Asset Pricing & Portfolio Management

Factor Models

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Contents

Financial Factor Models

Financial factor analysis explains returns with a small number of fundamental variables called factors or risk factor.
A factor model for equity returns (or excess equity returns $R_{j,t} - R_{f,t}$)

$$R_{j,t} = \beta_{0,j} + \beta_{1,j} F_{1,t} + \ldots + \beta_{p,j} F_{p,t} + \epsilon_{j,t}$$

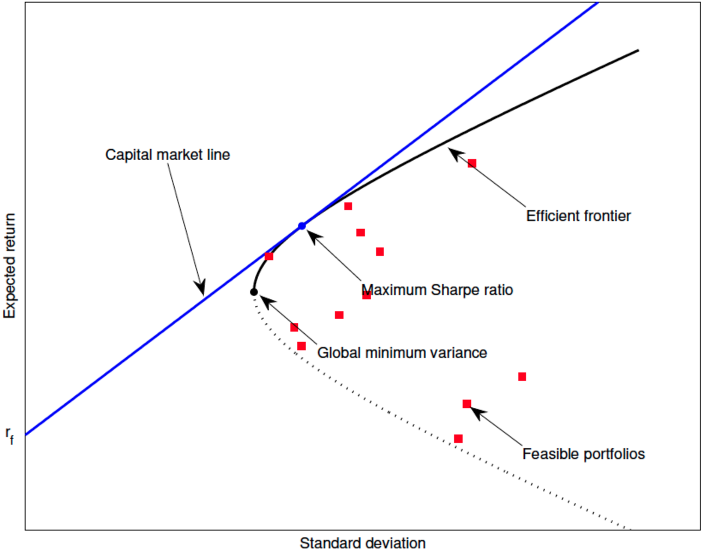
where

- $R_{j,t}$ is either the return or the excess return on the j^{th} asset at time t,
  - $R_{f,t}$ is the risk free rate at time t
  - $F_{1,t}, \ldots, F_{p,t}$ are variables, called factors or risk factors, that represent the "state of the financial markets and world economy" at time t.
  - $\epsilon_{1,t}, \ldots, \epsilon_{n,t}$ are uncorrelated, mean-zero error terms called the unique risks of the individual stocks
  - "Uncorrelated" means that all cross-correlation between the returns is due to the factors!
  - The factors are common to all returns $j = 1, \ldots, N$
  - $\beta_{k,j}$ is called factor loading and specifies the sensitivity of the j^{th} asset return to the k^{th} factor

Review of Portfolio Theory

- Risk/Return Trade-Off
  - Portfolio risk depends primarily on covariances
- Not stocks'individual volatilities
  - Diversification reduces risk
- But risk common to all firms cannot be diversified away
  - Hold the tangency portfolio
- The tangency portfolio has the highest expected return for a given level of risk (i.e., the highest Sharpe ratio)
  - Suppose all investors hold the same portfolio M; what must M be?
- The tangency portfolio is the market portfolio
  - Proxies for the market portfolio: S&P 500, Russell 2000, MSCI, etc.
- Value-weighted portfolio of broad cross-section of stocks

Review of Portfolio Theory



The Capital Asset Pricing Model

- The market CAPM model: The root of factor models
- Implications of the Market Portfolio
- Efficient portfolios are combinations of the market portfolio and T-Bills
- Expected returns of efficient portfolios satisfy (capital market line (efficient portfolios)):

$$E\left[R_p\right] = R_f + \frac{\sigma_p}{\sigma_m}\left(E\left[R_m\right] - R_f\right) \tag{1}$$

- This yields the required rate of return or cost of capital for efficient portfolios!
- Trade-off between risk and expected return
- Multiplier is the ratio of portfolio risk to market risk
- What about other (non-efficient) portfolios?

