Chapter 4, Part 1
Filtering in the Frequency Domain

Fourier Transform, Discrete Fourier Transform, 2D Fourier Transforms, Basics of Frequency Domain Filtering,
Low-pass, High-pass, Butterworth, Gaussian
Laplacian, High-boost, Homomorphic

FOURIER

Jean B. Joseph Fourier
(1768-1830)

*"An **arbitrary** function, continuous or with discontinuities, defined in a finite interval by an arbitrarily capricious graph can always be expressed as a sum of sinusoids"*

J.B.J. Fourier

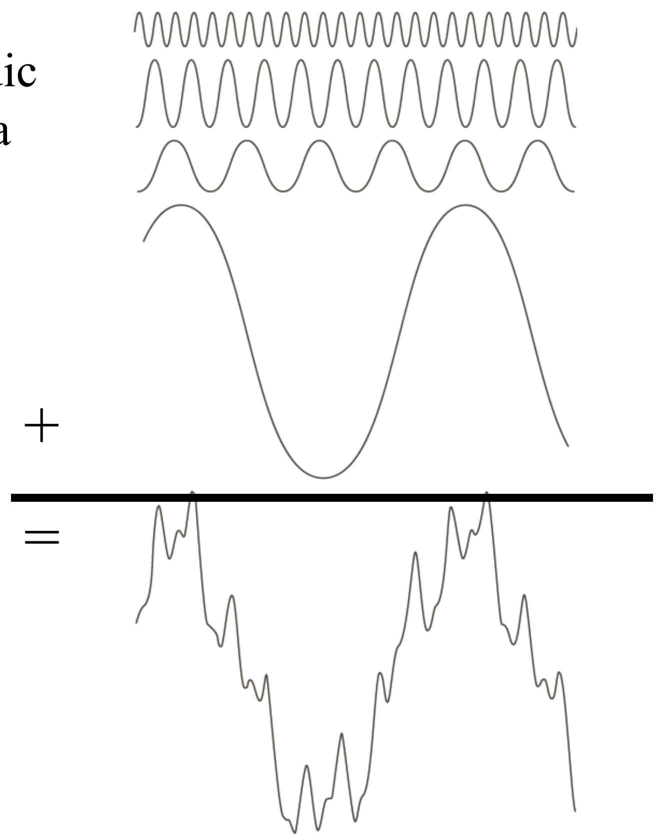
December 21, 1807

$$F[k] = \int f(t) e^{-j2\pi kt/N} dt \qquad f(t) = \frac{1}{2\pi} \sum_{i=0}^{N-1} F[k] e^{j2\pi kt/N}$$

Arbitrary capricious graph refers to any function not subject to any condit

The Fourier Series

Fourier series state that a periodic function can be represented by a weighted sum of sinusoids



Periodic and non-periodic functions can be represented by an integral of weighted sinusoids

FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

The Fourier Family

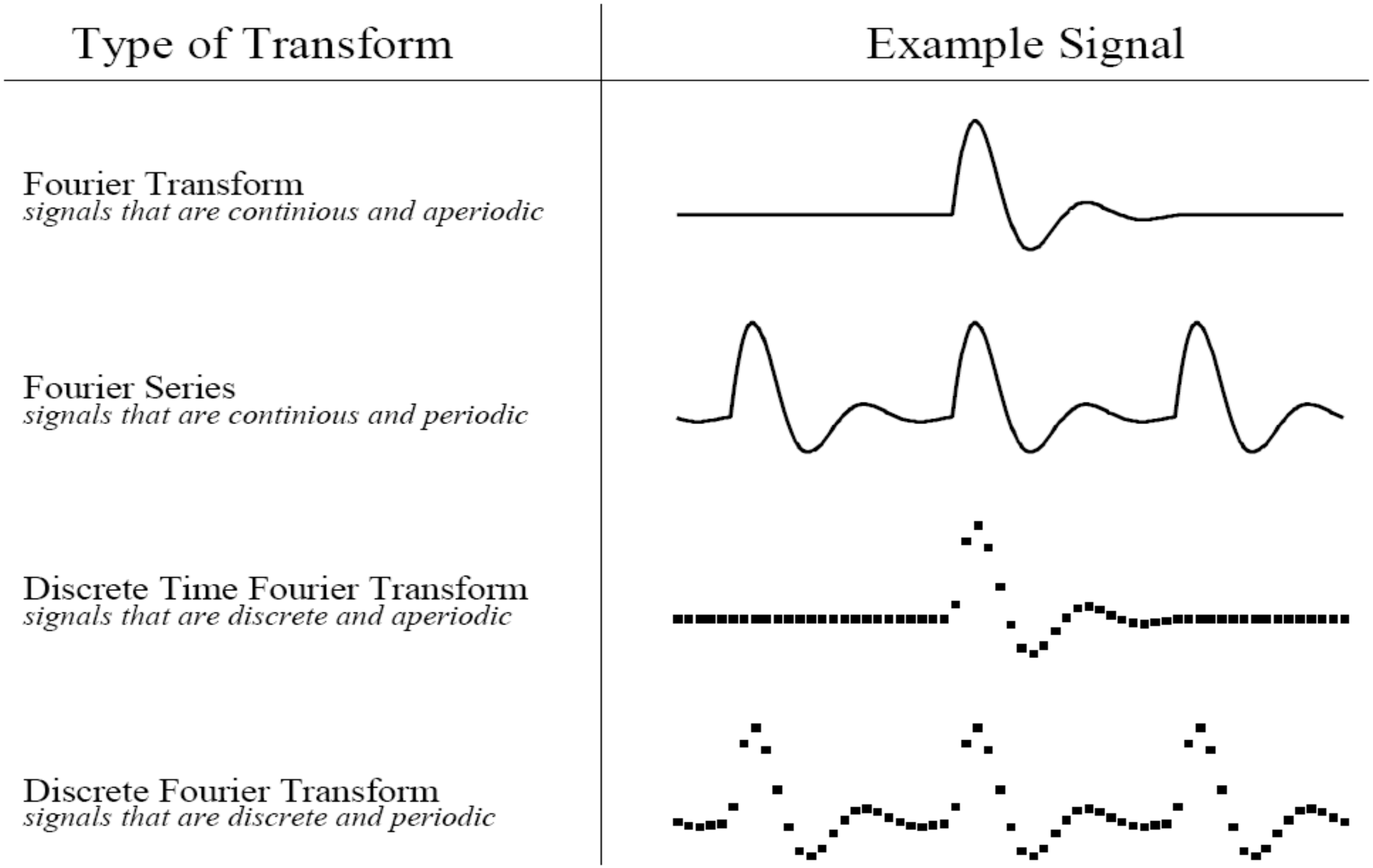


FIGURE 8-2
Illustration of the four Fourier transforms. A signal may be continuous or discrete, and it may be periodic or aperiodic. Together these define four possible combinations, each having its own version of the Fourier transform. The names are not well organized; simply memorize them.

Complex numbers

- A complex number C, is defined as $C = R + jI$
- In polar coordinates $C = |C| (\cos\theta + j\sin\theta)$
- In 1714, Roger Cotes published the formula:

$$j\theta = log\,(\cos\theta + j\sin\theta)$$

- Cotes never got the due credit for discovering this ground-breaking formula; it is instead named after Euler and written in reverse:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

- Therefore $C = |C|e^{j\theta}$

Complex numbers

$$C = R + jI,\ C = |C|e^{j\theta}$$

- Complex conjugate

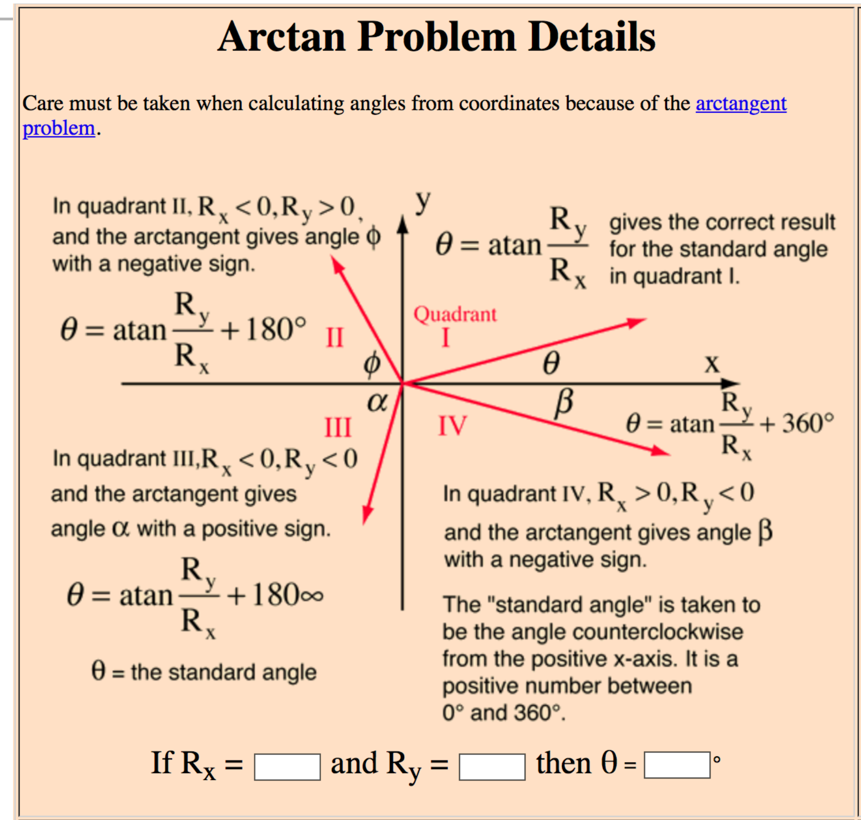
$$C^* = R - jI$$

- Magnitude

$$|C| = \sqrt{R^2 + I^2}$$

- Angle

$\theta = \arctan\left(\frac{I}{R}\right)$ (I quadrant, standard angle)



Fourier Series

- $f(t)$ is periodic with period T

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} \ ,$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-j\frac{2\pi n}{T}t}\, dt$$

for $n = 0, \pm 1, \pm 2, \ldots$

Impulses and Their Shifting Property

- Unit impulse

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

- Shifting property

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

Impulses and Their Shifting Property

- Unit discrete impulse

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

- Shifting property

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

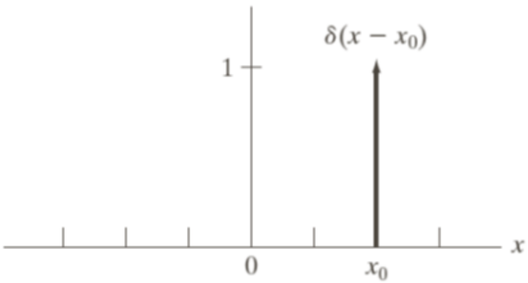


FIGURE 4.2
A unit discrete impulse located at $x = x_0$. Variable x is discrete, and δ is 0 everywhere except at $x = x_0$.

