Chapter 1

Questions to think about

1.1 General

1. Proof: Flips connect convex tilings in 2 dimensions.
2. Box tiling problem (2 dimensions): Say we have k rectangles with the ith rectangle having length m_i and breadth n_i. We have imperfectly tiled a box with these tiles—the box is covered, but some parts of the tiles stick out. When can we extend this to a tiling of a bigger box? Assume $\gcd(m_1, \ldots, m_k) = \gcd(n_1, \ldots, n_k) = 1$ (Why?).
Simpler: This is known for 2 rectangles; try to arrive at the result.

1.2 3 colour model

1. What does a random filling look like?
2. Begin with the set initial configuration with fixed boundary condition. What does a configuration look like after a large number of random switches?
Seems like the entire system is affected—colours change everywhere.

1.3 Aztecahedron

1. Run a Markov chain with flips and trits on the aztecahedron.

1.3.1 Paper [HLT23]

1. Does Theorem 1 generalise to the aztecahedron? The shape of the boundary will change things—can this be fit into a proof? Maybe code to find a counterexample.
Seems that we have that there exists a hypercube in any dimer configuration on the 3-dimensional aztechedron with two dimers.

2. In continuation of the above, does Theorem 1 generalise to other classes of shapes? Check the inequalities more deeply to see if they can be refined for this.

3. Does Conjecture 2 hold for the aztecahedron?

4. Can we also find many authorised vertices in the aztecahedron (Lemma 1)? This would allow us to look for authorised vertices (easy) instead of trits (hard).

5.

1.4 Paper [HLT23]

1. Conjecture 2.
   Plan: Try for $d = 2, 3, 4, 5$ either by hand or using a simulation.

Chapter 2

Survey

2.1 Local dimer dynamics in higher dimensions [HLT23]

2.1.1 Conjectures

Conjecture 1 *For all* 3*-dimensional shapes* $\mathbf{n}$*, the graphs* $\mathcal{D}(\mathbb{Q}_{\mathbf{n}}^3)$ *is connected.*

Conjecture 2 *Any set of* $2^{d-1}+1$ *disjoint dimers in* $\mathbb{Q}^d$ *admits an alternating cycle of length at most* $2d$ *for any* $d \geq 2$*.*

2.1.2 Theorems

Theorem 1 (Extraction of a dense $\mathbb{Q}$) *Let* $d \geq 2$ *and* $\mathbf{n}$ *be a* d*-dimensional shape. Then, for any dimer configuration* D *on* $\mathbb{Q}_{\mathbf{n}}^d$*, there exists* $\mathbf{x} \in \mathbb{Z}^d$ *such that th eunit cube* $\mathbf{x} + \mathbb{Q}^d \subseteq \mathbb{Q}_{\mathbf{n}}^d$ *contains at least* $2^{d-2}+1$ *dimers in* D*.*

Theorem 2 (Degree and component size) *Fix* $d \geq 3$ *and an even positive integer* n*. The minimum degree of* $\mathcal{D}_{2d-1}(\mathbb{Q}_n^d)$ *is at least* $n^{d-2}/(320d^6)$ *and each connected component contains at least* $2^{n^{d-2}/(320d^6)}$ *dimer configurations.*

Theorem 3 (Degree and component size in $\mathbb{Q}^d$) *Fix* $d \geq 3$*. The minimum degree of* $\mathcal{D}_{2d-1}(\mathbb{Q}^d)$ *is at least* $2^d/(d^4)$ *and each connected component contains at least* $2^{2^d/(d^4)}$ *dimer configurations.*

Theorem 4 (Ergodicity on $\mathbb{Q}^d$) *For every* $d \geq 2$*, the graph* $\mathcal{D}_{2d-2}(\mathbb{Q}^d)$ *is connected and has diameter at most* $(d-1)2^{d-1}$*.*

Theorem 5 (Diameter lower bound) *For all* $d, l, n \geq 2$ *with* n *even, the graph* $\mathcal{D}_l(\mathbb{Q}_n^d)$ *has diameter at least*

$$\frac{n^{d-1}(n^2-1)}{6l^2}.$$

Theorem 6 (Ergodicity on $\mathbb{T}_{m,n}$) *For all positive integers m, n with mn even, the graph $\mathcal{D}_3(\mathbb{T}_{m,n})$ is connected and has diameter at most $2mn$.*

Lemma 1 *Fix $d \geq 3$, $n \geq 4$. Then, there are at least $n^{d-2}/(20d^2)$ authorised vertices in any dimer configuration on $\mathbb{Q}_n^d$.*

2.2 Uniformly positive correlations in the dimer model and phase transition in lattice permutations in $\mathbb{Z}^d$, $d > 2$, via reflection positivity [Tag19]

2.3 T-systems, networks and dimers [DF14]

2.3.1 Excerpts

- In the case of type A, the T-system equation is also known as the octahedron recurrence, and appears to be central in a number of combinatorial objects, such as the lambda-determinant and the Alternating Sign Matrices [24][8], the puzzles for computing Littlewood-Richardson coefficients [20], generalizations of Coxeter-Conway frieze patterns [5][1][3], and the domino tilings of the Aztec diamond [12][25]

2.4 Q-systems as Cluster Algebras II: Cartan Matrix of Finite Type and the Polynomial Property [FK09]

A new interpretation for the T-system arose from realizing that the corresponding discrete evolution could be viewed as a particular mutation in a suitably defined cluster algebra

2.5 Arctic curves of the octahedron equation [DFSG14]

2.6 Double-dimers, the Ising model and the hexahedron recurrence [KP13]

2.7 Perfect Matchings and the Octahedron Recurrence [Spe04]

Chapter 3

Readings

3.1 Done

1. Local dimer dynamics in higher dimensions [HLT23]

3.2 In progress

3.3 Next up

1. Uniformly positive correlations in the dimer model and phase transition in lattice permutations in $\mathbb{Z}^d$, $d > 2$, via reflection positivity [Tag19]
2. The Probabilistic Method [NA08]
3. Lectures on Dimers [Ken09]

3.4 Later

1. Double-dimers, the Ising model and the hexahedron recurrence [KP13]
2. Combinatorial group theory (Conway tiling groups)

Chapter 4

Keywords

1. Double dimer model
2. Ising model
3. Hexahedron recurrence, octahedron recurrence
4. Pfaffian
5. Kasteleyn determinant
6. lattice permutations
7. phase transition

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