Synthetic Differential Geometry
within
Homotopy Type Theory
I

Hirokazu NISHIMURA

Institute of Mathematics, University of Tsukuba

Tsukuba, Ibaraki 305-8571

Japan

March 28, 2018

Abstract

Both syntheticc differential geometry and homotopy type theory prefer synthetic arguments to analytical ones. This paper gives a first step towards developing synthetic differential geometry within homotopy type theory. Model theory of this approach will be discussed in a subsequent paper.

1   Introduction

Homotopy type theory (cf. [11]), born at the crossroads of type theory and homotopy theory in the first decade of this century ([1] and [10]) inspired by [2], is expected to give a solid foundation to mathematics. A large portion of classical homotopy theory has already been developed within homotopy type theory with new formulations and new proofs of celebrated classical results such as the Freudenthal suspension theorem, the van Kampen theorem and the Whitehead theorem being discovered by the intimate collaboration of men and the proof assistant system COQ in the process of developing.

Synthetic differential geometry is developed synthetically by using nilpotent infinitesimals. For standard textbooks on synthetic differential geometry the reader is referred to [5] and [7]. The principal objective in this paper is to develop synthetic differential geometry within homotopy type theory. Since both theories prefer synthetic arguments to analytic ones, there is a tremendous affinity between them. In the next section (§2) we will set up the foundation for types of nilpotent infinitesimals and announce the homotopical generalized Kock-Lawvere axiom. After enjoying elementary differential calculus up to the

1

Taylor expansion (cf. [3] and [4]) in §3, we will discuss microlinearity in §4 and tangency in in §5 by using the machinery of set truncation. §6 is devoted to strong differences. It culminates in a streamlined presentation of the general Jacobi identity discussed in [8] and [9]. The last section (§7) deals with vector fields on a microlinear type. In a subsequent paper we will discuss model theory of this approach.

2 Nilpotent Infinitesimals

Axiom 1 *The type $\mathbb{R}$ is a set which is a $\mathbb{Q}$-algebra, where $\mathbb{Q}$ is the type of rational numbers.*

Definition 2 *A finitely presented $\mathbb{R}$-algebra of the form*

$$\mathbb{R}\left[X_1, ..., X_n\right] / \left(X_1^{m_1}, ..., X_n^{m_n}, f_1\left(X_1, ..., X_n\right), ..., f_k\left(X_1, ..., X_n\right)\right)$$

with f_i's being polynomials in $X_1, ..., X_n$ with coefficients in $\mathbb{R}$ is called a <u>Weil algebra</u>. It should be recalled that finitely presented $\mathbb{R}$-algebras are to be defined by higher induction in homotopy type theory.

Notation 3 *Given a Weil algebra $\mathfrak{W}$, we denote by $\mathrm{Spec}_{\mathbb{R}}\mathfrak{W}$ the type of homomorphisms of $\mathbb{R}$-algebras from the $\mathbb{R}$-algebra $\mathfrak{W}$ to the $\mathbb{R}$-algebra $\mathbb{R}$ By way of example, the type*

$$\mathrm{Spec}_{\mathbb{R}}\mathbb{R}\left[X\right]/(X^2)$$

is equivalent to the subtype

$$D :\equiv \left\{d : \mathbb{R} \mid d^2 = 0\right\}$$

of the type $\mathbb{R}$, while the type

$$\mathrm{Spec}_{\mathbb{R}}\mathbb{R}\left[X, Y\right]/(X^2, Y^2, XY)$$

is equivalent to the subtype

$$D\left(2\right) :\equiv \left\{(d_1, d_2) : D^2 \mid d_1 d_2 = 0\right\}$$

of the type D^2.

Definition 4 *Given a Weil algebra $\mathfrak{W}$, the type $\mathrm{Spec}_{\mathbb{R}}\mathfrak{W}$ is called the <u>infinitesimal type</u> associated to the Weil algebra $\mathfrak{W}$.*

Definition 5 *The diagram of infinitesimal types resulting from a finite limit diagram of Weil algebras by application of the contravariant functor $\mathrm{Spec}_{\mathbb{R}}$ is called a <u>quasi-colimit diagram of infinitesimal types</u>.*

Axiom 6 *(Homotopical Generalized Kock-Lawvere Axiom) Given a Weil algebra $\mathfrak{W}$, the canonical homomorphism of $\mathbb{R}$-algebras from the $\mathbb{R}$-algebra $\mathfrak{W}$ to the $\mathbb{R}$-algebra $\mathrm{Spec}_{\mathbb{R}} \mathfrak{W} \to \mathbb{R}$*

$$\lambda_{x:\mathfrak{W}} \lambda_{f:\mathrm{Spec}_{\mathbb{R}} \mathfrak{W}} f(x)$$

is an equivalence, namely,

$$\mathfrak{W} \simeq \mathrm{Spec}_{\mathbb{R}} \mathfrak{W} \to \mathbb{R}$$

Remark 7 *Under Axiom 6, a finite diagram of infinitesimal types is a quasi-colimit diagram iff the diagram resulting from it by application of the contravariant functor $\to \mathbb{R}$ is a limit diagram.*

We recall the notion of a simplicial small object introduced in §4 of [8].

Notation 8 *(Simplicial infinitesimal types) Given $n : \mathbb{N}$ and a finite set $\mathfrak{p}$ of lists of natural numbers i with $1 \le i \le n$, we denote by $D^n \{\mathfrak{p}\}$ a set*

$$\left\{ (d_1, ..., d_n) : D^n \mid \prod_{(i_1,...,i_k):\mathfrak{p}} d_{i_1}...d_{i_k} = 0 \right\}$$

By way of example, we have

$$D(2) = D^2 \{(1,2)\}$$
$$D(3) = D^3 \{(1,2),(1,3),(2,3)\}$$

while both $D^2 \{(1)\}$ and $D^2 \{(2)\}$ are equivalent to D via the equivalences $\lambda_{d:D}(0,d)$ and $\lambda_{d:D}(d,0)$ respectively.

Axiom 9 *The type $\mathbb{R}$ is a set endowed with a structure of a unitary commutative ring such that*

$$\prod_{f::D\to\mathbb{R}} \mathrm{isContr} \left(\sum_{a:\mathbb{R}} \prod_{d:D} f(d) =_{\mathbb{R}} f(0) + ad \right) \tag{1}$$

where D stands for the subtype

$$\sum_{x:\mathbb{R}} x^2 =_{\mathbb{R}} 0$$

of $\mathbb{R}$.

3 Elementary Differential Calculus

Notation 10 *Given $f : \mathbb{R} \to \mathbb{R}$ and $x : \mathbb{R}$, we write*

$$f'(x) : \mathbb{R}$$

for one of the propositionally identical $a : \mathbb{R}$ abiding by

$$\prod_{d:D} (\lambda_{y:D} f(x+y))(d) =_{\mathbb{R}} (\lambda_{y:D} f(x+y))(0) + ad$$

Proposition 11 *We have*

$$\prod_{f:\mathbb{R}\to\mathbb{R}} \text{isContr}\left(\sum_{g:\mathbb{R}\to\mathbb{R}} \prod_{x:\mathbb{R}} \prod_{d:D} f\left(x+d\right) =_{\mathbb{R}} f\left(x\right) + g\left(x\right)d\right)$$

Proof. This follows from the above axiom by the principle of unique choice (§3.9 of [11]). ■

Notation 12 *Given $f : \mathbb{R} \to \mathbb{R}$, we write*

$$f' : \mathbb{R} \to \mathbb{R}$$

for one of the propositionally identical $g : \mathbb{R} \to \mathbb{R}$ abiding by

$$\prod_{x:\mathbb{R}} \prod_{d:D} f\left(x+d\right) =_{\mathbb{R}} f\left(x\right) + g\left(x\right)d$$

Given $n : \mathbb{N}$, we can define
$$f^{(n)} : \mathbb{R} \to \mathbb{R}$$

inductively on n.

Proposition 13 *We have*

$$\prod_{f,g::\mathbb{R}\to\mathbb{R}} \prod_{x:\mathbb{R}} (fg)'\left(x\right) = f'\left(x\right)g\left(x\right) + f\left(x\right)g'\left(x\right)$$

Proof. Let $d : D$. We have

$$
\begin{aligned}
f(x+d)g(x+d) &= (f(x) + f'(x)d)\,(g(x) + g'(x)d) \\
&= f(x)g(x) + \left(f'\left(x\right)g\left(x\right) + f\left(x\right)g'\left(x\right)\right)d + f'(x)g'(x)d^2 \\
&= f(x)g(x) + \left(f'\left(x\right)g\left(x\right) + f\left(x\right)g'\left(x\right)\right)d \\
&\quad [\text{since } d^2 \text{ vanishes}]
\end{aligned}
$$

so that the desired conclusion follows. ■

Proposition 14 *We have*

$$\prod_{f,g::\mathbb{R}\to\mathbb{R}} \prod_{x:\mathbb{R}} (g\circ f)'\left(x\right) = g'\left(f\left(x\right)\right)f'\left(x\right)$$

Proof. Let $d : D$. We have

$$
\begin{aligned}
g(f(x+d)) &= g\left(f(x) + f'(x)d\right) \\
&= g\left(f(x)\right) + g'\left(f\left(x\right)\right)\left(f'\left(x\right)d\right) \\
&\quad [\text{since } f'(x)d : D] \\
&= g\left(f(x)\right) + \left(g'\left(f\left(x\right)\right)f'\left(x\right)\right)d
\end{aligned}
$$

so that the desired conclusion follows. ■

Notation 15 *Given* $n : \mathbb{N}$, *we write*

$$\mathrm{List}_D(n)$$

for the type of lists of elements in D with length n. Thus the type $\mathrm{List}_D(n)$ consists of $(d_1, ..., d_n)$'s with $d_i : D$ ($1 \leq i \leq n$). In particular, $\mathrm{List}_D(0)$ consists only of (). Given $m, n : \mathbb{N}$, we define

$$\mathrm{Sym}_{n,m} : \mathrm{List}_D(n) \to \mathbb{R}$$

by induction on n. We decree that

$$\mathrm{Sym}_{0,0} :\equiv \lambda_{x:\mathrm{List}_D(0)} 1$$

and

$$\mathrm{Sym}_{0,m+1} :\equiv \lambda_{x:\mathrm{List}_D(0)} 0$$

whatever m may be. We decree that

$$\mathrm{Sym}_{n+1,0} :\equiv \lambda_{x:\mathrm{List}_D(n+1)} 1$$

and that

$$\mathrm{Sym}_{n+1,m+1}(d_1, ..., d_{n+1})$$
$$:\equiv \mathrm{Sym}_{n,m+1}(d_1, ..., d_n) + d_{n+1}\mathrm{Sym}_{n,m}(d_1, ..., d_n)$$

whatever m may be.

It is easy to see that

Lemma 16 *Given $m, n : \mathbb{N}$, we have*

$$\mathrm{Sym}_{n,m} = \lambda_{x:\mathrm{List}_D(n)} 0$$

provided that $n < m$.

Now we have the infinitesimal Taylor expansion theorem.

Theorem 17 *Given $f : \mathbb{R} \to \mathbb{R}$, $x : \mathbb{R}$ and $n : \mathbb{N}$, we have*

$$f\left(x + \mathrm{Sym}_{n,1}(d_1, ..., d_n)\right)$$
$$= f(x) + f'(x)\,\mathrm{Sym}_{n,1}(d_1, ..., d_n) + f''(x)\,\mathrm{Sym}_{n,2}(d_1, ..., d_n) + ...$$
$$+ f^{(i)}\mathrm{Sym}_{n,i}(d_1, ..., d_n) + ... + f^{(n)}\mathrm{Sym}_{n,n}(d_1, ..., d_n)$$

Proof. By induction on n. If $n = 0$, the theorem holds trivially. We have

$f \left(x + \text{Sym}_{n+1,1} (d_1, ..., d_{n+1}) \right)$

$= f \left(x + \text{Sym}_{n,1} (d_1, ..., d_n) + d_{n+1} \text{Sym}_{n,0} (d_1, ..., d_n) \right)$

$= f \left(x + \text{Sym}_{n,1} (d_1, ..., d_n) + d_{n+1} \right)$

$= f \left(x + \text{Sym}_{n,1} (d_1, ..., d_n) \right) + d_{n+1} f' \left(x + \text{Sym}_{n,1} (d_1, ..., d_n) \right)$

$= f (x) + f' (x) \text{Sym}_{n,1} (d_1, ..., d_n) + ... + f^{(n)} (x) \text{Sym}_{n,n} (d_1, ..., d_n) +$

$d_{n+1} \left\{ f' (x) + f'' (x) \text{Sym}_{n,1} (d_1, ..., d_n) + ... + f^{(n+1)} (x) \text{Sym}_{n,n} (d_1, ..., d_n) \right\}$

$= f (x) + f' (x) \left(\text{Sym}_{n,1} (d_1, ..., d_n) + d_{n+1} \right) +$

$f'' (x) \left(\text{Sym}_{n,2} (d_1, ..., d_n) + d_{n+1} \text{Sym}_{n,1} (d_1, ..., d_n) \right) + ... +$

$f^{(n+1)} (x) d_{n+1} \text{Sym}_{n,n} (d_1, ..., d_n)$

$= f (x) + f' (x) \text{Sym}_{n+1,1} (d_1, ..., d_{n+1}) + f'' (x) \text{Sym}_{n+1,2} (d_1, ..., d_{n+1}) + ... +$

$f^{(n+1)} (x) \text{Sym}_{n+1,n+1} (d_1, ..., d_{n+1})$

∎

The familiar form of the Taylor expansion theorem goes as follows:

Corollary 18 *We assume that the ring $\mathbb{R}$ is an algebra over the rationals $\mathbb{Q}$. Given $f : \mathbb{R} \to \mathbb{R}$, $x : \mathbb{R}$ and $n : \mathbb{N}$, we have*

$f \left(x + \text{Sym}_{n,1} (d_1, ..., d_n) \right)$

$= f (x) + f' (x) \text{Sym}_{n,1} (d_1, ..., d_n) + \dfrac{1}{2} f'' (x) \left(\text{Sym}_{n,1} (d_1, ..., d_n) \right)^2 + ... +$

$\dfrac{1}{i!} \left(\text{Sym}_{n,1} (d_1, ..., d_n) \right)^i + ... + \dfrac{1}{n!} \left(\text{Sym}_{n,1} (d_1, ..., d_n) \right)^n$

Proof. This follows directly from the theorem simply by observing that

$$i! \text{Sym}_{n,i} (d_1, ..., d_n) = \left(\text{Sym}_{n,1} (d_1, ..., d_n) \right)^i \quad (1 \leq i \leq m)$$

∎

Definition 19 *An $\mathbb{R}$-module E is called <u>Euclidean</u> if it abides by the following condition:*

$$\prod_{f :: D \to E} \text{isContr} \left(\sum_{a : E} \prod_{d : D} f (d) = f (0) + ad \right)$$

Given $X : \mathcal{U}$ and an $\mathbb{R}$-module $E (x)$ for each $x : X$, the type $\prod_{x:X} E (x)$ is naturally an $\mathbb{R}$-module. It is easy to see that

Proposition 20 *If the $\mathbb{R}$-module $E (x)$ is Euclidean for each $x : X$, then the $\mathbb{R}$-module $\prod_{x:X} E (x)$ is also Euclidean.*

Proof. By the function extensionality axiom (Axiom 2.9.3 of [11]) and the principle of unique choice (§3.9 of [11]). ∎

Notation 21 *Given an $\mathbb{R}$-module E, a Euclidean $\mathbb{R}$-module F and $f : E \to F$, we write*

$$f' : E \to E \to F$$

for one of the propositionally identical f' abiding by

$$\prod_{x:E} \prod_{a:E} \prod_{d:D} f(x + ad) = f(x) + f'(x,a)d$$

Proposition 22 *Given an $\mathbb{R}$-module E, a Euclidean $\mathbb{R}$-module F and $f : E \to F$, we have*

$$\prod_{x:E} \prod_{a:E} \prod_{b:E} f'(x, a+b) = f'(x,a) + f'(x,b)$$

and

$$\prod_{x:E} \prod_{a:E} \prod_{r:\mathbb{R}} f'(x, ra) = rf'(x,a)$$

In other words,

$$f'(x) : E \to F$$

is a homomorphism of $\mathbb{R}$-modules.

