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UNIT-3
PARTIAL DERIVATIVES

→ Function of two or more variables

The $f(x,y)$ is real valued function depending on two different independent variables x & y.

→ limit and continuity of function of several variables.

~~limit~~ $\lim_{(x,y)\to(a,b)} f(x,y) = l$

① $\lim_{x\to a} \{ \lim_{y\to b} f(x,y) \}$ 4

② $\lim_{y\to b} \{ \lim_{x\to a} f(x,y) \}$

If both the limits are equal we can say that $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

If $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ by taking different path then

in these case put $y=mx$ or $y=mx^n$ and then Calculate the limit.
If these these limit ~~as~~ also same as previous two then limit of function exists otherwise or not.

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→ Continuity of function of two variables

A $f(x,y)$ is said to be function if
① $f(a,b)$ exists
② $\lim_{(x,y)\to(a,b)} f(x,y)$ exists

③ $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

→ Find limit of the following function:

① $\lim_{(x,y)\to(1,2)} \frac{x^2+y}{3x+y^2}$

$\lim_{x\to1} \left\{ \lim_{y\to2} \frac{x^2+y}{3x+y^2} \right\}$

$\lim_{x\to1} \left\{ \frac{x^2+2}{3x+4} \right\}$

$= \frac{1+2}{3+4} = \frac{3}{7}$

$\lim_{(x,y)\to(1,2)} \frac{x^2+y}{3x+y^2} = \frac{3}{7}$

② $\lim\limits_{(x,y)\to(0,0)} \frac{x-y}{x+y}$

$\lim\limits_{x\to 0} \left\{ \lim\limits_{y\to 0} \frac{x-y}{x+y} \right\}$ $\qquad$ $\lim\limits_{x\to 0} \left\{ \lim\limits_{x\to 0} \frac{x-y}{x+y} \right\}$

$= -1$

$= 1$

$\lim\limits_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ does not exist

③ $\lim\limits_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^4+y^2}$

$\lim\limits_{x\to 0} \left\{ \lim\limits_{y\to 0} \frac{x^4-y^4}{x^4+y^2} \right\}$

$\lim\limits_{x\to 0} \left\{ \frac{x^4}{x^4} \right\}$

$= 1$

$\lim\limits_{y\to 0} \left\{ \lim\limits_{x\to 0} \frac{x^4-y^4}{x^4+y^2} \right\}$

$\lim\limits_{y\to 0} \left\{ \frac{-y^4}{y^2} \right\}$

$\lim\limits_{y\to 0} \left\{ -y^2 \right\}$

$= 0$

(4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{y-x^2}$

$\lim\limits_{x\to 0} \left\{ \lim\limits_{y\to 0} \dfrac{xy}{y-x^2} \right\} = 0$

$\lim\limits_{y\to 0} \left\{ \lim\limits_{x\to 0} \dfrac{xy}{y^2-x^2} \right\} = 0$

put $y = mx$

$\lim\limits_{x\to 0} \dfrac{x(mx)}{m^2x^2-x^2}$

$\lim\limits_{x\to 0} \dfrac{mx^2}{x^2(m^2-1)}$

$= \dfrac{m}{m^2-1} \neq 0$

$\lim \dfrac{xy}{y^2-x^2}$ does not exist.

Test the Continuity of

1. $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

$$= \lim_{x\to 0}\left\{\lim_{y\to 0} \frac{xy}{\sqrt{x^2+y^2}}\right\}$$

$$= 0 \quad \text{— (1)}$$

$$\lim_{y\to 0}\left\{\lim_{x\to 0} \frac{xy}{\sqrt{x^2+y^2}}\right\}$$

$$= 0 \quad \text{— (2)}$$

$f(0,0) = 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x(mx)}{\sqrt{x^2+m^2x^2}}$$

$$\lim_{x\to 0} \frac{mx^2}{x\sqrt{1+m^2}}$$

$$0 \quad \text{— (IV)}$$

eq (1), (II), (3) (IV)

$f(x,y)$ is continuous at $(0,0)$.

② $f(x,y) = \frac{2xy}{x^2+y^2}$

test $\lim_{(x,y)\to(0,0)} f(x,y)$ exist or not

$(x,y) \to (0,0)$

$$\lim_{x\to 0} \left\{ \lim_{y\to 0} f(x,y) \right\}$$

$$\lim_{x\to 0} \left\{ \lim_{y\to 0} \frac{2xy}{x^2+y^2} \right\}$$

0 — ①

$$\lim_{x\to 0} \left\{ \lim_{x\to 0} \frac{2xy}{x^2+y^2} \right.$$

$= 0$ — ②

$y = mx$

$$\lim_{x\to 0} f(x) = \frac{2m}{1+m^2} \quad \text{— ③}$$

③ Find the limit & test the continuity of

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x+y} & , \text{if } x+y \neq 0 \\ & , \text{if } x+y = 0 \end{cases}$$

$$\lim_{x\to 0} \left\{ \lim_{y\to 0} \frac{x^3-y^3}{x+y} \right\} = \lim_{x\to 0} \left\{ \frac{x^3}{x} \right\} = 0 \quad \text{— ①}$$

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$$\lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x^3y^3}{x^8+y^8}\right\}=\lim_{y\to 0}\left\{\frac{-y^3}{y}\right\}=0 \quad \text{--- (2)}$$

$f(0,0)=0$ --- (3)

$y=mx$

$$\lim_{x\to 0}\left\{\frac{x^3 m^3x^3}{x+mx}\right\}=\lim_{x\to 0}\frac{x^3(1-m^3)}{x(1+m)}=0 \quad \text{--- (4)}$$

→ Partial derivatives

Let $f(x,y)$ be the function of two independent variables x & y. We treat y as constant and find the derivative of $f(x,y)$ with respect to x and vice versa then the derivative is called partial derivative.

Partial derivative is used in fluid dynamics, electricity, physical sciences, propability etc.

