2) a)

$20 \times 20 = 400$ pixeis $\begin{cases} 200 \text{ pretos} \to 0 \\ 200 \text{ brancos} \to 1 \end{cases}$

$P(0) = \frac{200}{400} = \frac{1}{2} = P(1)$

$A_x = \{0, 1\}$

$H(X) = \sum_{x \in A_x} P(x) \cdot \log_2\left(\frac{1}{P(x)}\right) =$

$= -\left[\frac{1}{2}\log_2\left(\frac{1}{2}\right) + \frac{1}{2}\log_2\left(\frac{1}{2}\right)\right] =$

$= -\left[-\frac{1}{2} - \frac{1}{2}\right] = 1$

$I(X) = \frac{1}{\#A_x} \sum^{\#A_x} -\log_2\left(\frac{1}{\#A_x}\right) =$

$= \log_2(2) = 1$

b) 400 pixeis $\begin{cases} 240 \to 0 \\ 160 \to 1 \end{cases}$ $P(0) = \frac{3}{5}$ $P(1) = \frac{2}{5}$

$H(x) = -\left[\frac{3}{5}\log_2\left(\frac{3}{5}\right) + \frac{2}{5}\log_2\left(\frac{2}{5}\right)\right] = 0{,}97$ bits/pixel

$I(x) = \log_2(2) = 1$

Conc: A entropia diminui pois há menos incerteza sobre a cor de um dado pixel

c) $P(0,0) = 0{,}6$; $P(1,1) = 0{,}2$; $P(0,1) = 0{,}1$; $P(1,0) = 0{,}1$

$G_n = -\sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_n=1}^{m} P(X = a_{i_1}, \ldots, X = a_{i_n}) \log_2 P(X = a_{i_1}, \ldots, X = a_{i_n})$

$m = \#Alfa$

$H(S) = \lim_{n \to \infty} \frac{1}{n} G_n$

traduzindo: $G = -\sum_{i=1}^{2} \sum_{j=1}^{2} P(X = a_i, Y = a_j) \times \log_2(P(X = a_i, Y = a_j))$

$G = -\sum^{2} P(x = a_i, 0) \log_2 P(X = a_i, 0) - \sum^{2} P(X = a_i, 1) \log_2 P(X = a_i, 1)$

$= -[0{,}6\log_2 0{,}6 + 0{,}2\log_2(0{,}2) + 0{,}1\log_2(0{,}1) + 0{,}1\log_2(0{,}1)]$

$= 1{,}570\ldots$

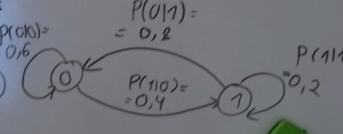
$H(S) = \lim_{n \to 2} \frac{1}{2} G = 0{,}785\ldots$

A entropia diminui ainda mais porque a distribuição de 0s é dependente da de 1s

d) $P(0) = \frac{2}{3}$ $P(1) = \frac{1}{3}$

$H = H(1)P(1) + P(2)H(2)$

$H(1) = -P(1|1)\log_2 P(1|1) - P(2|1)\log_2 P(2|1)$



$H(0) = -[P(0|0)\log_2 P(0|0) + P(1|0)\log_2 P(1|0)] =$

$= -[0{,}6\log_2 0{,}6 + 0{,}4\log_2 0{,}4] = 0{,}9705\ldots$

$H(1) = -[P(1|1)\log_2 P(1|1) + P(0|1)\log_2 P(0|1)] =$

$= -[0{,}2\log_2 0{,}2 + 0{,}8\log_2 0{,}8] = 0{,}721\ldots$

$H = H(1)P(1) + H(0)P(0) = 0{,}8879\ldots$

③

x/y	0	1
0	$1/3$	$1/3$
1	0	$1/3$

$P(X=0) = \sum_i P(X=0, Y=y_i) = \frac{1}{3} + 0 = \frac{1}{3}$

$P(X=1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

ⓐ $H(X) = -\left[\frac{1}{3}\log_2 \frac{1}{3} + \frac{2}{3}\log_2 \frac{2}{3}\right] = 0{,}91\ldots$

$P(Y=0) = \frac{2}{3}$; $P(Y=1) = \frac{1}{3}$

$H(Y) = -\left[\frac{2}{3}\log_2\left(\frac{2}{3}\right) + \frac{1}{3}\log_2\left(\frac{1}{3}\right)\right] = H(X)$