Fourier Transform

$$\Im(f(t)) = F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}\, dt$$

- Inverse Fourier transform

$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}\, d\mu$$

- Using Euler formula:

$$F(\mu) = \int_{-\infty}^{\infty} f(t)[\cos(2\pi\mu t) - j\sin(2\pi\mu t)]\, dt$$

Fourier Transform

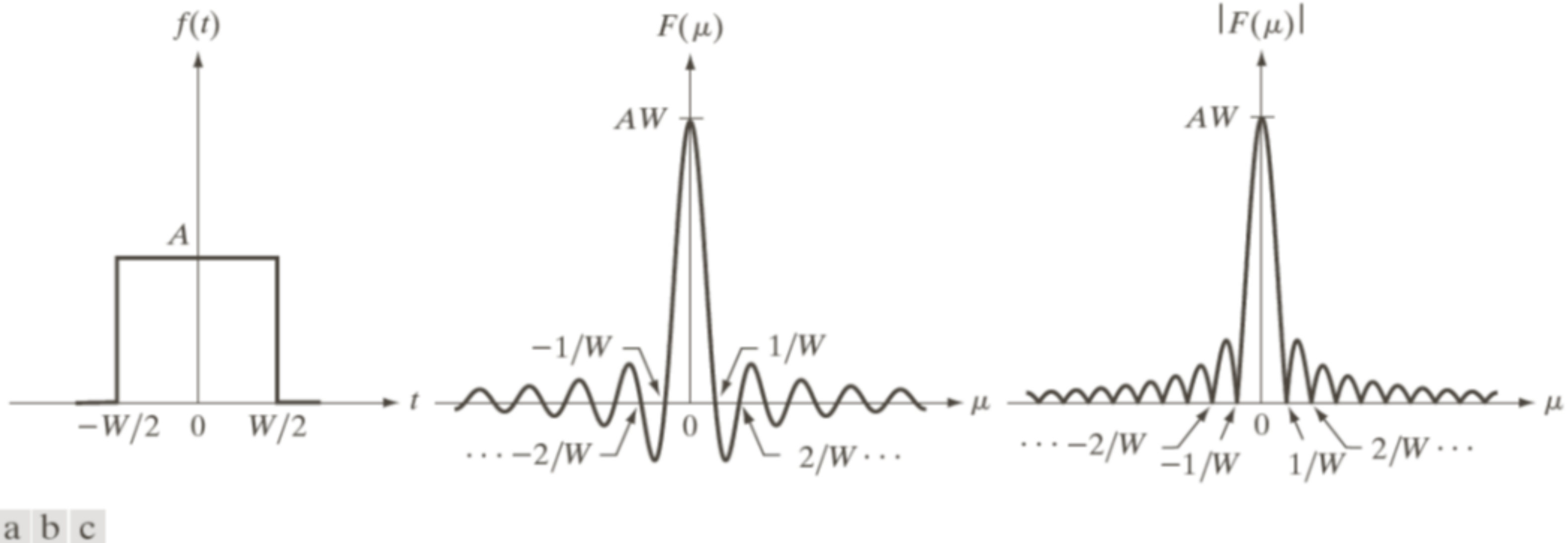


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

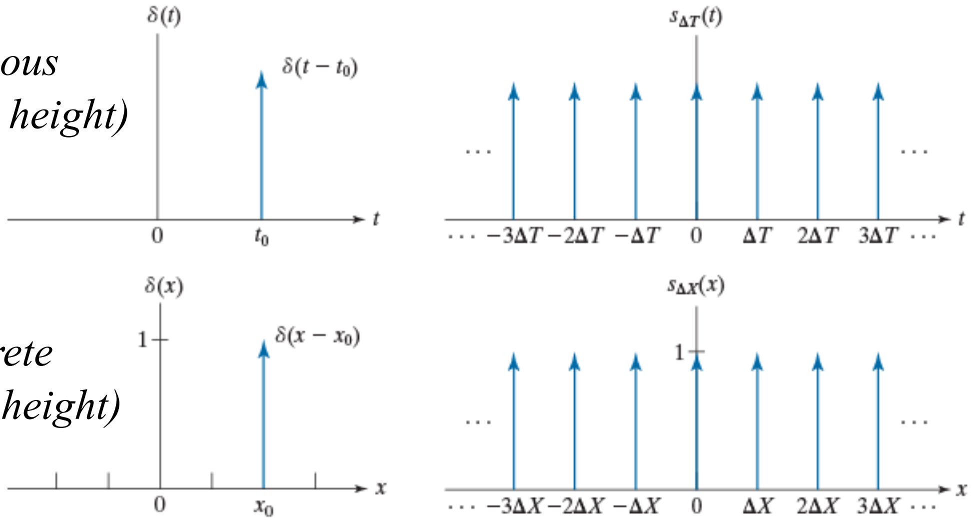
Impulse Train and Its Transform

- Impulse train

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

Continuous (infinite height)

Discrete (unit height)



a b
c d

FIGURE 4.3 (a) Continuous impulse located at $t = t_0$. (b) An impulse train consisting of continuous impulses. (c) Unit discrete impulse located at $x = x_0$. (d) An impulse train consisting of discrete unit impulses.

- Fourier transform of impulse train

- $\Im(s_{\Delta T}(t)) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$

Also an impulse train

Convolution

- Convolution of two continuous functions $f(t)$ and $h(t)$

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

- Convolution theorem

$$f(t) \star h(t) \Leftrightarrow H(\mu)F(\mu)$$
$$f(t)h(t) \Leftrightarrow H(\mu) \star F(\mu)$$

Sampling

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

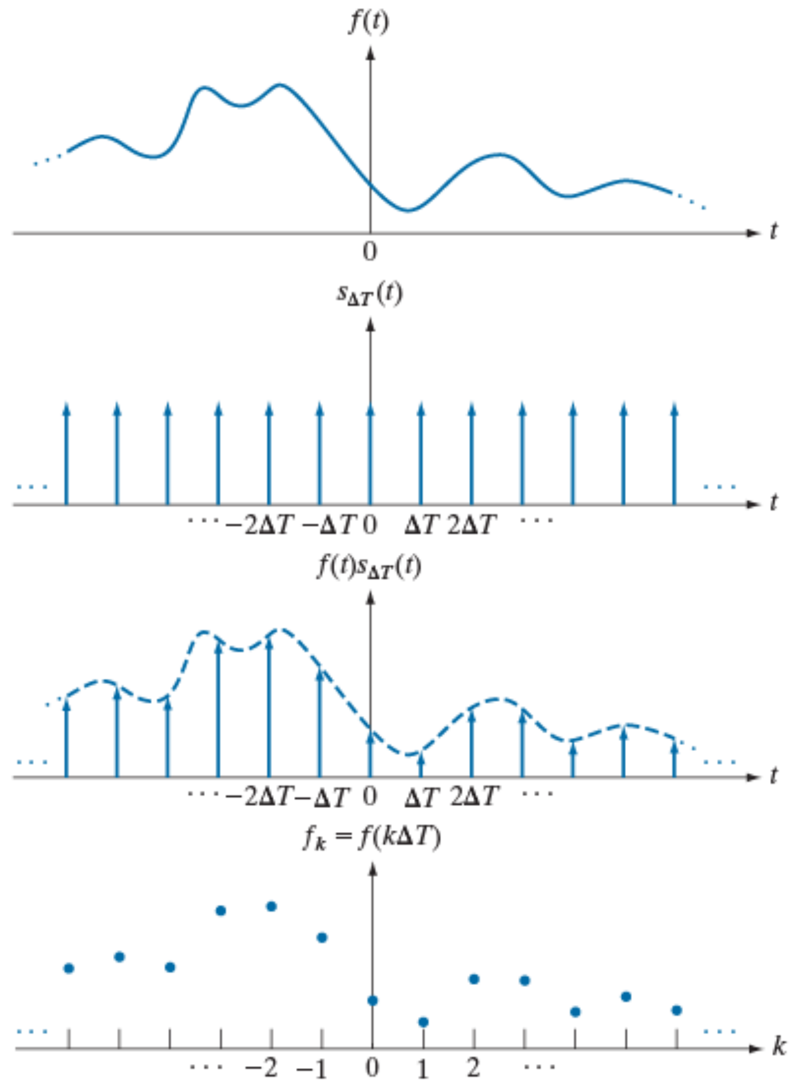


FIGURE 4.5 (a) A continuous function. (b) Train of impulses used to model sampling. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of impulses. (The dashed line in (c) is shown for reference. It is not part of the data.)

Fourier Transform of Sampled Functions

- We had:

$$\Im(s_{\Delta T}(t)) = S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

- Then

$$\tilde{F}(\mu) = \Im\left(\tilde{f}(t)\right) = \Im(f(t)s_{\Delta T}(t))$$

$$= F(\mu) \star S(\mu)$$

$$= \int_{-\infty}^{\infty} F(\tau)S(\mu - \tau)d\tau$$

Fourier Transform of Sampled Functions

$$= \int_{-\infty}^{\infty} F(\tau)S(\mu - \tau)d\tau$$

$$= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau)\delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta T})$$

Fourier Transform of Sampled Functions

$$\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

- $\tilde{F}(\mu)$ is an infinite, periodic sequence of copies of $F(\mu)$
- Separation between copies is $\frac{1}{\Delta T}$
- $\tilde{F}(\mu)$ is continuous

Fourier Transform of Sampled Functions

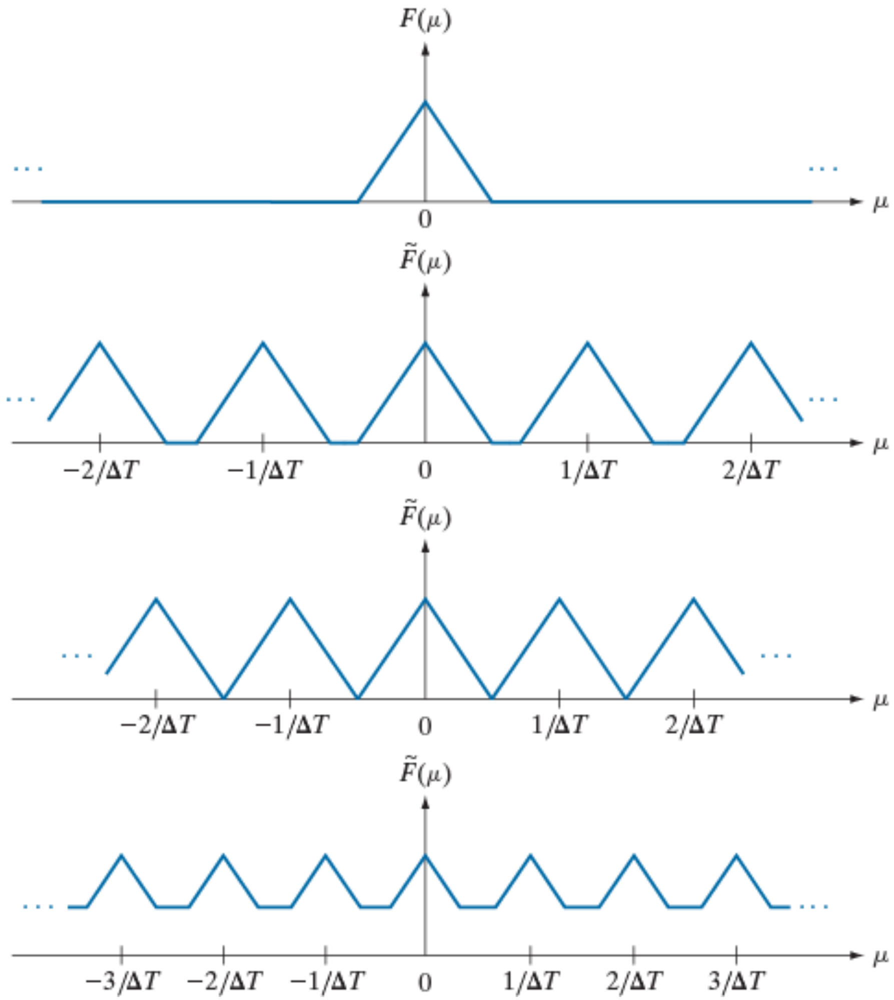


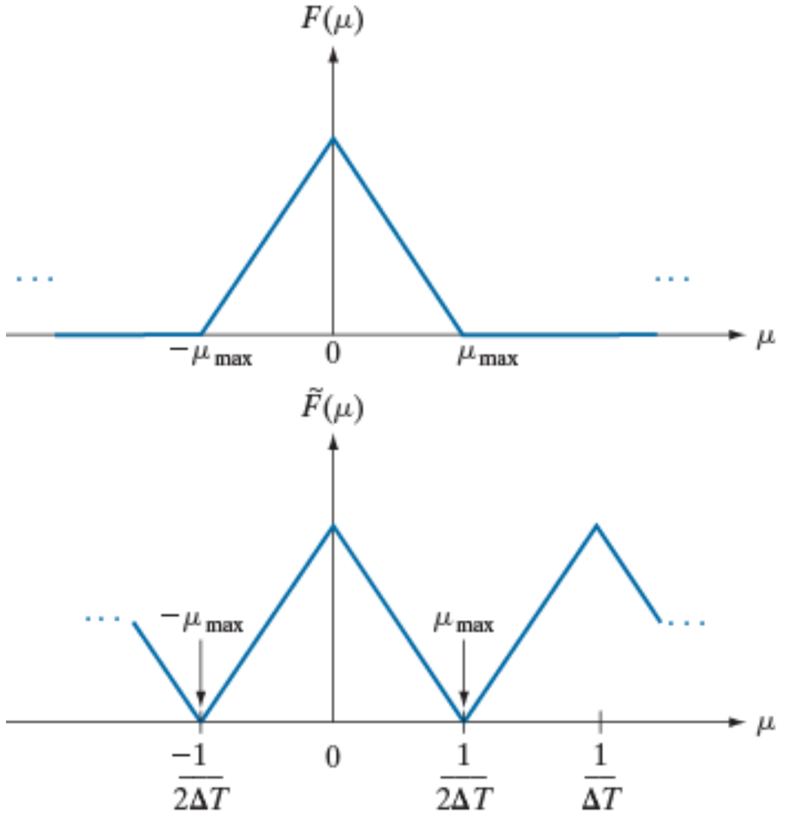
FIGURE 4.6 (a) Illustrative sketch of the Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled functions under the conditions of over-sampling, critically sampling, and under-sampling, respectively.

