Series.

If $u_1, u_2 \dots u_n \dots$ is an infinite sequence of real numbers, then the sum of the terms of the sequence $u_1 + u_2 + \dots + u_n + \dots \infty$ is called an infinite series. denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$

For convergence / divergence of $\sum u_n$

1) Find $S_n = u_1 + u_2 + \dots + u_n$
$= \text{sum of first } n \text{ terms.}$

2) Find $\lim_{n \to \infty} S_n$

3) If $\lim_{n \to \infty} S_n =$ finite then $\sum u_n$ converges.
$= \pm\infty$ then $\sum u_n$ diverges.
$=$ does not have unique limit / limit does not exist.
$\Rightarrow \sum u_n$ oscillates.

Apply the following tests for checking the convergence / divergence for $\sum u_n$ in that order.

1) The n^{th} term test for divergence. For $\sum u_n$
If $\lim_{n \to \infty} u_n \neq 0$ then $\sum u_n$ diverges. But if $\lim_{n \to \infty} u_n = 0$ then $\sum u_n$ may or may not converge.

2) Geometric Series.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$= \frac{a(r^n - 1)}{r-1} \quad \text{if } r > 1$$

$$= \frac{a(1-r^n)}{1-r}, \quad \text{if } r < 1$$

① When $|r| < 1$, $\lim_{n \to \infty} r^n = 0$

then $S_\infty = \frac{a}{1-r} = \text{finite} \Rightarrow \sum u_n$ converges.

② When $|r| > 1$ $\lim_{n \to \infty} r^n = \infty$

then $S_\infty = \infty \Rightarrow \sum u_n$ diverges.

★ Important Result.

The series $\sum \frac{1}{n^p}$ converges if $p > 1$

diverges if $p \leq 1$

e.g Discuss the convergence of. $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$

— $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1} \right] = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$

Both are geometric series with $a = 1$ & common ratios $\frac{1}{2}$ & $\frac{1}{6}$ respectively. ∴ The series converges &

$$S = \frac{a_1}{1-r_1} - \frac{a_2}{1-r_2} = \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}} = 2 - \frac{6}{5} = \frac{4}{5}.$$

3) Comparison Test (Applicable if u_n contains polynomials in 'n')

For $\sum u_n$ & $\sum v_n$ (both are series of positive terms) &

if $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = l$ finite & non zero real number then both

the series $\sum u_n$ & $\sum v_n$ converge or diverge together.

(Choose $\sum v_n$ in the form of $\sum \frac{1}{n^p}$ such that

$\lim\limits_{n\to\infty} \frac{u_n}{v_n}$ becomes finite)

e.g Test the convergence of the series $\sum\limits_{n=1}^{\infty} \frac{1}{1+2^2+3^2+\cdots+n^2}$

1) Here $u_n = \frac{1}{1+2^2+3^2+\cdots+n^2}$

$$= \frac{6}{n(n+1)(2n+1)}$$

$$= \frac{6}{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

Take $v_n = \frac{1}{n^3}$

Now $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} = 6 \neq 0$

$\therefore$ By Comparison test both the series $\sum u_n$ & $\sum v_n$ converge or diverge together. But $\sum v_n = \sum \frac{1}{n^3}$ which is convergent as $p = 3 > 1$ by p series test.

$\therefore$ $\sum u_n$ also converges.

2) Test the convergence of the series $\sqrt{\frac{1}{2^3}} + \sqrt{\frac{2}{3^3}} + \sqrt{\frac{3}{4^3}} + \cdots$

— Here $u_n = \sqrt{\frac{n}{(n+1)^3}}$

$$= \frac{n^{1/2}}{(n+1)^{3/2}}$$

$$= \frac{n^{1/2}}{n^{3/2}\left(1+\frac{1}{n}\right)^{3/2}}$$

$$= \frac{1}{n\left(1+\frac{1}{n}\right)^{3/2}}$$

Take $v_n = \frac{1}{n}$ Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{3/2}} = 1 \neq 0$

$\therefore$ By comparision test both the series $\sum u_n$ & $\sum v_n$ converge or diverge together. But $\sum v_n = \sum \frac{1}{n}$ which is divergent according to p- series test.

$\therefore$ $\sum u_n$ also diverges.

3) $\sum_{n=1}^{\infty} \left(\sqrt{n^4+1} - \sqrt{n^4-1} \right)$

— let $u_n = \sqrt{n^4+1} - \sqrt{n^4-1} \times \frac{\sqrt{n^4+1} + \sqrt{n^4-1}}{\sqrt{n^4+1} + \sqrt{n^4-1}}$ (Doing rationalization)

$\therefore$ $u_n = \frac{(n^4+1) - (n^4-1)}{\sqrt{n^4+1} + \sqrt{n^4-1}}$

$$= \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}} = \frac{2}{n^2\left(\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}\right)}$$

Take $v_n = \frac{1}{n^2}$ & proceed accordingly.

finally $\sum u_n$ converges.

4) D' Alembert's Ratio test.

