Asset Pricing & Portfolio Management

Risk-Based Portfolios

Jorge Miguel Bravo, PhD

NOVA IMS

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Agenda

1. Introduction to Risk-Based Portfolios
2. Global minimum variance portfolio (GMVP)
3. Inverse volatility portfolio (IVP)
4. Risk parity portfolio (RPP) or equal risk portfolio (ERP)
5. Most diversified portfolio (MDP)
6. Maximum decorrelation portfolio (MDCP)
7. Final remarks

- **Readings**:
  - Bruder, B., & Roncalli, T. (2012). Managing risk exposures using the risk budgeting approach. University Library of Munich, Germany.
  - Qian, E. (2005). Risk parity portfolios: Efficient portfolios through true diversification. PanAgora Asset Management.

Introduction to Risk-based portfolios

- Risk-based portfolios try to bypass the high sensitivity of Markowitz's mean-variance portfolio to the estimation errors of the expected returns by not making use of the expected returns altogether. They are based only on the covariance matrix (Ardia et al. 2017)

- We will explore the following risk-based portfolios:
  - global minimum variance portfolio (GMVP)
  - inverse volatility portfolio (IVP)
  - risk parity portfolio (RPP) or equal risk portfolio (ERP)
  - most diversified portfolio (MDP)
  - maximum decorrelation portfolio (MDCP).

Portfolio optimization & portfolio diversification

Example 1

- We consider an investment universe of 5 assets
- (μ_i, σ_i) are respectively equal to $(8\%, 12\%)$, $(7\%, 10\%)$, $(7.5\%, 11\%)$, $(8.5\%, 13\%)$ and $(8\%, 12\%)$
- The correlation matrix is $\mathcal{C}_5(\rho)$ with $\rho = 60\%$

The optimal mean-variance portfolio $x^\star$ such that $\sigma(x^\star) = 10\%$ is equal to:

$$x^\star = \begin{pmatrix} 23.97\% \\ 6.42\% \\ 16.91\% \\ 28.73\% \\ 23.97\% \end{pmatrix}$$

Portfolio optimization & portfolio diversification

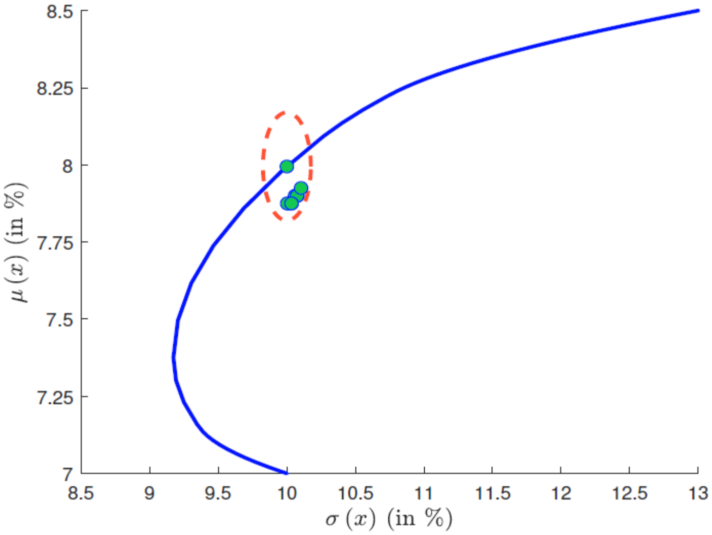


Figure: Optimized portfolios versus optimal diversified portfolios

Portfolio optimization & portfolio diversification

Table: Some equivalent mean-variance portfolios

x_1	23.97		5	5	35	35	50	5	5	10
x_2	6.42	25		25	10	25	10	30		25
x_3	16.91	5	40		10	5	15		45	10
x_4	28.73	35	20	30	5	35	10	35	20	45
x_5	23.97	35	35	40	40		15	30	30	10
$\mu(x)$	7.99	7.90	7.90	7.90	7.88	7.90	7.88	7.88	7.88	7.93
$\sigma(x)$	10.00	10.07	10.06	10.07	10.01	10.07	10.03	10.00	10.03	10.10

$\Rightarrow$ These portfolios have very different compositions, but lead to very close mean-variance features

Some of these portfolios appear more balanced and more diversified than the optimized portfolio

Weight budgeting versus risk budgeting

Let $x = (x_1, \dots, x_n)$ be the weights of n assets in the portfolio. Let $\mathcal{R}(x_1, \dots, x_n)$ be a coherent and convex risk measure. We have:

$$\begin{aligned}
\mathcal{R}(x_1, \dots, x_n) &= \sum_{i=1}^{n} x_i \cdot \frac{\partial\, \mathcal{R}(x_1, \dots, x_n)}{\partial\, x_i} \\
&= \sum_{i=1}^{n} \mathcal{RC}_i(x_1, \dots, x_n)
\end{aligned}$$

