

Black hole surrounded by a magnetic vortex in $f(R)$ gravity

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Abstract: Modified Gravity Theories (MGTs) are extensions of General Relativity (GR) in its standard formulation. In light of this premise, we employ Heisenberg's non-perturbative approach to promote the three-dimensional Einstein-Hilbert theory to a modified $f(R)$ gravity. Within this context, we investigate a cosmological system composed of a black hole (BH) surrounded by Maxwell-Higgs vortices, forming the BH-vortex system. In the case of linear $f(R)$ gravity derived from quantum metric fluctuations, one shows the existence of a three-dimensional ring-like BH-vortex system with quantized magnetic flux. Within this system, one notes the BH at $r = 0$ and its event horizon at $r = r_0$, while the magnetic vortices are at $r \in (r_0, \infty)$. A remarkable result is the constancy of the Bekenstein-Hawking temperature (T_H), regardless of MGTs and vortex parameters. This invariance of T_H suggests that the BH-vortex system reaches thermodynamic stability. Unlike the standard theory of Maxwell-Higgs vortices in flat spacetime, in $f(R)$ gravity, the vortices suffer the influence of the BH's event horizon. This interaction induces perturbations in the magnetic vortex profile, forming cosmological ring-like magnetic structures.

Keywords: Modified Gravity Theory. Metric Fluctuation. Black Holes. Magnetic Vortices.

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I. INTRODUCTION

Recently, we noted a growing interest in Modified Gravity Theories (MGTs). Generally speaking, these theories arise from the premise that the universe is suffering an accelerated expansion [1, 2], which can be explained through corrections to the equations of motion in General Relativity (GR) [3]. Initially, one explanation for this universal acceleration is the hypothesis that additional contributions from the scalar curvature may exist in the standard formulation of GR, leading to MGTs, e.g., $f(R)$ theory [4–7]. Specifically, one of the motivations for $f(R)$ gravity is the study of possible new gravitational effects related to late-time cosmic expansion, see Refs. [8–13]. These remarks have provided extensive studies on MGTs as promising frameworks for addressing emerging challenges to standard GR [14]. Among these theories are $f(R)$ [15–17] and its variants as $f(R, T)$ gravity [18–21], models involving couplings between curvature and matter [22], and proposals based on modified geometries [23, 24]. Thus, this jungle of MGTs has found significant applications in cosmological and astrophysical contexts [25–27].

In the framework of MGTs, a fundamental question arises: which of these theories is most appropriate for describing systems composed of gravitating compact objects? Traditionally, one chooses the MGT by basing on constraints in weak-field regimes derived from classical tests of GR applied to models similar to the solar system [28–32]. Thus, we propose an alternative approach that departs from this conventional perspective. Our goal is to determine the most suitable gravity theory for studying a system composed of a black hole and magnetic vortex (BH-vortex). Towards this purpose, one adopts an approach based on quantum fluctuations of the metric within the Einstein-Hilbert theory to identify the MGT capable of adequately describing this system. Summarizing the quantum fluctuation approach considers a metric that incorporates classical ($g^{\mu\nu}$) and quantum ($\delta\hat{g}^{\mu\nu}$) contributions [33–35]. This approach offers significant advantages as it naturally introduces modifications to Einstein’s gravity, allowing, among other aspects, the emergence of non-minimal couplings between gravity and matter. By adopting this methodology, we assume that the expected value of the metric satisfies certain physical conditions, such as the non-vanishing of the average quantum contribution and the ability to express the metric $g^{\mu\nu}$ in terms of a second-rank symmetric tensor [33–35]. These assumptions lead to a theoretical scenario in which the modified gravity of the $f(R)$ emerges as a fundamental structure for

describing BH-vortex systems. Grounded in this perspective, our objective in this work is to analyze a self-gravitating three-dimensional BH-vortex system in $f(R)$ gravity. Fig. 1 illustrates the system under consideration.



Figure 1: Illustration of the BH-vortex system.

In several contexts, one adopts widely the MGTs [36–39]; however, one knows little about the self-gravitating topological structures¹. A significant class of topological structures are the topological vortices, which emerge from a complex scalar field theory coupled to an Abelian gauge field [40–43]. Initially, Higgs used a massive scalar field in the mechanism that would later bear his name [44]. Subsequently, the investigation of models coupled to an Abelian gauge field was introduced by Nielsen and Olesen [45], adopting a flat spacetime. Generally speaking, vortex models preserve the Higgs mechanism and gauge invariance and are particularly relevant once these models are relativistic generalizations of the phenomenological Ginzburg-Landau theory for superconductivity [46]. Thus, the following question naturally arises: would it be possible for a BH-vortex system to emerge in an MGT? We aim to address this question in the development of this article.

Magnetic vortices arise in condensed matter physics [47, 48]; however, their application

¹ Topological structures are nonlinear field configurations with stability due to topological properties. These objects arise particularly in field theory and particle physics. Generally speaking, one classifies these structures as kinks, vortices, and monopoles [40–43].

has expanded to describe vortex lines in cosmic strings [45, 48]. Recently, vortices have gained increasing interest due to their reinterpretation within a cosmological framework, particularly in the BHs description [49, 50]. In this context, vortices are intrinsically related to the BH dynamic, enabling the emission or capture of magnetic matter fluxes [50]. The study of the BH-vortex system can thus provide a topological interpretation for the stability of BHs and their implications in Hawking radiation [49]. Furthermore, one performs several studies on BHs due to their relevance in contemporary physics [51] and experimental findings [52]. Thus, investigations on these objects have grown significantly, see Refs. [53–56]. In simple terms, BHs arise as solutions from the Einstein field equations [57]. The first BH solution was announced by K. Schwarzschild [58], describing static and spherical BH. Afterward, Oppenheimer and Snyder showed that Schwarzschild BH solution represents the final state of the collapse of a massive and spherical star [59].

Although widely studied the non-perturbative vortex solutions, the investigation of BH-vortex cosmological systems remains largely unexplored. That is because the description of vortices in a flat spacetime presents a high level of complexity in their description. Additionally, from a gravitational perspective, Einstein’s gravity is considered trivial, i.e., outside localized sources, spacetime in the vacuum is locally flat [60, 61]. Thus, meaningful discussions have emerged over time. Among these studies, the BTZ BH solution in an AdS_3 background highlights an interesting topic in standard Einstein gravity within a three-dimensional spacetime [62, 63]. Besides, Cardoni et al. [64] examined and announced the solutions for spherical scalar hair black holes in Einstein gravity. Naturally, one adopts a real scalar and complex scalar field with interaction by allowing the emergence of topological vortex in BHs within Einstein’s gravity. Taking that into account, we announce for the first time the study of a BH-vortex system (Fig. 1) in the MGTs framework.

We organized the manuscript into four distinct sections. We begin applying the Heisenberg formalism to the BH-vortex system (Sec. II), providing the most suitable MGT framework. This analysis leads to an $f(R)$ theory to study the BH-vortex system. Subsequently, in Sec. III, one explores the BH-vortex system within the $f(R)$ gravity. One divides this section into three subsections. We begin by outlining the general framework for generating the BH-vortex system; then, we investigate the BH solution and Hawking radiation in $f(R)$ gravity. Finally, we investigate the vortices surrounding the BH. In Sec. IV, one announces our main findings.