The Capital Asset Pricing Model

- Implications of M as the Market Portfolio
  - For any asset, define its market betas:

$$\beta_i \circeq \frac{Cov\left[R_i, R_m\right]}{Var\left[R_m\right]} \tag{2}$$

- Then the Sharpe-Lintner CAPM implies that (security market line)

$$E\left[R_i\right] = R_f + \beta_i \left(E\left[R_m\right] - R_f\right) \tag{3}$$

- Risk/reward relation is linear!
- Beta is the correct measure of risk, not sigma (except for efficient portfolios), only rewards systematic risk
- Beta measures sensitivity of stock to market movements

The Capital Asset Pricing Model

- The Security Market Line

$$E\left[R_i\right] = R_f + \beta_i \left(E\left[R_m\right] - R_f\right) \tag{4}$$

- The security market line yields a measure of risk: beta
  - This provides a method for estimating a firm's cost of capital
  - The CAPM also provides a method for evaluating portfolio managers
- Implications:

$$\begin{aligned}
\beta_i &= 1 \Longrightarrow E\left[R_i\right] = E\left[R_m\right] \\
\beta_i &> 1 \Longrightarrow E\left[R_i\right] > E\left[R_m\right] \\
\beta_i &= 0 \Longrightarrow E\left[R_i\right] = R_f \\
\beta_i &< 0 \Longrightarrow E\left[R_i\right] < R_f
\end{aligned}$$

The Capital Asset Pricing Model

- What About Arbitrary Portfolios of Stocks?

$$
\begin{aligned}
R_{P,t} &= \sum_{j=1}^{N} w_j R_{j,t} = w_1 R_{1,t} + \ldots + w_N R_{N,t} && (5) \\
Cov\left[R_P, R_m\right] &= Cov\left[w_1 R_{1,t} + \ldots + w_N R_{N,t}, R_m\right] && (6) \\
&= w_1 Cov\left[R_{1,t}, R_m\right] + \ldots + w_N Cov\left[R_{N,t}, R_m\right] && (7) \\
\frac{Cov\left[R_P, R_m\right]}{Var\left[R_m\right]} &= w_1 \frac{Cov\left[R_{1,t}, R_m\right]}{Var\left[R_m\right]} + \ldots + w_N \frac{Cov\left[R_{N,t}, R_m\right]}{Var\left[R_m\right]} && (8) \\
\beta_P &= w_1 \beta_1 + w_2 \beta_2 + \ldots + w_N \beta_N && (9)
\end{aligned}
$$

- Therefore, for any arbitrary portfolio of stocks

$$
E\left[R_P\right] = R_f + \beta_P \left(E\left[R_m\right] - R_f\right) \qquad (10)
$$

Implementing the CAPM

- Parameter Estimation:
  - Security market line must be estimated
  - One unknown parameter: β
  - Given return history, β can be estimated by linear regression:

$$R_{i,t} = R_f + \beta_i \left(R_{m,t} - R_f\right) + \epsilon_{i,t},$$

with $\mathbb{E}\left(\epsilon_{i,t}\right) = 0,\ Cov\left[R_{m,t} - R_f, \epsilon_{i,t}\right] = 0$

$$\begin{aligned} R_{i,t} - R_f &= \alpha_i + \beta_i \left(R_{m,t} - R_f\right) + \epsilon_{i,t} && (11) \\ CAPM &\implies \alpha_i = 0 \end{aligned}$$

- Alpha is the correct measure of performance, not total return
- Alpha takes into account the differences in risk among managers
- Alternatively

$$\begin{aligned} R_{i,t} &= R_f + \beta_i R_{m,t} + \epsilon_{i,t} && (12) \\ CAPM &\implies \alpha_i = R_f \left(1 - \beta_i\right) \end{aligned}$$

Macroeconomic factor model with single market factor

- Consider a simple example consisting of a single known factor (i.e., the market index)
- The model is

$$\mathbf{x}_t = \boldsymbol{\alpha} + \boldsymbol{\beta}\mathbf{f}_t + \boldsymbol{\epsilon}_t, \quad t = 1, \ldots, T \tag{13}$$

  where the explicit factor $\mathbf{f}_t$ is the S&P 500 index
- To estimate the parameters – intercept $\boldsymbol{\alpha}$ and the loading beta $\boldsymbol{\beta}$ – we can do a simple least squares (LS) regression

$$\underset{\boldsymbol{\alpha},\boldsymbol{\beta}}{\text{minimize}} \quad \sum_{t=1}^{T} \left\| \mathbf{x}_t - \boldsymbol{\alpha} - \boldsymbol{\beta}\mathbf{f}_t \right\|^2 \tag{14}$$

