

Post-adiabatic waveform-generation framework for asymmetric precessing binaries

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Recent years have seen rapid progress in calculations of gravitational waveforms from asymmetric compact binaries containing spinning secondaries. Here we outline a complete waveform-generation scheme, through first post-adiabatic order (1PA) in gravitational self-force theory, for generic secondary spin and generic (eccentric, precessing) orbital configurations around a generic Kerr primary. We emphasize the utility of a Fermi-Walker frame in parametrizing the secondary spin, and we analyse precession and nutation effects in the spin-orbit dynamics. We also explain convenient gauge choices within the waveform-generation scheme, and the gauge invariance of the resulting waveform. Finally, we highlight that, thanks to recent results due to Grant and Witzany et al., all relevant spin effects at 1PA order can now be computed without evaluating local self-forces or torques.

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I. INTRODUCTION

Gravitational self-force (SF) theory originally grew out of the need to model extreme-mass-ratio inspirals (EMRIs) for the planned gravitational-wave detector LISA [1]. After decades of progress [2], it has now matured into a practical framework for building fast, accurate gravitational waveform models [3–6]. This development is exemplified by the FastEMRIWaveforms (FEW) software package, which can generate LISA-length EMRI waveforms on a timescale of $\sim 10\text{ms}$ [3, 7–10]. Moreover, while it was traditionally motivated by EMRIs, with mass ratios as extreme as $\sim 1 : 10^7$ [11], SF theory exhibits good agreement with fully nonlinear numerical relativity simulations even at mass ratios $\sim 1 : 10$ [5, 12–21], and it is now recognized as one of the most viable methods of directly modelling intermediate-mass-ratio inspirals [22] as well as an important tool for calibrating effective models that cover the entire binary parameter space [23–25].

In the context of compact-binary models, SF theory is based on an asymptotic expansion of the spacetime metric in powers of the binary mass ratio $\epsilon \equiv m_2/m_1$, where m_1 and m_2 are the masses of the heavier, primary and lighter, secondary object, respectively. The zeroth order in this expansion represents the Kerr spacetime of the primary black hole as if it were in isolation, with the secondary treated as a source of small perturbations on that background.

This expansion for small mass ratios means that SF models are adapted to asymmetric binaries in which the primary is significantly heavier than the secondary. Currently, other waveform models have limited accuracy for such systems [26–28], even in the range of mass ratios that can be observed by present-day ground-based detectors (such as the system GW191219_163120, with mass ratio $\approx 1 : 26$ [29]). Self-force models are also naturally adapted to another challenging area of parameter space: highly precessing systems in which the two objects' spin vectors rapidly precess around the orbital angular momentum. Although they are perturbative in the mass ratio, SF models can account for the primary object's spin, and its precession around the orbital angular momentum, non-perturbatively because the expansion is performed around the fully nonlinear background Kerr spacetime of the primary.

Systems exhibiting both of these features—high mass asymmetry and spin precession—are currently subject to the largest modeling uncertainties [28, 30]. Discrepancies between waveform models have led to substantial parameter biases even for mildly asymmetric precessing binaries [31]. For EMRIs specifically, spin precession is likely to be ubiquitous [32], which has motivated the EMRI modeling community's long-term goal of developing SF models for generic, precessing orbital configurations around Kerr black holes [33]. The need for improved models of less extreme precessing binaries provides additional motivation for this goal.

It is also widely expected that meeting the accuracy requirements of future gravitational-wave detectors [33–41] will require carrying the SF expansion to second perturbative order in the mass ratio [22, 42], necessitating second-order calculations for generic configurations on a Kerr background. Moreover, at second order in SF theory, the spin of the secondary object enters [43] and contributes significantly to the waveform phase [44]. Over the past several years, this importance of the secondary spin has spurred rapid progress toward SF waveform models for spinning secondaries on generic orbits around spinning primaries [44–73]. Our aim in this paper is to describe a complete waveform-generation framework for that generic scenario, focusing on the impact of the secondary's spin.

A. Multiscale waveform generation with a nonspinning secondary

Typically, in modern SF calculations and waveform models, the small-mass-ratio expansion is formulated as a multiscale expansion based on the quasi-periodic behavior of asymmetric binaries [74–76]. This multiscale expansion enables rapid waveform generation by allowing one to pre-compute all the necessary, expensive inputs for the waveform in an offline step; the online waveform generation then comprises a cheap, fast evolution through pre-computed data [3].