ⓑ $H(X|Y) = -\sum_j P(Y=y_j) \sum_i P(x_i|y_j) \log_2 P(x_i|y_j)$

$$H(X|Y) = -\sum_j P(y_j) \sum_i \frac{P(x_i, y_j)}{P(y_j)} \log_2\left(\frac{P(x_i, y_j)}{P(y_j)}\right) =$$

$$-\left[\frac{2}{3}\left(\frac{P(0,0)}{2/3}\log_2\left(\frac{P(0,0)}{2/3}\right) + \frac{P(1,0)}{2/3}\log_2 \frac{P(1,0)}{2/3}\right) + \right.$$

$$\left. + \frac{1}{3}\left(\frac{P(0,1)}{1/3}\log_2\left(\frac{P(0,1)}{1/3}\right) + \frac{P(1,1)}{1/3}\log_2 \frac{P(1,1)}{1/3}\right)\right] = \frac{1}{3}$$

$$H(Y|X) = -\left[\frac{1}{3}\left(\frac{1/3}{1/3}\log_2\left(\frac{1/3}{1/3}\right) + \frac{0}{1/3}\right) + \frac{2}{3}\left(\frac{1/3}{2/3}\log_2\left(\frac{1/3}{2/3}\right)\right.\right.$$

$$\left.\left. + \frac{1/3}{2/3}\log_2\left(\frac{1/3}{2/3}\right)\right)\right] = \frac{1}{3}$$

ⓒ $H(X,Y) = -3 \cdot \frac{1}{3}\log_2\left(\frac{1}{3}\right) + 0 = 1{,}58\ldots$

ⓓ $I(X;Y) = D_{KL}(P(X,Y), P(X)P(Y)) =$

$$= \sum_{x \in A_x}\sum_{y \in A_y} P(x,y) \log_2 \frac{P(x,y)}{P(x)P(y)} \longrightarrow \frac{P(x|y)}{P(y)}$$

$$I(X;Y) = \frac{1}{3}\log_2\left(\frac{1/3}{1/3 \cdot 1/3}\right) + 0 + \frac{1}{3}\log_2\left(\frac{1/3}{2/3 \cdot 2/3}\right) + \frac{1}{3}\log_2\left(\frac{1/3}{1/3 \cdot 2/3}\right)$$

$= \log_2(3/2) = 0{,}58\ldots$

ou $I(X;Y) = I(Y;X) = H(Y) - H(Y|X) =$

$= H(X) - H(X|Y) = 0{,}91 - \frac{1}{3} = 0{,}577\ldots$

⑤ $S = \{12; 14; 16; 18; 20; 22; 22; 20; 18; 16; 14; 12\}$

$\#S = 12$ $A_x = \{12, 14, 16, 18, 20, 22\}$

A prob de cada símbolo é $1/6$

ⓐ $H(S) = -6 \cdot \frac{1}{6}\log_2\left(\frac{1}{6}\right) = 2{,}58$

ⓑ $\boxed{12; 2; 2; 2; 2; 2; 0; -2; -2; -2; -2; -2}$

$P(12) = \frac{1}{12}$; $P(2) = \frac{5}{12}$; $P(0) = \frac{1}{12}$; $P(-2) = \frac{5}{12}$

$$H(S) = -\left[\frac{2}{12}\log_2\left(\frac{1}{12}\right) + \frac{10}{12}\log_2\left(\frac{5}{12}\right)\right] = 1{,}65$$

6 $Alfa = \{1, 2\}$

$P(1|1) = 0{,}8$; $P(2|1) = 0{,}2$

$P(1|2) = 0{,}6$; $P(2|2) = 0{,}4$

$P(1,2) = P(2,1) \Rightarrow$ var i.i.d.

$\Downarrow$

$H = -P(1)\log_2 P(1) - P(2)\log_2 P(2)$

a) Se i.i.d.

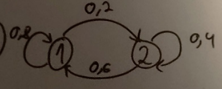
$P(1) = \dfrac{P(1|2)}{P(2|1)+P(1|2)} = \dfrac{0{,}6}{0{,}8} = 0{,}75$

$P(2) = \dfrac{P(2|1)}{P(2|1)+P(1|2)} = \dfrac{0{,}2}{0{,}8} = 0{,}25$

$H = -P(1)\log_2(P(1)) - P(2)\log_2(P(2)) \simeq$

$\simeq 0{,}8112\ldots \Rightarrow$ Falso

b)



