Asset Pricing & Portfolio Management

Markowitz's Modern Portfolio Theory

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Contents

Readings: Yiyon & Palomar (2016), Chapter 5; Pfaff (2016), Chapter 5

Introduction

- The modern portfolio theory (MPT) is a practical method for selecting investments in order to maximize their overall returns within an acceptable level of risk
- American economist Harry Markowitz pioneered this theory in his paper "Portfolio Selection", published in 1952. He was later awarded a Nobel Prize in economics
- MPT key principles
  - Investors are risk-averse and demand a reward for engaging in risky investments. The reward is taken as a **risk premium**, the difference between the expected rate of return and that available on alternative risk-free investments
  - There are investors' personal trade-offs between portfolio risk and expected return
  - We cannot evaluate the risk of an asset separate from the portfolio of which it is a part, i.e., the proper way to measure the risk of an individual asset is to assess its impact on the volatility of the entire portfolio of investments
  - **Diversification** → An investor can reduce portfolio unsystematic risk by holding combinations of instruments that are not perfectly positively correlated; systemic or market risk is generally unavoidable

The Markowitz framework

- We consider a universe of n assets
- $\mathbf{w} = (w_1, \ldots, w_n)$ is the vector of weights in the portfolio
- The portfolio is fully invested:

$$\sum_{i=1}^{n} w_i = \mathbf{w}^\top \mathbf{1}_n = 1$$

- $R = (R_1, \ldots, R_n)$ is the vector of asset returns where R_i is the return of asset i
- The return of the portfolio is equal to:

$$R(w) = \sum_{i=1}^{n} w_i R_i = w^\top R$$

- $\mu = \mathbb{E}[R]$ and $\Sigma = \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right]$ are the vector of expected returns and the covariance matrix of asset return

The Markowitz framework

- The expected return of the portfolio is:

$$\mu(w) = \mathbb{E}[R(w)] = \mathbb{E}\left[w^\top R\right] = w^\top \mathbb{E}[R] = w^\top \mu$$

whereas its variance is equal to:

$$\begin{aligned}
\sigma^2(w) &= \mathbb{E}\left[(R(w) - \mu(w))(R(w) - \mu(w))^\top\right] \\
&= \mathbb{E}\left[\left(w^\top R - w^\top \mu\right)\left(w^\top R - w^\top \mu\right)^\top\right] \\
&= \mathbb{E}\left[w^\top (R - \mu)(R - \mu)^\top w\right] \\
&= w^\top \mathbb{E}\left[(R - \mu)(R - \mu)^\top\right] w \\
&= w^\top \Sigma w
\end{aligned}$$

Portfolio variance

- Basic properties of individual returns

$$\begin{aligned} \text{Mean} &= \mathbb{E}\left(R_j\right) = \mu_j \\ \text{Variance} &= Var\left(R_j\right) = \mathbb{E}\left[\left(R_j - \mu_j\right)^2\right] = \sigma_j^2 \\ \text{Standard deviation} &= \sqrt{Var\left(R_j\right)} = \sigma_j \end{aligned}$$

- The portfolio variance using matrix notation is

$$\sigma^2\left(w\right) = w^\top \Sigma w \tag{1}$$

with Σ the covariance matrix of asset return

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_N^2 \end{bmatrix}$$

Portfolio variance

- Alternatively, the portfolio variance can be computed as

$$\begin{aligned}\sigma^2(w) &= \sum_{j=1}^{N} w_j^2\sigma_j^2 + \sum_{j\neq i}^{N} w_j w_i \sigma_{ji} \\ &= \sum_{j=1}^{N} w_j^2\sigma_j^2 + \sum_{j\neq i}^{N} w_j w_i \rho_{ji}\sigma_j\sigma_i\end{aligned} \tag{2}$$

- Two asset example

$$\begin{aligned}\sigma^2(w) &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2\end{aligned} \tag{3}$$