If we neglect the secondary's spin, then generic inspiraling orbits around Kerr black holes are approximately tri-periodic, with three independent, slowly evolving frequencies $\Omega^i = (\Omega^r, \Omega^\theta, \Omega^\phi)$ [77]. Each frequency has an associated phase $\dot{\psi}^i = \int \Omega^i dt$ describing the phase of the secondary's radial, azimuthal, or polar motion. Calculations are performed in the background Kerr geometry with the primary's spin axis orthogonal to the equatorial plane $\theta = \pi/2$, such that spin-orbit precession manifests itself as precession of the secondary's orbital plane (meaning $\Omega^\theta \neq \Omega^r$).

The multiscale expansion is adapted to this tri-periodicity, leading to waveforms with the form of slowly varying mode amplitudes multiplying tri-periodic phase factors [76, 78]:

$$h = \sum_{\mathbf{k} \in \mathbb{Z}^3} \left[\varepsilon \dot{h}_{\mathbf{k}}^{(1)}(\dot{\pi}_i, \theta, \phi) + \varepsilon^2 \dot{h}_{\mathbf{k}}^{(2)}(\dot{\pi}_i, \delta m_1, \delta \chi_1, \theta, \phi) \right. \\ \left. + \mathcal{O}(\varepsilon^3) \right] e^{-i \mathbf{k}_i \dot{\psi}^i}, \quad (1)$$

where (θ, ϕ) are angles on the sphere at future null infinity, $\mathbf{k} = (k_r, k_\theta, k_\phi)$ are integers, $\varepsilon \equiv 1$ is a bookkeeping parameter used to count powers of the mass ratio ϵ , and $\dot{\pi}_i$ are three independent, slowly evolving orbital parameters (such as suitably defined eccentricity, semi-latus rectum, and orbital inclination). δm_1 and $\delta \chi_1$ are slowly evolving corrections to the mass and spin of the primary, describing the effect of absorption through its horizon [75]. The waveform's time dependence is contained in its dependence on the binary's variables $\dot{\psi}^i$, $\dot{\pi}_i$, δm_1 , and $\delta \chi_1$, which are governed by simple ordinary differential equations [76]:

$$\frac{d\dot{\psi}^i}{dt} = \Omega^i(\dot{\pi}_k), \quad (2)$$

$$\frac{d\dot{\pi}_i}{dt} = \varepsilon \left[F_i^{(0)}(\dot{\pi}_k) + \varepsilon F_i^{(1)}(\dot{\pi}_k, \delta m_1, \delta \chi_1) + \mathcal{O}(\varepsilon^2) \right], \quad (3)$$

and

$$\frac{d\delta m_1}{dt} = \varepsilon \dot{\mathcal{E}}_{\mathcal{H}}^{(1)}(\dot{\pi}_i) + \mathcal{O}(\varepsilon^2), \quad (4)$$

$$\frac{d\delta \chi_1}{dt} = \varepsilon \dot{\mathcal{L}}_{\mathcal{H}}^{(1)}(\dot{\pi}_i) + \mathcal{O}(\varepsilon^2), \quad (5)$$

where $\dot{\mathcal{E}}_{\mathcal{H}}^{(1)}$ and $\dot{\mathcal{L}}_{\mathcal{H}}^{(1)}$ are the leading-order fluxes of energy and angular momentum into the primary's horizon.

We use rings over variables that cleanly separate slow evolution from rapid oscillations, avoiding oscillations in the mode amplitudes $\dot{h}_{\mathbf{k}}^{(n)}$ and in the right-hand side of Eqs. (2) and (3).

In practical models, all of the inputs, $\dot{h}_{\mathbf{k}}^{(n)}$, $F_i^{(n)}$, $\dot{\mathcal{E}}_{\mathcal{H}}^{(n)}$, and $\dot{\mathcal{L}}_{\mathcal{H}}^{(n)}$, are pre-computed on a grid of J_i values.¹ Waveforms are then rapidly generated by solving Eqs. (2)–(5) and summing the modes in Eq. (1). Equations (2)–(5) can be solved rapidly because all dependence on the phases $\dot{\psi}^i$ has been eliminated through the choice of variables $(\psi^i, \dot{\pi}_i)$, avoiding the need to resolve oscillations on the orbital time scales $\sim 2\pi/\Omega^i$ [79].