II. HEISENBERG'S APPROACH IN A GRAVITATIONAL BACKGROUND

Within the framework of this work, we will consider a quantization formalism in a three-dimensional gravitational background. In this framework, let us treat the metric as an ordinary field, and we will employ Heisenberg's non-perturbative approach. This approach requires promoting the quantities $\Gamma^\rho_{\mu\nu}$, $R^\rho_{\lambda\mu\nu}$, and $R_{\mu\nu}$ to operators, i.e., $\hat{\Gamma}^\rho_{\mu\nu}$, $\hat{R}^\rho_{\lambda\mu\nu}$, and $\hat{R}_{\mu\nu}$, respectively. Inevitably, this assumption allows us to reformulate Einstein's equation as

$$\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{R} = \kappa^2\hat{T}_{\mu\nu}. \quad (1)$$

It is relevant to highlight that the definitions of these quantities remain consistent even when promoted to operators. Hence, the quantities $\hat{\Gamma}^\rho_{\mu\nu}$, $\hat{R}^\rho_{\lambda\mu\nu}$, and $\hat{R}_{\mu\nu}$ are defined as natural extensions of their classical counterparts to the quantum domain, i.e.,

$$\hat{R}_{\mu\nu} = \hat{R}^\rho_{\mu\rho\nu}, \quad (2)$$

$$\hat{R}^\rho_{\sigma\mu\nu} = \frac{\partial\hat{\Gamma}^\rho_{\sigma\nu}}{\partial x^\mu} - \frac{\hat{\Gamma}^\rho_{\sigma\mu}}{\partial x^\nu} + \hat{\Gamma}^\rho_{\tau\mu}\hat{\Gamma}^\tau_{\sigma\nu} - \hat{\Gamma}^\rho_{\tau\nu}\hat{\Gamma}^\tau_{\sigma\mu}, \quad (3)$$

$$\hat{\Gamma}^\rho_{\mu\nu} = \frac{1}{2}\hat{g}^{\rho\sigma}\left(\frac{\partial\hat{g}_{\mu\sigma}}{\partial x^\nu} + \frac{\partial\hat{g}_{\nu\sigma}}{\partial x^\mu} - \frac{\partial\hat{g}_{\mu\nu}}{\partial x^\sigma}\right). \quad (4)$$

Adopting Heisenberg's approach, one arrives at an infinite set of equations governing all Green's functions. We express these equations as

$$\begin{aligned} \langle \mathcal{Q} | \hat{g}(x_1) \cdot (\hat{G}_{\mu\nu} - \kappa^2\hat{T}_{\mu\nu}) | \mathcal{Q} \rangle &= 0; \\ \langle \mathcal{Q} | \hat{g}(x_1) \cdot \hat{g}(x_2) \cdot (\hat{G}_{\mu\nu} - \kappa^2\hat{T}_{\mu\nu}) | \mathcal{Q} \rangle &= 0; \\ &\vdots \\ \langle \mathcal{Q} | \hat{g}(x_1) \cdot \dots \cdot \hat{g}(x_i) \cdot (\hat{G}_{\mu\nu} - \kappa^2\hat{T}_{\mu\nu}) | \mathcal{Q} \rangle &= 0, \end{aligned} \quad (5)$$

where $|\mathcal{Q}\rangle$ is a quantum eigenstate².

To analyze the system of equations (5), we decompose the metric operator $\hat{g}_{\mu\nu}$ into classical and quantum parts, assuming the premise that the expected value of the quantum part $\langle \hat{g}_{\mu\nu} \rangle$ non-trivial [33–35, 67]. By doing so, we compute the expected value of the

² A quantum eigenstate is a concept arising from quantum mechanics, referring to a state in which an observable has a well-defined value [65, 66]. Mathematically, one characterizes a quantum eigenstate by three essential properties: orthogonality, completeness, and Hermiticity [65, 66].

Lagrangian to first-order corrections in $\delta\hat{g}^{\mu\nu}$ for a three-dimensional Einstein-Hilbert-like action, viz.,

$$S = \int d^3x \sqrt{-g} \left[-\frac{1}{16\pi} R + \mathcal{L}_{\text{matter}} \right]. \quad (6)$$

Within this structure, $\mathcal{L}_{\text{matter}}$ is the Matter Lagrangian density.

A. On the decomposition of the metric

Hereafter, let us assume that the quantum metric $\mathcal{G}_{\mu\nu}$ has classical and quantum contributions, i.e.,

$$\mathcal{G}_{\mu\nu} = g_{\mu\nu} + \hat{g}_{\mu\nu}, \quad (7)$$

whose quantum contribution is non-trivial, namely,

$$\langle \hat{g}_{\mu\nu} \rangle \neq 0. \quad (8)$$

These assumptions lead to quantum corrections in the Einstein-Hilbert theory [33–35, 67].

Considering first-order corrections, one obtains

$$S(g + \hat{g}) = \int d^3x \left[\mathcal{L}_g(g) + \frac{\delta \mathcal{L}_g}{\delta g^{\mu\nu}} \hat{g}_{\mu\nu} + \sqrt{-g} \mathcal{L}_{\text{matter}}(g) + \frac{\delta \sqrt{-g} \mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}} \hat{g}^{\mu\nu} \right], \quad (9)$$

where

$$\mathcal{L}_g(g) = -\frac{1}{16\pi} \sqrt{-g} R, \quad (10)$$

i.e., \mathcal{L}_g is the classical Einstein-Hilbert Lagrangian density.

Therefore, the Einstein-Hilbert theory (9), corrected with first-order quantum fluctuation terms of the metric, results in the effective theory

$$S = \int d^3x \left[\mathcal{L}_g - \sqrt{-g} \mathcal{G}_{\mu\nu} K^{\mu\nu} + \frac{1}{2} \sqrt{-g} T_{\mu\nu} K^{\mu\nu} + \sqrt{-g} \mathcal{L}_{\text{matter}} \right]. \quad (11)$$

From this presumption, one defines the expectation value of the metric as $K_{\mu\nu} = \langle \hat{g}^{\mu\nu} \rangle$. Therefore, $K^{\mu\nu}$ is a tensor derived from the quantum contribution of the metric³ [67]. Thus, its choice should ensure it is sufficiently small.

³ $K_{\mu\nu}$ is a symmetric second-rank tensor ($K_{\mu\nu} = K_{\nu\mu}$) and non-trivial.

B. The case $K^{\mu\nu} = \langle \hat{g}^{\mu\nu} \rangle = \alpha g^{\mu\nu}$

Naturally, the simplest case that satisfies the criteria for the expectation value of the metric is $K^{\mu\nu} = \alpha g^{\mu\nu}$, where $|\alpha| \ll 1$ [68]. Consequently, in light of this criteria, the minimally coupled Einstein-Hilbert action to the matter sector is corrected by quantum metric fluctuations, leading to the action

$$S = \int d^3x \left[\mathcal{L}_g - \frac{1}{16\pi} \sqrt{-g} f(R) + \frac{1}{2} \sqrt{-g} T^\mu{}_\mu + \sqrt{-g} \mathcal{L}_{\text{matter}} \right]. \quad (12)$$

We are interested in the emergence of a BH-vortex system in this work. Therefore, it is necessary to implement the condition $T^\mu{}_\mu = 0$ ⁴ to study the BH-vortex system [50]. Thereby, only quantum metric fluctuations will introduce quantum corrections to the geometric part of the theory. In this case, the corrected Einstein-Hilbert action will be

$$S = \int d^3x \sqrt{-g} \left[-\frac{1}{16\pi} [R + f(R)] + \mathcal{L}_{\text{matter}} \right]. \quad (13)$$

Therefore, we obtain a modified $f(R)$ gravity. In this context, one notes that MGT naturally emerges as a consequence of the quantum fluctuations of the metric.

