Unit-5 Sequence & Series

1) Sequence

$a_1, a_2, a_3, \dots a_n$

$a_n = a(n+1)d$

$S_n = \frac{n}{2}[2a + (n-1)d]$

An ordered set of real numbers as $U_1, U_2, U_3 \dots U_n$ is called sequence and is denoted by $\{U_n\}$

If the no. of terms in $\{U_n\}$ are finite then that is called finite sequence otherwise infinite

2) Limit of Sequence: for $\forall \varepsilon > 0$ there exist $\exists n \to |U_n - l| < \varepsilon$ for every finite l. Then We can say that limit of $\{U_n\}$ exist and is denoted by $\lim_{n\to\infty} U_n = l$.

3) Continuous function for sequence: If $\lim_{n\to\infty} U_n = l$

then $f(U_n) \to f(l)$ i.e. $f(U_n)$ is continuous function at $f(l)$.

4) Convergent Sequence: If $\lim_{n\to\infty} U_n = l$ exist and

l is finite then ~~seq~~ $\{U_n\}$ is called convergent sequence.

$$\{U_n\} = \left\{\frac{1}{1+\frac{1}{n}}\right\}$$

Here

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} = \frac{1}{1+1/n}$$

$$= \frac{1}{1+0} = 1 \quad \text{which is finite}$$

Hence, $\{U_n\}$ is cgt.

converges (cgs) to 1.

5) Divergent Sequence

If $\lim_{n\to\infty} U_n = \infty$

$\{U_n\} = \{2n+1\}$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} (2n+1)$$

$$= \infty$$

$\{2n+1\}$ is divergent and diverges to ∞

6) Oscillating sequence

If $\lim_{n\to\infty} U_n$ is not unique then

Sequence U_n is called oscillating sequence

$$\{U_n\} \{(-1)^n + \frac{1}{2^n}\}$$

$$\lim_{n\to\infty} U_n = 1, \; n \text{ is even}$$
$$\qquad\qquad\; -1, \; n \text{ is odd}$$

Here, limit is not unique therefore $\{U_n\} = \{(-1)^n + \frac{1}{2^n}\}$ is oscillating.

7) Monotonic Sequence:

1) A sequence U_n is called monotonically increasing sequence if $U_{n+1} \geq U_n$

Eg. $\{1, 2, 3, 4 \ldots\}$

2) A sequence U_n is called monotonically decreasing sequence if $U_{n+1} \leq U_n$.

Eg. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\}$

8) Alternating Sequence: A $\{U_n\}$ is said to AS if it has alternate terms +ve & -ve.

eg. $1, -2, 3, -4 \ldots$

9) Bounded Sequence: A sequence U_n is said to be BS if $m \leq U_n \leq M$ where M, m are any real numbers.

Note: 1) Every convergent sequence is bounded but the converse is not true.

2) A monotonic increasing sequence is convergent if it is bounded above and divergent if it is not bounded above.

3) A monotonic decreasing sequence is convergent if it is bounded below and divergent if it is not bounded below.

4) If sequence is $\{U_n\}$ and $\{V_n\}$ converges to l_1 & l_2 resp. then

i) $\{U_n + V_n\}$ converges to $l_1 + l_2$

ii) $\{U_n - V_n\}$ cgs to $l_1 - l_2$

iii) $\{U_n \times V_n\}$ cgs to $l_1 \times l_2$

iv) $\left\{\frac{U_n}{V_n}\right\}$ cgs to $\frac{l_1}{l_2}$

Examples :

(1) Test the convergence of sequence $\left\{\frac{n^2+n}{2n^2-n}\right\}$

$$\lim_{n\to\infty} = \frac{n^2+n}{2n^2-n} = \lim_{n\to\infty} \frac{n^2(1+1/n)}{n^2\left(2-\frac{1}{n}\right)}$$

$$= \frac{1+0}{2-0}$$

$$= \frac{1}{2} \text{ finite.}$$

This sequence is convergent and converges to $1/2$.