Proof. Given $d : D$, we have

$$
\begin{aligned}
f\left(x + (a+b)\,d\right) &= f\left((x + ad) + bd\right) \\
&= f\left(x + ad\right) + f'\left(x + ad, b\right)d \\
&= f\left(x\right) + f'\left(x,a\right)d + \left\{f'\left(x,b\right) + \left(\lambda_{y:E} f'\left(y,b\right)\right)'\left(x,a\right)d\right\}d \\
&= f\left(x\right) + \left(f'\left(x,a\right) + f'\left(x,b\right)\right)d + \left(\lambda_{y:E} f'\left(y,b\right)\right)'\left(x,a\right)d^2 \\
&= f\left(x\right) + \left(f'\left(x,a\right) + f'\left(x,b\right)\right)d \\
&\quad \text{[since } d^2 \text{ vanishes]}
\end{aligned}
$$

while we have

$$
\begin{aligned}
f\left(x + (ra)\,d\right) &= f\left(x + a\left(rd\right)\right) \\
&= f(x) + f'\left(x,a\right)\left(rd\right) \\
&\quad \text{[since } rd : D] \\
&= f(x) + \left(rf'\left(x,a\right)\right)d
\end{aligned}
$$

so that the desired conclusion follows. ∎

Notation 23 *Given an $\mathbb{R}$-module E, a Euclidean $\mathbb{R}$-module F and $f : E \to F$, we have*

$$f' : E \to E \to F$$

Since the $\mathbb{R}$-module $E \to F$ is Euclidean by Proposition , we have

$$(f')' : E \to E \to E \to F$$

We will often write f'' in place of $(f')'$.

It is easy to see that

Proposition 24 *Given an $\mathbb{R}$-module E, a Euclidean $\mathbb{R}$-module F and $f : E \to F$, we have*

$$\prod_{x:E} \prod_{a_1, a_2:E} \prod_{b:E} f'' (x, a_1 + a_2, b) = f'' (x, a_1, b) + f'' (x, a_2, b)$$

$$\prod_{x:E} \prod_{a:E} \prod_{b_1, b_2:E} f'' (x, a, b_1 + b_2) = f'' (x, a, b_1) + f'' (x, a, b_2)$$

$$\prod_{x:E} \prod_{a:E} \prod_{b:E} \prod_{r:\mathbb{R}} f'' (x, ra, b) = r f'' (x, a, b)$$

and

$$\prod_{x:E} \prod_{a:E} \prod_{b:E} \prod_{r:\mathbb{R}} f'' (x, a, rb) = r f'' (x, a, b)$$

In short, $f''(x)$ is bilinear.

Proof. By Proposition 22. ∎
We can say more.

Proposition 25 *Given an $\mathbb{R}$-module E, a Euclidean $\mathbb{R}$-module F and $f : E \to F$, we have*

$$\prod_{x:E} \prod_{a:E} \prod_{b:E} f'' (x, a, b) = f'' (x, b, a)$$

Proof. Given $d_1, d_2 : D$, we compute

$$f(x + ad_1 + bd_2) - f(x + ad_1) - f(x + bd_2) + f(x)$$
$$= f(x + ad_1 + bd_2) - f(x + bd_2) - f(x + ad_1) + f(x)$$

in two different ways. On the one hand, we have

$$f(x + ad_1 + bd_2) - f(x + ad_1) - f(x + bd_2) + f(x)$$
$$= (f(x + ad_1 + bd_2) - f(x + ad_1)) - (f(x + bd_2) - f(x))$$
$$= f' (x + ad_1, b) d_2 - f' (x, b) d_2$$
$$= (f' (x + ad_1) - f' (x)) (b) d_2$$
$$= (f'' (x, a) d_1) (b) d_2$$
$$= f'' (x, a, b) d_1 d_2$$

8

On the other hand, we have

$$f(x + ad_1 + bd_2) - f(x + bd_2) - f(x + ad_1) + f(x)$$
$$= (f(x + ad_1 + bd_2) - f(x + bd_2)) - (f(x + ad_1) - f(x))$$
$$= f(x + bd_2, a)d_1 - f(x, a)d_1$$
$$= (f'(x + bd_2) - f'(x))(a)d_1$$
$$= (f''(x, b)d_2)(a)d_1$$
$$= f''(x, b, a)d_1 d_2$$

Therefore the desired conclusion follows. ■

4   Microlinearity

Definition 26 *The diagram of small objects resulting from a limit diagram of Weil algebras by application of the contravariant functor*

$$\mathrm{Spec}_{\mathbb{R}}$$

is called a <u>quasi-colimit diagram of small objects</u>. *Therefore, by Axiom* **??**, *a diagram $\mathcal{D}$ of small objects is a quasi-colimit diagram iff the exponentiation $\mathcal{D} \to \mathbb{R}$ of the diagram $\mathcal{D}$ over the type $\mathbb{R}$ is a limit diagram.*

Definition 27 *A type M is called* <u>microlinear</u> *provided that the exponentiation $\mathcal{D} \to \|M\|_0$ of any quasi-colimit diagram $\mathcal{D}$ of small objects over the set truncation $\|M\|_0$ of the type M is a limit diagram.of types.*

It is easy to see that

Proposition 28 *(cf. Proposition 1 of §2.3 in [7]) We have the following:*

1. *A type M is microlinear iff its set truncation $\|M\|_0$ is so.*

2. *The type $\mathbb{R}$ is microlinear.*

3. *If M is a microlinear set and X is an arbitrary type, then $X \to M$ is a microlinear set.*

4. *If M is the limit of a diagram $\mathcal{M}$ of microlinear sets, then M is a microlinear set.*

Proof. The first statement follows directly from the very definition of microlinearity. The second statement follows from the axiom. Let $\mathcal{D}$ be a quasi-colimit diagram of small objects. For the third statement, we note that the diagram

$$\mathcal{D} \to X \to M$$

is equivalent to the diagram

$$X \to \mathcal{D} \to M$$

which is a limit diagram because of the assumption that $\mathcal{D} \to M$ is a limit diagram. For the fourth statement, we note that the diagram

$$\mathcal{D} \to \mathcal{M}$$

is a limit diagram of diagrams of types over the diagram $\mathcal{M}$ so that the diagram

$$\mathcal{D} \to M$$

is a limit diagram, because, roughly speaking, double limits commute. ∎

5 Tangency

Notation 29 *Given a microlinear type M and $x : M$, the type $\mathbf{T}_x M$ of tangent vectors to M at x stands for the subtype*

$$\{t : D \to \|M\|_0 \mid t(0) = |x|_0\}$$

of the type

$$D \to \|M\|_0$$

We recall that.

Lemma 30 *(cf. Proposition 6 of §2.2 in [7]) The following diagram is a quasi-colimit diagram:*

$$
\begin{array}{ccc}
1 & \to & D \\
\downarrow & & \downarrow \lambda_{d:D}(0,d) \\
D & \xrightarrow{\lambda_{d:D}(d,0)} & D(2)
\end{array}
$$

Corollary 31 *Let M be a microlinear set with $x : M$. Given $t_1, t_2 : D \to M$ with $t_1(0) = t_2(0) = x$, there exists $\mathfrak{l}_{(t_1,t_2)} : D(2) \to M$ such that*

$$\mathfrak{l}_{(t_1,t_2)} \circ (\lambda_{d:D}(d,0)) = t_1$$
$$\mathfrak{l}_{(t_1,t_2)} \circ (\lambda_{d:D}(0,d)) = t_2$$

The above lemma has the following variant.

Lemma 32 *The following diagram is a quasi-colimit diagram:*

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & \swarrow & \downarrow & \searrow & \\
D & & D & & D \\
 & \searrow & \downarrow & \swarrow & \\
 & & D(3) & &
\end{array}
$$

where the lower three arrows stand from left to right for

$$\lambda_{d:D}(d,0,0)$$
$$\lambda_{d:D}(0,d,0)$$
$$\lambda_{d:D}(0,0,d)$$

respectively.

Corollary 33 *Let M be a microlinear set with $x : M$. Given $t_1, t_2 : D \to M$ with $t_1(0) = t_2(0) = x$, there exists $\mathfrak{l}_{(t_1,t_2,t_3)} : D(3) \to M$ such that*

$$\mathfrak{l}_{(t_1,t_2,t_3)} \circ (\lambda_{d:D}(d,0,0)) = t_1$$
$$\mathfrak{l}_{(t_1,t_2,t_3)} \circ (\lambda_{d:D}(0,d,0)) = t_2$$
$$\mathfrak{l}_{(t_1,t_2,t_3)} \circ (\lambda_{d:D}(0,0,d)) = t_3$$

Definition 34 *Given a microlinear type M with $x : M$, we define addition and scalar multiplication on $\mathbf{T}_x M$ as follows:For $t, t_1, t_2 : \mathbf{T}_x M$ and $\alpha : \mathbb{R}$, $t_1 + t_2$ and αt are defined to be*

$$t_1 + t_2 :\equiv \lambda_{d:D}\mathfrak{l}_{(t_1,t_2)}(d,d)$$
$$\alpha t :\equiv \lambda_{d:D} t(\alpha d)$$

Theorem 35 *Let M be a microlinear type with $x : M$. Given $\alpha, \beta : \mathbb{R}$ and $t, t_1, t_2, t_3 : \mathbf{T}_x(M)$, we have*

$$(t_1 + t_2) + t_3 = t_1 + (t_2 + t_3) \tag{2}$$
$$t_1 + t_2 = t_2 + t_1 \tag{3}$$
$$1t = t \tag{4}$$
$$(\alpha + \beta)t = \alpha t + \beta t \tag{5}$$
$$\alpha(t_1 + t_2) = \alpha t_1 + \alpha t_2 \tag{6}$$
$$(\alpha\beta)t = \alpha(\beta t) \tag{7}$$

In a word, the type $\mathbf{T}_x(M)$ is an $\mathbb{R}$-module.

Proof. We deal with the six properties in order.

1. It is easy to see that

$$\left(\lambda_{(d_1,d_2):D(2)}\mathfrak{l}_{(t_1,t_2,t_3)}(d_1,d_2,0)\right) \circ (\lambda_{d:D}(d,0)) = \lambda_{d:D}t_1(d)$$
$$\left(\lambda_{(d_1,d_2):D(2)}\mathfrak{l}_{(t_1,t_2,t_3)}(d_1,d_2,0)\right) \circ (\lambda_{d:D}(0,d)) = \lambda_{d:D}t_2(d)$$

so that
$$\mathfrak{l}_{(t_1,t_2)} = \lambda_{(d_1,d_2):D(2)}\mathfrak{l}_{(t_1,t_2,t_3)}(d_1,d_2,0)$$

and consequently

$$t_1 + t_2 = \lambda_{d:D}\mathfrak{l}_{(t_1,t_2,t_3)}(d,d,0)$$

It is easy to see that

$$\left(\lambda_{(d_1,d_2):D(2)}\mathfrak{l}_{(t_1,t_2,t_3)}(d_1,d_1,d_2)\right) \circ (\lambda_{d:D}(d,0)) = \lambda_{d:D}\mathfrak{l}_{(t_1,t_2,t_3)}(d,d,0) = t_1 + t_2$$
$$\left(\lambda_{(d_1,d_2):D(2)}\mathfrak{l}_{(t_1,t_2,t_3)}(d_1,d_1,d_2)\right) \circ (\lambda_{d:D}(0,d)) = \lambda_{d:D}\mathfrak{l}_{(t_1,t_2,t_3)}(0,0,d) = t_3$$

so that
$$\mathfrak{l}_{(t_1+t_2,t_3)} = \lambda_{(d_1,d_2):D(2)}\mathfrak{l}_{(t_1,t_2,t_3)}(d_1,d_1,d_2)$$

and consequently

$$(t_1 + t_2) + t_3 = \lambda_{d:D} \mathfrak{l}_{(t_1,t_2,t_3)} (d,d,d) \tag{8}$$

On the other hand, it is easy to see that

$$\left(\lambda_{(d_1,d_2):D(2)} \mathfrak{l}_{(t_1,t_2,t_3)} (0,d_1,d_2)\right) \circ \left(\lambda_{d:D} (d,0)\right) = \lambda_{d:D} t_2 (d)$$
$$\left(\lambda_{(d_1,d_2):D(2)} \mathfrak{l}_{(t_1,t_2,t_3)} (0,d_1,d_2)\right) \circ \left(\lambda_{d:D} (0,d)\right) = \lambda_{d:D} t_3 (d)$$

so that

$$\mathfrak{l}_{(t_2,t_3)} = \lambda_{(d_1,d_2):D(2)} \mathfrak{l}_{(t_1,t_2,t_3)} (0,d_1,d_2)$$

and consequently

$$t_2 + t_3 = \lambda_{d:D} \mathfrak{l}_{(t_1,t_2,t_3)} (0,d,d)$$

It is easy to see that

$$\left(\lambda_{(d_1,d_2):D(2)} \mathfrak{l}_{(t_1,t_2,t_3)} (d_1,d_2,d_2)\right) \circ \left(\lambda_{d:D} (d,0)\right) = \lambda_{d:D} \mathfrak{l}_{(t_1,t_2,t_3)} (d,0,0) = t_1$$
$$\left(\lambda_{(d_1,d_2):D(2)} \mathfrak{l}_{(t_1,t_2,t_3)} (d_1,d_2,d_2)\right) \circ \left(\lambda_{d:D} (0,d)\right) = \lambda_{d:D} \mathfrak{l}_{(t_1,t_2,t_3)} (0,d,d) = t_2 + t_3$$

so that

$$\mathfrak{l}_{(t_1,t_2+t_3)} = \lambda_{(d_1,d_2):D(2)} \mathfrak{l}_{(t_1,t_2,t_3)} (d_1,d_2,d_2)$$

and consequently

$$t_1 + (t_2 + t_3) = \lambda_{d:D} \mathfrak{l}_{(t_1,t_2,t_3)} (d,d,d) \tag{9}$$

It follows from (8) and (9) that (2) obtains.