III. THE BH-VORTEX SYSTEM IN MODIFIED $f(R)$ GRAVITY

From this stage onward, let us explore the BH and magnetic vortex solutions in modified $f(R)$ gravity. This study is particularly relevant and helps us understand the interaction between the magnetic vortex and the BH. Although this research branch is still relatively unexplored, this study provides us valuable insights into the physical aspects of the BH-vortex system in MGT.

A. The overall framework of the BH-Vortex system

MGTs are extensively studied, e.g., see Refs. [69–72]. Specifically, one employs the gravity $f(R)$ theories with coupling matter fields by adopting various dimensions to examine static and spherically symmetric field solutions [73, 74]. This approach also allows us to

⁴ This constraint ensures energy and moment contributions only at regions where the structures are localized.

understand how MGTs, defined as functions of the Ricci scalar (R), influence gravitating matter. Moreover, as previously mentioned, the Einstein-Hilbert theory corrected by quantum metric fluctuations takes the form presented in Eq. (13), which leads us to consider the Maxwell-Higgs action in $f(R)$ gravity, viz.,

$$S = \int d^3x \sqrt{-g} \left[-\frac{1}{16\pi}[R + f(R)] + \frac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{4}(|\phi|^2 - \nu^2)^2 \right], \quad (14)$$

with R the Ricci scalar, ϕ a complex scalar field, and $F_{\mu\nu}$ the electromagnetic field tensor, defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is the gauge field. The parameter ν denotes the Vacuum Expectation Value (VEV). Furthermore, the notation $D_\mu\phi$ is the covariant derivative, which minimally couples the matter field ϕ to the gauge field A_μ . One defines the covariant derivative as

$$D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi. \quad (15)$$

Here, e is a minimal coupling between the gauge and matter fields⁵.

To obtain static and topological structures in three-dimensional spacetime [50, 60], let us adopt the metric

$$ds^2 = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2d\theta^2, \quad (16)$$

where $h(r)$ is the metric function, r and θ are, respectively, the radial and angular variables⁶.

Magnetic topological vortices are gauge-invariant and rotationally symmetric structures [40–43]. Thus, one defines the gauge field as

$$\mathbf{A}_i(\mathbf{r}) = -\varepsilon_{ij}\hat{x}^j\frac{a(r)}{er}, \quad (17)$$

which leads us to structures with magnetic flux (Φ_B) determined by

$$\Phi_B = \iint_S \mathbf{B} \cdot d\mathbf{S} = -\frac{2\pi}{e} \int_0^\infty a'(r)dr = \frac{2\pi}{e}[a(0) - a(\infty)]. \quad (18)$$

Let us concentrate on the study of topological vortices. To fulfill this purpose, we will adopt the topological conditions [40–42]

$$a(0) = 0 \quad \text{and} \quad a(\infty) = -\beta \quad \text{with} \quad \beta \in \mathbb{Z}^+. \quad (19)$$

⁵ For simplicity, let us adopt the natural units system, which leads us to assume $\hbar = c = e = 1$.

⁶ For more details on this metric, see Refs. [50, 60]. In Refs. [50, 60], the authors used this metric to examine the vortex solution in Einstein's gravity.

Assuming the topological condition announced in Eq. (19), one obtains magnetic vortices with magnetic flux given by

$$\Phi_B = \frac{2\pi\beta}{e}. \quad (20)$$

Accordingly, one concludes that the topological vortices radiate a quantized magnetic flux.

For the existence of magnetic vortices, the $U(1)$ symmetry must, a priori, be omnipresent. Thus, as outlined in prior research [40–42, 45], let us adopt the *ansatz*

$$\phi(r, \theta) = g(r)e^{in\theta}, \quad (21)$$

where $n \in \mathbb{Z}^+$. Furthermore, n is the winding number⁷ concerning the vortex [76–79]. The functions $g(r)$ and $a(r)$ are, respectively, the field variables for the matter (ϕ) and gauge (A_μ) sector.

Now, allow us to investigate the BH-vortex system and examine the resulting metric signature in $f(R)$ gravity. To attain our target, we will consider the equation of motion derived from varying the action (14) concerning the metric. This variation leads to

$$R_{\mu\nu}(1 + f_R) - \frac{1}{2}g^{\mu\nu}[R + f(R)] + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla_\beta\nabla^\beta)f_R = T_{\mu\nu}, \quad (22)$$

where f_R is the total derivative of the function $f(R)$ concerning R .

Additionally, one defines the stress-energy tensor⁸ as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \quad (23)$$

B. The BH solution, Hawking radiation, and $f(R)$ gravity

To determine the profile of metric functions in gravity $f(R)$ theory, we will focus our study on cases where the effects of the presence of vortices are minimally perceptible. In other words, let us analyze regions outside the BH-Vortice system, i.e., sectors where the effects

⁷ The winding number is a quantity concerning the complex scalar field associated with the vortex. Supposing a vortex at $r = \bar{r}_0$, one can interpret the winding number as the amount of turns the function (or field) wraps around the vortex (at $r = \bar{r}_0$) along a closed trajectory. Therefore, the winding number plays a significant role, allowing us to understand the topological aspects of the vortex, e.g., the magnetic field [75].

⁸ This quantity will be essential in the following sections to examine the BH-vortex metric and the system's energy.

are negligible. In this regime, one can assume $T_{\mu\mu} = 0$ [Eq. (23)]. This simplification allows us to obtain using Eq. (22), three independent differential equations, corresponding to the components tt , rr , and $\theta\theta$, i.e.,

$$2rh(r)f_R'' + [rh'(r) + 2h(r)]f_R' + [rh''(r) + h'(r)]f_R - h'(r) = -rf(R); \quad (24)$$

$$[rh'(r) + 2h(r)]f_R' + [rh''(r) + h'(r)]f_R - h'(r) = -rf(R); \quad (25)$$

$$2rh(r)f_R'' + 2rh'(r)f_R' + 2h'(r)f_R - rh''(r) = -rf(R), \quad (26)$$

which leads us to

$$2rh'(r)f_R' + 2h'(r)f_R - rh''(r) + rf(R) = 0. \quad (27)$$

Additionally, through algebraic manipulations of Eqs. (24) and (25), we concluded that $f_R'' = 0$. It is important to highlight that the prime notation is the derivative concerning the radial variable r .

1. The case $f(R) = \alpha R$

Now, let us assume the case in which the metric satisfies the condition that the effects of the structure are minimally perceptible outside the BH-vortex system. Seen from this standpoint, the modified gravity must take the form $f(R) = \alpha R$, which allows for expressing

$$R = -\frac{2h'(r)}{r} - h''(r) \quad \text{and} \quad f(R) = -\alpha \left[\frac{2h'(r)}{r} + h''(r) \right] \quad (28)$$

which leads us to

$$(1 + \alpha)rh''(r) = 0 \quad \text{with} \quad \alpha \neq 1. \quad (29)$$

Consequently, Eq. (29) provides us with the solution

$$h(r) = r \mp r_0, \quad (30)$$

for all $\alpha \ll 1$.