In Eqs. (2)–(5), numeric labels in parentheses denote the post-adiabatic (PA) order at which the term enters the evolution of the orbital phases [74, 76], and $F_i^{(n)}$ are referred to as n PA forcing functions. To understand this order counting, note Eqs. (2)–(5) imply that the orbital phases, and hence the waveform phase [21], have an asymptotic expansion of the form

$$\dot{\psi}^i = \frac{1}{\varepsilon} \left[\dot{\psi}_{(0)}^i(\varepsilon t) + \varepsilon \dot{\psi}_{(1)}^i(\varepsilon t) + \mathcal{O}(\varepsilon^2) \right]. \quad (6)$$

Although this expansion has limited accuracy compared to a direct solution of Eqs. (2)–(5) [5], it provides a simple estimate of the impact of any given term in the dynamics: n PA forcing functions directly contribute to the n PA phase $\dot{\psi}_{(n)}^i$.

In a 1PA model, the residual phase error in Eq. (6) scales as $\mathcal{O}(\varepsilon)$, showing that 1PA models should achieve gravitational-wave phase errors much smaller than 1 radian for mass ratios much smaller than unity. Detailed analyses for binaries with a nonspinning secondary [74, 76] show what is required to achieve this 1PA accuracy. The 0PA forcing function $F_i^{(0)}$ can be computed from the dissipative self-force exerted by the first-order (linear in ε) metric perturbation $h_{\alpha\beta}^{(1)}$, while $F_i^{(1)}$ requires the complete (conservative and dissipative) first-order self-force as well as the dissipative self-force exerted by the second-order (quadratic in ε) metric perturbation $h_{\alpha\beta}^{(2)}$.

Fully relativistic, generic 0PA waveforms were first generated in Ref. [78], building on decades of progress [4, 77, 80–89]. Efforts are now underway to extend those results to build a fast 0PA model with complete coverage of the parameter space.

Some 1PA effects, particularly the effects of the conservative first-order self-force, have been thoroughly studied [2] and have been computed for a small sample of

generic orbits [87]. However, due to the challenge of computing $h_{\alpha\beta}^{(2)}$, until recently the only complete 1PA waveform model was restricted to the simplest case of quasicircular orbits around a nonspinning primary black hole [5, 42]. We report extensions of that model in a series of companion papers [90–92].

B. Addition of the secondary spin

Recent work has gone a long way toward incorporating a spinning secondary into the type of framework laid out above. See, for example, Refs. [63, 65, 71] for nearly complete treatments. In this paper we consolidate and extend those studies, presenting a unified framework for multiscale waveform generation with a spinning secondary. Such a framework is greatly simplified by the fact that at 1PA order, the secondary spin χ_2 only enters linearly; this is because, for a compact object, spin scales as mass squared. Concretely, we show the secondary spin only enters the 1PA waveform in the following ways:

1. The second-order amplitudes, $h_{\mathbf{k}}^{(2)}$, in the waveform (1) are modified by the addition of a linear-in- χ_2 term $h_{\mathbf{k}}^{(2-\chi_2)}$, and this term is multiplied by an additional phase factor $e^{-iq\dot{\psi}_s}$, where $\dot{\psi}_s$ denotes a precession angle of the secondary spin and $q = 0, \pm 1$. However, this modification to $h_{\mathbf{k}}^{(2)}$ can probably be neglected in most circumstances because it only contributes $\mathcal{O}(\varepsilon)$ to the waveform phase.
2. The orbital frequencies Ω^i in Eq. (2) are modified by a linear-in- χ_2 correction $\Omega_{(1-\chi_2)}^i$. This represents the *conservative* effect of the secondary spin.
3. The 1PA forcing functions $F_i^{(1)}$ are modified by the addition of a linear-in- χ_2 correction $F_i^{(1-\chi_2)}$. This represents the *dissipative* effect of the secondary spin.

However, there is gauge freedom in the multiscale expansion, which can be used to move 1PA contributions between $F_i^{(1-\chi_2)}$ and $\Omega_{(1-\chi_2)}^i$. This freedom can be used to eliminate $\Omega_{(1-\chi_2)}^i$, for example, altering $F_i^{(1-\chi_2)}$ in the process. As part of our analysis, we characterize this freedom and the invariance of the 1PA waveform.

