UNIT 3 DAMPED HARMONIC MOTION

Structure

3.1 INTRODUCTION

In Unit 1 you learnt that SHM is a universal phenomenon. Now you also know that in the ideal case the total energy of a harmonic oscillator remains constant in time and the displacement follows a sine curve. This implies that once such a system is set in motion it will continue to oscillate forever. Such oscillations are said to befree or ***undamped.*** Do you know of any physical system in the real world which experiences no damping? Probably there is none. You must have observed that oscillations of a swing, a simple or torsional pendulum and a spring-mass system when left to themselves, die down gradually. Similarly, the amplitude of oscillation of charge in an LCR circuit or of the coil in a suspended type galvanometer becomes smaller and smaller. This implies that every oscillating system loses some energy as time elapses. The question now arises: Where does this energy go? To answer this, we note that when a body oscillates in a medium it experiences resistance to its motion. This means that damping force comes into play. Damping force can arise within the body itself, as well as due to the surrounding medium (air or liquid). The work done by the oscillating system against the damping forces leads to dissipation of energy of the system. That is, the energy of an oscillating body is used up in overcoming damping. But in some engineering systems we knowingly introduce damping. A familiar example is that of brakes—we increase friction to reduce the speed of a vehicle in a short time. In general, damping causes wasteful loss of energy. Therefore, we invariably try to minimise it.

Many a time it is desirable to maintain the oscillations of a system. For this we have to feed energy from an outside agency to make up for the energy losses due to damping. Such oscillations are called ***forced oscillations.*** You will learn various aspects of such oscillations in the next unit.

In this unit you will learn to establish and solve the equation of motion of a damped harmonic oscillator. Damping may be quantified in terms of logarithmic decrement, relaxation time and quality factor. You will also learn to compute expressions for the logarithmic decrement, power dissipated in one cycle and the quality factor.

Objectives

After going through this unit you will be able to

- establish the differential equation for a damped harmonic oscillator and solve it analyse the effect of damping on amplitude, energy and period of oscillation

- highlight differences between weakly damped, critically damped and over-damped systems
- derive expressions for power dissipated in one oscillation
- compute relaxation time and quality factor of a damped oscillator, and

draw analogies between different physical systems.

3.2 DIFFERENTIAL EQUATION OF A DAMPED OSCILLATOR

While considering the motion of a damped oscillator, some of the questions that come to our mind are : Will Eq.(1.2) still hold? If not, what modification is necessary? How to describe damped motion quantitatively? To answer these questions we again consider the spring-mass system of Unit 1. Let us imagine that the mass moves horizontally in a viscous medium, say inside a lubricated cylinder, as shown in Fig.3.1. As the mass moves, it will experience a drag, which we denote by F_d. The question now arises : How to predict the magnitude of this damping force? Usually, it is difficult to quantify it exactly. However, we can make a reasonable estimate based on our experience. For oscillations of sufficiently small amplitude, it is fairly reasonable to model the damping force after Stokes' law. That is, we take F_d to be proportional to velocity and write

$$F_d = -\gamma v \tag{3.1}$$

The force experienced by a body falling freely in a viscous medium is given by

$F_d = 6\pi\eta rv$

This is known as Stokes' law. Here η is the coefficienl of viscosity of the medium and r is radius of body—assumed to be spherical, and v is its velocity.

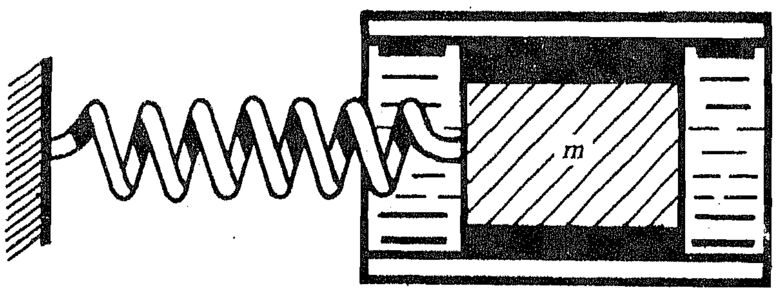

Fig. 3.1 A damped spring-mass system.

The negative sign signifies that the damping force opposes motion. The constant of proportionality γ is called the *damping coefficient,* Numerically, it is equal to force per unit velocity and is measured in $\frac{N}{ms^{-1}} = \frac{kgms^{-2}}{ms^{-1}} = kg\,s^{-1}$

We will now establish the differential equation which describes the oscillatory motion of a damped harmonic oscillator. Let us take the x-axis to be along the length of the spring. We define the origin of the axis $(x = 0)$ as the equilibrium position of the mass. Imagine that the mass (in the spring-mass system) is pulled longitudinally and then released. It gets displaced from its equilibrium position. At any instant, the forces acting on the spring-mass system are :

(i) *a restoring force* : $-kx$ where k is the spring factor, and

(ii) a *damping force* : $-\gamma v$, where $v = \frac{dx}{dt}$ is the instantaneous velocity of the oscillator. This means that for a damped harmonic oscillator, the equation of motion must include the restoring force as well as the damping force. Hence, in this case Eq.(1.2) is modified to

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt} \tag{3.2}$$

After rearranging terms and dividing throughout by m, the equation of motion of a **damped oscillator** takes the form

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \qquad (3.3)$$

where $\omega_0^2 = k/m$ and $2b = \gamma/m$. (You will note that a factor of 2 has been introduced in the damping term as it helps us to obtain a neat expression for the solution of this equation.) The constant b has the dimensions of

$$\frac{\text{force}}{\text{velocity} \times \text{mass}} = \frac{MLT^{-2}}{LT^{-1}M} = T^{-1}$$

Hence, its unit is s^{-1}, which is the same as that of ω_0.

You will note that like Eq. (1.3), Eq. (3.3) is a linear second order homogeneous differential equation with constant coefficients. If there were no damping, the second term in Eq. (3.3) will be zero and the general solution of the resulting equation will be given by Eq. (1.5), i.e. $x = A\cos(\omega_0 t + \phi)$. On the other hand, if there is damping of no restoring force, the third term in Eq.(3.3) will be zero. Then the general solution of the resulting equation is given by $x(t) = Ce^{-2bt} + D$ where C and D are constants. (You can show this by substituting the assumed solution in Eqs. (3.3).) This means that the displacement will decrease exponentially in the absence of any restoring force. Thus we expect that the general solution of Eq. (3.3) will represent an oscillatory motion whose amplitude **decreases** with time.

3.3 SOLUTIONS OF THE DIFFERENTIAL EQUATION

How does damping influence the amplitude of oscillation? To discover this we have to solve Eq. (3.3) when both the restoring force and the damping force are present. The general solution, as discussed above, should involve both exponential and harmonic **terms.** Let us therefore take a solution of the form

$$x(t) = a\exp(\alpha t) \qquad (3.4)$$

where a and α are unknown constants.

Differentiating Eq. (3.4) twice with respect to time, we get

$$\frac{dx}{dt} = a\alpha\exp(\alpha t)$$

and

$$\frac{d^2x}{dt^2} = a\alpha^2\exp(\alpha t).$$

Substituting these expressions in Eq. (3.3), we get

$$(\alpha^2 + 2b\alpha + \omega_0^2)\, a\exp(\alpha t) = 0 \qquad (3.5)$$

For this equation to hold at all times, we should either have

$$a = 0$$

which is trivial, or

$$\alpha^2 + 2b\alpha + \omega_0^2 = 0 \qquad (3.6)$$

The roots of the equation $ax^2 + bx + c = 0$ **are given by** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

This equation is quadratic in α. Let us call the two roots α_1 and α_2:

$$\alpha_1 = -b + (b^2 - \omega_0^2)^{1/2} \qquad (3.7\text{ a})$$

and

$$\alpha_2 = -b - (b^2 - \omega_0^2)^{1/2} \qquad (3.7\text{ b})$$

These roots determine the motion of the oscillator. Obviously α has dimensions of inverse time. Did you not expect it from the form of $\exp(\alpha t)$?