Macroeconomic factor model with single market factor

- The solution to the previous LS fitting is

$$
\begin{aligned}
\hat{\beta} &= \frac{cov\left(\mathbf{x}_t, \mathbf{f}_t\right)}{var\left(\mathbf{f}_t\right)\epsilon_t} \\
\hat{\alpha} &= \bar{x} - \hat{\beta}\bar{f} \\
\epsilon_i &= x_i - \alpha_i \mathbf{1} - \alpha_i \mathbf{f}, \quad i = 1, ..., N \\
\hat{\sigma}_i^2 &= \frac{1}{T-2}\hat{\epsilon}_t^T \hat{\epsilon}_t \\
\Sigma &= var\left(\mathbf{f}_t\right)\hat{\beta}\hat{\beta}^T + \Phi \\
\Phi &= diag\left(\hat{\sigma}_1^2, ..., \hat{\sigma}_N^2\right)
\end{aligned}
$$

Macroeconomic factor model with single market factor

- Alternatively, we can do the fitting using a more compact matrix notation
$$\mathbf{X}^T = \boldsymbol{\alpha}\mathbf{1}^T + \boldsymbol{\beta}\mathbf{f}^T + \mathbf{E}^T \tag{15}$$
- The OLS fitting is
$$\underset{\boldsymbol{\alpha},\boldsymbol{\beta}}{\text{minimize}} \quad \left\| \mathbf{X}^T - \boldsymbol{\alpha}\mathbf{1}^T - \boldsymbol{\beta}\mathbf{f}^T \right\|_F^2 \tag{16}$$
- More conveniently, we can define $\Gamma = [\alpha, \beta]$ and the extended factors $\tilde{\mathbf{F}} = [\mathbf{1}, \mathbf{f}]$. The LS formulation can then be written as
$$\underset{\Gamma}{\text{minimize}} \quad \left\| \mathbf{X}^T - \Gamma\tilde{\mathbf{F}}^T \right\|_F^2 \tag{17}$$
with solution
$$\Gamma = \mathbf{X}^T\tilde{\mathbf{F}}\left(\tilde{\mathbf{F}}^T\tilde{\mathbf{F}}\right)^{-1} \tag{18}$$

Macroeconomic factor models

- Macroeconomic factor models use macroeconomic variables such as
  - Return on the market portfolio
  - Growth rate of the GDP
  - Interest rate on short term Treasury bills or changes in this rate
  - Inflation rate or changes in this rate
  - Interest rate spreads; e.g., difference between long-term Treasury bonds and long-term corporate bonds
- Base: The efficient market hypothesis (another piece of theory in finance) implies that stock prices change because of new information; i.e., stock returns will be influenced by unpredictable changes in macroeconomic variables.
- The factors in a macroeconomic model are not the macroeconomic variables themselves, but rather the residuals when changes in the macroeconomic variables are predicted by a times series model, such as, multivariate AR models

Fundamental factor models: Fama-French

- The Fama-French (FF) Three-factor Model is an extension of the Capital Asset Pricing Model (CAPM)
- The FF model aims to describe stock returns through three factors:
  - (1) Excess return of the market portfolio (from the CAPM model),
  - (2) the outperformance of small-cap companies relative to large-cap companies, and
  - (3) the outperformance of high book-to-market value companies versus low book-to-market value companies
- The rationale behind the model is that high value and small-cap companies tend to regularly outperform the overall market.

Fama-French Three-factor Model

- Developed by University of Chicago professors Eugene Fama and Kenneth French

Expected Rate of Return = Risk-free Rate + Market Risk Premium + SMB + HML

- The multifactor model with the market (R_m), size (SMB), and value (HML) factors is called the 3–factor Fama-French model (Fama and French, 1993)
  - **Small minus big (SMB)**
    Difference in returns on a portfolio of small stocks and a portfolio of large stocks (size refer to the size of the market value)
  - **Hight minus low (HML)**
    Difference in returns of a portfolio of high book(‡)-to-market value (BE/ME) stocks and a portfolio of low BE/ME stocks