Homogenous function:

A $f(x,y)$ is said to be homogenous of degree n if $f(Kx,Ky)=K^n f(x,y)$

~~for~~ e.g

(1) $f(x,y)=\dfrac{x^6+y^6}{x^4-y^4}$ degree 2

$$=\frac{K^6x^6+K^6y^6}{K^4x^4-K^4y^4} = \frac{K^6(x^6+y^6)}{K^4(x^4-y^4)}$$

$$=K^2 f(x,y)$$

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Euler's theorem

If $f(x,y)$ is a homongenous function n and x, y are having continuous partial derivatives then:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y)$$

① If $u = \log(x^3+y^3+z^3-3xyz)$ then P.T

① $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

② $\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]^2 u = \frac{-9}{(x+y+z)^2}$

① $u = \log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz)$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy)$$

Now, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

$$\frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz+3y^2-3xz+3z^2-3xy)$$

$$\frac{3}{x^3+y^3+z^3-3xyz}\,(x^2+y^2+z^2-xy-yz-zx)$$

Now, $(x^2+y^2+z^2-xy-yz-zx)(x+y+z)$
$=x^3+y^3+z^3-3xyz$

$$=\frac{3}{(x^2+y^2+z^2-xy-yz-zx)(x+y+z)}\,(x^2+y^2+z^2-xy-yz-zx)$$

$$=\frac{3}{x+y+z}=R.H.S$$

(2) $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$

$$\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$3\left(\frac{-1}{(x+y+z)^2}\right)+z\left(\frac{-1}{(x+y+z)^2}\right)+z\left(\frac{-1}{(x+y+z}\right)$$

$$\frac{9}{(x+y+z)^3}=R.H.S$$

Ex If $u=(x-y)^4+(y-z)^4+(z-x)^4$, then find the value of $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$

$u=(x-y)^4+(y-z)^4+(z-x)^4$

$\frac{\partial u}{\partial x}=4(x-y)^3-4(z-x)^3$

$\frac{\partial u}{\partial y}=-4(x-y)^3+4(y-z)^3$

$\frac{\partial u}{\partial z}=-4(y-z)^3+4(z-x)^3$

$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$

$=4(x-y)^3-4(\cancel{z}-x)^3-4(x-y)^3+4(y-z)^3-4(y-z)^3+4(z-x)^3$

$=0$

Ex If $u=(x-y)(y-z)(z-x)$ then prove that

(1) $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

(2) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=3u$

→ $u=(x-y)(y-z)(z-x)$

$\frac{\partial u}{\partial x}=(y-z)\left[(x-y)(-1)+(z-x)(1)\right]$

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$= (y-z)(-x+y+z-x)$

$= (y-z)(y+z-2x)$

$= y^2+yz-2xy-3y-z^2+2xz$ — ①

$$\frac{\partial u}{\partial y} = z-x\left[(x-y)\overset{(+)}{+}(y-z)(-1)\right]$$

$$= \cancel{z-x\left[x-y+y+z\right]}$$

$$= z-x\left[x-y \div y+z\right]$$

$$= z-x\left[x-2y+z\right]$$

$= zx-2yz+z^2-x^2+2yx \div zx$ ÷ ②

$$\frac{\partial u}{\partial z} = (x-y)\left[(y-z)+(z-x)\right]$$

(2) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$

$u(x,y,z) = (x-y)(y-z)(z-x)$

$u(Kx,Ky,Kz) = (Kx-Ky)(Ky-Kz)(Kz-Kx)$

$= K^3 (x-y)(y-z)(z-x)$

$= K^3 u(x,y,z)$

Hence u is homogeneous function of degree 3

$\therefore$ by euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$$

Ex If $\sin u = \frac{x^2y^2}{x+y}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$

Proof: Given $\sin u = \frac{x^2y^2}{x+y}$

let $f(x,y) = \frac{x^2y^2}{x+y}$

$f(Kx,Ky) = \frac{K^2x^2\,K^2y^2}{Kx+Ky}$

$= \frac{K^4(x^2y^2)}{K(x+y)} = K^3 f(x,y)$

f is homogeneous function by degree 3

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial x} = 3f(x,y)$

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$\Rightarrow x \frac{\partial}{\partial x} \sin u + y \frac{\partial}{\partial y} \sin u = 3 \sin u$

$\Rightarrow x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = 3 \sin u$

$\Rightarrow \cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 3 \sin u$

$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3 \tan u$

Ex If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

$x \frac{\partial u}{\partial y} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$

$f(x) = \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right)$

$f(kx, ky, kz) = k\left(\frac{x}{y}\right) + k\left(\frac{y}{z}\right) + k\left(\frac{z}{x}\right)$

$= k\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right)$

$= , (f(x, y, z)$

$f(x)$ is homogenous

Ex If $u = f(x-y, y-z, z-x)$ then find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

Let $x - y = x_1$

Ex $u = \sin^{-1}\left[\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ then P.T $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$

$\sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

Let $f(x,y) = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

f is homogenous of degree 1/2

$x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = \dfrac{1}{2} f(x,y)$

$x \cdot \dfrac{\partial \sin u}{\partial x} + y\dfrac{\partial f}{\partial y} = \dfrac{1}{2} f(x,y)$

$x \cdot \dfrac{\partial \sin u}{\partial x} + y \cdot \dfrac{\partial \sin u}{\partial y} = \dfrac{1}{2}\sin u$

$x \cdot \cos u \cdot \dfrac{\partial u}{\partial x} + y \cdot \cos u \dfrac{\partial u}{\partial y} = \dfrac{1}{2}\sin u$

$$\boxed{x \cdot \dfrac{\partial u}{\partial x} + y \cdot \dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u}$$

Ex.

Ex- If $u = \tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$ then P.T

$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \sin 2u$

$u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right) \Rightarrow \tan u - \frac{x^3+y^3}{x-y}$

$f(x,y) = \frac{x^3+y^3}{x-y}$

$f(kx,ky) = \frac{k^3(x^3+y^3)}{k(x-y)}$

$= k^2(f(x,y))$

f is homogenous function of degree 2

$\therefore x \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = 2f(x,y)$

$\Rightarrow x \frac{\partial \tan x}{\partial x} + y \cdot \frac{\partial \tan u}{\partial y} = 2\tan u$

$x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y \cdot \sec^2 u \frac{\partial u}{\partial y} = 2\tan u$

$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{2\cos y \cdot \sin u}{\cos u} = 2\cos u \sin u$

$= \sin 2u$

Note:-

if $u = f(x,y,z)$

$f(x,y,z)$ is homogenous function of degree 'n'

then

$x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z} =$

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Result: If $u(x,y)$ is homogeneous ~~functi~~ function of degree n, in x & y with all first & second derivatives continuous, then $x^2 \frac{\partial^2 x}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$

