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Improved algorithms for the projection of points on NURBS curves and surfaces

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Abstract

Improved algorithms for the metric projection of points on NURBS curves and surfaces are given. The employment of new exclusion criteria within the subdivision strategy increases the robustness and leads to a considerable reduction of computation time.

Keywords: Projection; Nearest point; NURBS curve; NURBS surface

1. Introduction

Algorithms for the computation of nearest points and their corresponding parameters have numerous applications in CAGD and related topics. There are several approaches to attack the problem: In (Dyllong and Luther, 1999) and (Ma and Hewitt, 2003), as in our considerations, the curve or surface is subdivided directly. Zhou et al. (1993) convert the problem into a polynomial system of equations. Turnbull and Cameron (1998), Henshaw (2002), Piegl and Tiller (1997) either apply heuristic methods or cover special cases. Johnson and Cohen (2002) use the common Branch and Bound approach, which also appears within the framework of this paper.

While subdividing the curve or surface it is necessary to decide, which segments possibly contain the solution. The criterion of Dyllong and Luther (1999) for this decision, in some cases, exclude segments containing the solution. Fig. 1 illustrates such a situation. This paper presents new criteria, and proves that they never exclude the solution.

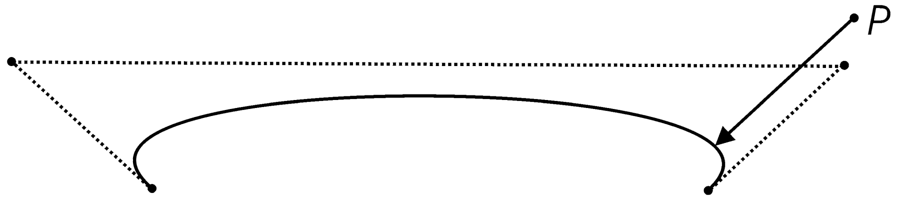

Fig. 1. Wrong exclusion.

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Finally, numerical tests demonstrate that, in comparison with (Ma and Hewitt, 2003), an improved performance has been obtained.

2. Definitions

For order p we consider the knot vectors of the form

$$u_0 = \cdots = u_{p-1} \leqslant \cdots \leqslant u_{m-p+1} = \cdots = u_m$$

with $u_i < u_{i+p}$, defining the normalised B-splines N_i^p. A NURBS curve is given by

$$\gamma(u) = \frac{\sum_{i=0}^{m-p} N_i^p(u) w_i P_i}{\sum_{i=0}^{m-p} N_i^p(u) w_i}, \quad u \in [u_0, u_m],$$

where $w_i \in \mathbf{R}^+$ are the weights and $P_i \in \mathbf{R}^3$ are the control points.

For a NURBS surface of order (p, q) let knot vectors $U = (u_0, \ldots, u_m)$ and $V = (v_0, \ldots, v_n)$ be given. Then we define

$$F(u, v) = \frac{\sum_{i=0}^{m-p} \sum_{j=0}^{n-q} N_i^p(u) N_j^q(v) w_{i,j} P_{i,j}}{\sum_{i=0}^{m-p} \sum_{j=0}^{n-q} N_i^p(u) N_j^q(v) w_{i,j}}, \quad u \in [u_0, u_m],\ v \in [v_0, v_n],$$

where the u-dependent B-splines are defined on the knot vector U and the others on V. Again, $w_{i,j} \in \mathbf{R}^+$ are the weights and $P_{i,j} \in \mathbf{R}^3$ are the control points. We assume that the multiplicity of the interior knots is less than the corresponding degree, which ensures the C^1-continuity of the curves and surfaces.

In addition to the curve or surface we are also given a query point $P \in \mathbf{R}^3$. The task is to compute a parameter which realises the minimum distance between the query point and the curve or surface, that is a parameter x with $\|P - \gamma(x)\| = \operatorname{dist}(P, \gamma)$ or $\|P - F(x)\| = \operatorname{dist}(P, F)$, respectively.

3. Outline of the algorithm

We localise the solution by subdividing the curve or surface recursively. In every step, we split the object into two parts at an interior knot. If there are no interior knots, the split is performed at the arithmetic mean of the boundary parameters in one direction.