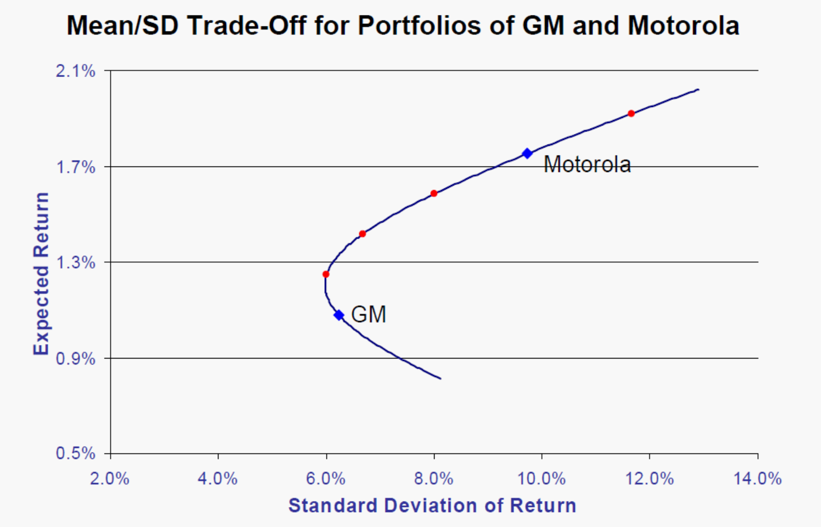
Markowitz's mean-variance trade-off

Example: From 1946 – 2001, Motorola had an average monthly return of 1.75% and a std dev of 9.73%. GM had an average return of 1.08% and a std dev of 6.23%. Their correlation is 0.37. How would a portfolio of the two stocks perform?

$$\begin{aligned} \mathrm{E}[R_p] &= \omega_{\mathrm{GM}}\, 1.08 + \omega_{\mathrm{MOT}}\, 1.75 \\ \mathrm{Var}[R_p] &= \omega_{\mathrm{GM}}^2\, 6.23^2 + \omega_{\mathrm{MOT}}^2\, 9.73^2 + \\ &\qquad 2\omega_{\mathrm{GM}}\omega_{\mathrm{MOT}}\, (0.37 \times 6.23 \times 9.73) \end{aligned}$$

w_{Mot}	w_{GM}	$E[R_P]$	$var(R_P)$	$stdev(R_P)$
0	**1**	1.08	38.8	6.23
0.25	**0.75**	1.25	36.2	6.01
0.50	**0.50**	1.42	44.6	6.68
0.75	**0.25**	1.58	64.1	8.00
1	**0**	1.75	94.6	9.73
1.25	**-0.25**	1.92	136.3	11.67

Markowitz's mean-variance trade-off



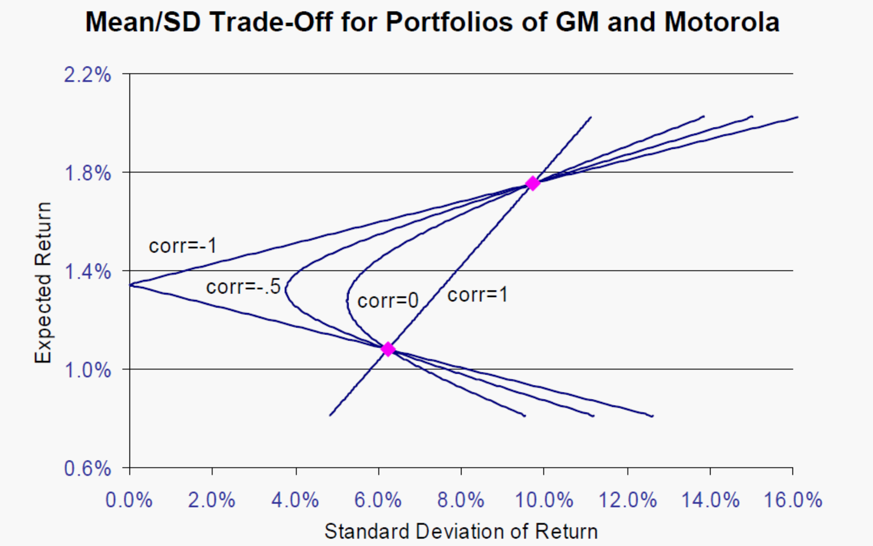
Markowitz's mean-variance trade-off

Example (cont): Suppose the correlation between GM and Motorola changes. What if it equals –1.0? 0.0? 1.0?

$$\begin{aligned} \mathsf{E}[R_p] &= \omega_{\mathsf{GM}} 1.08 + \omega_{\mathsf{MOT}} 1.75 \\ \mathsf{Var}[R_p] &= \omega^2_{\mathsf{GM}} 6.23^2 + \omega^2_{\mathsf{MOT}} 9.73^2 + \\ & \qquad 2\omega_{\mathsf{GM}}\omega_{\mathsf{MOT}} (\rho_{\mathsf{GM,MOT}} \times 6.23 \times 9.73) \end{aligned}$$

