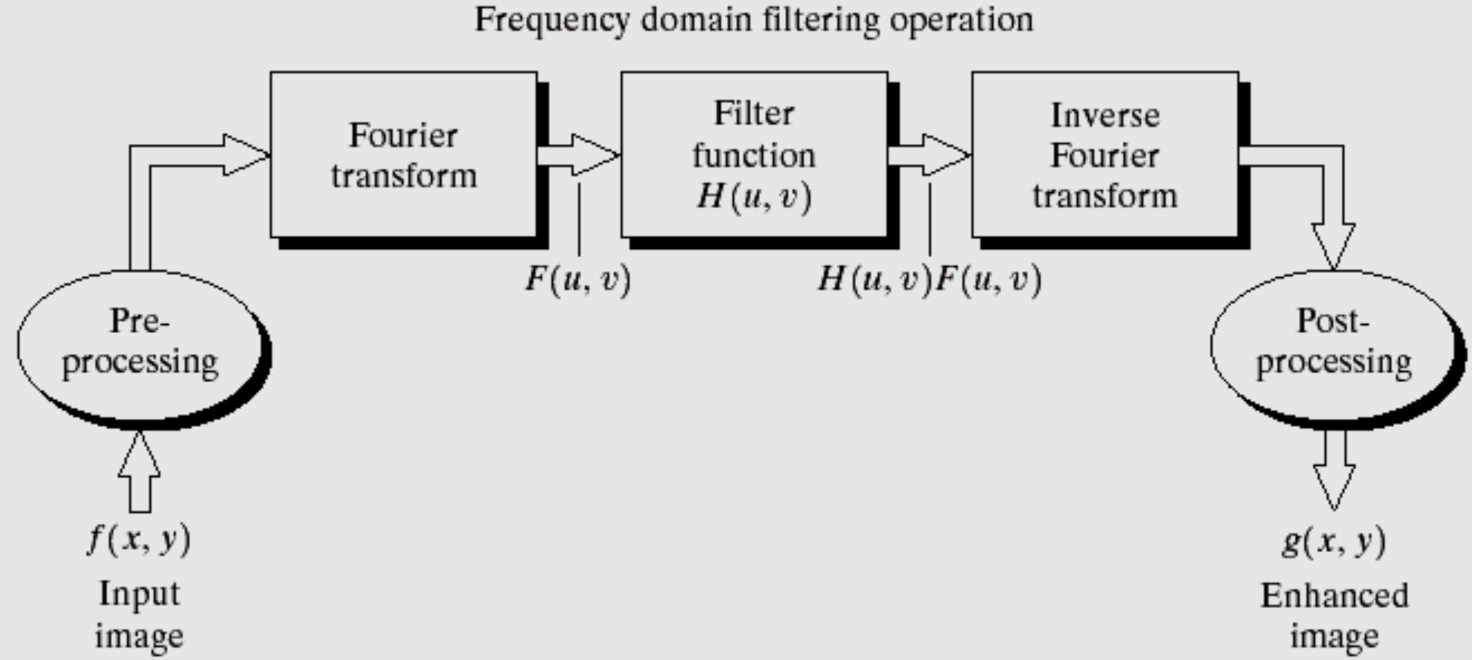
Chapter 4 (Part 2) start updating here
Filtering in the Frequency Domain

Fourier Transform, Discrete Fourier Transform, 2D Fourier Transforms, Basics of Frequency Domain Filtering, **Low-pass, High-pass, Butterworth, Gaussian, Laplacian, High-boost, Homomorphic**

Basics of Frequency Domain Filtering



Basic steps for filtering in the frequency domain.

Image domain ⟷ *Fourier domain*

convolution ⟷ *multiplication*

Frequency Domain Filtering Workflow

Given input image $f(x, y)$ of size MxN and a real symmetric filter function $H(u, v)$:

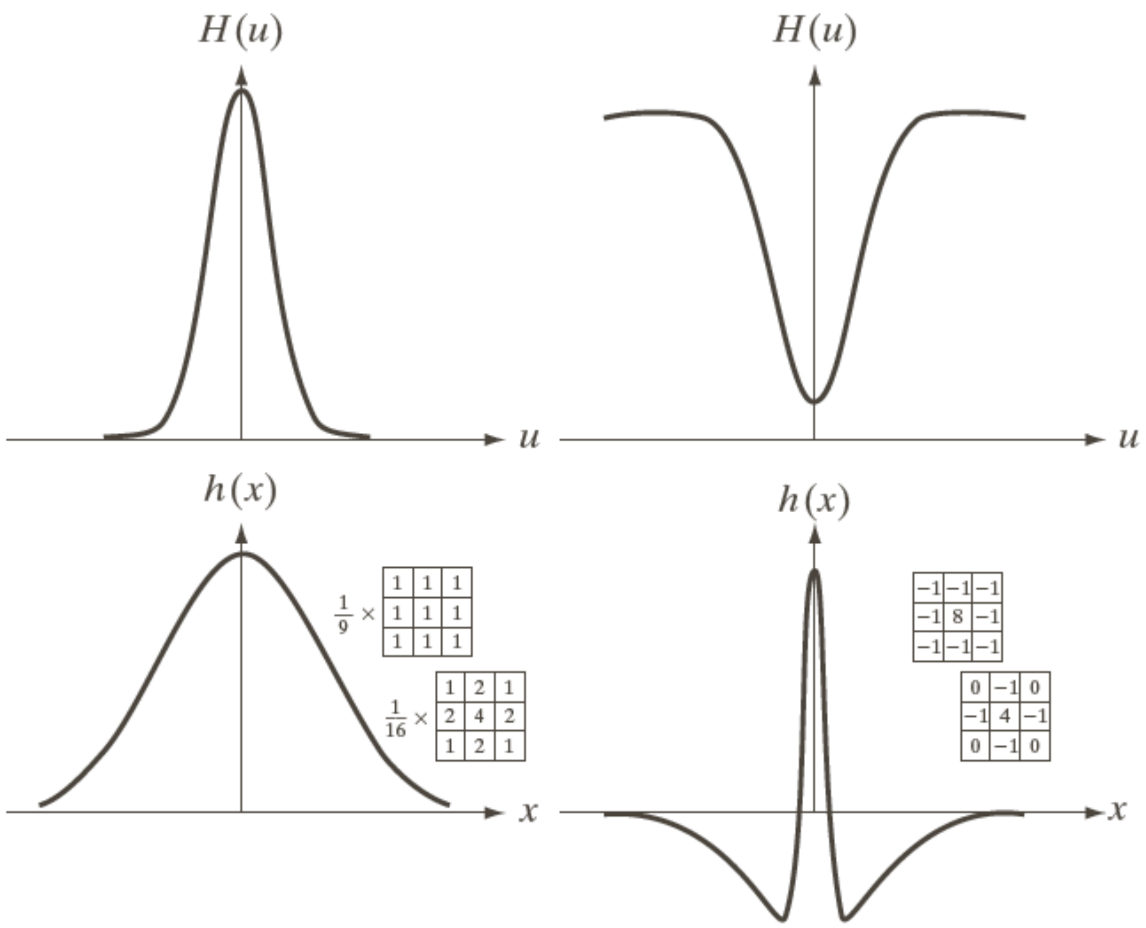
1) Pad the image with zeros to size PxQ. Typically, P=2M and Q=2N $\Longrightarrow f_P(x, y)$
2) Multiply $f_P(x, y)$ by $(-1)^{x+y}$ to center its transformation
3) Compute DFT of the image $\Longrightarrow F(u, v)$
4) Form the product $G(u, v) = H(u, v)F(u, v)$
5) Compute IDFT and take the real part
$$g_P(x, y) = \text{real}(\Im^{-1}[G(u, v)])(-1)^{x+y}$$
6) Obtain $g(x, y)$ by extracting the MxN region from the top left quadrant

Frequency Domain Filtering Guidelines

- Low frequencies correspond to slowly changing intensities in the spatial domain, e.g. monotonous regions
- High frequencies correspond to sharp transitions, such as edges and noise
- *Low-pass filtering* suppresses high frequencies
  $\Longrightarrow$ *smoothing* (blurring)
- *High-pass filtering* suppresses low frequencies
  $\Longrightarrow$ *sharpening*
- Low-pass and high-pass filters are naturally related via:

$$H_{HP}(u,v) = 1 - H_{LP}(u,v)$$

Blurring and Sharpening Filters Across Domains



a	c
b	d

FIGURE 4.37 (a) A 1-D Gaussian lowpass filter in the frequency domain. (b) Spatial lowpass filter corresponding to (a). (c) Gaussian highpass filter in the frequency domain. (d) Spatial highpass filter corresponding to (c). The small 2-D masks shown are spatial filters we used in Chapter 3.

Ideal Low-Pass Filters

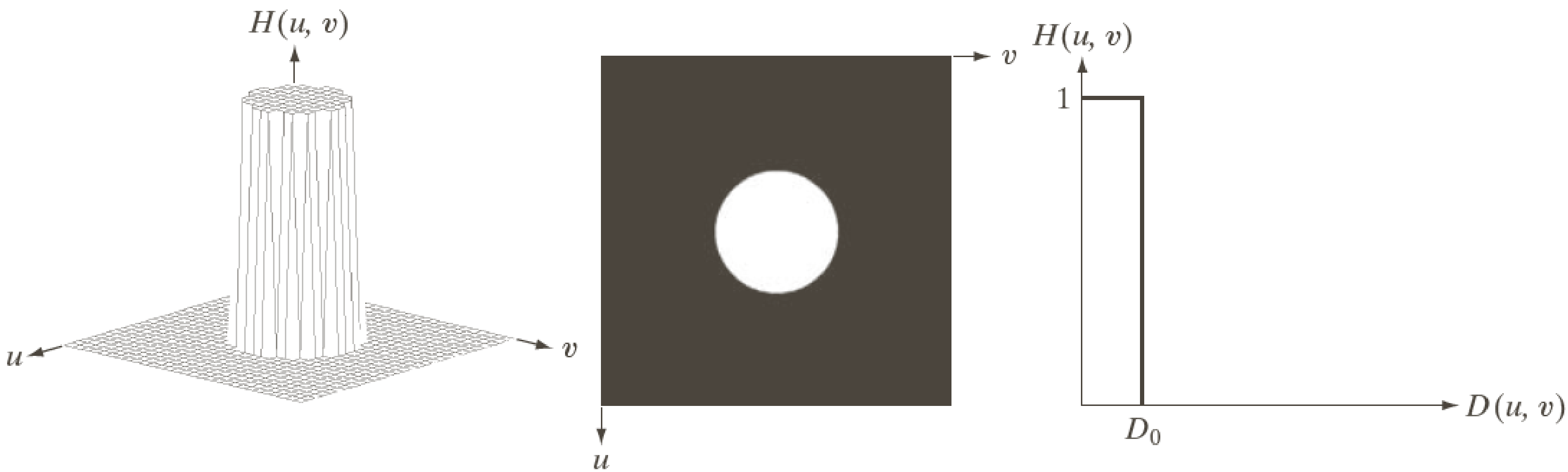
- *Ideal* in the sense of strictly preserving all the frequencies up to the threshold and suppressing the rest
- Not so *ideal* in practice
- Defined as:

$$H(u,v) = \begin{cases} 1 & if\ D(u,v) \leq D_0 \\ 0 & if\ D(u,v) \succ D_0 \end{cases}$$

