CHAPTER 4

NUMERICAL INTEGRATION

1 Statement of the problem

Numerical integration, often referred to as quadrature, is a fundamental technique in numerical analysis used to approximate the value of definite integrals, particularly when analytical solutions are difficult or impossible to obtain. This method is essential in various fields such as engineering, physics, and statistics, where functions may only be known at discrete points or are complex in nature. The primary goal of numerical integration is to estimate the area under a curve defined by a function $f(x)$ over a specified interval $[a, b]$

$$Area = \int_a^b f(x)dx$$

This chapter presents computational approaches for estimating definite integrals of single-variable functions. It covers the *Newton-Cotes integration formulae*, including the basic *Trapezoidal rule* (first-order approximation) and higher-order methods: *Simpson's $\frac{1}{3}$ rule* (second-order), *Simpson's $\frac{3}{8}$ rule* (third-order), *Boole's rule* (fourth-order), and *Weddle's rule* (sixth-order). The chapter also examines Romberg integration, which combines Richardson extrapolation with the Trapezoidal rule for improved accuracy. A detailed analysis of error terms in these quadrature formulae is provided, discussing how truncation errors vary with different methods and step sizes.

2 Numerical methods

2.1 Newton – Cotes integration formula

Newton-Cotes integration formulas are a family of numerical methods used for approximating definite integrals. These formulas are based on the idea of evaluating the integrand at equally spaced points and using polynomial interpolation to estimate the area under the curve.

For a function $f(x)$ defined on the interval $[a, b]$, we partition the interval into n equal subintervals of width

$$h = \frac{b-a}{n}$$

with equally spaced nodes $x_i = a + ih$ for $i = 0, 1, \ldots, n$.

The Newton-Cotes formula approximates the definite integral by replacing $f(x)$ with an interpolating polynomial $P_n(x)$ of degree n

$$\int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx = \int_a^b \sum_{i=0}^{n} w_i f(x_i)$$

where the weights w_i represent integrals of Lagrange basis polynomials $L_i(x)$ and they are determined by

$$w_i = \int_a^b L_i(x)\,dx = \int_a^b \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}\,dx, \quad i = 0, 1, \ldots, n$$

Ideally, the approximation converges to the exact integral as the number of quadrature points tends to infinity $n \to \infty$.

2.2 Trapezoidal rule

The trapezoidal rule is a Newton-Cotes quadrature formula that uses linear polynomials ($n = 1$) to approximate the integral of a function $f(x)$ as shown in Figure 4.1. For the interval $[a, b]$, we construct the Lagrange basis polynomials as

$$
\begin{aligned}
L_0(x) &= \frac{x - x_1}{x_0 - x_1} = -\frac{x - b}{h} \\
L_1(x) &= \frac{x - x_0}{x_1 - x_0} = \frac{x - a}{h}
\end{aligned}
$$

where $h = b - a$ is the interval length, with $x_0 = a$ and $x_1 = b$. The corresponding weights are computed by integrating these basis polynomials

$$
\begin{aligned}
w_0 &= \int_a^b L_0(x)\,dx = \frac{1}{h}\int_a^b (b - x)\,dx = \frac{h}{2} \\
w_1 &= \int_a^b L_1(x)\,dx = \frac{1}{h}\int_a^b (x - a)\,dx = \frac{h}{2}
\end{aligned}
$$

This yields the basic trapezoidal rule

$$
\int_a^b f(x)\,dx \approx \frac{h}{2}\big[f(a) + f(b)\big]. \tag{4.1}
$$

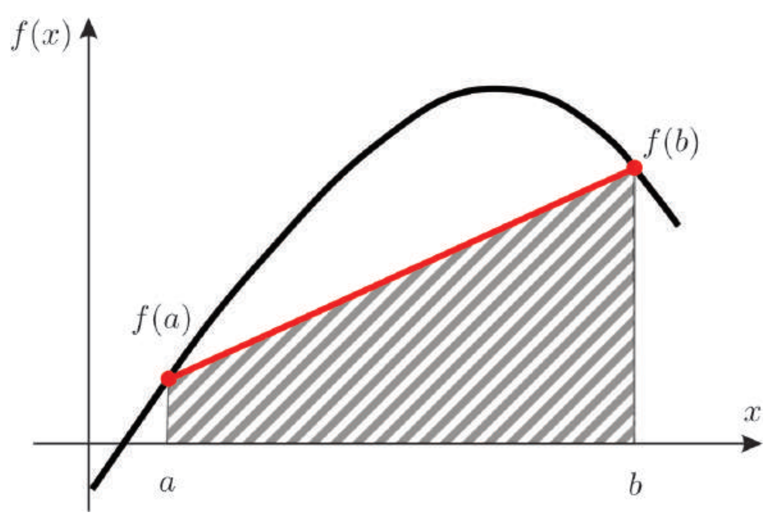

Figure 4.1: Basic trapezoidal rule approximation

This basic trapezoidal rule approximates the area under $f(x)$ by a trapezoid (Figure 4.1). For better accuracy over large intervals, we use the *composite* trapezoidal rule, which divides $[a, b]$ into n subintervals

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right), \tag{4.2}$$

where $h = \frac{b-a}{n}$ is the uniform step size, $x_i = a+ih$ for $i = 1, 2, \ldots, n-1$ are interior points The term $2\sum_{i=1}^{n-1} f(x_i)$ accounts for the values of the function at each interior point, multiplied by 2 because each interior point contributes to two trapezoids. Figure 4.2 illustrates the approximation of an integral using the composite trapezoidal rule. This numerical integration method can be interpreted as the exact integration of the piecewise linear interpolant of the function $f(x)$ at the nodes x_i (where $i = 0, 1, \ldots, n$).

