# Part II

# Second part

# CHAPTER 1

# FINITE DIFFERENCES

# 1 Statement of the problem

In many areas of applied mathematics and computational science, it is common to work with functions known only at discrete points rather than in continuous analytic form. In such scenarios, finite difference methods serve as essential tools for approximating derivatives and constructing numerical algorithms.

Finite differences provide discrete analogues of derivatives by evaluating changes in function values over fixed intervals. These methods are particularly important when

- The function is given in tabular form,
- Symbolic differentiation is impractical,
- Numerical solutions to differential equations are required.

Assuming the function $f(x)$ is sampled at equally spaced points $x_0, x_1, x_2, \dots$ with constant spacing $h$, we define the following difference operators:

- Forward Difference Operator ( $\Delta$ ) — based on the change between a point and the next;
- Backward Difference Operator ( $\nabla$ ) — based on the change between a point and the previous;
- Central Difference Operator ( $\delta$) — based on the average rate of change around a central point.

Each operator is designed for specific scenarios

- Forward and backward differences are best suited for computations at or near the endpoints of datasets;
- Central differences offer higher-order accuracy and are preferred for interior points.

These difference operators form the foundation of various numerical methods in interpolation, differentiation, and the numerical solution of ordinary and partial differential equations.

# 2 Forward difference operator

The forward difference operator is a fundamental tool in numerical analysis, used to approximate the rate of change of a function based on values at discrete, equally spaced points. Suppose a function $f(x)$ is given at points $x_0, x_1, x_2, \dots, x_n$, where $x_r = x_0 + rh$, and $h$ is the uniform spacing or step size. The first forward difference at $x_r$ is defined as

$$\Delta f(x_r) = f(x_{r+1}) - f(x_r),$$

which represents the change in function value over one step $h$. Higher-order forward differences are computed recursively as follows

– The second forward difference is defined by

$$\Delta^2 f(x_r) = \Delta f(x_{r+1}) - \Delta f(x_r),$$

which reflects the change in the first differences.

– The third forward difference is given by

$$\Delta^3 f(x_r) = \Delta^2 f(x_{r+1}) - \Delta^2 f(x_r),$$

capturing the rate of change of the second differences.

– More generally, the $k-$th forward difference is expressed as

$$\Delta^k f(x_r) = \Delta^{k-1} f(x_{r+1}) - \Delta^{k-1} f(x_r).$$

where each difference depends only on the previous order.

These values are organized in a forward difference table, where the first column contains the function values, and each subsequent column contains the differences of the previous one. For example, a typical forward difference table for $n+1$ points might appear as:

| $x_r$ | $f(x_r)$ | $\Delta f(x_r)$ | $\Delta^2 f(x_r)$ | $\Delta^3 f(x_r)$ | $\cdots$ |
|---|---|---|---|---|---|
| $x_0$ | $f(x_0)$ | | | | |
| $x_1$ | $f(x_1)$ | $\Delta f(x_0)$ | | | |
| $x_2$ | $f(x_2)$ | $\Delta f(x_1)$ | $\Delta^2 f(x_0)$ | | |
| $x_3$ | $f(x_3)$ | $\Delta f(x_2)$ | $\Delta^2 f(x_1)$ | $\Delta^3 f(x_0)$ | |
| $x_4$ | $f(x_4)$ | $\Delta f(x_3)$ | $\Delta^2 f(x_2)$ | $\Delta^3 f(x_1)$ | $\cdots$ |

Each diagonal in the table corresponds to higher-order differences originating at the top. The forward difference method is particularly effective for interpolation near the beginning of the data set, such as in Newton's forward interpolation formula. It also provides a numerical approximation to derivatives. For example, the first derivative can be estimated by

$$f'(x_r) \approx \frac{\Delta f(x_r)}{h},$$

and the second derivative by

$$f''(x_r) \approx \frac{\Delta^2 f(x_r)}{h^2},$$

assuming $h$ is small and the function is smooth. The accuracy of these approximations depends on the choice of step size $h$; smaller $h$ improves accuracy but may also amplify numerical errors due to limited floating-point precision. Forward differences form the basis for many numerical algorithms, including interpolation, numerical differentiation, and finite difference methods for solving differential equations.