- D_0 is the *cutoff frequency*
- $D(u,v)$ is the distance from the point (*u*,*v*) to the filter center:

$$D(u,v) = [(u - P/2)^2 + (v - Q/2)^2]^{1/2}$$

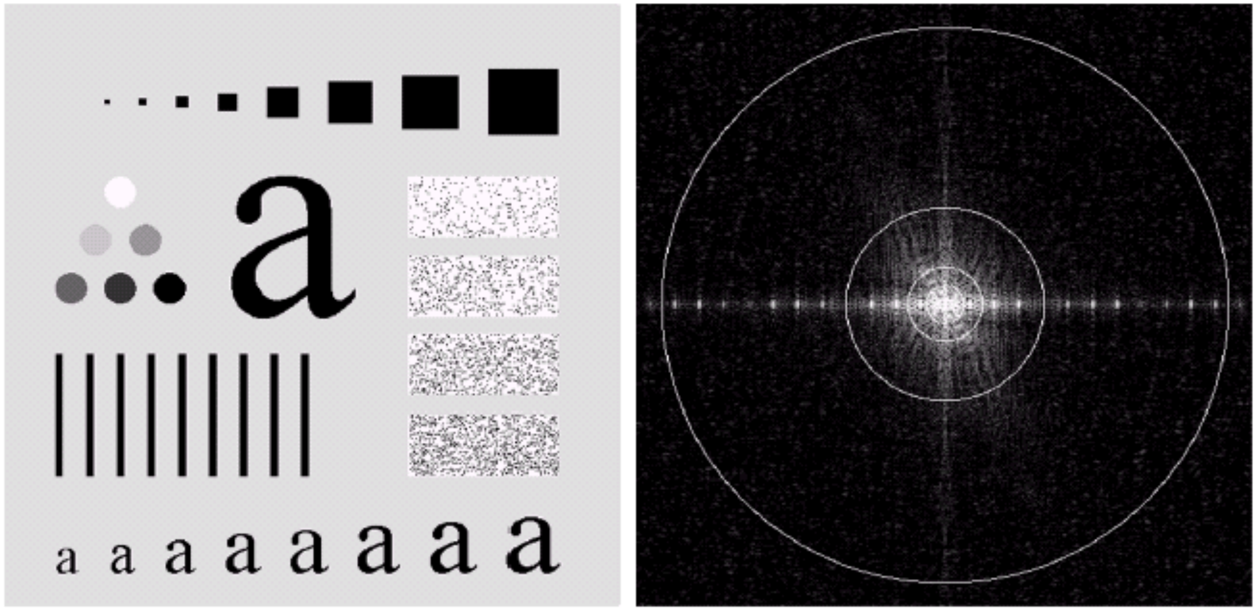
Ideal Low-Pass Filters



a b c

FIGURE 4.40 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Ideal Low-Pass Filters

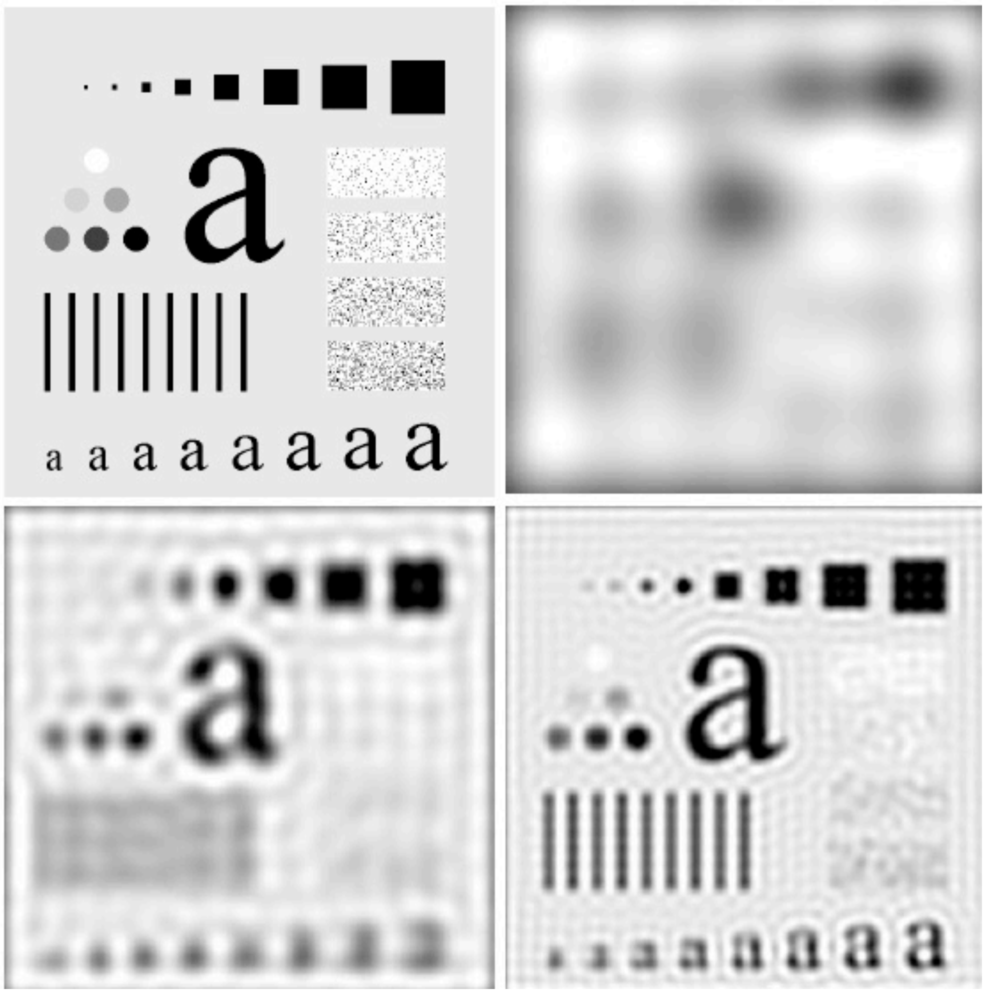


a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

- Note the concentration of image energy inside the inner circle.
- What happens if we low-pass filter it with cut-off freq. at the position of these circles? (see next slide)

Ideal Low-Pass Filters



Original	$D_0 = 10$
$D_0 = 30$	$D_0 = 60$

Note:
the filter with $D_0 = 10$
still retains **87%** of the
image energy
(since most energy
is in low frequencies)

Note that the narrower the filter in the frequency domain is the more severe are the blurring and ringing!

Ideal Low-Pass Filters

a greylevel profile of a horizontal scan line through the center

H(u,v) of Ideal Low-Pass Filter (ILPF) with radius 5

h(x,y) is the corresponding spatial filter

the center component is responsible for blurring

the concentric components are responbile for ringing

input image containing 5 bright impulses

result of convolution of input with h(x,y)

diagonal scan line through the filtered image center

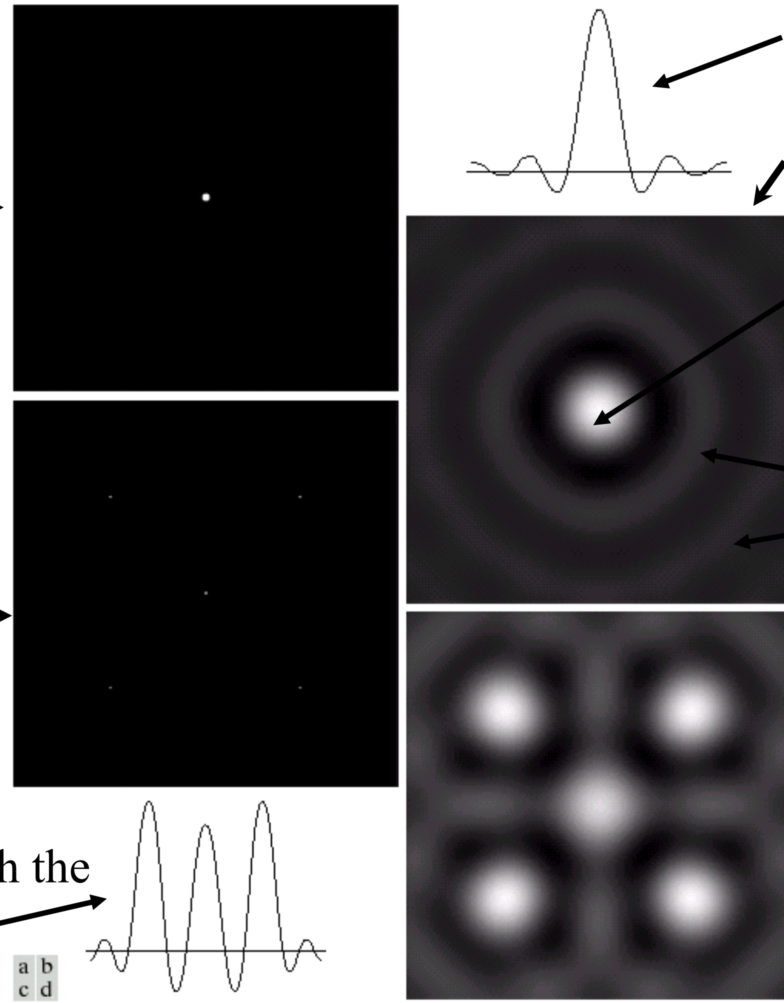


FIGURE 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

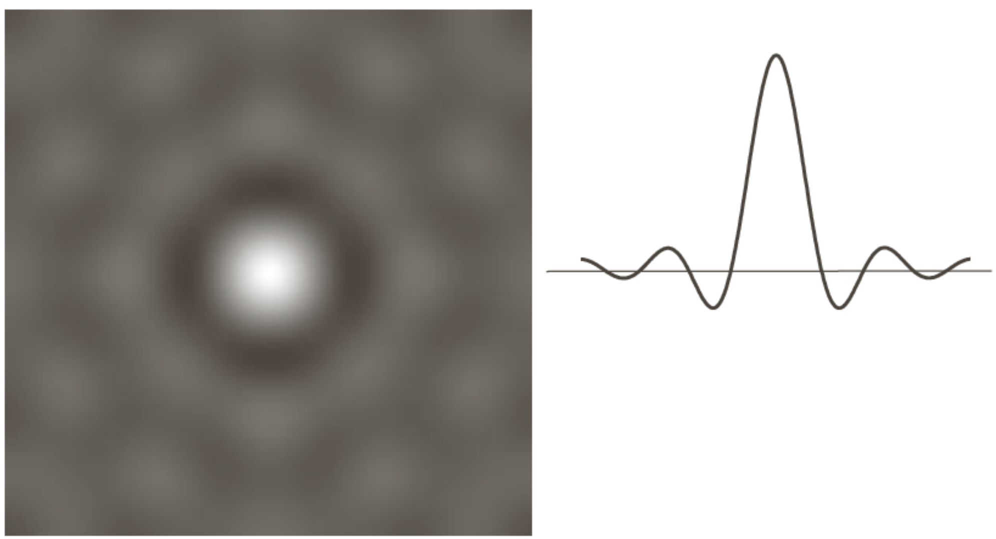
notice blurring and ringing!

