Unit IV (Application of PD)

⇒ Tangent Plane & Normal line

Equation of tangent plane at $P(x_0, y_0, z_0)$ to the surface $f(x,y,z) = 0$ is

$(x - x_0) f_x(x_0, y_0, z_0) + (y - y_0) f_y(x_0, y_0, z_0) + (z - z_0) f_z(x_0, y_0, z_0) = 0$

where $f_x(x_0, y_0, z_0) = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0, z_0)}$

$f_y(x_0, y_0, z_0) = \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0, z_0)}$

& $f_z(x_0, y_0, z_0) = \left.\frac{\partial f}{\partial z}\right|_{(x_0, y_0, z_0)}$

Normal line

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}$$

1. find the equation of tangent plane and normal line to the surface $xyz = 6$ at $(1, 2, 3)$

$xyz = 6$ at $(1, 2, 3)$

$f(x, y, z) = xyz - 6$

$f_x = yz \Rightarrow f_x(1, 2, 3) = 6$

$f_y = xz \Rightarrow f_y(1, 2, 3) = 3$

$f_z = xy \Rightarrow f_z(1, 2, 3) = 2$

4)

$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-4}{1}.$$

3) $x^2+y^2+z^2=3$ at the point $(1,1,1)$

$\cancel{x+y+z-3=0}$

$f(x,y,z) = \cancel{(1,1,1)}\ x^2+y^2+z^2-3$

$f(x) = 2x \Rightarrow f(1,1,1) = 2$

$f(y) = 2y \Rightarrow f(1,1,1) = 2$

$f(z) = 2z \Rightarrow f(1,1,1) = 2.$

$(x-1)f_x + (y-1)f_y + (z-1)f_z = 0$

$(x-1)2 + (y-1)2 + (z-1)2 = 0$

$2x-2+2y-2+2z-2=0$

$2x+2y+2z-6=0$

$x+y+z-3=0$

Normal line

$$\frac{x-1}{f_x(1,1,1)} = \frac{y-1}{f_y(1,1,1)} = \frac{z-1}{f_z(1,1,1)}.$$

$(x-1) f_x(1,2,3) + (y-2) f_y(1,2,3) + (z-3) f_z(1,2,3) = 0$

$(x-1)(6) + (y-2)(3) + (z-3)(2) = 0$

$6x - 6 + 3y - 6 + 2z - 6 = 0$

$\Rightarrow 6x + 3y + 2z - 18 = 0$

Eqⁿ of Normal line

$$\frac{x-1}{f_x(1,2,3)} = \frac{y-2}{f_y(1,2,3)} = \frac{z-3}{f_z(1,2,3)}$$

$$= \frac{x-1}{6} = \frac{y-2}{3} = \frac{z-3}{2}$$

2. $x^2 + y^2 + z - 9 = 0$ at $(1,2,4)$

$f(x,y,z) = x^2 + y^2 + z - 9$

$f_x = 2x + \cancel{y^2} + \cancel{z} \Rightarrow f(1,2,4) = 2$

$= 2$

$f_y = \cancel{x^2} + 2y + \cancel{z} \Rightarrow f(1,2,4) = \cancel{1+8+4}$

$= 4$

$f_z = \cancel{x^2} + \cancel{y^2}\ 1 \Rightarrow f(1,2,4) = 1 \quad = 1$

$(x-1) f_x(1,2,3) + (y-2) f_y(1,2,3) + (z-4) f_z(1,2,3) = 0$

$(x-1)\,2 + (y-2)\,4 + (z-4)(1)$

$= \cancel{10x - 10 + 13y - 26 + 5z - 20}$

$2x - 2 + 4y - 8 + z - 4 \Rightarrow 2x + 4y + z = 14$

$= \frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$

4) $2x^2 + y^2 + 2z = 3$ at $(2, 1, -3)$

$f(x, y, z) = 2x^2 + y^2 + 2z - 3$

$f(x) = 4x \Rightarrow f(2, 1, -3) = 8$
$f(y) = 2y \Rightarrow f(2, 1, -3) = 2$
$f(z) = 2 \Rightarrow f(2, 1, -3) = 2.$

$(x-2) f(x) + (y-1) f(y) + (z+3) f(z) = 0$

$(x-2) 8 + (y-1) 2 + (z+3)(2)$

$8x - 16 + 2y - 2 + 2z + 6 = 0$
$8x + 2y + 2z - 12 = 0$

Normal line

$$\frac{x-2}{8} = \frac{y-1}{2} = \frac{z+3}{2}.$$

5) $\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 3$ at $(-2, 1, -3)$

Ans. $3x - 6y + 2z + 18 = 0$

$$\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-2/3}$$

Local Extreme Values
(Maximum & minimum Values)

⇒ Working Rule to determine extreme values of a function $f(x,y)$.