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2) $\{\tanh n\}$

$U_n = \tanh n$

$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \tanh n$

$= \lim_{n\to\infty} \frac{\sinh x}{\cosh x}$

$= \lim_{n\to\infty} \frac{\left\{\frac{e^n + e^{-n}}{2}\right\}}{\left\{\frac{e^n - e^{-n}}{2}\right\}}$

$= \lim_{n\to\infty} \frac{e^n + e^{-n}}{e^n - e^{-n}}$

$= \lim_{n\to\infty} \frac{e^n\left(1 + \frac{1}{e^{2n}}\right)}{e^n\left(1 - \frac{1}{e^{2n}}\right)}$

$= 1$

$\therefore \{\tanh n\}$ is cgt

3) $\{2^n - (-1)^n\}$

$\lim_{n\to\infty} U_n = \lim_{n\to\infty} 2^n = \infty$

$\{2^n\}$ is dgt

3) $\{2 - (-1)^n\}$

$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \{2 - (-1)^n\}$

$= \begin{cases} 2 - 1 = 1, & n \text{ is even} \\ 2 - (-1) = -1, & n \text{ is odd} \end{cases}$

i.e. $\lim_{n\to\infty} U_n$ is not unique.

$\{2 - (-1)^n\}$ is oscillatory.

4) ~~$U_n = 1 + \frac{1}{3} + \frac{1}{3} +$~~ show that sequence U_n whose nth term is $U_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$ is monotonic increasing & bounded. Is it convergent?

$\Rightarrow$ $U_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$

$U_{n+1} = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} \cdots \frac{1}{3^n} + \frac{1}{3^{n+1}}$

$U_{n+1} - U_n = \frac{1}{3^{n+1}} > 0$

$U_{n+1} - U_n > 0 \Rightarrow U_{n+1} > U_n$
$\{U_n\}$ is monotonic increasing

Now, $U_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n}$

$$= \frac{1\left(1 - \frac{1}{3^{n+1}}\right)}{1 - \frac{1}{3}}$$

$$= \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right) < \frac{3}{2}$$

$\{U_n\}$ is bounded above by $\frac{3}{2}$ — (2)

By (1), (2), $\{U_n\}$ is ~~bounded~~ cgt.

6) Show that sequence U_n whose nth term is $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \dots \frac{1}{n!}$ is monotonic increasing and bounded. Is it convergent?

$\Rightarrow$ $U_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!}$

$U_{n+1} = \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{1}{(n+1)!}$

$U_{n+1} - U_n = \frac{1}{(n+1)!} > 0$

$U_{n+1} > U_n$

$\{U_n\}$ is ~~bounded and~~ increasing — (1)

$\therefore U_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$

$< 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$

$< \dfrac{1\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}$

$< 2\left(1 - \frac{1}{2^n}\right) < 2$. —— (2)

$\therefore$ Un is bounded above 2

$\therefore$ Hence it is cgt.

7) Show that the sequence $U_n = \left\{\frac{n}{n^2+1}\right\}$ is monotonic decreasing and bounded. Is it convergent?

$\Rightarrow \{U_n\} = \left\{\dfrac{n}{n^2+1}\right\}$

$U_n = \dfrac{n}{n^2+1}$

$U_{n+1} = \dfrac{(n+1)}{(n+1)^2+1}$

$U_{n+1} - U_n = \dfrac{(n+1)}{(n+1)^2+1} - \dfrac{n}{n^2+1}$

$= \dfrac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)}$

$$\frac{n^3+n+n^2+1-(n^3+2n^2+2n)}{n^4+2n^3+2n^2+n^2+2n+2}$$

$$= \frac{-n-n^2+1}{n^4+2n^3+4n^2+2n+2}$$

$$= \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0$$

$U_{n+1} - U_n$

$\{U_n\}$ is monotonic decreasing sequence

$$\Rightarrow U_n = \frac{n}{n^2+1}$$

$$= \frac{1/n}{1+1/n^2} \to 0$$

$\therefore$ U_n is bounded below $\{$ By (1), (2) U_n is convergent

Series

If $U_1, U_2, U_3, \cdots U_n$ so on is infinite sequence then sum of terms of the sequence $U_1 + U_2 + \ldots + U_n + \ldots \infty$ is called an infinite series. It is denoted