Sampling Theorem

- Signal $F(\mu)$ can be recovered uniquely from $\tilde{F}(\mu)$ if $\frac{1}{\Delta T} > 2\mu_{\text{max}}$
- Here $\frac{1}{\Delta T}$ is the Nyquist rate

a
b

FIGURE 4.7
(a) Illustrative sketch of the Fourier transform of a band-limited function.
(b) Transform resulting from critically sampling that band-limited function.



Recovery of $F(\mu)$ from $\tilde{F}(\mu)$

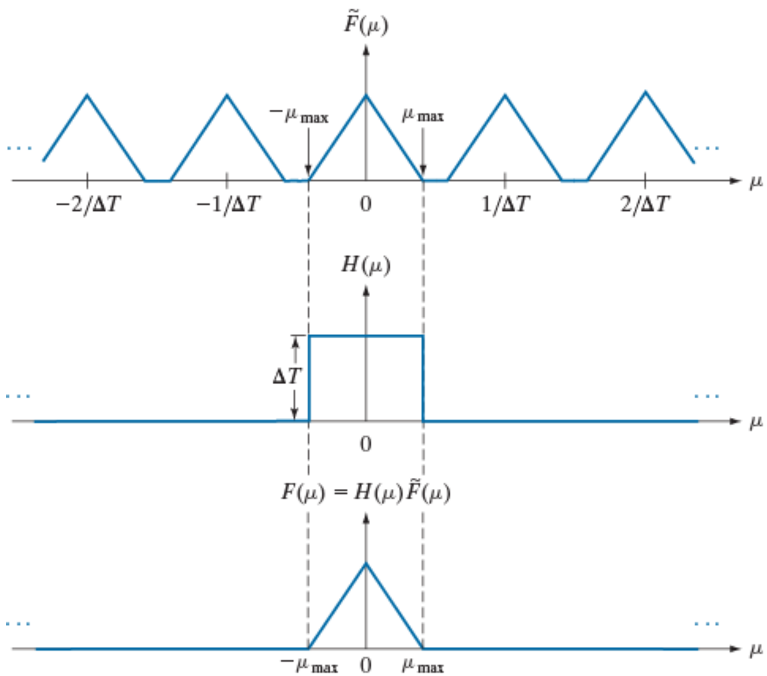
$$H(\mu) = \begin{cases} \Delta T, & -\mu_{\max} < \mu < \mu_{\max} \\ 0, & \text{otherwise} \end{cases}$$

$$F(\mu) = H(\mu)\tilde{F}(\mu)$$

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t}\, d\mu$$

a
b
c

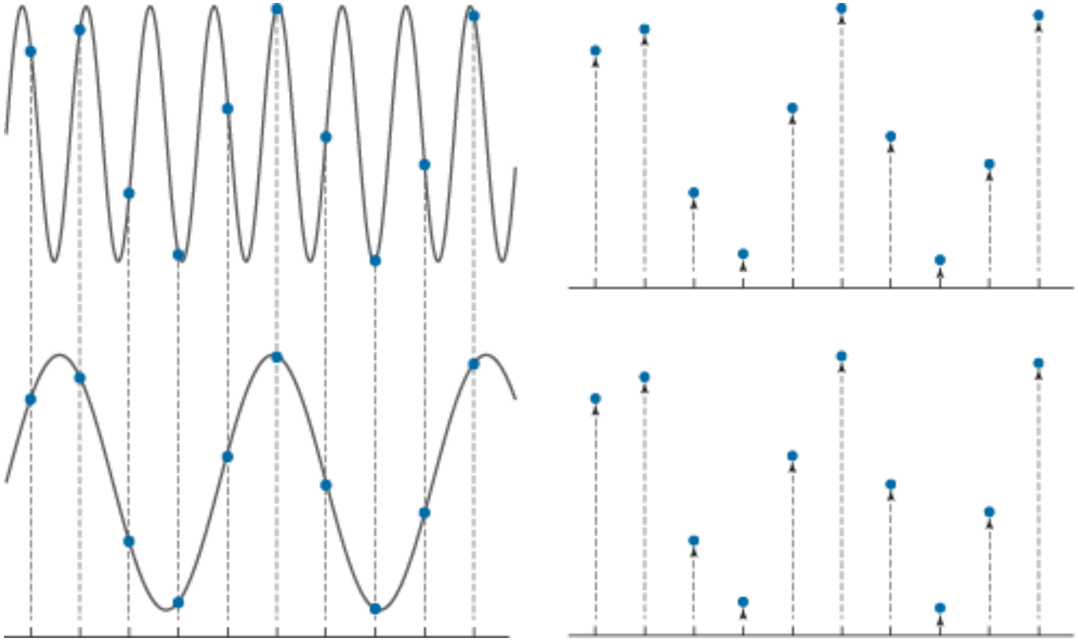
FIGURE 4.8 (a) Fourier transform of a sampled, band-limited function. (b) Ideal lowpass filter transfer function. (c) The product of (b) and (a), used to extract one period of the infinitely periodic sequence in (a).



Problems

a	b
c	d

FIGURE 4.9 The functions in (a) and (c) are totally different, but their digitized versions in (b) and (d) are identical. Aliasing occurs when the samples of two or more functions coincide, but the functions are different elsewhere.



Aliasing

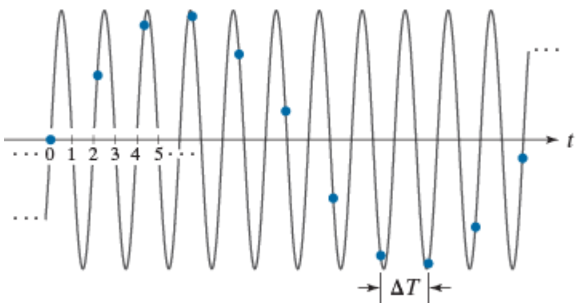


FIGURE 4.11 Illustration of aliasing. The under-sampled function (dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

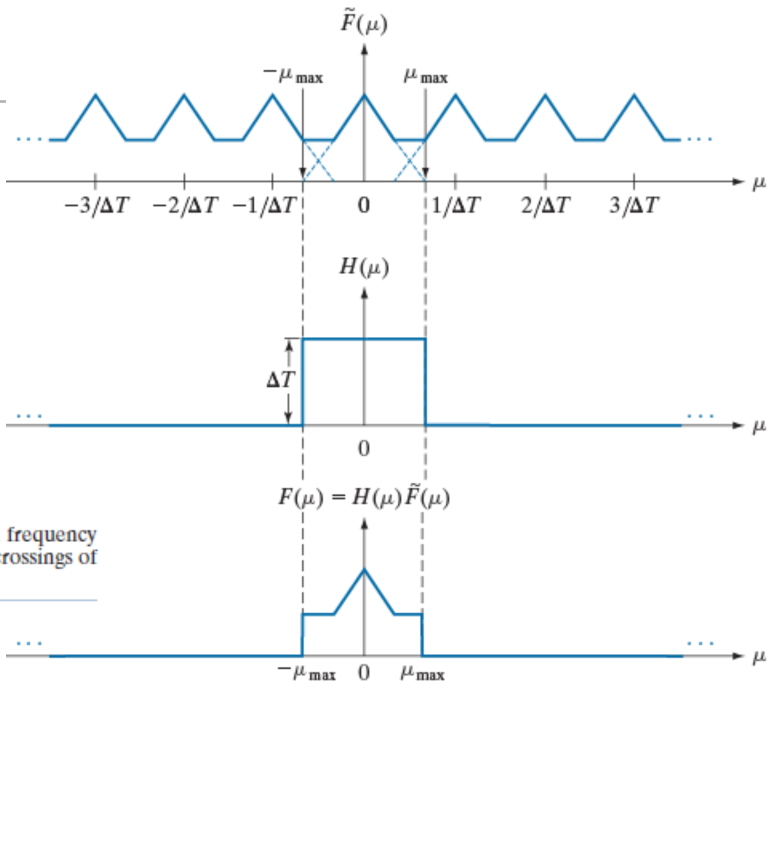


FIGURE 4.10 (a) Fourier transform of an under-sampled, band-limited function. (Interference between adjacent periods is shown dashed). (b) The same ideal lowpass filter used in Fig. 4.8. (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, consequently, of $f(t)$.

Discrete Fourier Transform (DFT)

Let $\{f_n\}$ be M samples of $f(t)$. Then the DFT of $\{f_n\}$, $\{F_m\}$, is as follows:

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}, m = 0,1,2, \ldots M-1$$

- Inverse Discrete Fourier Transform

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M}, n = 0,1,2, \ldots M-1$$

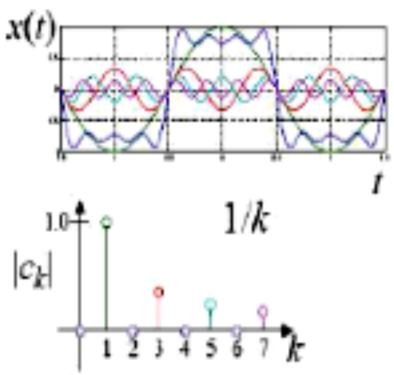
$\{F_m\}$ is a sequence of M complex discrete values

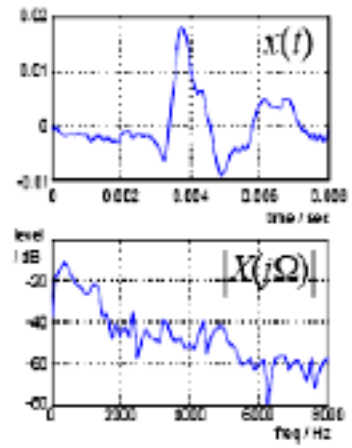
Discrete Fourier Transform (DFT)

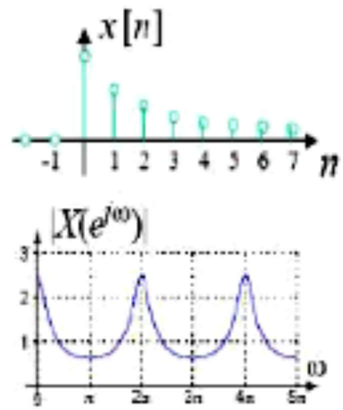
- From now on $f_n = f(x)$ and $F_m = F(u)$ (still discrete!)
- Applicable to any finite set of samples taken uniformly (does not depend on sampling interval ΔT)
- Both forward and inverse DFT are infinitely periodic with period M ($f(x) = f(x + kM)$)
- Discrete convolution

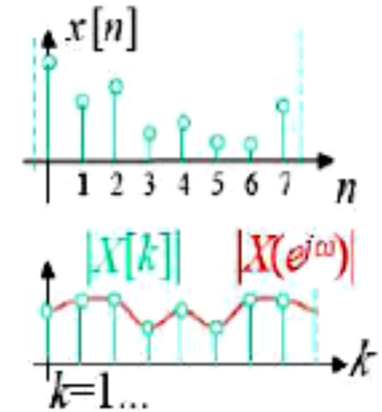
$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m), x = 0,1,\dots,M-1$$

SUMMARY









Fourier series:

- time: periodic continuous
- frequency: discrete

Fourier transform:

- time: non-periodic, continuous
- frequency: continuous

DTFT:

- time: discrete
- frequency: periodic, continuous

DFT:

- time: “periodic”, discrete
- frequency: periodic, discrete

$T \rightarrow \infty \Rightarrow, \omega_0 \rightarrow 0$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

Uniform sampling in [0,2π]

2-D Impulses and Their Shifting Property

- Unit impulse

$$\delta(t,z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t,z)\,dt\,dz = 1$$

- Shifting property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t,z)\,dt\,dz = f(0,0)$$

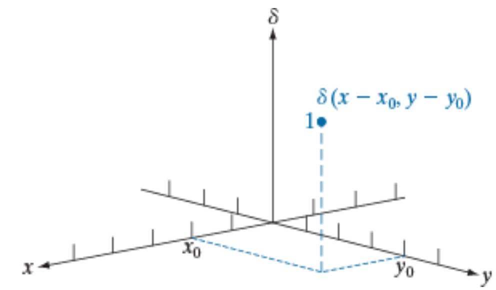
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t-t_0,z-z_0)\,dt\,dz = f(t_0,z_0)$$

2-D Impulses and Their Shifting Property

- Unit discrete impulse

$$\delta(x,y)=\begin{cases}1 & \text{if } x=y=0\\ 0 & \text{otherwise}\end{cases}$$

FIGURE 4.13
2-D unit discrete impulse. Variables x and y are discrete, and δ is zero everywhere except at coordinates (x_0, y_0), where its value is 1.



