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TANGENT BUNDLES, MONOIDAL THEORIES AND WEIL ALGEBRAS

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In any discussion of differential geometry (for instance, see [4]), a key role is played by the *tangent space* TM to a given space (say, smooth manifold) M; this is equipped with a projection map, $p_M \colon TM \to M$, and various other structures.

These structures have also been studied from a category theoretic point of view, in which T becomes a functor in its own right, and the projection p and the other structure become natural transformations. These notions have been given the name *tangent structure*. The precise structure of the tangent bundles then varies depending on the choices made in relation to the 'other structure'.

Rosický [6], for instance, detailed a set of choices which resulted in the tangent bundles having the structure of abelian groups, motivated by examples from computer science. Cockett and Cruttwell [3] later gave a set of choices resulting in the tangent bundles having the structure of commutative monoids.

In this thesis, we present a different perspective. We first introduce the Weil algebras, which have a well-established role in the discussion of differential geometry (for instance, see [5] or [7]). However, we follow up by restricting to a subcategory we shall call $\mathbf{Weil_1}$, and describe a way to characterise the Weil algebras of this (sub)category using (co)graphs (see [2] for more details). We develop this idea even further by showing how the maps between such Weil algebras can also be described using the language of graphs.

This part of the thesis concludes with a demonstration that to give a tangent structure to some category $\mathcal{M}$ is equivalent to giving a strong monoidal functor

$$F \colon \mathbf{Weil_1} \to \mathrm{End}(\mathcal{M})$$

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preserving certain limits; or more broadly, a strong monoidal functor

$$F\colon \mathbf{Weil_1} \to \mathcal{G}$$

for some monoidal category $\mathcal{G}$. This also suggests that we may think of $\mathbf{Weil_1}$ as a sort of monoidal theory (see [1]) for tangent structures. Further, the choice of structure for the tangent bundles (monoid/group/vector space) is determined by the choice of the set of coefficients for the Weil algebras.

In the penultimate chapter of the thesis, we discuss a method for extending the definition of tangent structure to include such things as higher-order derivatives and more complicated relationships between these derivatives. Namely, we propose a candidate category $\mathbf{Weil}_\infty$ intended to capture all finite-order derivatives. We then detail a methodical approach to characterise each Weil algebra in this category as an appropriate limit.

We conclude this chapter by suggesting that an 'extended' tangent structure could be given as a strong monoidal functor

$$F\colon \mathbf{Weil}_\infty \to \mathcal{G}$$

preserving the limits described.

In the final chapter, we give a brief discussion of what is needed to have an appropriate notion of 'addition' of tangent vectors, and then give some examples of possible subcategories other than $\mathbf{Weil_1}$ and $\mathbf{Weil}_\infty$ which may also prove to be useful in the broader discussion of tangent structure. We conclude by suggesting some ideas for extending the ideas described beyond the category $\mathbf{Weil}_\infty$.

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