$$\sum_{n=1}^{\infty} U_n$$

i) Consider the infinite series $\sum_{n=1}^{\infty} U_n = U_1 + U_2 + \cdots + U_n + \cdots$ and let the sum of whose first n terms be $S_n = U_1 + U_2 + \cdots + U_n$, then 3 possibities arise :-
i) If $\lim_{n\to\infty} S_n$ is finite then $\sum_{n=1}^{\infty} U_n$ is cgt

ii) If $\lim_{n\to\infty} S_n$ is $\pm$ infinite then series is dgt.

iii) If $\lim_{n\to\infty} S_n$ is not unqiue then series is oscillating.

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⇒ Test for convergence

1) n^{th} term test

If $\lim_{n\to\infty} U_n = 0$ then $\sum_{n=1}^{\infty} U_n$ is cgt, where $U_n > 0$

2) Geometric series.

Let $\sum_{n=1}^{\infty} U_n = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ be geometric series where $S_n = \frac{a(1-r^n)}{1-r}$, if $r<1$

$= \frac{a(r^n - 1)}{r-1}$ if $r<1$

$\sum_{n=1}^{\infty} U_n$ is

→ cgt if $|r| < 1$

→ dgt if $r \geq 1$

→ Oscillatory if $r \leq -1$

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Examples:

1) Prove that $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\dots$ converges and find its sum.

$\Rightarrow$ ~~$r=\frac{2}{3}$, $a=1$~~ proof: given series is geometric series where $r=\frac{2}{3}$, $a=1$

$|r|=\frac{2}{3}<1$ $\therefore$ Series is convergent

$$S_n = \frac{1\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}}$$

$$= \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-\left(\frac{2}{3}\right)^n}{1/3}$$

$$=3$$

2) Test convergence of $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\dots$

$\Rightarrow$ proof: $a=5$, $r=-\frac{2}{3}$

$\therefore$ $|r|=|-2/3|<1$ series is cgt.

(3) Harmonic Series:

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\dots$$

Harmonic series is dgt in any case.

(4) p-series test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

i) Cgt if $p > 1$

ii) dgt if $p \leq 1$.

(5) Comparison Test.

If ΣU_n & ΣV_n are series of +ve terms such that $\lim_{n \to \infty} \frac{U_n}{V_n} = l$ (l is finite & non-zero)

then both series converge & diverge together

Examples:

1) Test the convergence of the series : $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

$$S_n = \frac{\sqrt{n}}{n^2+1}$$

$$= \frac{n^{1/2}}{n^2+1} \Rightarrow \frac{1}{\left(\frac{n^2+1}{n^{1/2}}\right)} = \frac{1}{n^{3/2} + n^{-1/2}}$$

$$= \frac{1}{n^{3/2}\left(1+n^{-2}\right)}$$

$$V_n = \frac{1}{n^{3/2}}, \quad U_n = \frac{1}{n^3\left(1+1/n^2\right)}$$

$$\lim_{n \to \infty} \frac{U_n}{V_n} = \lim_{n \to \infty} \frac{\frac{1}{n^{3/2}(1+1/n^2)}}{(1/n^{3/2})}$$

$$\lim_{n\to\infty} \frac{1}{1+1/n^2} = 1$$

$$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

where $p = 3/2 > 1$

$\therefore$ By comparision test
as V_n is cgt then u_n is dgt.

Q. $$\sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)}$$

$$U_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$U_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{n(2-1/n)}{n^3\left(1+1/n\right)\left(1+\frac{2}{n}\right)}$$

$$= \frac{2-1/n}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let $V_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} ; \lim_{n\to\infty} \frac{2n-1}{n(n+1)(n+2)} n^2$$

$$\lim_{n\to\infty} \frac{n^3(2-1/n)}{n^3(1+1/n)(1+2/n)} = \frac{2}{(1)(1)} = 2$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 \text{ (finite \& non zero)}$$

where $p = 3/2 > 1$. $\Rightarrow \sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is p series

$\sum_{n=1}^{\infty} v_n$ is cgt

hence by comparision test $\sum u_n$ is also convergent.