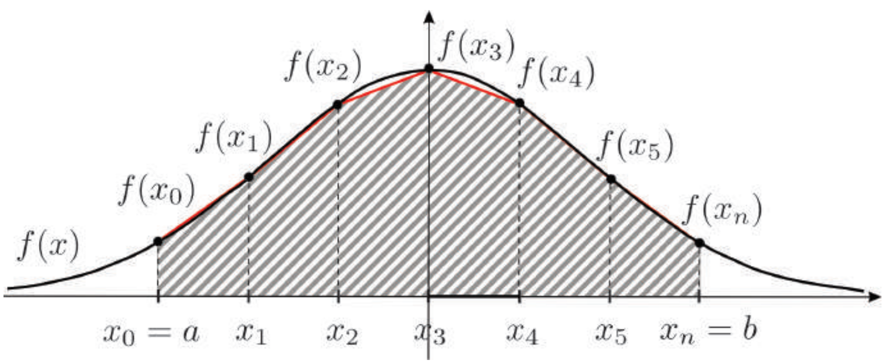

Figure 4.2: The composite trapezoidal rule

Implementation of the composite trapezoidal rule is straightforward. Algorithm 34 and Figure 4.3 illustrate how to implement numerical integration using the composite trapezoidal rule.

Example: Trapezoidal rule

Let's estimate the integral

$$\int_0^{\pi/2} \sin(x)\,dx = 1$$

using the composite trapezoidal rule with $n = 10$ subintervals.

Define parameters

$$a = 0, \quad b = \frac{\pi}{2}, \quad n = 10$$

$$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{10} \approx 0.15708$$

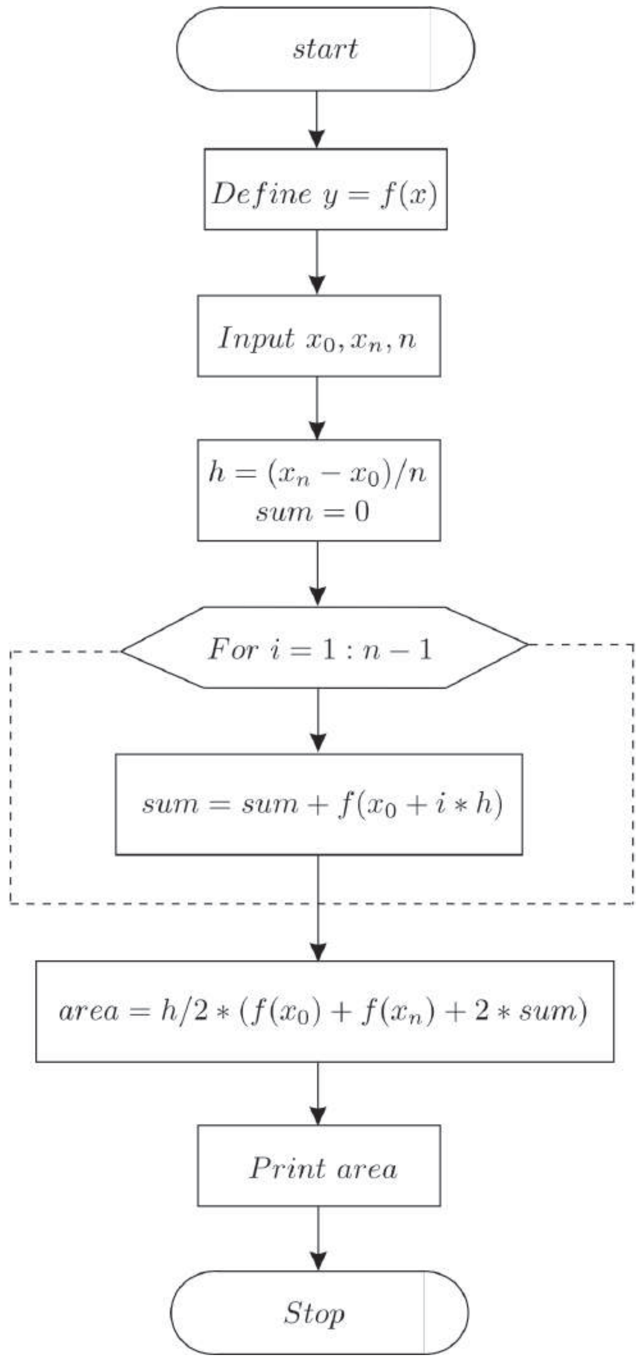

Figure 4.3: Block diagram of the composite trapezoidal rule

Algorithm 34 Trapezoidal rule numerical integration

Input:
$f(x)$ ▷ Function to be integrated
x_0, x_n ▷ Integration bounds $[x_0, x_n]$
n ▷ Number of intervals

Initialize:
$h \leftarrow (x_n - x_0)/n$ ▷ Step size
$\text{sum} \leftarrow 0$ ▷ Initialize accumulator

for $i \leftarrow 1$ to $n-1$ **do**
  $x_i \leftarrow x_0 + i \cdot h$ ▷ Compute current point
  $\text{sum} \leftarrow \text{sum} + f(x_i)$ ▷ Accumulate function values
end for

Compute:
$\text{area} \leftarrow \frac{h}{2}[f(x_0) + 2 \cdot \text{sum} + f(x_n)]$ ▷ Final approximation

Output:
area ▷ Numerical integral value

Compute function values

$$\begin{aligned} f(x) &= \sin(x) \\ f(a) &= \sin(0) = 0 \\ f(b) &= \sin(\pi/2) = 1 \end{aligned}$$