In the case where both directions have interior knots or none of them has, the direction is chosen such that the two adjacent corners with the smallest distance to the query point remain in the same part. These heuristics provide a better chance to eliminate one of the segments. Fig. 2 shows the two possibilities, of which we prefer the left one.

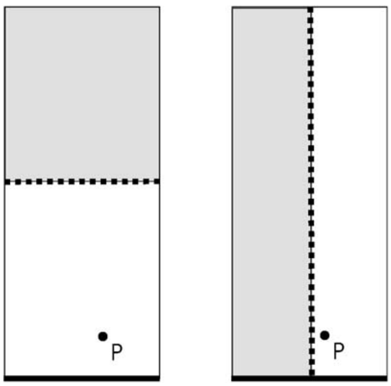

Fig. 2. Split direction.

After each split, we try to eliminate each of the parts by analysing the relationship between the query point and the control polygon. The efficiency of the algorithm substantially depends on the ability to exclude a large number of parts, which do not contain a solution.

The subdivision process stops as soon as a flatness condition is satisfied. We measure the distance of the control points from a line or plane, respectively, and check whether it is less than a prescribed tolerance. Only then we invoke the Newton iteration for each of these segments. In order to get a better chance of convergence, we apply the globalised Newton method as described in (Polak, 1997).

4. Elimination criteria

The most simple criterion uses an upper bound M for the distance between the query point and the curve (surface). Because of the endpoint interpolation it can be obtained from the ends (vertices) of the control polygon (control net). If the minimal distance between its bounding box and the query point is greater than M, we can eliminate the segment.

Even though this method is sufficient for a working algorithm, it is too time-consuming. Fig. 3 illustrates the reason. The left segment clearly does not contain the solution and cannot be eliminated. Both segments have to be subdivided within the next step. As we see here, additional criteria are necessary.

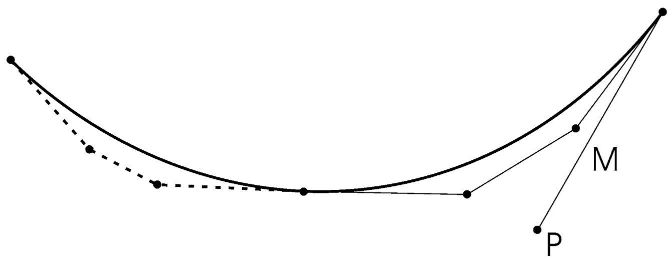

Fig. 3. Weakness of the standard elimination.

4.1. Elimination criteria for curves

Endpoint interpolation and the convex hull property are sufficient for a good criterion. They lead to an elimination if the closest point of the convex hull is an endpoint of the control polygon. Conditions for this are given by the following theorem.

Theorem 4.1. *Let γ be a NURBS curve and $P \in \mathbf{R}^3$ the query point. If*

$$\langle P_i - P_0, P_0 - P\rangle > 0, \quad \text{for all } i = 1, \ldots, m - p \tag{4.1}$$

then $\|\gamma(u_0) - P\| = \operatorname{dist}(P, \gamma)$.

The conditions, that are illustrated in Fig. 4, denote that the control points $P_1, \ldots, P_{m-p}$ lie behind the dashed line.

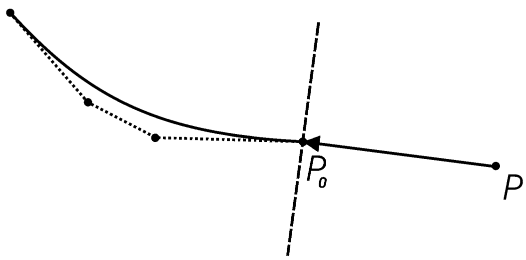

Fig. 4. Curve segment elimination.

