Asset Pricing & Portfolio Management

Black-Litterman model

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Agenda

1. Introduction
2. Black-Litterman model
3. Illustrative example

- **Readings**:
  - Black, F. and Litterman, R. (1990). "Asset Allocation: Combining Investors Views with Market Equilibrium." Fixed Income Research, Goldman, Sachs & Company, September.
  - Black, F. and Litterman, R. (1991). "Global Asset Allocation with Equities, Bonds, and Currencies." Fixed Income Research, Goldman, Sachs & Company, October.
  - Black, F. and Litterman, R. (1992). "Global Portfolio Optimization." Financial Analysts Journal, September/October, 28-43.
  - Roncalli, Thierry (2013). Introduction to Risk Parity and Budgeting, Chapman & Hall/CRC, Ch. 1

Introduction

- The Black-Litterman asset allocation model, created by Fischer Black and Robert Litterman, is a sophisticated portfolio construction method that overcomes the problem of unintuitive, highly-concentrated portfolios, input-sensitivity, and estimation error maximization
- The Black-Litterman model (1990, 1991, 1992) uses a Bayesian approach to combine the subjective views of an investor regarding the expected returns of one or more assets with the market equilibrium vector of expected returns (the prior distribution) to form a new, mixed estimate of expected returns
- The Black Litterman model combines
  - the CAPM (see Sharpe (1964)),
  - reverse optimization (see Sharpe (1974)),
  - mixed estimation (see Theil (1971, 1978)),
  - the universal hedge ratio / Black's global CAPM (see Black (1989a, 1989b) and Litterman (2003)), and
  - mean-variance optimization (see Markowitz (1952))

Black-Litterman model

Tactical asset allocation (TAA) model

How to incorporate portfolio manager's views in a strategic asset allocation (SAA)?

The Black-Litterman model starts with an initial Modern Portfolio Theory allocation, the model computes the implied risk premia and then deduces the optimized portfolio which is coherent with the bets of the portfolio manager
Two-step approach:

1. Initial allocation $\Rightarrow$ implied risk premia (Sharpe)
2. Portfolio optimization $\Rightarrow$ coherent with the bets of the portfolio manager (Markowitz)

Black-Litterman model

- The Black-Litterman model starts with a neutral equilibrium portfolio for the prior estimate of returns
- The model relies on General Equilibrium theory to state that if the aggregate portfolio is at equilibrium, each sub-portfolio must also be at equilibrium
- It can be used with any utility function which makes it very flexible
- In practice most practitioners use the Quadratic Utility function and assume a risk free asset, and thus the equilibrium model simplifies to the Capital Asset Pricing Model (CAPM)
- CAPM model is of the form

$$E(r_i) = r_f + \beta r_m + \epsilon_i \tag{1}$$

where

- r_f = risk free rate.
- $r_m = (r_m - r_f)$ = excess return of the market portfolio
- $\beta = \frac{Cov(r_i, r_m)}{\sigma^2(r_m)}$ is the systematic risk coefficient
- ϵ = residual, or asset specific (idiosyncratic) excess return

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Let us consoder the Markowitz quadratic optimization problem:

$$
\begin{aligned}
x^{\star} \quad &= \quad \arg\min \frac{1}{2} x^{\top} \Sigma x - \gamma x^{\top} \left(\mu - r\mathbf{1}_n\right) \\
\text{u.c.} \quad & \begin{cases} \mathbf{1}_n^{\top} x = 1 \\ x \in \Omega \end{cases}
\end{aligned}
$$

where

- $\left(\mu - r\mathbf{1}_n\right)$: Vector of equilibrium excess returns for each asset
- $\gamma = \phi^{-1}$
- $\phi =$ the risk aversion coefficient

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If we omit the constraint $\mathbf{1}_n^\top x = 1$, the solution, the first-order condition is:

$$\Sigma x - \gamma \left(\mu - r\mathbf{1}_n\right) = \mathbf{0}_n$$

The solution is:

$$x^\star = \gamma \Sigma^{-1} \left(\mu - r\mathbf{1}_n\right)$$

- In the Markowitz model, the unknown variable is the vector x
- If the initial allocation x_0 is given, Black and Litterman (1992) assume that this allocation corresponds to an optimal solution, implying that:

$$\tilde{\mu} = r\mathbf{1}_n + \frac{1}{\gamma}\Sigma x_0 \tag{2}$$

- $\tilde{\mu}$ is the vector of expected returns which is coherent with x_0

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We deduce that:

$$\begin{aligned} \tilde{\pi} &= \tilde{\mu} - r \\ &= \frac{1}{\gamma}\Sigma x_0 \end{aligned} \tag{3}$$