- Shifting property

$$\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} f(x,y)\delta(x,y)=f(0,0)$$

$$\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} f(x,y)\delta(x-x_0,y-y_0)=f(x_0,y_0)$$

2-D Continuous Fourier Transform

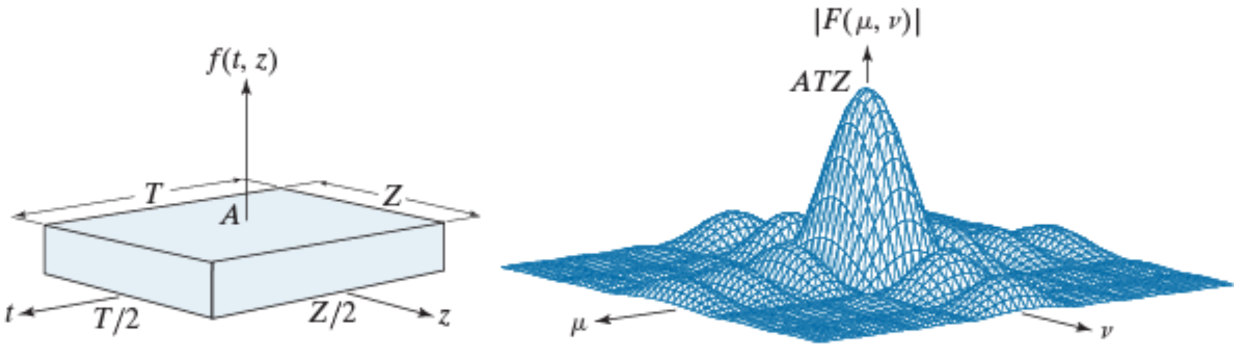
$$F(\mu, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+vz)}\,dt\,dz$$

- Inverse Fourier Transform

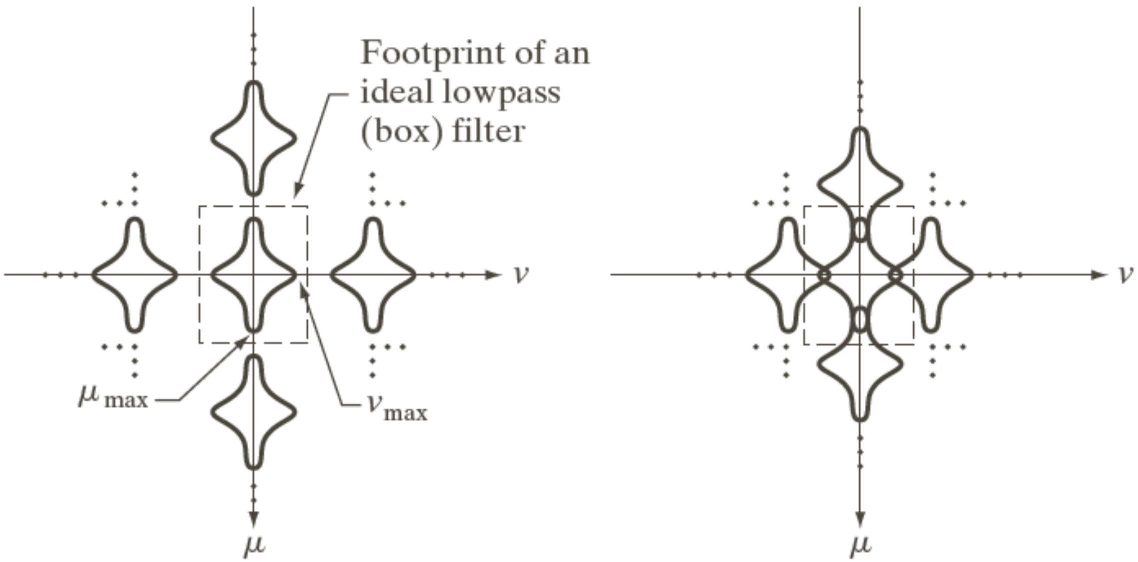
$$f(t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu,v)e^{j2\pi(\mu t+vz)}\,d\mu\,dv$$

a b

FIGURE 4.14 (a) A 2-D function and (b) a section of its spectrum. The box is longer along the t-axis, so the spectrum is more contracted along the μ-axis.



Aliasing in Images

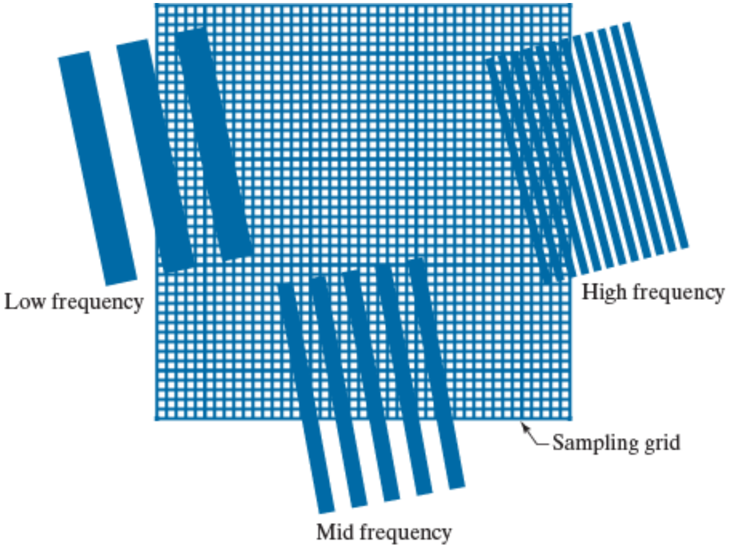


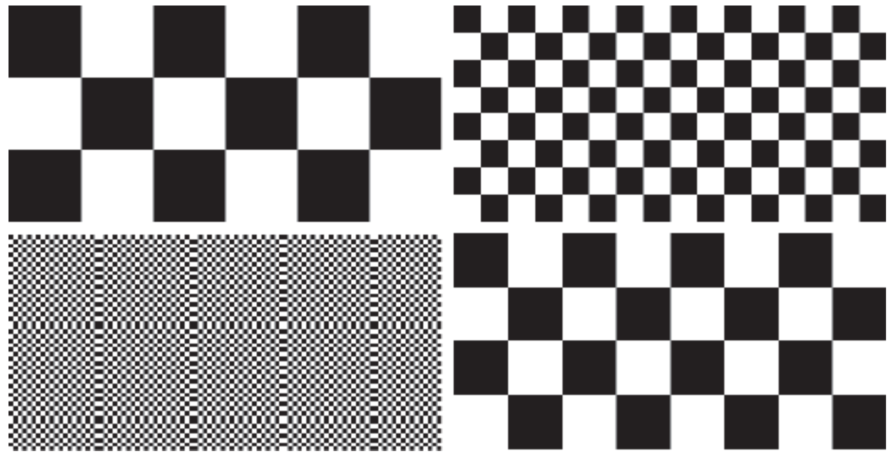
a b

FIGURE 4.16 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) an under-sampled, band-limited function.

- To avoid aliasing $\frac{1}{\Delta T} > 2\mu_{\max}$ and $\frac{1}{\Delta Z} > 2v_{\max}$

FIGURE 4.17
Various aliasing effects resulting from the interaction between the frequency of 2-D signals and the sampling rate used to digitize them. The regions outside the sampling grid are continuous and free of aliasing.





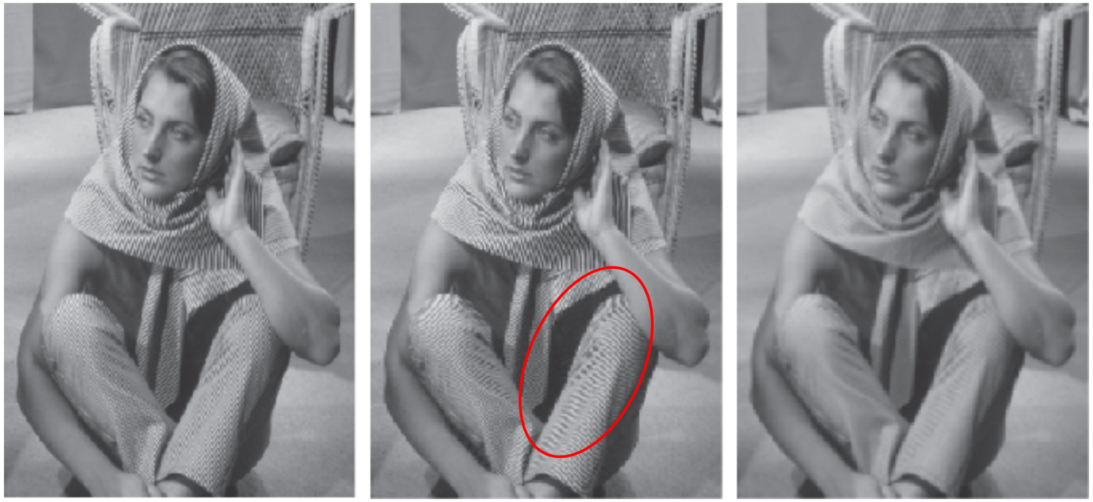
a b
c d

FIGURE 4.18 Aliasing. In (a) and (b) the squares are of sizes 16 and 6 pixels on the side. In (c) and (d) the squares are of sizes 0.95 and 0.48 pixels, respectively. Each small square in (c) is one pixel. Both (c) and (d) are aliased. Note how (d) masquerades as a “normal” image.

When the size of the squares is reduced to slightly less than one pixel, a severely aliased image results, see (c) where the squares are approx. 0.95x0.95 pixels).

Reducing the size of the squares less than 0.5 pixels on the side yields the image in (d)!! The aliased result looks like a normal checkerboard pattern (the image would normally result from a checkerboard image whose squares are 12 pixels on the side!)

Aliasing can create results that may be visually quite misleading!