Our purpose is to obtain BH-like compact objects. In pursuit of this, we consider only physically admissible solutions [Eq. (30)] from the theory, i.e., $h(r) = r - r_0$. Indeed, the function $h(r) = r - r_0$ represents a simple and linear form of a curvilinear coordinate function in three-dimensional spacetime. In this framework, one can interpret the parameter

r_0 as the BH mass. Thus, this function constitutes a modification of the radial term of the Schwarzschild-like metric. By analogy with the Schwarzschild-like BH, this metric function profile implies that the temporal term assumes negative values for $r < r_0$, suggesting the existence of an event horizon at $r = r_0$. Furthermore, in the limit $r \rightarrow r_0$, the temporal contribution vanishes, ensuring an essential feature of an event horizon [80].

Once we have the emergence of a BH-like compact object, it becomes necessary to evaluate the stability of this cosmological object. For this assessment, let us adopt the metric function solution $h(r) = r - r_0$ [Eq. (30)] and investigate the quadratic invariant of the Ricci tensor and the Kretschmann scalar. Through direct calculation for the metric (16), the quadratic invariant of the Ricci tensor is

$$R^{\mu\nu} R_{\mu\nu} = \frac{1}{2r^2} h'(r)[h'(r) + rh''(r)] = \frac{1}{2r^2}. \quad (31)$$

Additionally, one defines the Kretschmann scalar (K) by the quadratic invariant of the Riemann curvature tensor ($R_{\mu\nu\tau\sigma}$). Thus, for our conjecture, the Kretschmann scalar is

$$K = R^{\mu\nu\tau\sigma} R_{\mu\nu\tau\sigma} = \frac{h'(r)^2}{r^2} = \frac{1}{r^2}. \quad (32)$$

Analyzing the Kretschmann scalar (see Fig. 2), we noted the existence of a singularity at the origin. Asymptotically, the gravitational field approaches zero curvature, indicating that the BH (and the vortex) does not affect the curvature. Besides, the results presented in Fig. 2 ensure a BH at $r = 0$.

Considering the metric function profile described in Eq. (30), one concludes that the $f[R(r)]$ gravity produced by the correction is

$$f[R(r)] = -\frac{2\alpha}{r}. \quad (33)$$

2. On the Bekenstein-Hawking temperature

In our system, we have a three-dimensional BH emitting Hawking radiation. Thus, it is essential to study this radiation. To fulfill this, let us use the Hamilton-Jacobi formalism through the tunneling approach to examine the thermodynamics concerning the BH [81–85]. This approach is fruitful once it allows us to calculate the Hawking temperature for the black hole surrounded by magnetic vortices, using the metric function presented in Eq. (30).

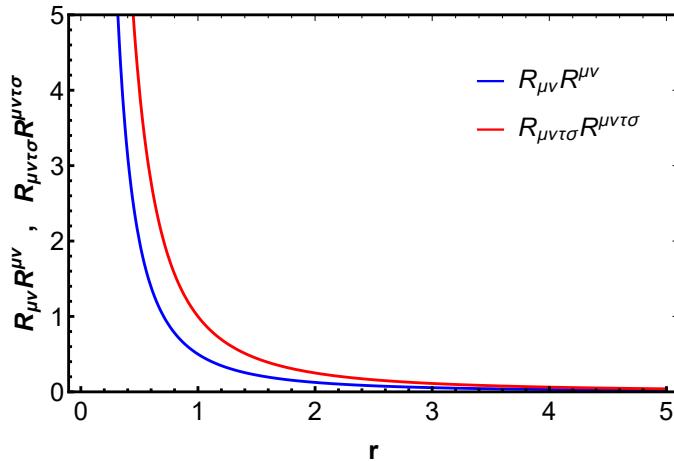


Figure 2: Quadratic invariant of the Ricci tensor (blue curve) and the Kretschmann scalar (red curve) versus radial distance.

The fundamental conception of the tunneling method involves calculating the probability of particles interacting near the event horizon and escaping through a quantum tunneling process. This interpretation becomes feasible when Hawking radiation is understood as an intrinsic emission process of the black hole. In our system, one can interpret this process as an emission resulting from the spontaneous creation of particle pairs near the event horizon. Thus, this emission is driven by the matter sector, which undergoes decay into the black hole. Consequently, particles with negative energy remain inside the black hole, contributing to its mass decrease. At the same time, particles with positive energy escape the horizon through a “tunnel” toward infinity, influencing the energetic changes of the vortex. These changes perturb the massive scalar field solutions and modify the matter sector near the event horizon.

We are ready to study the tunneling probability related to the BH temperature. A significant advantage of this approach for studying the thermodynamics of BH is that thermodynamic properties are intrinsically bounded to the geometry, allowing for its broad application to various spacetime manifolds [86–89]. To perform this investigation, one knows that, near the BH event horizon, only the temporal and radial terms of the metric remain relevant, while the angular part is redshifted [90]. Thus, for our metric (16), one obtains

$$ds^2 = -(r - r_0)dt^2 + \frac{1}{r - r_0}dr^2. \quad (34)$$

Let us apply a perturbation to the massive scalar sector ϕ around the BH background. That allows us to obtain

$$\nabla^\mu \nabla_\mu \phi - m^2 \phi = 0. \quad (35)$$

This expression corresponds to the Klein-Gordon equation in natural unit system [91], where m is the mass of the field ϕ . By considering the decomposition of the Klein-Gordon equation (35) into spherical harmonics, we come to

$$-\frac{\partial^2 \phi}{\partial t^2} + (r - r_0)^2 \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{2} \frac{\partial}{\partial r} (r - r_0)^2 \frac{\partial \phi}{\partial r} - m^2 (r - r_0) \phi = 0. \quad (36)$$

Assuming that the particle creation process in the BH is semiclassical in the BH background, the WKB approximation [66] informs us that the *ansatz* for the field ϕ [81–85] is

$$\phi(t, r) = e^{\Theta(t, r)}, \quad (37)$$

which leads us to

$$\left(\frac{\partial \Theta}{\partial t} \right)^2 - (r - r_0)^2 \left(\frac{\partial \Theta}{\partial r} \right)^2 - m^2 (r - r_0) = 0, \quad (38)$$

such that the particle-like solutions are

$$\Theta(t, r) = -\omega t + W(r) \quad (39)$$

In this scenario, ω is a constant representing the energy of the emitted radiation [50].

By substituting Eq. (39) into Eq. (38), boil down to the function $W(r)$, i.e.,

$$W(r) = \pm \int \frac{\sqrt{\omega^2 - m^2(\bar{r} - r_0)}}{\bar{r} - r_0} d\bar{r}. \quad (40)$$

The positive sign is associated with outgoing particles, while the negative solution corresponds to incoming particles. Hence, we will focus on the outgoing particles responsible for emitting radiation as they cross the event horizon. Thus, solving Eq. (40) and considering only the outgoing particles (i.e., the positive sign), one concludes that

$$W(r) = 2\pi i\omega + (\text{real contribution}), \quad (41)$$

As a result, the probability of a particle escaping the BH via the tunneling process is

$$\Gamma \sim \exp(-2im\Theta) = \exp(-4\pi\omega). \quad (42)$$

Recalling that the tunneling probability (42) is related to the Boltzmann factor $\exp(-\omega/T)$ [92], it follows that the Hawking temperature will be

$$T_H = \frac{\omega}{2im\Theta} = \frac{1}{4\pi} = 0.0795. \quad (43)$$

Therefore, one concludes that for the BH with an event horizon at $r = r_0$, the Bekenstein-Hawking temperature (T_H) remains constant, independent of the $f(R)$ gravity (α) parameter or the vortex (λ and ν). This constancy in T_H suggests that the BH-vortex system is stable and maintains thermodynamically stable.