(1) If $f(x,y) = x^2y + xy^2$ then find $f_x(1,2)$ & $f_y(1,2)$

$\frac{\partial f}{\partial x} = 2xy + (1)y^2$

$f_x = 2xy + y^2$

$f_x(1,2) = 8$

(2) $\frac{\partial f}{\partial y} - x^2 + 2xy = f_y$

$f_y(1,2) = 5$

(2) If $u = (1-2xy+y^2)^{-1/2}$ then show that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u^3y^2$

$u = (1-2xy+y^2)^{-1/2}$

$\frac{\partial u}{\partial x} = \frac{-1}{2}(1-2xy+y^2)^{-1/2-1}(0-2y)$

$= y(1-2xy+y^2)^{-3/2}$ — (1)

$\frac{\partial u}{\partial y} = \frac{-1}{2}(1-2xy+y^2)^{-1/2-1}(0-2x+2y)$

$= (x-y)(1-2xy+y^2)^{-3/2}$ — (2)

$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}$

$$= xy(1-2xy+y^2)^{-3/2} - y(x-y)(1-2xy+y)^{-3/2}$$

$$= \left[(1-2xy+y^2)^{-1/2}\right]^3 \{\cancel{xy} - \cancel{xy} + y^2\}$$

$$= y^2 u^3$$

$$= R.H.S$$

(5) If $u = \log(\tan x + \tan y + \tan z)$ then show that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$

Proof $\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \sec^2 x$

$\frac{\partial u}{\partial y} =$ " $\sec^2 y$

$\frac{\partial u}{\partial z} =$ " $\sec^2 z$

$$= \frac{1}{\tan x + \tan y + \tan z}\left[\sin 2x \sec^2 x + \sin 2y \sec^2 y + \sin 2z \sec^2 z\right]$$

$$= \frac{1}{\tan x + \tan y + \tan z}\left[\frac{2\sin x \cos x}{\cos^2 x} + \frac{2\sin y \cos y}{\cos^2 y} + \frac{2\sin z \cos z}{\cos^2 z}\right]$$

$$= 2$$

$f\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

(4) If $u = \frac{e^{x+y+z}}{e^x + e^y + e^z}$ then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 2u$

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} =$

$$\frac{\partial u}{\partial x} = \frac{(e^x + e^y + e^z)(e^{x+y+z}) - (e^{x+y+z})(e^x)}{(e^x + e^y + e^z)^2}$$

$$= \frac{\cancel{e^x + e^y + e^z}}{\cancel{(e^x +}} \frac{e^{x+y+z}}{(e^x + e^y + e^z)} \left[\cancel{e^x} + e^y + e^z - \cancel{e^x}\right]$$

$$= \frac{e^{x+y+z} (e^y + e^z)}{(e^x + e^y + e^z)^2} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{e^{x+y+z}}{(e^x + e^y + e^z)} (e^x + e^z) \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{e^{x+y+z}}{(e^x + e^y + e^z)} (e^x + e^y) \quad \text{--- (3)}$$

(1) + (2) + (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{e^x + e^y + e^z}{(e^x + e^y + e^z)^{\cancel{2}}} \left[2(e^x + e^y + e^z)\right]$$

$$= 2u$$

(6) Find the Value of n for which $u = kt^{-1/2} e^{-\left(\frac{x^2}{na^2 t}\right)}$ statisfies eq^n $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$

$$u = kt^{-1/2} e^{-\left(\frac{x^2}{na^2 t}\right)}$$

$$\frac{\partial u}{\partial t} = k\left[t^{-1/2} e^{-\left(\frac{x^2}{na^2 t}\right)} \left(\frac{-1}{t^2}\left(\frac{-x^2}{na^2}\right)\right) + e^{\frac{-x^2}{na^2 t}} \left(\frac{-1}{2} t^{-3/2}\right)\right]$$

$$= kt^{-1/2} e^{-\left(\frac{x^2}{na^2 t}\right)} \left(\frac{x^2}{na^2 t^2}\right) - \frac{1}{2} k e^{\frac{-x^2}{na^2 t}} t^{-3/2}$$

$$= u\left(\frac{x^2}{na^2 t^2}\right) - \frac{1}{2t} u$$

$$= u\left[\frac{x^2}{na^2 t^2} - \frac{1}{2t}\right] \quad \text{---} (1)$$

$$u = kt^{-1/2} e^{-\left(\frac{x^2}{na^2 t}\right)}$$

$$\frac{du}{dx} = kt^{-1/2} e^{-\left(\frac{x^2}{na^2 t}\right)} \left(\frac{-2x}{na^2 t}\right)$$

$$= \frac{-2kt^{-1/2}}{na^2 t} e^{-\left(\frac{x^2}{na^2 t}\right)}$$

$$\frac{d^2 u^2}{dx^2} = \frac{-2kt^{-1/2}}{na^2 t}\left[x \cdot e^{\frac{-x^2}{na^2 t}} \left(\frac{-2x}{na^2 t}\right) + e^{\frac{-x^2}{na^2 t}}\right]$$

$$= \frac{-2kt^{-1/2}}{na^2 t} e^{\frac{-x^2}{na^2 t}} \left[\frac{-2x^2}{na^2 t} + 1\right]$$

$$= \frac{-2u}{na^2 t}\left[\frac{-2x^2}{na^2 t} + 1\right]$$

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$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{-2ua^2}{na^2t}\left[\frac{-2x^2}{na^2t}+1\right]$$

$$= \frac{-2u}{nt}\left[\frac{-2x^2}{na^2t}+1\right] \quad \text{— (2)}$$

we have $\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$

$$\Rightarrow u\left[\frac{x^2}{na^2t^2}-\frac{1}{2t}\right] = \frac{-2u}{nt}\left[\frac{-2x^2}{na^2t}+1\right]$$

$$\Rightarrow \frac{x^2}{na^2t^2}-\frac{1}{2t} = \frac{4x^2}{n^2a^2t}-\frac{2}{nt}$$

$$\Rightarrow \frac{4x^2}{n^2a^2t} - \frac{2}{nt}$$

$$\Rightarrow \frac{2tx^2-na^2t^2}{2na^2t^3} = \frac{4ntx^2-2n^2a^2t}{n^3a^2t^3}$$