2. It is easy to see that

$$\left(\lambda_{(d_1,d_2):D(2)} \mathfrak{l}_{(t_1,t_2)} (d_2,d_1)\right) \circ \left(\lambda_{d:D} (d,0)\right) = \lambda_{d:D} t_2 (d)$$
$$\left(\lambda_{(d_1,d_2):D(2)} \mathfrak{l}_{(t_1,t_2)} (d_2,d_1)\right) \circ \left(\lambda_{d:D} (0,d)\right) = \lambda_{d:D} t_1 (d)$$

so that

$$\mathfrak{l}_{(t_2,t_1)} = \lambda_{(d_1,d_2):D(2)} \mathfrak{l}_{(t_1,t_2)} (d_2,d_1)$$

Therefore we have

$$t_2 + t_1 = \lambda_{d:D} \mathfrak{l}_{(t_2,t_1)} (d,d)$$
$$= \lambda_{d:D} \mathfrak{l}_{(t_1,t_2)} (d,d)$$
$$= \lambda_{d:D} (t_1 + t_2) (d)$$

so that (3) obtains.

3. It is easy to see that, for any $d : D$, we have

$$(1t) (d) = t (1d) = t (d)$$

so that (4) obtains.

4. It is easy to see that

$$\left(\lambda_{(d_1,d_2):D(2)}t\left(\alpha d_1 + \beta d_2\right)\right) \circ \left(\lambda_{d:D}\left(d,0\right)\right) = \lambda_{d:D}t\left(\alpha d\right) = \lambda_{d:D}\left(\alpha t\right)\left(d\right)$$

$$\left(\lambda_{(d_1,d_2):D(2)}t\left(\alpha d_1 + \beta d_2\right)\right) \circ \left(\lambda_{d:D}\left(0,d\right)\right) = \lambda_{d:D}t\left(\beta d\right) = \lambda_{d:D}\left(\beta t\right)\left(d\right)$$

so that

$$\mathfrak{l}_{(\alpha t, \beta t)} = \lambda_{(d_1,d_2):D(2)}t\left(\alpha d_1 + \beta d_2\right)$$

Therefore, for any $d : D$, we have

$$\begin{aligned}
\left(\alpha + \beta\right)t &= \lambda_{d:D}t\left(\left(\alpha + \beta\right)d\right) \\
&= \lambda_{d:D}t\left(\alpha d + \beta d\right) \\
&= \lambda_{d:D}\mathfrak{l}_{(\alpha t, \beta t)}\left(d,d\right) \\
&= \lambda_{d:D}\left(\alpha t + \beta t\right)\left(d\right)
\end{aligned}$$

so that (5) obtains.

5. It is easy to see that

$$\left(\lambda_{(d_1,d_2):D(2)}\mathfrak{l}_{(t_1,t_2)}\left(\alpha d_1, \alpha d_2\right)\right) \circ \left(\lambda_{d:D}\left(d,0\right)\right) = \lambda_{d:D}t_1\left(\alpha d\right) = \lambda_{d:D}\left(\alpha t_1\right)\left(d\right)$$

$$\left(\lambda_{(d_1,d_2):D(2)}\mathfrak{l}_{(t_1,t_2)}\left(\alpha d_1, \alpha d_2\right)\right) \circ \left(\lambda_{d:D}\left(0,d\right)\right) = \lambda_{d:D}t_2\left(\alpha d\right) = \lambda_{d:D}\left(\alpha t_2\right)\left(d\right)$$

so that

$$\mathfrak{l}_{(\alpha t_1, \alpha t_2)} = \lambda_{(d_1,d_2):D(2)}\mathfrak{l}_{(t_1,t_2)}\left(\alpha d_1, \alpha d_2\right)$$

Therefore, for any $d : D$, we have

$$\begin{aligned}
\alpha\left(t_1 + t_2\right) &= \alpha\left(\lambda_{d:D}\mathfrak{l}_{(t_1,t_2)}\left(d.d\right)\right) \\
&= \lambda_{d:D}\mathfrak{l}_{(t_1,t_2)}\left(\alpha d.\alpha d\right) \\
&= \lambda_{d:D}\mathfrak{l}_{(\alpha t_1, \alpha t_2)}\left(d.d\right) \\
&= \lambda_{d:D}\left(\alpha t_1 + \alpha t_2\right)\left(d\right)
\end{aligned}$$

so that (6) obtains.

6. It is easy to see that

$$\begin{aligned}
\left(\alpha\beta\right)t &= \lambda_{d:D}t\left(\alpha\beta d\right) \\
&= \lambda_{d:D}\left(\beta t\right)\left(\alpha d\right) \\
&= \lambda_{d:D}\left(\alpha\left(\beta t\right)\right)\left(d\right)
\end{aligned}$$

so that (7) obtains.

∎

We recall that

Lemma 36 *(cf.Proposition 7 of §2.2 in [7]) The following is a quasi-colimit diagram:*

$$D \quad \begin{matrix} \xrightarrow{\lambda_{d:D}\left(d,0\right)} \\ \xrightarrow{\lambda_{d:D}\left(0,d\right)} \\ \xrightarrow{\lambda_{d:D}\left(0,0\right)} \end{matrix} \quad D \times D \quad \xrightarrow{\lambda_{(d_1,d_2):D\times D}d_1 d_2} \quad D$$

Corollary 37 *Let M be a microlinear set. Given*

$$\theta : D \times D \to M$$

in accordance with

$$\theta \circ (\lambda_{d:D}(d,0)) = \theta \circ (\lambda_{d:D}(0,d)) = \theta \circ (\lambda_{d:D}(0,0))$$

there exists a homotopically unique

$$t : D \to M$$

in accordance with

$$t \circ \left(\lambda_{(d_1,d_2):D \times D} d_1 d_2\right) = \theta$$

Theorem 38 *(cf. Proposition 2 of §3.1.1 in [7]) Let M be a microlinear type. For any $x : M$, the $\mathbb{R}$-module $\mathbf{T}_x(M)$ is Euclidean.*

Proof. It is easy to see that

$$\left(\lambda_{(d_1,d_2):D \times D}\left(\varphi(d_1) - \varphi(0)\right)(d_2)\right) \circ (\lambda_{d:D}(d,0)) = \lambda_{d:D} \left|x\right|_0$$
$$\left(\lambda_{(d_1,d_2):D \times D}\left(\varphi(d_1) - \varphi(0)\right)(d_2)\right) \circ (\lambda_{d:D}(0,d)) = \lambda_{d:D} \left|x\right|_0$$
$$\left(\lambda_{(d_1,d_2):D \times D}\left(\varphi(d_1) - \varphi(0)\right)(d_2)\right) \circ (\lambda_{d:D}(0,0)) = \lambda_{d:D} \left|x\right|_0$$

Therefore, by dint of Corollary 37, there exists $t : D \to \|M\|_0$ such that

$$\lambda_{(d_1,d_2):D \times D}\left(\varphi(d_1) - \varphi(0)\right)(d_2) = \lambda_{(d_1,d_2):D \times D} t\,(d_1 d_2)$$

which is no other than

$$\prod_{d:D} \varphi(d) - \varphi(0) = dt$$

This completes the proof. ∎

6   Strong Differences

We recall that

Lemma 39 *(The first Lemma of §3.4 in [7]) The following diagram is a quasi-colimit diagram:*

$$
\begin{array}{ccc}
D^2\{(1,2)\} & \xrightarrow{\;\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\;} & D^2 \\
{\scriptstyle\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)}\downarrow & & \downarrow{\scriptstyle\lambda_{(d_1,d_2):D^2}(d_1,d_2,d_1 d_2)} \\
D^2 & \xrightarrow[\;\lambda_{(d_1,d_2):D^2}(d_1,d_2,0)\;]{} & D^3\{(1,3),(2,3)\}
\end{array}
$$

Corollary 40 *Let M be a microlinear set. Given $\theta_1, \theta_2 : D^2 \to M$ with*

$$\theta_1 \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right) = \theta_2 \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$

there exists $\mathfrak{m}_{(\theta_1,\theta_2)} : D^3\{(1,3),(2,3)\} \to M$ with

$$\mathfrak{m}_{(\theta_1,\theta_2)} \circ \left(\lambda_{(d_1,d_2):D^2}(d_1,d_2,0)\right) = \theta_2$$
$$\mathfrak{m}_{(\theta_1,\theta_2)} \circ \left(\lambda_{(d_1,d_2):D^2}(d_1,d_2,d_1 d_2)\right) = \theta_1$$

Now we define strong differences.

Definition 41 *Let M be a microlinear set. Given $\theta_1, \theta_2 : D^2 \to M$ with*

$$\theta_1 \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right) = \theta_2 \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$

we define $\theta_1 \overset{\cdot}{-} \theta_2 : D \to M$ to be

$$\theta_1 \overset{\cdot}{-} \theta_2 :\equiv \lambda_{d:D}\mathfrak{m}_{(\theta_1,\theta_2)}(0,0,d)$$

We recall that

Proposition 42 *(cf. Proposition 8 of §3.4 in [7]) Let M be a microlinear set. Given $\theta_1, \theta_2 : D^2 \to M$ with*

$$\theta_1 \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right) = \theta_2 \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$

we have

$$\theta_1 \circ \left(\lambda_{(d_1,d_2):D^2}(d_2,d_1)\right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$
$$= \theta_2 \circ \left(\lambda_{(d_1,d_2):D^2}(d_2,d_1)\right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$

and

$$\theta_1 \circ \left(\lambda_{(d_1,d_2):D^2}(d_2,d_1)\right) \overset{\cdot}{-} \theta_2 \circ \left(\lambda_{(d_1,d_2):D^2}(d_2,d_1)\right)$$
$$= \theta_1 \overset{\cdot}{-} \theta_2$$

Proof. The first identity should be obvious. For the second identity, it suffices to note that

$$\mathfrak{m}_{\left(\theta_1\circ\left(\lambda_{(d_1,d_2):D^2}(d_2,d_1)\right),\theta_2\circ\left(\lambda_{(d_1,d_2):D^2}(d_2,d_1)\right)\right)} = \lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}}\mathfrak{m}_{(\theta_1,\theta_2)}(d_2,d_1,d_3)$$

∎

Definition 43 *Let M be a microlinear set. We give two definitions:*

- *Given $\theta_1, \theta_2 : D^2 \to M$ with*

$$\theta_1 \circ (\lambda_{d:D}(0,d)) = \theta_2 \circ (\lambda_{d:D}(0,d))$$

we define $\theta_1 \underset{1}{+} \theta_2 : D^2 \to M$ to be

$$\theta_1 \underset{1}{+} \theta_2 :\equiv \lambda_{(d_1,d_2):D^2}\left(\lambda_{d_1:D}\lambda_{d_2:D}\theta_1(d_1,d_2) + \lambda_{d_1:D}\lambda_{d_2:D}\theta_2(d_1,d_2)\right)(d_1)(d_2)$$

- *Given $\alpha : \mathbb{R}$ and $\theta : D^2 \to M$, we define $\alpha \underset{1}{\cdot} \theta$ to be*

$$\alpha \underset{1}{\cdot} \theta :\equiv \lambda_{(d_1,d_2):D^2} \left(\alpha \left(\lambda_{d_1:D} \lambda_{d_2:D} \theta \left(d_1, d_2 \right) \right) \right) \left(d_1 \right) \left(d_2 \right)$$

Lemma 44 *The diagram consisting of*

$$\mathcal{N} :\equiv D^5 \left\{ (1,2), (1,4), (1,5), (2,4), (2,5), (4,5) \right\}$$
$$\mathcal{L}_{11} :\equiv D^2, \mathcal{L}_{12} :\equiv D^2, \mathcal{L}_{21} :\equiv D^2, \mathcal{L}_{22} :\equiv D^2$$
$$\mathcal{P}_1 :\equiv D^2 \left\{ (1,2) \right\}, \mathcal{P}_2 :\equiv D^2 \left\{ (1,2) \right\}$$
$$\mathcal{Q}_1 :\equiv D, \mathcal{Q}_2 :\equiv D$$
$$\lambda_{(d_1,d_2):D^2} \left(d_1, 0, d_2, d_1 d_2, 0 \right) : \mathcal{L}_{11} \to \mathcal{N}, \lambda_{(d_1,d_2):D^2} \left(d_1, 0, d_2, 0, 0 \right) : \mathcal{L}_{21} \to \mathcal{N}$$
$$\lambda_{(d_1,d_2):D^2} \left(0, d_1, d_2, 0, d_1 d_2 \right) : \mathcal{L}_{12} \to \mathcal{N}, \lambda_{(d_1,d_2):D^2} \left(0, d_1, d_2, 0, 0 \right) : \mathcal{L}_{22} \to \mathcal{N}$$
$$\lambda_{(d_1,d_2):D^2\{(1,2)\}} \left(d_1, d_2 \right) : \mathcal{P}_1 \to \mathcal{L}_{11}, \lambda_{(d_1,d_2):D^2\{(1,2)\}} \left(d_1, d_2 \right) : \mathcal{P}_1 \to \mathcal{L}_{21}$$
$$\lambda_{(d_1,d_2):D^2\{(1,2)\}} \left(d_1, d_2 \right) : \mathcal{P}_2 \to \mathcal{L}_{12}, \lambda_{(d_1,d_2):D^2\{(1,2)\}} \left(d_1, d_2 \right) : \mathcal{P}_2 \to \mathcal{L}_{22}$$
$$\lambda_{d:D} \left(0, d \right) : \mathcal{Q}_1 \to \mathcal{L}_{11}, \lambda_{d:D} \left(0, d \right) : \mathcal{Q}_1 \to \mathcal{L}_{12}$$
$$\lambda_{d:D} \left(0, d \right) : \mathcal{Q}_2 \to \mathcal{L}_{21}, \lambda_{d:D} \left(0, d \right) : \mathcal{Q}_2 \to \mathcal{L}_{22}$$

is a quasi-colimit diagram.