Conservative 1PA effects of the secondary spin have been well studied (e.g., in [51, 60, 61, 63, 71, 73]), and the frequency corrections $\Omega_{(1-\chi_2)}^i$ were derived by Witzany [50, 71, 72]. Dissipative 1PA effects have also received much attention (e.g., in [53, 55, 58, 59, 65, 69, 71]) but are not yet complete. Recent work has highlighted the possibility of obtaining all relevant dissipative effects by computing the rates of change of quantities that would be constant for a test body [65]: spin-corrected versions of the energy, angular momentum, and Carter constant,

¹ The background spin must also be included as an axis on this parameter-space grid, while the background mass only provides an overall scale. The corrections δm_1 and $\delta\chi_1$ do not need to be included as axes because $h_{\mathbf{k}}^{(2)}$ and $F_i^{(1)}$ are linear in these corrections, meaning one can pre-compute coefficients of δm_1 and $\delta\chi_1$ as functions of J_i and of the background spin.

along with a fourth quantity known as the Rüdiger constant [57, 93, 94]. The spin-corrected energy and angular momentum satisfy balance laws, which allow one to compute their evolution (at linear order in χ_2) from asymptotic fluxes at infinity and the horizon [53].

Grant [70] recently showed that the 1PA evolution of the spin-corrected Carter constant can also be obtained from asymptotic fluxes, extending the analogous, classic formulas [88, 95] for the 0PA evolution of the Carter constant. Finally, Ref. [69] showed that the evolution of the Rüdiger constant is redundant at 1PA order. This is because it depends functionally on the other quasi-conserved quantities and a component of the secondary's spin, commonly labeled χ_{\parallel} , which they have shown does not evolve at 1PA order.

Our analysis solidifies and extends Ref. [69]'s conclusion that the evolution of χ_{\parallel} can be neglected at 1PA order. We also show how to incorporate Grant's evolution equation for the spin-corrected Carter constant into the complete waveform-generation scheme. The essential step here is to combine Grant's formulas with those of Witzany et al. [72] for the conservative sector of the linear-in-spin dynamics and those of Isoyama et al. [88] for asymptotic fluxes.

C. Outline and conventions

In Sec. II we review SF theory with a spinning compact secondary.² In Sec. III we highlight a convenient parameterization of the secondary's spin and derive equations for its precession and nutation. We show, in particular, that the spin's two independent components (parallel and orthogonal to the orbital angular momentum) are exactly constant at linear order in the spin.

In Sec. IV we present the multiscale expansion describing the evolving inspiral and metric of a spinning secondary on a generic orbit in a Kerr background spacetime. We elucidate the gauge freedom in this expansion, among other things. In Sec. V we summarize the waveform-generation scheme. We analyze the gauge invariance of the waveform and characterize the impact of the secondary spin precession. In Sec. VI, we describe how the results of Grant [70], Witzany et al. [72], and Isoyama et al. [88] can be combined to obtain a complete, practical prescription for the calculation of 1PA spin effects in the dynamics. In Sec. VII we summarise our findings, outline the path to complete 1PA waveforms, and discuss other avenues for future extensions.

Our analysis in Secs. IV–VI applies away from orbital resonances [74, 76, 96], while Secs. II and III apply generally, both away from and across resonances.

We work in geometric units with $G = c = 1$. We denote the individual masses as m_i with $m_1 \gg m_2$. We

use $\varepsilon \equiv 1$ as a counting parameter of the small mass ratio $\epsilon \equiv m_2/m_1$. χ_i are the dimensionless spins S_i/m_i^2 , where S_i are the spin angular momenta of the two bodies. For astrophysical compact objects, $S_i \sim m_i^2$ and so in the small-mass-ratio expansion $S_1 \sim \varepsilon^0$ and $S_2 \sim \varepsilon^2$. For each dimensionless spin this implies $\chi_i \sim \varepsilon^0$. If the body is a Kerr black hole, then we have the more precise restriction $0 \leq |\chi_i| \leq 1$.

II. SELF-FORCE THEORY WITH A SPINNING SECONDARY

In this section we review the formulation of SF theory with a spinning secondary at second order in perturbation theory. We will ultimately use a multiscale formulation of the problem in later sections, but in this section for simplicity we adopt the self-consistent formulation [97] as extended in Ref. [75].