**Algorithm 23** Forward difference operator

**Require:** A set of data points $(x_0, f_0), (x_1, f_1), \ldots, (x_n, f_n)$, where $f_i = f(x_i)$, and spacing between $x_i$ may be equal or unequal

**Ensure:** Computation of the forward difference $\Delta^k f(x_i)$ for $i = 0, 1, \ldots, n-k$

**Step 1: Define the forward difference operator**

The first forward difference is given by

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$$

**Step 2: Compute first-order forward differences**

**for** $i \leftarrow 0\ n-1$ **do**

  Compute $\Delta f(x_i) = f_{i+1} - f_i$

  If data is unequally spaced, record spacing $h_i = x_{i+1} - x_i$

**end for**

**Step 3: Compute higher-order forward differences**

Use the recursive formula

$$\Delta^k f(x_i) = \Delta^{k-1} f(x_{i+1}) - \Delta^{k-1} f(x_i)$$

For example:

$$\Delta^2 f(x_i) = \Delta(\Delta f(x_i)) = f_{i+2} - 2f_{i+1} + f_i$$

$$\Delta^3 f(x_i) = f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i$$

Continue until desired order $k$ is reached.

**Step 4: Construct the forward difference table (optional)**

Organize $f_i$, $\Delta f(x_i)$, $\Delta^2 f(x_i)$, etc., into a difference table for visualization or interpolation use.

**Step 5: Output the computed values**

Return the list of forward differences

$$\Delta f(x_0), \Delta f(x_1), \ldots, \Delta f(x_{n-1})$$

Optionally return higher-order differences $\Delta^2 f(x_0), \ldots, \Delta^k f(x_{n-k})$

***Example: Computing Forward Differences***

Consider the following set of equally spaced data points:

| $x_i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x_i)$ | 2 | 4 | 7 | 11 |

**Step 1: Confirm uniform spacing**

We check the spacing between the $x_i$ values:

$$h = x_{i+1} - x_i = 1$$

Since the spacing $h$ is constant, we can apply the standard forward difference method.

**Step 2: Compute first-order forward differences**

$$\Delta f(x_0) = f(x_1) - f(x_0) = 4 - 2 = 2$$
$$\Delta f(x_1) = f(x_2) - f(x_1) = 7 - 4 = 3$$
$$\Delta f(x_2) = f(x_3) - f(x_2) = 11 - 7 = 4$$

**Step 3: Compute second-order forward differences**

$$\Delta^2 f(x_0) = \Delta f(x_1) - \Delta f(x_0) = 3 - 2 = 1$$
$$\Delta^2 f(x_1) = \Delta f(x_2) - \Delta f(x_1) = 4 - 3 = 1$$

**Step 4: Compute third-order forward difference**

$$\Delta^3 f(x_0) = \Delta^2 f(x_1) - \Delta^2 f(x_0) = 1 - 1 = 0$$

**Step 5: Construct the forward difference table**

| $x_i$ | $f(x_i)$ | $\Delta f(x_i)$ | $\Delta^2 f(x_i)$ | $\Delta^3 f(x_i)$ |
|---|---|---|---|---|
| 1 | 2 | | | |
| 2 | 4 | 2 | | |
| 3 | 7 | 3 | 1 | |
| 4 | 11 | 4 | 1 | 0 |

**Conclusion:**
Since the third-order forward difference is zero, the data can be exactly modeled by a polynomial of degree two (i.e., a quadratic function).

**Convergence rate and accuracy**
The accuracy of the forward difference method depends on the step size $h$ and the smoothness of the function being approximated. For a function $f(x)$ that is sufficiently differentiable, the first-order forward difference for approximating the derivative is given by

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} = \frac{\Delta f(x)}{h},$$

which is a first-order accurate formula with truncation error

$$\text{Error} = \frac{h}{2} f''(\xi), \quad \text{for some } \xi \in [x, x+h].$$

This implies that the error is directly proportional to the step size $h$, and reducing $h$ improves the accuracy. However, excessively small $h$ can introduce round-off errors in numerical implementations. For higher-order derivatives, such as the second derivative, the forward difference approximation becomes

$$f''(x) \approx \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} + \mathcal{O}(h),$$

where the error now depends on higher derivatives of the function. The method is consistent and converges to the true derivative as $h \to 0$, provided the function is sufficiently smooth. Therefore, the forward difference method is reliable for numerical differentiation when applied with appropriate step sizes and to functions that possess continuous derivatives.