			Std dev of portfolio		
w_{Mot}	w_{GM}	$E[R_P]$	corr = -1	corr = 0	corr = 1
0	1	1.08%	6.23%	6.23%	6.23%
0.25	0.75	1.25	2.24	5.27	7.10
0.50	0.50	1.42	1.75	5.78	7.98
0.75	0.25	1.58	5.74	7.46	8.85
1	0	1.75	9.73	9.73	9.73

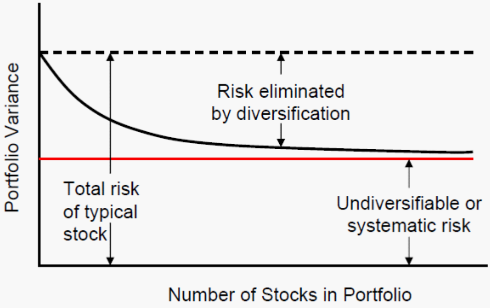
Markowitz's mean-variance trade-off



Markowitz's mean-variance trade-off

Eventually, Diversification Benefits Reach A Limit:

- Remaining risk known as **systematic** or **market risk**
- Due to common factors that cannot be diversified
- Example: S&P 500
- Other sources of systematic risk may exist:
  - Credit
  - Liquidity
  - Volatility
  - Business Cycle
  - Value/Growth
- Provides motivation for **linear factor models**



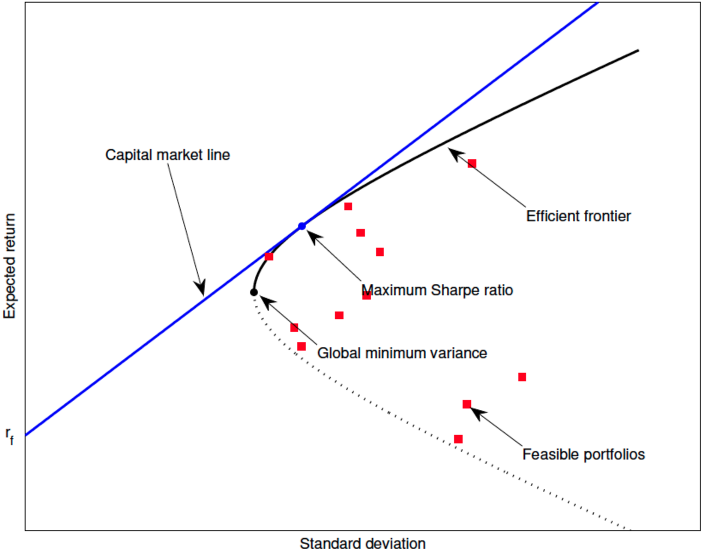
Markowitz's mean-variance trade-off

- MPT assumes returns are covariance stationary and jointly normally distributed over the investment horizon
- This implies that means, variances and covariances of returns are constant over the investment horizon and completely characterize the joint distribution of returns
- MPT assumes that investors are risk averse
- This means that given two portfolios that offer the same expected return $\mu_p = w^\top \boldsymbol{\mu}$, investors will prefer the less risky one (i.e., the one with lower variance $\sigma_p^2 = w^\top \Sigma w$ or standard deviation $\sigma_p = \sqrt{w^\top \Sigma w}$)

Markowitz's mean-variance trade-off

- The exact trade-off will not be the same for all investors
- Different investors will evaluate the trade-off differently based on individual risk aversion characteristics (preferences)
- Usually, the higher the mean return the higher the variance and vice-versa
- Thus, the investor is faced with two objectives to be optimized: it is a multi-objective optimization problem
- They define a fundamental mean-variance tradeoff curve (Pareto curve)
- A solution is called **Pareto optimal** if none of the objective functions can be improved without degrading some of the other objective values
- The choice of a specific point in this tradeoff curve depends on how agressive or risk-averse the investor is