$$\Rightarrow \frac{t(2x^2-na^2t)}{2n} = \frac{tn(4x^2-2nta^2)}{n^3}$$

$$\Rightarrow \frac{2x^2-na^2t}{2} = \frac{4x^2-2nta^2}{n}$$

$$\Rightarrow \frac{2x^2-na^2t}{2} = \frac{2(x^2-2na^2t)}{n}$$

$$\boxed{n=4}$$

~~If $f(x,y) = x^2 y + x y^2$ then find $f_x(1,2)$ & $f_y(1,2)$~~

(1) If $u = e^{x^y}$, find $\dfrac{\partial^2 u}{\partial y \cdot \partial x}$

$$u = e^{x^y}$$

$$\frac{\partial^2 u}{\partial y \cdot \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(e^{x^y}\right)\right]$$

$$= \frac{\partial}{\partial y}\left[e^{x^y}\, y x^{y-1}\right]$$

$$= \frac{\partial}{\partial y}\left[y x^{y-1} e^{x^y}\right]$$

$$= \frac{\partial}{\partial y}\left[y x^{y-1} u\right]$$

$$= y\left[\frac{\partial}{\partial y}\left(x^{y-1} u\right)\right] + u x^{y-1} (1)$$

$$= y\left[x^{y-1} \log x \, u + x^{y-1} \frac{\partial u}{\partial x}\right] + u x^{y-1}$$

$$= y x^{y-1} \log x \, e^{x^y} + x^{y-1} \frac{\partial}{\partial y}\left(e^{x^y}\right) + u x^{y-1}$$

$$= y x^{y-1} \log x \, e^{x^y} + y x^{y-1} e^{x^y} x^y \log x + e^{x^y} x^{y-1}$$

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$= e^{x^y} x^{y-1} [y\log x + y x^y \log x + 1]$

② If $z^3 - zx - y = 0$ then find $\frac{\partial^2 z}{\partial x \partial y}$

Sol^n $z^3 - 2x - y = 0$

Differentiate ~~posit~~ partially w.r.t y

$$3z^2 \frac{\partial z}{\partial y} - \left(x \frac{\partial z}{\partial y} + z(0)\right) - 1 = 0$$

$$3z^2 \frac{\partial z}{\partial y} - x \frac{\partial z}{\partial y} - 1 = 0$$

$$\frac{\partial z}{\partial y} = \frac{1}{3z^2 - x}$$

Differentiate with respect to x

$$\frac{\partial^2 z}{\partial x y} = \frac{\partial}{\partial x}\left[\frac{1}{3z^2 - x}\right]$$

$$\frac{(3z^2 - x)(0) - 1 \frac{\partial}{\partial x}(3z^2 - x)}{(3z^2 - x)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-\left(6z \frac{\partial z}{\partial x} - 1\right)}{(3z^2 - x)^2}$$

$$\left[3z^2 \frac{\partial z}{\partial x} - \left(2(1) + x \frac{\partial z}{\partial x}\right) \frac{\partial z}{\partial x}\right.$$

$$\left.\frac{\partial z}{\partial x} = \frac{2}{3z^2 - x} = 0\right]$$

$$= \frac{-6z^2 + 3z^2 u}{(3z^2 - x)^3}$$

$$= \frac{-3z^2 - x}{(3z^2 - x)^3}$$

3) If $u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$ then Show that $\frac{\partial^2 u}{\partial x \partial y} =$

$$\frac{1}{(1+x^2+y^2)^{3/2}}$$

$\Rightarrow$ Diff partially w.r to y

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)\right]$$

$$= \frac{1}{1 + \frac{x^2y^2}{1+x^2+y^2}}\left[\frac{\partial}{\partial y}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)\right]$$

$$= \frac{1+x^2+y^2}{1+x^2+y^2+x^2y^2}\left[\frac{\sqrt{1+x^2+y^2}(x) - xy\left(\frac{-1\cdot 2y}{2(1+x^2+y^2)^{3/2}}\right)}{(1+x^2+y^2)}\right]$$

$$= \frac{1}{1+x^2+y^2+x^2y^2}\left[\frac{x(1+x^3+y^2)^2 + xy^2}{(1+x^2+y^2)^{3/2}}\right]$$

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→ Total derivatives

Chain rule

If $z = f(u)$

$x \& y$

i.e $u = \phi(x, y)$ then

$$\frac{dz}{dx} = \frac{dz}{\partial u} = \frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{\partial u} = \frac{\partial u}{\partial y}$$

1) If $u = y^2 - 4ua$, $x = at^2$, $y = 2at$ find $\frac{du}{dt}$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= (-4a)(2at) + 2y(2a)$$

$$= -8a^2t + 4ay$$

Subsitute $y = 2at$

$$\frac{du}{dt} = -8a^2t + 4(2at)(a)$$

$$= -8a^2t + 8a^2t$$

$$= 0$$

2) If $u = \sin\left(\frac{x}{y}\right)$ where $x = e^t$, $y = t^2$, find $\frac{du}{dt}$

$$\Rightarrow \frac{du}{dt} = \frac{du}{\partial x} \cdot \frac{dx}{dt} + \frac{du}{\partial y} \cdot \frac{dy}{dt}$$

$$= \cos\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) \cdot (-e^t) + \cos\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)(2t)$$

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$$\frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right)\left(\frac{1}{t^2}\right)(e^t) + \cos\left(\frac{e^t}{t^2}\right)\left(\frac{-e^t}{t^4}\right)(2t)$$

$$\frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right)\left(\frac{1}{t^2}\right)(e^t) - \cos\left(\frac{e^t}{t^2}\right)\left(\frac{e^t}{t^4}\right)(2t)$$

3) If $u = x^2y^3$, $x = \log t$, $y = e^t$, find $\frac{du}{dt}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{dt}$$

$$= 2x\cdot 3y^2\cdot\frac{1}{t} + e^t\,3y^2x^2$$

$$= 2(\log t)\,3(e^t)^3\cdot\frac{1}{t} + e^t\cdot 3(e^t)^2(\cancel{\log} 1/t)^2$$

$$= e^{3t}\log t\left[\frac{2}{t} + 3\log t\right]$$

4) If $u = xy + yz + zx$ where $x = \frac{1}{t}$, $y = e^t$, $z = e^{-t}$ then find $\frac{du}{dt}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{\partial z}{dt}$$

$$= y + yz \times \frac{1}{t^2} + (x+z)e^t + (-xy - y)e^t)$$

$$= e^t + e^t(e^{-t})x\frac{1}{t^2} + \left(\frac{1}{t} + e^{-t}\right)e^t + \left(-\frac{1}{t}e^t - e^t\right)e^t$$

$$= \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{-t}}{t} + 1 + \frac{e^{-2t}}{t} - e^{2t}$$

$$= \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{-t}}{t} + 1 - \frac{e^{2t}}{t} - e^{2t}$$