Original

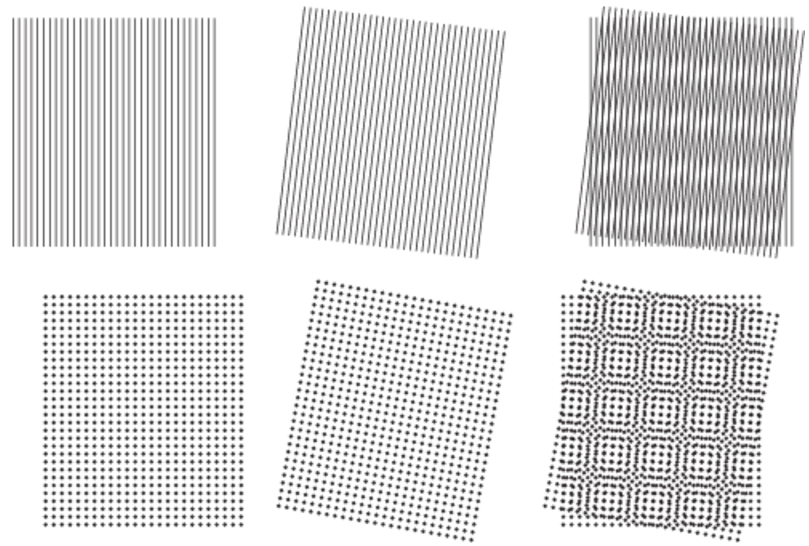
Original subsampled and upsampled

Blurred original subsampled and upsampled

a b c

FIGURE 4.19 Illustration of aliasing on resampled natural images. (a) A digital image of size 772 × 548 pixels with visually negligible aliasing. (b) Result of resizing the image to 33% of its original size by pixel deletion and then restoring it to its original size by pixel replication. Aliasing is clearly visible. (c) Result of blurring the image in (a) with an averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

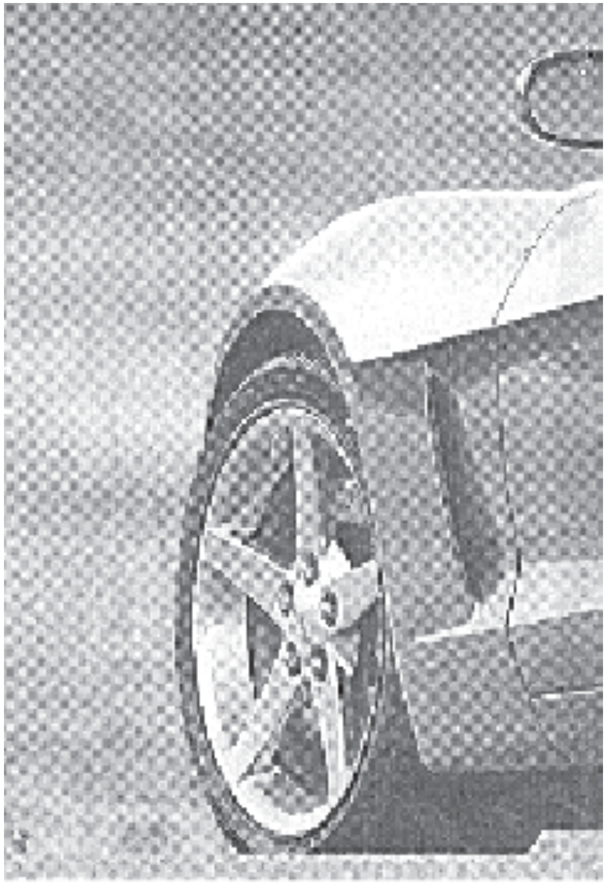
Chapter 4
Filtering in the Frequency Domain



a b c
d e f

FIGURE 4.20 Examples of the moiré effect. These are vector drawings, not digitized patterns. Superimposing one pattern on the other is analogous to multiplying the patterns.

FIGURE 4.21
A newspaper image digitized at 75 dpi. Note the moiré-like pattern resulting from the interaction between the $\pm 45°$ orientation of the half-tone dots and the north-south orientation of the sampling elements used to digitized the image.



2-D Discrete Fourier Transform

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

- Inverse discrete Fourier transform

$$f(x,y) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(u,v)\, e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

where $f(x,y)$ is a digital image of size MxN

2-D DFT - Translation

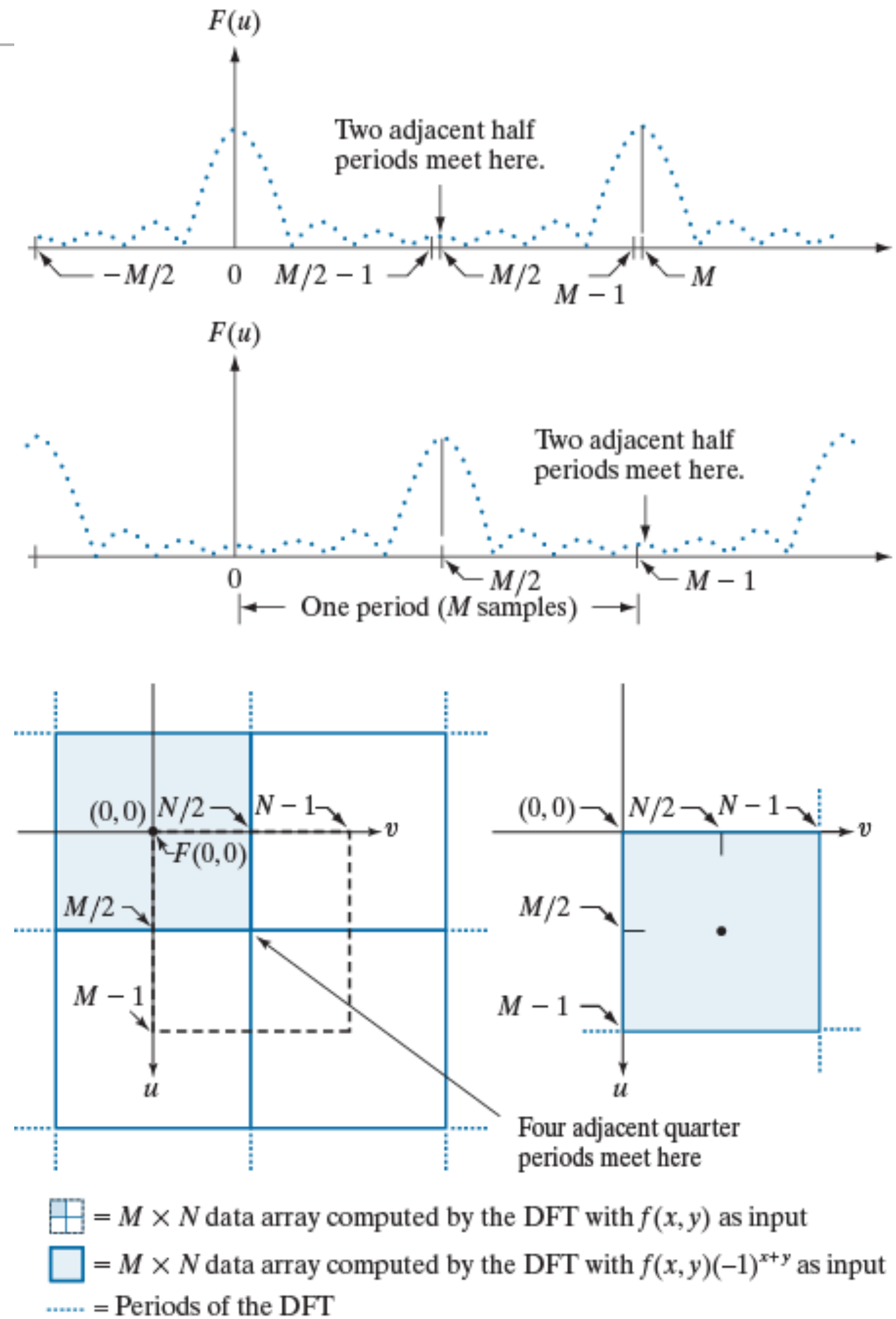
$$f(x,y)e^{j2\pi(\frac{u_0 x}{M}+\frac{v_0 y}{N})} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)\, e^{-j2\pi(\frac{u x_0}{M}+\frac{v y_0}{N})}$$

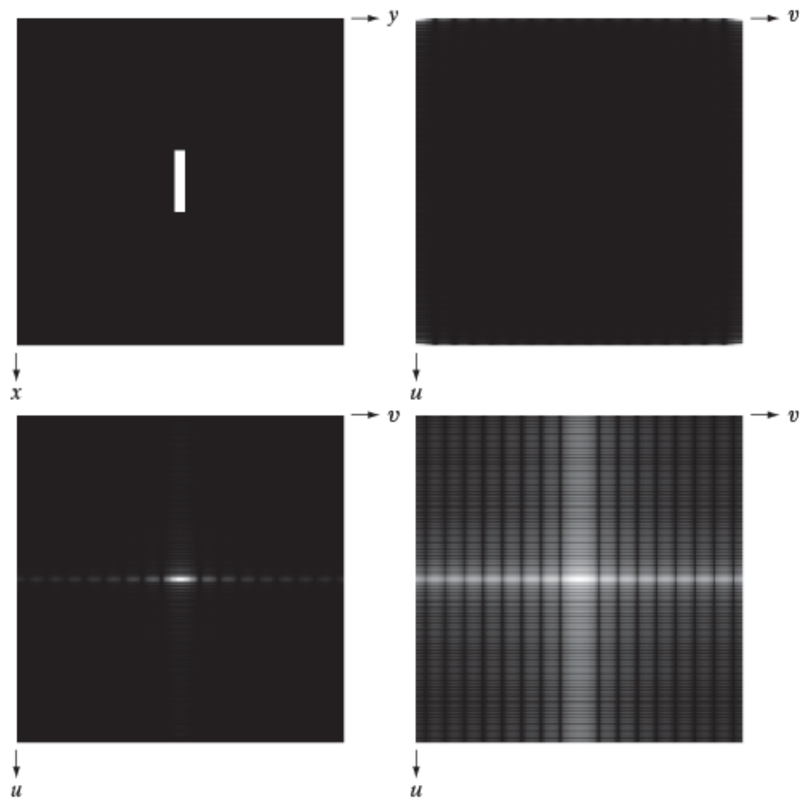
- Special case when $u_0 = M/2$ and $v_0 = N/2$,
  $$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$$
- This shifts $F(0,0)$ to the center of the frequency rectangle.

a b
c d

FIGURE 4.22 Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The area within the dashed rectangle is the data array, $F(u,v)$, obtained with Eq. (4-67) with an image $f(x,y)$ as the input. This array consists of four quarter periods. (d) Shifted array obtained by multiplying $f(x,y)$ by $(-1)^{x+y}$ before computing $F(u,v)$. The data now contains one complete, centered period, as in (b).



Centering the Fourier Transform



a b
c d

FIGURE 4.23
(a) Image.
(b) Spectrum, showing small, bright areas in the four corners (you have to look carefully to see them).
(c) Centered spectrum.
(d) Result after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The right-handed coordinate convention used in the book places the origin of the spatial and frequency domains at the top left (see Fig. 2.19).

2-D DFT: Zero-Frequency Term

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$F(0,0) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = MN\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

$= \mathrm{MN}\overline{f(x,y)}$, where $\overline{f(x,y)}$ denotes the average value of $f(x,y)$

- $F(0,0)$ is sometimes called the *dc component* ('dc' for direct current, current of zero frequency)

Fourier Spectrum and Phase Angle

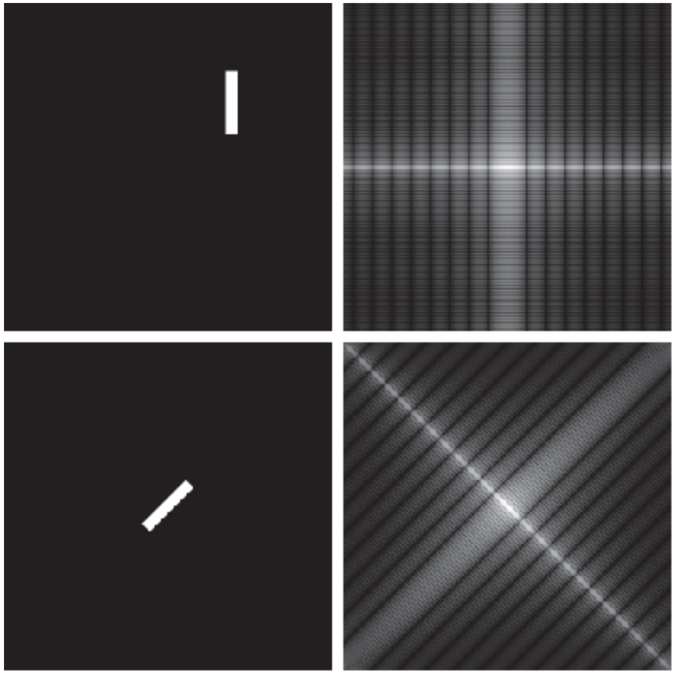
$F(u,v) = |F(u,v)|\, e^{j\phi(u,v)}$

where the magnitude $|F(u,v)| = |R^2(u,v) + \mathrm{I}^2(u,v)|^{\frac{1}{2}}$ is called the *Fourier* (or *frequency*) *spectrum,* and

$\phi(u,v) = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]$ is the *phase angle.*