Efficient frontier



Efficient frontier

- The **investment opportunity set** is the set of portfolio expected return and portfolio standard deviation values for all possible portfolios whose weights sum to one
- If we assume that investors choose portfolios to maximize expected return subject to a target level of risk, or, equivalently, to minimize risk subject to a target expected return, then we can simplify the asset allocation problem by only concentrating on the set of **efficient portfolios**
- Markowitz assumed that investors wish to find portfolios that have the best expected return-risk trade-off
- Markowitz characterized these efficient portfolios in two equivalent ways:
  - **A.** Investors seek to find portfolios that maximize portfolio expected return for a given level of risk as measured by portfolio variance, $\sigma^2_{p,0}$. The constrained maximization problem to find an efficient portfolio as

$$w^\star = \arg\max w^\top \mu \tag{4}$$

$$\text{s.t. } \sigma_p^2 = w^\top \Sigma w \leq \sigma^2_{p,0} \text{ and } \mathbf{w}^\top \mathbf{1} = 1$$

where $\mathbf{w}^\top \mathbf{1} = 1$ is the capital budget constraint

Efficient frontier

- Markowitz characterized these efficient portfolios in two equivalent ways:
  - **B.** The dual problem: the investor's problem of maximizing portfolio expected return subject to a target level of risk has an equivalent dual representation in which investors minimize the risk of the portfolio subject to a target expected return level, $\mu_{p,0}$

$$w^\star = \arg\min w^\top \Sigma w \qquad (5)$$
$$\text{s.t. } \mu_p = w^\top \boldsymbol{\mu} \leq \mu_{p,0} \text{ and } w^\top \mathbf{1} = 1$$

- To find efficient portfolios of risky assets in practice, the dual problem is most often solved because of computational conveniences and the fact that investors are more willing to specify target expected returns rather than target risk levels
- To solve the constrained minimization problem (5), we first form the Lagrangian function

$$\mathcal{L}(x, \lambda_1, \lambda_2) = w^\top \Sigma w + \lambda_1 \left(w^\top \boldsymbol{\mu} - \mu_{p,0} \right) + \lambda_2 \left(w^\top \mathbf{1} - 1 \right) \qquad (6)$$

Markowitz's trick

- Markowitz transforms the two original non-linear optimization problems into a quadratic optimization problem:

$$w^{\star}(\phi) = \arg\max w^{\top}\mu - \frac{\phi}{2} w^{\top}\Sigma w \quad \text{s.t.} \quad w^{\top}\mathbf{1} = 1 \tag{7}$$

where ϕ is a risk-aversion parameter:
- $\phi = 0 \Rightarrow$ we have $\mu(w^{\star}(0)) = w^{+}$
- If $\phi = \infty$, the optimization problem becomes

$$w^{\star}(\infty) = \arg\min \frac{1}{2} w^{\top}\Sigma w \quad \text{s.t.} \quad w^{\top}\mathbf{1} = 1 \tag{8}$$

$\Longrightarrow$ we have $\sigma(w^{\star}(\infty)) = \sigma^{-}$. This is the minimum variance portfolio (MVP)

-
  - A higher ϕ value means the investor will have high defense against risk at the expense of lower returns and keeping a lower value will place higher emphasis on maximising returns, neglecting the risk associated with it
  - This is often referred to as quadratic risk utility: $U(R) = E(R) - \frac{\phi}{2} Var(R)$

Markowitz's mean-variance portfolio: Analytical solution

- The Lagrange function is:

$$\mathcal{L}\left(w;\lambda_0\right) = w^\top \mu - \frac{\phi}{2} w^\top \Sigma w + \lambda_0 \left(w^\top \mathbf{1} - 1\right)$$

The first-order conditions are:

$$\begin{cases} \partial_w \mathcal{L}\left(w;\lambda_0\right) = \mu - \phi\Sigma w + \lambda_0 \mathbf{1} = \mathbf{0} \\ \partial_{\lambda_0} \mathcal{L}\left(w;\lambda_0\right) = w^\top \mathbf{1} - 1 = 0 \end{cases}$$

We obtain:

$$w = \phi^{-1}\Sigma^{-1}\left(\mu + \lambda_0 \mathbf{1}_n\right)$$

Because $w^\top \mathbf{1} - 1 = 0$, we have:

$$\mathbf{1}_n^\top \phi^{-1}\Sigma^{-1}\mu + \lambda_0 \left(\mathbf{1}_n^\top \phi^{-1}\Sigma^{-1}\mathbf{1}_n\right) = 1$$