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$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}$$

5. If $z = xy^2 + x^2y$, $x = at^2$, $y = 2at$, find $\frac{dz}{dt}$

Ans $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} = 2a^3t^3(8+5t)$

6. If $z=\sin^{-1}(x-y)$, $(x-y)$, $x=3t$, $y=4t^3$ prove that

$$\frac{\partial z}{\partial t}=\frac{3}{\sqrt{1-t^2}}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$=\frac{1}{\sqrt{1-(x-y)^2}}(3)+\frac{-1}{\sqrt{1-(x-y)^2}}(12t^2)$$

$$=\frac{3}{\sqrt{1-(3t-4t^3)^2}}\frac{-12t^2}{\sqrt{1-(3t-4t^3)^2}}$$

$$=\frac{1}{\sqrt{1-(9t^2-24t^4+16t^6)}}\left[3-12t^2\right]$$

$$=\frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-10t^6}}$$

$p(t)=1-9t^2+24t^4-16t^6$

$t=1 \Rightarrow p(1)=0$

$t-1$ is one of factor

$$\begin{array}{c|ccccccc} & -16 & 0 & 24 & 0 & -9 & 0 & 1 \\ 1 & 0 & -16 & -16 & 8 & 8 & -1 & -1 \\ \hline & -16 & -16 & 8 & 8 & -1 & -1 & 0 \\ & t^5 & t^4 & t^3 & t^2 & t & & \end{array}$$

$(t-1)(-16t^5-16^4+8t^3+8t^2-t-1)$

$=(t-1)(t+1)(-16t^4+8t^2-1)$

$=(t-1)(t+1)\{(-16t^4-8t^2+1)\}$

$=(t-1)(t+1)\{(4t^2-1)^2\}$

$$= \frac{3(1-4t^2)}{\sqrt{(t-1)(t+1)\{-(4t^2-1)^2\}}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(t-1)(t+1)(1-4t^2)^2}}$$

$$= \frac{3}{\sqrt{1-t^2}} \quad = R.H.S$$

Composite function of two variables

If $z = f(x, y)$, where $x = \phi(u, v)$ & $y = \psi(u, v)$ then z is called a fun of u, v & is called composite function of two variables u & v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \&$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

① If $z = f(u, v)$, $u = \log(x^2+y^2)$, $v = y/x$ then show that $x\frac{\partial z}{\partial y} = y\frac{\partial z}{\partial x} = (1+v^2)\frac{\partial z}{\partial v}$ ②

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\frac{1}{x^2+y^2}2x + \frac{\partial z}{\partial v}\left(\frac{-y}{x^2}\right)$$

$$y\frac{\partial z}{\partial x} = \frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} \frac{-y^2}{x^2} = \frac{\partial z}{\partial v} \quad \text{— ①}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial v}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial y}\frac{2y}{x^2+y^2} + \frac{\partial z}{\partial v}\frac{1}{x}$$

$$x\frac{\partial z}{\partial y} = \frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{— ②}$$

(2) - (1)

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \cancel{\frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u}} + \frac{\partial z}{\partial v} \cancel{- \frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u}} + \frac{y^2}{x^2}\frac{\partial z}{\partial v}$$

$$= \left(1+\frac{y^2}{x^2}\right)\frac{\partial z}{\partial x}$$

$$= \left(1+v^2\right)\frac{\partial z}{\partial x}$$

$$= R.H.S$$

(2) If $w=\phi(u,v)$, $u=x^2y^2-2xy$, $v=y$ then P.T $\frac{\partial w}{\partial v}=0$ is equivalent to $(x+y)\frac{\partial w}{\partial x}+(x-y)\frac{\partial w}{\partial y}=0$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial v}{\partial u}$$

$$= \frac{\partial w}{\partial u}(2x-2y) + \frac{\partial w}{\partial u}(0)$$

$$= 2(x-y)\frac{\partial w}{\partial u}$$

$$(x+y)\frac{\partial w}{\partial x} = 2(x^2-y^2)\frac{\partial w}{\partial y} \quad \text{--- (1)}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial u}\frac{\partial w}{\partial y}$$

$$= \frac{\partial w}{\partial u}(-2y-2x) + \frac{\partial w}{\partial u}(1)$$

$$= -2(x+y)\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$

$$(x-y)\frac{\partial w}{\partial y} = -2(x^2-y^2)\frac{\partial w}{\partial u} + \overset{(x-y)}{\frac{\partial w}{\partial v}} \quad \text{--- (2)}$$

(1) + (2)

$$(x+y)\frac{\partial w}{\partial x} + (x-y)\frac{\partial w}{\partial y}$$

$$= 2(x^2-y^2)\frac{\partial w}{\partial u} - 2(x^2-y^2)\frac{\partial w}{\partial u} + (x-y)\frac{\partial w}{\partial v} = 0$$

$$= (x-y)\frac{\partial w}{\partial v} = 0$$

$$= \frac{\partial w}{\partial v} = 0$$

(3) If $z = f(x,y)$ & $x = e^u + e^{-v}$ and $y = e^{-u} + e^v$, show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial u} = x\frac{\partial z}{\partial u} - y\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}(-e^{-u})$$

$$= e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial v}{\partial v} \quad \text{--- (2)}$$

(1) − (2)

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$e^{u} - (e - v)\frac{\partial z}{\partial x} + (-e^{-u} - e^{v})\frac{\partial z}{\partial y}$

$(e^{u} + e^{-v})\frac{\partial z}{\partial u} - (e^{-u} + e^{v})\frac{\partial z}{\partial y}$

$= (x)\frac{\partial z}{\partial x} - (y)\frac{\partial z}{\partial y}$

4) If $z = f(u,v)$ & $u = x\cos\theta - y\sin\theta$, $v = x\sin\theta + y\cos\theta$ then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial y}$

$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial u}{\partial x}$

$= x\frac{\partial z}{\partial u}$ — (1)

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$

$y\frac{\partial z}{\partial y}$ — (2)

(1) + (2)

$x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial y}$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(-\sin\theta) + \frac{\partial z}{\partial v}(\cos\theta)$

$= -\sin\theta\frac{\partial z}{\partial u} + \cos\theta\frac{\partial z}{\partial v}$ — (1)

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(-\cos\theta) + \frac{\partial z}{\partial v}(-\sin\theta)$

$= -\cos\theta\frac{\partial z}{\partial u} - \sin\theta\frac{\partial z}{\partial v}$ — (2)

(1) + (2)

$$x\frac{\partial z}{\partial x} = x(-\sin\theta)\frac{\partial z}{\partial u} + x\cos\theta\frac{\partial z}{\partial v}$$

$$y\frac{\partial z}{\partial y} = y(-\cos\theta)\frac{\partial z}{\partial u} + y(-\sin\theta)\frac{\partial z}{\partial v}$$

$$= (-x\sin\theta - y\cos\theta)\frac{\partial z}{\partial u} + (x\cos\theta - y\sin\theta)\frac{\partial z}{\partial v}$$

$$= -(x\sin\theta + y\cos\theta)\frac{\partial z}{\partial u} + (x\cos\theta - y\sin\theta)\frac{\partial z}{\partial v}$$

5) If $u = f(x-y, y-z, z-x)$ $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

$x-y = p$, $y-z = q$, $z-x = r$.