Interior points summation

$$\text{sum} = \sum_{i=1}^{9} \sin(x_i), \quad x_i = 0 + i \cdot h$$

i	$f(x_i)$
1	$\sin(0.15708) \approx 0.15643$
2	$\sin(0.31416) \approx 0.30902$
$\vdots$	$\vdots$
9	$\sin(1.41372) \approx 0.98769$

Compute the approximation

$$\begin{aligned} \text{Area} &\approx \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right] \\ &\approx \frac{0.15708}{2}[0 + 2(5.85342) + 1] \\ &\approx 0.99794 \end{aligned}$$

The exact value is 1, while our approximation is 0.99794 (correct to 2 decimal places). The observed error is $|1 - 0.99794| \approx 0.00206$ ($O(10^{-3})$). Theoretical error bound for the trapezoidal rule:

$$|E| \leq \frac{b-a}{12n^2} \max_{x\in[a,b]} |f''(x)|,$$

For $f(x) = \sin(x)$, $f''(x) = -\sin(x)$, so:

$$|E| \leq \frac{(\pi/2)(0.15708)^2}{12} \approx 0.00323$$

The error decreases by a factor of 4 when n doubles, confirming the $\mathcal{O}(h^2)$ convergence rate.

2.3 Simpson's one-third rule

Simpson's rule (named after the British mathematician Thomas Simpson (1710–1761), though the method was known earlier by others like Cavalieri and Gregory) is a Newton-Cotes quadrature formula that approximates definite integrals by integrating a quadratic interpolating polynomial. Also commonly referred to as the "1/3 rule" due to the $h/3$ factor in its formula, this numerical integration technique offers significantly better accuracy than the trapezoidal rule for smooth functions.

The rule is derived by approximating the integrand $f(x)$ with a second-degree polynomial $P_2(x)$ that passes through three equally spaced points. For a single application of Simpson's rule, we require three nodes (see Figure 4.4)

$$x_0 = a, \quad x_1 = \frac{a+b}{2} \text{ (the midpoint)}, \quad x_2 = b$$

with two subintervals ($n = 2$) and spacing $h = \frac{b-a}{2}$.

For the given three points x_0, x_1 and x_2, the quadratic Lagrange basis polynomials are

$$\begin{aligned}
L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-\frac{a+b}{2})(x-b)}{(a-\frac{a+b}{2})(a-b)},\\
L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-a)(x-b)}{(\frac{a+b}{2}-a)(\frac{a+b}{2}-b)},\\
L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-a)(x-\frac{a+b}{2})}{(b-a)(b-\frac{a+b}{2})}.
\end{aligned}$$

Using the substitution L_0, L_1 and L_2, the weights become

$$\begin{aligned}
w_0 &= \int_a^b L_0 dx = \frac{h}{3},\\
w_1 &= \int_a^b L_1 dx = \frac{4h}{3},\\
w_2 &= \int_a^b L_2 dx = \frac{h}{3}.
\end{aligned}$$

This yields Simpson's rule

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{2} w_i f(x_i) = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{4.3}$$

For more accuracy, especially over larger intervals, we use the composite Simpson's rule by dividing $[a, b]$ into n subintervals (where n must be even):

$$h = \frac{b-a}{n}.$$

The composite formula is

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1\\ i \text{ odd}}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\ i \text{ even}}}^{n-2} f(x_i) + f(x_n)\right], \tag{4.4}$$

where $x_0 = a$, $x_n = b$, $x_i = a + ih$ for $i = 1, \dots, n-1$, the 4/2 pattern alternates weights for odd/even indices.

Example: Simpson's 1/3 rule

Let's approximate the integral

$$I = \int_0^8 \sqrt{x}\,dx$$

using Simpson's 1/3 rule with different numbers of intervals ($n = 2, 4, 8$) and compare the results with the exact solution.

The exact value of the integral is

$$I_{\text{exact}} = \int_0^8 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_0^8 = \frac{2}{3} \times 8^{3/2} = \frac{2}{3} \times 22.6274 \approx 15.0849.$$

Simpson's 1/3 rule approximates the integral using quadratic interpolation

$$I \approx \frac{h}{3}\left[f(x_0) + 4\sum_{\text{odd } i} f(x_i) + 2\sum_{\text{even } i} f(x_i) + f(x_n)\right]$$

where $h = \frac{b-a}{n}$ and $x_i = a + ih$.

For the case with 2 intervals ($n = 2$), the step size is $h = \frac{8-0}{2} = 4$. The evaluation points are $x_0 = 0$, $x_1 = 4$, and $x_2 = 8$. The function evaluations give $f(0) = 0$, $f(4) = 2$, and $f(8) = \sqrt{8} \approx 2.8284$. The approximation becomes

$$I_2 \approx \frac{4}{3}\left[0 + 4(2) + 2.8284\right] = 14.4379$$

. The absolute error is $|15.0849 - 14.4379| = 0.6470$, giving a relative error of $\frac{0.6470}{15.0849} \times 100\% \approx 4.29\%$.