Fama-French Three-factor Model

- The mathematical representation of the Fama-French three-factor model is:

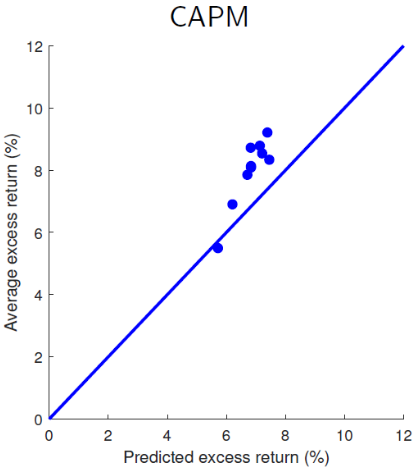
$$E\left[R_{i,t}\right] = \beta_1 E\left[R_m\right] + \beta_2 E\left[SMB\right] + \beta_3 E\left[HML\right] \tag{19}$$

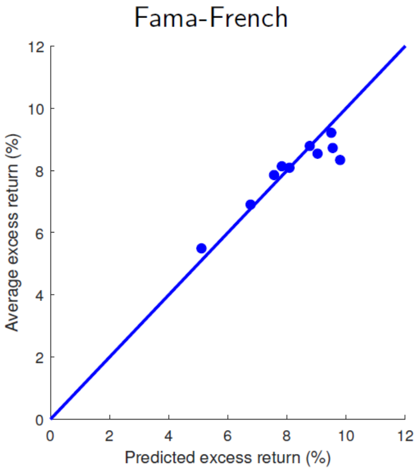
- where the β_i estimates are calculated from the regression

$$R_{i,t} = \alpha_i + \beta_{i,m} R_{m,t} + \beta_{i,SMB} SMB_{i,t} + \beta_{i,HML} HML_{i,t} + u_{i,t} \tag{20}$$

Performance of the Fama-French model

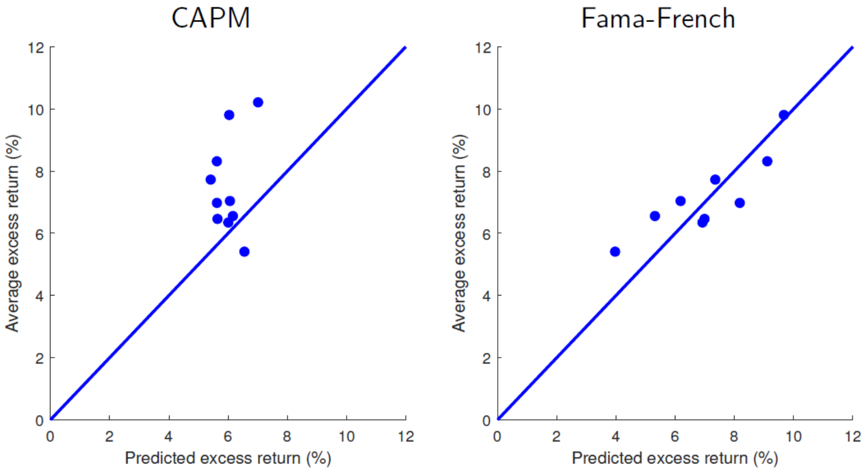
Size portfolios 1963-2018





Performance of the Fama-French model

Book-to-market portfolios 1963-2018



Fama-French Five-factor Model

- Fama and French's initial framework has since undergone many alterations and evolutions as other researchers added their own factors and put their own spin on the duo's insights
- Fama and French updated their model with two more factors to further capture asset returns:
  - **Robust minus weak (RMW)**
    Compares the returns of firms with high, or robust, operating profitability, and those with weak, or low, operating profitability
  - **Conservative minus aggressive (CMA)**
    Gauges the difference between companies that invest aggressively and those that do so more conservatively (investment pattern)