Markowitz's mean-variance portfolio: Analytical solution

It follows that:

$$\lambda_0 = \frac{1 - \mathbf{1}_n^\top \phi^{-1} \Sigma^{-1} \mu}{\mathbf{1}_n^\top \phi^{-1} \Sigma^{-1} \mathbf{1}_n}$$

The closed-form solution is then:

$$w^\star(\phi) = \frac{\Sigma^{-1} \mathbf{1}_n}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n} + \frac{1}{\phi} \cdot \frac{\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right) \Sigma^{-1} \mu - \left(\mathbf{1}_n^\top \Sigma^{-1} \mu\right) \Sigma^{-1} \mathbf{1}_n}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}$$

Markowitz's portfolio with practical constraints

- A general Markowitz's portfolio with practical constraints could be:

$$w^{\star}(\phi) = \arg\max \; w^{\top}\boldsymbol{\mu} - \frac{\phi}{2} w^{\top}\Sigma w \tag{9}$$

$$\begin{aligned} \text{subject to } w^{\top}\mathbf{1} &= 1 && \text{budget} \\ \mathbf{w} &\geq \mathbf{0} && \text{no short-sales} \\ \|\mathbf{w}\|_{l} &\leq \gamma && \text{leverage} \\ \|\mathbf{w} - \mathbf{w}_0\| &\leq \tau && \text{turnover} \\ \|\mathbf{w}\|_{\infty} &\leq u && \text{max position} \\ \|\mathbf{w}\|_{0} &\leq K && \text{sparsity} \end{aligned}$$

where:

- $\gamma \geq 1$ controls the amount of shorting and leveraging; $\gamma = 1$ means no shorting; $\gamma > 1$ allows some shorting as well as leverage in the longs, e.g., $\gamma = 1.5$ would allow the portfolio $w = (1.25, -0.25)$
- $\tau > 0$ controls the turnover (i.e., transaction costs in the rebalancing)
- u limits the position in each stock
- K controls the cardinality of the portfolio (to select a small set of stocks)

Global minimum variance portfolio (GMVP)

- Recall the risk minimization formulation:

$$w^\star = \arg\min \quad \frac{1}{2} w^\top \Sigma w \tag{10}$$

$$\begin{aligned} \text{s.t.} \quad \mu_p &= w^\top \boldsymbol{\mu} \leq \mu_{p,0} \\ w^\top \mathbf{1} &= 1 \end{aligned}$$

- The global minimum variance portfolio (GMVP) ignores the expected return and focuses on the risk only:

$$w^\star = \arg\min \quad \frac{1}{2} w^\top \Sigma w \tag{11}$$

$$\mathbf{w} \geq \mathbf{0} \qquad \text{no short-sales}$$

- The GMVP solution is

$$\mathbf{w}_{GMVP} = w^\star(\infty) = \frac{1}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} \Sigma^{-1}\mathbf{1} \tag{12}$$

Global minimum variance portfolio (GMVP)

with leverage constraints

- The GMVP is typically considered with no-short constraints $\mathbf{w} \geq \mathbf{0}$:

$$w^{\star} = \arg\min \frac{1}{2} w^{\top} \Sigma w \tag{13}$$

$$\begin{aligned} \text{subject to } w^{\top}\mathbf{1} &= 1 \quad \text{budget} \\ \mathbf{w} &\geq \mathbf{0} \quad \text{no short-sales} \end{aligned}$$

- However, if short-selling is allowed, one needs to limit the amount of leverage to avoid impractical solutions with very large positive and negative weights that cancel out. A sensible GMVP formulation with leverage is

$$w^{\star} = \arg\min \frac{1}{2} w^{\top} \Sigma w \tag{14}$$

$$\begin{aligned} \text{subject to } w^{\top}\mathbf{1} &= 1 \quad \text{budget} \\ \|\mathbf{w}\|_{l} &\leq \gamma \quad \text{leverage} \end{aligned}$$

where $\gamma = 1$ means no shorting (so equivalent to $\mathbf{w} \geq \mathbf{0}$)

The tangency portfolio

Markowitz

There are many optimized portfolios $\Rightarrow$ there are many optimal portfolios

Tobin

One optimized portfolio dominates all the others if there is a risk-free asset

Capital asset pricing model (CAPM)

- We consider a combination of the risk-free asset and a portfolio w:

$$R\left(y\right) = \left(1-\alpha\right) r_f + \alpha R\left(w\right)$$

where:

  - r_f is the return of the risk-free asset
  - $y = \begin{pmatrix} \alpha w \\ 1-\alpha \end{pmatrix}$ is a vector of dimension $(n+1)$
  - $\alpha \geq 0$ is the proportion of the wealth invested in the risky portfolio

- It follows that:

$$\mu\left(y\right) = \left(1-\alpha\right) r_f + \alpha\mu\left(w\right) = r_f + \alpha\left(\mu\left(w\right) - r_f\right)$$

and

$$\sigma^2\left(y\right) = \alpha^2\sigma^2\left(w\right)$$

We deduce that:

$$\mu\left(y\right) = r_f + \frac{\left(\mu\left(w\right) - r_f\right)}{\sigma\left(w\right)}\sigma\left(y\right) \tag{15}$$

CAPM - The tangency portfolio

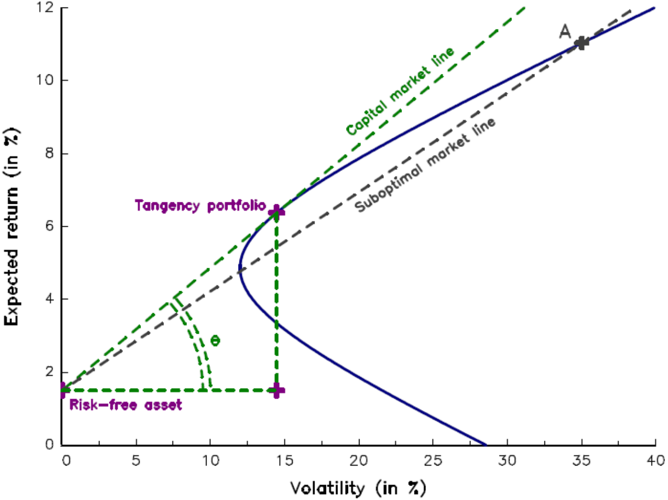


Figure 3: The capital market line ($r = 1.5\%$)

CAPM - The tangency portfolio

- Let $\mathrm{SR}\left(w \middle| r_f\right)$ be the Sharpe ratio of portfolio w:

$$\mathrm{SR}\left(w \middle| r_f\right) = \frac{\mu\left(w\right) - r_f}{\sigma\left(w\right)}$$

We obtain:

$$\frac{\mu\left(y\right) - r_f}{\sigma\left(y\right)} = \frac{\mu\left(w\right) - r_f}{\sigma\left(w\right)} \Leftrightarrow \mathrm{SR}\left(y \middle| r_f\right) = \mathrm{SR}\left(w \middle| r_f\right)$$

- The tangency portfolio is the one that maximizes the angle θ (the Sharpe ratio) or equivalently $\tan\theta$:

$$\tan\theta = \mathrm{SR}\left(w \middle| r_f\right) = \frac{\mu\left(w\right) - r_f}{\sigma\left(w\right)}$$

Maximum Sharpe ratio portfolio (MSRP)

- Markowitz's mean-variance framework provides portfolios along the Pareto-optimal frontier and the choice depends on the risk-aversion of the investor
- Consider now that in addition to the the n risky assets there is one risk-free asset (T-Bill) with return r_f
- In this case, all efficient portfolios are combinations of the riskless asset and a unique portfolio of risky assets called the tangency portfolio
- The tangency portfolio is the portfolio of risky assets that has the highest Sharpe ratio
- Sharpe (1966) first proposed the maximization of the Sharpe ratio:

$$w^{\star} = \arg\max \frac{w^{\top}\boldsymbol{\mu} - r_f}{\sqrt{w^{\top}\Sigma w}} \tag{16}$$

$$\begin{aligned} \text{s.t. } w^{\top}\mathbf{1} &= 1 \quad \text{budget} \\ \mathbf{w} \geq \mathbf{0} & \quad\quad \text{no short-sales} \end{aligned}$$

The capital market line (CML)

- If all investors hold the tangency portfolio M, then that portfolio must include the risky assets in exactly the same proportion as they exist in the market $\rightarrow$ Portfolio M is the **market portfolio**
- The efficient frontier is a straight line that passes through points $(0, r_f)$ and $(\sigma_M, E(R_M))$

$$E\left(R_P\right) = r_f + \left(\frac{w^\top \boldsymbol{\mu} - r_f}{\sqrt{w^\top \Sigma w}}\right)\sigma_P \tag{17}$$