Why Not Ideal?

- Filtering with ILPF introduces *ringing*
- Consider:

multiplication (frequency) $\Leftrightarrow$ convolution (spatial)

box filter (frequency) $\Leftrightarrow$ sinc function (spatial)



a b

FIGURE 4.43 (a) Representation in the spatial domain of an ILPF of radius 5 and size 1000×1000. (b) Intensity profile of a horizontal line passing through the center of the image.

Butterworth Low-Pass Filters

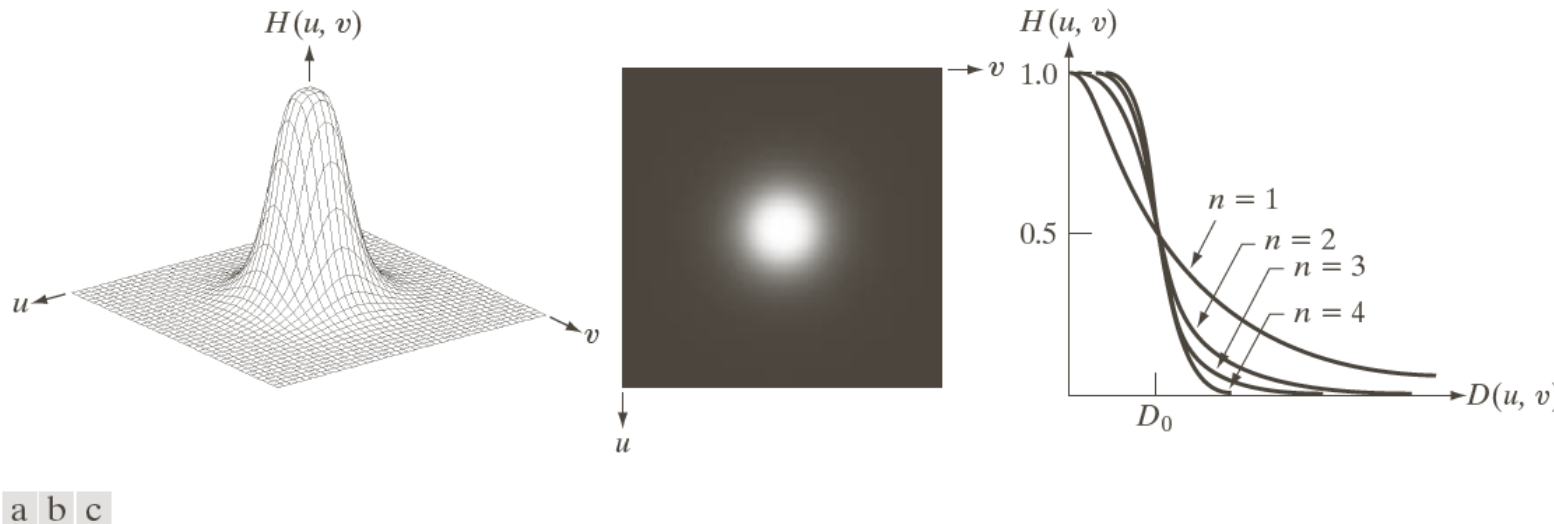


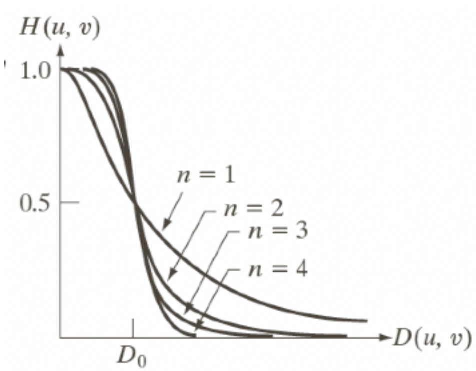
FIGURE 4.44 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

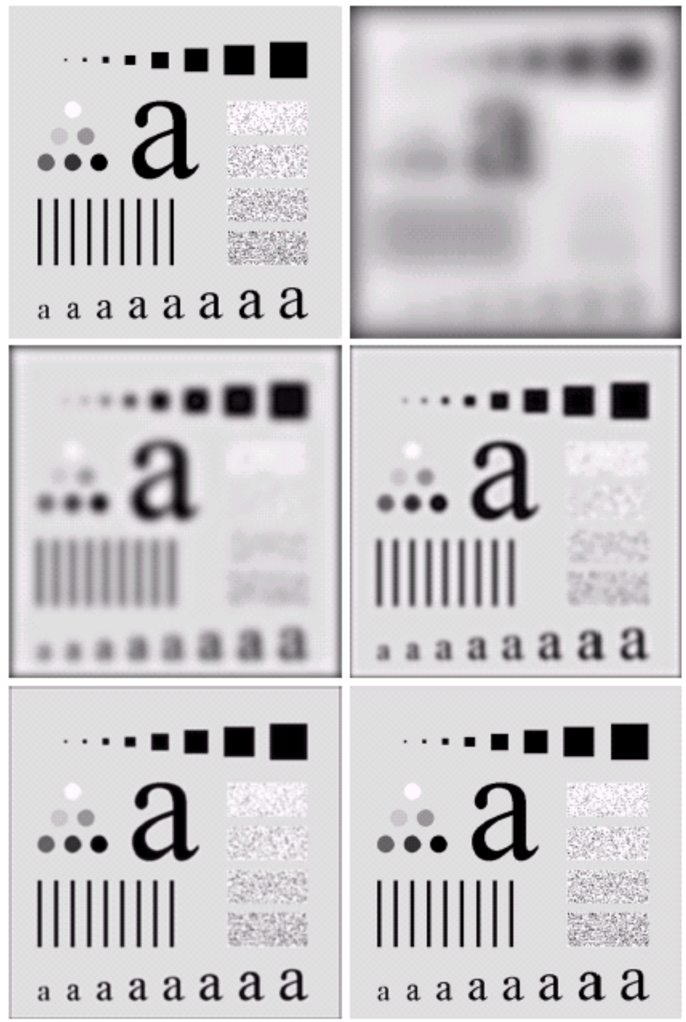
$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$$

n – order of the filter

Butterworth Low-Pass Filters

Filtering with BLPF with **n=2** and increasing cut-off as was done with the Ideal LPF





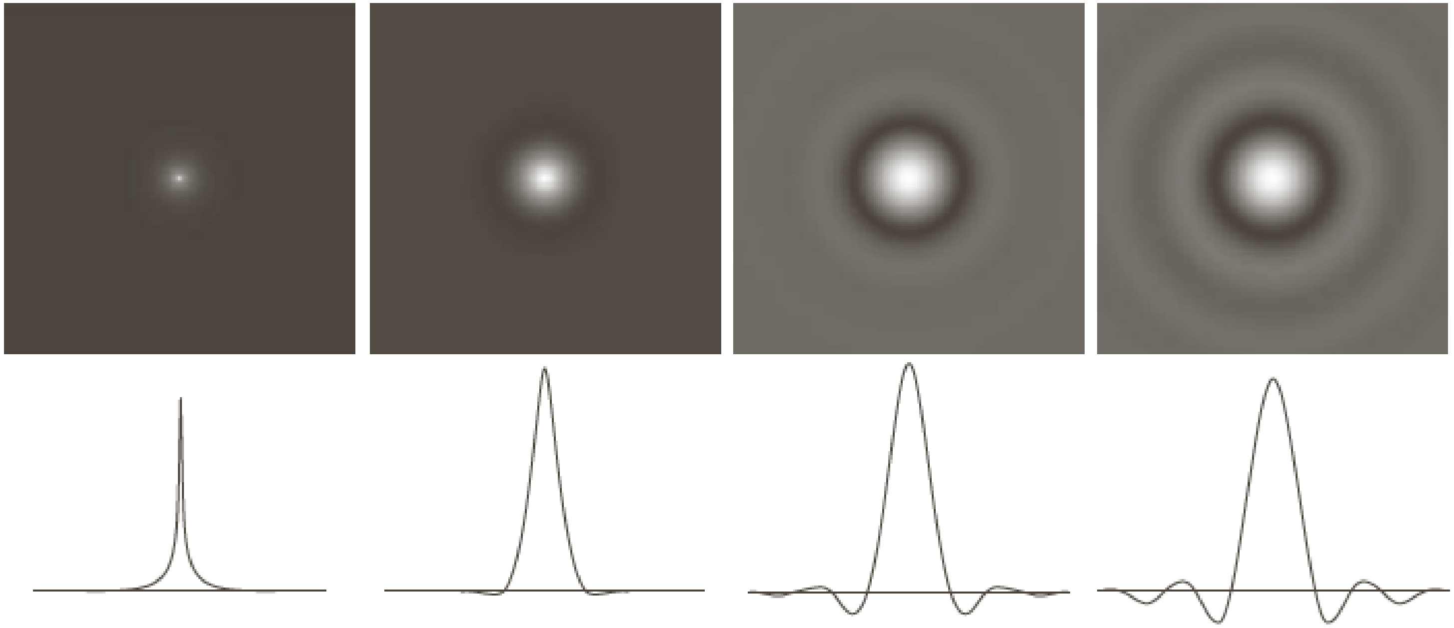
a b
c d
e f

FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

Note the smooth transition in blurring achieved as a function of increasing cutoff but no ringing is present in any of the filtered images with this particular BLPF (with n=2)

this is attributed to the smooth transition between low and high frequencies

Butterworth Filter Order



a b c d

FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000×1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

No ringing for n=1, imperceptible ringing for n=2, ringing increases for higher orders (getting closer to Ideal LPF).