Q $\sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)}$

$$u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{n(2-1/n)}{n^3(1+\frac{1}{n})(1+\frac{2}{n})}$$

$$= \frac{2-1/n}{n^2(1+\frac{1}{n})(1+\frac{2}{n})}$$

let $v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2n-1}{n(n+1)(n+2)} n^2$$

$$\lim_{n\to\infty} \frac{n^3(2-1/n)}{n^3(1+1/n)(1+2/n)} = \frac{2}{(1)(1)} = 2$$

finite & non zero

Now $\sum V_n = \sum \frac{1}{n^2}$

$p = 2 > 1$

$\sum v_n$ is cgt $\Rightarrow$ $\sum u_n$ is cgt

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Q. $\frac{2}{1} + \frac{3}{8} + \frac{4}{27} + \frac{5}{64}$

$u_n = \frac{1+n}{n^3} = \frac{n\left(\frac{1}{n}+1\right)}{n^3} = \frac{1+1/n}{n^2}$

$v_n = \frac{1}{n^2}$ (should be p series)

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1+n}{n^3}(n^2)$

$= \lim_{n\to\infty} \frac{n^3(1+1/n)}{n^3} = 1 =$ finite & non zero.

Now, $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^p}$

$p = 2 > 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is cgt

$\sum_{n=1}^{\infty} v_n$ is cgt

hence, by comparision test $\sum_{n=1}^{\infty} u_n$ is also cgt

Q. $\frac{1}{1+\sqrt{2}} + \frac{2}{1+2\sqrt{3}} + \frac{3}{1+3\sqrt{4}} + \cdots$

$u_n = \frac{n}{1+n\sqrt{n+1}}$ $\quad \sum u_n$ is dgt

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$

$u_n = \frac{n}{(2n-1)(2n+1)}$

$v_n = \frac{1}{n}$

Σv_n is dgt

1, 3, 5, 7 $\quad a_n = a + (n-1)d$

$= 1 + (n-1)(2)$

$= 1 + 2n - 2$

$= 2n - 1$

Q $\frac{1}{3^p} + \frac{1}{5^p} + \frac{1}{7^p} + \dots$

$u_n = \frac{1}{(2n+1)^p} = \frac{1}{n^p(2 + 1/n)^p}$

$v_n = \frac{1}{n^p}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^p(2+1/n)^p} n^p$

$= \frac{1}{2^p}$ = finite & non-zero.

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(n+2)\sqrt{n}}{(n+1)\sqrt{n}}$

$= \lim_{n\to\infty} \frac{n(1+2/n)}{n(1+1/n)} = 1$; finite & non zero.

now, $\Sigma v_n = \Sigma \frac{1}{n^{1/2}}$

$p = \frac{1}{2} < 1$

Q. $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$

$$u_n = \frac{2n+1}{(n+1)^2} = \frac{n(2+1/n)}{n^2(1+1/n)^2} = \frac{(2+1/n)}{n(1+1/n)^2}$$

let $v_n = 1/n$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2n+1}{(n+1)^2} \, n$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(2+1/n)\cancel{n}}{\cancel{n^2}(1+1/n)^2} = 2 = \text{finite \& non zero}$$

Now, $\sum v_n = \sum 1/n \quad n=1$

$\sum v_n$ is dgt.

$\sum u_n$ is dgt.