Proof. By Axiom 6 we are sure that, given

$$\gamma^{11}, \gamma^{12}, \gamma^{21}, \gamma^{22} : D^2 \to \mathbb{R}$$

there exist

$$a^{11}, a^{11}_1, a^{11}_2, a^{11}_{12}, a^{12}, a^{12}_1, a^{12}_2, a^{12}_{12}, a^{21}, a^{21}_1, a^{21}_2, a^{21}_{12}, a^{22}, a^{22}_1, a^{22}_2, a^{22}_{12} : \mathbb{R}$$

such that

$$\lambda_{(d_1,d_2):D^2} \gamma^{11} \left(d_1, d_2 \right)$$
$$= \lambda_{(d_1,d_2):D^2} \left(a^{11} + a^{11}_1 d_1 + a^{11}_2 d_2 + a^{11}_{12} d_1 d_2 \right)$$
$$\lambda_{(d_1,d_2):D^2} \gamma^{12} \left(d_1, d_2 \right)$$
$$= \lambda_{(d_1,d_2):D^2} \left(a^{12} + a^{12}_1 d_1 + a^{12}_2 d_2 + a^{12}_{12} d_1 d_2 \right)$$
$$\lambda_{(d_1,d_2):D^2} \gamma^{21} \left(d_1, d_2 \right)$$
$$= \lambda_{(d_1,d_2):D^2} \left(a^{21} + a^{21}_1 d_1 + a^{21}_2 d_2 + a^{21}_{12} d_1 d_2 \right)$$
$$\lambda_{(d_1,d_2):D^2} \gamma^{22} \left(d_1, d_2 \right)$$
$$= \lambda_{(d_1,d_2):D^2} \left(a^{22} + a^{22}_1 d_1 + a^{22}_2 d_2 + a^{22}_{12} d_1 d_2 \right)$$

The conditions

$$\gamma^{11} \circ \left(\lambda_{(d_1,d_2)::D^2\{(1,2)\}} \left(d_1, d_2 \right) \right) = \gamma^{21} \circ \left(\lambda_{(d_1,d_2)::D^2\{(1,2)\}} \left(d_1, d_2 \right) \right)$$
$$\gamma^{12} \circ \left(\lambda_{(d_1,d_2)::D^2\{(1,2)\}} \left(d_1, d_2 \right) \right) = \gamma^{22} \circ \left(\lambda_{(d_1,d_2)::D^2\{(1,2)\}} \left(d_1, d_2 \right) \right)$$

imply that

$$a^{11} = a^{21}, a_1^{11} = a_1^{21}, a_2^{11} = a_2^{21}$$
$$a^{12} = a^{22}, a_1^{12} = a_1^{22}, a_2^{12} = a_2^{22}$$

The conditions

$$\gamma^{11} \circ (\lambda_{d:D} (0, d)) = \gamma^{12} \circ (\lambda_{d:D} (0, d))$$
$$\gamma^{21} \circ (\lambda_{d:D} (0, d)) = \gamma^{22} \circ (\lambda_{d:D} (0, d))$$

imply that

$$a^{11} = a^{12}, a_2^{11} = a_2^{12}$$
$$a^{21} = a^{22}, a_2^{21} = a_2^{22}$$

Therefore we have

$$a^{11} = a^{12} = a^{21} = a^{22}$$
$$a_1^{11} = a_1^{21}, a_1^{12} = a_1^{22}$$
$$a_2^{11} = a_2^{12} = a_2^{21} = a_2^{22}$$

which implies that there exist

$$b, b_1, b_2, b_3, b_{13}, b_{23}, b_4, b_5 : \mathbb{R}$$

such that

$$\left(\lambda_{(d_1, d_2, d_3, d_4, d_5):N} b + b_1 d_1 + b_2 d_2 + b_3 d_3 + b_{13} d_1 d_3 + b_{23} d_2 d_3 + b_4 d_4 + b_5 d_5 \right)$$
$$\circ \left(\lambda_{(d_1, d_2):D^2} (d_1, 0, d_2, d_1 d_2, 0) \right)$$
$$= \gamma^{11}$$
$$\left(\lambda_{(d_1, d_2, d_3, d_4, d_5):N} b + b_1 d_1 + b_2 d_2 + b_3 d_3 + b_{13} d_1 d_3 + b_{23} d_2 d_3 + b_4 d_4 + b_5 d_5 \right)$$
$$\circ \left(\lambda_{(d_1, d_2):D^2} (d_1, 0, d_2, 0, 0) \right)$$
$$= \gamma^{21}$$
$$\left(\lambda_{(d_1, d_2, d_3, d_4, d_5):N} b + b_1 d_1 + b_2 d_2 + b_3 d_3 + b_{13} d_1 d_3 + b_{23} d_2 d_3 + b_4 d_4 + b_5 d_5 \right)$$
$$\circ \left(\lambda_{(d_1, d_2):D^2} (0, d_1, d_2, 0, d_1 d_2) \right)$$
$$= \gamma^{12}$$
$$\left(\lambda_{(d_1, d_2, d_3, d_4, d_5):N} b + b_1 d_1 + b_2 d_2 + b_3 d_3 + b_{13} d_1 d_3 + b_{23} d_2 d_3 + b_4 d_4 + b_5 d_5 \right)$$
$$\circ \left(\lambda_{(d_1, d_2):D^2} (0, d_1, d_2, 0, 0) \right)$$
$$= \gamma^{22}$$

This completes the proof. ∎

Corollary 45 *Let M be a microlinear set. Given $\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} : D^2 \to M$ with*

$$\theta_{11} \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1, d_2) \right) = \theta_{21} \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1, d_2) \right)$$
$$\theta_{12} \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1, d_2) \right) = \theta_{22} \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1, d_2) \right)$$
$$\theta_{11} \circ \left(\lambda_{d:D} (0, d) \right) = \theta_{12} \circ \left(\lambda_{d:D} (0, d) \right)$$
$$\theta_{21} \circ \left(\lambda_{d:D} (0, d) \right) = \theta_{22} \circ \left(\lambda_{d:D} (0, d) \right)$$

there exists

$$\mathfrak{n}_{(\theta_{11},\theta_{12},\theta_{21},\theta_{22})} : \mathcal{N} \to M$$

such that

$$\mathfrak{n}_{(\theta_{11},\theta_{12},\theta_{21},\theta_{22})} \circ \left(\lambda_{(d_1,d_2):D^2} (d_1, 0, d_2, d_1 d_2, 0) \right) = \theta_{11}$$
$$\mathfrak{n}_{(\theta_{11},\theta_{12},\theta_{21},\theta_{22})} \circ \left(\lambda_{(d_1,d_2):D^2} (d_1, 0, d_2, 0, 0) \right) = \theta_{21}$$
$$\mathfrak{n}_{(\theta_{11},\theta_{12},\theta_{21},\theta_{22})} \circ \left(\lambda_{(d_1,d_2):D^2} (0, d_1, d_2, 0, d_1 d_2) \right) = \theta_{12}$$
$$\mathfrak{n}_{(\theta_{11},\theta_{12},\theta_{21},\theta_{22})} \circ \left(\lambda_{(d_1,d_2):D^2} (0, d_1, d_2, 0, 0) \right) = \theta_{22}$$

Proposition 46 *Let M be a microlinear set. Given $\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} : D^2 \to M$ with*

$$\theta_{11} \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1, d_2) \right) = \theta_{21} \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1, d_2) \right)$$
$$\theta_{12} \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1, d_2) \right) = \theta_{22} \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1, d_2) \right)$$
$$\theta_{11} \circ \left(\lambda_{d:D} (0, d) \right) = \theta_{12} \circ \left(\lambda_{d:D} (0, d) \right)$$
$$\theta_{21} \circ \left(\lambda_{d:D} (0, d) \right) = \theta_{22} \circ \left(\lambda_{d:D} (0, d) \right)$$

we have

$$\left(\theta_{11} \underset{1}{+} \theta_{12} \right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1, d_2) \right) = \left(\theta_{21} \underset{1}{+} \theta_{22} \right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1, d_2) \right)$$

and

$$\left(\theta_{11} \underset{1}{+} \theta_{12} \right) \overset{.}{-} \left(\theta_{21} \underset{1}{+} \theta_{22} \right) = \left(\theta_{11} \overset{.}{-} \theta_{21} \right) + \left(\theta_{12} \overset{.}{-} \theta_{22} \right)$$

Proof. Since

$$\lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}} \mathfrak{m}_{\left(\theta_{11}\underset{1}{+}\theta_{12}, \theta_{21}\underset{1}{+}\theta_{22} \right)} (d_1, d_2, d_3)$$

$$= \lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}} \mathfrak{n}_{(\theta_{11},\theta_{12},\theta_{21},\theta_{22})} (d_1, d_1, d_2, d_3, d_3)$$

and

$$\lambda_{(d_1,d_2):D^2\{(1,2)\}} \mathfrak{l}_{\left(\theta_{11}\overset{.}{-}\theta_{21}, \theta_{12}\overset{.}{-}\theta_{22} \right)} (d_1, d_2)$$

$$= \lambda_{(d_1,d_2):D^2\{(1,2)\}} \mathfrak{n}_{(\theta_{11},\theta_{12},\theta_{21},\theta_{22})} (0, 0, 0, d_1, d_2)$$

we have

$$\lambda_{d:D}\left(\left(\theta_{11}\underset{1}{+}\theta_{12}\right)\overset{\cdot}{-}\left(\theta_{21}\underset{1}{+}\theta_{22}\right)\right)(d)$$

$$=\lambda_{d:D}\mathfrak{m}_{\left(\theta_{11}\underset{1}{+}\theta_{12},\theta_{21}\underset{1}{+}\theta_{22}\right)}(0,0,d)$$

$$=\lambda_{d:D}\mathfrak{n}_{(\theta_{11},\theta_{12},\theta_{21},\theta_{22})}(0,0,0,d,d)$$

$$=\lambda_{d:D}\mathfrak{l}_{\left(\theta_{11}\overset{\cdot}{-}\theta_{21},\theta_{12}\overset{\cdot}{-}\theta_{22}\right)}(d,d)$$

$$=\lambda_{d:D}\left(\left(\theta_{11}\overset{\cdot}{-}\theta_{21}\right)+\left(\theta_{12}\overset{\cdot}{-}\theta_{22}\right)\right)(d)$$

This completes the proof. ∎

Proposition 47 *Let M be a microlinear set. Given $\alpha:\mathbb{R}$ and $\theta_1,\theta_2:D^2\to M$ with*

$$\theta_1\circ\left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)=\theta_2\circ\left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$

we have

$$\left(\alpha\underset{1}{\cdot}\theta_1\right)\circ\left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)=\left(\alpha\underset{1}{\cdot}\theta_2\right)\circ\left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$

and

$$\left(\alpha\underset{1}{\cdot}\theta_1\right)\overset{\cdot}{-}\left(\alpha\underset{1}{\cdot}\theta_2\right)=\alpha\left(\theta_1\overset{\cdot}{-}\theta_2\right)$$

Proof. Since

$$\lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}}\mathfrak{m}_{\left(\alpha\underset{1}{\cdot}\theta_1,\alpha\underset{1}{\cdot}\theta_2\right)}(d_1,d_2,d_3)$$

$$=\lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}}\mathfrak{m}_{(\theta_1,\theta_2)}(\alpha d_1,d_2,\alpha d_3)$$

we have

$$\lambda_{d:D}\left(\alpha\left(\theta_1\overset{\cdot}{-}\theta_2\right)\right)(d)$$

$$=\lambda_{d:D}\mathfrak{m}_{(\theta_1,\theta_2)}(0,0,\alpha d)$$

$$=\lambda_{d:D}\mathfrak{m}_{\left(\alpha\underset{1}{\cdot}\theta_1,\alpha\underset{1}{\cdot}\theta_2\right)}(0,0,d)$$

$$=\lambda_{d:D}\left(\left(\alpha\underset{1}{\cdot}\theta_1\right)\overset{\cdot}{-}\left(\alpha\underset{1}{\cdot}\theta_2\right)\right)(d)$$

∎

Lemma 48 *The diagram*

$$D^4\{(1,3),(2,3),(1,4),(2,4),(3,4)\}$$

$$\nearrow \qquad \uparrow \qquad \nwarrow$$

$$D^2 \qquad\qquad D^2 \qquad\qquad D^2$$

$$\nwarrow \qquad \uparrow \qquad \nearrow$$

$$D^2\{(1,2)\}$$

19

is a quasi-colimit diagram, where the lower three arrows are all

$$\lambda_{(d_1,d_2)::D^2\{(1,2)\}}(d_1,d_2)$$

while the upper three arrows are

$$\lambda_{(d_1,d_2)::D^2}(d_1,d_2,0,0)$$
$$\lambda_{(d_1,d_2)::D^2}(d_1,d_2,d_1d_2,0)$$
$$\lambda_{(d_1,d_2)::D^2}(d_1,d_2,d_1d_2,d_1d_2)$$

from left to right

Proof. By Axiom 6 we are sure that, given

$$\gamma^1,\gamma^2,\gamma^3:D^2\to\mathbb{R}$$

there exist

$$a^1,a_1^1,a_2^1,a_{12}^1,a^2,a_1^2,a_2^2,a_{12}^2,a^3,a_1^3,a_2^3,a_{12}^3:\mathbb{R}$$

such that

$$\lambda_{(d_1,d_2):D^2}\gamma^1(d_1,d_2)$$
$$=\lambda_{(d_1,d_2):D^2}\left(a^1+a_1^1d_1+a_2^1d_2+a_{12}^1d_1d_2\right)$$
$$\lambda_{(d_1,d_2):D^2}\gamma^2(d_1,d_2)$$
$$=\lambda_{(d_1,d_2):D^2}\left(a^2+a_1^2d_1+a_2^2d_2+a_{12}^2d_1d_2\right)$$
$$\lambda_{(d_1,d_2):D^2}\gamma^3(d_1,d_2)$$
$$=\lambda_{(d_1,d_2):D^2}\left(a^3+a_1^3d_1+a_2^3d_2+a_{12}^3d_1d_2\right)$$