Gaussian Low-Pass Filters

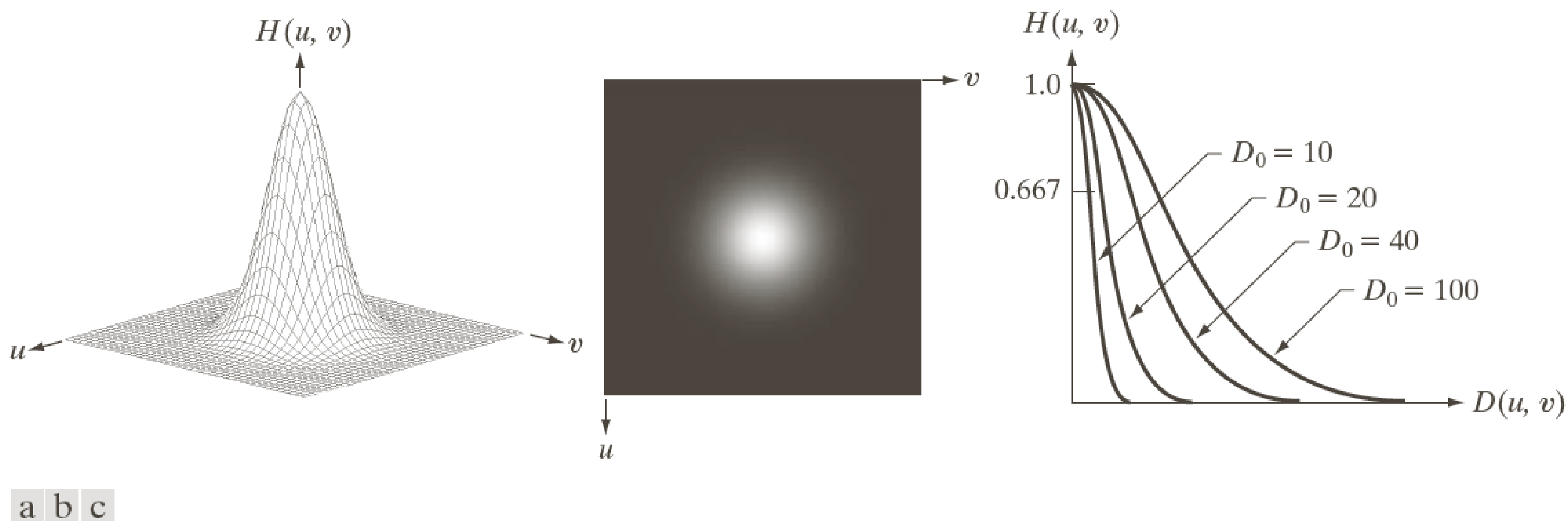


FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

$$H(u,v) = e^{-D^2(u,v)/2{D_0}^2}$$

Does this filter suffer from ringing artefacts?

+ IDFT of the Gaussian filter is also Gaussian $\Longrightarrow$ **no ringing!**

– transition less sharp than BLFP

Gaussian Low-Pass Filters

Remarks:

1. Note the smooth transition in blurring achieved as a function of increasing cutoff frequency.

2. Less smoothing than BLPFs since the latter have tighter control over the transitions between low and high frequencies.

The price paid for tighter control by using BLP is possible ringing.

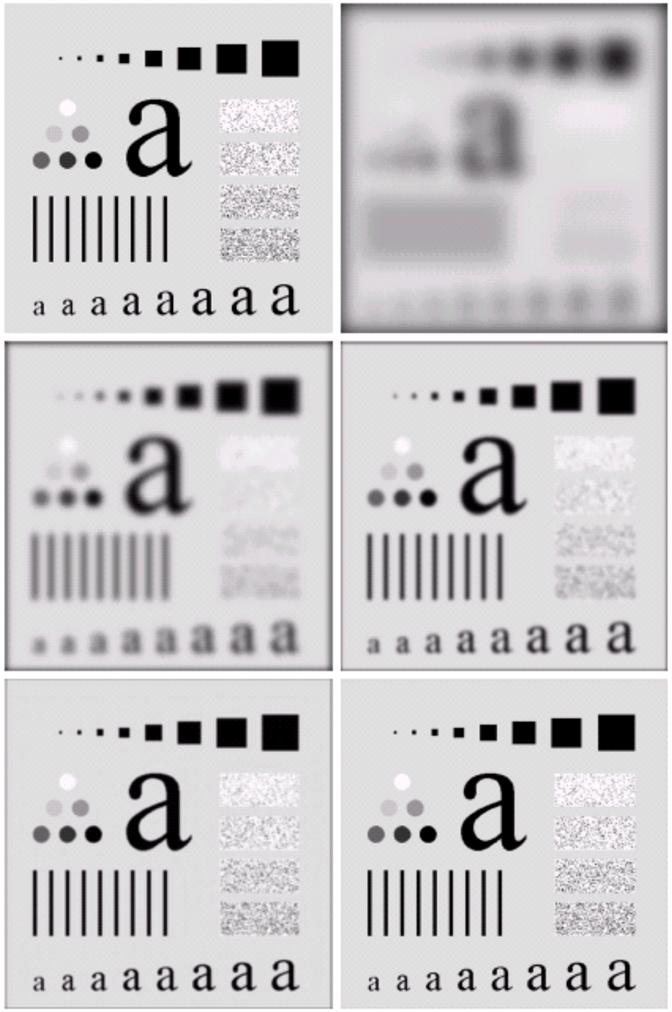


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

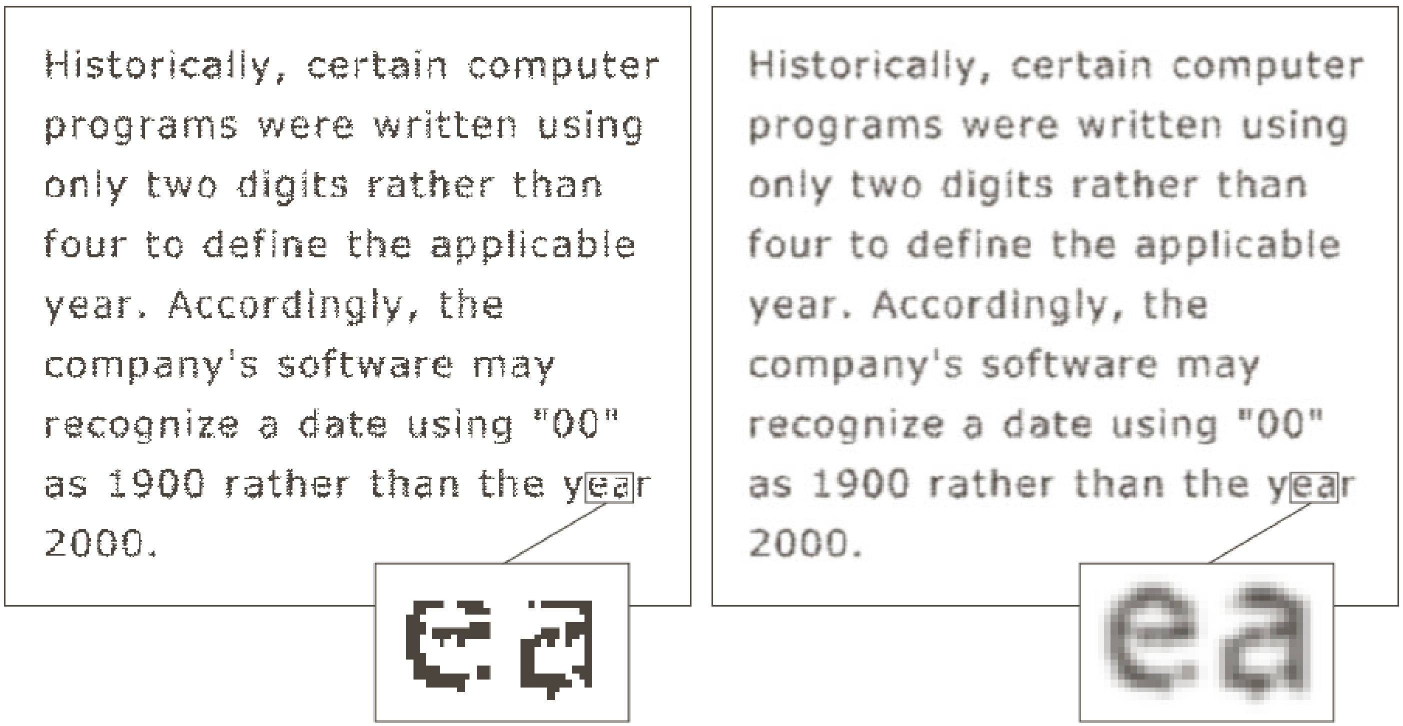
a b
c d
e f

3. No ringing!

4. GLPF is preferable if no image artifacts are acceptable (e.g. in medical imaging)

Low-Pass Filtering Examples

Applications: fax transmission, duplicated documents and old records.



a b

FIGURE 4.49 (a) Sample text of low resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

GLPF with D_0=80 is used.

Low-Pass Filtering Examples

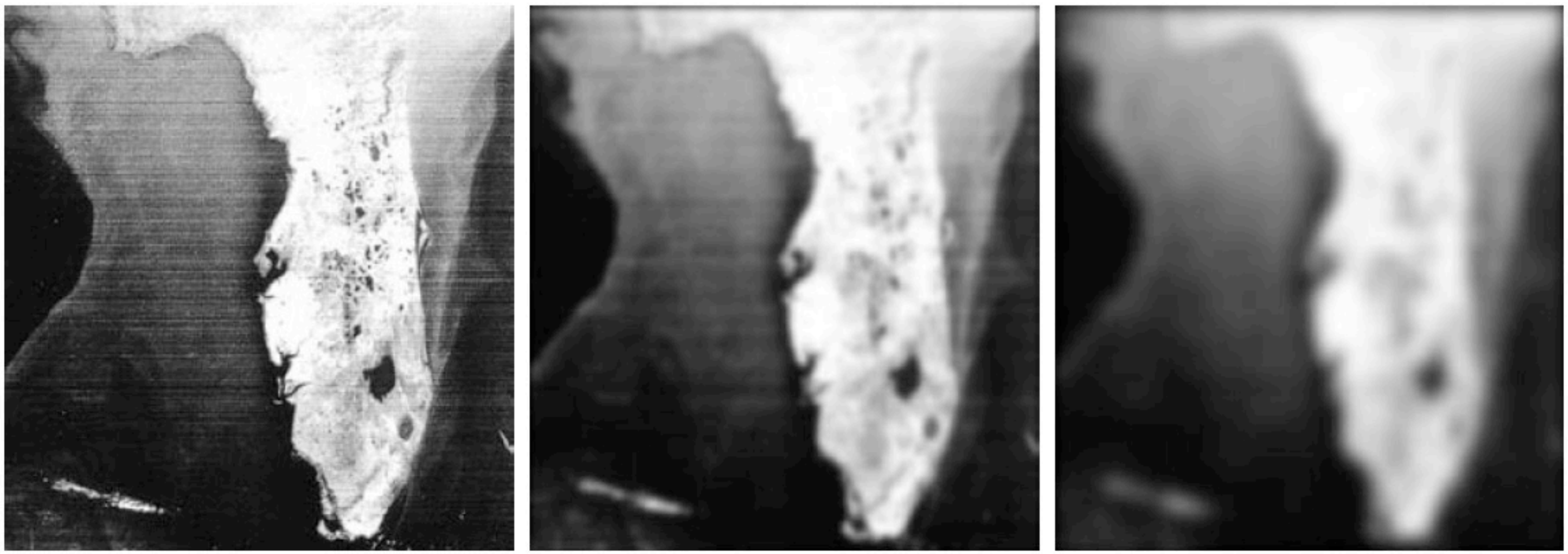
A LPF is also used in printing, e.g. to smooth fine skin lines in faces.

a b c

FIGURE 4.50 (a) Original image (784 × 732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

Low-Pass Filtering Examples

(a) a Very High Resolution Radiometer (VHRR) image showing part of the Gulf of Mexico (dark) and Florida (light) taken from NOAA satellite. Note horizontal scan lines caused by sensors.



a b c

FIGURE 4.51 (a) Image showing prominent horizontal scan lines. (b) Result of filtering using a GLPF with $D_0 = 50$. (c) Result of using a GLPF with $D_0 = 20$. (Original image courtesy of NOAA.)