$u = f(p, q, r)$

$$\frac{\partial u}{\partial x} = f\left(\frac{\partial p}{\partial r}, \frac{\partial q}{\partial x}, \frac{\partial r}{\partial r}\right) = f(1, 0, -1)$$

$$\frac{\partial u}{\partial y} = f(-1, 1, 0) \quad \& \quad \frac{\partial u}{\partial z} = f(0, 1, 1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = f(0, 0, 0)$$

① If $z=f(x,y)$ where $x^2=au+bv$, $y^2=au-bv$ then P.T

$$u\frac{\partial z}{\partial x}+v\frac{\partial z}{\partial x}=\frac{1}{2}\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)$$

proof $$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$x^2=au+bv$ $\qquad$ $y^2=au-bv$

$2x\frac{\partial x}{\partial u}=a$ $\qquad$ $\partial y\frac{\partial y}{\partial u}=a$

$\frac{\partial x}{\partial u}=\frac{a}{2x}$ $\qquad$ $\frac{\partial y}{\partial u}=\frac{a}{2y}$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\left(\frac{a}{2x}\right)+\frac{\partial z}{\partial y}\cdot\left(\frac{a}{2y}\right)$$

$$=\frac{a}{2}\left[\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}\right] \quad \text{—— (1)}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$x^2=au+bv$ $\qquad$ $y^2=au-bv$

$2x\frac{\partial x}{\partial v}=b$ $\qquad$ $2y\frac{\partial y}{\partial v}=-b$

$\frac{\partial x}{\partial v}=\frac{b}{2x}$ $\qquad$ $\frac{\partial u}{\partial v}=\frac{-b}{2y}$

$$\Rightarrow \frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\left(\frac{b}{2x}\right)+\frac{\partial z}{\partial y}\left(\frac{-b}{2y}\right)$$

$$=\frac{b}{2}\left[\frac{1}{x}\frac{\partial z}{\partial x}-\frac{1}{y}\frac{\partial z}{\partial y}\right] \quad \text{—— (2)}$$

$$u\left(\frac{a}{2}\left[\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}\right]+v\left[\frac{b}{2}\left[\frac{1}{x}\frac{\partial z}{\partial x}-\frac{1}{y}\frac{\partial z}{\partial y}\right]\right.\right.$$

$$=\left(\frac{ua}{2}+\frac{vb}{2}\right)\frac{1}{x}\frac{\partial z}{\partial x}+\left(\frac{ua}{2}-\frac{vb}{2}\right)\frac{1}{y}\frac{\partial z}{\partial y}$$

$$=\left(\frac{ua+bv}{2}\right)\frac{1}{x}\frac{\partial z}{\partial x}+\left(\frac{au-bv}{2}\right)\frac{1}{y}\frac{\partial z}{\partial y}$$

$$=\frac{x^2}{2}\times\frac{1}{x}\frac{\partial z}{\partial x}+\left(\frac{y^2}{2}\right)\frac{1}{y}\frac{\partial z}{\partial y}$$

$$=\frac{x}{2}\frac{\partial z}{\partial x}+\frac{y}{2}\frac{\partial z}{\partial y}$$

$$=\frac{1}{2}\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=R.H.S$$

2) If $u=f(x,y)$ where $x=r\cos\theta$ & $y=r\sin\theta$

P.T $\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta \quad -①$$

$$\frac{\partial u}{\partial\theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

$$=\frac{\partial u}{\partial x}(-r\sin\theta)+\frac{\partial u}{\partial y}(r\cos\theta) \quad -②$$

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$$L.H.S = \left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right)^2$$

$$\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\left(-r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}\right)^2$$

$$= (a+b)^2 = a^2 + 2ab + b^2$$

$$\frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\left(\frac{\partial u}{\partial x}\cos\theta\right)\left(\frac{\partial u}{\partial y}\sin\theta\right) +$$

$$\frac{1}{r^2}\left(r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2} + r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2} + 2\,(r\cos^2\theta\frac{\partial^2 u}{\partial y^2}\right.$$

$$\left. - \left(r\sin\theta\frac{\partial u}{\partial x}\right)\right)$$

$$= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial u}{\partial x}\cos\theta\frac{\partial u}{\partial y}\sin\theta + \frac{\partial^2 u}{\partial x^2}\sin^2\theta +$$

$$\frac{\partial^2 u}{\partial y^2}\cos^2\theta - 2\frac{\partial u}{\partial y}\cos\theta$$

$$\frac{\partial^2 u}{\partial x^2}(\sin^2\theta + \cos^2\theta) + \frac{\partial^2 u}{\partial y^2}(\sin^2\theta + \cos^2\theta)\; 2\left(\frac{\partial u}{\partial x}\cos\theta\cdot\frac{\partial u}{\partial y}\sin\theta\right.$$

$$\left. - \frac{\partial u}{\partial y}\cos\theta\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