Thus, the two possible solutions of Eq. (3.3) are

$$x_1(t) = a_1\exp[-\{b - (b^2 - \omega_0^2)^{1/2}\}t]$$

and

$$x_2(t) = a_2\exp[-\{b + (b^2 - \omega_0^2)^{1/2}\}t] \qquad (3.8)$$

Since Eq. (3.3) is linear, the principle of superposition is applicable. Hence, the general **solution** is **obtained** by the superposition of x_1 and x_2:

$$x(t) = \exp(-bt)\,[a_1\exp\{(b^2 - \omega_0^2)^{1/2}t\} + a_2\exp\{-(b^2 - \omega_0^2)^{1/2}t\}] \qquad (3.9)$$

You will note the quantity $(b^2 - \omega_0^2)$ can be negative, zero or positive respectively depending on whether b is less than, equal. to or greater than ω_0 respectively. We, therefore, have three possibilities :

(i) If $b > \omega_0$. we say that the system is over damped,

(ii) If $b = \omega_0$, we have a critically damped system,

(iii) If $b < \omega_0$, we have an under-damped system.

Each of these conditions gives a different solution, which describes a particular behaviour.

We will now discuss these solutions in order of their increasing importance.

3.3.1 Heavy Damping

When resistance to motion is very strong, the system is said to be heavily damped. Can you name a heavily damped system of practical interest? Springs joining wagons of a train constitute the most important heavily damped system. In your physics laboratory, vibrations of a pendulum in a viscous medium such as thick oil and motion of the coil of a dead beat galvanometer are heavily damped systems.

Mathematically, a system is said to be heavily damped if $b > \omega_0$. Then the quantity $(b^2 - \omega_0^2)$ is positive definite. If we put

$$\beta = \sqrt{b^2 - \omega_0^2}$$

the general solution for damped oscillator given by Eq. (3.9) reduces to

$$x\,(t) = \exp(-bt)\,[a_1 \exp(\beta t) + a_2 \exp(-\beta t)]. \tag{3.10}$$

This represents non-oscillatory behaviour. Such a motion is called dead-beat. The actual displacement will, however, be determined by the initial conditions. Let us suppose that to begin with the oscillator is at its equilibrium position, i.e $x = 0$ at $t = 0$. Then we give it a sudden kick so that it acquires a velocity v_0, i.e $v = v_0$ at $t = 0$. Then from Eq. (3.10) we have

$$a_1 + a_1 = 0$$

and $$-\,b\,(a_1 + a_2) + \beta\,(a_1 - a_2) = v_0$$

These equations may be solved to give

$$a_1 = -\,a_2 = \frac{v_0}{2\beta}$$

On substituting these results in Eq. (3.10), we can write the solution in compact form:

$$x\,(t) = \frac{v_0}{2\beta} \exp(-\,bt)\,[\exp(\beta t) - \exp(-\,\beta t)]$$

$$= \frac{v_0}{\beta} \exp(-\,bt) \sinh \beta t \tag{3.11}$$

where singh $\beta t = [\exp(\beta t) - \exp(-\,\beta t)]$ 2 is hyperbolic sine function. From Eq. (3.11) it is clear that x (t) will be determined by the interplay of an increasing hyperbolic function and a decaying exponential. These are plotted separately in Fig 3.2(a). Fig. 3.2(b) shows the plot of Eq. (3.11) for a heavily damped system when it is suddenly disturbed from its equilibrium position. You will note that initially the displacement increases with time. But soon the exponential term becomes important and displacement begins to decrease gradually.

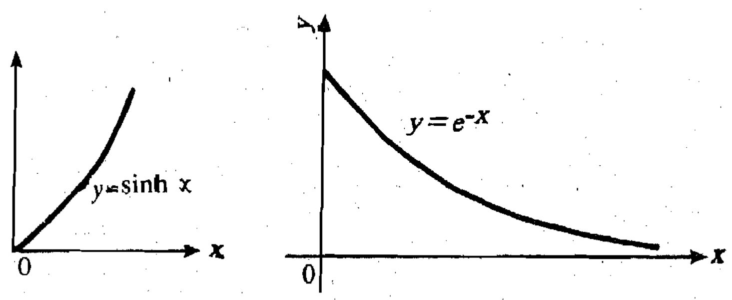

Fig. 3.2a Plot of sinh x and exp $(-\,x)$.

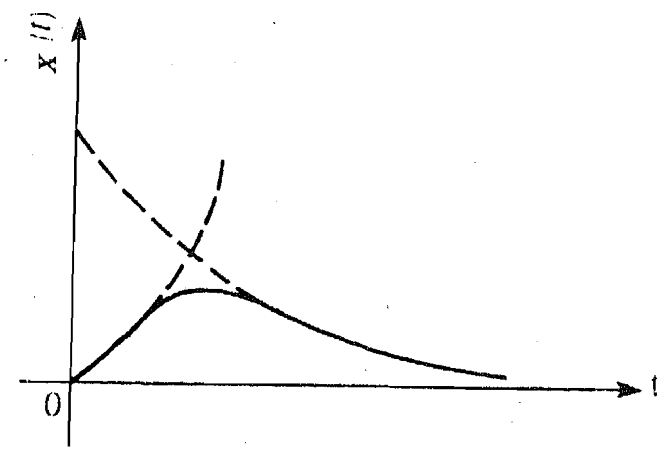

Fig. 3.2b Plot of Eq. (3.11) for a heavily damped system.

3.3.2 Critical Damping

You may have observed that on hitting an isolated road bump, a car bounces up and down and the occupants feel uncomfortable. To minimise this discomfort, the bouncing caused by the road bumps must be damped very rapidly and the automobile is restored to equilibrium quickly. For this we use critically damped shock absorbers. Critical damping is also useful in recording instruments such as a galvanometer (pointer type as well as suspended coil type) which experience sudden impulses. We require the pointer to move to the correct position in minimum time and stay there without executing oscillations. Similarly, a ballistic galvanometer coil is required to return to zero displacement immediately.

Mathematically, we say that a system is critically damped if b is equal to the natural frequency, ω_0, of the system. This means that $b^2 - \omega_0^2 = 0$, so that Eq **(3.9)** reduces to

$$x(t) = (a_1 + a_2)\exp(-bt) = a\exp(-bt) \qquad (3.12)$$

where $a = a_1 + a_2$.

Let us pause for a minute and recall that the solution of the differential equation for SHM involves two arbitrary constants which are fixed by giving the initial conditions. But **Eq.** (3.12) has only one constant. Does this mean that it is not a complete solution? It is important to understand how this happens. The reason is simple : the quadratic equation for α (Eq. 3.6) has equal roots. So, the two terms in Eq. (3.9) give the same time dependence and reduce to one term. It can be easily verified that in this case the general solution of Eq. (3.3) is

$$x(t) = (p + qt)\exp(-bt) \qquad (3.13\,a)$$

where p and q are constants. p has the dimensions of length and q those of velocity. These can be determined easily from the initial conditions.

Let us assume that the system is disturbed from its mean equilibrium position by a sudden impulse. (The coil of a suspended type galvanometer receives some electric charge at $t = 0$.) That is, at $t = 0$, $x(0) = 0$ and $\left.\frac{dx}{dt}\right|_{t=0} = v_0$. This gives $p = 0$ and $q = v_0$, so that the complete solution is

$$x(t) = v_0 t\exp(-bt) \qquad (3.13b)$$

Like π, e is a transedental and is equal to 2.718.

Fig. 3.3 illustrates the displacement time graph of a critically damped system described by Eq.(3.13b). At maximum displacement, $\left.\frac{dx}{dt}\right|_{x=x_{max}} = 0$ and $\left.\frac{d^2x}{dt^2}\right|_{x=x_{max}} < 0$.