Let $b = (b_1, \dots, b_n)$ be a vector of budgets such that $b_i \geq 0$ and $\sum_{i=1}^{n} b_i = 1$. We consider two allocation schemes:

1. Weight budgeting (WB)
$$x_i = b_i$$
2. Risk budgeting (RB)
$$\mathcal{RC}_i = b_i \cdot \mathcal{R}(x_1, \dots, x_n)$$

Application to the volatility risk measure

Let Σ be the covariance matrix of the assets returns. We note x the vector of the portfolio's weights:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

It follows that the portfolio volatility is equal to:

$$\sigma(x) = \sqrt{x^\top \Sigma x}$$

Remark

The decomposition of risk can be defined based on other risk measures, like VaR and CVaR.

Computation of the marginal volatilities

The vector of marginal volatilities (marginal risk contributions) is equal to:

$$\begin{aligned}
\mathcal{MRC}_i &= \frac{\partial\,\sigma\left(x\right)}{\partial\,x} = \begin{pmatrix} \dfrac{\partial\,\sigma\left(x\right)}{\partial\,x_1} \\ \vdots \\ \dfrac{\partial\,\sigma\left(x\right)}{\partial\,x_n} \end{pmatrix} \\
&= \frac{\partial}{\partial\,x}\left(x^\top \Sigma x\right)^{1/2} \\
&= \frac{1}{2}\left(x^\top \Sigma x\right)^{1/2-1}\left(2\Sigma x\right) \\
&= \frac{\Sigma x}{\sqrt{x^\top \Sigma x}}
\end{aligned}$$

It follows that the marginal risk contribution of Asset i is given by:

$$\mathcal{MRC}_i = \frac{\partial\,\sigma\left(x\right)}{\partial\,x_i} = \frac{\left(\Sigma x\right)_i}{\sqrt{x^\top \Sigma x}} = \sum_{j=1}^{n} \frac{\rho_{i,j}\sigma_i\sigma_j x_j}{\sigma\left(x\right)} = \sigma_i \sum_{j=1}^{n} x_j \frac{\rho_{i,j}\sigma_j}{\sigma\left(x\right)}$$

Computation of the risk contributions

We deduce that the risk contribution of the i^{th} asset is then:

$$\begin{aligned}
\mathcal{RC}_i &= x_i \cdot \frac{\partial\, \sigma(x)}{\partial\, x_i} \\
&= \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \\
&= \sigma_i x_i \sum_{j=1}^{n} x_j \frac{\rho_{i,j}\sigma_j}{\sigma(x)}
\end{aligned}$$

The Euler allocation principle

We verify that the volatility satisfies the full allocation property:

$$\begin{aligned}\sum_{i=1}^{n} \mathcal{RC}_i &= \sum_{i=1}^{n} \sigma_i x_i \sum_{j=1}^{n} x_j \frac{\rho_{i,j}\sigma_j}{\sigma(x)} = \frac{1}{\sigma(x)} \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \rho_{i,j}\sigma_i\sigma_j \\ &= \frac{\sigma^2(x)}{\sigma(x)} = \sigma(x)\end{aligned}$$

An alternative proof uses the definition of the dot product:

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = a^\top b$$

Indeed, we have:

$$\sum_{i=1}^{n} \mathcal{RC}_i = \sum_{i=1}^{n} \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = \frac{1}{\sqrt{x^\top \Sigma x}} \sum_{i=1}^{n} x_i \cdot (\Sigma x)_i = \frac{1}{\sqrt{x^\top \Sigma x}} x^\top \Sigma x = \sigma(x)$$

Definition of the risk contribution

Definition

The marginal risk contribution of Asset i is:

$$\mathcal{MRC}_i = \frac{\partial\, \sigma(x)}{\partial\, x_i} = \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

The absolute risk contribution of Asset i is:

$$\mathcal{RC}_i = x_i \frac{\partial\, \sigma(x)}{\partial\, x_i} = \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

The relative risk contribution of Asset i is:

$$\mathcal{RRC}_i = \frac{\mathcal{RC}_i}{\sigma(x)} = \frac{x_i \cdot (\Sigma x)_i}{x^\top \Sigma x}$$

The Euler allocation principle

Remark

We have $\sum_{i=1}^{n} \mathcal{RC}_i = \sigma(x)$ *and* $\sum_{i=1}^{n} \mathcal{RRC}_i = 100\%$.