C. The topological vortex

We will focus on the study of magnetic vortices. To accomplish this, let us analyze the equations of motion within the context of modified $f(R)$ gravity (33), taking into account the quantum correction to the metric (13). In this framework, one notes that the equations of motion derived through the variation of the action concerning the scalar field, the gauge field, and the metric are, respectively,

$$r(r - r_0)g''(r) + (2r - r_0)g'(r) + \lambda(g^2 - \nu^2)g(r) - [n - a(r)]^2g(r) = 0; \quad (44)$$

$$-r(r - r_0)a''(r) - r_0a'(r) + r[n - a(r)]g(r)^2 = 0; \quad (45)$$

$$g'(r) = \frac{1}{r}a'(r). \quad (46)$$

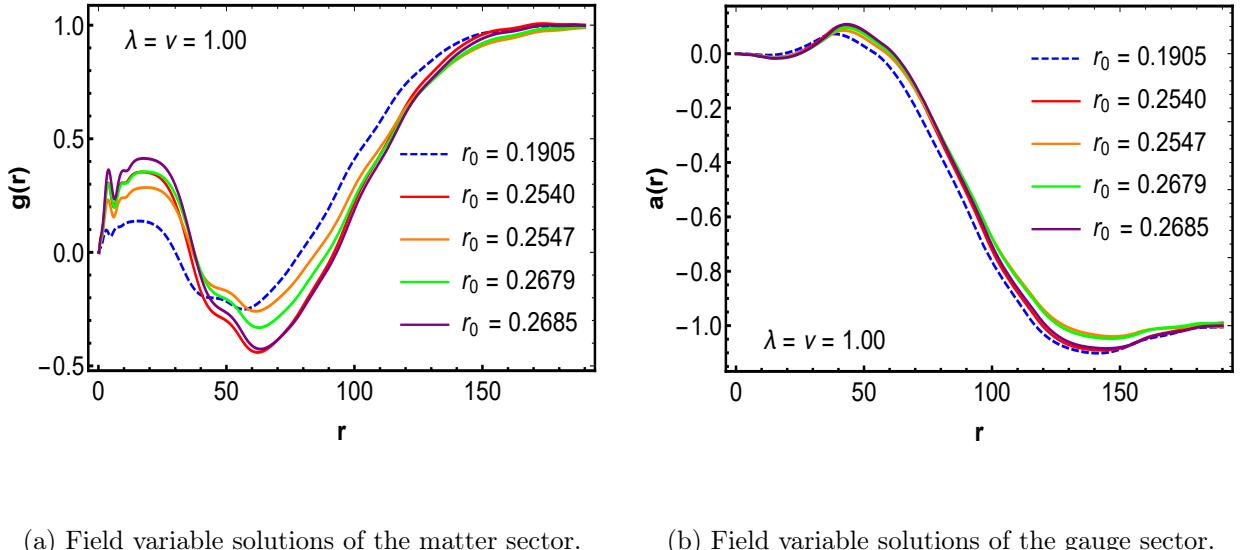
Note that the set of linearly independent equations [(44)-(46)] is also invariant concerning the parameter α . This behavior arises because cosmological magnetic vortices are classical objects not influenced by small quantum fluctuations originating from the quantum correction to the metric. Conversely, these vortices are significantly affected by the BH radius. That occurs because, as the event horizon expands, the vortex matter begins to collapse into the BH, causing increasing distortions in the magnetic vortex. We expose a representation of the collapse of the magnetic vortex matter into the BH in Fig. 1.

By adopting the equations of motion that describe the topological vortices [Eqs. (44)-(46)], we are ready to investigate the numerical solution of these vortices, taking into account the topological conditions expressed in (19) along with

$$g(0) = 0 \quad \text{and} \quad g(\infty) = \nu. \quad (47)$$

Here, ν is the VEV. For our analysis, we adopt, for simplicity, $\lambda = \beta = \nu = n = 1$. Additionally, to examine the numerical solutions of the system of equations [(44)-(46)], we will use the numerical interpolation method, which allows us to estimate the solutions for the differential equations [Eqs. (44)-(46)]. That will provide us with the solutions for the field variables $g(r)$ and $a(r)$ that satisfy the Eqs. [(44)-(46)]. Generally speaking, we apply the numerical interpolation method and discretize the function's domain. For instance, by considering the differential equations [(44)-(46)] defined in the range of the radial variable r , this range is subdivided into discrete steps. Thus, we discretize the position of the system by dividing the position range into two hundred thousand discrete points. Therefore, the solutions of the equations are investigated within the range $[0, 200]$, using the discretization $r_1, r_2, \dots, r_{200000}$, where $r_1 = 0$ and $r_{200000} = 200$ ⁹.

Using the numerical interpolation approach [93, 94], let us investigate the numerical vortex solutions, i.e., the solutions for the field variables $g(r)$ and $a(r)$. We display the numerical solutions for the field variables $g(r)$ and $a(r)$, respectively, in Figs. 3(a) and 3(b).



(a) Field variable solutions of the matter sector.

(b) Field variable solutions of the gauge sector.

Figure 3: The numerical vortex solutions.

From the analysis of the numerical vortex solutions in $f(R)$ gravity with the varying event horizon of the BH, one can note that the vortices are sensitive to the presence of the

⁹ For further details on the numerical method, see Refs. [93, 94].

compact cosmological object. This compact structure generates perturbations in the vortex core, resulting in the formation of vortex structures with the $g(r)$ sector experiencing abrupt changes near the event horizon, see Fig. 3(a). Indeed, this behavior is a consequence of the collapse of the magnetic vortex matter into the BH.

Analyzing the quadratic invariant of the Ricci scalar and the Kretschmann scalar, we noted the BH core at $r = 0$. The presence of the BH promotes a local minimum near the event horizon, followed by an absolute maximum in the gauge sector before rapidly stabilizing to the asymptotic value $a(r \rightarrow \infty) \rightarrow -\beta$, as shown in Fig. 3(b).

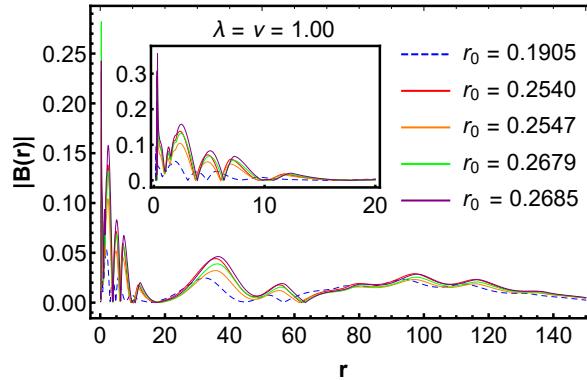
Finally, evaluating the magnetic field¹⁰ and energy density¹¹ of the BH-vortex system, we noted the existence of various magnetic oscillations near the event horizon of the BH, with energy density localized in the vortex. Naturally, the magnetic vortices formed in $f(R)$ gravity are of the ring-like structures. We exposed the ring-like vortices in Figs. 4[(a)-(d)] and 5[(a)-(d)].