Proof. Because of the convex hull property for all $u \in [u_0, u_m]$ there exist $\lambda_i > 0$ with $\sum_{i=0}^{m-p} \lambda_i = 1$, so that $\gamma(u) = \sum_{i=0}^{m-p} \lambda_i P_i$. Then

$$\begin{aligned}\left\|\gamma(u)-P\right\|^2 &= \left\|\sum_{i=0}^{m-p}\lambda_i P_i - P_0 + P_0 - P\right\|^2 \\ &= \|P_0-P\|^2 + \left\|\sum_{i=0}^{m-p}\lambda_i (P_i-P_0)\right\|^2 + 2\sum_{i=0}^{m-p}\lambda_i\langle P_i-P_0, P_0-P\rangle \\ &\overset{(4.1)}{\geqslant} \|P_0-P\|^2 = \left\|\gamma(u_0)-P\right\|^2. \qquad \square\end{aligned}$$

To apply this, we determine the closest endpoint of the curve and select it as P_0. If the condition is fulfilled, P_0 is considered a possible solution and the segment is not subdivided further. In Section 5, this criterion will be compared with the one by Ma and Hewitt (2003).

4.2. Elimination criteria for surfaces

An analogous criterion regarding the corners of surfaces is given by:

Theorem 4.2. *Let F be a NURBS surface and $P \in \mathbf{R}^3$ the query point. If*

$$\langle P_{i,j}-P_{0,0}, P_{0,0}-P\rangle > 0, \quad \text{for all } (i,j) \neq (0,0),\ 0 \leqslant i \leqslant m-p,\ 0 \leqslant j \leqslant n-q,$$

then $\|F(u_0,v_0)-P\| = \operatorname{dist}(P,F)$.

Proof. Similar to the proof of Theorem 4.1. □

The criterion is always applied to the nearest corner, as depicted in Fig. 5. If, apart from $P_{0,0}$, all control points are behind the dashed plane, then $P_{0,0}$ is the point of minimal distance. The gray wedge with its cusp in $P_{0,0}$ contains all query points for which this segment can be excluded.

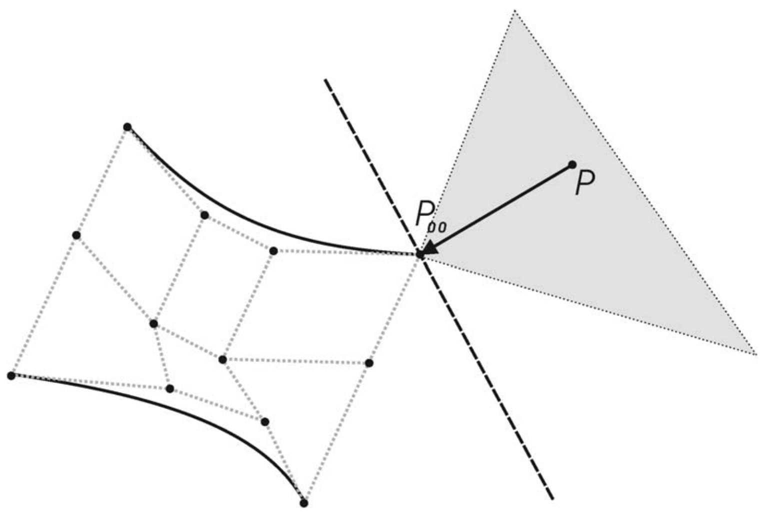

Fig. 5. Surface elimination criterion.

4.3. Tangent cones

Tangent cones are a frequently used tool for the solution of global problems in CAGD. It seems that they have not been applied to point projections yet.

For our purpose in this setting, we define a cone to be a closed angle space around a direction $z \in \mathbf{R}^3 \setminus \{0\}$ with angle $\alpha \in [0, \frac{\pi}{2}]$. A tangent cone T of a C^1-curve contains all its derivatives. A partial tangent cone T_u (T_v) of a C^1-surface contains partial derivatives with respect to u (v).

For rational Bézier surfaces Saito et al. (1995) give a set of vectors with a convex hull containing the derivatives. It is well known that for B-spline surfaces such a set is given by the first order differences of the control net in the corresponding direction. One possible construction of a bounding cone is described by Sederberg and Meyers (1988).

NURBS surfaces are not covered by this. However, after some subdivision steps, the pieces will have Bézier form, because we subdivide them at the interior knots.