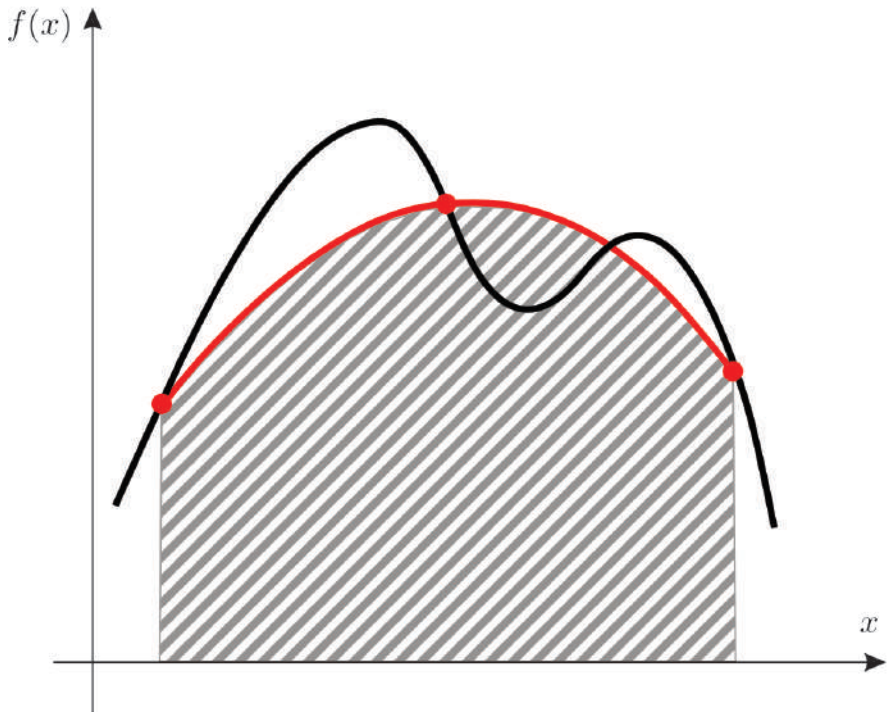

Figure 4.4: Simpson's 1/3 rule definition

With 4 intervals ($n = 4$), the step size reduces to $h = 2$. The evaluation points are $x_i = 0, 2, 4, 6, 8$. The function values are $f(0) = 0$, $f(2) = \sqrt{2} \approx 1.4142$, $f(4) = 2$, $f(6) = \sqrt{6} \approx 2.4495$, and $f(8) \approx 2.8284$. The approximation calculates to

$$I_4 \approx \frac{2}{3}\left[0 + 4(1.4142 + 2.4495) + 2(2) + 2.8284\right] = 15.0695$$

. The absolute error reduces to 0.0154 with a relative error of 0.10%.

For 8 intervals ($n = 8$), h decreases further to 1. The points are $x_i = 0, 1, \ldots, 8$ with key evaluations $f(1) = 1$, $f(3) = \sqrt{3} \approx 1.7321$, $f(5) \approx 2.2361$, and $f(7) \approx 2.6458$. The approximation yields

$$I_8 \approx \frac{1}{3}\left[0 + 4(1 + 1.7321 + 2.2361 + 2.6458) + 2(1.4142 + 2 + 2.4495) + 2.8284\right] = 15.0779$$

. The absolute error is just 0.0070 with a relative error of 0.046%.

The error ratio when doubling intervals from $n = 2$ to $n = 4$ is about 42 (close to $2^4 = 16$ expected from $O(h^4)$ convergence), demonstrating the rapid convergence characteristic of Simpson's 1/3 rule. The exact integral value is 15.0849, and using $n = 4$ intervals provides an excellent balance between accuracy (0.10% error) and computational efficiency. For higher precision, $n = 8$ intervals reduces the error to just 0.046%.

The implementation of the composite Simpson's rule follows a similar structure to the trapezoidal rule but with a distinct weighting pattern for the function evaluations. For n subintervals (where n must be even), the weights alternate between 4 for odd-indexed nodes

$(x_1, x_3, \ldots, x_{n-1})$ and 2 for even-indexed nodes $(x_2, x_4, \ldots, x_{n-2})$, while the endpoints ($x_0$ and x_n) receive weight 1. In practice, this can be efficiently implemented by first multiplying all interior node values by 2 (excluding the endpoints), then applying an additional factor of 2 to the odd-indexed nodes (converting their weights from 2 to 4), and finally summing all weighted values and multiplying the total by $h/3$ to obtain the integral approximation, where $h = (b-a)/n$ is the uniform spacing between nodes as given in Figure 4.5. This two-step weighting approach allows for straightforward implementation while maintaining the rule's $\mathcal{O}(h^4)$ accuracy for sufficiently smooth functions.

2.4 Simpson's three-eighth rule

Simpson's 3/8 rule is a Newton-Cotes quadrature formula that approximates definite integrals using cubic polynomial interpolation. While less commonly used than Simpson's 1/3 rule, it provides comparable accuracy and is particularly useful when the number of subintervals is divisible by three.

Given a function $f(x)$ on $[a, b]$, we partition the interval into three equal subintervals with four points

$$x_0 = a, \quad x_1 = a + h, \quad x_2 = a + 2h, \quad x_3 = b \quad \text{with} \quad h = \frac{b-a}{3}.$$

The cubic interpolating polynomial $P_3(x)$ passes through $(x_i, f(x_i))$ for $i = 0, 1, 2, 3$. The Lagrange basis polynomials are

$$\begin{aligned}
L_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)},\\
L_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)},\\
L_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)},\\
L_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}.
\end{aligned}$$

The quadrature weights are obtained by integrating the basis polynomials

$$\begin{aligned}
w_0 &= \int_a^b L_0(x)dx = \frac{3h}{8}\\
w_1 &= \int_a^b L_1(x)dx = \frac{9h}{8}\\
w_2 &= \int_a^b L_2(x)dx = \frac{9h}{8}\\
w_3 &= \int_a^b L_3(x)dx = \frac{3h}{8}
\end{aligned}$$

