H_0 tension in Tsallis and Rényi statistics

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Motivated by recent attempts to study the implications of Tsallis and Rényi statistics in gravitational, cosmological, and astrophysical systems, the possible relationships between H_0 tension and generalized statistics are explored. It is obtained that, in the light of H_0 tension, the energy-time uncertainty relations in Tsallis and Rényi statistics constrain the values of Tsallis and Rényi parameters. Hence, the way to find the footprints of non-extensivity in the universe is paved.

I. INTRODUCTION

Despite the pre-expectations, Hubble parameter measurements unveil a meaningful and nonignorable difference between the reports obtained from the Cosmic Microwave Background (CMB) and Type Ia Supernova (SN) which are $H_{\mathrm{CMB}} = 67.66 \pm 0.42 \mathrm{Km/s/Mpc}$ [1] and $H_{\mathrm{SN}} = 74.03 \pm 1.42 \mathrm{Km/s/Mpc}$ [2], respectively. This controversial issue in the current value of the Hubble parameter is called the H_0 (Hubble) tension and has attracted a lot of attempts to itself and has raised the hopes of finding even new physics [3, 4]. Recently, it has been argued that the Heisenberg uncertainty principle (HUP), the energy-time uncertainty relation, and their constraints on the precision of observations may pave the way to eliminate the tension [5].

On the other hand, the quantum aspects of gravity, leading to generalized uncertainty principles (GUP) and relations, should be highlighted in the early universe and thus induce their footprint on CMB. Therefore, any CMB observation (and its accuracy) should be bounded by the generalized uncertainty relation instead of the energy-time uncertainty relation [6]. Motivated by this expectation and the dominant role of quantum mechanics in describing SN and the current universe, a permanent difference between H_{CMB} and H_{SN} is shown to be expected, its value depending on the values of the parameters of quantum gravity [6]. Indeed, the H_0 tension may be employed to bound the free parameters of quantum gravity [6, 7].

Ordinary statistical mechanics is based on the Shannon entropy whose special case is the Gibbs entropy. Therefore, extensive and additive systems are well described in this framework. Consequently, non-extensive systems such as those including long-range interactions such as gravity are proposed to be studied using generalized statistics like Tsallis and Rényi statistics providing generalized forms of Shannon entropy [8, 9]. On the other hand, the Bekesntein entropy is non-extensive, which strengthens the idea of using generalized statistics

to study gravitational systems [11–17]. In this regard, it has also been shown that quantum aspects of gravity inspire the Rényi entropy [18], Tsallis entropy [19], and, generally deviation from Gaussian statistics [20]. Correspondingly, the study of different cosmological, gravitational, and astrophysical systems in the frameworks of Tsallis and Rényi entropies has been intensified [11–17, 21–35] revealing at least theoretical abilities in eliminating various problems.

Having the entropies of Tsallis and Rényi in hand, modifications to HUP and energy-time uncertainty relation (ETUR) are obtained [21]. On the other hand, due to the addressed deep connection between (quantum) gravity and non-extensivity as well as the dominant role of quantum gravity in the early universe, such modified forms of HUP and ETUR may be used to find the relevant accuracies (instead of HUP and GUP) and perform measurements at least for focusing on CMB. Indeed, the main goal is to investigate whether this tension is related to non-extensivity. To achieve this aim, focusing on Tsallis and Rényi entropies and the corresponding ETURs, relationships between the H_0 tension and the values of non-extensive parameters are derived and analyzed in the second and third sections, respectively. A summary is also provided in the last section.

II. TSALLIS

The Tsallis entropy content (S_q^T) of a distribution including the W number of states is defined as [10, 11]

$$S_q^T = \frac{1}{1-q} \sum_{i=1}^{W} (P_i^q - P_i), \tag{1}$$

that reduces to $\frac{W^{1-q}-1}{1-q}$ when all states have the same probability [10, 11]. For black holes with area A, it leads to [22]

$$S_q^T = \frac{1}{1-q} [\exp\left(\frac{(1-q)A}{4}\right) - 1], \tag{2}$$

generating new Tsallis holographic dark energy and shedding light on the way towards Kaniadakis holographic

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dark energy, the considerable models to justify the current universe based on relating the vacuum energy of quantum fields to the dark energy [22]. The corresponding uncertainty principle and uncertainty relation are also obtained as [21]

$$\Delta x \Delta p \geq \frac{1}{2} \exp[-\frac{(\Delta x)^2}{\sigma^2}], \qquad (3)$$

and

$$\Delta t \Delta E \geq \frac{1}{2} \exp[-\frac{(\Delta t)^2}{\sigma^2}]. \qquad (4)$$

respectively. Here, $\sigma \equiv \sqrt{\frac{1}{\delta \pi}}$ where $\delta \equiv 1 - q$ and easily, HUP and ETUR are obtained when $q \to 1$ [21]. Based on Eq. (4), for the minimum energy (ΔE_m) extractable from a measurement during the time interval $\Delta t = \frac{1}{H}$, one finds

$$\Delta E_m \simeq \frac{H}{2} \exp[-\frac{1}{\sigma^2 H^2}]. \qquad (5)$$

Considering two independent observations done on SN and CMB so that the same ΔE_m is extracted during $\Delta t_{SN} = \frac{1}{H_{SN}}$ and $\Delta t_{CMB} = \frac{1}{H_{CMB}}$, then we will reach

$$q_{SN} \simeq 1 \cdot 2 \, q_{CMB} - 157 \cdot 24, \qquad (6)$$

as the relationship between the non-extensivity parameters of SN and CMB. Hence, measurements designed to detect ΔE_m from SN and CMB during time intervals H_{SN}^{-1} and H_{CMB}^{-1}, respectively, pave the way to understand the trace of Tsallis statistics. Ordinary statistical mechanics is based on Shannon entropy (Gibbs entropy is a special case of the Shannon entropy) and therefore, it satisfies the HUP bounds. Thus, since quantum mechanics and ordinary statistical mechanics seem sufficient to study SN [36], we have $q_{SN} = 1$ and thus $q_{CMB} \simeq 131 \cdot 03$.