If they satisfy

$$\gamma^1\circ\left(\lambda_{(d_1,d_2)::D^2\{(1,2)\}}(d_1,d_2)\right)$$
$$=\gamma^2\circ\left(\lambda_{(d_1,d_2)::D^2\{(1,2)\}}(d_1,d_2)\right)$$
$$=\gamma^3\circ\left(\lambda_{(d_1,d_2)::D^2\{(1,2)\}}(d_1,d_2)\right)$$

then we have

$$a^1=a^2=a^3$$
$$a_1^1=a_1^2=a_1^3$$
$$a_2^1=a_2^2=a_3^3$$

Therefore there exist

$$b,b_1,b_2,b_{12},b_3,b_4:\mathbb{R}$$

such that

$$\left(\lambda_{(d_1,d_2,d_3,d_4):D^4\{(1,3),(2,3),(1,4),(2,4),(3,4)\}}b + b_1d_1 + b_2d_2 + b_{12}d_1d_2 + b_3d_3 + b_4d_4\right)$$
$$\circ \left(\lambda_{(d_1,d_2):D^2}\gamma^1(d_1,d_2)\right)$$
$$= \gamma^1$$

$$\left(\lambda_{(d_1,d_2,d_3,d_4):D^4\{(1,3),(2,3),(1,4),(2,4),(3,4)\}}b + b_1d_1 + b_2d_2 + b_{12}d_1d_2 + b_3d_3 + b_4d_4\right)$$
$$\circ \lambda_{(d_1,d_2)::D^2}(d_1,d_2,d_1d_2,0)$$
$$= \gamma^2$$

$$\left(\lambda_{(d_1,d_2,d_3,d_4):D^4\{(1,3),(2,3),(1,4),(2,4),(3,4)\}}b + b_1d_1 + b_2d_2 + b_{12}d_1d_2 + b_3d_3 + b_4d_4\right)$$
$$\circ \lambda_{(d_1,d_2)::D^2}(d_1,d_2,d_1d_2,d_1d_2)$$
$$= \gamma^3$$

This completes the proof. ∎

Corollary 49 *Let M be a microlinear set. Given $\theta_1, \theta_2, \theta_3 : D^2 \to M$ with*

$$\theta_1 \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$
$$= \theta_2 \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$
$$= \theta_3 \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$

there exists

$$\mathfrak{m}_{(\theta_1,\theta_2,\theta_3)} : D^4\{(1,3),(2,3),(1,4),(2,4),(3,4)\} \to M$$

such that

$$\mathfrak{m}_{(\theta_1,\theta_2,\theta_3)} \circ \left(\lambda_{(d_1,d_2)::D^2}(d_1,d_2,d_1d_2,d_1d_2)\right) = \theta_1$$
$$\mathfrak{m}_{(\theta_1,\theta_2,\theta_3)} \circ \left(\lambda_{(d_1,d_2)::D^2}(d_1,d_2,d_1d_2,0)\right) = \theta_2$$
$$\mathfrak{m}_{(\theta_1,\theta_2,\theta_3)} \circ \left(\lambda_{(d_1,d_2)::D^2}(d_1,d_2,0,0)\right) = \theta_3$$

Proposition 50 *(The primordial general Jacobi identity) Let M be a micro-linear set. Given $\theta_1, \theta_2, \theta_3 : D^2 \to M$ with*

$$\theta_1 \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$
$$= \theta_2 \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$
$$= \theta_3 \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$

we have

$$\left(\theta_1 \overset{\cdot}{-} \theta_2\right) + \left(\theta_2 \overset{\cdot}{-} \theta_3\right) = \theta_1 \overset{\cdot}{-} \theta_3$$

In particular, we have

$$\left(\theta_1 \overset{\cdot}{-} \theta_2\right) + \left(\theta_2 \overset{\cdot}{-} \theta_1\right) = 0$$

Proof. Since

$$\lambda_{(d_1,d_2):D^2\{(1,2)\}}\mathfrak{l}_{\left(\theta_1\overset{.}{-}\theta_2,\theta_2\overset{.}{-}\theta_3\right)}(d_1,d_2)$$

$$= \lambda_{(d_1,d_2):D^2\{(1,2)\}}\mathfrak{m}_{(\theta_1,\theta_2,\theta_3)}(0,0,d_1,d_2)$$

and

$$\lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}}\mathfrak{m}_{(\theta_1,\theta_3)}(d_1,d_2,d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}}\mathfrak{m}_{(\theta_1,\theta_2,\theta_3)}(d_1,d_2,d_3,d_3)$$

we have

$$\lambda_{d:D}\left(\left(\theta_1\overset{.}{-}\theta_2\right)+\left(\theta_2\overset{.}{-}\theta_3\right)\right)(d)$$

$$= \lambda_{d:D}\mathfrak{l}_{\left(\theta_1\overset{.}{-}\theta_2,\theta_2\overset{.}{-}\theta_3\right)}(d,d)$$

$$= \lambda_{d:D}\mathfrak{m}_{(\theta_1,\theta_2,\theta_3)}(0,0,d,d)$$

$$= \lambda_{d:D}\mathfrak{m}_{(\theta_1,\theta_3)}(0,0,d)$$

$$= \lambda_{d:D}\left(\theta_1\overset{.}{-}\theta_3\right)(d)$$

This completes the proof. ■

Now we define relative strong differences.

Definition 51 *Let M be a microlinear set. We give three definitions:*

- *Given $\theta_1,\theta_2 : D^3 \to M$ with*

$$\theta_1 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1,d_2,d_3)\right) = \theta_2 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1,d_2,d_3)\right)$$

we define $\theta_1 \overset{.}{\underset{3}{-}} \theta_2 : D^2 \to M$ to be

$$\theta_1 \overset{.}{\underset{3}{-}} \theta_2$$

$$:\equiv \lambda_{(d_1,d_2):D^2}\left(\left(\begin{array}{c}\left(\lambda_{(d_1,d_2):D^2}\lambda_{d_3:D}\theta_1\,(d_1,d_2,d_3)\right)\overset{.}{-}\\\left(\lambda_{(d_1,d_2):D^2}\lambda_{d_3:D}\theta_2\,(d_1,d_2,d_3)\right)\\\circ\left(\lambda_{(d_1,d_2):D^2}\,(d_2,d_1)\right)\end{array}\right)\right)(d_2)(d_1)$$

- *Given $\theta_1,\theta_2 : D^3 \to M$ with*

$$\theta_1 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}}(d_1,d_2,d_3)\right) = \theta_2 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}}(d_1,d_2,d_3)\right)$$

we define $\theta_1 \overset{.}{\underset{2}{-}} \theta_2 : D^2 \to M$ to be

$$\theta_1 \overset{.}{\underset{2}{-}} \theta_2 :\equiv \theta_1 \circ \left(\lambda_{(d_1,d_2,d_3):D^3}(d_2,d_3,d_1)\right) \overset{.}{\underset{3}{-}} \theta_2 \circ \left(\lambda_{(d_1,d_2,d_3):D^3}(d_2,d_3,d_1)\right)$$

- *Given $\theta_1, \theta_2 : D^3 \to M$ with*

$$\theta_1 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right) = \theta_2 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right)$$

  we define $\theta_1 \underset{1}{\overset{\cdot}{-}} \theta_2 : D^2 \to M$ to be

$$\theta_1 \underset{1}{\overset{\cdot}{-}} \theta_2 :\equiv \theta_1 \circ \left(\lambda_{(d_1,d_2,d_3):D^3} (d_3, d_1, d_2)\right) \underset{3}{\overset{\cdot}{-}} \theta_2 \circ \left(\lambda_{(d_1,d_2,d_3):D^3} (d_3, d_1, d_2)\right)$$

Lemma 52 *The diagram*

$$
D^{n+m_1+m_2}\left\{ \begin{array}{c} (n+i, n+m_1+j) \mid \\ 1 \leq i \leq m_1, 1 \leq j \leq m_2 \end{array} \right\}
$$

$$
D^{n+m_1+m_2}\left\{ \begin{array}{c} (n+1),..., \\ (n+m_1) \end{array} \right\} \qquad \nearrow \qquad \nwarrow \qquad D^{n+m_1+m_2}\left\{ \begin{array}{c} (n+m_1+1 \\ (n+m_1+ \end{array} \right.
$$

$$\nwarrow \qquad D^n \qquad \nearrow$$

$$(10)$$

with the four arrows being the canonical injections is a quasi-colimit diagram.

Corollary 53 *Let M be a microlinear set. Given $\theta_1, \theta_2 : D^{n+m_1+m_2} \to M$,*

$$\theta_1 \mid D^{n+m_1+m_2}\{(n+i, n+m_1+j) \mid 1 \leq i \leq m_1, 1 \leq j \leq m_2\}$$
$$= \theta_2 \mid D^{n+m_1+m_2}\{(n+i, n+m_1+j) \mid 1 \leq i \leq m_1, 1 \leq j \leq m_2\}$$

obtains iff both

$$\theta_1 \mid D^{n+m_1+m_2}\{(n+1),...,(n+m_1)\} = \theta_2 \mid D^{n+m_1+m_2}\{(n+1),...,(n+m_1)\}$$

and

$$\theta_1 \mid D^{n+m_1+m_2}\{(n+m_1+1),...,(n+m_1+m_2)\} = \theta_2 \mid D^{n+m_1+m_2}\{(n+m_1+1),...,(n+m_1+m_2)\}$$

obtain.

Proposition 54 *Let M be a microlinear set and $\theta_1, \theta_2, \theta_3, \theta_4 : D^3 \to M$. Then we have the following:*

- *Let us suppose that the identities*

$$\theta_1 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right) = \theta_2 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right)$$
$$\theta_3 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right) = \theta_4 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right)$$

  hold so that the strong differences

$$\theta_1 \underset{1}{\overset{\cdot}{-}} \theta_2$$

$$\theta_3 \underset{1}{\overset{\cdot}{-}} \theta_4$$

23

are to be defined. If the identies

$$\theta_1 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1,d_2,d_3) \right) = \theta_3 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1,d_2,d_3) \right)$$
$$(11)$$

$$\theta_2 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1,d_2,d_3) \right) = \theta_4 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1,d_2,d_3) \right)$$
$$(12)$$

obtain, then the identity

$$\left(\theta_1 \underset{1}{\overset{\cdot}{-}} \theta_2 \right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1,d_2) \right) = \left(\theta_3 \underset{1}{\overset{\cdot}{-}} \theta_4 \right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1,d_2) \right)$$

obtains so that the strong difference

$$\left(\theta_1 \underset{1}{\overset{\cdot}{-}} \theta_2 \right) \overset{\cdot}{-} \left(\theta_3 \underset{1}{\overset{\cdot}{-}} \theta_4 \right)$$

is to be defined.

- *Let us suppose that the identities*

$$\theta_1 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1,d_2,d_3) \right) = \theta_2 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1,d_2,d_3) \right)$$
$$\theta_3 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1,d_2,d_3) \right) = \theta_4 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1,d_2,d_3) \right)$$

hold so that the strong differences

$$\theta_1 \underset{2}{\overset{\cdot}{-}} \theta_2$$

$$\theta_3 \underset{2}{\overset{\cdot}{-}} \theta_4$$

are to be defined. If the identies

$$\theta_1 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(2,3)\}} (d_1,d_2,d_3) \right) = \theta_3 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(2,3)\}} (d_1,d_2,d_3) \right)$$
$$(13)$$

$$\theta_2 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(2,3)\}} (d_1,d_2,d_3) \right) = \theta_4 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(2,3)\}} (d_1,d_2,d_3) \right)$$
$$(14)$$

obtain, then the identity

$$\left(\theta_1 \underset{2}{\overset{\cdot}{-}} \theta_2 \right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1,d_2) \right) = \left(\theta_3 \underset{2}{\overset{\cdot}{-}} \theta_4 \right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1,d_2) \right)$$

obtains so that the strong difference

$$\left(\theta_1 \underset{2}{\overset{\cdot}{-}} \theta_2 \right) \overset{\cdot}{-} \left(\theta_3 \underset{2}{\overset{\cdot}{-}} \theta_4 \right)$$

is to be defined.

- *Let us suppose that the identities*

$$\theta_1 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1,d_2,d_3)\right) = \theta_2 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1,d_2,d_3)\right)$$

$$\theta_3 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1,d_2,d_3)\right) = \theta_4 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1,d_2,d_3)\right)$$

  hold so that the strong differences

$$\theta_1 \underset{3}{\overset{\cdot}{-}} \theta_2$$

$$\theta_3 \underset{3}{\overset{\cdot}{-}} \theta_4$$

  are to be defined. If the identies

$$\theta_1 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}}(d_1,d_2,d_3)\right) = \theta_3 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}}(d_1,d_2,d_3)\right) \tag{15}$$

$$\theta_2 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}}(d_1,d_2,d_3)\right) = \theta_4 \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}}(d_1,d_2,d_3)\right) \tag{16}$$

  obtain, then the identity

$$\left(\theta_1 \underset{3}{\overset{\cdot}{-}} \theta_2\right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right) = \left(\theta_3 \underset{3}{\overset{\cdot}{-}} \theta_4\right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$

  obtains so that the strong difference

$$\left(\theta_1 \underset{3}{\overset{\cdot}{-}} \theta_2\right) \overset{\cdot}{-} \left(\theta_3 \underset{3}{\overset{\cdot}{-}} \theta_4\right)$$

  is to be defined.