4.4. Further elimination criteria for surfaces

We will need the following observation:

Lemma 4.3. *Let $\gamma : I \to \mathbf{R}^3$ be a C^1-curve with tangent cone T. Then for all $t_0, t_1 \in I$ with $t_1 > t_0$*

$$\gamma(t_1) - \gamma(t_0) \in T \cup \{0\}.$$

Proof. We have

$$\gamma(t_1) - \gamma(t_0) = \int_{t_0}^{t_1} \gamma'(t)\,\mathrm{d}t = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{t_1 - t_0}{n} \gamma'\left(t_0 + (t_1 - t_0)\frac{i}{n}\right). \tag{4.2}$$

Every sum lies in $T \cup \{0\}$ and hence also the limit lies in $T \cup \{0\}$. □

Theorem 4.4. *Let $P \in \mathbf{R}^3$ be the query point, F a regular NURBS surface and T_v a tangent cone of F with respect to v. If*

$$\langle P_{i,0} - P, w\rangle > 0, \quad \textit{for all } i = 1, \ldots, m - p,\ w \in T_v \tag{4.3}$$

then the nearest points lie on the parameter line $v = 0$ and none of them is a critical point of the distance function. An analogous statement holds true for the points $P_{i,m-p}$, $P_{0,i}$ and $P_{n-q,i}$ and the parameter lines $v = 1$, $u = 0$, $u = 1$.

Proof. By Lemma 4.3 there exists a $w \in T_v \cup \{0\}$, so that

$$\left\|F(u, v) - P\right\|^2 = \left\|F(u, 0) + w - P\right\|^2.$$

As $F(u, 0)$ lies in the convex hull of $P_{i,0}$ there exist $\lambda_i > 0$ with $\sum \lambda_i = 1$, so that $\sum \lambda_i P_{i,0} = F(u, 0)$. Then we obtain

$$\begin{aligned}\left\|F(u, v) - P\right\|^2 &= \left\|F(u, 0) - P\right\|^2 + 2\big\langle F(u, 0) - P, w\big\rangle + \|w\|^2 \\ &= \left\|F(u, 0) - P\right\|^2 + 2\sum \lambda_i \langle P_{i,0} - P, w\rangle + \|w\|^2 \\ &\geqslant \left\|F(u, 0) - P\right\|^2.\end{aligned}$$

Let $F(u_*, 0)$ be one of the global minima and λ_i the corresponding coefficients. The necessary condition $\langle F(u_*, 0) - P, \partial_v F(u_*, 0)\rangle = 0$ for critical points of the distance function cannot be satisfied, because

$$\big\langle F(u_*, 0) - P, \partial_v F(u_*, 0)\big\rangle = \sum \lambda_i \big\langle P_{i,0} - P, \partial_v F(u_*, 0)\big\rangle > 0. \qquad \square$$

If the tangent cone is given by the axis z and the angle α, the condition

$$\angle(P_{i,0} - P, z) < \frac{\pi}{2} - \alpha$$

is sufficient for the elimination of a segment.

Fig. 6 shows a typical snapshot of the remaining segments in the parameter space during the subdivision. The first pattern shows the situation in which only the distance criterion is used, the second belongs to the criteria of Ma and Hewitt (2003) and the last one to the new criteria presented in this paper.

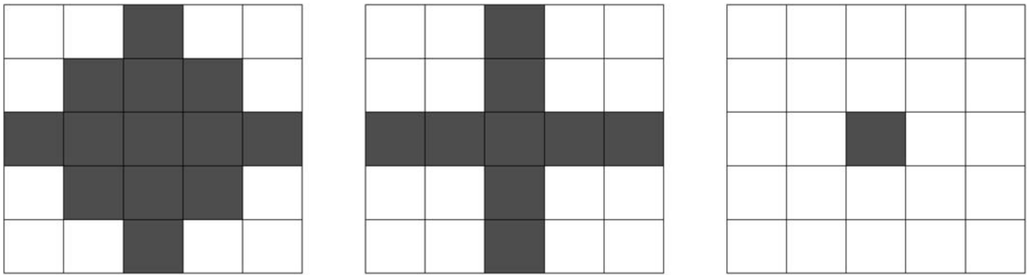

Fig. 6. Elimination pattern.