IV. SUMMARY AND CONCLUSION

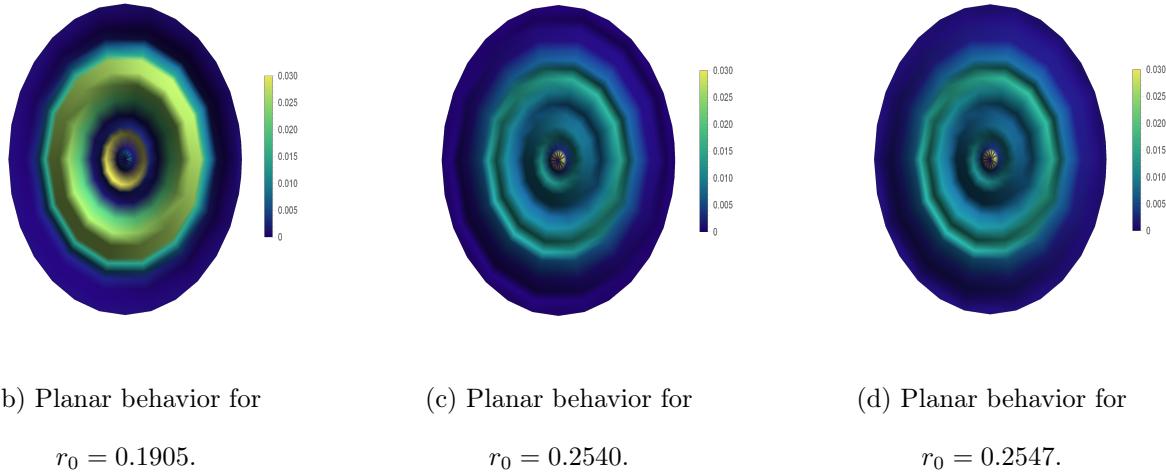
We employed Heisenberg's non-perturbative approach to promote the three-dimensional Einstein-Hilbert theory to a modified $f(R)$ gravity. In this conjecture, we studied a BH-vortex system. Within this framework, one shows the existence of a static three-dimensional black hole surrounded by magnetic ring-like vortices responsible for generating quantized magnetic flux and an anisotropic energy density near the event horizon, i.e., $r = r_0$. An analytical analysis of the quadratic invariant of the Ricci scalar ($R_{\mu\nu}R^{\mu\nu}$) and the Kretschmann scalar ($R_{\mu\nu\tau\sigma}R^{\mu\nu\tau\sigma}$), announced a BH at $r = 0$. Meanwhile, the vortices surround the black hole with their innermost rings close to $r \approx r_0$. By examining the thermodynamics of the black hole originating from vortex matter collapsing into the black hole, we found that the Bekenstein-Hawking temperature remains constant, independent of the parameters governing the MGT and the magnetic vortex. This result suggests that the BH-vortex system is thermodynamically stable. Lastly, the vortices are directly influenced by the event horizon, causing perturbations in the magnetic profile of the vortices and leading to the formation of

¹⁰ One defines the intensity of the magnetic field of the vortex as $B = -F_{21} \equiv \frac{a'(r)}{er}$ [95].

¹¹ The energy density of the BH-vortex system is the 00-component of the stress-energy tensor [00-component of Eq. (23)] [40–42].

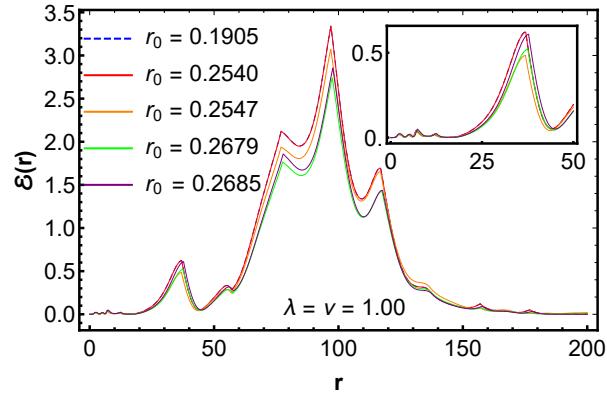
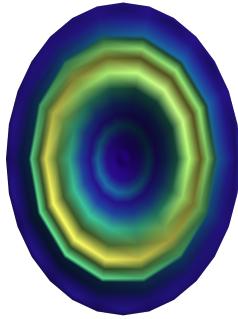


(a) Magnetic field intensity versus the vortex radius.

(b) Planar behavior for
 $r_0 = 0.1905$.(c) Planar behavior for
 $r_0 = 0.2540$.(d) Planar behavior for
 $r_0 = 0.2547$.Figure 4: Magnetic field of the vortex vs. r .

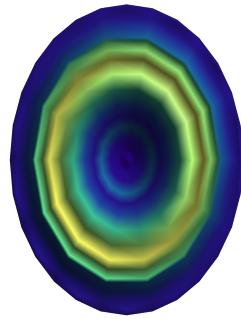
cosmological structures with magnetic ring-like profiles.

Starting from the premise that outside of the BH-vortex system $T_{\mu\mu} = 0$, one obtains that $h(r) = r - r_0$, with $f(R) = -2\alpha r^{-1}$, where α is the quantum fluctuation of the metric. Thus, we can reinterpret r_0 as a parameter associated with the BH mass. Naturally, this function represents a modification of the radial term in the Schwarzschild metric. By analogy to the Schwarzschild-like black hole, the profile of this metric function implies that the temporal term takes negative values for $r < r_0$, indicating the existence of an event horizon at $r = r_0$. Additionally, when $r \rightarrow r_0$, the temporal factor vanishes, which characterizes an essential feature of an event horizon. These results corroborate with the results for the quadratic invariant of the Ricci scalar and the Kretschmann scalar, which ensure the existence of a black hole at $r = 0$, with an event horizon at $r = r_0$.

(a) Energy density versus r .

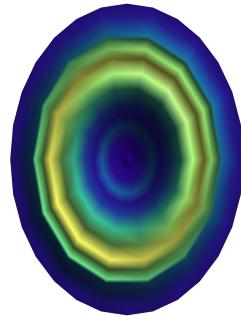
(b) Planar behavior for

$$r_0 = 0.1905.$$



(c) Planar behavior for

$$r_0 = 0.2540.$$



(d) Planar behavior for

$$r_0 = 0.2547.$$

Figure 5: Energy density of the BH-Vortex vs. r .

Examining the magnetic vortices around the three-dimensional BH, we noted that the field variables [$g(r)$ and $a(r)$] describing the vortex are highly sensitive to the presence of the compact object (BH) at $r = 0$ and their dynamics are altered with alters of the event horizon (r_0). Consequently, these vortices experience significant perturbations near the event horizon. This behavior naturally arises from the collapse of matter present in the magnetic vortex into the black hole. We concluded that the magnetic field of the vortex exhibits a ring-like profile, showing various magnetic oscillations near the event horizon with energy density is anisotropically distributed.

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CONFLICTS OF INTEREST/COMPETING INTEREST

The authors declared that there is no conflict of interest in this manuscript.

DATA AVAILABILITY

No data was used for the research described in this article.

- [1] J. L. Tonry, B. P. Schmidt, B. Barris, P. Candia, et al., *Cosmological Results from High- z Supernovae*, Ap. J **594** (2003) 1.
- [2] R. A. Knop, G. Aldering, R. Amanullah, P. Astier, et al., *New Constraints on Ω_M , Ω_Λ , and w from an Independent Set of 11 High-Redshift Supernovae Observed with the Hubble Space Telescope*, Ap. J **598** (2003) 102.
- [3] G. J. Olmo, *Post-Newtonian constraints on $f(R)$ cosmologies in metric formalism*, Phys. Rev. D **72** (2005) 083505.
- [4] S. Capozziello, *Curvature Quintessence*, Int. J. Mod. Phys. D **11** (2002) 483.
- [5] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, *Is cosmic speed-up due to new gravitational physics?*, Phys. Rev. D **70** (2004) 043528.
- [6] D. N. Vollick, *$1/R$ curvature corrections as the source of the cosmological acceleration*, Phys. Rev. D **68** (2003) 063510.
- [7] S. M. Carroll, A. De Felice, V. Duvvuri, D. A. Easson, M. Trodden and M. S. Turner, *Cosmology of generalized modified gravity models*, Phys. Rev. D **71**, 063513 (2005)
- [8] S. Capozziello, M. De Laurentis and O. Luongo, *Connecting early and late universe by $f(R)$ gravity*, Int. J. Mod. Phys. D **24** (2015) 1541002.