Application

Example 2

We consider three assets. We assume that their expected returns are equal to zero whereas their volatilities are equal to 30%, 20% and 15%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.80 & 1.00 & \\ 0.50 & 0.30 & 1.00 \end{pmatrix}$$

We consider the portfolio x, which is given by:

$$x = \begin{pmatrix} 50\% \\ 20\% \\ 30\% \end{pmatrix}$$

Application

Using the relationship $\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$, we deduce that the covariance matrix is[1]:

$$\Sigma = \begin{pmatrix} 9.00 & 4.80 & 2.25 \\ 4.80 & 4.00 & 0.90 \\ 2.25 & 0.90 & 2.25 \end{pmatrix} \times 10^{-2}$$

It follows that the variance of the portfolio is:

$$\begin{aligned} \sigma^2(x) &= 0.50^2 \times 0.09 + 0.20^2 \times 0.04 + 0.30^2 \times 0.0225 + \\ &\quad 2 \times 0.50 \times 0.20 \times 0.0480 + 2 \times 0.50 \times 0.30 \times 0.0225 + \\ &\quad 2 \times 0.20 \times 0.30 \times 0.0090 \\ &= 4.3555\% \end{aligned}$$

The volatility is then $\sigma(x) = \sqrt{4.3555\%} = 20.8698\%$.

[1]The covariance term between assets 1 and 2 is equal to $\Sigma_{1,2} = 80\% \times 30\% \times 20\%$ or $\Sigma_{1,2} = 4.80\%$

Application

The computation of the marginal volatilities gives:

$$\frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \frac{1}{20.8698\%} \begin{pmatrix} 6.1350\% \\ 3.4700\% \\ 1.9800\% \end{pmatrix} = \begin{pmatrix} 29.3965\% \\ 16.6269\% \\ 9.4874\% \end{pmatrix}$$

Application

Finally, we obtain the risk contributions by multiplying the weights by the marginal volatilities:

$$x \circ \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \begin{pmatrix} 50\% \\ 20\% \\ 30\% \end{pmatrix} \circ \begin{pmatrix} 29.3965\% \\ 16.6269\% \\ 9.4874\% \end{pmatrix} = \begin{pmatrix} 14.6982\% \\ 3.3254\% \\ 2.8462\% \end{pmatrix}$$

We verify that the sum of risk contributions is equal to the volatility:

$$\sum_{i=1}^{3} \mathcal{RC}_i = 14.6982\% + 3.3254\% + 2.8462\% = 20.8698\%$$

Application

Table: Risk decomposition of the portfolio's volatility (Example 2)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RRC}_i$
1	50.00	29.40	14.70	70.43
2	20.00	16.63	3.33	15.93
3	30.00	9.49	2.85	13.64
$\sigma(x)$			20.87	

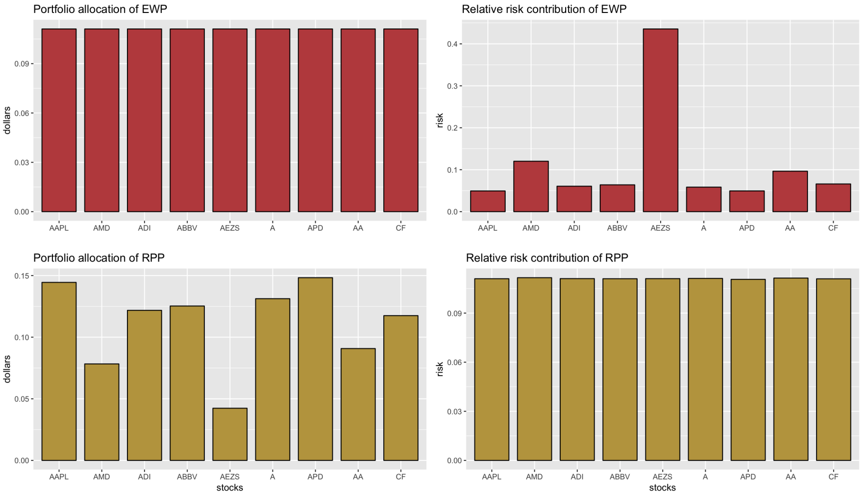
The Risk Parity (RPP) or Equal Risk Contribution (ERC) Portfolio

- The Markowitz mean-variance portfolio has some limitations
  - it focuses on the risk of the portfolio as a whole and less on the risk diversification (i.e., concentrates risk too much in few assets, this was observed in the 2008 financial crisis)
  - it is highly sensitive to the estimation errors in the parameters (i.e., small estimation errors in the parameters may change completely the designed portfolio)
- Since the global financial crisis in 2008, risk management has become more important than performance management in portfolio optimization
- The risk parity portfolio design has been receiving significant attention from both the theoretical and practical sides because it
  - diversifies the risk, instead of the capital, among the assets
  - is less sensitive to parameter estimation errors
- Today, pension funds and institutional investors are using this approach in the development of smart indexing and the redefinition of long-term investment policies