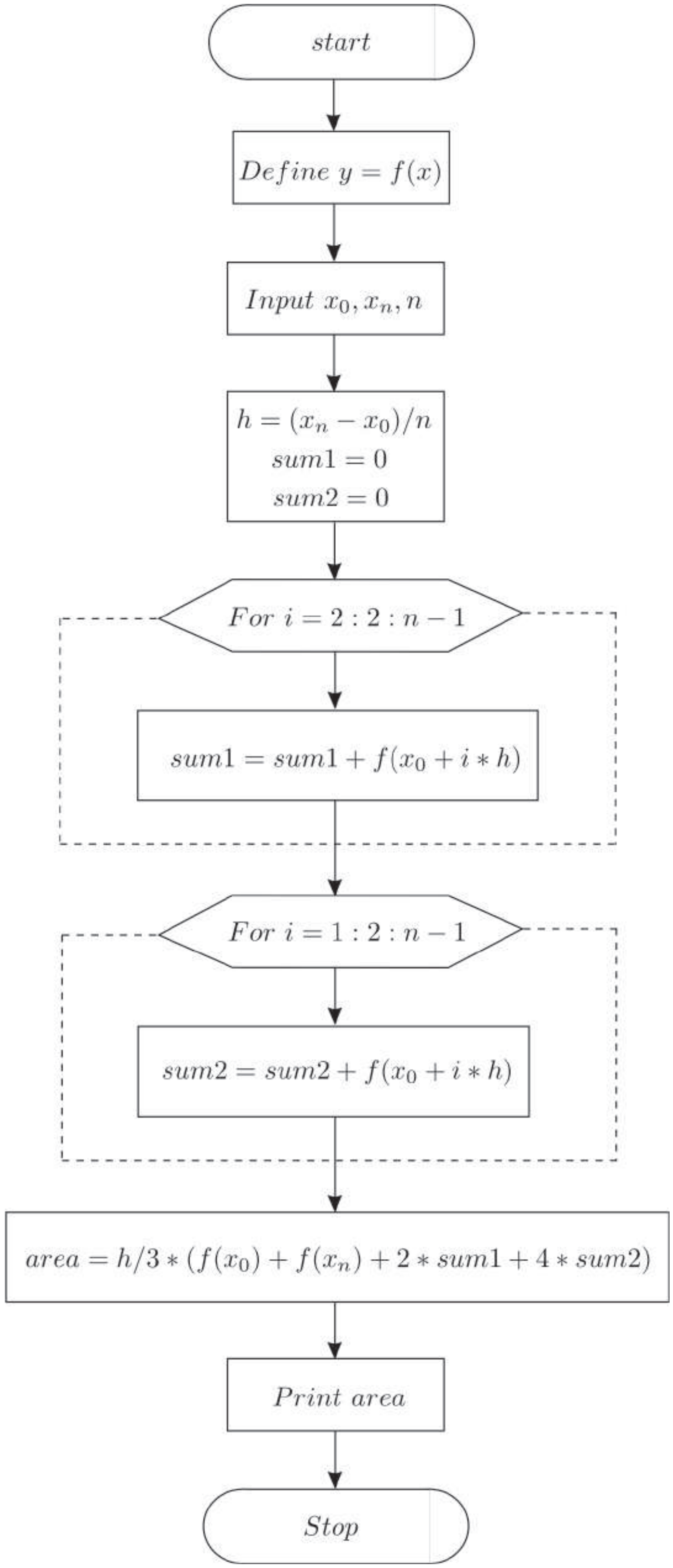

Figure 4.5: Simpson's 1/3 rule algorithm

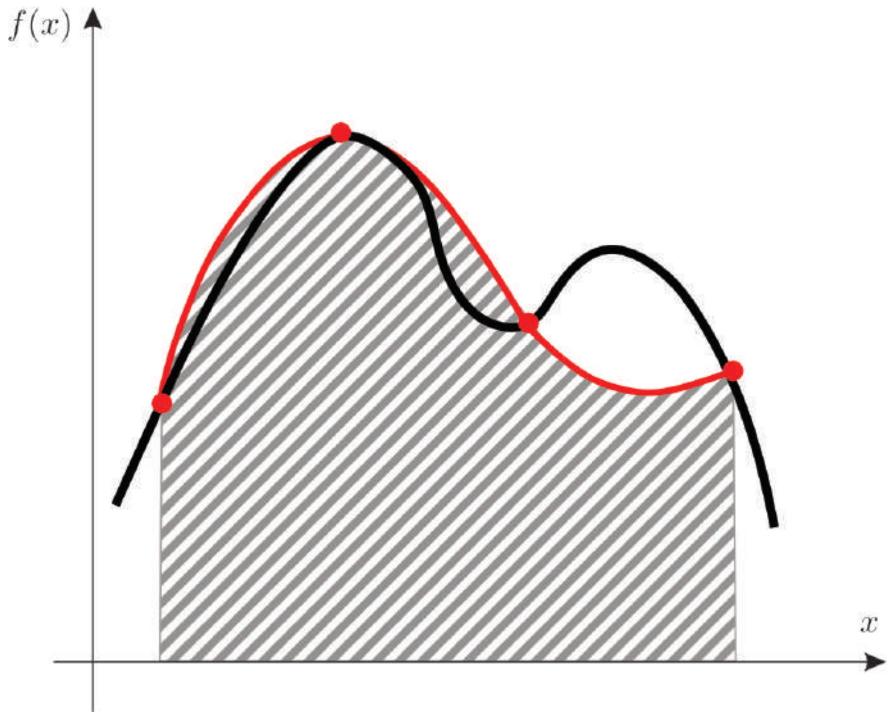

Figure 4.6: Simpson's 3/8 rule definition

This leads to the 3/8 rule formula

$$\int_a^b f(x)dx \approx \frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right]. \tag{4.5}$$

For improved accuracy, we can apply the 3/8 rule compositionally over n subintervals (where n is divisible by 3)

$$h=\frac{b-a}{n}, \quad x_i=a+ih \quad (i=0,\ldots,n) \tag{4.6}$$

The composite formula is

$$\int_a^b f(x)dx \approx \frac{3h}{8}\left[f(x_0)+3\sum_{\substack{i=1\\ i\not\equiv 0 \pmod 3}}^{n-1} f(x_i)+2\sum_{\substack{i=3\\ i\equiv 0 \pmod 3}}^{n-3} f(x_i)+f(x_n)\right]. \tag{4.7}$$

Example: Simpson's 3/8 rule

Let's approximate the integral

$$I=\int_0^6 \frac{dx}{1+x^2}$$

using Simpson's 3/8 rule with 6 intervals ($n=6$).