1. Solve $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ and obtain the stationary point (a,b).

2. Obtain $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$ & $t = \frac{\partial^2 f}{\partial y^2}$

3. i) If $rt - s^2 > 0$ & $r < 0$ (or $t < 0$) then $f(x,y)$ has maximum value at (a,b) which is $f(a,b)$

ii) If $rt - s^2 > 0$ & $r > 0$ (or $t > 0$) then $f(x,y)$ has minimum value at (a,b) which is $f(a,b)$.

iii) If $rt - s^2 < 0$ then function has neither minimum nor maximum value.

iv. If $rt - s^2 = 0$ then no conclusion can be made about the function and further investigation is required about the given function.

1) Discuss maxima and minima of the function $x^2 + y^2 + 6x + 12 = 0$

$f(x, y) = x^2 + y^2 + 6x + 12.$

$\frac{\partial f}{\partial x} = 2x + 6 = 0$

$\Rightarrow 2x = -6$

$x = -3$

$\frac{\partial f}{\partial y} = 2y = 0 \Rightarrow y = 0$

Stationary point $(-3, 0)$

$r = \frac{\partial^2 f}{\partial x^2} = 2$; r at $(-3, 0) \Rightarrow r = 2$

$s = \frac{\partial^2 f}{\partial x \partial y} = 0$; s at $(-3, 0) = 2$

$t = \frac{\partial^2 f}{\partial y^2} = 2$; t at $(-3, 0) = 2$

$rt - s^2 = (2)(2) - 0 = 4 > 0$

r at $(-3, 0) = 2 > 0$

$f(x, y)$ has min value

& $f_{min}(-3, 0) = (-3)^2 + 0 + 6(-3) + 12$

$= 9 - 18 + 12 = 0$

2) $f(x,y) = 3x^2 - y^2 + x^3$

$\frac{\partial f}{\partial x} = 6x + 3x^2$

$\frac{\partial f}{\partial x} = 0 \Rightarrow 6x + 3x^2 = 0$

$\Rightarrow 3x(2+x)$

$\Rightarrow 3x = 0$ or $2 + x = 0$

$\Rightarrow x = 0, -2.$

$\frac{\partial f}{\partial y} = 0$

$\Rightarrow -2y = 0$

$y = 0.$

Stationary point $(0,0)$, $(-2,0)$

1) At $(0,0)$

$r = \frac{\partial^2 f}{\partial x^2} = 6 + 6x$

$s = \frac{\partial^2 f}{\partial x \partial y} = 0$

$t = \frac{\partial^2 f}{\partial y^2} = -2$

$rt - s^2 = -2(6+6x) - 0$

$= -12 - 12x$

$rt - s^2 \Big|_{(0,0)} = -12 < 0$

$\therefore$ $f(x,y)$ has neither max or min value at $(0,0)$

ii) At $(-2,0)$

$$rt - s^2 = -12 - 12x$$

$$rt - s^2 \big|_{(-2,0)} = -12 - 12(-2)$$
$$= -12 + 24$$
$$= 12 > 0$$

$$r = 6 + 6x$$

$$r \big|_{(2,0)} = 6 + 6(-2)$$
$$= 6 - 12$$
$$= -6 < 0$$

$\therefore$ $f(x,y)$ has max value at $(-2,0)$

$$f_{max} = f(-2,0)$$

$$= 3(-2)^2 - 0^2 + (-2)^3$$
$$= 3(4) - 8$$
$$= 12 - 8$$
$$= 4.$$

3). Show that minimum value of $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$ is $3a^2$

$$\frac{\partial f}{\partial x} = y - \frac{a^3}{x^2} = 0$$

$$\Rightarrow y = \frac{a^3}{x^2}$$

$$\frac{\partial f}{\partial y} = x - \frac{a^3}{y^2} = 0$$

$$x = \frac{a^3}{y^2}$$

Solving (1) & (2)