- The location of the tangency portfolio, and the sign of the Sharpe ratio, depends on the relationship between the risk-free rate and the expected return on the global minimum variance portfolio
  - If $w^\top \boldsymbol{\mu} - r_f > 0$ which is the usual case, then the tangency portfolio with have a positive Sharpe ratio
  - If $w^\top \boldsymbol{\mu} - r_f < 0$ (stock prices falling and the economy is in a recession), then the tangency portfolio will have a negative Sharpe slope. In this case, efficient portfolios involve shorting the tangency portfolio and investing the proceeds in T-Bills

Tobin's "separation theorem"

- James Tobin ... in a 1958 paper said if you hold risky securities and are able to borrow - buying stocks on margin - or lend - buying risk-free assets - and you do so at the same rate, then the efficient frontier is a single portfolio of risky securities plus borrowing and lending....

- **Tobin's Separation Theorem**

Theorem

Investors can separate the problem into first finding that optimal combination of risky securities and then deciding whether to lend or borrow, depending on your attitude toward risk. It then showed that if there's only one portfolio plus borrowing and lending, it's got to be the market

- The **optimal portfolio** is defined as the efficient portfolio that maximizes the investor expected utility

CAPM - The Security Market Line (SML)

- In equilibrium, what is the relation between expected return and risk for individual stocks?
  - Individual stocks are below CML
  - This relation is named Security Market Line (SML)
- On the efficient frontier, we have:

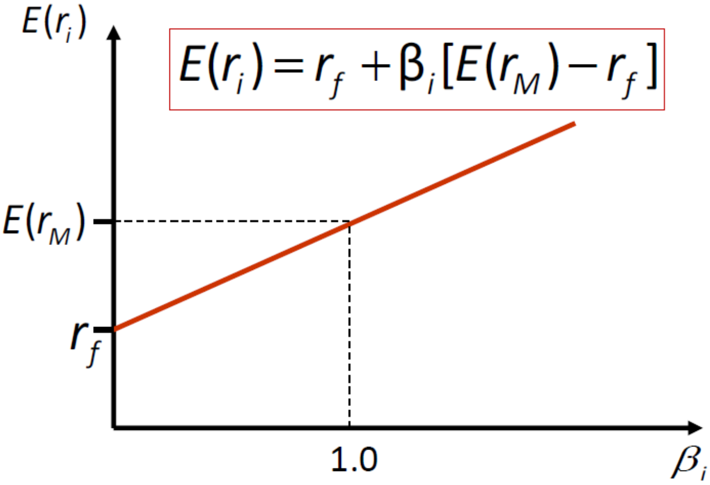
$$\mu(y) = r + \frac{\sigma(y)}{\sigma(w^\star)}\left(\mu(w^\star) - r_f\right)$$

- We consider a portfolio z with a proportion w invested in the asset i and a proportion $(1-w)$ invested in the tangency portfolio $w^\star$, it can shown the the Security Market Line (SML) is given by

$$\begin{aligned} E(R_i) &= r_f + \beta_i\left(E\left(R_M - r_f\right)\right) \\ \beta_i &= \frac{Cov(R_i, R_M)}{Var(R_M)} \end{aligned}$$

- β_i measures the responsiveness of a stock to movements in the market portfolio (i.e., systematic risk)

CAPM - The Security Market Line (SML)



Drawbacks of Markowitz's formulation

- Variance is not a good measure of risk in practice since it penalizes both the unwanted high losses and the desired low losses: the solution is to use alternative measures for risk, e.g., VaR and CVaR,
- Variance is highly sensitive to parameter estimation errors (i.e., to the covariance matrix Σ and especially to the mean vector $\boldsymbol{\mu}$): solution is robust optimization and improved parameter estimation,
- it only considers the risk of the portfolio as a whole and ignores the risk diversification (i.e., concentrates risk too much in few assets, this was observed in the 2008 financial crisis): solution is the risk parity portfolio.
- Alternative measures for risk
  - Downside risk (DR)
  - Value-at-Risk (VaR)
  - Conditional VaR (CVaR) or Expected Shortfall (ES)
  - Drawdown (DD)

Downside risk (DR)