This occurs at time $t = 1/b$:

$$x_{max} = v_0 t e^{-1} = 0{,}368\,\frac{v_0}{b} = 0.736\,\frac{m v_0}{\gamma}$$

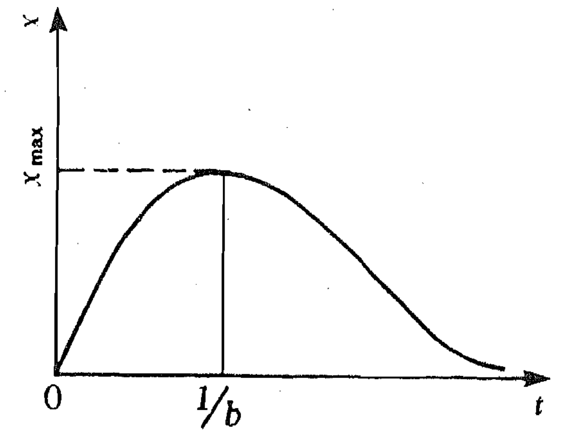

Fig. 3.3 Displacement-time graph for a critically damped system described by Eq. (3.13b).

3.3.3 Weak or light damping

When $b < \omega_0$ we refer to it as a case of weak damping. This implies that $(b^2 - \omega_0^2)$ is a negative quantity, i.e. $(b^2 - \omega_0^2)^{1/2}$ is imaginary. Let us rewrite it as

$$(b^2 - \omega_0^2)^{1/2} = \sqrt{-1}\,(\omega_0^2 - b^2)^{1/2}$$
$$= \pm i\omega_d$$

where $\quad i = \sqrt{-1}$ and

$$\omega_d = (\omega_0^2 - b^2)^{1/2} = \left[\frac{k}{m} - \frac{\gamma^2}{4m^2}\right]^{1/2} \tag{3.14}$$

is a real positive quantity. You will note that for no damping ($b = 0$), ω_d reduces to ω_0, the natural frequency of the oscillator.

On combining Eqs. (3.9) and (3.14) we find that the displacement now has the form

$$x(t) = \exp(-bt)\,[a_1 \exp(i\omega_d t) + a_2 \exp(-i\omega_d t)] \tag{3.15}$$

To compare the behaviour of a damped oscillator with that of a free oscillator, we should recast Eq.(3.15) so that the displacement varies sinusoidally. To do this, we write the complex exponential in terms of sine and cosine functions. This gives

$$x(t) = \exp(-bt)\,[a_1(\cos\omega_d t + i\sin\omega_d t)$$
$$- a_2(\cos\omega_d t - i\sin\omega_d t)]$$

$\exp(\pm ix) = \cos x \pm i \sin x.$

On collecting coefficients of $\cos\omega_d t$ and $\sin\omega_d t$, we obtain

$$x(t) = \exp(-bt)\,[(a_1 + a_2)\cos\omega_d t + i(a_1 - a_2)\sin\omega_d t)] \tag{3.16}$$

Let us now put

$$a_1 + a_2 = a_0\cos\phi$$

and

$$-i(a_1 - a_2) = a_0\sin\phi \tag{3.17}$$

where a_0 and ϕ are arbitrary constants. These are given by

$$a_0 = 2\sqrt{a_1 a_2}$$

and

$$\tan\phi = -i\,\frac{a_1 - a_2}{a_1 + a_2} \tag{3.18}$$

From the second of these results we note that $\tan\phi$ is a complex quantity. Does this mean that ϕ is also complex? How can we interpret a complex angle? To know this, we use the identity

$$\sec^2\phi = 1 + \tan^2\phi$$

and calculate $\cos\phi$. The result is

$$\cos\phi = \frac{a_1 + a_2}{2\sqrt{a_1 a_2}}$$

This means that $\cos\phi$, and hence ϕ, is real.

Substituting Eq. (3.17) into Eq. (3.16) we find that the expression within the parentheses is cosine of the sum of two angles. Hence, the general solution of Eq. *(3.3)* for a weakly damped oscillator ($b < \omega_0$) is

$$x(t) = a_0 \exp(-bt)\cos(\omega_d t + \phi) \qquad (3.19)$$

with ω_d as given by Eq. *(3.14)*. You will note that the solution given by Eq. *(3.19)* describes sinusoidal motion with frequency ω_d which remains the same throughout the motion. This property is crucial for the use of oscillators in accurate time-pieces. How is the amplitude modified vis-a-vis an ideal SHM? You will note that the amplitude decreases exponentially with time at a rate governed by b. So ***we con say that the motion of a weakly damped system is not simple harmonic.***

The damped oscillatory behaviour described by Eq. (3.19) is plotted in Fig. **3.4** for the particular case of $\phi = 0$. Since the cosine function varies between $+1$ and -1, we observe that the displacement-time curve lies between $a_0 \exp(-bt)$ and $-a_0 \exp(-bt)$. Thus, we may conclude that ***damping results in decrease of amplitude and angulur frequency.***

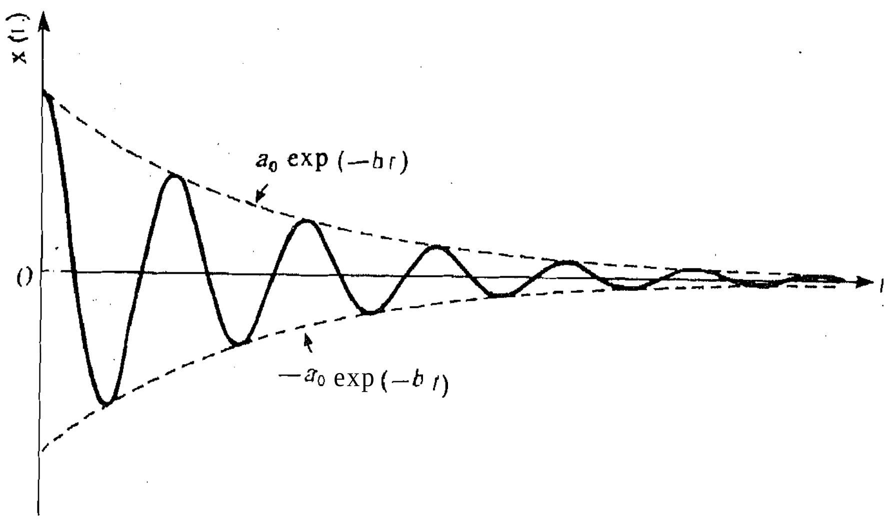

Fig. 3.4 Displacement-time graph for a weakly damped harmonic oscillator.

How does damping influence the period of oscillation? You can discover this effect by noting that the period of oscillation is given by

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{(\omega_0^2 - b^2)^{1/2}} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}}$$

If $b > 0$, $\omega_d < \omega_0$. This means that the period of vibration of a damped oscillator is more than that of an ideal oscillator. Did you not expect it since damping forces resist motion?

SAQ 1

The amplitude of vibration of a damped spring-mass system decreases from 10 cm to 2.5 cm in 200 s. If this oscillator performs 100 oscillations in this time, compare the periods with and without damping.

We have discussed solutions of the differential equation for a damped oscillator for heavy, critical and weak dampings. In the fallowing discussion we shall concentrate only on weakly damped systems.

3.4 AVERAGE ENERGY OF A WEAKLY DAMPED OSCILLATOR

In Unit 1 we calculated the average energy of an undamped oscillator. The question now arises: How does damping influence the average energy of a weakly damped oscillator? To answer this we note that in the presence of damping the amplitude of oscillation decreases with the passage of time. This means that energy is dissipated in overcoming resistance to motion. From Unit 1 we recall that at any time, the total energy of a harmonic oscillator is made up of kinetic and potential components. We can still use the same definition and write

$$E(t) = K.E(t) + U(t)$$

$$= \tfrac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2} kx^2 \tag{3.20}$$

where (dx/dt) denotes instantaneous velocity.