Proof. We deal only with the first statement, safely leaving the second and third ones to the reader. We have to show that

$$\left(\theta_1 \underset{1}{\overset{\cdot}{-}} \theta_2\right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right) = \left(\theta_3 \underset{1}{\overset{\cdot}{-}} \theta_4\right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}}(d_1,d_2)\right)$$

which is, by dint of Corollary 53 with respect to the quasi-colimit diagram

$$
\begin{array}{ccccc}
 & & D^2\{(1,2)\} & & \\
 & \nearrow & & \nwarrow & \\
D^2\{(1)\} & & & & D^2\{(2)\} \\
 & \nwarrow & & \nearrow & \\
 & & D^2\{(1),(2)\} & & \\
\end{array}
$$

with the four arrows being the canonical injections (Lemma 52 with $n = 0$, $m_1 = 1$ and $m_2 = 1$), tantamount to showing that

$$\left(\theta_1 \underset{1}{\overset{\cdot}{-}} \theta_2\right) \mid D^2\{(1)\} = \left(\theta_3 \underset{1}{\overset{\cdot}{-}} \theta_4\right) \mid D^2\{(1)\} \tag{17}$$

$$\left(\theta_1 \underset{1}{\overset{\cdot}{-}} \theta_2\right) \mid D^2\{(2)\} = \left(\theta_3 \underset{1}{\overset{\cdot}{-}} \theta_4\right) \mid D^2\{(2)\} \tag{18}$$

25

Due to the quasi-colimit diagram

$$D^3 \{(1,2),(1,3)\}$$

$$D^3 \{((1))\} \qquad \nearrow \qquad \nwarrow \qquad D^3 \{(2),(3)\}$$

$$\nwarrow \qquad \nearrow$$

$$D^3 \{(1),(2),(3)\}$$

with the four arrows being the canonical injections (Lemma 52 with $n = 0$, $m_1 = 1$ and $m_2 = 2$), the condition (11) is equivalent by dint of Corollary 53 to the conditions

$$\theta_1 \mid D^3 \{(1)\} = \theta_3 \mid D^3 \{(1)\} \tag{19}$$

$$\theta_1 \mid D^3 \{(2),(3)\} = \theta_3 \mid D^3 \{(2),(3)\} \tag{20}$$

while the condition (12) is equivalent to the conditions

$$\theta_2 \mid D^3 \{(1)\} = \theta_4 \mid D^3 \{(1)\} \tag{21}$$

$$\theta_2 \mid D^3 \{(2),(3)\} = \theta_4 \mid D^3 \{(2),(3)\} \tag{22}$$

In order to show that (17) obtains, we note that the quasi-colimit diagram (cf. Lemma 2.1 in [8])

$$D^4 \{(2,4),(3,4)\}$$

$$D^3 \qquad \nearrow \qquad \nwarrow \qquad D^3$$

$$\nwarrow \qquad \nearrow$$

$$D^3 \{(2,3)\}$$

with the upper arrows being

$$\lambda_{(d_1,d_2,d_3):D^3} (d_1,d_2,d_3,0)$$

$$\lambda_{(d_1,d_2,d_3):D^3} (d_1,d_2,d_3,d_2 d_3)$$

from left to right and the lower arrow being the canonical injections is to be restricted to the quasi-colimit diagram

$$D^4 \{(1),(2,4),(3,4)\}$$

$$D^3 \{(1)\} \qquad \nearrow \qquad \nwarrow \qquad D^3 \{(1)\}$$

$$\nwarrow \qquad \nearrow$$

$$D^3 \{(1),(2,3)\}$$

so that the conditions (**??**) and (21) imply (17). It is easy to see that

$$\left(\left(\theta_1 \underset{1}{\overset{\cdot}{-}} \theta_2 \right) \mid D^2 \{(2)\} \right) \circ (\lambda_{d:D} (d,0))$$

$$= \left(\theta_1 \mid D^3 \{(2),(3)\} \right) \circ (\lambda_{d:D} (d,0,0)) = \left(\theta_2 \mid D^3 \{(2),(3)\} \right) \circ (\lambda_{d:D} (d,0,0))$$

$$\left(\left(\theta_3 \underset{1}{\overset{\cdot}{-}} \theta_4 \right) \mid D^2 \{(2)\} \right) \circ (\lambda_{d:D} (d,0))$$

$$= \left(\theta_3 \mid D^3 \{(2),(3)\} \right) \circ (\lambda_{d:D} (d,0,0)) = \left(\theta_4 \mid D^3 \{(2),(3)\} \right) \circ (\lambda_{d:D} (d,0,0))$$

obtain so that (20) and (22) imply (18). This completes the proof. ∎

Lemma 55 *The diagram consisting of*

$$\mathcal{G} \equiv D^8 \left\{ \begin{array}{c} (2,4),(3,4),(1,5),(3,5),(1,6),(2,6),(4,5),(4,6),(5,6), \\ (1,7),(2,7),(3,7),(4,7),(5,7),(6,7),(7,8), \\ (1,8),(2,8),(3,8),(4,8),(5,8),(6,8) \end{array} \right\}$$

$$\mathcal{H}_{123} \equiv D^3, \mathcal{H}_{132} \equiv D^3, \mathcal{H}_{213} \equiv D^3, \mathcal{H}_{231} \equiv D^3, \mathcal{H}_{312} \equiv D^3, \mathcal{H}_{321} \equiv D^3$$

$$\mathcal{K}^1_{123,132} \equiv D^3\{(2,3)\}, \mathcal{K}^1_{231,321} \equiv D^3\{(2,3)\}, \mathcal{K}^2_{231,213} \equiv D^3\{(1,3)\},$$

$$\mathcal{K}^2_{312,132} \equiv D^3\{(1,3)\}, \mathcal{K}^3_{312,321} \equiv D^3\{(1,2)\}, \mathcal{K}^3_{123,213} \equiv D^3\{(1,2)\}$$

$$f_{123} \equiv \lambda_{(d_1,d_2,d_3):D^3}(d_1,d_2,d_3,0,0,0,0,0) : \mathcal{H}_{123} \to \mathcal{G}$$

$$f_{132} \equiv \lambda_{(d_1,d_2,d_3):D^3}(d_1,d_2,d_3,d_2d_3,0,0,0,0) : \mathcal{H}_{132} \to \mathcal{G}$$

$$f_{213} \equiv \lambda_{(d_1,d_2,d_3):D^3}(d_1,d_2,d_3,0,0,d_1d_2,0,0) : \mathcal{H}_{213} \to \mathcal{G}$$

$$f_{231} \equiv \lambda_{(d_1,d_2,d_3):D^3}(d_1,d_2,d_3,0,d_1d_3,d_1d_2,0,0) : \mathcal{H}_{231} \to \mathcal{G}$$

$$f_{312} \equiv \lambda_{(d_1,d_2,d_3):D^3}(d_1,d_2,d_3,d_2d_3,d_1d_3,0,d_1d_2d_3,0) : \mathcal{H}_{312} \to \mathcal{G}$$

$$f_{321} \equiv \lambda_{(d_1,d_2,d_3):D^3}(d_1,d_2,d_3,d_2d_3,d_1d_3,d_1d_2,0,d_1d_2d_3) : \mathcal{H}_{321} \to \mathcal{G}$$

$$h^1_{123} \equiv \lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}}(d_1,d_2,d_3) : \mathcal{K}^1_{123,132} \to \mathcal{H}_{123}$$

$$h^1_{132} \equiv \lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}}(d_1,d_2,d_3) : \mathcal{K}^1_{123,132} \to \mathcal{H}_{132}$$

$$h^1_{231} \equiv \lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}}(d_1,d_2,d_3) : \mathcal{K}^1_{231,321} \to \mathcal{H}_{231}$$

$$h^1_{321} \equiv \lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}}(d_1,d_2,d_3) : \mathcal{K}^1_{231,321} \to \mathcal{H}_{321}$$

$$h^2_{231} \equiv \lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}}(d_1,d_2,d_3) : \mathcal{K}^2_{231,213} \to \mathcal{H}_{231}$$

$$h^2_{213} \equiv \lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}}(d_1,d_2,d_3) : \mathcal{K}^2_{231,213} \to \mathcal{H}_{213}$$

$$h^2_{312} \equiv \lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}}(d_1,d_2,d_3) : \mathcal{K}^2_{312,132} \to \mathcal{H}_{312}$$

$$h^2_{132} \equiv \lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}}(d_1,d_2,d_3) : \mathcal{K}^2_{312,132} \to \mathcal{H}_{132}$$

$$h^3_{312} \equiv \lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1,d_2,d_3) : \mathcal{K}^3_{312,321} \to \mathcal{H}_{312}$$

$$h^3_{321} \equiv \lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1,d_2,d_3) : \mathcal{K}^3_{312,321} \to \mathcal{H}_{321}$$

$$h^3_{123} \equiv \lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1,d_2,d_3) : \mathcal{K}^3_{123,213} \to \mathcal{H}_{123}$$

$$h^3_{213} \equiv \lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1,d_2,d_3) : \mathcal{K}^3_{123,213} \to \mathcal{H}_{213}$$

is a quasi-colimit diagram.

Proof. By Axiom 6 we are sure that, given

$$\gamma^{123}, \gamma^{132}, \gamma^{213}, \gamma^{231}, \gamma^{312}, \gamma^{321} : D^3 \to \mathbb{R}$$

there exist

$$a^{123}, a_1^{123}, a_2^{123}, a_3^{123}, a_{12}^{123}, a_{13}^{123}, a_{23}^{123}, a_{123}^{123} : \mathbb{R}$$
$$a^{132}, a_1^{132}, a_2^{132}, a_3^{132}, a_{12}^{132}, a_{13}^{132}, a_{23}^{132}, a_{123}^{132} : \mathbb{R}$$
$$a^{213}, a_1^{213}, a_2^{213}, a_3^{213}, a_{12}^{213}, a_{13}^{213}, a_{23}^{213}, a_{123}^{213} : \mathbb{R}$$
$$a^{231}, a_1^{231}, a_2^{231}, a_3^{231}, a_{12}^{231}, a_{13}^{231}, a_{23}^{231}, a_{123}^{231} : \mathbb{R}$$
$$a^{312}, a_1^{312}, a_2^{312}, a_3^{312}, a_{12}^{312}, a_{13}^{312}, a_{23}^{312}, a_{123}^{312} : \mathbb{R}$$
$$a^{321}, a_1^{321}, a_2^{321}, a_3^{321}, a_{12}^{321}, a_{13}^{321}, a_{23}^{321}, a_{123}^{321} : \mathbb{R}$$

such that

$$\lambda_{(d_1,d_2,d_3):D^3} \gamma^{123}(d_1, d_2, d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3} \left(\begin{array}{c} a^{123} + a_1^{123} d_1 + a_2^{123} d_2 + a_3^{123} d_3 + \\ a_{12}^{123} d_1 d_2 + a_{13}^{123} d_1 d_3 + a_{23}^{123} d_2 d_3 + a_{123}^{123} d_1 d_2 d_3 \end{array} \right)$$

$$\lambda_{(d_1,d_2,d_3):D^3} \gamma^{132}(d_1, d_2, d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3} \left(\begin{array}{c} a^{132} + a_1^{132} d_1 + a_2^{132} d_2 + a_3^{132} d_3 + \\ a_{12}^{132} d_1 d_2 + a_{13}^{132} d_1 d_3 + a_{23}^{132} d_2 d_3 + a_{123}^{132} d_1 d_2 d_3 \end{array} \right)$$

$$\lambda_{(d_1,d_2,d_3):D^3} \gamma^{213}(d_1, d_2, d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3} \left(\begin{array}{c} a^{213} + a_1^{213} d_1 + a_2^{213} d_2 + a_3^{213} d_3 + \\ a_{12}^{213} d_1 d_2 + a_{13}^{213} d_1 d_3 + a_{23}^{213} d_2 d_3 + a_{123}^{213} d_1 d_2 d_3 \end{array} \right)$$

$$\lambda_{(d_1,d_2,d_3):D^3} \gamma^{231}(d_1, d_2, d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3} \left(\begin{array}{c} a^{231} + a_1^{231} d_1 + a_2^{231} d_2 + a_3^{231} d_3 + \\ a_{12}^{231} d_1 d_2 + a_{13}^{231} d_1 d_3 + a_{23}^{231} d_2 d_3 + a_{123}^{231} d_1 d_2 d_3 \end{array} \right)$$

$$\lambda_{(d_1,d_2,d_3):D^3} \gamma^{312}(d_1, d_2, d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3} \left(\begin{array}{c} a^{312} + a_1^{312} d_1 + a_2^{312} d_2 + a_3^{312} d_3 + \\ a_{12}^{312} d_1 d_2 + a_{13}^{312} d_1 d_3 + a_{23}^{312} d_2 d_3 + a_{123}^{312} d_1 d_2 d_3 \end{array} \right)$$

$$\lambda_{(d_1,d_2,d_3):D^3} \gamma^{321}(d_1, d_2, d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3} \left(\begin{array}{c} a^{321} + a_1^{321} d_1 + a_2^{321} d_2 + a_3^{321} d_3 + \\ a_{12}^{321} d_1 d_2 + a_{13}^{321} d_1 d_3 + a_{23}^{321} d_2 d_3 + a_{123}^{321} d_1 d_2 d_3 \end{array} \right)$$

If it holds that

$$\gamma^{123} \circ h_{123}^1 = \gamma^{132} \circ h_{132}^1$$
$$\gamma^{231} \circ h_{231}^1 = \gamma^{321} \circ h_{321}^1$$
$$\gamma^{231} \circ h_{231}^2 = \gamma^{213} \circ h_{213}^2$$
$$\gamma^{312} \circ h_{312}^2 = \gamma^{132} \circ h_{132}^2$$
$$\gamma^{312} \circ h_{312}^3 = \gamma^{321} \circ h_{321}^3$$
$$\gamma^{123} \circ h_{123}^3 = \gamma^{213} \circ h_{213}^3$$

then we have

$$a^{123} = a^{132}, a^{123}_1 = a^{132}_1, a^{123}_2 = a^{132}_2, a^{123}_3 = a^{132}_3, a^{123}_{12} = a^{132}_{12}, a^{123}_{13} = a^{132}_{13}$$
$$a^{231} = a^{321}, a^{231}_1 = a^{321}_1, a^{231}_2 = a^{321}_2, a^{231}_3 = a^{321}_3, a^{231}_{12} = a^{321}_{12}, a^{231}_{13} = a^{321}_{13}$$
$$a^{231} = a^{213}, a^{231}_1 = a^{213}_1, a^{231}_2 = a^{213}_2, a^{231}_3 = a^{213}_3, a^{231}_{12} = a^{213}_{12}, a^{231}_{23} = a^{213}_{23}$$
$$a^{312} = a^{132}, a^{312}_1 = a^{132}_1, a^{312}_2 = a^{132}_2, a^{312}_3 = a^{132}_3, a^{312}_{12} = a^{132}_{12}, a^{312}_{23} = a^{132}_{23}$$
$$a^{312} = a^{321}, a^{312}_1 = a^{321}_1, a^{312}_2 = a^{321}_2, a^{312}_3 = a^{321}_3, a^{312}_{13} = a^{321}_{13}, a^{312}_{23} = a^{321}_{23}$$
$$a^{123} = a^{213}, a^{123}_1 = a^{213}_1, a^{123}_2 = a^{213}_2, a^{123}_3 = a^{213}_3, a^{123}_{13} = a^{213}_{13}, a^{123}_{23} = a^{213}_{23}$$

$$a^{123} = a^{231}, a^{123}_1 = a^{231}_1, a^{123}_2 = a^{231}_2, a^{123}_3 = a^{231}_3, a^{123}_{23} = a^{231}_{23}$$
$$a^{132} = a^{321}, a^{132}_1 = a^{321}_1, a^{132}_2 = a^{321}_2, a^{132}_3 = a^{321}_3, a^{132}_{23} = a^{321}_{23}$$
$$a^{231} = a^{312}, a^{231}_1 = a^{312}_1, a^{231}_2 = a^{312}_2, a^{231}_3 = a^{312}_3, a^{231}_{13} = a^{312}_{13}$$
$$a^{213} = a^{132}, a^{213}_1 = a^{132}_1, a^{213}_2 = a^{132}_2, a^{213}_3 = a^{132}_3, a^{213}_{13} = a^{132}_{13}$$
$$a^{312} = a^{123}, a^{312}_1 = a^{123}_1, a^{312}_2 = a^{123}_2, a^{312}_3 = a^{123}_3, a^{312}_{12} = a^{123}_{12}$$
$$a^{321} = a^{213}, a^{321}_1 = a^{213}_1, a^{321}_2 = a^{213}_2, a^{321}_3 = a^{213}_3, a^{321}_{12} = a^{213}_{12}$$

which is tantamout to

$$a^{123} = a^{132} = a^{213} = a^{231} = a^{312} = a^{321}$$
$$a^{123}_1 = a^{132}_1 = a^{213}_1 = a^{231}_1 = a^{312}_1 = a^{321}_1$$
$$a^{123}_2 = a^{132}_2 = a^{213}_2 = a^{231}_2 = a^{312}_2 = a^{321}_2$$
$$a^{123}_3 = a^{132}_3 = a^{213}_3 = a^{231}_3 = a^{312}_3 = a^{321}_3$$
$$a^{123}_{12} = a^{132}_{12} = a^{312}_{12} \text{ and } a^{213}_{12} = a^{231}_{12} = a^{321}_{12}$$
$$a^{123}_{13} = a^{132}_{13} = a^{213}_{13} \text{ and } a^{231}_{13} = a^{312}_{13} = a^{321}_{13}$$
$$a^{123}_{23} = a^{213}_{23} = a^{231}_{23} \text{ and } a^{132}_{23} = a^{312}_{23} = a^{321}_{23}$$