**Advantages**

- Simple and intuitive: Easy to understand and implement, especially for uniformly spaced data.
- Ideal for tabulated data: Well-suited for discrete, tabular values typically encountered in experimental or empirical datasets.
- Foundation for interpolation: Serves as the basis for Newton's Forward Interpolation Formula and other finite difference schemes.
- Non-iterative computation: Once the difference table is constructed, the values can be reused for multiple evaluations without recomputation.
- Efficient for low-degree polynomials: Performs well when approximating functions that are close to linear or quadratic in behavior.

**Disadvantages**

- Requires equal spacing: The method assumes that the data points are evenly spaced, making it unsuitable for arbitrary spacing unless adapted.
- Numerical instability for large tables: Higher-order differences may introduce numerical instability and amplify round-off errors, particularly in noisy datasets.
- Reduced accuracy near boundaries: The method becomes less accurate near the end of the dataset, especially for extrapolation purposes.
- Sensitive to data irregularities: Sudden changes, oscillations, or measurement errors in the data can distort difference values and reduce reliability.
- Local approximation only: Forward differences approximate the derivative locally and may not reflect the global behavior of the underlying continuous function.

# 3 Backward difference operator

The backward difference operator is a key tool in finite difference calculus, especially useful when approximating derivatives near the end of a dataset or when working backward in time or space. Like the forward difference method, it assumes that the function $f(x)$ is known at equally spaced points $x_0, x_1, x_2, \ldots, x_n$, where $x_r = x_0 + rh$ and $h$ is the uniform step size.

The first backward difference at a point $x_r$ is defined as

$$\nabla f(x_r) = f(x_r) - f(x_{r-1}),$$

which measures the change in function values in the reverse direction over a step size $h$. Higher-order backward differences are defined recursively:

– The second backward difference:

$$\nabla^2 f(x_r) = \nabla f(x_r) - \nabla f(x_{r-1}) = f(x_r) - 2f(x_{r-1}) + f(x_{r-2}),$$

– The third backward difference:

$$\nabla^3 f(x_r) = \nabla^2 f(x_r) - \nabla^2 f(x_{r-1}),$$

– The general form:

$$\nabla^k f(x_r) = \nabla^{k-1} f(x_r) - \nabla^{k-1} f(x_{r-1}).$$

These differences are systematically organized in a backward difference table, where the function values form the first column, and each subsequent column contains differences calculated backward from the bottom up.

| $x_r$ | $f(x_r)$ | $\nabla f(x_r)$ | $\nabla^2 f(x_r)$ | $\nabla^3 f(x_r)$ |
|---|---|---|---|---|
| $x_0$ | $f(x_0)$ | $\nabla f(x_1)$ | $\nabla^2 f(x_2)$ | $\nabla^3 f(x_3)$ |
| $x_1$ | $f(x_1)$ | $\nabla f(x_2)$ | $\nabla^2 f(x_3)$ | |
| $x_2$ | $f(x_2)$ | $\nabla f(x_3)$ | | |
| $x_3$ | $f(x_3)$ | | | |

The backward difference method is particularly useful in Newton's Backward Interpolation Formula and is suitable for approximating derivatives near the end of the data set. For small $h$, the first and second derivatives can be estimated as:

$$f'(x_r) \approx \frac{\nabla f(x_r)}{h}, \quad f''(x_r) \approx \frac{\nabla^2 f(x_r)}{h^2}.$$

**Algorithm 24** Backward difference operator

**Require:** A set of tabulated function values $(x_0, f_0), (x_1, f_1), \ldots, (x_n, f_n)$, where $f_i = f(x_i)$, and $x_i$ are equally or unequally spaced

**Ensure:** Backward difference values $\nabla^k f(x_i)$ for $i = k, \ldots, n$, where $k$ is the order of the difference

**Step 1: Define the first-order backward differenced**

For $i = 1, 2, \ldots, n$, the first backward difference is

$$\nabla f(x_i) = f(x_i) - f(x_{i-1})$$

**Step 2: Compute first-order differences**

**for** $i \leftarrow 1\ n$ **do**

  Compute $\nabla f(x_i) = f_i - f_{i-1}$

  If spacing is non-uniform, store $h_i = x_i - x_{i-1}$

**end for**

**Step 3: Compute higher-order backward differences**

Use the recursive relation

$$\nabla^k f(x_i) = \nabla^{k-1} f(x_i) - \nabla^{k-1} f(x_{i-1})$$

For example:

$$\nabla^2 f(x_i) = \nabla f(x_i) - \nabla f(x_{i-1}) = f_i - 2f_{i-1} + f_{i-2}$$

$$\nabla^3 f(x_i) = \nabla^2 f(x_i) - \nabla^2 f(x_{i-1}) = f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}$$