$(x, y) = (a, a)$

$r = \dfrac{\partial^2 f}{\partial x^2} = -a^3(-2x^{-3})$

$s = \dfrac{\partial^2 f}{\partial x \partial y} = 1$

$t = \dfrac{\partial^2 f}{\partial y^2} = -a^2(-2y^{-3})$

$rt - s^2 = \dfrac{2a^3}{x^3} \cdot \dfrac{2a^3}{y^3} = \dfrac{4a^6}{x^3 y^3}$

$rt - s^2 \Big|_{(a,a)} = \dfrac{4a^6}{a^6 a^3} - 1 = 3 > 0$

$r \Big|_{(a,a)} = \dfrac{2a^3}{a^3} = 2 > 0$

$rt - s^2 > 0$ & $r > 0$

$f(x, y)$ has min value at (a, a)

$f_{min} = f(a, a) = a^2 + \dfrac{a^3}{a} + \dfrac{a^3}{a}$

$= a^2 + a^2 + a^2$

$= 3a^2$

4) $f(x, y) =$

$x^3 + y^3 - 3axy, \ a > 0$

5)

$$\frac{\partial f}{\partial x} = 0 \Rightarrow x = 3x^2 - 3ay$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow y = a$$

$$f_{min} = -a^3$$

5) $f(x,y) = x^3 + 3ay^2 - 3x^2 - 3y^2 + 4.$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x = 0$$

$$= 3x(x-2) + 3y^2 = 0$$

$$y^2 = -x(x-2)$$

$$\frac{\partial f}{\partial y} = 6xy - 6y = 0$$

$$xy = y$$

$$xy - y = 0$$

$$y(x-1) = 0$$

$$y = 0, \; x = 1$$

$y = 0$

$\rightarrow 0 = -x(x-2)$

$x = 0, 2$

$\left[(0,0), (2,0)\right]$

$x = 1$

$y^2 = -1(1-2)$

$y^2 = -1(-1) = 1$

$y = \pm 1$

$\left[(1,1), (1,-1)\right]$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x - 6$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 6$$

$$rt - s^2 = (6x-6)(6x-6) - 36y^2$$

$$36x^2 - 72x + 36 - 36y^2$$

Points	r	s	t	$rt - s^2$	Conclusion
(0,0)	-6	0	-6	36	max
(2,0)	6	0	6	36	min
(1,1)	0	6	0	-36	neither max nor min
(1,-1)	0	-6	0	36	〃

	f_{max}
P 1)	-4
P 2)	0
P 3)	–
P 4)	–

6 $x^3 + y^3 - 63(x+y) + 12xy$

$f_{max} = 784$

$f_{min} = -216$

Method of Lagrange Multipliers

Let $f(x,y,z)$ be the function of three variable i.e x, y, z and the variable be $\phi(x,y,z) = 0$ connected by the relation.

Suppose we wish to find the values of x, y, z for which $f(x,y)$ is stationary i.e max or min. For this purpose we write an auxillary eqn $f(x,y,z) + \lambda'$ $\phi(x,y,z) = 0$.

Differentiate w.r.t $x, y, z \Rightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

Q1 Find the minimum value of $x^2 + y^2$ subject to the condition i.e $ax + by = c$

$\Rightarrow$ $x^2 + y^2 + \lambda(ax + by - c) = 0$

w.r.t x

$2x + \lambda a = 0$

$\lambda = \frac{-2x}{a}$ — (1)

w.r.t y

$2y + \lambda b = 0$

$\lambda = \frac{-2y}{b}$ — (2)

$$-\frac{2x}{a} = -\frac{2y}{b}$$

$$\Rightarrow x = \frac{ya}{b} \quad \text{—— (3)}$$

put in $ax + by - c = 0$

$$a \times \frac{ya}{b} + by - c = 0$$

$$\frac{a^2y + b^2y - bc}{b} = 0$$

$$a^2y + b^2y - bc = 0$$

$$y(a^2 + b^2) = bc$$

$$y = \frac{bc}{a^2+b^2} \quad \text{—— (4)}$$

put (4) in (3)