This means that there exist

$$b, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_{12}, b_{13}, b_{23}, b_{123}, b_{14}, b_{25}, b_{36} : \mathbb{R}$$

such that

$$\left(\lambda_{(d_1,d_2,d_3,d_4,d_5,d_6,d_7,d_8):G}\left(\begin{array}{c} b + b_1d_1 + b_2d_2 + b_3d_3 + b_4d_4 + b_5d_5+ \\ b_6d_6 + b_7d_7 + b_8d_8+ \\ b_{12}d_1d_2 + b_{13}d_1d_3 + b_{23}d_2d_3 + b_{123}d_1d_2d_3+ \\ b_{14}d_1d_4 + b_{25}d_2d_5 + b_{36}d_3d_6 \end{array}\right)\right) \circ f_{123}$$
$$= \gamma^{123}$$

$$\left(\lambda_{(d_1,d_2,d_3,d_4,d_5,d_6,d_7,d_8):G}\left(\begin{array}{c} b + b_1d_1 + b_2d_2 + b_3d_3 + b_4d_4 + b_5d_5+ \\ b_6d_6 + b_7d_7 + b_8d_8+ \\ b_{12}d_1d_2 + b_{13}d_1d_3 + b_{23}d_2d_3 + b_{123}d_1d_2d_3+ \\ b_{14}d_1d_4 + b_{25}d_2d_5 + b_{36}d_3d_6 \end{array}\right)\right) \circ f_{132}$$
$$= \gamma^{132}$$

$$\left(\lambda_{(d_1,d_2,d_3,d_4,d_5,d_6,d_7,d_8):G}\left(\begin{array}{c} b + b_1d_1 + b_2d_2 + b_3d_3 + b_4d_4 + b_5d_5+ \\ b_6d_6 + b_7d_7 + b_8d_8+ \\ b_{12}d_1d_2 + b_{13}d_1d_3 + b_{23}d_2d_3 + b_{123}d_1d_2d_3+ \\ b_{14}d_1d_4 + b_{25}d_2d_5 + b_{36}d_3d_6 \end{array}\right)\right) \circ f_{213}$$
$$= \gamma^{213}$$

$$\left(\lambda_{(d_1,d_2,d_3,d_4,d_5,d_6,d_7,d_8):G}\left(\begin{array}{c} b + b_1d_1 + b_2d_2 + b_3d_3 + b_4d_4 + b_5d_5+ \\ b_6d_6 + b_7d_7 + b_8d_8+ \\ b_{12}d_1d_2 + b_{13}d_1d_3 + b_{23}d_2d_3 + b_{123}d_1d_2d_3+ \\ b_{14}d_1d_4 + b_{25}d_2d_5 + b_{36}d_3d_6 \end{array}\right)\right) \circ f_{231}$$
$$= \gamma^{231}$$

$$\left(\lambda_{(d_1,d_2,d_3,d_4,d_5,d_6,d_7,d_8):G}\left(\begin{array}{c} b + b_1d_1 + b_2d_2 + b_3d_3 + b_4d_4 + b_5d_5+ \\ b_6d_6 + b_7d_7 + b_8d_8+ \\ b_{12}d_1d_2 + b_{13}d_1d_3 + b_{23}d_2d_3 + b_{123}d_1d_2d_3+ \\ b_{14}d_1d_4 + b_{25}d_2d_5 + b_{36}d_3d_6 \end{array}\right)\right) \circ f_{312}$$
$$= \gamma^{312}$$

$$\left(\lambda_{(d_1,d_2,d_3,d_4,d_5,d_6,d_7,d_8):G}\left(\begin{array}{c} b + b_1d_1 + b_2d_2 + b_3d_3 + b_4d_4 + b_5d_5+ \\ b_6d_6 + b_7d_7 + b_8d_8+ \\ b_{12}d_1d_2 + b_{13}d_1d_3 + b_{23}d_2d_3 + b_{123}d_1d_2d_3+ \\ b_{14}d_1d_4 + b_{25}d_2d_5 + b_{36}d_3d_6 \end{array}\right)\right) \circ f_{321}$$
$$= \gamma^{321}$$

This completes the proof. ∎

Corollary 56 *Let M be a microlinear set. Given*

$$\theta_{123}, \theta_{132}, \theta_{213}, \theta_{231}, \theta_{312}, \theta_{321} : D^3 \to M$$

with

$$\theta_{123} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right) = \theta_{132} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right)$$

$$\theta_{231} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right) = \theta_{321} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right)$$

$$\theta_{231} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1, d_2, d_3)\right) = \theta_{213} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1, d_2, d_3)\right)$$

$$\theta_{312} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1, d_2, d_3)\right) = \theta_{132} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1, d_2, d_3)\right)$$

$$\theta_{312} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}} (d_1, d_2, d_3)\right) = \theta_{321} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}} (d_1, d_2, d_3)\right)$$

$$\theta_{123} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}} (d_1, d_2, d_3)\right) = \theta_{213} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}} (d_1, d_2, d_3)\right)$$

there exists

$$\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})} : \mathcal{G} \to M$$

such that

$$\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})} \circ f_{123} = \theta_{123}$$

$$\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})} \circ f_{132} = \theta_{132}$$

$$\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})} \circ f_{213} = \theta_{213}$$

$$\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})} \circ f_{231} = \theta_{231}$$

$$\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})} \circ f_{312} = \theta_{312}$$

$$\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})} \circ f_{312} = \theta_{312}$$

Theorem 57 *(The general Jacobi identity) Let M be a microlinear set and $\theta_{123}, \theta_{132}, \theta_{213}, \theta_{231}, \theta_{312}, \theta_{321} : D^3 \to M$. If the identities*

$$\theta_{123} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right) = \theta_{132} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right)$$

$$\theta_{231} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right) = \theta_{321} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3)\right)$$

$$\theta_{231} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1, d_2, d_3)\right) = \theta_{213} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1, d_2, d_3)\right)$$

$$\theta_{312} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1, d_2, d_3)\right) = \theta_{132} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1, d_2, d_3)\right)$$

$$\theta_{312} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}} (d_1, d_2, d_3)\right) = \theta_{321} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}} (d_1, d_2, d_3)\right)$$

$$\theta_{123} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}} (d_1, d_2, d_3)\right) = \theta_{213} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}} (d_1, d_2, d_3)\right)$$

obtain so that the six relative strong differences

$$\theta_{123} \overset{\cdot}{\underset{1}{-}} \theta_{132}$$

$$\theta_{231} \overset{\cdot}{\underset{1}{-}} \theta_{321}$$

$$\theta_{231} \overset{\cdot}{\underset{2}{-}} \theta_{213}$$

$$\theta_{312} \overset{\cdot}{\underset{2}{-}} \theta_{132}$$

$$\theta_{312} \overset{\cdot}{\underset{3}{-}} \theta_{321}$$

$$\theta_{123} \overset{\cdot}{\underset{3}{-}} \theta_{213}$$

are to be defined, then the identities

$$\theta_{123} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1, d_2, d_3) \right) = \theta_{231} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1, d_2, d_3) \right) \tag{23}$$

$$\theta_{132} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1, d_2, d_3) \right) = \theta_{321} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1, d_2, d_3) \right) \tag{24}$$

$$\theta_{231} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(2,3)\}} (d_1, d_2, d_3) \right) = \theta_{312} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(2,3)\}} (d_1, d_2, d_3) \right) \tag{25}$$

$$\theta_{213} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(2,3)\}} (d_1, d_2, d_3) \right) = \theta_{132} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(2,3)\}} (d_1, d_2, d_3) \right) \tag{26}$$

$$\theta_{312} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}} (d_1, d_2, d_3) \right) = \theta_{123} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}} (d_1, d_2, d_3) \right) \tag{27}$$

$$\theta_{321} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}} (d_1, d_2, d_3) \right) = \theta_{213} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3),(2,3)\}} (d_1, d_2, d_3) \right) \tag{28}$$

obtain so that the three strong differences

$$\left(\theta_{123} \underset{1}{\overset{\cdot}{-}} \theta_{132} \right) \overset{\cdot}{-} \left(\theta_{231} \underset{1}{\overset{\cdot}{-}} \theta_{321} \right)$$

$$\left(\theta_{231} \underset{2}{\overset{\cdot}{-}} \theta_{213} \right) \overset{\cdot}{-} \left(\theta_{312} \underset{2}{\overset{\cdot}{-}} \theta_{132} \right)$$

$$\left(\theta_{312} \underset{3}{\overset{\cdot}{-}} \theta_{321} \right) \overset{\cdot}{-} \left(\theta_{123} \underset{3}{\overset{\cdot}{-}} \theta_{213} \right)$$

are to be defined (by dint of Proposition 54) and to sum up only to vanish, namely,

$$\left(\left(\theta_{123} \underset{1}{\overset{\cdot}{-}} \theta_{132} \right) \overset{\cdot}{-} \left(\theta_{231} \underset{1}{\overset{\cdot}{-}} \theta_{321} \right) \right) + \left(\left(\theta_{231} \underset{2}{\overset{\cdot}{-}} \theta_{213} \right) \overset{\cdot}{-} \left(\theta_{312} \underset{2}{\overset{\cdot}{-}} \theta_{132} \right) \right) +$$

$$\left(\left(\theta_{312} \underset{3}{\overset{\cdot}{-}} \theta_{321} \right) \overset{\cdot}{-} \left(\theta_{123} \underset{3}{\overset{\cdot}{-}} \theta_{213} \right) \right)$$

$$= 0 \tag{29}$$

Proof. The proof is divided into the proof of (23)-(28) and that of (29).

1. Since the identity

   $$\theta_{231} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1, d_2, d_3) \right) = \theta_{213} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}} (d_1, d_2, d_3) \right)$$

   obtains by assumption, we have

   $$\theta_{231} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1, d_2, d_3) \right) = \theta_{213} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1, d_2, d_3) \right)$$

   Since the identity

   $$\theta_{123} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}} (d_1, d_2, d_3) \right) = \theta_{213} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}} (d_1, d_2, d_3) \right)$$

obtains by assumption, we have

$$\theta_{123}\circ\big(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}}\,(d_1,d_2,d_3)\big) = \theta_{213}\circ\big(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}}\,(d_1,d_2,d_3)\big)$$

Therefore we have

$$\theta_{123}\circ\big(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}}\,(d_1,d_2,d_3)\big) = \theta_{231}\circ\big(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}}\,(d_1,d_2,d_3)\big)$$

which is no other than (23). The remaining five identities (24)-(28) can be dealt with by the same token.

2. Since the identities

$$\lambda_{(d_1,d_2,d_3):D^3}\theta_{123}\,(d_1,d_2,d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}\,(d_1,d_2,d_3,0,0,0,0,0)$$
$$\lambda_{(d_1,d_2,d_3):D^3}\theta_{132}\,(d_1,d_2,d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}\,(d_1,d_2,d_3,d_2d_3,0,0,0,0)$$
$$\lambda_{(d_1,d_2,d_3):D^3}\theta_{213}\,(d_1,d_2,d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}\,(d_1,d_2,d_3,0,0,d_1d_2,0,0)$$
$$\lambda_{(d_1,d_2,d_3):D^3}\theta_{231}\,(d_1,d_2,d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}\,(d_1,d_2,d_3,0,d_1d_3,d_1d_2,0,0)$$
$$\lambda_{(d_1,d_2,d_3):D^3}\theta_{312}\,(d_1,d_2,d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}\,(d_1,d_2,d_3,d_2d_3,d_1d_3,0,d_1d_2d_3,0)$$
$$\lambda_{(d_1,d_2,d_3):D^3}\theta_{321}\,(d_1,d_2,d_3)$$
$$= \lambda_{(d_1,d_2,d_3):D^3}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}\,(d_1,d_2,d_3,d_2d_3,d_1d_3,d_1d_2,0,d_1d_2d_3)$$

obtain so that the identities

$$\lambda_{(d_1,d_2):D^2}\left(\theta_{123}\underset{1}{\overset{\cdot}{-}}\theta_{132}\right)(d_1,d_2)$$
$$=\lambda_{(d_1,d_2):D^2}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}(d_1,0,0,-d_2,0,0,0,0)$$

$$\lambda_{(d_1,d_2):D^2}\left(\theta_{231}\underset{1}{\overset{\cdot}{-}}\theta_{321}\right)(d_1,d_2)$$
$$=\lambda_{(d_1,d_2):D^2}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}(d_1,0,0,-d_2,0,0,0,-d_1d_2)$$

$$\lambda_{(d_1,d_2):D^2}\left(\theta_{231}\underset{2}{\overset{\cdot}{-}}\theta_{213}\right)(d_1,d_2)$$
$$=\lambda_{(d_1,d_2):D^2}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}(0.d_1,0,d_2,0,0,0,0)$$

$$\lambda_{(d_1,d_2):D^2}\left(\theta_{312}\underset{2}{\overset{\cdot}{-}}\theta_{132}\right)(d_1,d_2)$$
$$=\lambda_{(d_1,d_2):D^2}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}(0,d_1,0,0,d_2,0,d_1d_2,0)$$

$$\lambda_{(d_1,d_2):D^2}\left(\theta_{312}\underset{3}{\overset{\cdot}{-}}\theta_{321}\right)(d_1,d_2)$$
$$=\lambda_{(d_1,d_2):D^2}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}(0,0,d_1,0,0,-d_2,d_1d_2,-d_1d_2)$$

$$\lambda_{(d_1,d_2):D^2}\left(\theta_{123}\underset{3}{\overset{\cdot}{-}}\theta_{213}\right)(d_1,d_2)$$
$$=\lambda_{(d_1,d_2):D^2}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}(0,0,d_1,0,0,-d_2,0,0)$$

take place, we have

$$\lambda_{d:D}\left(\left(\theta_{123}\underset{1}{\overset{\cdot}{-}}\theta_{132}\right)\overset{\cdot}{-}\left(\theta_{231}\underset{1}{\overset{\cdot}{-}}\theta_{321}\right)\right)(d)$$
$$=\lambda_{d:D}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}(0,0,0,0,0,0,0,d)$$

$$\lambda_{d:D}\left(\left(\theta_{231}\underset{2}{\overset{\cdot}{-}}\theta_{213}\right)\overset{\cdot}{-}\left(\theta_{312}\underset{2}{\overset{\cdot}{-}}\theta_{132}\right)\right)(d)$$
$$=\lambda_{d:D}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}(0,0,0,0,0,0,-d,0)$$

$$\lambda_{d:D}\left(\left(\theta_{312}\underset{3}{\overset{\cdot}{-}}\theta_{321}\right)\overset{\cdot}{-}\left(\theta_{123}\underset{3}{\overset{\cdot}{-}}\theta_{213}\right)\right)(d)$$
$$=\lambda_{d:D}\mathfrak{k}_{(\theta_{123},\theta_{132},\theta_{213},\theta_{231},\theta_{312},\theta_{321})}(0,0,0,0,0,0,d,-d)$$

This completes the proof.