Repeat until the desired order $k$ is reached

**Step 4 (Optional): Construct the backward difference table**

Organize the values $f(x_i)$, $\nabla f(x_i)$, $\nabla^2 f(x_i)$, etc., in tabular form to support interpolation or trend detection

**Step 5: Output the results**

Return the list of backward differences

$$\nabla f(x_1), \nabla f(x_2), \ldots, \nabla f(x_n)$$

Optionally return higher-order differences

$$\nabla^k f(x_k), \ldots, \nabla^k f(x_n)$$

***Example: Computing Backward Differences***

Given the equally spaced data points:

| $x_i$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(x_i)$ | 1 | 2 | 4 | 8 |

**Step 1: Confirm step size**
Since the values of $x_i$ are uniformly spaced:

$$h = x_i - x_{i-1} = 1 \quad \text{(constant)}$$

We can apply the backward difference method.

**Step 2: Compute first-order backward differences**

$$\begin{aligned}
\nabla f(x_1) &= f(x_1) - f(x_0) = 2 - 1 = 1 \\
\nabla f(x_2) &= f(x_2) - f(x_1) = 4 - 2 = 2 \\
\nabla f(x_3) &= f(x_3) - f(x_2) = 8 - 4 = 4
\end{aligned}$$

**Step 3: Compute second-order backward differences**

$$\begin{aligned}
\nabla^2 f(x_2) &= \nabla f(x_2) - \nabla f(x_1) = 2 - 1 = 1 \\
\nabla^2 f(x_3) &= \nabla f(x_3) - \nabla f(x_2) = 4 - 2 = 2
\end{aligned}$$

**Step 4: Compute third-order backward difference**

$$\nabla^3 f(x_3) = \nabla^2 f(x_3) - \nabla^2 f(x_2) = 2 - 1 = 1$$

**Step 5: Backward difference table**

| $x_i$ | $f(x_i)$ | $\nabla f(x_i)$ | $\nabla^2 f(x_i)$ | $\nabla^3 f(x_i)$ |
|---|---|---|---|---|
| 0 | 1 | | | |
| 1 | 2 | 1 | | |
| 2 | 4 | 2 | 1 | |
| 3 | 8 | 4 | 2 | 1 |

**Conclusion:**
The third-order backward difference is nonzero, indicating that the data is *not modeled by a polynomial of degree 2*. In fact, the pattern of first differences doubling suggests an exponential relationship of the form

$$f(x) = 2^x$$

**Convergence rate and accuracy**
The backward difference approximation for the first derivative is given by

$$f'(x_r) \approx \frac{f(x_r) - f(x_{r-1})}{h} = \frac{\nabla f(x_r)}{h}.$$

This is a first-order approximation, with truncation error

$$\text{Error} = \frac{h}{2} f''(\xi), \quad \text{for some } \xi \in [x_{r-1}, x_r].$$

For the second derivative

$$f''(x_r) \approx \frac{f(x_r) - 2f(x_{r-1}) + f(x_{r-2})}{h^2} + \mathcal{O}(h).$$

The method converges as $h \to 0$ and is best suited for functions that are smooth and differentiable. Like the forward method, very small $h$ increases round-off error risks.

**Advantages**

– Effective near the end of a data table, where forward differences may not be applicable.

– Compatible with Newton's Backward Interpolation Formula.

– Same structure and ease of use as forward differences for uniformly spaced data.

– Computationally simple and non-iterative.

**Disadvantages**

– Like forward differences, assumes equal spacing.

– Sensitive to noise and error accumulation in higher-order differences.

– Less effective in central regions of data where central difference methods are more accurate.

– Extrapolation using backward differences is less stable and more error-prone than interpolation.

# 4 Central difference operator

The central difference operator is a finite difference technique used in numerical analysis to approximate derivatives of a function using values symmetrically located around a point. This method provides improved accuracy over forward or backward difference operators, especially for the first and second derivatives, as it balances contributions from both directions.