A. Field equations and effective metric

The Einstein field equations governing the compact binary's metric ($\mathbf{g}_{\mu\nu}$) are

$$G_{\mu\nu}(\mathbf{g}) = 8\pi T_{\mu\nu}, \quad (7)$$

where $T_{\mu\nu}$ is an effective stress-energy tensor for the secondary object, described below. Taking the usual black hole perturbation theory approach, we solve the field equations by expanding the binary's metric in powers of ε :

$$\mathbf{g}_{\mu\nu} = g_{\mu\nu} + \varepsilon h_{\mu\nu}^{(1)} + \varepsilon^2 h_{\mu\nu}^{(2)} + \mathcal{O}(\varepsilon^3), \quad (8)$$

where $g_{\mu\nu}$ is the spacetime of the primary black hole as if it were in isolation, and $h_{\mu\nu}^{(n)}$ is the n^{th} -order metric perturbation.

Since it absorbs gravitational radiation during the binary's evolution, the primary black hole itself slowly evolves. This can be accounted for in one of two ways. The first option is to take $g_{\mu\nu}$ to be a Kerr metric with time-dependent mass and spin parameters. In this first approach, the Einstein tensor of the background metric, $G_{\mu\nu}(g)$, is small but nonzero. The second option is to split the mass and spin into constant terms ($m_1^{(0)}$ and $\chi_1^{(0)}$) and small, dynamical corrections (δm_1 and $\delta \chi_1$); $g_{\mu\nu}$ is then taken to be a Kerr metric with mass $m_1^{(0)}$ and dimensionless spin $\chi_1^{(0)}$, and the perturbations $h_{\mu\nu}^{(n)}$ then include terms that are n^{th} order in δm_1 and $\delta \chi_1$. Both approaches, which yield equivalent total metrics, are described in Sec. II of Ref. [75]. Here we adopt the second approach, which implies $G_{\mu\nu}(g) = 0$ and allows us to exploit the exact Killing symmetries of the fixed Kerr background. We return to the treatment of the primary in Ref. [90], where we specialize to the case of a slowly spinning primary with $\chi_1^{(0)} = 0$ and $\delta \chi_1 \neq 0$.

² The spin of the primary body is automatically assimilated at the level of the metric in SF theory.

Because the secondary compact object is much smaller than the curvature scale of $g_{\mu\nu}$, it can be represented in the field equations as a ‘gravitational skeleton’, meaning a point-particle stress-energy tensor $T_{\mu\nu}$ equipped with the object’s multipole moments. More concretely, through second order in ε , the secondary object can be described as a spinning point particle under the pole-dipole approximation [43, 44, 98]. For a material body whose size is much smaller than the curvature scale of $g_{\mu\nu}$, this pole-dipole approximation follows from a Mathisson-Dixon multipole expansion of the body’s physical stress-energy tensor [99–101] (as extended by Harte to the physical case of a gravitating material body [102, 103]). More generally, for black holes as well as material bodies, the pole-dipole approximation has been derived via matched asymptotic expansions [43, 98, 104]. In either approach, the monopole and dipole terms in the stress-energy are

$$T^{\alpha\beta} = \varepsilon T_{(m)}^{\alpha\beta} + \varepsilon^2 T_{(d)}^{\alpha\beta} + \mathcal{O}(\varepsilon^3), \quad (9)$$

where $T_{(m)}^{\alpha\beta}$ is a mass-monopole term and $T_{(d)}^{\alpha\beta}$ is a spin-dipole term; quadrupole and higher moments would appear at higher orders in ε . Explicitly, the two contributions are

$$T_{(m)}^{\alpha\beta} = m_2 \int d\hat{\tau}' \frac{\delta^4[x^\mu - z^\mu(\tau')]}{\sqrt{-\hat{g}'}} \hat{u}^\alpha(\tau') \hat{u}^\beta(\tau'), \quad (10a)$$

$$T_{(d)}^{\alpha\beta} = (m_2)^2 \hat{\nabla}_\rho \int d\hat{\tau}' \frac{\delta^4[x^\mu - z^\mu(\tau')]}{\sqrt{-\hat{g}'}} \hat{u}^{(\alpha}(\tau') \hat{S}^{\beta)\rho}(\tau'), \quad (10b)$$

where δ^4 is the four-dimensional Dirac delta function, z^μ is the object’s effective center-of-mass worldline, $\hat{S}^{\alpha\beta}$ is the object’s dimensionless (mass-normalized) spin tensor, and primes are used to indicate evaluation at $z^\mu(\tau')$.