- [9] T. Clifton and P. K. S. Dunsby, *On the emergence of accelerating cosmic expansion in $f(R)$ theories of gravity*, Phys. Rev. D **91** (2015) 103528.
- [10] M. Eingorn, J. Novák and A. Zhuk, *$f(R)$ gravity: scalar perturbations in the late Universe*, Eur. Phys. J. C **74** (2014) 3005.
- [11] R. Saffari and S. Rahvar, *$f(R)$ gravity: From the Pioneer anomaly to cosmic acceleration*, Phys. Rev. D **77** (2008) 104028.
- [12] S. Capozziello, V. F. Cardone and V. Salzano, *Cosmography of $f(R)$ gravity*, Phys. Rev. D **78** (2008) 063504.
- [13] T. P. Sotiriou, *Constraining $f(R)$ gravity in the Palatini formalism*, Class. Quantum Grav. **23** (2006) 1253.
- [14] C. A. S. Almeida, F. C. E. Lima, S. S. Mishra, G. J. Olmo and P. K. Sahoo, *Thick brane in mimetic-like gravity*, Nucl. Phys. B **1009** (2024) 116747.
- [15] A. de Felice and S. Tsujikawa, *$f(R)$ gravity*, Living Rev. Relativ. **13** (2010) 3.
- [16] S. Nojiri and S. D. Odintsov, *Unified cosmic history in modified gravity: from $F(R)$ theory to Lorentz non-invariant models*, Phys. Rep. **505** (2011) 59.
- [17] S. Capozziello, A. Stabile and A. Troisi, *Newtonian limit of $f(R)$ gravity*, Phys. Rev. D **76** (2007) 104019.
- [18] T. Harko, F. S. N. Lobo, S. Nojiri and S. D. Odintsov, *$f(R, T)$ gravity*, Phys. Rev. D **84** (2011) 024020.
- [19] T. B. Gonçalves, J. L. Rosa and F. S. N. Lobo, *Cosmology in scalar-tensor $f(R, T)$ gravity*, Phys. Rev. D **105** (2022) 064019.
- [20] J. L. Rosa, *Junction conditions and thin shells in perfect-fluid $f(R, T)$ gravity*, Phys. Rev. D **103** (2021) 104069.
- [21] J. L. Rosa, A. S. Lobão Jr. and D. Bazeia, *Impact of compactlike and asymmetric configurations of thick branes on the scalar-tensor representation of $f(R, T)$ gravity*, Eur. Phys. J. C **82** (2022) 191.
- [22] T. Harko and F. S. N. Lobo, *Extensions of $f(R)$ Gravity: Curvature-Matter Couplings and Hybrid Metric-Palatini Theory*, (Cambridge Monographs on Mathematical Physics, University Printing House, Cambridge, United Kingdom, 2019).
- [23] L. Heisenberg, *Review on $f(Q)$ gravity*, Phys. Rep. **1066** (2024) 1–74.
- [24] Y.-F. Cai, S. Capozziello, M. De Laurentis and E. N. Saridakis, *$f(T)$ teleparallel gravity and*

- cosmology*, Rep. Prog. Phys. **79** (2016) 106901.
- [25] R. Saito, D. Yamauchi, S. Mizuno, J. Gleyzes and D. Langlois, *Modified gravity inside astrophysical bodies*, J. Cosmol. Astropart. Phys. **06** (2015) 2015.
- [26] A. V. Astashenok, S. D. Odintsov and V.K. Oikonomou, *Compact stars with dark energy in general relativity and modified gravity*, Phys. Dark Universe **42** (2023) 101295.
- [27] A.-C. Davis, R. Gregory, R. Jha and J. Muir, *Astrophysical black holes in screened modified gravity*, J. Cosmol. Astropart. Phys. **08** (2014) 33.
- [28] T. Chiba, *1/R gravity and scalar-tensor gravity*, Phys. Lett. B **575** (2003) 1.
- [29] A. L. Erickcek, T. L. Smith and M. Kamionkowski, *Solar system tests do rule out 1/R gravity*, Phys. Rev. D **74** (2006) 121501.
- [30] T. Chiba, T. L. Smith and A. L. Erickcek, *Solar System constraints to general f(R) gravity*, Phys. Rev. D **75** (2007) 124014.
- [31] S. Nojiri and S. D. Odintsov, *Modified non-local-f(R) gravity as the key for the inflation and dark energy*, Phys. Lett. B **659** (2008) 821.
- [32] S. Capozziello, A. Stabile and A. Troisi, *Spherical symmetry in f(R)-gravity*, Class. Quantum Grav. **25** (2008) 085004.
- [33] V. Dzhunushaliev, V. Folomeev, B. Kleihaus and J. Kunz, *Modified gravity from the quantum part of the metric*, Eur. Phys. J. C **74** (2014) 2743.
- [34] V. Dzhunushaliev, V. Folomeev, B. Kleihaus and J. Kunz, *Modified gravity from the nonperturbative quantization of a metric*, Eur. Phys. J. C **75** (2015) 157.
- [35] V. Dzhunushaliev, *Dynamical f(R) Gravities*, Int. J. Mod. Phys. D **21** (2012) 1250042.
- [36] V. I. Zhdanov, O. S. Stashko and Yu. V. Shtanov, *Spherically symmetric configurations in the quadratic gravity*, Phys. Rev. D **110** (2024) 024056.
- [37] S. Nojiri and S. D. Odintsov, *Black holes and their shadows in f(R) gravity*, Physics of the Dark Universe **47** (2025) 101785.
- [38] G. Giacomozzi and S. Zerbini, *Direct smooth reconstruction of inflationary models in f(R) gravity*, Physics of the Dark universe **44** (2024) 101431.
- [39] G. G. L. Nashed and S. Capozziello, *Anisotropic compact stars in f(R) gravity*, Eur. Phys. J. C **81** (2021) 481.
- [40] T. Vachaspati, *Kinks and Domain Walls: An Introduction to Classical and Quantum Solitons*, (Cambridge University Press, Cambridge, England, 2006).