For the positive term series $\sum u_n$ if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then

(1) $\sum u_n$ converges if $l < 1$ $(l<1)$

(2) $\sum u_n$ diverges if $l > 1$ $(l>1)$

(3) The test fails for $l=1$

★ This test is applicable if u_n contains factorial (!)

e.g Test the convergence of the series. $\sum \frac{n!(2)^n}{n^n}$

1) let $u_n = \frac{n!(2)^n}{n^n}$ then $u_{n+1} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!\,2^{n+1}}{(n+1)^n\,(n+1)} \times \frac{n^n}{n!\,2^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\,\cancel{n!}\cdot\cancel{2^n}\cdot 2 \times n^n}{(n+1)^n\,\cancel{(n+1)}\,\cancel{n!}\,\cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1$$

∴ By Ratio test $\sum u_n$ converges.

2) $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \frac{4}{1+2^4} + \cdots$

Here $u_n = \frac{n}{1+2^n}$ $\quad u_{n+1} = \frac{n+1}{1+2^{n+1}}$, $\quad \frac{u_{n+1}}{u_n} = \frac{n+1}{1+2^{n+1}} \times \frac{1+2^n}{n}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \times \frac{\cancel{2^n}\left(1+\frac{1}{2^n}\right)}{\cancel{2^n}\left(2+\frac{1}{2^n}\right)} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)\frac{\left(1+\frac{1}{2^n}\right)}{\left(2+\frac{1}{2^n}\right)}$$

as $n\to\infty$ $\quad \frac{1}{2^n}\to 0$

$$= \frac{(1+0)(1+0)}{(2+0)} = \frac{1}{2} < 1$$

∴ By ratio test $\sum u_n$ converges.

3) Test the convergence of the series

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 2}{3\cdot 5}\right)^2 + \left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2 + \cdots\cdots \infty$$

— Here $u_n = \left[\frac{1\cdot 2\cdot 3\cdots\cdots n}{3\cdot 5\cdot 7\cdots\cdots(2n+1)}\right]^2$

$$u_{n+1} = \left[\frac{1\cdot 2\cdot 3\cdots\cdots n(n+1)}{3\cdot 5\cdot 7\cdots\cdots (2n+1)(2n+3)}\right]^2$$

$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(2n+3)^2}$ Now $\lim_{n\to\infty} \frac{(n+1)^2}{(2n+3)^2} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)^2}{n^2\left(2+\frac{3}{n}\right)^2}$

$$= \frac{(1+0)^2}{(2+0)^2} = \frac{1}{4} < 1$$

∴ By ratio test $\sum u_n$ converges.

5) Cauchy's Root test.

For the positive term series $\sum u_n$ if $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l$ then

(1) $\sum u_n$ converges if $l < 1$

(2) $\sum u_n$ diverges if $l > 1$

(3) The test fails if $l = 1$

★ This test is applicable if u_n has power 'n'.

e.g.

1) $\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \cdots\cdots + \left(\frac{n}{2n+1}\right)^n + \cdots\cdots$

— Here $u_n = \left(\frac{n}{2n+1}\right)^n$

Now $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+1}$

$$= \lim_{n\to\infty} \frac{n}{n\left(2+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{\left(2+\frac{1}{n}\right)} = \frac{1}{2} < 1$$

∴ $\sum u_n$ converges.

2) Test the convergence of the series

$$\frac{1^3}{3} + \frac{2^3}{3^2} + \frac{3^3}{3^3} + \frac{4^3}{3^4} + \cdots$$

— let $u_n = \frac{n^3}{3^n}$ $\quad (u_n)^{\frac{1}{n}} = \left(\frac{n^3}{3^n}\right)^{\frac{1}{n}} = \frac{n^{3/n}}{3}$

Now $\lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{n^{3/n}}{3} = \lim\limits_{n\to\infty} \frac{\left(n^{\frac{1}{n}}\right)^3}{3}$

(let $\lim\limits_{n\to\infty} (n)^{\frac{1}{n}}$ which is in the form $(\infty)^0$ indeterminate form)

To evaluate use L'Hôspital's rule

let $y = (n)^{\frac{1}{n}}$

$\therefore \log y = \frac{1}{n} \log n = \frac{\log n}{n}$

Now $\lim\limits_{n\to\infty} \log y = \lim\limits_{n\to\infty} \frac{\log n}{n}$ $\left(\frac{\infty}{\infty} \text{ type}\right)$

$= \lim\limits_{n\to\infty} \frac{\frac{1}{n}}{1}$ (Applying L' Hôspital's rule)

$= \lim\limits_{n\to\infty} \frac{1}{n}$

$= 0$

$\therefore \lim\limits_{n\to\infty} y = e^0 = 1$ $\quad \therefore \lim\limits_{n\to\infty} (n)^{\frac{1}{n}} = 1$

$\therefore \lim\limits_{n\to\infty} \frac{\left(n^{\frac{1}{n}}\right)^3}{3} = \frac{1}{3} < 1$

$\therefore$ By Cauchy's root test $\sum u_n$ converges.