(b) scan lines are removed in smoothed image by a GLP with D_0=*30*
(c) a large lake in southeast Florida is more visible when more aggressive smoothing is applied (GLP with D_0=*10).*

High-Pass Filters in Frequency Domain

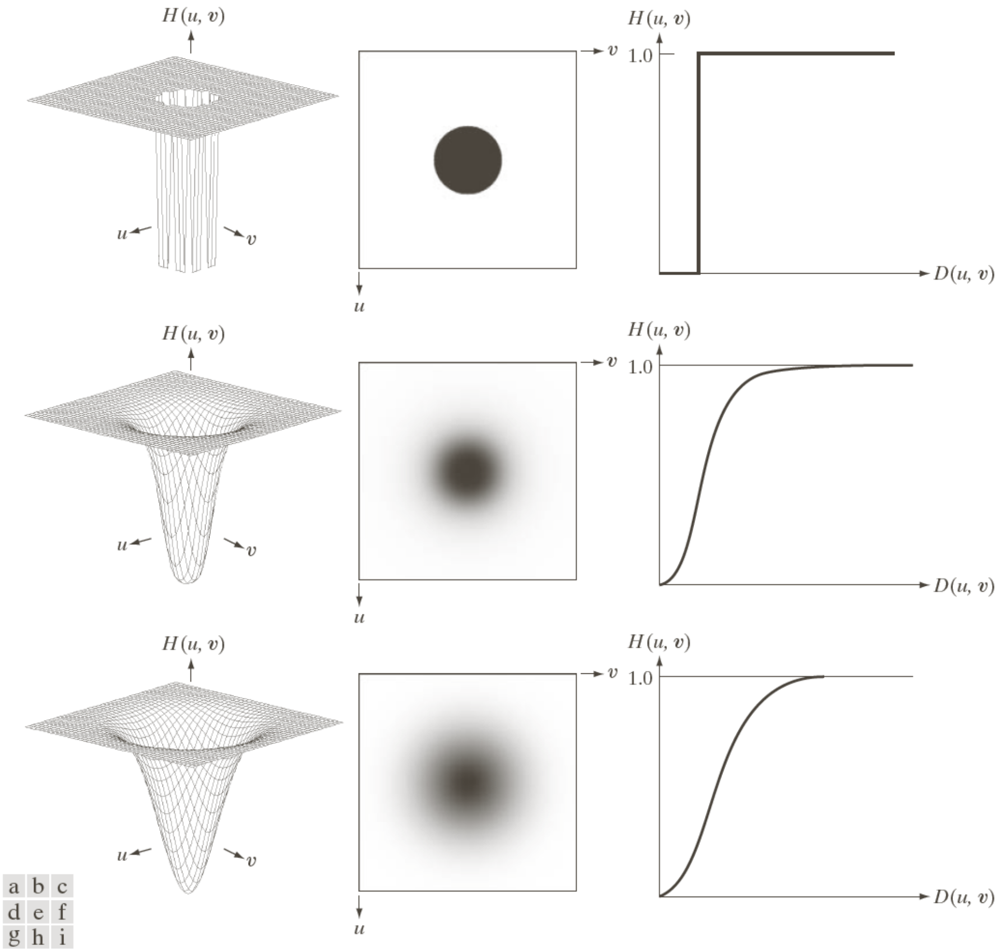


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Ideal

$$H(u,v) = \begin{cases} 0 & if \;\; D(u,v) \leq D_0 \\ 1 & if \;\; D(u,v) \succ D_0 \end{cases}$$

Butterworth

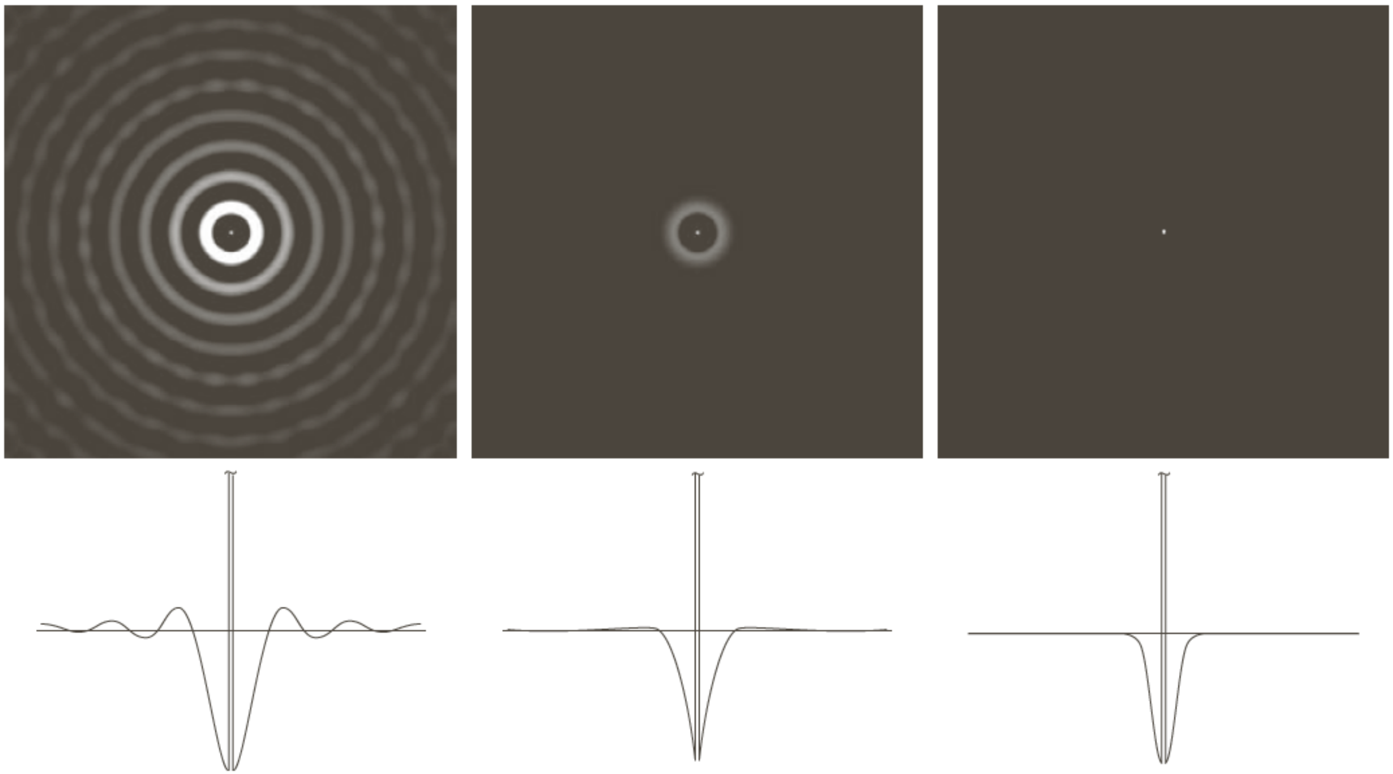
$$H(u,v) = \frac{1}{1 + [D_0/D(u,v)]^{2n}}$$

Gaussian

$$H(u,v) = 1 - e^{-D^2(u,v)/2{D_0}^2}$$

Spatial Representations of High-Pass Filters

Ideal HPFs are expected to suffer from the same ringing effects as Ideal LPF, see part (a) below



a b c

FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

High-Pass Filters in Frequency Domain

a b c

FIGURE 4.54 Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60$, and 160.

1. Ideal high-pass filters enhance edges but suffer from ringing artefacts, just like Ideal LPF.
2. As all the frequencies below the cutoff are suppressed, monotonous regions disappear