Risk Parity Portfolio

- From "dollar" to risk diversification
- Risk parity is an approach to portfolio management that focuses on allocation of risk rather than allocation of capital
- The risk parity approach asserts that when asset allocations are adjusted to the same risk level, the portfolio can achieve a higher Sharpe ratio and can be more resistant to market downturns
- While the minimum variance portfolio tries to minimize the variance (with the disadvantage that a few assets may be the ones contributing most to the risk), the risk parity portfolio tries to constrain each asset (or asset class, such as bonds, stocks, real estate, etc.) to contribute equally to the portfolio overall volatility
- The term "risk parity" was coined by Edward Qian from PanAgora Asset Management (Qian 2005)

From “dollar” to risk diversification



The RPP portfolio

Definition

- Let Σ be the covariance matrix of asset returns
- The risk measure corresponds to the volatility risk measure
- The ERC or RPP portfolio is the unique portfolio x such that the risk contributions are equal:

$$\mathcal{RC}_i = \mathcal{RC}_j \Leftrightarrow \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = \frac{x_j \cdot (\Sigma x)_j}{\sqrt{x^\top \Sigma x}}$$

ERC = Equal Risk Contribution
RPP = Risk Parity Portfolio

Risk Parity Portfolio (RPP)

- Goal: to allocate the weights so that all the assets contribute the same amount of risk, effectively "equalizing" the risk
- The risk parity portfolio (RPP) or equal risk contribution portfolio (ERC) equalizes the risk contributions:

$$\mathcal{RC}_i = \frac{\sigma\left(\mathbf{x}\right)}{N} \tag{1}$$

or

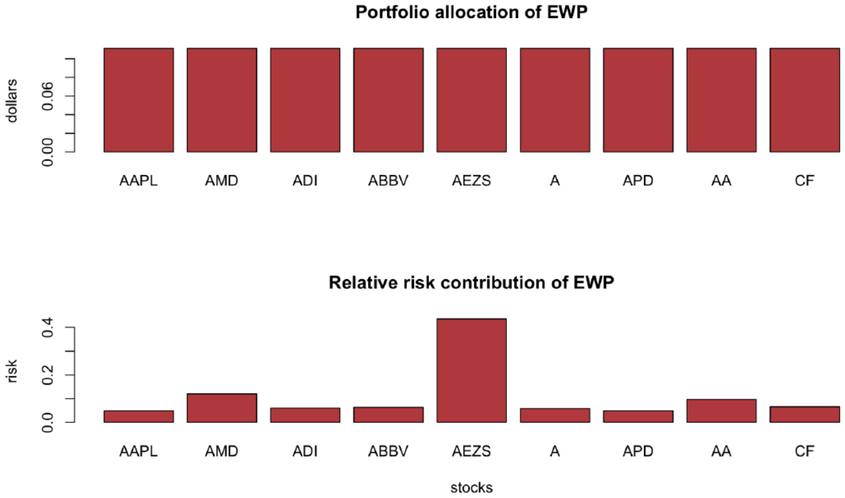
$$\mathcal{RRC}_i = \frac{1}{N} \tag{2}$$

- Note the parallel with the equal weight portfolio (EWP) (aka uniform portfolio):

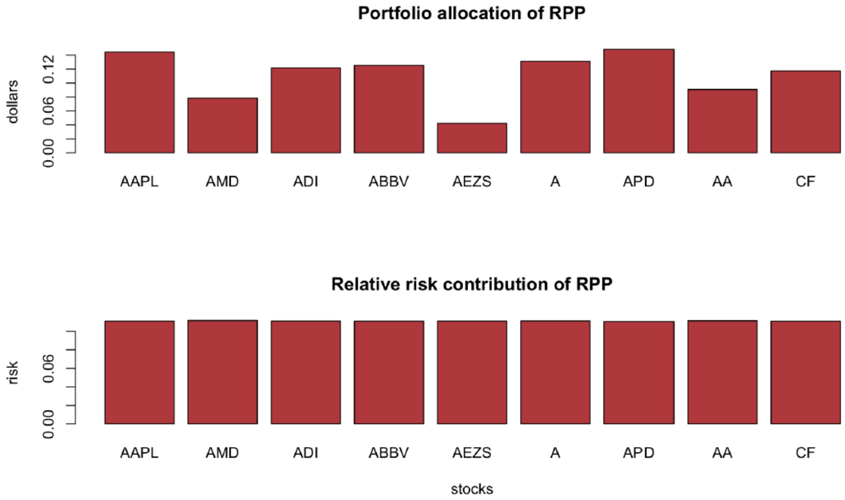
$$x_i = \frac{1}{N}$$

- While the EWP equalizes the capital allocation $x_i = 1/N$, the RPP equalizes the risk allocation $\mathcal{RRC}_i = 1/N$.