$P(u,v) = |F(u,v)|^2 = |R^2(u,v) + \mathrm{I}^2(u,v)|$ is the *power spectrum.*

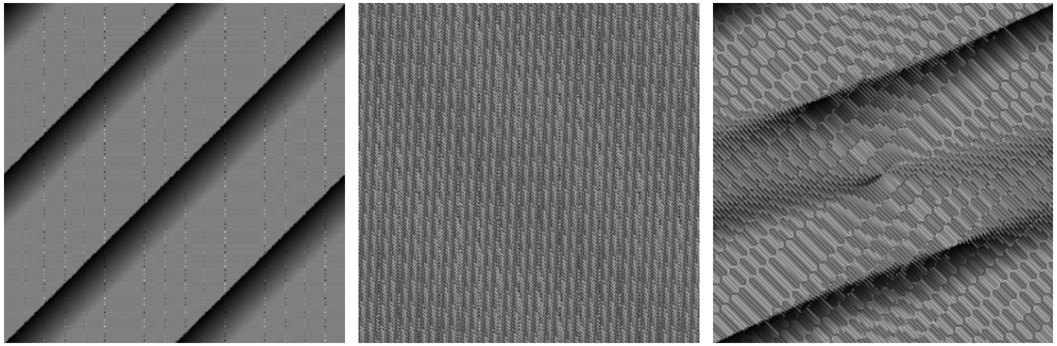
Translation and Rotation



a b
c d

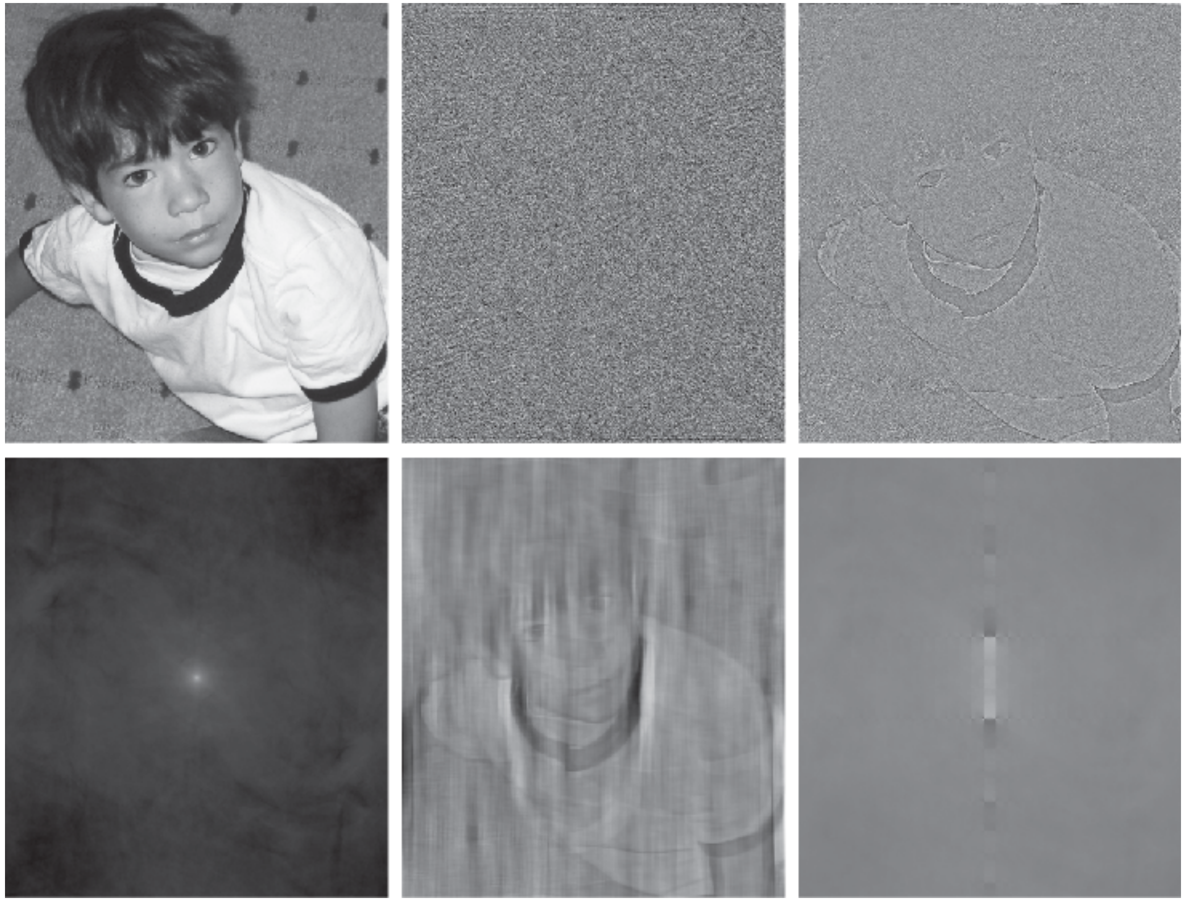
FIGURE 4.24
(a) The rectangle in Fig. 4.23(a) translated.
(b) Corresponding spectrum.
(c) Rotated rectangle.
(d) Corresponding spectrum.
The spectrum of the translated rectangle is identical to the spectrum of the original image in Fig. 4.23(a).

Phase Angle



a b c

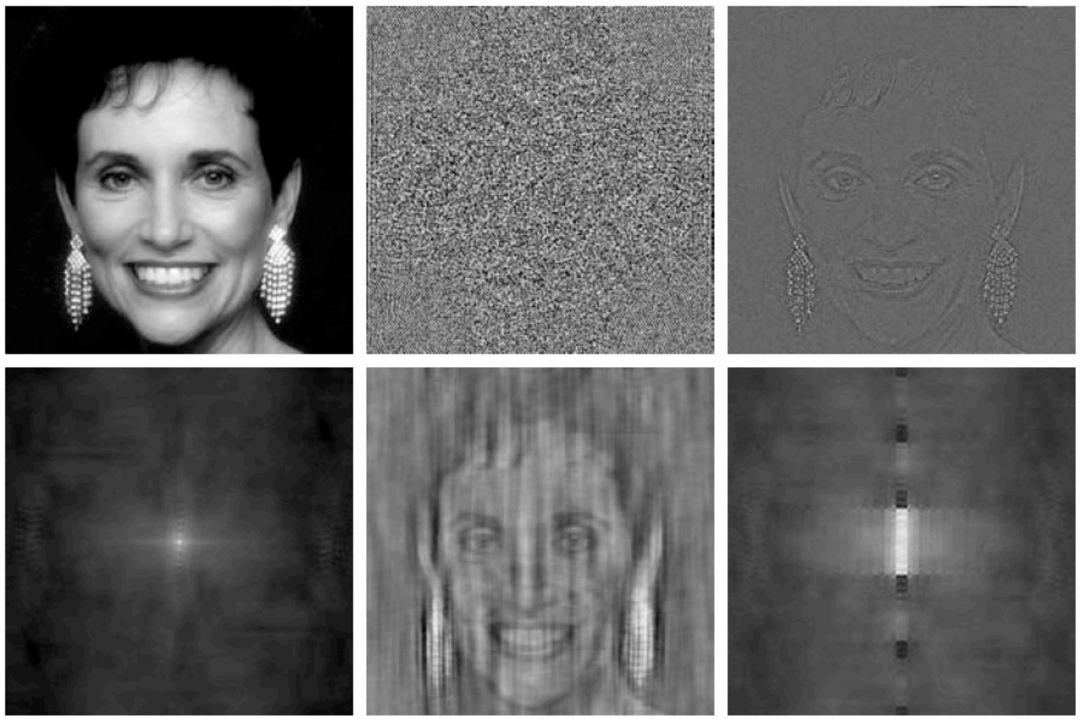
FIGURE 4.25 Phase angle images of (a) centered, (b) translated, and (c) rotated rectangles.



a b c
d e f

FIGURE 4.26 (a) Boy image. (b) Phase angle. (c) Boy image reconstructed using only its phase angle (all shape features are there, but the intensity information is missing because the spectrum was not used in the reconstruction). (d) Boy image reconstructed using only its spectrum. (e) Boy image reconstructed using its phase angle and the spectrum of the rectangle in Fig. 4.23(a). (f) Rectangle image reconstructed using its phase and the spectrum of the boy's image.

Spectrum and Phase Angle in Reconstruction



a b c
d e f

~~FIGURE 4.27~~ (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in ~~Fig. 4.24(~~a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

Chapter 4
Filtering in the Frequency Domain

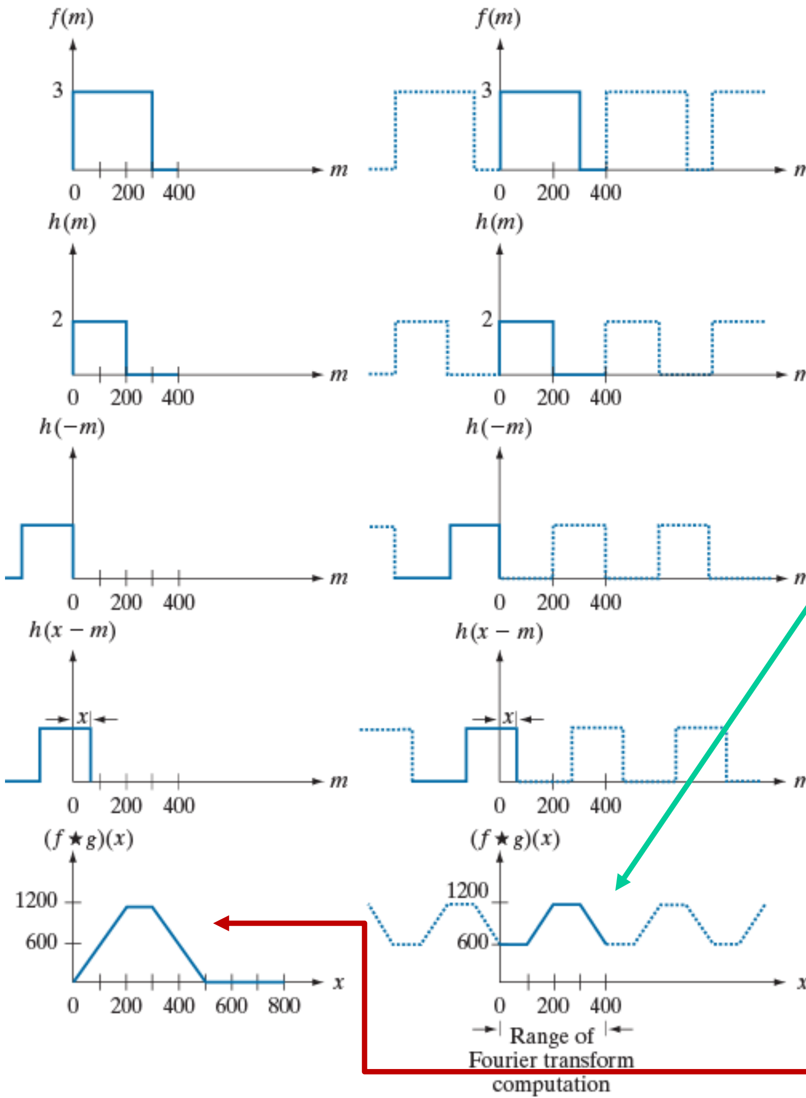


FIGURE 4.27 Left column: Spatial convolution computed with Eq. (3-44), using the approach discussed in Section 3.4. Right column: Circular convolution. The solid line in (j) is the result we would obtain using the DFT, or, equivalently, Eq. (4-48). This erroneous result can be remedied by using zero padding.

What happens if we convolve f and h without zero padding?

f and h are 400-point sequences
DFT assumes periodic functions (left and right)
The convolution of two periodic functions is itself periodic
However, the closeness of the periods makes them interfere and cause wraparound (aliasing) errors!
Wrong convolution results

Remedy:
Append enough zeros to f and h ($P \geq A+B-1$, *A and B are the size of f and h, respectively*)
Compute the convolution and the result will match the left side as expected

Original image

Blurred without padding
Note: blurring is not uniform, i.e., the top white edge is blurred but the sides are not!