For a weakly damped harmonic oscillator, the instantaneous displacement is given by Eq. (3.19):

$$x(t) = a_0 \exp(-bt) \cos(\omega_d t + \phi)$$

By differentiating it with respect to time, we get instantaneous velocity:

$$\frac{dx(t)}{dt} = v = -a_0 \exp(-bt)\,[b \cos(\omega_d t + \phi) + \omega_d \sin(\omega_d t + \phi)] \tag{3.21}$$

Hence, kinetic energy of the oscillator is

$$K.E = \tfrac{1}{2} m \left(\frac{dx}{dt}\right)^2 = \tfrac{1}{2} m a_0^2 \exp(-2bt)\,[b \cos(\omega_d t + \phi) + \omega_d \sin(\omega_d t + \phi)]^2$$

$$= \tfrac{1}{2} m a_0^2 \exp(-2bt)\,[b^2 \cos^2(\omega_d t + \phi) + \omega_d^2 \sin^2(\omega_d t + \phi) + b\,\omega_d \sin 2(\omega_d t + \phi)] \tag{3.22a}$$

Similarly, the potential energy of the oscillator is

$$U = \tfrac{1}{2} kx^2 = \tfrac{1}{2} m\omega_0^2 x^2$$

since $k = m\omega_0^2$

On substituting for x, we get

$$U = \tfrac{1}{2} m a_0^2 \omega_0^2 \exp(-2bt) \cos^2(\omega_d t + \phi) \tag{3.22b}$$

Hence, the total energy of the oscillator at any time t is given by

$$E(t) = \tfrac{1}{2} m a_0^2 \exp(-2bt)\,[(b^2 + \omega_0^2) \cos^2(\omega_d t + \phi) + \omega_d^2 \sin^2(\omega_d t + \phi) + b\omega_d \sin 2(\omega_d t + \phi)] \tag{3.23}$$

When damping is small, the amplitude of oscillation does not change much over one oscillation. So we may take the factor $\exp(-2bt)$ as essentially constant. Further, since $\langle \sin^2(\omega_d t + \phi) \rangle = \langle \cos^2(\omega_d t + \phi) \rangle = \tfrac{1}{2}$ and $\langle \sin(\omega_d t + \phi) \rangle = 0$, the energy of a weakly damped oscillator when averaged over one cycle is given by

$$\langle E \rangle = \tfrac{1}{2} m a_0^2 \exp(-2bt) \langle [(b^2 + \omega_0^2) \cos^2(\omega_d t + \phi) + \omega_d^2 \sin^2(\omega_d t + \phi) + b\omega_d \sin 2(\omega_d t + \phi)] \rangle$$

$$= \tfrac{1}{2} m a_0^2 \exp(-2bt) \left[\frac{b^2 + \omega_0^2}{2} + \frac{\omega_d^2}{2}\right]$$

$$= \tfrac{1}{2} m a_0^2 \omega_0^2 \exp(-2bt) \tag{3.24a}$$

From Unit 1 we recall that $E_0 = \tfrac{1}{2} m a_0^2 \omega_0^2$ is the total energy of an undamped oscillator. Hence, we can write

$$\langle E \rangle = E_0 \exp(-2bt) \tag{3.24b}$$

This shows that the ***average energy of a weakly damped oscillator decreases exponentially with time.*** This is illustrated in Fig. *3.5.* From Eq.(3.24 *b)* you will also observe that the rate of decay of energy depends on the value of b; larger the value of b, faster will be the decay.

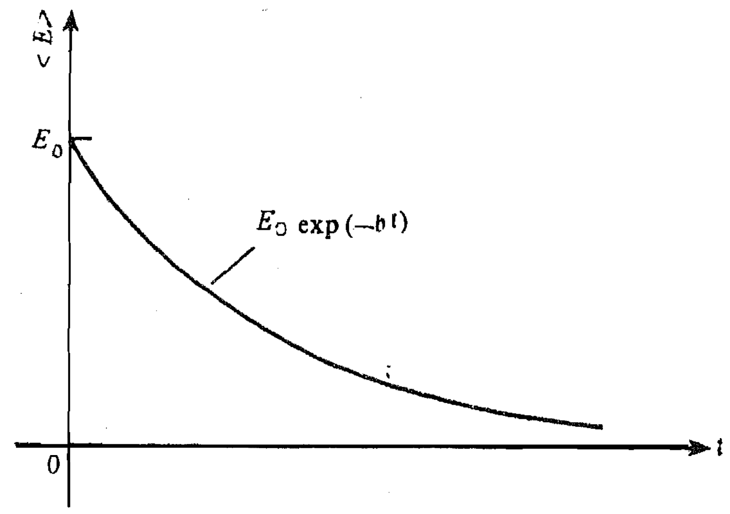

Fig. 3.5 Time variation of average energy for a weakly damped **system.**

3.4.1 Average Power Dissipated Over One Cycle

Since energy of a damped oscillator does not remain constant in time, $\left(\frac{dE}{dt}\right)$ is not zero. In fact, it is negative. The rate of loss of energy at any time gives instantaneous power dissipated. From Eq. *(3.20)* we can write

$$\frac{dE}{dt} = P(t) = \left[m\frac{d^2x}{dt^2} + kx \right] \frac{dx}{dt}$$

On combining this result with Eq. *(3.2)* we find that power dissipated by a damped oscillator is given by

$$P(t) = -\gamma \left(\frac{dx}{dt}\right)^2$$

This relation shows that the rate of doing work against the frictional force is directly proportional to the square of instantaneous velocity. On substituting for $\left(\frac{dx}{dt}\right)$ from Eq. (3.21), we obtain

$$P(t) = -\gamma a_0^2 \exp(-2bt) [b^2 \cos^2(\omega_d t + \phi) + \omega_d^2 \sin^2(\omega_d t + \phi) + b\omega_d \sin(2\omega_d t + \phi)]$$

Hence, the average power dissipated over one cycle is given by

$$\begin{aligned} \langle P \rangle &= -\tfrac{1}{2} \gamma a_0^2 \omega_0^2 \exp(-2bt) \\ &= -\frac{\gamma}{m} \langle E \rangle \\ &= -2b \langle E \rangle \end{aligned} \qquad (3.25)$$

The negative sign here signifies that power is dissipated.

3.5 METHODS OF CWARACTERISING DAMPED SYSTEMS

We now know that in the visccus damping model, a damped oscillator is characterised by *y* and *wo.* We also know that this model applies to vastly different physical systems. Therefore, you may ask: Are there other ways of characterising damped oscillations? Experience tells us that in certain cases it is more convenient to use other parameters to characterise damped motion. In all cases we can relate these to γ and ω_0. We will now discuss these briefly.

3.5.1 Logarithmic Decrement

The most convenient way to determine the amount of damping present in a system is to measure the rate at which ***amplitude*** of oscillation dies away. Let us consider the damped vibration shown graphically in Fig. 3.6. Let a_0 and a_1 be the first two successive amplitudes of oscillation separated by one period.

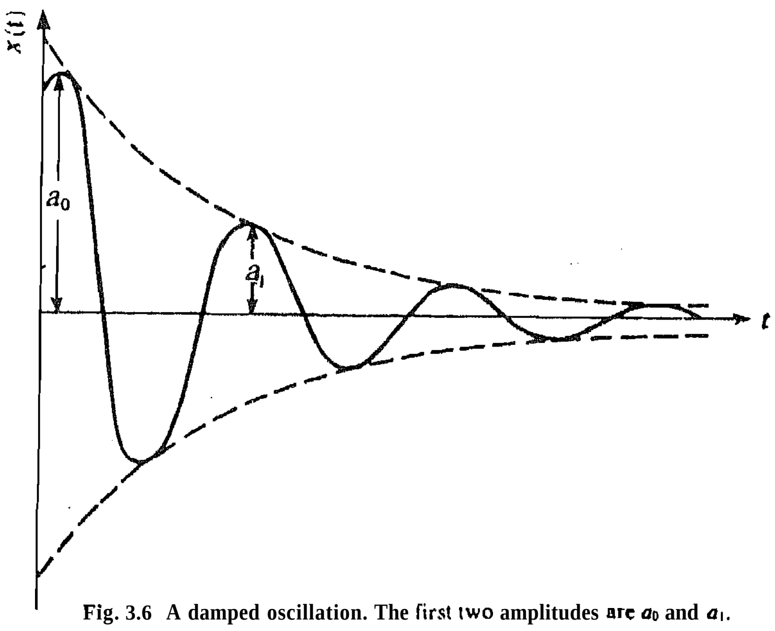

Fig. 3.6 A damped oscillation. The first two amplitudes are a_0 and a_1.