- The idea of downside risk is that the left-handside of the return distribution involves risk while the right-handside contains the better investment opportunities
- One example is the semi-variance, already considered by Markowitz (1959)
- The semi-variance measures the variability of the returns below the mean; It is defined as

$$SV = \mathbb{E}\left[\left((\mu - R)^{+}\right)^{2}\right] \tag{18}$$

  where $\mu = \mathbb{E}(R)$ is the mean return and $(\cdot)^{+} = \max(0; \cdot)$
- The semi-variance is a special case of the more general lower partial moments (LPM):

$$LPM = \mathbb{E}\left[\left((\xi - R)^{+}\right)^{\alpha}\right] \tag{19}$$

  where ξ is is termed the disaster level.
- The parameter α reflects the investor's feeling about the relative consequences of falling short of ξ by various amounts:
  - the value $\xi = 1$ (which suits a neutral investor) separates risk-seeking $(0 < \alpha < 1)$ from risk-averse $(\alpha > 1)$ behavior with regard to returns below the target ξ

Mean Semi-Variance portfolio (MSVP)

- The mean - semi-variance portfolio (MSVP) formulation is the convex QP problem

$$\underset{w}{\text{maximize}} \quad w^\top \boldsymbol{\mu} - \frac{\lambda}{T} \sum_{t=1}^{T} \left(\left(w^\top \boldsymbol{\mu} - w^\top R_t \right)^+ \right)^2 \tag{20}$$

$$\begin{aligned} \text{subject to } w^\top \mathbf{1} &= 1 \quad \text{budget} \\ \mathbf{w} &\geq \mathbf{0} \quad \text{no short-sales} \end{aligned}$$

Performance criteria

- **Annual return**: the (geometric) annualized return
- **Annual volatility**: the annualized standard deviation of returns
- **Max drawdown**: the maximum drawdown is defined as the maximum loss from a peak to a trough of a portfolio
- **Sharpe ratio**: the annualized Sharpe ratio, the ratio between the (geometric) annualized return and the annualized standard deviation;
- **Sterling ratio**: the return over average drawdown computed e.g., as

$$\text{Sterling Ratio} = \frac{\text{Annualized return}}{\text{Max drawdown}} \tag{21}$$

- **Sortino ratio**: variation of the Sharpe ratio that differentiates harmful volatility from total overall volatility by using the asset's standard deviation of negative portfolio returns—downside deviation—instead of the total standard deviation of portfolio returns

$$\text{Sortino Ratio} = \frac{R_P - r_F}{\sigma_D} \tag{22}$$

where σ_D is the standard deviation of the downside

Performance criteria

- **Omega ratio**: the probability weighted ratio of gains over losses for some threshold return target (Keating and Shadwick, 2002); The ratio is calculated as:

$$\Omega\left(r\right)=\frac{\int_{r}^{\infty}\left(1-F\left(x\right)\right)dx}{\int_{-\infty}^{r}F\left(x\right)dx} \tag{23}$$

  where $F\left(x\right)$ is the cumulative probability distribution function and r is the target return threshold defining what is considered a gain versus a loss; if $r=0$, $\Omega\left(0\right)$ is known as Gain-Loss-Ratio

- **Return over Turnover (ROT)**: is defined as the sum of cummulative return over the sum of turnover.

$$ROT=\frac{\text{Cumulative } PnL\left(T\right)}{\sum_{i=1}^{T}B\left\|w_i-w_{i-1}\right\|} \tag{24}$$

Performance criteria

- **Value-at-Risk (VaR):** calculates the maximum loss expected (or worst case scenario) on an investment, over a given time period and given a specified degree of confidence α
the VaR is a quantile of the loss distribution, and in particular the $\alpha-$quantile for which

$$VaR_{\alpha}(L) = \inf \{ l \in \mathbb{R} : P(L > l) \leq 1 - \alpha \} \tag{25}$$

In words: the VaR is the "loss value for which the probability of observing a larger loss, given the available information, is equal to $1 - \alpha$

- **Expected Shortfall** (ES), also called conditional value at risk (CVaR), average value at risk (AVaR), expected tail loss (ETL)): measures the average value of a loss in an investment that exceeds the $VaR_{\alpha}(L)$

$$ES_{\alpha}(CVaR) = E[LL \geq VaR_{\alpha}(L)] \tag{26}$$

Markowitz's Modern Portfolio Theory

THANK YOU for your attention

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