Blurred with padding
Note: uniform dark borders as a result of zero-padding, as expected

a b c

FIGURE 4.31 (a) A simple image. (b) Result of blurring with a Gaussian lowpass filter without padding. (c) Result of lowpass filtering with zero padding. Compare the vertical edges in (b) and (c).

Further details about the discrepancy between (b) and (c) are given in the next slide.

Chapter 4
Filtering in the Frequency Domain

Padding creates a uniform boarder around each image of the periodic sequence and the convolution gives the correct result

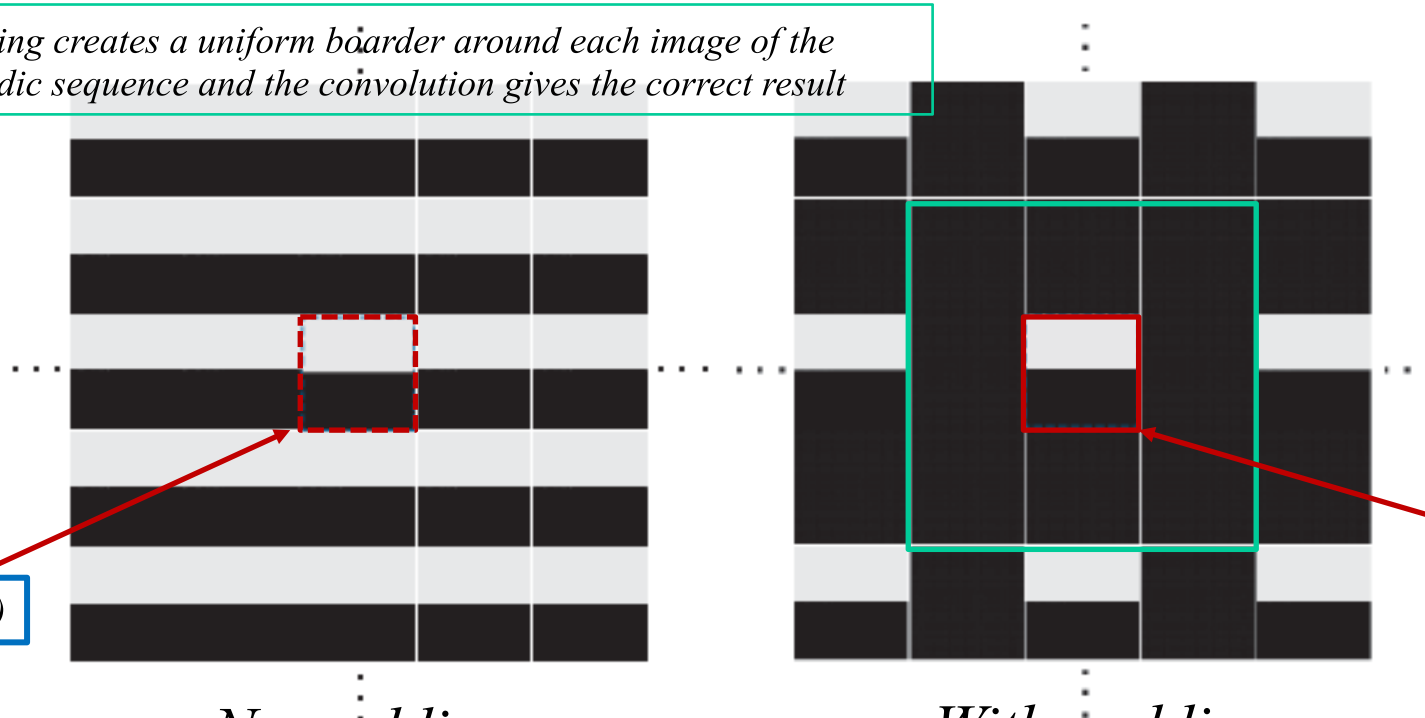


Fig. 4.31 (a)

Fig. 4.31 (c)

No padding

With padding

a b

FIGURE 4.32 (a) Image periodicity without image padding. (b) Periodicity after padding with 0's (black). The dashed areas in the center correspond to the image in Fig. 4.31(a). Periodicity is inherent when using the DFT. (The thin white lines in both images are superimposed for clarity; they are not part of the data.)

TABLE 4.3 Summary of DFT definitions and corresponding expressions.

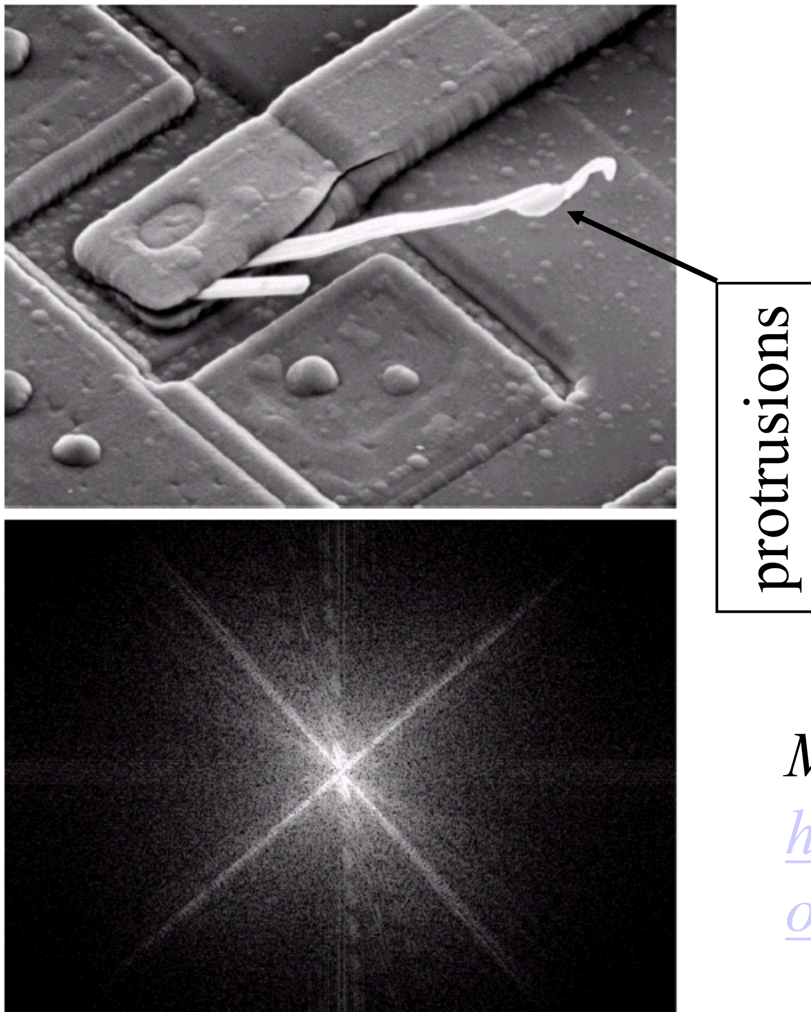
	Name	Expression(s)
1)	Discrete Fourier transform (DFT) of $f(x,y)$	$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
2)	Inverse discrete Fourier transform (IDFT) of $F(u,v)$	$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
3)	Spectrum	$\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$ $\quad R = \text{Real}(F);\ I = \text{Imag}(F)$
4)	Phase angle	$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
5)	Polar representation	$F(u,v) = \lvert F(u,v)\rvert e^{j\phi(u,v)}$
6)	Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
7)	Average value	$\bar{f} = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = \frac{1}{MN}F(0,0)$
8)	Periodicity (k_1 and k_2 are integers)	$F(u,v) = F(u+k_1M,v) = F(u,v+k_2N) = F(u+k_1,v+k_2N)$ $f(x,y) = f(x+k_1M,y) = f(x,y+k_2N) = f(x+k_1M,y+k_2N)$
9)	Convolution	$(f \bigstar h)(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$
10)	Correlation	$(f \star h)(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^*(m,n)h(x+m,y+n)$
11)	Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.
12)	Obtaining the IDFT using a DFT algorithm	$MNf^*(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$ This equation indicates that inputting $F^*(u,v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x,y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.

TABLE 4.4 Summmary of DFT pairs. The closed-form expressions in 12 and 13 are valid only for continuous variables. They can be used with discrete variables by sampling the continuous expressions.

	Name	DFT Pairs
1)	Symmetry properties	See Table 4.1
2)	Linearity	$af_1(x,y) + bf_2(x,y) \Leftrightarrow aF_1(u,v) + bF_2(u,v)$
3)	Translation (general)	$f(x,y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4)	Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$
5)	Rotation	$f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$ $r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1}(y/x) \quad \omega = \sqrt{u^2+v^2} \quad \varphi = \tan^{-1}(v/u)$
6)	Convolution theorem†	$f \star h)(x,y) \Leftrightarrow (F \bullet H)(u,v)$ $(f \bullet h)(x,y) \Leftrightarrow (1/MN)[(F \star H)(u,v)]$
7)	Correlation theorem†	$(f \,☆\, h)(x,y) \Leftrightarrow (F^* \bullet H)(u,v)$ $(f^* \bullet h)(x,y) \Leftrightarrow (1/MN)[(F \,☆\, H)(u,v)]$
8)	Discrete unit impulse	$\delta(x,y) \Leftrightarrow 1$ $1 \Leftrightarrow MN\delta(u,v)$
9)	Rectangle	$\text{rec}[a,b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
10)	Sine	$\sin(2\pi u_0x/M + 2\pi v_0y/N) \Leftrightarrow \dfrac{jMN}{2}[\delta(u+u_0, v+v_0) - \delta(u-u_0, v-v_0)]$
11)	Cosine	$\cos(2\pi u_0x/M + 2\pi v_0y/N) \Leftrightarrow \dfrac{1}{2}[\delta(u+u_0, v+v_0) + \delta(u-u_0, v-v_0)]$
	The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.	
12)	Differentiation (the expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.	$\left(\dfrac{\partial}{\partial t}\right)^m\left(\dfrac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m(j2\pi\nu)^n F(\mu,\nu)$ $\dfrac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu,\nu); \quad \dfrac{\partial^n f(t,z)}{\partial z^m} \Leftrightarrow (j2\pi\nu)^n F(\mu,\nu)$
13)	Gaussian	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

† Assumes that $f(x,y)$ and $h(x,y)$ have been properly padded. Convolution is associative, commutative, and distributive. Correlation is distributive (see Table 3.5). The products are elementwise products (see Section 2.6).

Characteristics of the Frequency Domain



a
b

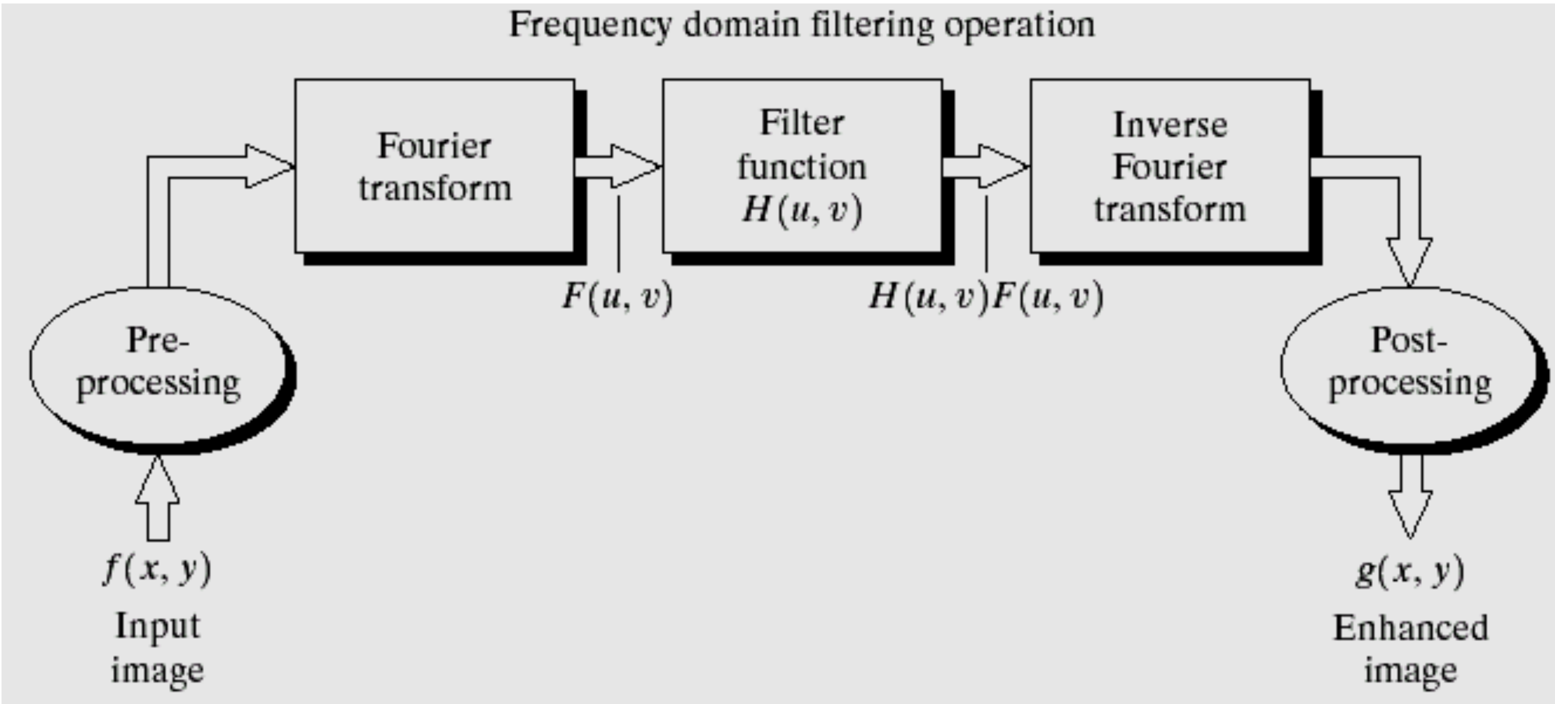
FIGURE 4.28 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