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8) If $z = f(u,v)$ and $u = x^2 - y^2$, $v = 2xy$ then show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(u^2+v^2)^{1/2}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\cdot 2x + \frac{\partial z}{\partial v}\cdot 2y$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial u}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-2y) + \frac{\partial z}{\partial v}\cdot(2x)$$

$$\left(\frac{\partial z}{\partial x}\right)^2 = \left(2x\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}\right)^2$$

$$\left(\frac{\partial z}{\partial y}\right)^2 = \left(-2y\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}\right)^2$$

$$\left(\frac{\partial z}{\partial x}\right)^2 = 4x^2\frac{\partial^2 z}{\partial u^2} + 4y^2\frac{\partial^2 z}{\partial v^2} + 8x\frac{\partial z}{\partial u}\cdot y\frac{\partial z}{\partial v}$$

$$\left(\frac{\partial z}{\partial y}\right)^2 = \left(4x^2\frac{\partial^2 z}{\partial u^2} + 4y^2\frac{\partial^2 z}{\partial u^2} - 8x\frac{\partial z}{\partial v}\frac{\partial z}{\partial u}\right)$$

$$4\left[x^2\frac{\partial^2 z}{\partial u^2} + y^2\frac{\partial^2 z}{\partial u^2}\right] + 4\left[x^2\frac{\partial^2 z}{\partial u^2} + y^2\frac{\partial^2 z}{\partial v^2}\right]$$

$$4\left\{4\frac{\partial^2 z}{\partial u^2}\left[x^2+y^2\right] + 4\frac{\partial^2 z}{\partial v^2}\left[x^2+y^2\right]\right.$$

$$= 4(x^2+y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$u^2+v^2 \Rightarrow u = x^2-y^2$

$u^2 = (x^2-y^2)^2 = x^4-2x^2y^2+y^4$

$y = 2xy \Rightarrow v^2 = 4x^2y^2$

$u^2+v^2 = x^4-2x^2y^2+y^4+4x^2y^2$

$= x^4+y^4+2x^2y^2$

$= (x^2+y^2)^2$

$(x^2+y^2) = (u^2+v^2)^{1/2}$

$$4(u^2+v^2)^{1/2}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

Impilict Differentation

Any func of the type $f(x,y)=c$ is called an impilict func where x, y is a func of x & c is any constant. if $f(x,y)-c$ then $\frac{dy}{dx} = \frac{-\frac{df}{dx}}{\frac{df}{dy}}$

① If $y\log(\cos x) = x\log(\sin y)$, find $\frac{dy}{dx}$

$f(x,y) = y\log(\cos x) - x\log(\sin y)$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{dx}}{\frac{\partial f}{dy}} = \frac{\frac{y}{\cos x}(-\sin x) - \log(\sin y)}{\log(\cos x) - \frac{x}{\sin y}\cos y}$$

$$= \frac{-y\tan x - \log(\sin y)}{\log(\cos x) - x\cot y}$$

② if $x^3+y^3 = 3axy$, find $\frac{dy}{dx}$

$$f(x,y) = \frac{-\frac{\partial f}{dx}}{\frac{df}{dy}} = \left(\frac{3x^2-3ay}{3y^2-3ax}\right)$$

③ if $y\sin x = x\cos y$, find $\frac{dy}{dx}$

$f(x,y) = y\sin x - x\cos y$

$$\frac{dy}{dx} = -\frac{\frac{df}{dx}}{\frac{df}{dy}} = -\left(\frac{y\cos x - \cos y}{\sin x + x\sin y}\right)$$

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(4) If $x^y + y^x = c$, find $\frac{dy}{dx}$

$f(x,y) = x^y + y^x - c$

$$\frac{dy}{dx} = \frac{-\frac{df}{dx}}{\frac{df}{dy}} = -\left[\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right]$$

(5) If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$

$f(x,y) = (\cos x)^y - (\sin y)^x$

$$\frac{dy}{dx} = \frac{-\frac{df}{dx}}{\frac{df}{dy}} = \left[\frac{y(\cos)^{y-1}(-\sin x) - (\sin y)^x \log(\sin y)}{(\cos x)^y \log(\cos x) - x(\sin y)\cos y}\right]$$

$\therefore \frac{y\tan x + \log \sin y}{\log \cos x - x\cos y}$

(6) If $u = \sin(x^2+y^2)$ & $a^2x^2 + b^2y^2 = c^2$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{du}{dx}\frac{dx}{dx} + \frac{du}{dy}\cdot\frac{dy}{dx} \quad \text{—— (A)}$$

$u = \sin(x^2+y^2)$

$$\frac{du}{dx} = \cos(x^2+y^2)(2x) \quad \text{—— (1)}$$

$$\frac{du}{dy} = \cos(x^2+y^2)(2y) \quad \text{—— (2)}$$

$$\frac{dx}{dx} = 1 \quad \text{—— (3)}$$

for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \text{where } f(x,y) = a^2x^2 + b^2y^2 - c^2$$

$$= -\left[\frac{2a^2x}{2b^2y}\right] \quad \text{—— (4)}$$

sub (1), (2), (3) & (4) — in (A)

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Taylor Series

- If $f(x+h)$ is even function of h which can be expanded into a convergent series of positive ascending integers power of h, then $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots + \frac{h^n}{n!}f^n(x) + \dots$

Sub $x=a$ & $h=x-a$ then $f(x) = a + (x-a) + \frac{(x-a)^2}{2!}f''(a) + \dots$

① P.T $f(mx) = f(x) + (m-1)x f'(x) + \frac{(m-1)^2}{2!}x^2 f''(x) + \dots$

$mx = mx - x + x$
$= x(m-1)x$
$= x+h$

$f(mx) = f(mx - x + x)$
$= f(x + (m-1)x)$
$= f(x) + (m-1)x f'(x) + \frac{(m-1)^2}{2!}x^2 f''(x) + \dots$

② P.T $f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{2!(1+x)^2} f''(x) - \frac{x^3}{3!(1+x)^3} f'''(x) + $

$\frac{x^2}{1+x} = \frac{x^2-1+1}{1+x} = \frac{(x-1)(x+1)+1}{1+x} = \frac{(x-1)\cancel{(x+1)}}{\cancel{1+x}} + \frac{1}{1+x}$

$= x - 1 + \frac{1}{1+x}$

$= x - \left[1 - \frac{1}{1+x}\right]$

$$= x - \left[\frac{\cancel{1}+x-\cancel{1}}{1+x}\right] = \boxed{x - \frac{x}{1+x}}$$

Sub $h = \frac{-x}{1+x}$

$$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{2!(1+x)^2} f''(x) - \frac{x^3}{3!(1+x)^2} + \ldots$$