You, will note that these amplitudes lie in the same **direction/quadrant.** If T is the period of oscillation, then using Eq. (3.19) for a weakly damped oscillator, we can write

$$a_1 = a_0 \exp(-bT)$$

so that
$$\frac{a_0}{a_1} = \exp(bT) = \exp(\gamma T/2m) \tag{3.26}$$

You will note that in the ratio a_0/a_1, the larger amplitude is in the numerator. **That** is why this ratio is called the ***decrement.*** It is denoted by the symbol d. You may now ask: Is the decrement same for ***any*** two consecutivk amplitudes? The answer is: yes, it is. To show this let us consider the ratio of the second and the third **amplitudes.** These are observed for $t = T$ and $t = 2T$, respectively in Eq. (3.19). **Then, we** can write

$$\frac{a_1}{a_2} = \frac{a_0 \exp(-bT)}{a_0 \exp(-2bT)} = \exp(bT)$$

So, we may conclude that for any two consecutive amplitudes **separated** by **one** period, we have

$$\frac{a_{n-1}}{a_n} = \exp(bT) = d \tag{3.27}$$

That is, ***decrement is the same for two successive amplitudes*** and we can write

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots = \frac{a_{n-1}}{a_n} = d \tag{3.28}$$

The logarithm of the ratio of successive amplitudes of oscillation separated by one period is called the logarithmic decrement. It is usually **denoted** by the symbol λ:

$$\lambda = \ln\left(\frac{a_{n-1}}{a_n}\right) = \frac{\gamma T}{2m} \tag{3.29 a}$$

This equation shows that we can measure A by knowing two successive amplitudes. But from an experimental point of view it is more convenient and accurate to compare amplitudes of oscillations separated by n periods. That is, we measure a_0/a_n. To compute this ratio, we first invert Eq. (3.29 a) to write

$$\frac{a_{n-1}}{a_n} = \exp(\lambda) \tag{3.29b}$$

The ratio a_0/a_n can now be written as

$$\frac{a_0}{a_n} = \left(\frac{a_0}{a_1}\right)\left(\frac{a_1}{a_2}\right)\left(\frac{a_2}{a_3}\right)\cdots\left(\frac{a_{n-1}}{a_n}\right) = [\exp(A)]''$$

$$= \exp(n\lambda) \tag{3.30}$$

since the ratio of any two consecutive amplitudes is the same.

Taking log of both sides, we get the required result:

$$\lambda = \frac{1}{n} \ln\left(\frac{a_0}{a_n}\right) \tag{3.31}$$

This shows that if we plot In (a_0/a_n) versus n for different values of n, we will obtain a straight line. The slope of the line gives us A.

SAQ 2

A damped harmonic oscillator has the first amplitude of 20 cm. It reduces to 2 cm after 100 oscillations, each of period 4.6 s. Calculate the logarithmic decrement and damping constant. Compute the number of oscillations in which the amplitude drops by 50%.

3.5.2 Relaxation Time

In physics we often measure decay of a quantity in terms of the fraction e^{-1} of the initial value. This gives us another way of expressing the damping effect by means of the time taken by the amplitude to decay to $e^{-1} = 0.368$ of its original value. This time is called the *relaxation* rime. To understand this, we recall that the amplitude of a damped oscillation is given by

$$a(t) = a_0 \exp(-bt)$$

If we denote the amplitude of oscillation after an interval of time τ by a $(t + \tau)$, we can write

$$a(t + \tau) = a_0 \exp[-b(t + \tau)]$$

By taking the ratio a $(t + \tau)/$ a (t), we obtain

$$\frac{a(t+\tau)}{a(t)} = \exp(-b\tau)$$

$$= \frac{1}{e} \text{ for } b\tau = 1 \tag{3.32}$$

This shows that for $b = \tau^{-1}$, the amplitude drops to $1/e = 0.368$ of its initial value. Using this result in Eq. (3.25), we get

$$\langle P \rangle = \frac{2\langle E \rangle}{\tau}$$

The relaxation time, τ, is therefore a measure of the rapidity with which motion is damped. (You will note that the negative sign occuring in Eq. (3.25) has been dropped here.)

3.5.3 The Quality Factor

Yet another way of expressing the damping effect is by means of the rate of decay of energy. From Eq. (3.24b) we note that the average energy of a weakly damped oscillator decays to $E_0 e^{-1}$ in time $t = \frac{1}{2b} = \frac{m}{\gamma}$ seconds. If ω_d is its angular frequency, then in this time the oscillator will vibrate through $\omega_d\, m/\gamma$ radians. ***The number of radians through which a weakly damped system oscillates as its average energy decays to $E_0 e^{-1}$ is a measure of the quality factor, Q :***

$$Q = \frac{\omega_d m}{\gamma} = \frac{\omega_d}{2b} = \frac{\omega_d \tau}{2} \tag{3.33}$$

You will note that Q is only a number and has no dimensions. In general, γ is small so that Q is very large. A tuning fork has Q of a thousand or so, whereas a rubber band exhibits a much lower (—10) Q. This is due to the internal friction generated by the coiling of the long chain of molecules in a rubber band. An undamped oscillator ($\gamma = 0$) has an infinite quality factor.

For a weakly damped mechanical oscillator, the quality factor can be expressed in terms of the spring factor and damping constant. For weak damping,

$$\omega_d \simeq \omega_0 - \sqrt{k/m}$$

Hence $\quad Q = \sqrt{km/\gamma^2}$

That is, the quality factor of a weakly damped oscillator is directly proportional to the square root of k and inversely proportional to γ.

We can rewrite Eq. (3.33) in a more physically meaningful form using Eq. (3.25):

$$Q = \frac{\omega_d}{2b} = \frac{2\pi}{T_d} \times \frac{\langle E \rangle}{\langle P \rangle}$$

$$= 2\pi \, \frac{\text{average energy stored in the system in one cycle}}{\text{average energy lost in one cycle}} \tag{3.34}$$

The quality factor is related to the fractional change in the frequency of an undamped oscillator. To establish this relation, we note that

$$\omega_d = \sqrt{\omega_0^2 - b^2}$$

or

$$\frac{\omega_d^2}{\omega_0^2} = 1 - \frac{b^2}{\omega_0^2}$$

$$\simeq 1 - \frac{1}{4Q^2}$$

where we have used Eq. (3.33). This result can be rewritten as

$$\frac{\omega_d}{\omega_0} = \left(1 - \frac{1}{4Q^2}\right)^{1/2}$$

$$= 1 - \frac{1}{8Q^2}$$

where in the binomial expansion we have retained terms upto first order in Q^2. Hence, the fractional change in ω_0 is $1/(8\,Q^2)$.

SAQ 4

The quality factor of a tuning fork of frequency 256 Hz is 10^3. Calculate the time in which its energy becomes 10% of its initial value.

3.6 EXAMPLES OF DAMPED SYSTEMS

You know that all harmonic oscillators in nature have some damping, which in general, is quite small. To enable you to appreciate the effect of damping, we will consider two specific cases: (i) Oscillations of charge in an LCR circuit, and (ii) motion of the coil in a suspension type galvanometer. These are of particular interest to us as the former has wide applications in radio engineering and the latter is used in the physics laboratory.