High-Pass Filters in Frequency Domain

a b c

FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

improved enhanced images with BHPFs

High-Pass Filters in Frequency Domain

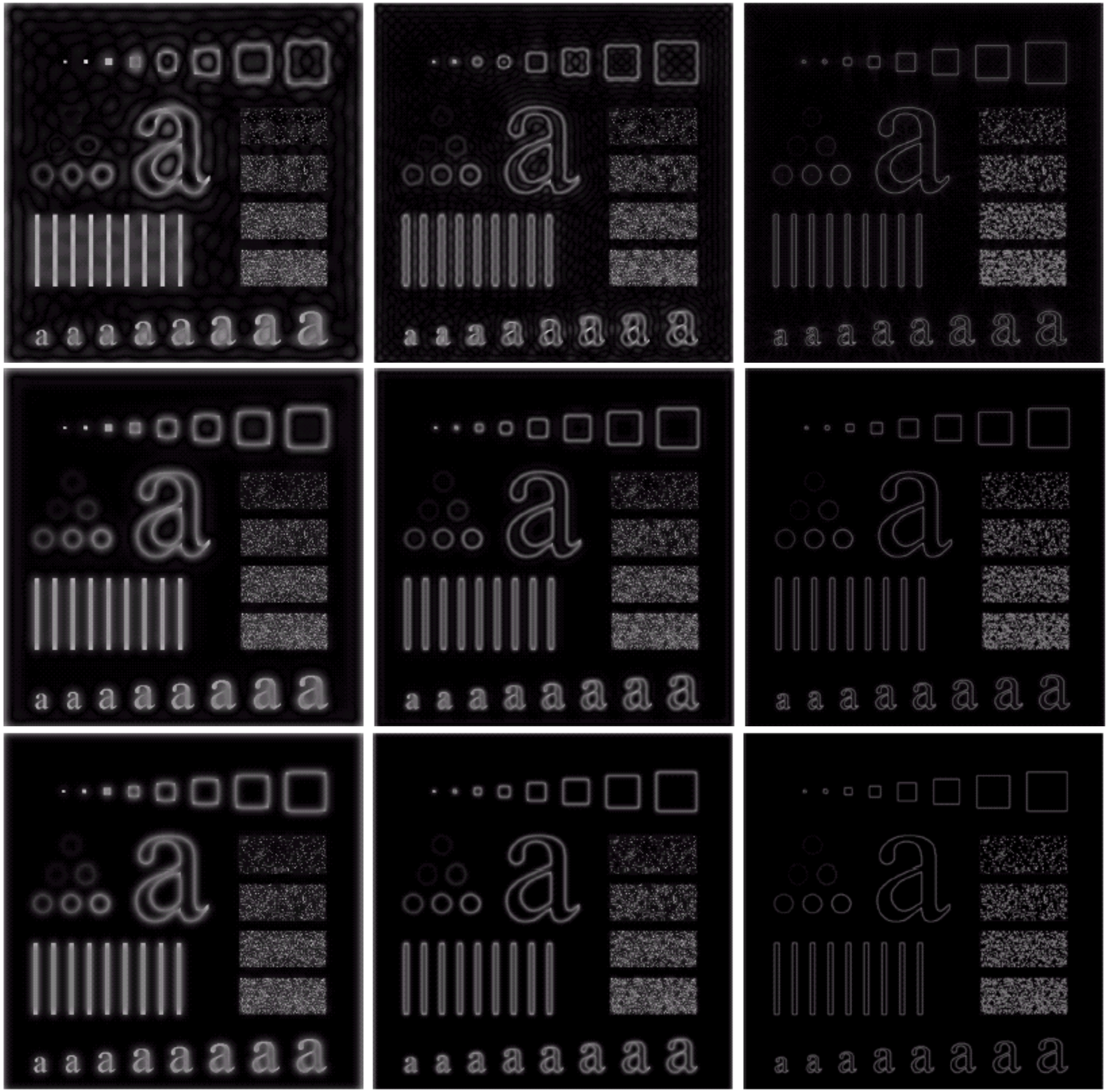
FIGURE 4.56 Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

even smoother results with GHPFs

High-Pass Filtering Example

a b c

FIGURE 4.57 (a) Thumb print. (b) Result of highpass filtering (a). (c) Result of thresholding (b). (Original image courtesy of the U.S. National Institute of Standards and Technology.)



Ideal HPF

BHPF

GHPF

a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

Laplacian in the Frequency Domain

- **Chapter 3**: edges can be enhanced by adding the Laplacian (second derivative) of the image on top of the original:
$$g(x,y) = f(x,y) + c\nabla^2 f(x,y)$$
- *Laplacian filter* can be implemented in the frequency domain:
$$H(u,v) = -4\pi^2 D^2(u,v)$$
- Laplacian image can then be obtained:
$$\nabla^2 f(x,y) = \mathfrak{I}^{-1}\{H(u,v)F(u,v)\}$$
- $c = -1$ should be chosen in this case, as $H(u,v) < 0$
- Results are <u>visually identical</u> to the equivalent spatial filter

Note: $f(x,y)$ and $c\nabla^2 f(x,y)$ <u>need to be normalized</u> to be of comparable scales!

Laplacian in the Frequency Domain

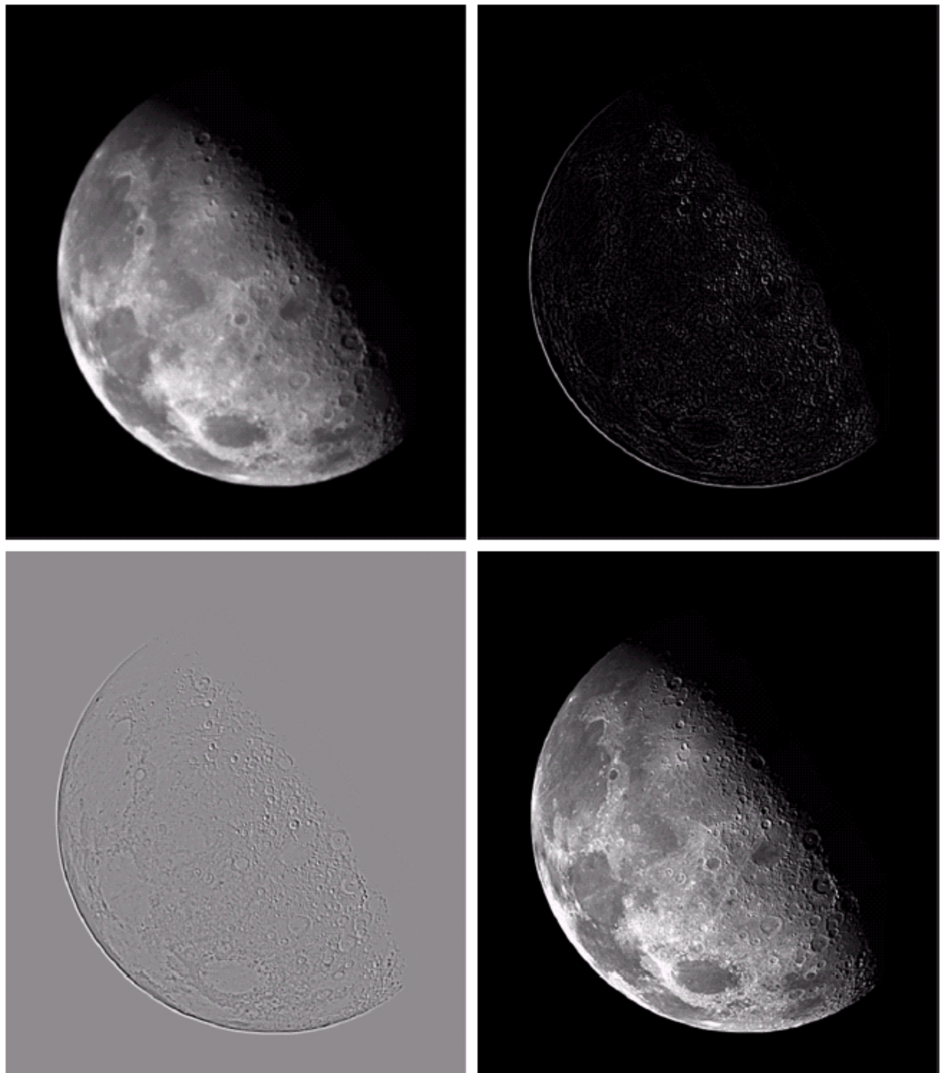
a b

FIGURE 4.58
(a) Original, blurry image.
(b) Image enhanced using the Laplacian in the frequency domain. Compare with Fig. 3.38(e).

Laplacian in the Frequency Domain (Chapter 3 comparison)

a b
c d

FIGURE 3.40 (a) Image of the North Pole of the moon. (b) Laplacian-filtered image. (c) Laplacian image scaled for display purposes. (d) Image enhanced by using Eq. (3.7-5). (Original image courtesy of NASA.)



result of filtering orig in frequency domain by Laplacian

previous result scaled

enhanced result obtained using

$$g(x,y) = f(x,y) - \nabla^2 f(x,y)$$

Unsharp Masking and Highboost Filtering

- The general expression is (from Chapter 3):

$$g(x, y) = f(x, y) + k * g_{mask}(x, y)$$

($k = 1$ for unsharp masking and $k > 1$ for highboost filtering)

- Obtain the mask by subtracting the low-pass filtered image:

$$g_{mask}(x, y) = f(x, y) - f_{LP}(x, y)$$

$$f_{LP}(x, y) = \Im^{-1}\{H_{LP}(u, v)F(u, v)\}$$

- Minor reformulation gives a *high-frequency-emphasis filter*:

$$g(x, y) = \Im^{-1}\{[1 + k * H_{HP}(u, v)]F(u, v)\}$$

$$g(x, y) = \Im^{-1}\{[k_1 + k_2 * H_{HP}(u, v)]F(u, v)\}$$

High-Frequency-Emphasis Filter

X-ray images cannot be focused in the same manner as a lens, so they tend to produce slightly blurred images with biased (towards black) grey levels → complement frequency domain filtering with spatial domain filtering!

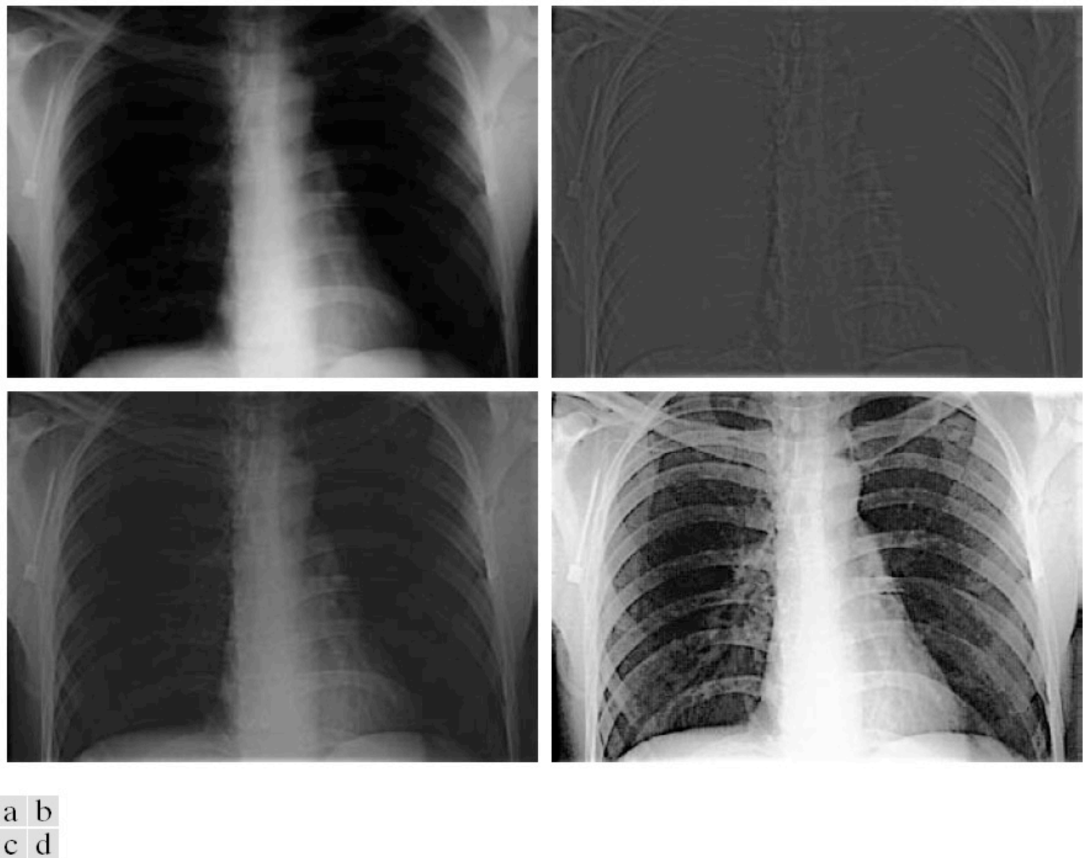


FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

$D_0 = 40$
$k_1 = 0.5$
$k_2 = 0.75$

Homomorphic Filtering

- Recall the *illumination-reflectance model*, in which the image can be expressed as the product of *illumination* and *reflectance* components:

$$f(x,y) = i(x,y)r(x,y)$$

- The product cannot be easily decomposed in the frequency domain:

$$FT[f(x,y)] \neq FT[i(x,y)] * FT[r(x,y)]$$

- However, the logarithmic transformation helps here:

$$z(x,y) = \ln(f(x,y)) = \ln(i(x,y)) + \ln(r(x,y))$$

- Then $Z(u,v) = FT[z(x,y)] = FT[\ln(f(x,y))] = FT[\ln(i(x,y))] + FT[\ln(r(x,y))] = F_i(u,v) + F_r(u,v)$
- Frequency domain filtering then becomes straightforward:

$$S(u,v) = H(u,v)Z(u,v) = H(u,v)F_i(u,v) + H(u,v)F_r(u,v)$$

$$g(x,y) = e^{\Im^{-1}\{S(u,v)\}} = e^{s(x,y)} = e^{i'(x,y)}e^{r'(x,y)} = i_0(x,y)r_0(x,y)$$

Homomorphic Filtering

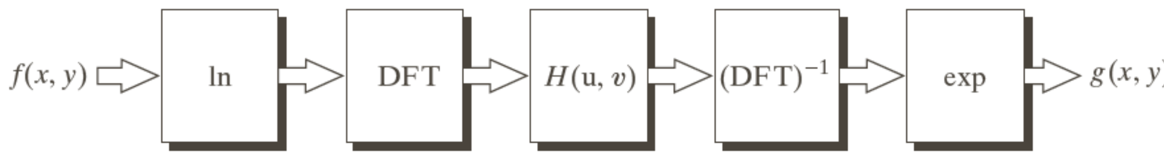


FIGURE 4.60 Summary of steps in homomorphic filtering.

if the gain of H(u,v) is set such as

$$\gamma_L \prec 1 \quad \text{and} \quad \gamma_H \succ 1$$

then H(u,v) tends to decrease the contribution of low-frequency (illumination) and amplify high frequency (reflectance)

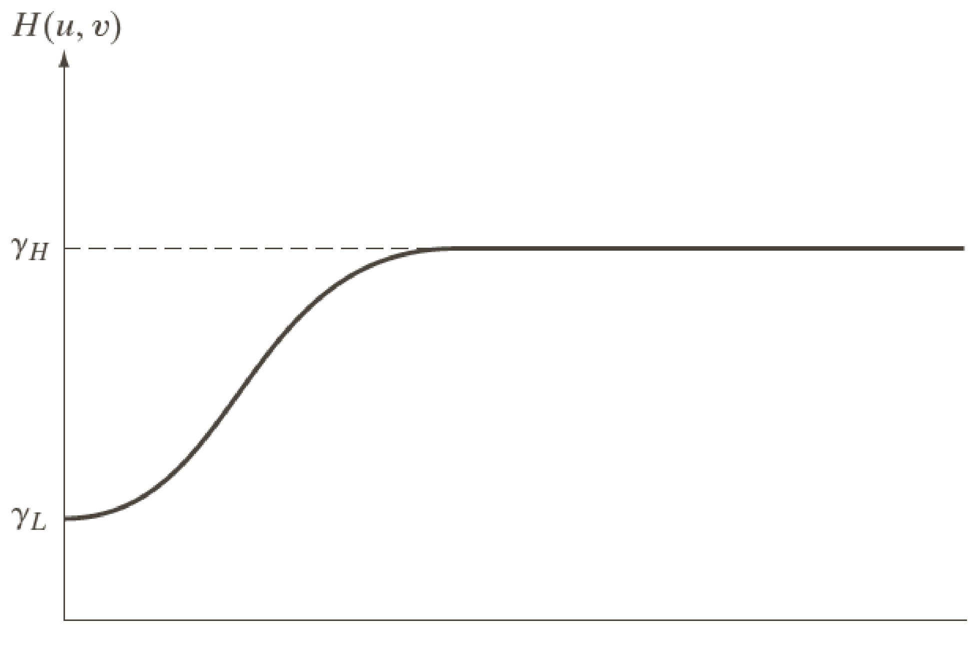


FIGURE 4.61 Radial cross section of a circularly symmetric homomorphic filter function. The vertical axis is at the center of the frequency rectangle and $D(u, v)$ is the distance from the center.

Result: simultaneous dynamic range compression and contrast enhancement

Homomorphic Filtering

- Homomorphic filtering approach allows for both low frequencies (≈ illumination) and high frequencies (≈ reflectance) to be adjusted simultaneously
- One possibility is a modified Gaussian high-pass filter (see Fig. 4.61):

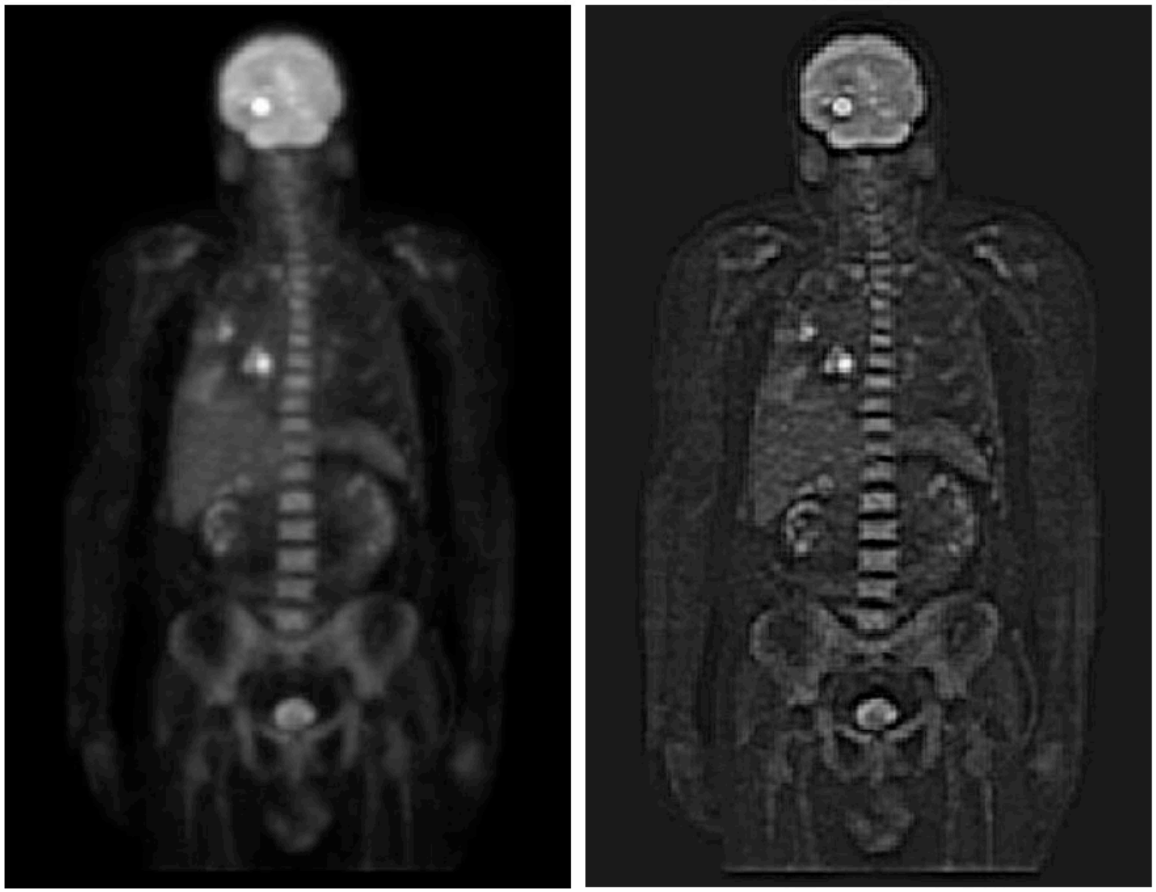
$$H(u,v) = (\gamma_H - \gamma_L)\left[1 - e^{-c\left[D^2(u,v)/{D_0}^2\right]}\right] + \gamma_L$$

- γ_H and γ_L define the filter behavior at the frequency extremes
- c controls the slope of the transition between the extremes
- E.g. $\gamma_L < 1$ and $\gamma_H > 1$

  $\Longrightarrow$ low frequencies attenuated, high frequencies boosted

  $\Longrightarrow$ contrast improved, as monotonous regions are less dominant

Homomorphic Filtering



a b

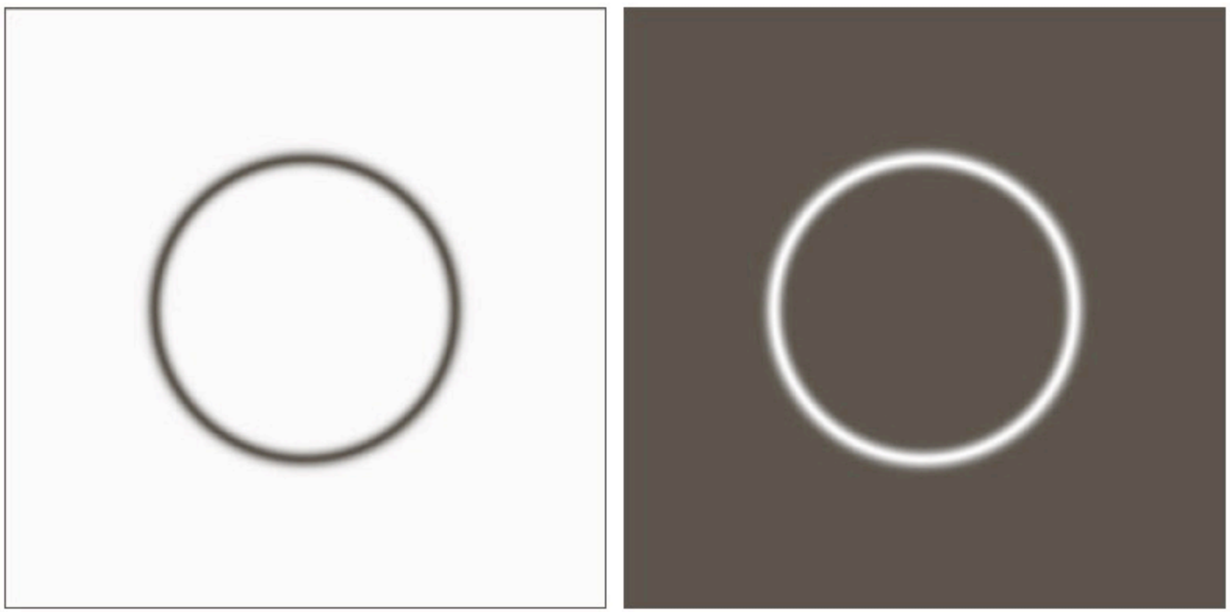
FIGURE 4.62 (a) Full body PET scan. (b) Image enhanced using homomorphic filtering. (Original image courtesy of Dr. Michael E. Casey, CTI PET Systems.)