For Simpson's 3/8 rule, we divide the interval $[0,6]$ into 6 equal subintervals with

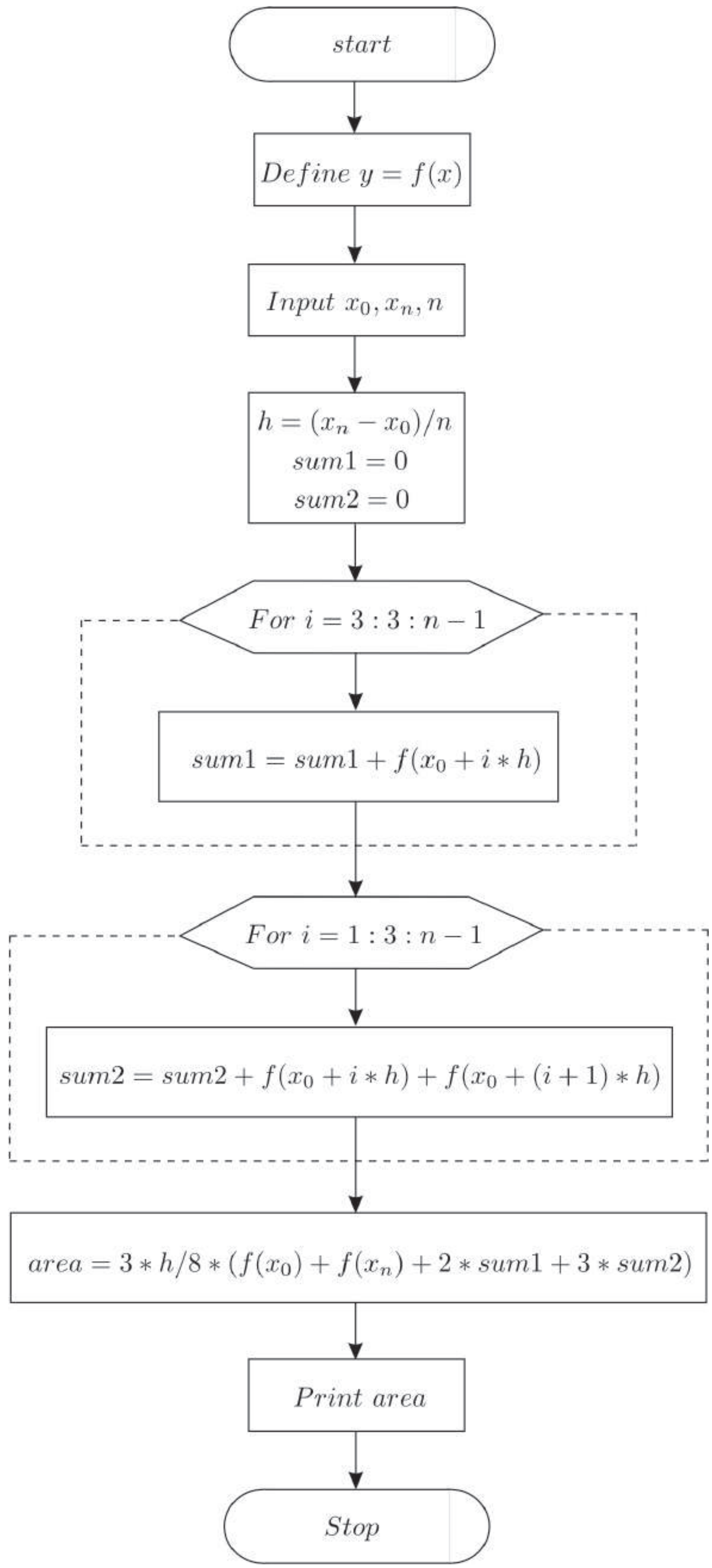

Figure 4.7: Simpson's 3/8 rule algorithm

step size $h = \frac{6-0}{6} = 1$, evaluation points $x = \{0, 1, 2, 3, 4, 5, 6\}$ and function values $y = \{1, 0.5, 0.2, 0.1, 0.0588, 0.0385, 0.0270\}$.

The composite formula for $n = 6$ is

$$I \approx \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\right]$$

Substituting the calculated values

$$I \approx \frac{3 \times 1}{8}\left[(1.0000 + 0.0270) + 3(0.5000 + 0.2000 + 0.0588 + 0.0385) + 2(0.1000)\right]$$

Calculating step-by-step, we get

$$I \approx \frac{3}{8} \times 3.6189 \approx 1.3571$$

The exact value of the integral is

$$f(x) = \arctan(6) - \arctan(0) \approx 1.4056$$

Absolute error is $|1.4056 - 1.3571| = 0.0485$ and relative error is $\frac{0.0485}{1.4056} \times 100\% \approx 3.45\%$

2.5 Boole's rule

Boole's rule is a Newton-Cotes quadrature formula that approximates definite integrals using 4th-degree polynomial interpolation. It provides higher accuracy than Simpson's rules by fitting a fourth-degree polynomial to five equally spaced points.

Given a function $f(x)$ on $[a, b]$, we divide the interval into 4 equal subintervals using 5 points

$$x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h,\ x_3 = a + 3h,\ x_4 = b$$

where $h = \frac{b-a}{4}$.