Fama-French Five-factor Model

The mathematical representation of the Fama-French five-factor model is:

$$E\left[R_{i,t}\right] = \beta_1 E\left[R_m\right] + \beta_2 E\left[SMB\right] + \beta_3 E\left[HML\right] + \tag{21}$$
$$\beta_4 E\left[RMW\right] + \beta_3 E\left[CMA\right] \tag{22}$$

where the β_i estimates are calculated from the regression

$$R_{i,t} = \alpha_i + \beta_{i,m} R_{m,t} + \beta_{i,SMB} SMB_{i,t} + \beta_{i,HML} HML_{i,t} + \tag{23}$$
$$\beta_{i,RMW} RMW_{i,t} + \beta_{i,CMA} CMA_{i,t} + u_{i,t} \tag{24}$$

The authors suggest two interpretations of the $\alpha_i = 0$ hypothesis

- - The first proposes that the mean-variance-efficient tangency portfolio, which prices all assets, combines the risk-free asset, the market portfolio, SMB, HML, RMW, and CMA
  - The more ambitious interpretation proposes (23) as the regression equation for a version of Merton's (1973) model in which up to four unspecified state variables lead to risk premiums that are not captured by the market factor

The Carhart (1997) model

The market, size, value, and momentum factors, together, form the Carhart (1997) model:

$$E\left[R_{i,t}\right] = \beta_m E\left[R_m\right] + \beta_{SMB} E\left[SMB\right] + \beta_{HML} E\left[HML\right] + \beta_{WML} E\left[WML\right] \tag{25}$$

with WML the Winner minus Loser factor

The portfolio with the stocks that had the lowest return is called the Looser portfolio. The portfolio with the stocks that had the highest return is called the Winner portfolio.

It assumes that the current winners will continue to outperform the current losers in the future.

Building a portfolio that is long on assets that have outperformed and short on assets that have underperformed

The β_i estimates are calculated from the regression

$$R_{i,t} = \alpha_i + \beta_{i,m} R_{m,t} + \beta_{i,SMB} SMB_{i,t} + \beta_{i,HML} HML_{i,t} + \beta_{WML} WML + u_{i,t} \tag{26}$$

Statistical factor models

- Let's now consider statistical factor models or implicit factor models where both the factors and the loadings are not available
- The common-factor variables $\{\mathbf{f}_t\}$ are hidden (latent) and their structure is deduced from analysis of the observed returns/data $\{\mathbf{x}_t\}$
- The primary methods for extraction of factor structure are:
  - Factor Analysis
  - Principal Components Analysis (PCA)
- Both methods model the , the covariance matrix of $\{\mathbf{x}_t,\ t = 1, \ldots, T\}$
- Recall the principal factor method for the model $\mathbf{X}^T = \boldsymbol{\alpha}\mathbf{1}^T + \boldsymbol{\beta}\mathbf{f}^T + \mathbf{E}^T$ with K factors

Statistical factor models

- Estimation procedures
- 1. PCA
  - sample mean: $\hat{\boldsymbol{\alpha}} = \overline{\mathbf{x}} = \frac{1}{T}\mathbf{X}^T\mathbf{1}^T$
  - de-meaned matrix: $\overline{\mathbf{X}} = \mathbf{X} - \mathbf{1}^T\overline{\mathbf{x}}^T$
  - sample covariance matrix: $\Sigma = \frac{1}{T-1}\overline{\mathbf{X}}^T\overline{\mathbf{X}}$
  - eigen-decomposition: $\Sigma = \hat{\Gamma}\Lambda\hat{\Gamma}^T$
    - $\Lambda =$diagonal matrix of eigenvalues; $\hat{\Gamma} =$ matrix of orthonormal eigenvectors of Σ
- 2.Estimates
  - $B = \hat{\Gamma}_1\Lambda_1^{1/2}$
  - $\Psi = diag\left(\Sigma - \mathbf{B}\mathbf{B}^T\right)$
  - $\Sigma = \mathbf{B}\mathbf{B}^T + \Psi$
- 3. Update the eigen-decomposition: $\Sigma - \Psi = \hat{\Gamma}\Lambda\hat{\Gamma}^T$
- 4. Repeat Steps 2-3 until convergence.

References

- Fama, E., French, K., (2012). Size, value, and momentum in international stock returns. Journal of Financial Economics 105, 457–472
- Fama, Eugene F. and Kenneth R. French (2015). A five-factor asset pricing model. Journal of Financial Economics, 116 1–22