Assume that a function $f(x)$ is known at equally spaced points $x_0, x_1, \ldots, x_n$, with a constant spacing $h$. The central difference operator makes use of values at $x_{r-1}$, $x_r$, and $x_{r+1}$ to compute approximate derivatives at the point $x_r$. The first-order central difference is given by

$$\delta f(x_r) = \frac{f(x_{r+1}) - f(x_{r-1})}{2h},$$

which estimates the first derivative using the average rate of change between the surrounding points. The second-order central difference is defined as

$$\delta^2 f(x_r) = \frac{f(x_{r+1}) - 2f(x_r) + f(x_{r-1})}{h^2},$$

providing an approximation for the second derivative. These expressions assume a smooth function and evenly spaced points. In general, higher-order central differences can be constructed by extending the stencil symmetrically, although their derivation requires more elaborate coefficients.

A central difference table, unlike the strictly lower or upper triangular arrangements of forward or backward difference tables, centers calculations around the middle data points. It may be represented as:

| $x_r$ | $f(x_r)$ | $\delta f(x_r)$ | $\delta^2 f(x_r)$ |
|---|---|---|---|
| $x_0$ | $f(x_0)$ | | |
| $x_1$ | $f(x_1)$ | $\delta f(x_1)$ | |
| $x_2$ | $f(x_2)$ | $\delta f(x_2)$ | $\delta^2 f(x_2)$ |
| $x_3$ | $f(x_3)$ | $\delta f(x_3)$ | $\delta^2 f(x_3)$ |
| $x_4$ | $f(x_4)$ | | |

Here, values for $\delta f(x_r)$ and $\delta^2 f(x_r)$ are only defined at internal points where the required surrounding function values exist. This operator is most effective in the central portion of a dataset and plays an important role in numerical differentiation, central interpolation methods, and finite difference schemes for solving differential equations.

**Algorithm 25** Central difference operator

**Require:** A set of tabulated values $(x_0, f_0), (x_1, f_1), \ldots, (x_n, f_n)$, where $f_i = f(x_i)$, and $x_i$ are either equally or unequally spaced

**Ensure:** Central differences $\delta f(x_i)$ for each interior point $x_i$, $i = 1, \ldots, n-1$

**Step 1: Define the central difference operator**

For interior points $x_i$, the first-order central difference is

$$\delta f(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}}$$

This uses data symmetrically from both sides of the point $x_i$.

**Step 2: Compute first-order central differences**

**for** $i \leftarrow 1 \; n-1$ **do**

  Compute the symmetric difference

$$\delta f(x_i) = \frac{f_{i+1} - f_{i-1}}{x_{i+1} - x_{i-1}}$$

  For equally spaced data: $h = x_{i+1} - x_i = x_i - x_{i-1}$, so

$$\delta f(x_i) = \frac{f_{i+1} - f_{i-1}}{2h}$$

**end for**

**Step 3: Handle boundaries (optional)**

Central differences are undefined at the endpoints $x_0$ and $x_n$, since both neighboring values are not available.

Use forward or backward differences instead

$$\delta f(x_0) \approx \frac{f_1 - f_0}{x_1 - x_0}, \quad \delta f(x_n) \approx \frac{f_n - f_{n-1}}{x_n - x_{n-1}}$$

**Step 4: Output results**

Return the central difference values $\delta f(x_1), \ldots, \delta f(x_{n-1})$

Optionally include first-order forward/backward differences at the boundaries for completeness.

***Example: Computing central differences***

Consider the following equally spaced data

| $x_i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f(x_i)$ | 3 | 6 | 11 | 18 | 27 |

**Step 1: Confirm uniform spacing**

We observe that

$$h = x_{i+1} - x_i = 1$$

Since the step size is constant, we proceed with the central difference formulas.

**Step 2: Compute first-order central differences**
For interior points $x_2, x_3, x_4$, the first-order central difference is given by

$$\delta f(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

$$\begin{aligned}
\delta f(x_2) &= \frac{f_3 - f_1}{2} = \frac{11 - 3}{2} = 4.0 \\
\delta f(x_3) &= \frac{f_4 - f_2}{2} = \frac{18 - 6}{2} = 6.0 \\
\delta f(x_4) &= \frac{f_5 - f_3}{2} = \frac{27 - 11}{2} = 8.0
\end{aligned}$$

**Step 3: Compute second-order central differences**
The second-order central difference uses

$$\delta^2 f(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

$$\begin{aligned}
\delta^2 f(x_3) &= \frac{f_4 - 2f_3 + f_2}{1} = 18 - 22 + 6 = 2.0 \\
\delta^2 f(x_4) &= \frac{f_5 - 2f_4 + f_3}{1} = 27 - 36 + 11 = 2.0
\end{aligned}$$