$H = P(1)H(1) + P(2)H(2)$

$H(1) = -P(1|1)\log_2 P(1|1) - P(2|1)\log_2 P(2|1) = 0{,}722\ldots$

$H(2) = -P(2|2)\log_2 P(2|2) - P(1|2)\log_2 P(1|2) = 0{,}971$

$\Rightarrow H = 0{,}284\ldots$

7 $X \in \{1, 2, 3\}$

$P(X=1) = \dfrac{a}{2}$

$P(X=2) = 1 - a$

$P(X=3) = \dfrac{a}{2}$

a) $a = 1 \Rightarrow H = -\left(\frac{1}{2}\log_2\left(\frac{1}{2}\right) + 0 + \frac{1}{2}\log_2\left(\frac{1}{2}\right)\right) = 1$

$a = \frac{1}{2} \Rightarrow H = -\left(\frac{1}{4}\log_2\left(\frac{1}{4}\right) + \frac{3}{4}\log_2\left(\frac{3}{4}\right) + \frac{1}{4}\log_2\left(\frac{1}{4}\right)\right) =$

$= 1{,}3\ldots$

$a = \frac{1}{3} \Rightarrow H = -\left(\frac{2}{6}\log_2\left(\frac{1}{6}\right) + \frac{5}{6}\log_2\left(\frac{5}{6}\right)\right) =$

$= 1{,}08\ldots \Rightarrow$ C $\left(a = \frac{1}{2}\right)$

b) $0 \leq H(X) \leq \log_2 \#A = 1{,}58 \Rightarrow$ B C

c) A

d) C $I(X;Y) = H(Y)$ se completamente dependentes; $= 0 \rightarrow$ se i.i.d

e) $a = 0 \Rightarrow H = -(\log_2 1) = 0 \Rightarrow$ C

8 $Y = \log_2(2X+2)$

a) $0 \leq H(X,Y) = H(X) + H(Y|X) \leq \overbrace{H(X) + H(Y)}^{\text{igualdade caso i.i.d.}}$

A e C

b) A e C

c) C $I(X;Y) = H(Y) - \underbrace{H(Y|X)}_{\substack{\text{0 pois são completamente dependentes,} \\ \text{há 0 incerteza}}}$

d) A B Desigualdade de Gibbs

e) B

10

$C \to$ codificador

$C^{-1} \to$ descodificador

$t \to t_1 t_2 t_3$

$r_1 r_2 r_3 \xrightarrow{C^{-1}}$ mediana$\{r_1 r_2 r_3\}$

$\Downarrow$

Apenas 2 dos 3 bits tem de ser transmitido corretamente

$P(\text{erro 1 bit}) = 0,1$

$P(\text{erro da mensagem de 3 bits}) =$

$= P(\text{erro 2 bits}) + P(\text{erro 3 bits}) \Leftrightarrow$ ← como são indep.

$= 3\left[P(\text{erro 1 bit})^2 P(\text{certo 1 bit})\right] + \left(P(\text{erro 1 bit})\right)^3 =$

$= 3\left[(0,1)^2(1-0,1)\right] + (0,1)^3 = 0,027 + 0,001 = 0,028$

Há $C(3,2)$ maneiras de ter 2 errados e 1 certo

11 $A_X = \{1, 2, \ldots, n, \ldots\}$ $\#A_X = \infty$

cara $\equiv h$ $P(t) = P(h) = \frac{1}{2}$

coroa $\equiv t$

$P(0t) \equiv$ Prob de sair coroa à $1^{\text{a}} = \frac{1}{2} = P(h)$

$P(1t) = P(t)P(h)$ pois são indep $= \frac{1}{4}$

$P(2t) = \underbrace{P(t)P(t)P(h)}_{P(t \to t \to h)} = \frac{1}{8}$

$\vdots$

$P(nt) = nP(t)P(h)$

a) Situação genérica:

$P(h) = f$

$P(t) = 1 - f$

$P(x) = \{(1-f)^1, (1-f)^2, \ldots, (1-f)^n\}$

$H(X) = -\sum_{n=1}^{\infty} (1-f)^n \log_2\left((1-f)^n\right)$

b) $P_X = \left\{\frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, \ldots\right\}$

$H(X) = \sum_{x \in A_X} P(x) \log_2 \frac{1}{P(x)} =$

$= \sum_{n=1}^{\infty} \frac{1}{2^n} \log_2 2^n =$

$\sum_{n=1}^{\infty} n r^n = \frac{r}{(1-r)^2}$

$= \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n =$

$= \frac{1/2}{(1-1/2)^2} = 2$

(12) $A_Y = \{4,5,6,7\}$ $A_X = \{AAAA, AAABA, BBBAB, \dots\}$ $C^n_k = \frac{k!}{k!(n-k)!}$