3.6.1 An LCR Circuit

In Unit 1 we observed that in an ideal LC circuit, charge executes SHM. Do you expect any change in this behaviour when a resistor is added? To answer this question we consider Fig. 3.7. If a current I flows through the circuit due to discharging/ charging of the capacitor, the voltage drop across the resistor is RI. Thus Eq. (1.36) Now modifies to

$$\frac{q}{C} = -L\frac{dI}{dt} \quad \frac{dq}{dt}$$

Eq. (3.35) may be rewritten as

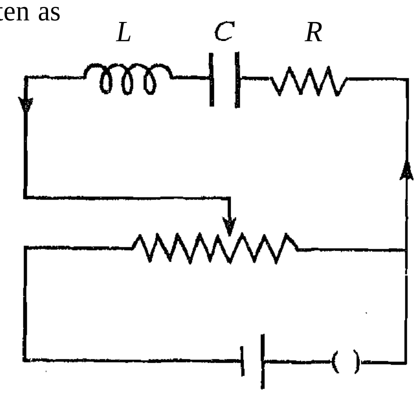

Fig. 3.7 An LCR circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \tag{3.36}$$

Comparing it with Eq. (3.2) we find.that L, R and $1/C$ are respectively analogous to m, y and k. This means that a resistor in an electric circuit has an exactly analogous effect as that of the viscous force in a mechanical system.

To proceed further, we divide Eq. (3.36) throughout by L obtaining

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt^2} + \frac{1}{LC}q = 0 \tag{3.37}$$

In this form, Eq. (3.37) is analogous to Eq. (3.3) and the two may be compared directly. This gives

$$\omega_0^2 = \frac{1}{LC}$$

and $$b = \frac{R}{2L} \tag{3.38}$$

We know that b has dimensions of time inverse. This means that R/L has the unit of s^{-1}, same as that of ω_0. That is why $\omega_0 L$ is measured in ohm.

With these analogies all the results of Section 3.3 apply to Eq. (3.37). For a weakly damped circuit, the charge on the capacitor plates at time t is

$$q(t) = q_0 \exp\left(-\frac{R}{2L}t\right)\cos(\omega_d t + \phi) \tag{3.39a}$$

with angular frequency

$$\omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{3.39b}$$

Eq. (3.39 a) shows that the charge amplitude $q_0 \exp\left(-\frac{R}{2L}t\right)$ will decay at a rate which depends on the resistance. Thus in an ***LCR*** circuit, resistance is the only dissipative element; an increase in R increases the rate of decay of the charge and decreases the frequency of oscillations.

When $1/LC >> R^2/4L$,

$$\omega_d^2 \cong \omega_0^2 = \frac{1}{LC}$$

or $$\omega_0 L = \frac{1}{\omega_0 C}$$

Since $\omega_0 L$ is measured in ohms, $1/\omega_0 C$ is also measured in ohms. These are respectively reftrred to as ***induriivr reactance and capacitive reactance.***

For $R = 0$, **Eq. (3.39** a) reduces to **Eq. (1.38)** and $\omega_d = \omega_0$. The Q value of a weakly daruped LCR circuit is

$$Q = \frac{\omega_d}{2b} \simeq \omega_0 \frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} \qquad (3.40)$$

This equation shows that for a purely inductive circuit (R = 0), quality factor will be infinite.

SAQ 5

In an LCR circuit, L = 2mH and C = 5μF. If R = IR, 40R and 100Ω, calculate the frequency of oscillation and the quality factor when the discharge is oscillatory.

3.6.2 A Suspension Type Galvanometer

A suspension type galvanometer consists of a current carrying coil suspended in a magnetic field. The field is produced by a horse-shoe magnet. The magnet is shaped so that the coil is aligned always along the magnetic lines of force. To ensure uniform strength, an iron cylinder is suspended between the poles of the magnet, as shown in Fig. **(3.8).** When we pass charge through the galvanometer coil, it rotates through some angle θ. Since the coil is mechanically a torsional pendulum, it experiences a restoring couple $-k_t\theta$ and a damping couple $-\gamma \frac{d\theta}{dt}$. Do you know how damping creeps in, in this case? It has origin in air friction and electromagnetic induction.

Part of the damping arises from the viscous drag of air. In general, it is small.

As the galvanometer coil rotates in the magnetic field, an induced e.m.f. is produced, which opposes its motion in accordance with Lenz's law. This so-called electromagnetic damping controls the motion of the coil when galvanometer is in use.

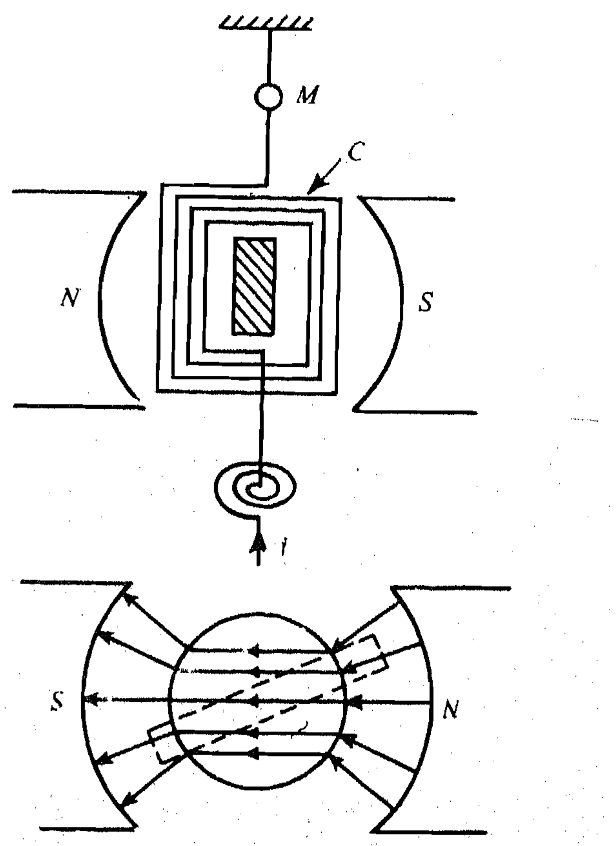

Fig. 3.8 A schematic representation of a suspension type galvanometer.

Hence, for the motion of the coil, Eq .(1.35) modifies to

$$I \frac{d^2\theta}{dt^2} = - k_t\theta - \gamma \frac{d\theta}{dt} \tag{3.41}$$

where I is moment of inertia of the coil about the axis of suspension. Comparing it with Eq. (3.2) we find that I and k_t are analogous to m and k respectively.

Dividing throughout by I and defining

$$2b = \gamma/I$$

and $$\omega_0^2 = k_t/I \tag{3.42}$$

we get $$\frac{d^2\theta}{dt^2} + 2b \frac{d\theta}{dt} + \omega_0^2 \theta = 0 \tag{3.43}$$

This equation is of the same form as Eq. *(3.3).* Hence, all results deduced earlier will apply to the motion of the coil described by Eq. *(3.43).*

For low damping, the solution of Eq. (3.43) is

$$\theta = \theta_0 \exp(-bt) \cos(\omega_d t + \phi) \tag{3.44}$$

where $\theta_0 \exp(-bt)$ is the amplitude of oscillation. Eq. *(3.44)* describes oscillatory motion with the period of oscillation T given by

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{(\omega_0^2 - b^2)^{1/2}} = \frac{2\pi}{\left[\frac{k_t}{I} - \frac{\gamma^2}{4I^2}\right]^{1/2}} \tag{3.45}$$

This explains why a weakly damped suspension type galvanometer is called a ***ballistic*** galvanometer. You will note that for damping to be small, we must decrease γ and increase I. The question now arises: How can we reduce γ? As mentioned earlier, air damping is usually small. Nevertheless, it will always be present. To reduce electromagnetic damping, we must minimise induced emf. To ensure this, we wind the coil over a non-conducting bamboo or ivory frame. If the frame is metallic, it is cut at one place, so that no current can flow through it.

The quality factor of a ballistic galvanometer is

$$Q = \frac{\omega_d}{2b} = \frac{I}{\gamma} \sqrt{\frac{k_t}{I} - \frac{\gamma^2}{4I^2}} \tag{3.46a}$$

If $\frac{k_t}{I} >> \frac{\gamma^2}{4I^2}$, this expression reduces to

$$Q = \sqrt{\frac{k_t I}{\gamma^2}} \tag{3.46b}$$

This relation shows that a lightly damped suspension type galvanometer will have high quality factor.