- [41] R. Rajaraman, *Solitons and Instantons. An Introduction to Solitons and Instantons in Quantum Field Theory* (North-Holland, Amsterdam, 1982).
- [42] N. S. Manton and P. Sutcliffe, *Topological Solitons*, (Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, England, 2004).
- [43] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects*, (Cambridge University Press, Cambridge, England, 2000).
- [44] P. W. Higgs, *Broken Symmetries and the Masses of Gauge Bosons*, Phys. Rev. Lett. **13** (1964) 508.
- [45] H. B. Nielsen and P. Olesen, *Vortex-line models for dual strings*, Nuc. Phys. B **61** (1973) 45.
- [46] D. Saint-James, G. Sarma and E. J. Thomas, *Type II super-conductivity*, (Pergamon Press, 1969).
- [47] A. Abrikosov, *On the magnetic properties os superconductors of the second group*, Sov. Phys. JETP **32** (1957) 1442.
- [48] J. Albert, *The Abrikosov vortex in curved space*, JHEP **09** (2021) 012.
- [49] G. Dvali, F. Kühnel and M. Zantedeschi, *Vortices in blach holes*, Phys. Rev. Lett. **129** (2022) 061302.
- [50] F. C. E. Lima, A. R. P. Moreira and C. A. S. Almeida, *Properties of blach hole vortex in Einstein's gravity*, Eur. Phys. J. Plus **138** (2023) 429.
- [51] V. P. Frolov and A. Zelnikov, *Introductions to Black Hole Physics*, (Oxford University Press, Oxford, New York, 2011)
- [52] K. Akiyama, A. Alberdi, R. Azulay, A. -K. Baczko, et al., *First M87 Event Horizon Telescope Results. IV. Imaging the Central Supermassive Black Hole*, The Astrophysical Journal Letters, **875:L4** (2019) 52.
- [53] S. Choudhury and M. Sami, *Large fluctuations and primordial black holes*, Phys. Rep. **1103** (2025) 1-276.
- [54] A. Ditta, F. Javed, A. Bouzenada, G. Mustafa, et al., *Thermal chemistry of Anti-de-Sitter black holes in Kalb-Ramond gravity*, Journal od High Energy Astrophysics **45** (2025) 62.
- [55] X. -D. Zhu, W. Liu and Di Wu, *Universal thermodynamic topological classes of rotating black holes*, Phys. Lett. B **860** (2025) 139163.
- [56] A. Kehagias, D. Perrone and A. Riotto, *Quasinormal modes and Love numbers of Kerr black holes from AdS₂ black holes*, JCAP **2023** (2023) 035.

- [57] W. Rindler, *Relativity: Special, General and Cosmological*, (2nd edn., Oxford University Press, Oxford, New York, 2006).
- [58] K. Schwarzschild, *On the influence of the mass of the spacetime curvature*, Goött. Nachr., **1916** 4 (1916) 41.
- [59] J. R. Oppenheimer and H. Snyder, *On continued gravitational contraction*, Phys. Rev. **56** (1939) 455.
- [60] A. Edery, *Non-singular vortices with positive mass in $(2+1)$ -dimensional Einstein gravity with AdS_3 and Minkowski background*, JHEP **2021** (2021) 166.
- [61] S. Deser, R. Jackiw and G. 't Hooft, *Three-dimensional Einstein gravity: Dynamics of flat space*, Ann. Phys. **152** (1984) 220.
- [62] M. Bañados, C. Teitelboim and J. Zanelli, *Black hole in three-dimensional spacetime*, Phys. Rev. Lett. **69** (1992) 1849.
- [63] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the $2+1$ black hole*, Phys. Rev. D **48** (1993) 1506.
- [64] M. Cardoni, P. Pani and M. Serra, *Scalar hairs and exact vortex solutions in 3D AdS gravity*, JHEP **2010** (2010) 91.
- [65] R. Shankar, *Principles of Quantum Mechanics*, (2nd Ed., Springer, 2005).
- [66] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, (2nd Ed., Cambridge University Press, 2011).
- [67] R. Yang, *Effects of quantum fluctuations of metric on the universe*, Physics of the Dark Universe **13** (2016) 87.
- [68] C. A. S. Almeida and F. C. E. Lima, *Effects of quantum fluctuations of the metric on a braneworld*, Eur. Phys. J. Plus **139** (2024) 544.
- [69] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, *Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution*, Phys. Rep. **692** (2017) 1-104.
- [70] C. G. Böhmer and E. Jensko, *Modified gravity: A unified approach* Phys. Rev. D **104** (2021) 024010.
- [71] K. Koyama, *Cosmological tests of modified gravity*, Rep. Prog. Phys. **79** (2016) 046902.
- [72] S. Capozziello, F. S. N. Lobo and J. P. Mimoso, *Energy conditions in modified gravity*, Phys. Lett. B **730** (2014) 280.
- [73] D. Bazeia, L. Losano, R. Menezes, G. J. Olmo and D. Rubiera-Garcia, *Thick brane in $f(R)$*

- gravity with Palatini dynamics*, Eur. Phys. J. C **75** (2015) 569.
- [74] V. I. Afonso, D. Bazeia, R. Menezes and A. Yu. Petrov, *f(R)-Brane*, Phys. Lett. B **658** (2007) 71.
- [75] J. C. Neu, *Vortices in complex scalar fields*, Physica D **43** (1990) 385.
- [76] F. C. E. Lima and C. A. S. Almeida, *Topological solitons in the sigma-cuscuton model*, Eur. Phys. J. C **83** (2023) 831.
- [77] F. C. E. Lima and C. A. S. Almeida, *New class of solutions in the non-minimal O(3)-sigma model*, Phys. Lett. B **829** (2022) 137042.
- [78] F. C. E. Lima, A. Yu. Petrov and C. A. S. Almeida, *Vortex solution in nonpolynomial scalar QED*, Phys. Rev. D **103** (2021) 096019.
- [79] F. C. E. Lima and C. A. S. Almeida, *Phase transitions in the logarithmic Maxwell O(3)-sigma model*, Eur. Phys. J. C **81** (2021) 1044.
- [80] R. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, (2nd ed., Addison-Wesley, 2019).
- [81] K. Srinivasan and T. Padmanabhan, *Particle production and complex path analysis*, Phys. Rev. D **60** (1999) 024007.
- [82] M. Angheben, M. Nadalini, L. Vanzo and S. Zerbini, *Hawking radiation as tunneling for extremal and rotating black holes*, JHEP **05** (2005) 014.
- [83] R. Kerner and R. B. Mann, *Tunnelling, temperature, and Taub-NUT black holes*, Phys. Rev. D **73** (2006) 104010.
- [84] P. Mitra, *Hawking temperature from tunnelling formalism*, Phys. Lett. B **648** (2007) 240.
- [85] E. T. Akhmedov, V. Akhmedova and D. Singleton, *Hawking temperature in the tunneling picture*, Phys. Lett. B **642** (2006) 124.
- [86] Q.-Q. Jiang, S.-Q. Wu and X. Cai, *Hawking radiation as tunneling from the Kerr and Kerr-Newman black holes*, Phys. Rev. D **73** (2006) 064003.
- [87] R. Kerner and R. B. Mann, *Fermions tunnelling from black holes*, Class. Quant. Grav. **25** (2008) 095014.
- [88] M. S. Ma and R. Zhao, *Corrected form of the first law of thermodynamics for regular black holes*, Class. Quant. Grav. **31** (2014) 245014.
- [89] D. A. Gomes, F. C. E. Lima and C. A. S. Almeida, *Correlations between emission events in Rainbow Gravity*, Ann. Phys. **428** (2021) 168436.

- [90] C. A. S. Silva and F. A. Brito, *Quantum tunneling radiation from self-dual black holes*, Phys. Lett. B **725** (2013) 456.
- [91] O. Klein and W. Gordon, *Die aktuelle Theorie der Elektronen*, Zeitschrift für Physik, **37** (1926) 895.
- [92] R. K. Pathria, *Statistical Mechanics*, (2nd ed. Oxford: Pergamon Press, 1996).
- [93] J. B. Scarborough, *Numerical Mathematical*, (Oxford and IBH Publishing, 1955).
- [94] R. L. Burden and J. D. Faires, *Numerical analysis*, (Brooks., 1997).
- [95] R. Jackiw, K. Lee and E. J. Weinberg, *Self-dual Chern-Simons solitons*, Phys. Rev. D **42** (1990) 3488.