Risk contribution of EWP



Risk contribution of RRP



The RPP portfolio

Example 3

- 3 assets
- Volatilities are respectively equal to 20%, 30% and 15%
- Correlations are set to 60% between the 1^{st} asset and the 2^{nd} asset and 10% between the first two assets and the 3^{rd} asset
- Budgets are set to 50%, 25% and 25%
- For the ERC (Equal Risk Contribution) portfolio, all the assets have the same risk budget

The concept of risk budgeting

Example 3

- 3 assets
- Volatilities are respectively equal to 20%, 30% and 15%
- Correlations are set to 60% between the 1st asset and the 2nd asset and 10% between the first two assets and the 3rd asset
- Budgets are set to 50%, 25% and 25%
- For the ERC (Equal Risk Contribution) portfolio, all the assets have the same risk budget

Weight budgeting (or traditional approach)

Asset	Weight	Marginal Risk	Risk Contribution	
			Absolute	Relative
1	**50.00%**	17.99%	9.00%	54.40%
2	**25.00%**	25.17%	6.29%	38.06%
3	**25.00%**	4.99%	1.25%	7.54%
Volatility			16.54%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution	
			Absolute	Relative
1	41.62%	16.84%	7.01%	**50.00%**
2	15.79%	22.19%	3.51%	**25.00%**
3	42.58%	8.23%	3.51%	**25.00%**
Volatility			14.02%	

ERC approach

Asset	Weight	Marginal Risk	Risk Contribution	
			Absolute	Relative
1	30.41%	15.15%	4.61%	**33.33%**
2	20.28%	22.73%	4.61%	**33.33%**
3	49.31%	9.35%	4.61%	**33.33%**
Volatility			13.82%	

The concept of risk budgeting

We have:

$$\sigma\left(50\%, 25\%, 25\%\right) = 16.54\%$$

The marginal risk for the first asset is:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,x_1} = \lim_{\varepsilon \to 0} \frac{\sigma\left(x_1 + \varepsilon, x_2, x_3\right) - \sigma\left(x_1, x_2, x_3\right)}{\left(x_1 + \varepsilon\right) - x_1}$$

If $\varepsilon = 1\%$, we have:

$$\sigma\left(0.51, 0.25, 0.25\right) = 16.72\%$$

We deduce that:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,x_1} \simeq \frac{16.72\% - 16.54\%}{1\%} = 18.01\%$$

whereas the true value is equal to:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,x_1} = 17.99\%$$

The concept of risk budgeting

Example 4

- 3 assets
- Volatilities are respectively 30%, 20% and 15%
- Correlations are set to 80% between the 1st asset and the 2nd asset, 50% between the 1st asset and the 3rd asset and 30% between the 2nd asset and the 3rd asset

Weight budgeting (or traditional) approach

Asset	Weight	Marginal Risk	Risk Contribution	
			Absolute	Relative
1	**50.00%**	29.40%	14.70%	70.43%
2	**20.00%**	16.63%	3.33%	15.93%
3	**30.00%**	9.49%	2.85%	13.64%
Volatility			20.87%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution	
			Absolute	Relative
1	31.15%	28.08%	8.74%	**50.00%**
2	21.90%	15.97%	3.50%	**20.00%**
3	46.96%	11.17%	5.25%	**30.00%**
Volatility			17.49%	

ERC approach

Asset	Weight	Marginal Risk	Risk Contribution	
			Absolute	Relative
1	19.69%	27.31%	5.38%	**33.33%**
2	32.44%	16.57%	5.38%	**33.33%**
3	47.87%	11.23%	5.38%	**33.33%**
Volatility			16.13%	

Risk Budgeting Portfolio (RBP)

- The RPP aims at allocating the total risk evenly across the assets
- More generally, the risk budgeting portfolio (RBP) allocates the risk according to the risk profile determined by the weights b (with $\mathbf{1}'\mathbf{b} = 1$ and $\mathbf{b} \geq 0$):

$$\mathcal{RC}_i = b_i \sigma(\mathbf{x}) \tag{3}$$

or

$$\mathcal{RRC}_i = b_i \tag{4}$$

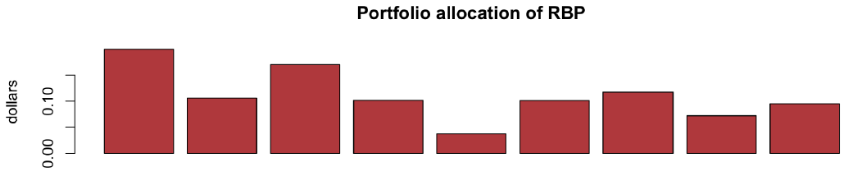
- We can rewrite $\mathcal{RRC}_i = \frac{x_i(\Sigma\mathbf{x})_i}{\mathbf{x}^\top\Sigma\mathbf{x}} = b_i$ simply as