Q. $\sum_{n=1}^{\infty} \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$

$$u_n = \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}} = \frac{n^{2/3}(2-1/n)^{1/3}}{n^{3/4}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{1/4}}$$

$$= \frac{(2-1/n)^{1/3}}{n^{1/12}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{1/4}} \quad ; \quad \text{let } v_n = \frac{1}{n^{1/12}}$$

$$= \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{2^{1/3}}{3^{1/4}}$$

Q $\frac{1}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{3}{5\cdot 7} + \frac{4}{7\cdot 9} + \ldots$

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Q. $\sum_{n=1}^{\infty} \frac{1}{1^2+2^2+3^2+\dots n^2}$

$\Sigma n^2 = 1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{n=1}^{\infty} \frac{6}{n(n+1)(2n+1)} = \frac{6}{n^3(1+1/n)(2+1/n)}$

$v_n = \frac{1}{n^3}$; $\lim_{n\to\infty} \frac{u_n}{v_n}$

$= \lim_{n\to\infty} \frac{6}{n(n+1)(2n+1)} = \cancel{\lim_{n\to\infty} \frac{6}{n(n+1)(2n+1)}}$

$= \lim_{n\to\infty} \frac{6n^3}{n^3(1+\frac{1}{n})(2+\frac{1}{n})} = \frac{6}{3} = 2$

Q. $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)\sqrt{n}}$

Let $u_n = \frac{n+2}{(n+1)\sqrt{n}} = \frac{n(1+2/n)}{n^{3/2}(1+\frac{1}{n})} = \frac{1+2/n}{n^{1/2}(1+1/n)}$

Let $v_n = \frac{1}{n^{1/2}}$

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(n+2)\sqrt{n}}{(n+1)\sqrt{n}}$

$= \lim_{n\to\infty} \frac{n(1+2/n)}{n(1+1/n)} = 1$ finite and non zero

now; $\Sigma v_n = \Sigma \frac{1}{n^{1/2}}$; $p = \frac{1}{2} < 1$

Σv_n is dgt $\quad \Sigma u_n$ is dgt.

D'Alembert's Ratio Test.

If ΣU_n is a +ve term series and $\lim_{n\to\infty} \frac{U_{n+1}}{U_n}$

then 1) ΣU_n is convergent if $l<1$
2) ΣU_n is divergent if $l<1$.

Test the convergence of following series:

1) $\sum_{n\leq 0}^{\infty} \frac{3^{2n}}{2^{3n}}$

$\Rightarrow U_n = \frac{3^{2n}}{2^{3n}}$, $U_{n+1} = \frac{3^{2(n+1)}}{2^{3(n+1)}}$

$= \frac{3^{2n+2}}{2^{3n+3}}$

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{3^{2n+2}}{2^{3n+3}} \cdot \frac{2^{3n}}{3^{2n}}$

$= \frac{3^2}{2^3} = \frac{9}{8} > 1$

By ratio test ΣU_n is dgt.

2) $\sum_{n=1}^{\infty} \frac{5^{n-1}}{n!}$

$U_n = \frac{5^{n-1}}{n!}$

$U_{n+1} = \frac{5^{(n-1)+1}}{(n+1)!}$

$= \frac{5^n}{(n+1)!}$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{5^n}{(n+1)!} \times \frac{n!}{5^{n-1}}$$

$$= \lim_{n\to\infty} \frac{5^n}{(n+1)} \frac{n!}{n!} \frac{\times 5}{5^n}$$

$$= \lim_{n\to\infty} \frac{5}{n+1}$$

$$= 0 < 1$$

$\therefore$ By Ratio test $\sum_{n=1} U_n$ is cgt.

3) $\sum_{n=1}^{\infty} \frac{2^n}{n^3+1}$

$$U_n = \frac{2^n}{n^3+1}$$

$$U_{n+1} = \frac{2^{n+1}}{(n+1)^3+1}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3+1}{2^n}$$

$$= \lim_{n\to\infty} \frac{2 \cdot 2^n \cdot n^3(1+1/n^3)}{2^n \cdot n^3\{(1+1/n)^3 + 1/n^3\}}$$

$$= \frac{2(1+0)}{1^3} = 2 > 1$$

$\therefore$ By ratio test $\sum U_n$ is dgt.

4) $\sum \frac{n!}{n^n}$

$$U_n = \frac{n!}{n^n}$$

$$U_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{(n+1)}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{n!\,(n+1)\cdot n^n}{n!\,(n+1)\,(n+1)^n}$$

$$= \lim_{n\to\infty} \frac{n^n}{n^n(1+1/n)^n} = \frac{1}{e} < 1.$$

$\therefore$ By ratio test $\sum_{n=1}^{\infty} U_n$ is dgt.