SAQ 6
The period of vibration of a galvanometer coil is 4 s. The amplitude of its vibration decreases to 1/10th of its original value in 46 s. Calculate the damping constant γ and quality factor.

3.7 SUMMARY

1 The differential equation of a damped harmonic oscillator is

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + \omega_0^2 x = 0$$

where $2b = \gamma/m$ and $\omega_0^2 = k/m$.
The solution of the equation for heavy damping is

$x(t) = \exp(-bt)\,[a_1 \exp(\beta t) + a_2 \exp(-\beta t)]$
where $\beta = \sqrt{b^2 - \omega_0^2}$.

For critical damping
$x(t) = (p + qt)\exp(-bt)$
and in case of weak damping
$x(t) = a_0 e^{-bt} \cos(\omega_d t + \phi)$

2 The amplitude and average energy of a weakly damped oscillator decrease exponentially with time:
$a = a_0 e^{-bt}$
and $\langle E \rangle = E_0 \exp(-2bt)$
where a_0 is the initial amplitude and E_0 is total initial energy.

3 The period of a weakly damped system is given by

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{(\omega_0^2 - b^2)^{1/2}} = \frac{2\pi}{\left(\frac{k}{m} - \frac{\gamma}{4m^2}\right)^{1/2}}$$

4 The logarithmic decrement is defined as the logarithm of the ratio of successive amplitudes separated by one period. It is given by

$$\lambda = \ln\left(\frac{a_{n-1}}{a_n}\right) = bT$$

5 The rate of loss of energy or power dissipated by a weakly damped harmonic oscillator over one cycle is
$\langle P \rangle = 2\langle E \rangle / \tau$

6 Q-factor of a weakly damped harmonic oscillator is given by
$Q = \omega_d \tau / 2$

7 The differential equation describing flow of charge in a LCR circuit is

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0$$

The effect of L, R and $1/C$ in an LCR circuit is respectively analogous to those of m, γ and k in a mechanical oscillator. In a weakly damped circuit, the charge oscillates **harmonically** :

$$q(t) = q_0 \exp\left(-\frac{R}{2L}t\right)\cos(\omega_d t + \phi)$$

and the frequency of oscillation is given by

$$\nu_d = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

For low R circuit

$$Q = \omega_0 L / R = \frac{1}{R}\sqrt{L/C}$$

8 The differential equation of a damped suspension type galvanometer is

$$I\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + k_t\theta = 0$$

For weak damping it describes ballistic motion given by
$\theta = \theta_0 \exp(-bt)\cos(\omega_d t + \phi)$

where $\omega_d = \sqrt{\frac{k_t}{I} - \frac{\gamma^2}{4I^2}}$

3.8 TERMINAL QUESTIONS

1 A simple pendulum has a period of 2 s and an amplitude of 5°. After 20 complete oscillations, its amplitude is reduced to 4°. Find the damping constant and time constant.

2 The quality factor of a sonometer wire is 4,000. The wire vibrates at a frequency of 300 Hz. Find the time in which the amplitude decreases to half of its initial value.

3 A box of mass 0.2 kg is attached to one end of a spring whose other end is fixed to a rigid support. When a mass of 0.8 kg is placed inside the box, the system performs 4 oscillation per second and the amplitude falls from 2 cm to 1 cm in 30 s. Calculate (i) the force constant, (ii) the relaxation time, and (iii) Q-factor.

4. In an LCR circuit L = 5 mH, C = 2 μF and R = 0.2 Ω. Will the discharge be oscillatory? If so, calculate the frequency and quality factor of the circuit. How long does the charge oscillation take to decay to half? What value of R will make the discharge just non-oscillatory?

5. The quality factor of a tuning fork of frequency 512 Hz is 6×10^4. Calculate the time in which its energy is reduced to e^{-1} of its energy in the absence of damping. How many oscillations will the tuning fork make in this time?

3.9 SOLUTIONS

SAQ's

1 $T_d = \dfrac{200\text{s}}{100} = 2\text{ s}$

Now $T_d = 2\text{s} = \dfrac{2\pi}{(\omega_0^2 - b^2)^{1/2}}$

so that $\omega_0^2 = \pi^2 + b^2$

Hence $T_0 = \dfrac{2\pi}{\omega_0} = \dfrac{2\pi}{(\pi^2 + b^2)^{1/2}}$

To compute b, we use the relation

$a = a_0 \exp(-bt)$

This may be rewritten as

$$b = \frac{1}{t} \ln\left(\frac{a_0}{a}\right)$$

$$= \frac{1}{200\text{ s}} \ln\left(\frac{10\text{ cm}}{2.5\text{ cm}}\right)$$

$$= \frac{2.3}{200\text{ s}} \log_{10} 4$$

$$= 6.9 \times 10^{-3}\text{ s}^{-1}$$

Substituting this value in (i), we get

$$T_0 = \frac{2\pi}{[\pi^2 + (6.9 \times 10^{-3})^2]^{1/2}} \simeq 2.0\text{ s} \simeq T_d$$

This means that the system is weakly damped.

2 We know that

$$\lambda = \frac{1}{n} \ln\left(\frac{a_0}{a_n}\right) \tag{i}$$

$$= \frac{1}{100} \ln 10$$

$$= \frac{2.3}{100} \log_{10} 10 = 2.3 \times 10^{-2}$$

Since $b = \frac{\Lambda}{T}$

we get

$$b = \frac{2.3 \times 10^{-2}}{4.6\ \text{s}}$$

$$= 5.0 \times 10^{-3}\ \text{s}^{-1}$$

Further, to calculate n for which the amplitude drops by 50%, we invert (i) to write

$$n = \frac{1}{\lambda} \ln\left(\frac{a_0}{a_n}\right)$$

$$= \frac{\ln 2}{2.3 \times 10^{-2}} = \frac{2.3 \log_{10} 2}{2.3 \times 10^{-2}}$$

$$= 30$$

3 Since

$$\Lambda = bT = \frac{1}{n} \ln\left(\frac{a_0}{a_n}\right)$$

we can write

$$b = \frac{1}{nT} \ln (a_0/a_n)$$

$$= \frac{1}{200\ \text{s}} \ln 4$$

$$= \frac{2.3 \times 0.6010}{200}\ \text{s}^{-1}$$

Hence

$$\tau = \frac{1}{b} = \frac{200}{2.3 \times 0.6010\ \text{s}^{-1}} = 145\ \text{s}$$

4 $Q = 10^3$ and $\nu = 256$ Hz

we know that for weak damping Q is given by

$$Q = \frac{\omega_0 \tau}{2} = \pi \nu \tau$$

On inverting this relation, we get

$$\tau = \frac{Q}{\pi \nu} = \frac{10^3}{256\pi\ \text{s}^{-1}} = 1.24\ \text{s}$$

Since $E = E_0 \exp(-2bt) = E_0 \exp(-2t/\tau)$, wc get for $E/E_0 = 1/10$

$$\frac{1}{10} = \exp\left(-\frac{2t}{\tau}\right)$$

Hence,

$$t = \frac{\tau}{2} \ln 10$$

$$= \frac{1.24\ \text{s}}{2} \times 2.3$$

$$= 1.4\ \text{s}$$

5 $L = 2 \times 10^{-3}$ H and $C = 5 \times 10^{-6}$ F

$$\therefore \frac{1}{LC} = \frac{1}{2 \times 10^{-3}\ \text{H} \times 5 \times 10^{-6}\ \text{F}} = 10^8\ \text{s}^{-2}$$

Case I. $R = 1\ \Omega$

$$\therefore \frac{R^2}{4L^2} = \frac{1\Omega^2}{4 \times (2 \times 10^{-3})^2 H^2} = 6 \times 10^4 \frac{\Omega^2}{H^2}$$

Thus,

$\frac{1}{LC} >> \frac{R^2}{4L^2}$ so that the discharge is oscillatory. The frequency of oscillation

$$\nu = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$= 1.6$ kHz

and quality factor of the circuit

$$Q = \frac{\omega_0 L}{R} = \frac{2\pi \times 1.6 \times 10^3 s^{-1} \times 2 \times 10^{-3} H}{1\Omega}$$

$= 20$

Case II. $R = 40\Omega$

In this case

$$\frac{R^2}{4L^2} = \frac{40 \times 40\ \Omega^2}{4 \times (2 \times 10^{-3})^2\ H^2} = 10^8 \frac{\Omega^2}{H^2}$$

Hence, $\frac{1}{LC} = \frac{R^2}{4L^2}$ and this is the case of critical damping.