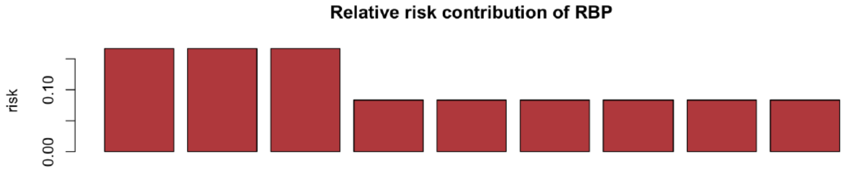
$$x_i(\Sigma\mathbf{x})_i = b_i\mathbf{x}^\top\Sigma\mathbf{x}$$

- Note that RPP is a special case of RBP with $b_i = 1/N$.
- We will consider the more general RBP and we will generally call it RPP with some abuse of terminology

Risk contribution of RBP

- Risk budgeting portfolio with budget $b \propto (2, 2, 2, 1, 1, 1, 1, 1, 1)$:





Properties of the risk based portfolio

Relationship with coefficient beta

We recall that the covariance between the returns of assets and portfolio x is equal to $\Sigma\mathbf{x}$

The beta β_i of asset i with respect to portfolio x is then defined as the ratio between the covariance term $(\Sigma x)_i$ and the variance $\mathbf{x}^\top\Sigma\mathbf{x}$ of the portfolio:

$$\beta_i = \frac{Cov\left(R_i, R\left(x\right)\right)}{\sigma^2\left(x\right)} = \frac{\left(\Sigma x\right)_i}{\sigma^2\left(x\right)} \tag{5}$$

This means that the risk contribution can be written as

$$\mathcal{RC}_i = \frac{x_i\left(\Sigma x\right)_i}{\sigma\left(x\right)} = \sigma\left(x\right) x_i \beta_i$$

Finding the RBP such that $\mathcal{RC}_i = \mathcal{RC}_j$ is equivalent to:

$$x_i \beta_i = x_j \beta_j$$

It follows that:

$$x_i = \frac{\beta_i^{-1}}{\sum_{j=1}^{n} \beta_j^{-1}} \tag{6}$$

Properties of the risk based portfolio

Relationship with coefficient beta

- We notice that:

$$\sum_{i=1}^{n} x_i \beta_i = \sum_{i=1}^{n} \frac{\mathcal{RC}_i}{\sigma(x)} = \frac{1}{\sigma(x)} \sum_{i=1}^{n} \mathcal{RC}_i = 1$$

  and:

$$\sum_{i=1}^{n} x_i \beta_i = \sum_{i=1}^{n} \left(\frac{1}{\sum_{j=1}^{n} \beta_j^{-1}} \right) = 1$$

  It follows that:

$$\frac{1}{\sum_{j=1}^{n} \beta_j^{-1}} = \frac{1}{n}$$

- We finally obtain:

$$x_i = \frac{1}{n\beta_i} \tag{7}$$

Properties of the risk based portfolio

Relationship with coefficient beta

Result

The weight of Asset i is proportional to the inverse of its beta:

$$x_i \propto \beta_i^{-1}$$

Remark

This solution is endogenous since x_i is a function of itself because $\beta_i = \beta\left(\mathbf{e}_i \mid x\right)$.

Solving the Risk Parity Portfolio

Naive diagonal formulation

- Assuming that the assets are uncorrelated, i.e., that Σ is diagonal, and simply using the volatilities $\sigma = diag\left(\Sigma\right)$, one obtains

$$\mathbf{x}_{NRPP} = \frac{\sigma^{-1}}{\mathbf{1}^{\top}\sigma^{-1}} \tag{8}$$

or, more generally,

$$\mathbf{x}_{NRPP} = \frac{\sqrt{\mathbf{b}} \circ \sigma^{-1}}{\mathbf{1}^{\top}\left(\sqrt{\mathbf{b}} \circ \sigma^{-1}\right)} \tag{9}$$

- However, for non-diagonal Σ or with other additional constraints or objective function terms, a closed-form solution does not exist and some optimization procedures have to be constructed
- The previous diagonal solution can always be used and is called naive risk budgeting portfolio