5. Numerical results

We compare the performance of the subdivision algorithm with different elimination criteria. Because time performance strongly depends on hardware and implementation, we will count the number of necessary subdivisions. This shows the difference consistently, because the evaluation of the criteria takes less time than the subdivision.

In every test, we project the points of a $10 \times 10 \times 10$ equidistant grid on the curve or surface. The grid is chosen to be, in every direction, three times larger than the bounding box. Figs. 8 and 10 contain the average number of subdivisions.

5.1. Curve test

Fig. 7 shows the curves, which are used in the test. Only the last one is not planar.

If the curve contains inflection points, the new criterion leads to an enhanced time performance.

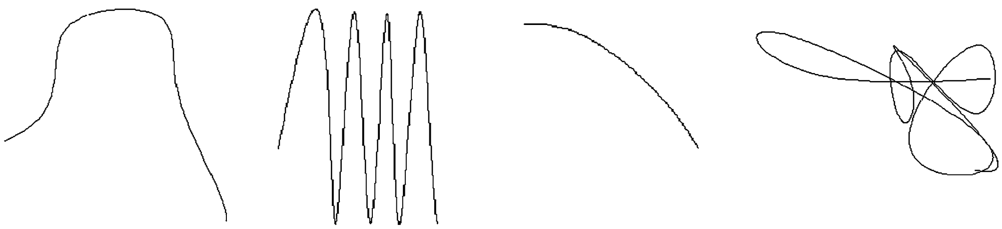

Fig. 7. Curve examples.

Example	Hewitt & Ma	Selimovic
1. Example	8.04	5.92
2. Example	19.59	13.81
3. Example	3.91	3.91
4. Example	18.07	16.37

Fig. 8. Subdivisions during the curve test.

5.2. Surface test

Surfaces of low geometrical complexity as arising in CAD models have been used for comparison. They are shown in Fig. 9.

The new criteria clearly decrease the number of necessary subdivisions. Taking a smaller grid of points reduces the difference a little, but in all further tests (which are not presented here) the new criteria have always been clearly superior.

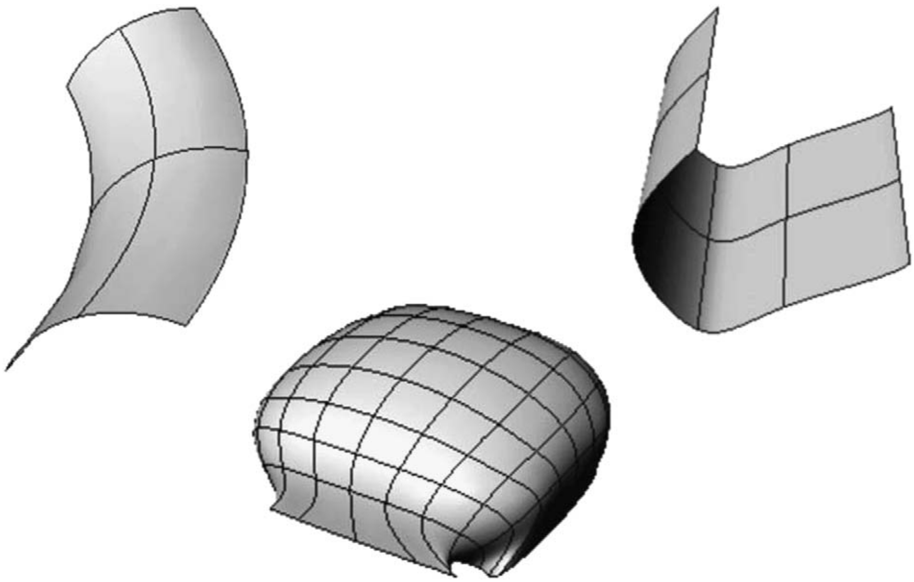

Fig. 9. Tested surfaces.

EXAMPLE	HEWITT & MA	SELIMOVIC
1. Blend	39.47	14.27
2. Translational	21.30	3.15
3. Patch	41.43	15.20

Fig. 10. Subdivisions during the surface test.

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