Case III. $R = 100\ \Omega$

Here

$$\frac{R^2}{4L^2} = \frac{100^2 \Omega^2}{4 \times (2 \times 10^{-3})^2\ H^2} = 6 \times 10^8 \frac{\Omega^2}{H^2}$$

That is, $\frac{R^2}{4L^2} > \frac{1}{LC}$ This corresponds to dead beat motion.

You will note that increasing resistance in the circuit increases damping.

6 $$T = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} = 4\ s$$

or

$$\omega_0^2 - b^2 = \frac{\pi^2}{4}$$

Also

$$\ln\left(\frac{a_0}{a_n}\right) = \ln 10 = bt$$

or

$$b = \frac{1}{t} \ln 10$$

$$= \frac{2.3}{46\ s} \log_{10} 10 = 0.05\ s^{-1}$$

Hence,

$$\omega_0^2 = (0.0025 + 2.4649)\ s^{-2}$$
$$= 2.467\ s^{-2}$$

$\Rightarrow \omega_0 = 1.57\ s^{-1}$

and

$$Q = \frac{\omega_0 T}{2} = \frac{\omega_0}{2b} = \frac{1.57}{0.1} = 15.7$$

Terminal Questions

1 Since $\theta = \theta_0 e^{-bt}$, we can write

$$b = \frac{1}{t} \ln \left(\frac{\theta_0}{\theta}\right)$$

substituting the given data, we get

$$b = \frac{1}{40\text{ s}} \ln \frac{5}{4}$$

$$= 5.57 \times 10^{-3}\text{s}^{-1}$$

and

$$\tau = \frac{1}{b} = 179.5\text{ s}$$

Since $Q = \frac{\omega_0 \tau}{2}$ we can write $\tau = \frac{2Q}{\omega_0} = \frac{2 \times 4000}{2\pi \times 300} = 4.24\text{ s}$

Now $a = a_0 e^{-bt} = a_0 e^{-t/\tau}$

$$\therefore t = \tau \ln \frac{a_0}{a} = 4.24\text{ s} \times \ln 2 = 2.94\text{ s}$$

3 (i) Here $\omega_0 = 2\pi\nu = 2 \times 3.14\text{ rad} \times 4\text{ s}^{-1} = 25\text{ rad s}^{-1}$
But $\omega_0 = \sqrt{k/m}$ or $k = m\omega_0^2 = 1\text{ kg} \times 25^2\text{ s}^{-2} = 625\text{ Nm}^{-1}$

(ii) $a = a_0 e^{-bt}$ or $0.01\text{ m} = 0.02\text{ m}\, e^{-30b}$,

$$\therefore b = \frac{\ln 2}{30} = 2.3 \times 10^{-2}\text{ s}^{-1}$$

Hence, relaxation time $\tau = \frac{1}{b} = \frac{1}{2.3 \times 10^{-2}\text{ s}^{-1}} = 43.5\text{ s}$

(iii) For a weakly damped system $Q = \frac{\omega_0 \tau}{2} = 25 \times 43.5 = 1088$

4 Here $\frac{1}{LC} = \frac{1}{5 \times 10^{-3}\text{H} \times 2 \times 10^{-6}\text{ F}} = 10^8\text{ s}^{-2}$

and $\frac{R^2}{4L^2} = \frac{(0.2)^2\,\Omega^2}{4 \times (5 \times 10^{-3})^2\text{ H}^2} = 400\ \frac{\Omega^2}{\text{H}^2}$

Since $\frac{1}{LC} > \frac{R^2}{4L^2}$, the discharge is oscillatory and has frequency

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = 1.59 \times 10^3\text{ Hz.}$$

The quality factor of the circuit is

$$Q = \frac{\omega_0 L}{R} = \frac{2\pi \times 1.59 \times 10^3\text{s}^{-1} \times 5 \times 10^{-3}\text{ H}}{0.2\,\Omega} = 250$$

Also

$$t = \frac{R}{2L} \ln\left(\frac{q_0}{q}\right) = \frac{0.2\,\Omega}{2 \times 5 \times 10^{-3}\text{ H}} \ln 2 = 14\text{ s.}$$

The discharge will be just non-oscillatory when

$$\frac{1}{LC} = \frac{R^2}{4L^2} \text{ or } R^2 = \frac{4L}{C} = \frac{4 \times 5 \times 10^{-3}\text{ H}}{2 \times 10^{-6}\text{ F}} = 10^4\text{ HF}^{-1} \text{ or}$$

$$R = 100\Omega.$$

5 The average energy of a damped harmonic oscillator at any time t is given by

$$\langle E \rangle = E_0 \exp(-2bt)$$
$$= E_0 \exp(-2t/\tau)$$

where $\tau = b^{-1}$ is the relaxation time.

When $t = \tau/2$, $\langle E \rangle = \frac{E_0}{e}$

Also, $Q = \frac{\omega_d \tau}{2}$

Hence,

$$\tau = \frac{2Q}{\omega_d} = \frac{2 \times 6 \times 10^4}{2\pi \times 512\ \mathrm{s}^{-1}} = \frac{3 \times 10^{4}}{256\pi\ \mathrm{s}^{-1}} = 37.3\ \mathrm{s}$$

Thus energy will reduce to $\frac{1}{e}$ of its initial value in 18.7 s.

The number of oscillations made by the tuning fork in this time is given by

$$\begin{aligned} n &= \nu_d \times t \\ &= 512 \times 18.7 \\ &= 95.7 \times 10^2 \end{aligned}$$

UNIT 4 FORGED OSCILLATIONS AND RESONANCE

Structure

4.1 INTRODUCTION

In the previous unit we studied how the presence of damping affects the amplitude and the frequency of oscillation of a system. However, in systems, such as a wall clock or an ideal LC circuit, oscillations do not seem to die out. To maintain oscillations we have to feed energy to the system from an external agent called a driver. In general, the frequencies of the'driver and the driven system may not match. But in steady-state, irrespective of its natural frequency, the system oscillates with the frequency of the applied periodic force. Such oscillations are called *forced* oscillations. However, when the frequency of the driving force exactly matches the natural frequency of the vibrating system a spectacular effect is observed; the amplitude of forced oscillations becomes very large and we say that *resonance* occurs. Do you know that Galileo was the first physicist who understood how and why resonance occurs?

Resonances are desirable in many mechanical and molecular phenomena. But resonance can be disastrous also; it can literally break an oscillating system apart. For instance, fast blowing wind may set a suspension bridge in oscillation. If the frequency of the fluctuating force produced by the wind matches the natural frequency of the bridge, it gains in amplitude and may ultimately collapse. In 1940, the Tacoma Narrows bridge in Washington State collapsed within 4 months of its being opened. Similarly, when the army marches on a suspension bridge, soldiers are instructed to break step to avoid resonant vibrations. In practice, isolated systems are rare. In solid state and molecular physics, two or more systems are coupled through interatomic forces. In an electric circuit we have inductive and capacitative couplings. The oscillations of such systems will be studied in the next unit.

In this unit we shall study, in detail, the response of a system when it is driven by an external harmonic force.

Objectives

After studying this unit you should be able to

- establish the differential equation of a system driven by a harmonic force and solve it
- analyse the response of the oscillator at different frequencies