**Step 4: Construct the central difference table**

| $x_i$ | $f(x_i)$ | $\delta f(x_i)$ | $\delta^2 f(x_i)$ |
|---|---|---|---|
| 1 | 3 | | |
| 2 | 6 | 4.0 | |
| 3 | 11 | 6.0 | 2.0 |
| 4 | 18 | 8.0 | 2.0 |
| 5 | 27 | | |

**Conclusion**
The second-order central differences are constant, indicating that the function follows a *quadratic trend*. *Note:* Central differences cannot be applied at boundary points $x_1$ and $x_5$, where forward or backward differences should be used instead. ■

**Convergence rate and accuracy**
The central difference operator for the first derivative is a second-order accurate method, with an error term proportional to $h^2$. The formula:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

has a truncation error:

$$\text{Error} = \frac{h^2}{6} f^{(3)}(\xi), \quad \text{for some } \xi \in [x-h, x+h]$$

This is more accurate than both forward and backward difference approximations, which are first-order. The second derivative approximation also has second-order accuracy:

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Summary:

- Central differences provide higher accuracy for the same $h$ compared to forward/backward differences.
- Suitable for smooth functions with symmetric data around the point of interest.
- Require knowledge of function values on both sides of the point.

**Advantages**

- Higher-order accuracy with the same step size as forward/backward schemes.
- More symmetric and stable approximation of derivatives.
- Effective for smooth data and central regions of a dataset.

**Disadvantages**

- Cannot be applied at the boundaries (needs both left and right points).
- Sensitive to asymmetry or noise in data.
- More computational steps when applied to uneven or sparse datasets.

# 5 Tasks

**1.** Given the following data:

| $x$ | $y$ |
|---|---|
| 10 | 1.1 |
| 20 | 2.0 |
| 30 | 4.4 |
| 40 | 7.9 |

Construct the forward difference table for the data provided. Using the table, calculate $\Delta^2 f(30)$.

**2.** For the given data:

| $x$ | $f(x)$ |
|---|---|
| 0 | 1.0 |
| 1 | 1.5 |
| 2 | 2.2 |
| 3 | 3.1 |
| 4 | 4.6 |

- Construct the forward difference table for the function values.
- Evaluate $\Delta^3 f(2)$ using the difference table.

**3.** Given the sequence values:

$$u_0 = 3,\ u_1 = 12,\ u_2 = 81,\ u_3 = 2000,\ u_4 = 100$$

- Construct the forward difference table for the sequence values.
- Calculate $\Delta^4 u_0$ using the difference table.

**4.** For a given sequence $y_0, y_1, y_2, y_3, \ldots$, show the following identity holds:

$$\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$$

Provide the proof by constructing the necessary forward difference tables for different values of $i$.

**5.** Given the function $y = x^3 + x^2 - 2x + 1$, calculate the values of $y$ for $x = 0, 1, 2, 3, 4, 5$ from the difference table.

- Construct the forward difference table for the given function.
- Using the forward difference table, predict the value of $y$ at $x = 6$.
- Verify that the value obtained by extending the table matches the value obtained by direct substitution into the function.

**6** For the function $f(x) = x^3 + 5x - 7$, construct the forward difference table for $x = -1, 0, 1, 2, 3, 4, 5$.

- Complete the forward difference table for the data.
- Use the table to estimate $f(6)$ by extending the table.

**7.** Given the function $f(x) = 2^x + 3x$, construct the forward difference table for the values $x = 0, 1, 2, 3, 4, 5$.

- Construct the forward difference table for the function values.
- Extend the table to estimate the value of $f(6)$.

**8.** For the function $f(x) = e^x$, evaluate the central difference approximation of the derivative at $x = 1$.

- Construct a central difference table for $x = 0, 1, 2, 3, 4$.
- Use the central difference formula to approximate $\frac{df}{dx}$ at $x = 1$.

**9.** For the sequence $s_0 = 5, s_1 = 10, s_2 = 20, s_3 = 40, s_4 = 80$, calculate the following:

- Construct the forward difference table for the sequence.
- Calculate $\Delta^4 s_0$ and explain its significance.

**10.** Given the quadratic function $f(x) = 2x^2 + 3x + 1$:

- Construct the forward difference table for the values of $f(x)$ at $x = -2, -1, 0, 1, 2, 3$.
- Use the table to calculate $\Delta^2 f(0)$.
- Show that $\Delta^2 f(x)$ is constant for a quadratic function.