∎

7 Vector Fields

The trinity of the three notions of vector fields in synthetic differential geometry, namely the identification of sections of tangent vector bundles, infinitesimal

flows and infinitesimal transformations discussed in §3.2.1 of [7], remains valid in the following sense.

Theorem 58 *Let M be a microlinear type. The following three types are mutually equivalent.*

- *the type of sections of the fibration $\lambda_{x:M} \mathbf{T}_x M : M \to \mathcal{U}$, namely,*

$$\prod_{x:M} \mathbf{T}_x M$$

- *the type of infinitesimal flows on $\|M\|_0$, namely, mappings $f : D \times \|M\|_0 \to \|M\|_0$ in accordance with*

$$\prod_{x:\|M\|_0} f(0, x) = x$$

- *the type of infinitesimal transformations of $\|M\|_0$, namely, mappings $X : D \to \|M\|_0 \to \|M\|_0$ in accordance with*

$$X_0 = \mathrm{id}_{\|M\|_0}$$

where we prefer to write X_d in place of $X(d)$ as in [7].

Proof. A section of the dependent type family $\lambda_{x:M} \mathbf{T}_x M$ can be identified with a mapping

$$\widetilde{f} : M \to D \to \|M\|_0$$

in accordance with

$$\prod_{x:M} \widetilde{f}(x, 0) = |x|_0$$

which can naturally be identified with a mapping

$$\widehat{f} : D \to M \to \|M\|_0$$

in accordance with

$$\prod_{x:M} \widehat{f}(0, x) = |x|_0$$

Since

$$M \to \|M\|_0 \simeq \|M\|_0 \to \|M\|_0$$

obtains naturally, $\widehat{f}$ can be identified with a mapping

$$f : D \to \|M\|_0 \to \|M\|_0$$

in accordance with

$$\prod_{x:\|M\|_0} f(0, x) = x$$

This has established the equivalence between the first and the second. The equivalence between the second and the third can be established more directly, which is safely left to the reader. ■

Notation 59 *Given a microlinear type M, the notaion $\mathfrak{X}(M)$ denotes one of the equivalent three types in the above theorem, but it usually means the third one in the rest of this paper unless specified otherwise.*

Proposition 60 *(cf. Proposition 3 of §3.2 in [7]) Let M be a microlinear type. Let $X : \mathfrak{X}(M)$. Then we have*

$$\prod_{(d_1,d_2):D(2)} X_{d_1} \circ X_{d_2} = X_{d_1+d_2}$$

Proof. It is easy to see that

$$\left(\lambda_{(d_1,d_2):D(2)} X_{d_1} \circ X_{d_2}\right) \circ \left(\lambda_{d:D}\,(d,0)\right) = \left(\lambda_{(d_1,d_2):D(2)} X_{d_1+d_2}\right) \circ \left(\lambda_{d:D}\,(d,0)\right)$$
$$\left(\lambda_{(d_1,d_2):D(2)} X_{d_1} \circ X_{d_2}\right) \circ \left(\lambda_{d:D}\,(0,d)\right) = \left(\lambda_{(d_1,d_2):D(2)} X_{d_1+d_2}\right) \circ \left(\lambda_{d:D}\,(0,d)\right)$$

so that the desired result follows by dint of Corollary 31. ∎

Proposition 61 *(cf. Proposition 6 of §3.2 in [7]) Let M be a microlinear type. Let $X, Y : \mathfrak{X}(M)$. Then we have*

$$\prod_{d:D} X_d \circ Y_d = (X+Y)_d$$
$$\prod_{d:D} Y_d \circ X_d = (X+Y)_d$$

Proof. It is easy to see that

$$\left(\lambda_{(d_1,d_2):D(2)} X_{d_1} \circ Y_{d_2}\right) \circ \left(\lambda_{d:D}\,(d,0)\right) = \lambda_{d:D} X_d$$
$$\left(\lambda_{(d_1,d_2):D(2)} X_{d_1} \circ Y_{d_2}\right) \circ \left(\lambda_{d:D}\,(0,d)\right) = \lambda_{d:D} Y_d$$

so that the first desired result follows by dint of Corollary 31.

$$\left(\lambda_{(d_1,d_2):D(2)} Y_{d_2} \circ X_{d_1}\right) \circ \left(\lambda_{d:D}\,(d,0)\right) = \lambda_{d:D} X_d$$
$$\left(\lambda_{(d_1,d_2):D(2)} Y_{d_2} \circ X_{d_1}\right) \circ \left(\lambda_{d:D}\,(0,d)\right) = \lambda_{d:D} Y_d$$

so that the second desired result follows by dint of Corollary 31. ∎

Definition 62 *Let M be a microlinear set. Given*

$$\theta_1 : D^{n_1} \to M \to M$$
$$\theta_2 : D^{n_2} \to M \to M$$

we define

$$\theta_1 * \theta_2 : D^{n_1+n_2} \to M \to M$$

to be

$$\theta_1 * \theta_2 :\equiv \lambda_{(d_1,...,d_{n_1+n_2}):D^{n_1+n_2}} \theta_1\,(d_{n_2+1},...,d_{n_1+n_2}) \circ \theta_2\,(d_1,...,d_{n_2})$$

It is easy to see that

Lemma 63 *Let M be a microlinear type. Given*

$$\theta_1 : D^{n_1} \to M \to M$$
$$\theta_2 : D^{n_2} \to M \to M$$
$$\theta_3 : D^{n_3} \to M \to M$$

we have

$$(\theta_1 * \theta_2) * \theta_3 = \theta_1 * (\theta_2 * \theta_3)$$

Remark 64 *Therefore, when various $\theta_i : D^{n_i} \to M \to M$ $(1 \leq i \leq m)$ are concatenated by $*$, we can omit parentheses, so that we can write*

$$\theta_1 * \ldots * \theta_m : D^{n_1 + \ldots + n_m} \to M \to M$$

Definition 65 *Let M be a microlinear type. Given $X, Y : \mathfrak{X}(M)$, we have*

$$(Y * X) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1, d_2) \right)$$
$$= (X * Y) \circ \left(\lambda_{(d_1,d_2):D^2} (d_2, d_1) \right) \circ \left(\lambda_{(d_1,d_2):D^2\{(1,2)\}} (d_1, d_2) \right)$$

so that we can define

$$[X, Y] :\equiv (Y * X) \overset{\cdot}{-} (X * Y) \circ \left(\lambda_{(d_1,d_2):D^2} (d_2, d_1) \right)$$

Lemma 66 *Let M be a microlinear type. Given $X_1, X_2, X_3 : \mathfrak{X}(M)$, we define*

$$\theta_{123} :\equiv X_3 * X_2 * X_1$$
$$\theta_{132} :\equiv (X_2 * X_3 * X_1) \circ \left(\lambda_{(d_1,d_2,d_3):D^3} (d_1, d_3, d_2) \right)$$
$$\theta_{231} :\equiv (X_1 * X_3 * X_2) \circ \left(\lambda_{(d_1,d_2,d_3):D^3} (d_2, d_3, d_1) \right)$$
$$\theta_{321} :\equiv (X_1 * X_2 * X_3) \circ \left(\lambda_{(d_1,d_2,d_3):D^3} (d_3, d_2, d_1) \right)$$

Then we have

$$\theta_{123} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3) \right) = \theta_{132} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3) \right)$$
$$\theta_{231} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3) \right) = \theta_{321} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}} (d_1, d_2, d_3) \right)$$
$$\theta_{123} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1, d_2, d_3) \right) = \theta_{231} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1, d_2, d_3) \right)$$
$$\theta_{132} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1, d_2, d_3) \right) = \theta_{321} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2),(1,3)\}} (d_1, d_2, d_3) \right)$$
$$[X_2, X_3] * X_1 = \theta_{123} \overset{\cdot}{\underset{1}{-}} \theta_{132}$$
$$(X_1 * [X_2, X_3]) \circ \left(\lambda_{(d_1,d_2):D^2} (d_2, d_1) \right) = \theta_{231} \overset{\cdot}{\underset{1}{-}} \theta_{321}$$
$$[X_1, [X_2, X_3]] = \left(\theta_{123} \overset{\cdot}{\underset{1}{-}} \theta_{132} \right) \overset{\cdot}{-} \left(\theta_{231} \overset{\cdot}{\underset{1}{-}} \theta_{321} \right)$$

Lemma 67 *Let M be a microlinear type. Given $X_1, X_2, X_3 : \mathfrak{X}(M)$, we define*

$$\theta_{123} :\equiv X_3 * X_2 * X_1$$
$$\theta_{132} :\equiv (X_2 * X_3 * X_1) \circ \left(\lambda_{(d_1,d_2,d_3):D^3}(d_1, d_3, d_2)\right)$$
$$\theta_{213} :\equiv (X_3 * X_1 * X_2) \circ \left(\lambda_{(d_1,d_2,d_3):D^3}(d_2, d_1, d_3)\right)$$
$$\theta_{231} :\equiv (X_1 * X_3 * X_2) \circ \left(\lambda_{(d_1,d_2,d_3):D^3}(d_2, d_3, d_1)\right)$$
$$\theta_{312} :\equiv (X_2 * X_1 * X_3) \circ \left(\lambda_{(d_1,d_2,d_3):D^3}(d_3, d_1, d_2)\right)$$
$$\theta_{321} :\equiv (X_1 * X_2 * X_3) \circ \left(\lambda_{(d_1,d_2,d_3):D^3}(d_3, d_2, d_1)\right)$$

Then we have

$$\theta_{123} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}}(d_1, d_2, d_3)\right) = \theta_{132} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}}(d_1, d_2, d_3)\right)$$
$$\theta_{231} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}}(d_1, d_2, d_3)\right) = \theta_{321} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(2,3)\}}(d_1, d_2, d_3)\right)$$
$$\theta_{231} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}}(d_1, d_2, d_3)\right) = \theta_{213} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}}(d_1, d_2, d_3)\right)$$
$$\theta_{312} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}}(d_1, d_2, d_3)\right) = \theta_{132} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,3)\}}(d_1, d_2, d_3)\right)$$
$$\theta_{312} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1, d_2, d_3)\right) = \theta_{321} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1, d_2, d_3)\right)$$
$$\theta_{123} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1, d_2, d_3)\right) = \theta_{213} \circ \left(\lambda_{(d_1,d_2,d_3):D^3\{(1,2)\}}(d_1, d_2, d_3)\right)$$

Theorem 68 *Let M be a microlinear type. Given $\alpha : \mathbb{R}$ and $X_1, X_2, X_3 : \mathfrak{X}(M)$, we have*

$$[X_1 + X_2, X_3] = [X_1, X_3] + [X_2, X_3] \tag{30}$$
$$[\alpha X_1, X_2] = \alpha[X_1, X_2] \tag{31}$$
$$[X_1, X_2] + [X_2, X_1] = 0 \tag{32}$$
$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0 \tag{33}$$

In a word, the Lie bracket $[\cdot, \cdot]$ is bilinear, antisymmetric, and satisfies the Jacobi identity.

Proof. The property (30) follows from Proposition 46. The property (**??**) follows from Proposition 47, The property (32) follows from Proposition 50. The property (33) follows from Theorem 57. ∎

References

[1] Awodey, Steve and Warren, Michael A., Homotopy theoretic models of identity types, Mathematical Proceedings of the Cambridge Philosophical Society **146** (2009), 45-55.

[2] Hofmann and Streicher, The groupoid interpretation of type theory, in Twenty-Five Years of Constructive Type Theory (Venice 1995), pp.83-111, Oxford University Press, New York, 1998.

[3] Kock, Anders, A simple axiomatics for differentiation, Mathematica Scandinavica **40** (1977), 183-193.

[4] Kock, Anders, Taylor series calculus for ring objects of line type, Journal of Pure and Applied Algebra **12** (1978), 271-293.

[5] Kock, Anders, Synthetic Differential Geometry (2nd edition), London Mathematical Society Lecture Note Series **333**, Cambridge University Press, 2006.

[6] Kock, Anders and Lavendhomme, René, Strong infinitesimal linearity, with applications to strong differences and affine connections, Cahiers de Topologie et Géométrie Différentielle Catégoriques **25** (1984), 311-324.

[7] Lavendhomme, René, Basic Concepts of Synthetic Differential Geometry, Kluwer Texts in the Mathematical Sciences **13**, Kluwer Academic Publishers, 1996.

[8] Nishimura, Hirokazu, Theory of microcubes, International Journal of Theoretical Physics **36** (1997), 1099-1131.

[9] Nishimura, Hirokazu, General Jacobi identity revisited, International Journal of Theoretical Physics **38** (1999), 2163-2174.

[10] Voevodsky, Vladimir, A very short note on the homotopy λ-calculus, unpublished note, 2006.

[11] The Univalent Foundations Program, Homotopy Type Theory-Univalent Foundations of Mathematics, Institute for Advanced Study, 2013.