The quartic Lagrange interpolating polynomial is

$$P_4(x) = \sum_{i=0}^{4} f(x_i)L_i(x)$$

where $L_i(x)$ are the Lagrange basis polynomials

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{4} \frac{x - x_j}{x_i - x_j}$$

The integral approximation becomes

$$\int_a^b f(x)dx \approx \sum_{i=0}^{4} w_i f(x_i)$$

where the weights w_i are integrals of the basis polynomials

$$w_i = \int_a^b L_i(x)dx$$

Using the substitution $\xi = \frac{x-a}{h}$ with $\xi \in [0, 4]$, the weights simplify to:

$$w_0 = \frac{14}{45}h, \quad w_1 = \frac{64}{45}h \quad w_2 = \frac{24}{45}h$$
$$w_3 = \frac{64}{45}h, \quad w_4 = \frac{14}{45}h.$$

Combining the weights gives

$$\int_a^b f(x)dx \approx \frac{2h}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right]$$

When integrating over larger intervals or when higher accuracy is needed, the composite version of Boole's rule can be applied. This involves subdividing the interval into n equal parts, where n must be a multiple of 4. The composite formula is expressed as

$$\int_a^b f(x)dx \approx \frac{2h}{45}\left[7(f(x_0) + f(x_N)) + 32\sum_{i=1, i\,odd}^{N-1} f(x_i) + 12\sum_{i=2, i\,even}^{N-2} f(x_i) + 14\sum_{i=4, i\,multiple\,of\,4}^{N} f(x_i)\right]$$

where the sums account for function evaluations at different indices based on their parity.

Example: Boole's rule

Let's consider the same problem we considered for Simpson's 3/8 rule for approximating an integral

$$I = \int_0^6 \frac{dx}{1+x^2}$$

using Boole's rule with $n = 4$ intervals (5 points).

For Boole's rule, we divide $[0, 6]$ into 4 equal subintervals, so step size is $h = \frac{6-0}{4} = 1.5$, evaluation points are $x = \{0, 1.5, 3.0, 4.5, 6.0\}$ and their function values are $y = \{1.0000, 0.3077, 0.1000, 0.0471, 0.0270\}$.

The Boole's rule formula is

$$I \approx \frac{2h}{45}\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\right]$$

Substituting values

$$I \approx \frac{3}{45}\left[7(1.0000) + 32(0.3077) + 12(0.1000) + 32(0.0471) + 7(0.0270)\right] \approx 1.3162$$

The exact value is

$$I = \arctan(6) - \arctan(0) \approx 1.4056$$

Absolute error is $|1.4056 - 1.3162| = 0.0894$ and relative error is $\frac{0.0894}{1.4056} \times 100\% \approx 6.36\%$

2.6 Weddle's rule

Weddle's rule is a sixth-order numerical integration method that approximates definite integrals by fitting a sixth-degree polynomial to seven equally spaced points. It is generally more accurate than Simpson's rule for certain types of integrals, especially when higher-order derivatives of the function are well-behaved. However, it requires at least seven consecutive values of the function to apply effectively.

Weddle's rule approximates the integral of a function $f(x)$ over the interval $[a, b]$ using values of the function at points that are spaced equally. The formula is given by

$$\int_a^b f(x)\,dx \approx \frac{3h}{10}[(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6) + (y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) + \ldots]$$

where $h = \frac{b-a}{n}$ and n must be a multiple of 6, $y_i = f(x_i)$ represents the function values at equally spaced points x_i.

Example: Weddle's rule

To illustrate Weddle's rule, consider approximating the integral

$$f(x) = \int_4^{5.2} \log(x)dx$$

Define the interval $a = 4, b = 5.2$; and calculate $h = \frac{b-a}{6} = 0.2; x_i = 4 + ih.$, the nodes x_i and corresponding function values are

i	$x_i = 4 + ih$	$y_i = \log(x_i)$	Weight
0	4.0	$\log(4.0) \approx 1.3863$	1
1	4.2	$\log(4.2) \approx 1.4351$	5
2	4.4	$\log(4.4) \approx 1.4816$	1
3	4.6	$\log(4.6) \approx 1.5261$	6
4	4.8	$\log(4.8) \approx 1.5686$	1
5	5.0	$\log(5.0) \approx 1.6094$	5
6	5.2	$\log(5.2) \approx 1.6487$	1

Apply Weddle's formula

$$I \approx \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

Substituting the values and calculate the approximation as

$$I \approx \frac{0.6}{10}\big[1.3863 + 5(1.4351) + 1.4816 + 6(1.5261) + 1.5686 + 5(1.6094) + 1.6487\big] \approx 1.8279$$

Exact solution (using integration by parts)

$$I = [x \log x - x]_4^{5.2} \approx (5.2 \log 5.2 - 5.2) - (4 \log 4 - 4) \approx 1.8280$$

Absolute error is $|1.8280 - 1.8279| = 0.0001$ and relative error is $\frac{0.0001}{1.8280} \times 100\% \approx 0.0055\%$.