Notice the ±45° components and the vertical component which is slightly off-axis to the left! It corresponds to the protrusion caused by thermal failure above.

More examples →

http://www.cs.unm.edu/~brayer/vision/fourier.html

Basics of Filtering in the Frequency Domain



~~FIGURE 4.5~~ Basic steps for filtering in the frequency domain.

Spatial domain ⟺ *Fourier domain*

convolution *multiplication*

Basic Filtering Examples

1. Removal of image average

- in the spatial domain?
- in the frequency domain:

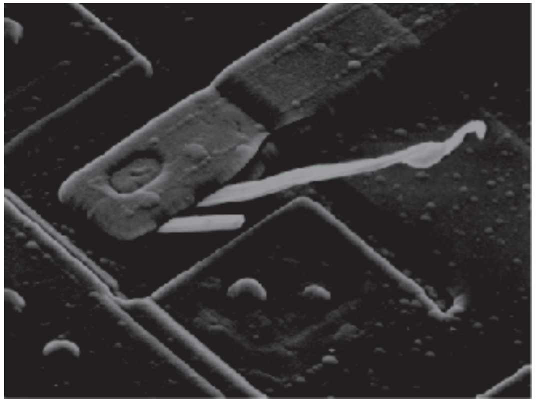
$$H(u,v) = \begin{cases} 0 & \text{if } (u,v) = (M/2, N/2) \\ 1 & \text{otherwise} \end{cases}$$

- the output is:

$$G(u,v) = H(u,v)\, F(u,v)$$

This is called the notch filter, i.e. a constant function with a whole at the origin.

FIGURE 4.29 Result of filtering the image in Fig. 4.28(a) with a filter transfer function that sets to 0 the dc term, $F(P/2, Q/2)$, in the centered Fourier transform, while leaving all other transform terms unchanged.



How is this image displayed if the average value is 0?!

Basic Filtering Examples

2. Low-pass, high-pass and offset high-pass filtering

The small offset a is added to the filter so as it will not completely eliminate F(0,0)

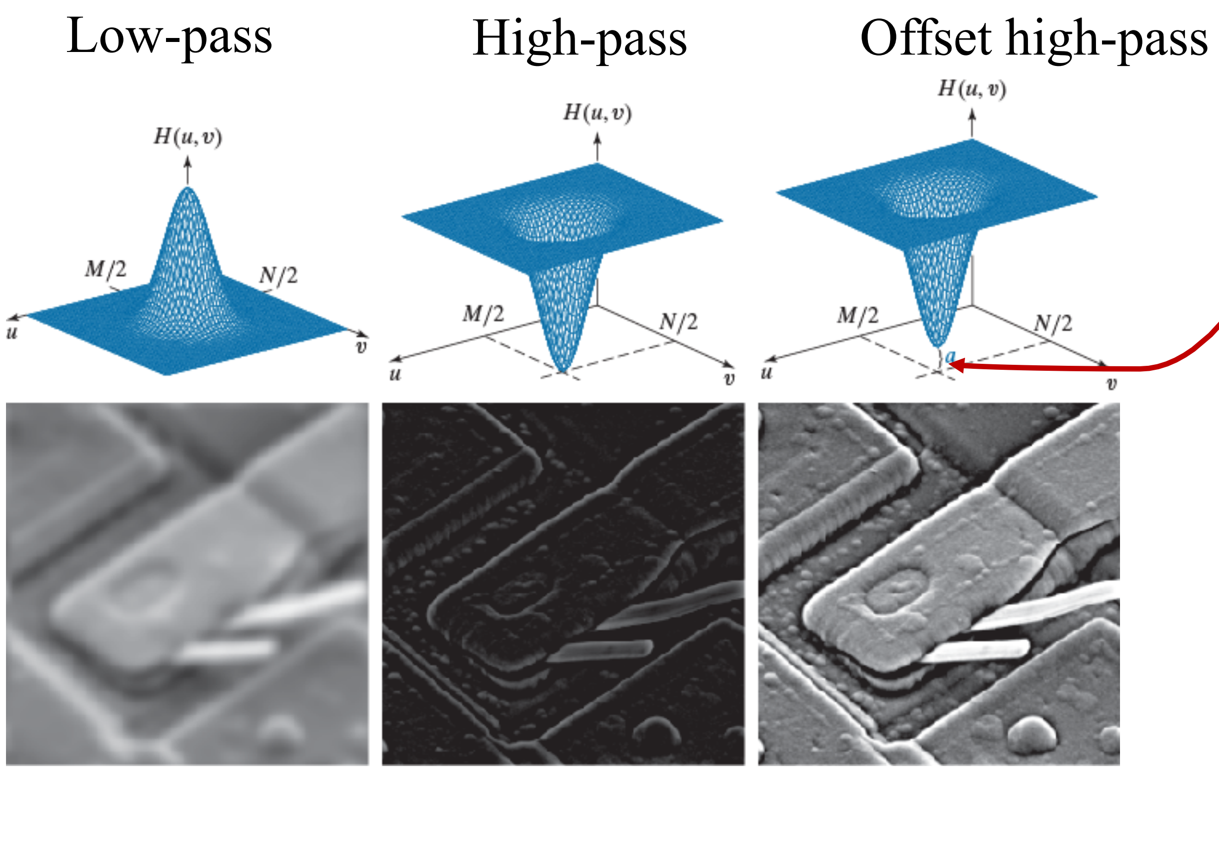


FIGURE 4.30 Top row: Frequency domain filter transfer functions of (a) a lowpass filter, (b) a highpass filter, and (c) an offset highpass filter. Bottom row: Corresponding filtered images obtained using Eq. (4-104). The offset in (c) is $a = 0.85$, and the height of $H(u,v)$ is 1. Compare (f) with Fig. 4.28(a).

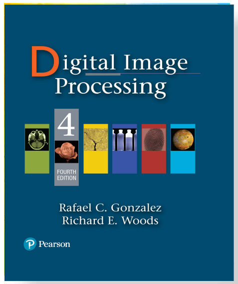
Basics of Filtering in the Frequency Domain

To filter an input image $f(x,y)$ of size MxN with a real symmetric filter $H(u,v)$, we need to do the following:

1) Pad image $f(x,y)$ with zeros to the size PxQ. Typically, P=2M and Q=2N $\Longrightarrow$ call the padded image $f_P(x,y)$
2) Multiply $f_P(x,y)$ by $(-1)^{x+y}$ to center its transformation
3) Compute DFT, $F(u,v)$
4) Compute the product $G(u,v) = H(u,v)F(u,v)$
5) Compute IDFT and take the real part

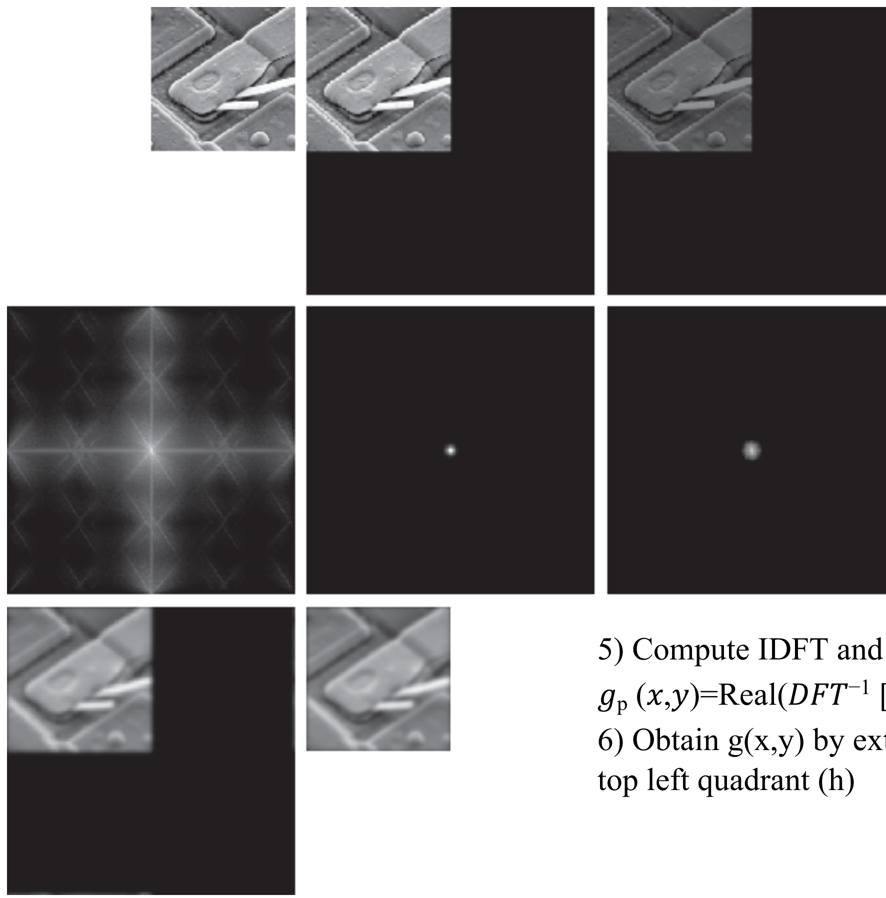
$$g_P(x,y) = \text{Real}(DFT^{-1}[G(u,v)])(-1)^{x+y}$$

6) Obtain g(x,y) by extracting the MxN region from top left quadrant



a b c
d e f
g h

FIGURE 4.35
(a) An $M \times N$ image, f. (b) Padded image, f_p of size $P \times Q$. (c) Result of multiplying f_p by $(-1)^{x+y}$. (d) Spectrum of F. (e) Centered Gaussian lowpass filter transfer function, H, of size $P \times Q$. (f) Spectrum of the product HF. (g) Image g_p, the real part of the IDFT of HF, multiplied by $(-1)^{x+y}$. (h) Final result, g, obtained by extracting the first M rows and N columns of g_p.



0) Original image (a)

1)Pad image $f(x, y)$ with zeros to the size PxQ. Typically, P=2M and Q=2N $\Longrightarrow f_P(x, y)$ (b)

2) Multiply $f_P(x, y)$ by $(-1)^{x+y}$ to center its transformation (c)

3) Compute DFT, $F(u, v)$ (d). (e) is a centered Gaussian LPF

4) Compute the product $G(u, v) = H(u, v)F(u, v)$ (f)

5) Compute IDFT and take the real part $g_p(x,y)$=Real(DFT^{-1} $[G(u,v)]$)$(-1)^{(x+y)}$ (g)

6) Obtain g(x,y) by extracting the MxN region from top left quadrant (h)