The variable $\tilde{\pi}$ is:

- the *risk premium priced* by the portfolio manager
- the '*implied risk premium*' of the portfolio manager
- the '*market risk premium*' when x_0 is the market portfolio

Black-Litterman model

The computation of $\tilde{\mu}$ needs to the value of Σ and the parameter γ or the risk aversion $\phi = \gamma^{-1}$

For the first parameter Σ, we generally use the empirical covariance matrix $\hat{\Sigma}$
Since in the optimal solution we have $\Sigma x_0 - \gamma \left(\tilde{\mu} - r\mathbf{1}_n\right) = \mathbf{0}_n$, we deduce that:

$$
\begin{aligned}
(*) \quad & \Leftrightarrow \quad \gamma \left(\tilde{\mu} - r\mathbf{1}_n\right) = \Sigma x_0 \\
& \Leftrightarrow \quad \gamma \left(x_0^\top \tilde{\mu} - r x_0^\top \mathbf{1}_n\right) = x_0^\top \Sigma x_0 \\
& \Leftrightarrow \quad \gamma \left(x_0^\top \tilde{\mu} - r\right) = x_0^\top \Sigma x_0 \\
& \Leftrightarrow \quad \gamma = \frac{x_0^\top \Sigma x_0}{x_0^\top \tilde{\mu} - r} \qquad (4)
\end{aligned}
$$

It follows that

$$\phi = \frac{x_0^\top \tilde{\mu} - r}{x_0^\top \Sigma x_0} = \frac{\mathrm{SR}\left(x_0 \mid r\right)}{\sqrt{x_0^\top \Sigma x_0}} = \frac{\mathrm{SR}\left(x_0 \mid r\right)}{\sigma\left(x_0\right)} \qquad (5)$$

where $\mathrm{SR}\left(x_0 \mid r\right)$ is the portfolio's expected Sharpe ratio

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We finally obtain:

$$\tilde{\mu} = r + \mathrm{SR}\left(x_0 \mid r\right) \frac{\Sigma x_0}{\sqrt{x_0^\top \Sigma x_0}} \tag{6}$$

and:

$$\tilde{\pi} = \mathrm{SR}\left(x_0 \mid r\right) \frac{\Sigma x_0}{\sqrt{x_0^\top \Sigma x_0}} \tag{7}$$

Black-Litterman model

Example 1

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

Black-Litterman model

Example 1

We consider Example 1 and we suppose that the initial allocation x_0 is $(40\%, 30\%, 20\%, 10\%)$

- The volatility of the portfolio is equal to:

$$\sigma(x_0) = 15.35\%$$

  The objective of the portfolio manager is to target a Sharpe ratio equal to 0.25

- We obtain $\phi = 1.63$
- If $r = 3\%$, the implied expected returns are:

$$\tilde{\mu} = \begin{pmatrix} 5.47\% \\ 6.68\% \\ 8.70\% \\ 9.06\% \end{pmatrix}$$

Black-Litterman model: The optimization problem

Black and Litterman assume that μ cannot be know with certainty
In particular, they assume μ is a Gaussian vector with expected returns $\tilde{\mu}$ and covariance matrix Γ:

$$\mu \sim \mathcal{N}(\tilde{\mu}, \Gamma)$$

The market risk premium $\tilde{\mu}$ is then the unconditional mathematical expectation of the asset returns R
The portfolio manager's views are given by this relationship:

$$P\mu = Q + \varepsilon \tag{8}$$

where P is a $(k \times n)$ matrix, Q is a $(k \times 1)$ vector and $\varepsilon \sim \mathcal{N}(0, \Omega)$ is a Gaussian vector of dimension k

- If the portfolio manager has two views, the matrix P has two rows $\Rightarrow$ k is then the number of views
- Ω is the covariance matrix of $P\mu - Q$, therefore it measures the uncertainty of the views

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With the specification (8), we can express the views in an absolute or relative way. We consider the three-asset case:

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

The portfolio manager has an absolute view on the expected return of the first asset:

$$\mu_1 = q_1 + \varepsilon_1$$

We have:

$$P = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \; Q = q_1, \; \varepsilon = \varepsilon_1 \text{ and } \Omega = \omega_1^2$$

If $\omega_1 = 0$, the portfolio manager has a very high level of confidence. If $\omega_1 \neq 0$, his view is uncertain

Black-Litterman model

- The portfolio manager has an absolute view on the expected return of the second asset:

$$\mu_2 = q_2 + \varepsilon_2$$

We have:

$$P = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},\ Q = q_2,\ \varepsilon = \varepsilon_2 \text{ and } \Omega = \omega_2^2$$