Algorithm 35 Boole's rule numerical integration

Input:
$f(x)$ ▷ Integrand function
x_0, x_n ▷ Integration interval endpoints
n ▷ Number of subintervals (must be divisible by 4)

Initialize:
$h \leftarrow (x_n - x_0)/n$ ▷ Step size calculation
sum $\leftarrow 0$ ▷ Initialize weighted sum

Compute Function Values:
for $i \leftarrow 0$ to n **do**
  $x_i \leftarrow x_0 + i \cdot h$ ▷ Compute current node
  $y_i \leftarrow f(x_i)$ ▷ Evaluate function at node
  Apply Weights:
  if $i = 0$ or $i = n$ **then**
    sum $\leftarrow$ sum $+ 7y_i$ ▷ Endpoint weight
  else if $i \bmod 2 = 1$ **then**
    sum $\leftarrow$ sum $+ 32y_i$ ▷ Odd index weight
  else if $i \bmod 4 = 2$ **then**
    sum $\leftarrow$ sum $+ 12y_i$ ▷ Midpoint weight
  else
    sum $\leftarrow$ sum $+ 14y_i$ ▷ Other even weight
  end if
end for

Output:
integral $\leftarrow \frac{2h}{45} \cdot$ sum ▷ Final integral computation
integral ▷ Approximated integral value

Algorithm 36 Weddle's rule numerical integration

Input:
$f(x)$ ▷ Integrand function to be evaluated
x_0, x_n ▷ Lower and upper bounds of integration
n ▷ Number of subintervals (must be divisible by 6)

Initialize:
$h \leftarrow \frac{x_n - x_0}{n}$ ▷ Calculate uniform step size
sum $\leftarrow 0$ ▷ Initialize weighted sum accumulator

Compute Weighted Sum:
for $i \leftarrow 0$ n **do**
  $x_i \leftarrow x_0 + i \cdot h$ ▷ Compute current node position
  $y_i \leftarrow f(x_i)$ ▷ Evaluate function at current node
  Apply Weddle's Weights:
  if $i = 0$ $i = n$ **then**
    sum $\leftarrow$ sum $+ y_i$ ▷ Endpoint weight $= 1$
  else if $i \bmod 6 = 3$ **then**
    sum $\leftarrow$ sum $+ 6y_i$ ▷ Central point weight
  else if $i \bmod 2 = 1$ **then**
    sum $\leftarrow$ sum $+ 5y_i$ ▷ Odd index weight
  else
    sum $\leftarrow$ sum $+ y_i$ ▷ Other even point weight
  end if
end for

Output:
integral $\leftarrow \frac{3h}{10} \cdot$ sum ▷ Final integral approximation
return integral ▷ Return computed integral value

3 Tasks

1. Test the implementation of the composite Simpson's rule by evaluating the integral:

$$\int_0^{\pi/2} \sin(x)\,dx = 1$$

using $N = 10$ subintervals, and compare its accuracy with the trapezoidal rule.

   – Simpson's method should demonstrate higher accuracy compared to the trapezoidal rule for the same number of intervals

   – The result should match the exact value 1 with at least 5 decimal digits of precision

   – The observed error should confirm the $\mathcal{O}(h^4)$ convergence rate characteristic of Simpson's rule

2. Approximate the integral of the function $f(x) = x^2 + 3x + 2$ over the interval $[1, 5]$ using:

   (a) A single trapezoid.

   (b) Four equal subintervals.

3. Evaluate the integral of $f(x) = e^x$ over $[0, 2]$ using Simpson's $1/3$ Rule with:

   (a) Two subintervals.

   (b) Four subintervals.

4. Use Simpson's $3/8$ rule to approximate the integral $\int_0^3 \sqrt{x}\,dx$. Divide the interval into:

   (a) Three subintervals.

   (b) Six subintervals.

5. Approximate the integral $\int_0^4 (x^4 - 2x^3 + x^2 + x + 1)\,dx$ using Boole's rule. Use five equally spaced points.

6. For $f(x) = x^3 - x^2 + x - 1$, approximate $\int_1^5 f(x)\,dx$ using:

   (a) Trapezoidal rule.

   (b) Simpson's $1/3$ rule.

   (c) Simpson's $3/8$ rule.

   Compare the results and discuss which method is more accurate for this function.

7. For $f(x) = \ln(x+1)$, approximate $I = \int_0^2 f(x)\,dx$: Using the Trapezoidal rule with four subintervals and Simpson's $1/3$ rule with four subintervals. Calculate and compare the errors for both methods with respect to the exact value of the integral.

8. The velocity of a car is given by $v(t) = t^2 + t + 1$, where t is in seconds and $v(t)$ is in m/s. Use numerical integration to find the total distance traveled by the car from $t = 0s$ to $t = 5s$: Using:

   (a) Simpson's $1/3$ rule with four subintervals.

   (b) Boole's rule.

9. For $f(x) = e^{-x^2}$, approximate $I = \int_0^1 f(x)\,dx$: Using:

   (a) Simpson's $1/3$ rule with four subintervals.

   (b) Boole's rule with five points.

   Compare the results and discuss which method provides higher accuracy.

10. Write a program to implement:

(a) The Trapezoidal rule for any function over a given interval.

(b) Simpson's rules (both 1/3 and 3/8).

Test your program by evaluating $I = \int_0^\pi \sin(x)\,dx$.

11. A city's energy consumption varies throughout the day due to factors like industrial activity, residential usage, and commercial operations. The power consumption rate $P(t)$ (in megawatts, MW) as a function of time t (in hours) is given by the following data:

t	0	2	4	6	8	10	12	14	16	18	20	22	24
$P(t)$	500	480	450	600	800	950	1000	980	920	850	700	550	500

The total energy consumed E (in megawatt-hours, MWh) over a 24-hour period is given by the integral of $P(t)$ with respect to time:

$$E = \int_0^{24} P(t)\,dt$$

Since the data is discrete, numerical integration methods must be used to approximate the integral.

(a) Write a program to approximate the integral using all numerical integration methods.

(b) Compare the results obtained from the different methods.

(c) Calculate the total energy consumption E using each method.

(d) Determine which method provides the most accurate result and justify your choice.

(e) Write a detailed report explaining your methodology, results, and recommendations. Include graphs, tables, and any relevant calculations.