5) $\sum_{n=1}^{\infty} \frac{n!\,(2)^n}{n^n}$

$$U_n = \frac{n!\,(2)^n}{n^n}$$

$$U_{n+1} = \frac{(n+1)!\,(2)^{(n+1)}}{(n+1)^{(n+1)}}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{(n+1)!\,(2)^{(n+1)}}{(n+1)^{(n+1)}} \cdot \frac{n^n}{n!\,(2)^n}$$

$= \frac{2}{e} < 1$

$e = 2.713$

$\therefore$ By ratio test it is cgt.

6) $\sum_{n=1}^{\infty} \frac{n^3}{(n-1)!}$

$U_n = \frac{n^3}{(n-1)!}$

$U_{n+1} = \frac{(n+1)^3}{((n-1)+1)!}$

$= \frac{(n+1)^3}{n!}$

$\frac{U_{n+1}}{U_n} = \frac{(n+1)^3}{n!} \cdot \frac{(n-1)!}{n^3}$

$= \frac{n^3+1+n^2+n}{\cancel{n!}} \cdot \frac{\cancel{(n-1)!}}{n^3}$

$\frac{n(n^2+n+1)+1}{n!} \cdot \frac{(n-1)!}{n^3}$

$\frac{n(n+1)^2+1}{\cancel{n!}} \cdot \frac{(n-1)!}{\cancel{n^3}}$

$\frac{(\cancel{n}+1)^2+1}{n^3} \cdot (n-1)!$

$= \frac{n^3+\cancel{n^2}+n-\cancel{n^2}-n-1+1}{n^3}$

$= \frac{n^3}{\cancel{n^3}} = 1$

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8) $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n+1}$

$U_n = \frac{2^{(n)}+1}{3^{(n)}+1}$

$U_{n+1} = \frac{2^{(n+1)}+1}{3^{(n+1)}+1}$

$\frac{U_{n+1}}{U_n} = \frac{2^{(n+1)}+1}{3^{(n+1)}+1} \cdot \frac{3^n+1}{2^n+1}$

$= \frac{2}{3} < 1$

ΣU_n is cgt.

9) $\sum^{\infty} \frac{5^n+a}{3^n+b}$; $a, b > 0$

$U_n = \frac{5^n+a}{3^n+b}$

$U_{n+1} = \frac{5^{(n+1)}+a}{3^{(n+1)}+b}$

$\frac{U_{n+1}}{U_n} = \frac{5^{(n+1)}+a}{3^{(n+1)}+b} \cdot \frac{3^n+b}{5^n+a}$

$= \frac{(5^n \times 5)+a}{(3^n \times 3)+b} \cdot \frac{3^n+b}{5^n+a}$

$= \frac{(5^n+a)5}{(3^n+b)3} \cdot \frac{3^n+b}{5^n+a}$

$= \frac{5}{3} > 1$. $\therefore$ dgt.

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10) $\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \dots$

$2!, 3!, 4! \dots$

$\cancel{a = 2!} \quad \cancel{d = 4}$

$U_n = \frac{(n+1)!}{3^n}$

$\cancel{a_n = a + (n-1)d}$

$\cancel{= 2! + (n-1)4}$

$U_{n+1} = \frac{(n+2)!}{3^{n+1}}$

$\cancel{= 2 + 4n - 4}$

$\cancel{U_n = \frac{4n-2}{3^n}}$

$\frac{U_{n+1}}{U_n} = \frac{(n+2)!}{3n+1} \cdot \frac{3^n}{(n+1)!}$

$= \frac{(n+2)!}{3^{n+1}} \cdot \frac{3^n}{(n+1)!}$

$= \frac{\cancel{(n+1)!} \times 1!}{\cancel{3^n} \times 3} \cdot \frac{\cancel{3^n}}{\cancel{(n+1)!}}$

$= \frac{1}{3}$