Solving the Risk Parity Portfolio

General non-convex formulation

- The simplest risk budgeting formulation with a simplex constraint set (i.e., $\mathbf{x}^\top \mathbf{1} = 1$ and $\mathbf{x} \geq \mathbf{0}$) is based on a convex reformulation of the problem so they are guaranteed to converge to the optimal risk budgeting solution
- They cannot be used if
  - we have other constraints like allowing shortselling or box constraints: $l_i \leq x_i \leq u_i$
  - on top of the risk budgeting constraints $x_i\left(\Sigma \mathbf{x}\right)_i = b_i \mathbf{x}^\top \Sigma \mathbf{x}$ we have other objectives like maximizing the expected return $\mathbf{x}^\top \boldsymbol{\mu}$ or minimizing the overall variance $\sqrt{\mathbf{x}^\top \Sigma \mathbf{x}}$
- For those more general cases, we need more sophisticated formulations, which unfortunately are not convex
- The idea is to try to achieve equal risk contributions $\mathcal{RC}_i = \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$ by penalizing the differences between the terms $x_i \cdot \left(\Sigma x\right)_i$

Solving the Risk Parity Portfolio

General nonconvex formulation

- There are many reformulations possible
- For illustrative purposes, one such formulation is

$$x^{\star} = \arg\min \sum_{i,j=1}^{N} \left(x_i \cdot (\Sigma x)_i - x_j \cdot (\Sigma x)_j \right)^2 - F(x) \qquad (10)$$

$$\begin{aligned} \text{subject to } x^{\top}\mathbf{1} &= 1 \quad \text{budget} \\ \mathbf{x} &\geq \mathbf{0} \quad \text{no short-sales} \\ \mathbf{x} &\in \mathcal{W} \quad \text{other constraints} \end{aligned}$$

where $F(x)$ denotes some additional objective function, e.g.,

$$F(x) = \lambda_{\mu} \mathbf{x}^{\top} \boldsymbol{\mu} - \lambda_{\mathcal{V}} \sqrt{\mathbf{x}^{\top} \Sigma \mathbf{x}}$$

λ_{μ} and $\lambda_{\mathcal{V}}$ are the trade-off weights for the expected return and the variance terms, respectively, and $\mathcal{W}$ denotes an arbitrary convex set of constraints.

Solving the Risk Parity Portfolio

General nonconvex formulation: alternative formulations

- Fixed flat vector of portfolio weights

$$R\left(x,\theta\right)=\sum_{i=1}^{N}\left(x_i\cdot\left(\Sigma x\right)_i-\theta\right)^2 \tag{11}$$

- Targeting the risk budgets

$$R\left(x,b\right)=\sum_{i=1}^{N}\left(x_i\cdot\left(\Sigma x\right)_i-b_i\right)^2 \tag{12}$$

- Targeting the risk budgets II

$$R\left(x,b_i,b_j\right)=\sum_{i,j=1}^{N}\left(\frac{x_i\cdot\left(\Sigma x\right)_i}{b_i}-\frac{x_j\cdot\left(\Sigma x\right)_j}{b_j}\right)^2 \tag{13}$$

Solving the Risk Parity Portfolio

General nonconvex formulation: alternative formulations

- Risk budgeting portfolio with variance

$$R(x) = \sum_{i=1}^{N} \left(x_i \cdot (\Sigma x)_i - b_i x^\top \Sigma x \right)^2 \tag{14}$$

- Relative risk budgeting portfolio with sd

$$R(x) = \sum_{i=1}^{N} \left(\frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} - b_i \sqrt{x^\top \Sigma x} \right)^2 \tag{15}$$

- Relative risk budgeting portfolio over fixed share

$$R(x) = \sum_{i=1}^{N} \left(\frac{x_i \cdot (\Sigma x)_i}{b_i} - \theta \right)^2 \tag{16}$$

Solving the Risk Parity Portfolio

General nonconvex formulation: alternative formulations

- Risk contribution over variance

$$R(x) = \sum_{i=1}^{N} \left(\frac{x_i \cdot (\Sigma x)_i}{x^\top \Sigma x} \right)^2 \tag{17}$$

Global minimum variance portfolio (GMVP)

- Recall the risk minimization formulation:

$$x^\star = \arg\min \ \sigma_p^2 = \mathbf{x}'\Sigma\mathbf{x} \tag{18}$$

$$\begin{aligned} \text{subject to } \mu_p &= \mathbf{x}^\top \boldsymbol{\mu} \leq \mu_{p,0} \\ \mathbf{x}^\top \mathbf{1} &= 1 \end{aligned}$$

- The GMVP can be seen as a particular case of Markowitz's mean-variance portfolio when the expected return is totally ignored:

$$x^\star_{GMVP} = \arg\min \ \sigma_p^2 = \mathbf{x}'\Sigma\mathbf{x} \tag{19}$$

$$\begin{aligned} \text{subject to } \mathbf{x}^\top \mathbf{1} &= 1 && \text{budget} \\ \mathbf{x} &\geq \mathbf{0} && \text{no short-sales} \end{aligned}$$

- The GMVP solution is

$$x^\star_{GMVP} = \frac{1}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}} \Sigma^{-1} \mathbf{1} \tag{20}$$

Inverse volatility portfolio (IVP)