Importantly, the stress-energy terms (10a) and (10b) take the form of a spinning particle in a certain *effective* vacuum metric $\hat{g}_{\alpha\beta}$ rather than in the external background $g_{\alpha\beta}$. The proper time $\hat{\tau}$, four-velocity $\hat{u}^\alpha \equiv \frac{dz^\alpha}{d\hat{\tau}}$, metric determinant \hat{g} , and covariant derivative $\hat{\nabla}_\alpha$ are all defined from $\hat{g}_{\alpha\beta}$. The effective metric itself is defined by subtraction of suitably defined, singular self-fields $h_{\alpha\beta}^{S(n)}$ from the physical metric. Given the physical metric from Eq. (8), which we can write as

$$g_{\mu\nu} = g_{\mu\nu} + \sum_{n \geq 0} \varepsilon^n h_{\mu\nu}^{(n)}, \quad (11)$$

the effective metric is then

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}^R, \quad (12)$$

in which we have defined

$$h_{\mu\nu}^R = \sum_{n \geq 0} \varepsilon^n \left(h_{\mu\nu}^{(n)} - h_{\mu\nu}^{S(n)} \right). \quad (13)$$

The appropriate singular/self-field $h_{\mu\nu}^{S(2)}$ for a spinning compact secondary was derived in Ref. [43]; we refer to Ref. [44] for further discussion.

With the expansions of the metric and stress-energy tensor, we can rewrite Eq. (7) through second order as

$$\begin{aligned} & \varepsilon \delta G_{\mu\nu}(h^{(1)}) + \varepsilon^2 \delta G_{\mu\nu}(h^{(2)}) \\ &= 8\pi \left(\varepsilon T_{\mu\nu}^{(m)} + \varepsilon^2 T_{\mu\nu}^{(d)} \right) \\ &\quad - \varepsilon^2 \delta^2 G_{\mu\nu}(h^{(1)}, h^{(1)}) + \mathcal{O}(\varepsilon^3), \end{aligned} \quad (14)$$

where $\delta G_{\mu\nu}$ is the linearized Einstein tensor and $\delta^2 G_{\mu\nu}$ is the quadratic term in the expansion of the Einstein tensor. We will not require more explicit expressions, but they can be found in Sec. II of Ref. [105], for example.

Equation (14) can be divided into a sequence of equations for each $h_{\mu\nu}^{(1)}$ as described in Sec. II of Ref. [75]. However, this is nontrivial due to the presence of the evolving mass and spin corrections ($\delta m_1, \delta \chi_1$) and because the trajectory z^μ and spin $\hat{S}^{\alpha\beta}$ satisfy ε -dependent dynamical equations. We will ultimately adopt a multi-scale expansion that cleanly splits Eq. (14) into hierarchical equations. To enable that expansion, we next turn to the dynamical equations for z^μ and $\hat{S}^{\alpha\beta}$.

B. MPD-Harte equations of motion

Through second order in ε , the secondary behaves as a test body in the effective metric $\hat{g}_{\mu\nu}$ [102, 106, 107], obeying the Mathisson-Papapetrou-Dixon (MPD) test-body equations [99, 108, 109]. The secondary’s quadrupole moment enters these equations at $\mathcal{O}(\varepsilon^2)$, but we allow ourselves to neglect its effects because they are purely conservative (at least for a small Kerr black hole [67, 110]) and therefore only influence the waveform at 2PA order. For the same reason, we neglect terms that are quadratic or higher order in the particle’s spin, which we denote $\mathcal{O}(s^2)$. We hence assume the pole-dipole approximation, under which the MPD equations in $\hat{g}_{\mu\nu}$ read

$$\frac{\hat{D}\hat{u}^\alpha}{d\hat{\tau}} = -\frac{m_2}{2} \hat{R}^\alpha{}_{\beta\gamma\delta} \hat{u}^\beta \hat{S}^{\gamma\delta} + \mathcal{O}(s^2), \quad (15a)$$

$$\frac{\hat{D}\hat{S}^{\gamma\delta}}{d\hat{\tau}} = \mathcal{O}(s^2), \quad (15b)$$

where $\hat{D}/d\hat{\tau} \equiv \hat{u}^\alpha \hat{\nabla}_\alpha$. We refer to these as the MPD-Harte equations since Harte extended them (subject to some caveats [44]) to gravitating material bodies rather than only test bodies in a fixed background. Reference [44] contains a critical assessment of their validity in our context.