The portfolio manager has two absolute views:

$$\begin{aligned} \mu_1 &= q_1 + \varepsilon_1 \\ \mu_2 &= q_2 + \varepsilon_2 \end{aligned}$$

We have:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},\ \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$$

Black-Litterman model

Suppose now that he thinks that the outperformance of the first asset with respect to the second asset is q:

$$\mu_1 - \mu_2 = q_{1|2} + \varepsilon_{1|2}$$

We have:

$$\begin{aligned} P &= \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \\ Q &= q_{1|2} \\ \varepsilon &= \varepsilon_{1|2} \\ \Omega &= \omega_{1|2}^2 \end{aligned}$$

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The Markowitz optimization problem becomes:

$$\begin{aligned} x^{\star}(\gamma) \quad &= \quad \arg\min \frac{1}{2} x^{\top} \Sigma x - \gamma x^{\top} (\bar{\mu} - r\mathbf{1}_n) \\ \text{u.c.} \quad & \mathbf{1}_n^{\top} x = 1 \end{aligned} \tag{9}$$

where $\bar{\mu}$ is the vector of expected returns conditional to the views:

$$\begin{aligned} \bar{\mu} &= \mathbb{E}[\mu \mid \text{views}] \\ &= \mathbb{E}[\mu \mid P\mu = Q + \varepsilon] \\ &= \mathbb{E}[\mu \mid P\mu - \varepsilon = Q] \end{aligned}$$

To compute $\bar{\mu}$, we consider the random vector $(\mu, \nu = P\mu - \varepsilon)$:

$$\begin{pmatrix} \mu \\ \nu = P\mu - \varepsilon \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \tilde{\mu} \\ P\tilde{\mu} \end{pmatrix}, \begin{pmatrix} \Gamma & \Gamma P^{\top} \\ P\Gamma & P\Gamma P^{\top} + \Omega \end{pmatrix} \right)$$

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Let us consider a Gaussian random vector defined as follows:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \Sigma_{x,x} & \Sigma_{x,y} \\ \Sigma_{y,x} & \Sigma_{y,y} \end{pmatrix} \right)$$

The conditional distribution of Y given $X = x$ is a multivariate normal distribution with mean $\mu_{y|x}$ and and covariance matrix $\Sigma_{y,y|x}$:

$$Y \mid X = x \sim \mathcal{N}\left(\mu_{y|x}, \Sigma_{y,y|x}\right)$$

where:

$$\mu_{y|x} = \mathbb{E}\left[Y \mid X = x\right] = \mu_y + \Sigma_{y,x}\Sigma_{x,x}^{-1}\left(x - \mu_x\right)$$

and:

$$\Sigma_{y,y|x} = \operatorname{cov}\left(Y \mid X = x\right) = \Sigma_{y,y} - \Sigma_{y,x}\Sigma_{x,x}^{-1}\Sigma_{x,y}$$

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We apply the conditional expectation formula:

$$\begin{aligned}\bar{\mu} &= \mathbb{E}\left[\mu \mid v = Q\right] \\ &= \mathbb{E}\left[\mu\right] + \operatorname{cov}\left(\mu, v\right)\operatorname{var}\left(v\right)^{-1}\left(Q - \mathbb{E}\left[v\right]\right) \\ &= \tilde{\mu} + \Gamma P^{\top}\left(P\Gamma P^{\top} + \Omega\right)^{-1}\left(Q - P\tilde{\mu}\right) \qquad (10)\end{aligned}$$

The conditional expectation $\bar{\mu}$ has two components:

1. The first component corresponds to the vector of implied expected returns $\tilde{\mu}$
2. The second component is a correction term which takes into account the *disequilibrium* $(Q - P\tilde{\mu})$ between the manager views and the market views

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The conditional covariance matrix is equal to:

$$\begin{aligned}\bar{\Sigma} &= \operatorname{var}(\mu \mid \nu = Q) \\ &= \Gamma - \Gamma P^\top \left(P \Gamma P^\top + \Omega\right)^{-1} P \Gamma\end{aligned}$$

Another expression is:

$$\begin{aligned}\bar{\Sigma} &= \left(I_n + \Gamma P^\top \Omega^{-1} P\right)^{-1} \Gamma \\ &= \left(\Gamma^{-1} + P^\top \Omega^{-1} P\right)^{-1}\end{aligned}$$

The conditional covariance matrix is a weighted average of the covariance matrix Γ and the covariance matrix Ω of the manager views.