- The aim of inverse volatility portfolio (IVP) is to control the portfolio risk (risk parity portfolio being a refined version (Qian 2005)).
- The IVP is defined as

$$\mathbf{x} = \frac{\sigma^{-1}}{\mathbf{1}^\top \sigma^{-1}}$$

  where $\sigma^2 = Diag\left(\Sigma\right)$

- Lower weights are given to high volatility assets and higher weights to low volatility assets
- IVP is also called "equal volatility" portfolio since the weighted constituent assets have equal volatility:

$$sd\left(x_i r_i\right) = x_i \sigma_i = \frac{1}{N}$$

- The GMVP when the covariance matrix is diagonal leads to an inverse-variance solution:

$$\mathbf{x} = \frac{\sigma^{-2}}{\mathbf{1}^\top \sigma^{-2}}$$

Most diversified portfolio (MDP)

- Maximum diversification portfolio tries to diversify the holdings across as many assets as possible
- The **diversification ratio** ($\mathcal{DR}$) is defined analogous to the Sharpe ratio ($\mathcal{SR}$) but substituting the weighted return for the weighted volatility:

$$\mathcal{DR} = \frac{\mathbf{x}^\top \sigma}{\sqrt{\mathbf{x}^\top \Sigma \mathbf{x}}} \tag{21}$$

- The term in the numerator is the weighted average volatility of the assets
- The term in the denominator is the volatility of the portfolio
- More diversification within a portfolio decreases the denominator and leads to a higher diversification ratio
- For long-only portfolios, it can be shown that $\mathcal{DR} \geq 1$
- For a single stock, $\mathcal{DR} = 1$.

Most diversified portfolio (MDP)

- The most diversified portfolio (MDP) is obtained as the maximization of DR (akin to the maximization of the Sharpe ratio):

$$x^{\star}_{MDP} = \arg\max \ \mathcal{DR} = \frac{\mathbf{x}^{\top}\sigma}{\sqrt{\mathbf{x}^{\top}\Sigma\mathbf{x}}} \tag{22}$$

$$\begin{aligned} \text{subject to } \mathbf{x}^{\top}\mathbf{1} &= 1 \quad \text{budget} \\ \mathbf{x} &\geq \mathbf{0} \quad \text{no short-sales} \end{aligned}$$

Most diversified portfolio (MDP)

- The MDP has some interesting properties:
  - the correlation of some portfolio $\mathbf{x}$ with the MDP $\mathbf{x}_{MDP}$ is proportional to the $\mathcal{DR}$ of the portfolio:
$$\rho = \frac{\mathcal{DR}(\mathbf{x})}{\mathcal{DR}(\mathbf{x}_{MDP})} \tag{23}$$
  - as a consequence, all the assets in the MDP have the same positive correlation to the MDP,
  - also, any stock not held by the MDP is more correlated to the MDP than any of the stocks that belong to it (this illustrates that all assets in the universe considered are effectively represented in the MDP, even if the portfolio does not physically hold them);
  - if all the stocks in the universe have the same volatility, then the MDP is equivalent to the GMVP;
  - the squared DR can be interpreted as the effective number of independent risk factors in the portfolio (Choueifaty et al. 2013),
  - as a consequence, the MDP has a DR equal to the square root of the effective number of independent risk factors available in the entire market (which is typically larger than the market index)

Maximum decorrelation portfolio (MDCP)

- The maximum decorrelation portfolio (MDCP) (Christoffersen et al. 2012) is closely related to GMVP and MDP, but applies to the case where an investor believes all assets have similar returns and volatility, but heterogeneous correlations.
- It is a Minimum Variance optimization that is performed on the correlation matrix rather than the covariance matrix
- The MDCP is formulated as

$$x^{\star}_{MDCP} = \arg\min \; \mathbf{x}^{\top} C \mathbf{x} \tag{24}$$

$$\text{subject to } \mathbf{x}^{\top}\mathbf{1} = 1 \quad \text{budget}$$

where $C \triangleq Diag\left(\Sigma\right)^{-\frac{1}{2}} \Sigma Diag\left(\Sigma\right)^{-\frac{1}{2}}$ is the correlation matrix

- Interestingly, when the weights derived from the MDCP are divided by their respective volatilities and re-standardized so that they sum to 1, we retrieve the MDP weights.
- The MDCP happen to
  - maximize the DR when all assets have equal volatility and
  - maximize the SR when all assets have equal risks and returns

Risk-based portfolios

Thank you for your attention

Jorge Miguel Bravo
jbravo@novaims.unl.pt
Universidade Nova de Lisboa, NOVA IMS & MagIC
Université Paris-Dauphine PSL
CEFAGE-UE & BRU-ISCTE-IUL
Scopus ID: 57193745638
ORCID: 0000-0002-7389-5103