$\gamma_L = 0.25 \quad \gamma_H = 2$

$c = 1 \quad D_0 = 80$

Selective Filtering Examples

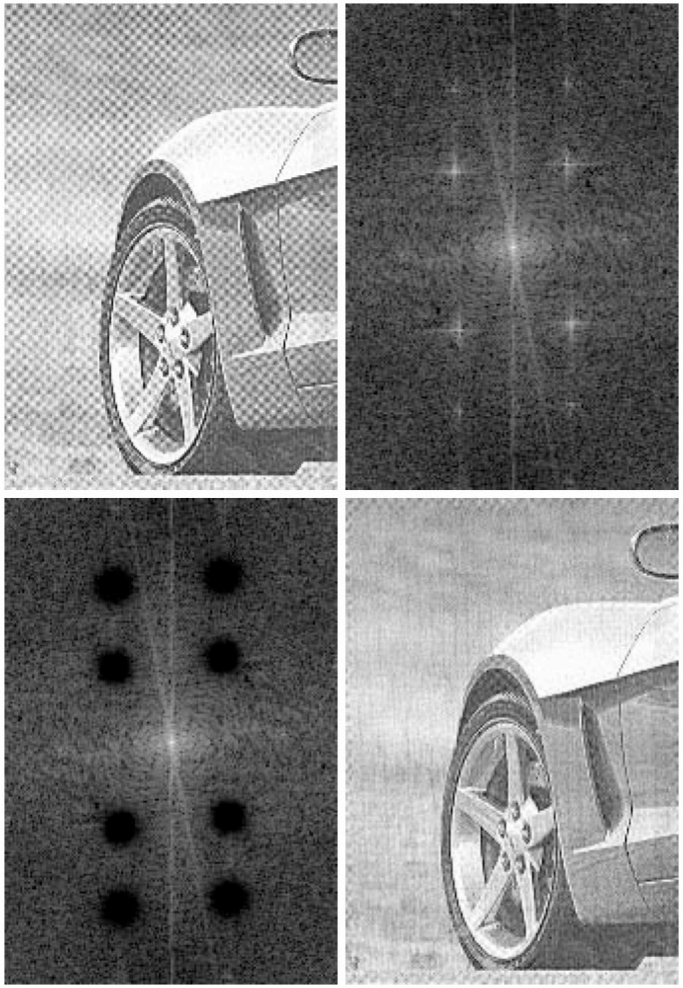
Band-reject filter *Band-pass filter*



a b

FIGURE 4.63 (a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.

Selective Filtering Examples

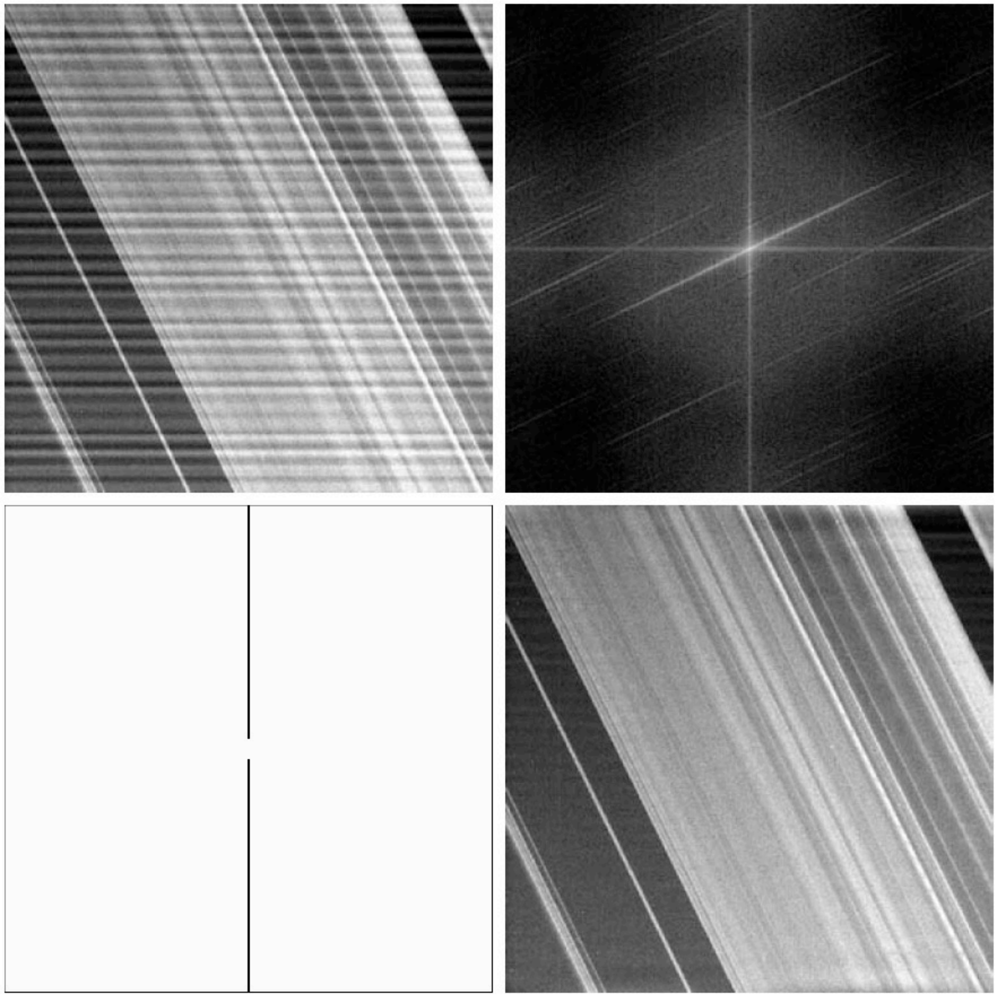


a	b
c	d

FIGURE 4.64 (a) Sampled newspaper image showing a moiré pattern. (b) Spectrum. (c) Butterworth notch reject filter multiplied by the Fourier transform. (d) Filtered image.

Selective Filtering Examples

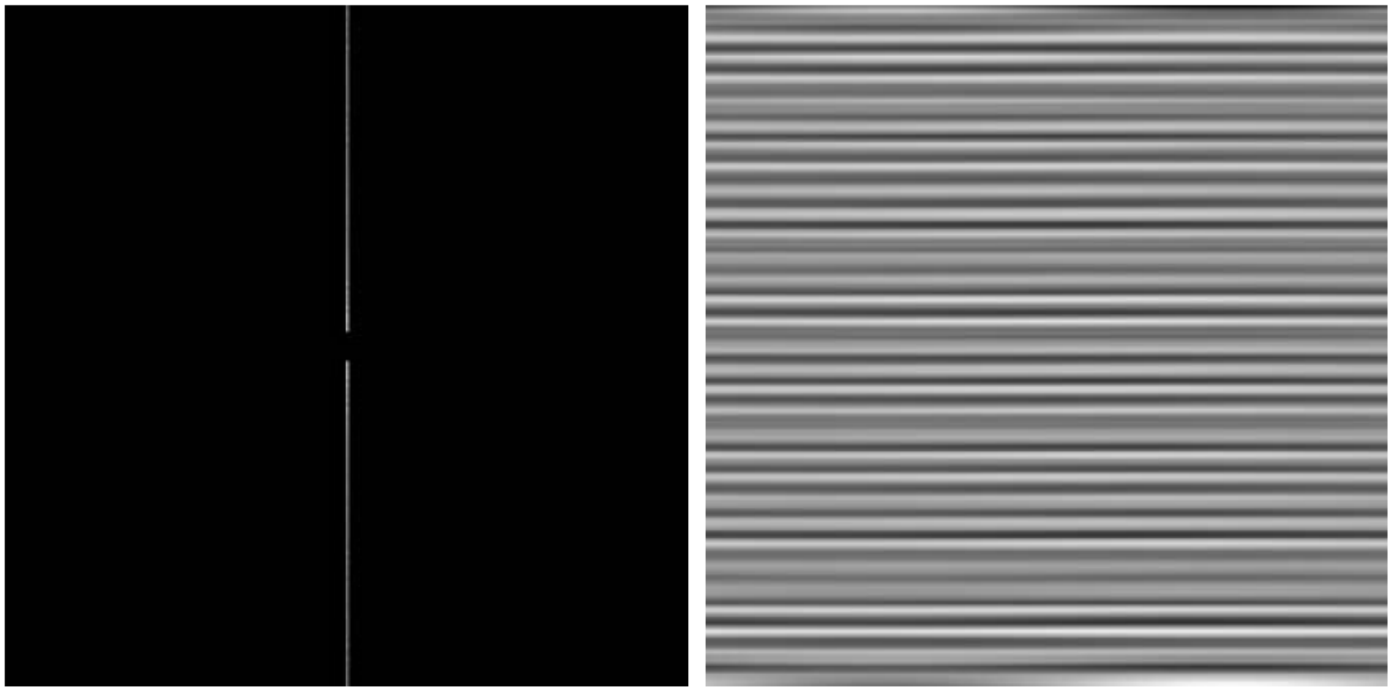
Original Saturn rings showing nearly period interference



a b
c d

FIGURE 4.65 (a) 674 × 674 image of the Saturn rings showing nearly periodic interference. (b) Spectrum: The bursts of energy in the vertical axis near the origin correspond to the interference pattern. (c) A vertical notch reject filter. (d) Result of filtering. The thin black border in (c) was added for clarity; it is not part of the data. (Original image courtesy of Dr. Robert A. West, NASA/JPL.)

Selective Filtering Examples



a b

FIGURE 4.66 (a) Result (spectrum) of applying a notch pass filter to the DFT of Fig. 4.65(a). (b) Spatial pattern obtained by computing the IDFT of (a).