$P(A) = P(B) = \frac{1}{2}$

$P(X = AAAA) = P(A)^4 = \frac{1}{16}$

$P(X = \underbrace{BAAAA}_{\ell(x)=y}) = P(B)P(A)^4 = \frac{1}{32}$

Portanto: $P(X) = \left(\frac{1}{2}\right)^y$

$K_n = \{K_7, K_6, K_5, K_4\}$

ou seja, há K_n elementos em A_X tal que: $P(x) = \left(\frac{1}{2^n}\right) \Rightarrow$

$$\Rightarrow H(X) = \sum_{n=4}^{7} \sum^{K_n} P(x)\log_2(P(x)) =$$

$$= \sum_{n=4}^{7} \sum^{K_n} \left(\frac{1}{2}\right)^n \log_2(2^n) = \sum_{n=4}^{7} \left(\frac{1}{2}\right)^n n \log_2(2)\, K_n =$$

$$= \sum_{n=4}^{7} \left(\frac{1}{2}\right)^n n K_n = \frac{4.2}{2^4} + \frac{5.8}{2^6} + \frac{6.20}{2^6} + \frac{7.40}{2^7} \approx 5{,}81$$

CA

sequências que existem em X com y elementos:

$y=4: C(4,4).2 = 2 \equiv K_4$

Maneiras de ordenar 0000

2 equipas

$y=5: (C(5,4) - C(4,4)) \cdot 2 = 8 \equiv K_5$

Maneiras de ordenar 10000 sem que apareçam 4 símbolos iguais primeiro

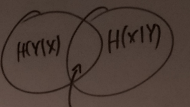
$y=6: 2C(6,4) - K_5 - K_4 = 20 \equiv K_6$

$y=7: 2C(7,4) - K_5 - K_4 - K_6 = 40 \equiv K_7$

$$H(Y) = \sum_{n=4}^{7} \frac{K_n}{\sum_{i \in K_n} i} \cdot \log_2\left(\frac{\sum_{i \in K_n} i}{K_n}\right) = \frac{2}{70}\log_2\frac{70}{2} + \frac{8}{70}\log_2\frac{70}{8} +$$

$$+ \frac{20}{70}\log_2\frac{70}{20} + \frac{40}{70}\log_2\frac{70}{40} \approx 1{,}48$$

$H(Y|X) = 0$ pois Y é dependente de X ($Y = len(X)$)



$I(X;Y)$

$I(X;Y) = H(Y) - H(Y|X) =$

$= H(X) - H(X|Y)$

$\Downarrow$

$I(X;Y) = H(Y) + 0 \approx 1{,}48$

$H(Y) - H(Y|X) = H(X) - H(X|Y) \Leftrightarrow$

$\Leftrightarrow 1{,}48 - 0 = 5{,}81 - H(X|Y) \Leftrightarrow$

$\Leftrightarrow H(X|Y) = 5{,}81 - 1{,}48 = 4{,}33$

(13) Provar que $H(P(X))$ é máxima quando $P(X=x_0)=P(X=x_1)=\dots=P(X=x_n)$, $X=\{x_0, x_1, \dots, x_n\}$,

ou seja: $P(x_i) = \frac{1}{\# A_X} = \frac{1}{n}$

- Sabemos que $\sum_{i=0}^{n} P(X=x_i) = 1$
- Queremos maximizar $\mathcal{L} = -\sum_{i=0}^{n} P(X=x_i) \log P(X=x_i) + \lambda \left(\sum_{i=0}^{n} P(X=x_i) - 1 \right)$

$$\frac{\partial \mathcal{L}}{\partial P(x_i)} = 0 \Leftrightarrow -\sum_{i=0}^{n} \left(1 \cdot \log P(x_i) + 1 \right) + \lambda n = 0 \Leftrightarrow -n \log P(x_i) - n + \lambda n = 0 \Leftrightarrow$$

$$\Leftrightarrow -\log P(x_i) - 1 + \lambda = 0 \Leftrightarrow P(x_i) = e^{-1+\lambda} \Rightarrow$$

$$\Rightarrow \sum_{i=0}^{n} P(x_i) = 1 \Leftrightarrow \sum_{i=0}^{n} e^{-1+\lambda} = 1 \Leftrightarrow n e^{-1+\lambda} = 1 \Leftrightarrow \log\left(\frac{1}{n}\right) + 1 = \lambda \Rightarrow$$

$$\Rightarrow P(x_i) = e^{-1 + (\log(\frac{1}{n}) + 1)} = e^{\log(1/n)} = \frac{1}{n}$$

cqd

logo, pelo principio da máxima entropia, $H(P(X))$ é máxima quando os acontecimentos são equiprováveis