There is also another expression for entropy introduced by Tsallis defined as [9, 11]

$$S_{q,\eta} = \sum_{i=1}^{W} P_i (\ln_q \frac{1}{P_i})^\eta, \qquad (7)$$

that leads to $(\ln_q W)^\eta$ for equal probabilities [11]. η is also a free parameter and the corresponding black hole entropy ($S_{1,\eta}$) takes the form [11]

$$S_{1,\eta} \propto A^\eta, \qquad (8)$$

generating Tsallis holographic dark energy [23]. For the corresponding uncertainty principle and uncertainty relation, we have [21]

$$\Delta x \Delta p \geq \frac{(\Delta x)^{2-2\eta}}{2}, \qquad (9)$$

$$\Delta t \Delta E \geq \frac{(\Delta t)^{2-2\eta}}{2},$$

that recovers HUP and ETUR at the desired limit $\eta = 1$ [21], and finally yields

$$\frac{\ln H_{SN}}{\ln H_{CMB}} = 1 \cdot 02 \simeq \frac{2\eta_{CMB} - 1}{2\eta_{SN} - 1}, \qquad (10)$$

as a relationship used to estimate the values of η_{CMB} and η_{SN} in terms of each other. Similar to Eq. (6), it is obvious that there will be a permanent difference between H_{SN} and H_{CMB} provided they obey Eq. (7) with distinct values of η. Again, if quantum mechanics and ordinary statistics (based on Gibbs entropy) are enough to survey SN and the related measurements [5, 6, 36], then we face the HUP or equally $\eta_{SN} = 1$ meaning that $\eta_{CMB} \simeq 1 \cdot 01$.

III. RÉNYI FRAMEWORK

The Rényi entropy is defined as [8, 10]

$$\mathcal{S} = \frac{1}{1-q} \ln \sum_{i=1}^{W} P_i^q, \qquad (11)$$

forming the backbone of Rényi holographic dark energy model [24], and it is inspired by the quantum aspects of gravity [18]. For the Rényi energy uncertainty relations, we have [18]

$$\Delta t \Delta E \geq \frac{1}{2}\big(1 + (1-q)\frac{\pi}{4}(\Delta t)^2\big), \qquad (12)$$

that finally leads to

$$(H_{SN} - H_{CMB})(\frac{4}{\pi} - \frac{1}{H_{CMB} H_{SN}}) \simeq \frac{q_{SN}}{H_{SN}} - \frac{q_{CMB}}{H_{CMB}},$$
$$\Rightarrow q_{SN} > \frac{H_{SN}}{H_{CMB}} q_{CMB}, \qquad (13)$$

where the last line is the result of $H_{SN} > H_{CMB}$. Moreover, since $\frac{1}{H_{CMB} H_{SN}} \ll 1$, one may be leaded to $H_{SN} - H_{CMB} \approx \frac{q_{SN}}{H_{SN}} - \frac{q_{CMB}}{H_{CMB}}$. On one hand, the quantum aspects of the early universe are supposed to be stored in CMB and indeed, the physics supporting CMB and SN are different [4]. Moreover, the Rényi entropy seems to be supported by the quantum aspects of gravity [18]. Therefore, even if we completely analyze SN using HUP (or equally, $q_{SN} = 0$ since SN is a phenomenon well described by quantum mechanics and ordinary statistical mechanics based on Shannon entropy [36]), then we are still motivated to use Eq. (12) to focus on CMB meaning that we will have $q_{SN} = 0$ and hence $q_{CMB} \approx H_{CMB}(H_{CMB} - H_{SN}) < 0$.

IV. SUMMARY AND CONCLUDING REMARKS

Recently, motivated by the long-range nature of gravity, the usage of generalized statistics to study gravity and the related phenomena has been proposed leading to interesting results [11, 22–35]. Moreover, it seems that the quantum aspects of gravity, if there are any, as well as the non-extensivity of the Bekenstein entropy strengthen the motivation, especially in the case of Tsallis and Rényi formalisms [12–18, 20]. Therefore, the uncertainty principles and relations allowed in the Tsallis and Rényi statistics seem reasonable for studying the SN and especially the CMB observations. On the other hand, finding out the reason for the discrepancy between the current values of the Hubble parameter measured using the SN and CMB data is a challenge in contemporary physics that may be a signal for new physics and perspectives on understanding gravity [4].

Equipped with them, it is tried to show that the non-extensive aspects of CMB and SN may be traced using the related observations. The idea is based on extracting the same amount of energy during observations. The key outcomes are the bridges between the values of non-extensivity parameters of CMB and SN in different statistics. Therefore, if quantum mechanics and ordinary statistical mechanics (based on Shannon entropy whose special case is the Gibbs entropy) are enough to analyze SN, then the survey helps us estimate the values of the non-extensivity parameter of CMB in each statistics. Finally, it should be noted if the non-extensivity of gravity is proved, a permanent difference between H_{CMB} and H_{SN} is expected unless the values of the non-extensivity parameters of CMB and SN are the same.

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