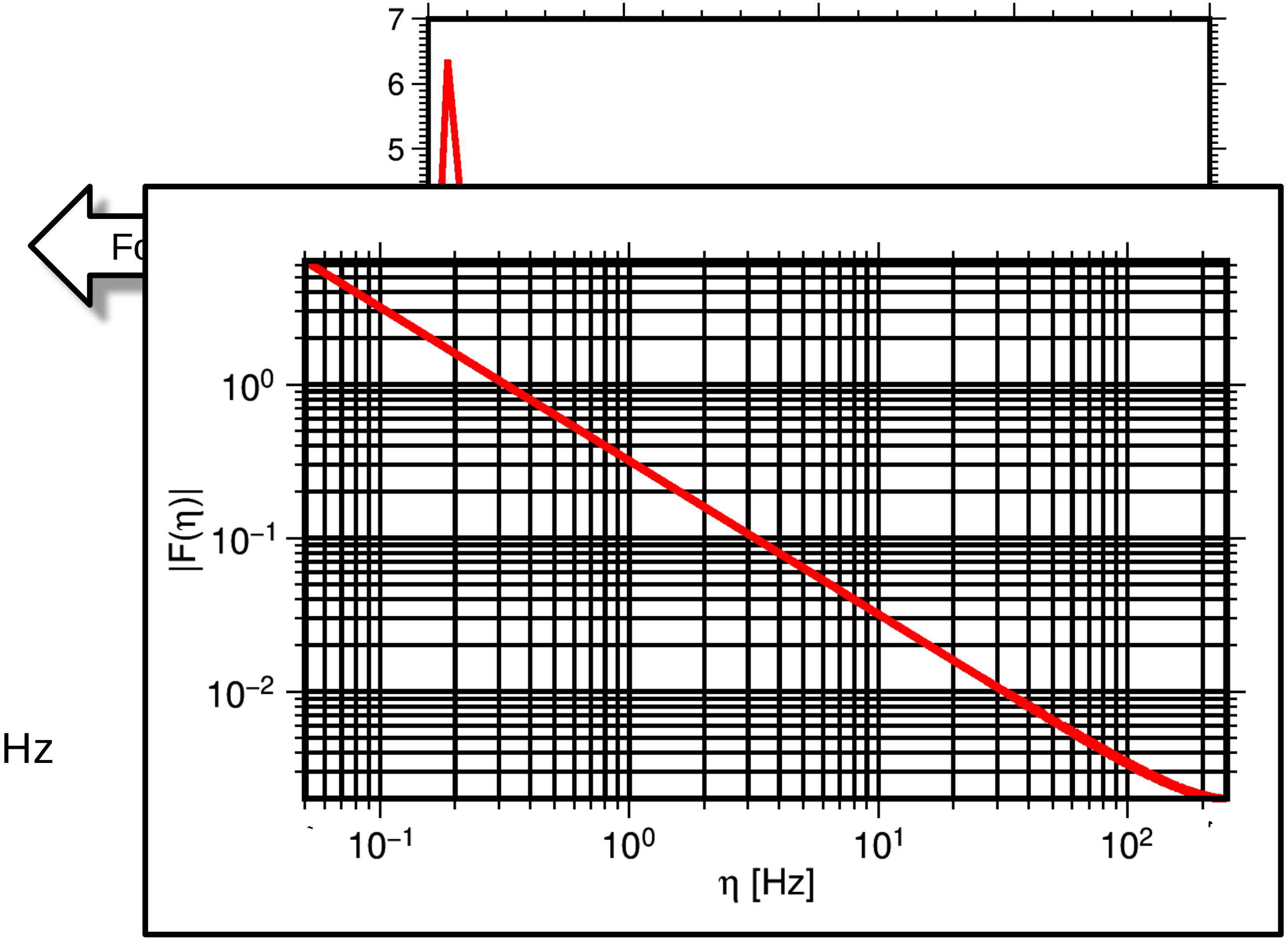
Example: Trend





N = 10001

$$\Delta t = \frac{20s}{10000}$$

Nyquist $\eta = \frac{1}{2\Delta t} = 250$ Hz

Lab

Lab

- Lab
  - Engineering point of view
  - Application of the time series tools
  - Interpretation!
  - Units are important

- Presentation
  - Explain your data
  - Should be for the other students
  - Explain your problems / pitfalls