Black-Litterman model

Choice of Σ

From a theoretical point of view, we have:

$$\Sigma = \bar{\Sigma} = \left(\Gamma^{-1} + P^\top \Omega^{-1} P\right)^{-1}$$

In practice, we use:

$$\Sigma = \hat{\Sigma}$$

Choice of Γ

We assume that:

$$\Gamma = \tau \Sigma$$

We can also target a tracking error volatility and deduce τ

- Remarks:
  - The difficulty lies in specifying the covariance matrix Γ. One solution is to define $\Gamma = \tau\Sigma$ and to calibrate τ in order to target a tracking error volatility (Meucci, 2005)
  - Tracking error volatility defines how volatile a managed portfolio's return is compared to a benchmark
  - It is calculated as the standard deviation of the return differences

Black-Litterman model: Numerical implementation

The five-step approach to implement the Black-Litterman model is:

1. We estimate the empirical covariance matrix $\hat{\Sigma}$ and set $\Sigma = \hat{\Sigma}$
2. Given the current portfolio, we compute the implied risk aversion $\phi = \gamma^{-1}$ and we deduce the vector $\tilde{\mu}$ of implied expected returns
3. We specify the views by defining the P, Q and Ω matrices
4. Given a matrix Γ, we compute the conditional expectation $\bar{\mu}$
5. We finally perform the portfolio optimization with $\hat{\Sigma}$, $\bar{\mu}$ and γ

Black-Litterman model

- We use Example 1 and impose that the optimized weights are positive
- The portfolio manager has an absolute view on the first asset and a relative view on the second and third assets:

$$
\begin{aligned}
P &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \\
Q &= \begin{pmatrix} q_1 \\ q_{2-3} \end{pmatrix} \\
\Omega &= \begin{pmatrix} \varpi_1^2 & 0 \\ 0 & \varpi_{2-3}^2 \end{pmatrix}
\end{aligned}
$$

- The portfolio manager believes that the expected return of the first asset is 4% whereas he believes that the third asset will outperform the second asset by 1% in average, i.e., $q_1 = 4\%$, $q_{2-3} = -1\%$
- $\varpi_1 = 10\%$ and $\varpi_{2-3} = 5\% \Longrightarrow$ the level of confidence is higher for the relative view than for the absolute view

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- Case #1: $\tau = 1$
- Case #2: $\tau = 1$ and $q_1 = 7\%$
- Case #3: $\tau = 1$ and $\varpi_1 = \varpi_{2-3} = 20\%$
- Case #4: $\tau = 10\%$
- Case #5: $\tau = 1\%$
- Note: We impose that the optimized weights are positive $x_i > 0$

Black-Litterman model

Table: Black-Litterman portfolios

	#0	#1	#2	#3	#4	#5
$x_1^\star$	40.00	33.41	51.16	36.41	38.25	39.77
$x_2^\star$	30.00	51.56	39.91	42.97	42.72	32.60
$x_3^\star$	20.00	5.46	0.00	10.85	9.14	17.65
$x_4^\star$	10.00	9.58	8.93	9.77	9.89	9.98
$\sigma\left(x^\star \middle\vert x_0\right)$	0.00	3.65	3.67	2.19	2.18	0.45

Black-Litterman model

- Portfolio #1 differs from the initial allocation x_0
- The weight of the first asset decreases from 40% to 33.41%
- This is coherent with the absolute view of the portfolio manager, who thinks that μ_1 is smaller than its market price (4% versus 5.47%)
- The weight difference between the second and third asset is 10% for the initial allocation whereas it becomes 46.1%
- This is also coherent with the relative view since the market believes, implicitly, that the third asset will outperform the second asset by 2.02%, whereas the portfolio manager thinks that this outperformance is only 1%
- We note that if we increase the uncertainty of the views (portfolio #3) or if the covariance Γ of the expected return is smaller (portfolios #4 and #5), the differences between the optimized portfolio x^* and the current portfolio x_0 are reduced

Black-Litterman model

To calibrate the parameter τ, we could target a tracking error volatility $\sigma^{\star}$, defined as

$$\sigma^{\star} = \sqrt{\left(x_{BL} - x_0\right)^{\top} \cdot \Sigma \cdot \left(x_{BL} - x_0\right)} \qquad (11)$$

- If we target $\sigma^{\star} = 2\%$, the optimized portfolio is between portfolios $\#4$ $\left(\sigma\left(x^{\star} \mid x_0\right) = 2.18\%\right)$ and $\#5$ $\left(\sigma\left(x^{\star} \mid x_0\right) = 0.45\%\right)$
- The optimal value of τ is between 10% and 1%
- Using a bisection algorithm, we obtain $\tau = 5.2\%$

The optimal portfolio $x^{\star}$ is:

$$x^{\star} = \begin{pmatrix} 36.80\% \\ 41.83\% \\ 11.58\% \\ 9.79\% \end{pmatrix}$$

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Thank you for your attention

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