③ Express $f'(x) = 2x^3 + 3x^2 - 8x + 7$ in terms of $(x-2)$

Sub $a = 2$

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2) + \ldots$$

$f(x) = 2x^3 + 3x^2 - 8x + 7 \Rightarrow f(2) = 19$

$f'(x) = 6x^2 + 6x - 8 \Rightarrow f'(2) = 28$

$f''(x) = 12x + 6 \Rightarrow f''(2) = 30$

$f'''(x) = 12 \Rightarrow f'''(2) = 12$

$$= 19 + (x-2)28 + \frac{(x-2)^2}{\cancel{2}} \times \cancel{30}^{15} + \frac{(x-2)^3}{\cancel{3}} \times \cancel{12}^{4} + \ldots$$

$$= 19 + 28x - 46 + x^2 - 4x + 4 \times 15$$

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(4) Expand $x^4 - 3x^3 + 2x^2 - x + 1$ in power of $(x-3)$

$a = 3$

(5) Using taylor Series
Evalulate upto four places of decimals

(1) $\sqrt{1.02}$ (2) $\sqrt{25.15}$ (3) $\sqrt{9.12}$ (4) $\sqrt{10}$ (5) $\sqrt{36.12}$

(1) $\sqrt{1.02}$

$f(x) = \sqrt{x}$

$f(x+h) = \sqrt{x+h}$

$x = 1, h = 0.02$

$f(x+h) = f(1.02) = \sqrt{1.02}$

$\sqrt{1.02} = f(1+0.02)$

$= f(1) + 0.02 f'(1) + \frac{(0.02)^2}{2!} f''(1) + \frac{(0.02)^3}{3!} f'''(1)$

$f(1) = \sqrt{\cancel{x}} = 1$

$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(1) = \frac{1}{2}$

$\Rightarrow f''(x) = \frac{1}{2}\left(\frac{1}{2}\right) x^{-1/2-1} \Rightarrow f''(1) = \frac{-1}{4}$

$\Rightarrow f'''(x) = \left(\frac{-1}{4}\right)\left(\frac{-3}{2}\right) x^{-3/2-1} \Rightarrow f'''(1) = \frac{3}{8}$

$\sqrt{1.02} = 1 + 0.02\left(\frac{1}{2}\right) + \frac{(0.02)^2}{2}\left(\frac{-1}{4}\right) + \frac{(0.02)^3}{6}\left(\frac{3}{6}\right)$

$= 1.0099$

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Maclorin's Series Expansion

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

OR

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots + \frac{x^n}{n!} y_n(0) + \dots$$

① Expand the following function

① e^x

$f(x) = e^x = f(0) = 1$

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$

$f'(x) = e^x \Rightarrow f'(0) = 1$

$f''(x) = e^x \Rightarrow f''(0) = 1$

$f(x) : e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$

$e^{ax} = 1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)}{3!} + \dots$

(ii) $\sin x$

$f(x) = \sin x \Rightarrow f(0) = 0$
$f'(x) = \cos x \Rightarrow f'(0) = 1$
$f''(x) = -\sin x \Rightarrow f''(0) = 0$
$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f(x) = \sin x$
$= 0 + 1(x) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \cdots$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$

(iii) $\cos x$
$f(x) = \cos x \Rightarrow f(0) = 1$
$f'(x) = -\sin x \Rightarrow f(0) = 0$
$f''(x) = -\cos x \Rightarrow f(0) = -1$
$f'''(x) = \sin x \Rightarrow f(0) = 0$

$f(x) = \cos x$
$= 1 + (0)(x) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \cdots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

(iv) $\tan x = f(x)$

$y(x) = \tan x \Rightarrow y(0) = 0$
$y_1(x) = \sec^2 x = 1 + \tan^2 x$
$= 1 + y^2 \Rightarrow y_1(0) = 1 + y^2(0) = 1$
$y_2(x) = 2yy_1 \rightarrow y_2(0) = 2y(0)y_1(0) = 2(0)(1) = 0$
$y_3(x) = 2[y_1 y_2 + y_1^2] \Rightarrow y_3(0) = 2[(0)(0) + 1] = 2$

$y = \tan x$

$= 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2)$

$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$

(v) $f(x) = \log(1+x) \Rightarrow f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$

$f'''(x) = +2(1+x^{-3}) = 2$

$f(x) = \log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \dots$

$= x - \frac{x^2}{2} + \frac{x^3}{3} \dots$

(1) Expand 5^x upto 3 non-zero terms of series

$f(x) = 5^x \Rightarrow f(0) = 5^0 = 1$

$f'(x) = 5^x \log 5 \Rightarrow f'(0) = \log 5$

$f''(x) = 5^x (\log 5)^2 = f''(0) = (\log 5)^2$

~~Sut~~ $f(x) = 5^x = 1 + x\log 5 + \frac{x^2}{2!}(\log 5)^2 + \dots$

② Obtain series $\log(1+x)$ and find the series $\log\left(\frac{1+x}{1-x}\right)$ and hence evalulate $\log\left(\frac{11}{9}\right)$.

$$f(x) = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$f(x) = \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots$$

$$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} \cdots\right]$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right]$$

put $x = 10$

$$\cancel{\log}\ \log\left(\frac{1+\frac{1}{10}}{1-\frac{1}{10}}\right) = \log\frac{1\mathit{1}}{9} = 2\left[\frac{1}{10} + \frac{1}{1000 \times 3}\right]$$

$$= \underline{0.20067}$$

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(3) If $x^3+y^3+xy-1=0$, ~~put~~ p.t $y=1-\frac{x}{3}-\frac{26}{81}x^3 \ldots$

$x^3+y^3+xy-1=0 \Rightarrow (0)+y^3(0)+0y(0)-1=0 \Rightarrow y(0)=1$

$3x^2+3y^2y_1+xy_1+y=0$

$3x^2+3y^2(0)+0y_1(0)+y(0)=0$

$= 0+3(1)^2y_1(0)+0+1=0$

$= 3y_1(0)+1=0$

$= \boxed{y_1(0)=-1/3}$

Diff w.r.t to x

$6x+3\left[y^2y_2+y_1^2 2y\right]+\left[xy_2+y_1(1)\right]+y_1=0$

put $x=0$

$6(0)+3\left[(1)^2 y_2(0)+\left(\frac{-1}{3}\right)^2 2(1)\right]+\left[0y_2(0)+\left(\frac{-1}{3}\right)+\left(\frac{-1}{3}\right)\right.$

$+\left(\frac{-1}{3}\right)=0$

$\boxed{y_2(0)=0}$

Diff (1) w.r.t x

$y_3(0)=\cancel{x}\ \frac{-52}{27}$

Sub all these values into the maclorin's Series

$y=1-\frac{x}{3}-\frac{26}{81}-x^3 \ldots$