$$x = \left(\frac{bc}{a^2+b^2}\right)\frac{a}{b}$$

$$= \frac{abc}{a^2b+b^3} = \frac{b(ac)}{b(a^2+b^2)} = \frac{ac}{a^2+b^2}$$

$\therefore$ for $x^2 + y^2$

$$\left(\frac{ac}{a^2+b^2}\right)^2 + \left(\frac{bc}{a^2+b^2}\right)^2$$

Find the minimum value of x^2yz^3 subject to condition $2x + y + 3z = a$

$$f(x,y,z) + \lambda \phi(x,y,z) = 0$$

$$x^2yz^3 + \lambda(2x+y+3z-a) = 0$$

w.r.t x

$$2xyz + \lambda(2) = 0$$

$$\lambda = -xyz^3$$

w.r.t y

$$x^2z^3 + \lambda(1) = 0$$

$$\lambda = -x^2z^3$$

w.r.t. z

$$3x^2yz^2 + \lambda(3) = 0$$

$$\lambda = \frac{-3x^2yz^2}{3}$$

$$= -x^2yz^2$$

$$\Rightarrow -xyz^3 = -x^2z^3 = -x^2yz^2$$

$$\Rightarrow yz = xz = xy$$

from $yz = xz$

$$\Rightarrow x = y \quad \text{--- (1)}$$

from $yz = xy$

$$\Rightarrow z = x \quad \text{--- (2)}$$

from (1) & (2)

$$2x + y + 3z = a$$

$$2x + x + 3x = a$$

$$6x = a$$

$$x = \frac{a}{6}$$

$\therefore y = \frac{a}{6}$ and $z = \frac{a}{6}$

$\therefore x^2 y z^3$

$= \left(\frac{a}{6}\right)^2 \left(\frac{a}{6}\right) \left(\frac{a}{6}\right)^3$

$= \left(\frac{a}{6}\right)^6$

Q3. Find min and max distance from the point (1, 2, 2) to the sphere i.e $x^2+y^2+z^2=36$

$\Rightarrow \quad D = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$

$= \sqrt{(x-1)^2+(y-2)^2+(z-2)^2}$

$= \sqrt{x^2+1-2x+y^2-4y+4+z^2+4-4z}$

$D^2 = x^2+y^2+z^2-2x-4y-4z+1+4+4$

$D^2 = x^2+y^2+z^2-2x-4y-4z+9$

$\therefore$ or $D^2 = (x-1)^2+(y-2)^2+(z-2)^2 + \lambda(x^2+y^2+z^2-36)=0$

w.r.t x:

$2(x-1)+\lambda(2x)=0$

$\lambda = -\frac{2(x-1)}{2x}$

$\lambda = \frac{1-x}{x}$

w.r.t y

$2(y-2) + \lambda(2y) = 0$

$$\lambda = \frac{2(y-2)}{2y}$$

$$\lambda = \frac{y-2}{y}$$

w.r.t z

$2(z-2) + \lambda(2z) = 0$

$$\lambda = \frac{2(z-2)}{2z}$$

$$\lambda = \frac{z-2}{z}$$

$$\frac{1}{x} - 1 = \frac{2}{y} - 1 = \frac{2}{z} - 1$$

$$\frac{1}{x} = \frac{2}{y} = \frac{2}{z}$$

$y = 2x$ & $z = 2x$

$x^2 + y^2 + z^2 = 36$

$x^2 + (2x)^2 + (2x)^2 = 36$

$x^2 + 4x^2 + 4x^2 = 36$

$9x^2 = 36$

$x^2 = 4$

$x = \pm 2.$

$y = \pm 4$, $z = \pm 4.$

$\therefore$ $D = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$

$\therefore$ with $x = 2$, $y = 4$, $z = 4$

$D = \sqrt{(2-1)^2 + (4-2)^2 + (4-2)^2}$

$D = 3.$

with $x = -2,\ y = -4,\ z = -4$

$$D = \sqrt{(-2-1)^2 + (-4-2)^2 + (-4-2)^3}$$

$$= 9$$

$\therefore$ $max = 9$

$min = 3.$