In Eq. (15) we have imposed the Tulczyjew-Dixon spin supplementary condition (SSC),

$$\hat{p}^\alpha \hat{S}_{\alpha\beta} = 0, \quad (16)$$

where here (and throughout this paper) indices on hatted quantities are lowered with the effective metric. Recall

we have defined $\hat{S}_{\alpha\beta}$ as the mass-normalised effective spin tensor of the secondary, such that

$$(\chi_2)^2 = \frac{1}{2} \hat{S}_{\alpha\beta} \hat{S}^{\alpha\beta}. \quad (17)$$

The quantity \hat{p}^μ is the secondary's linear momentum in the effective spacetime. We may also define an effective (dimensionless) spin vector,

$$\hat{S}^\mu = -\frac{1}{2} \hat{\epsilon}^\mu_{\alpha\beta\gamma} \hat{u}^\alpha \hat{S}^{\beta\gamma}, \quad (18)$$

with inverse relation

$$\hat{S}^{\mu\nu} = -\hat{\epsilon}^{\mu\nu\alpha\beta} \hat{S}_\alpha \hat{u}_\beta. \quad (19)$$

Note that the hat on the Levi-Civita tensor is important as it indicates dependence on the metric determinant of $\hat{g}_{\alpha\beta}$ as opposed to the metric determinant of $g_{\alpha\beta}$ or $\mathbf{g}_{\alpha\beta}$. The Tulczyjew-Dixon SSC implies that $\hat{p}^\alpha = m_2 \hat{u}^\alpha + \mathcal{O}(s^2)$, and $\hat{u}^\alpha \hat{S}_\alpha = 0$, where the higher-order spin terms only contribute conservative effects at $\mathcal{O}(\varepsilon^2)$ and hence may be neglected. Given our definitions and Eq. (15), the spin vector must satisfy the parallel transport equation

$$\frac{\hat{D}\hat{S}^\alpha}{d\hat{\tau}} = \mathcal{O}(s^2). \quad (20)$$

It is more practical to re-express the equations of motion in Eq. (15) in terms of the *background* metric and its regular perturbations, expanding in powers of ε . The equations of motion become those of a self-accelerated and self-torqued spinning body in the background space-time [44]:

$$\begin{aligned} \frac{Du^\mu}{d\tau} &= -\frac{1}{2} P^{\mu\nu} (g_\nu{}^\lambda - h_\nu^R{}^\lambda) (2h_{\lambda\rho;\sigma}^R - h_{\rho\sigma;\lambda}^R) u^\rho u^\sigma \\ &\quad - \frac{m_2}{2} R^\mu{}_{\alpha\beta\gamma} \left(1 - \frac{1}{2} h_{\rho\sigma}^R u^\rho u^\sigma\right) u^\alpha \hat{S}^{\beta\gamma} \\ &\quad + \frac{m_2}{2} P^{\mu\nu} (2h_{\nu(\alpha;\beta)\gamma}^R - h_{\alpha\beta;\nu\gamma}^R) u^\alpha \hat{S}^{\beta\gamma} \\ &\quad + \mathcal{O}(\varepsilon^3, s^2), \end{aligned} \quad (21a)$$

$$\frac{D\hat{S}^{\mu\nu}}{d\tau} = u^{(\rho} \hat{S}^{\sigma)}[\mu g^{\nu]\lambda} (2h_{\lambda\rho;\sigma}^R - h_{\rho\sigma;\lambda}^R) + \mathcal{O}(\varepsilon^2, s^2)], \quad (21b)$$

where $D/d\tau \equiv u^\mu \nabla_\mu$, $u^\mu \equiv dz^\mu/d\tau$, $P^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$, τ is the proper time as measured in $g_{\mu\nu}$, and ∇_μ and semi-colons both denote the covariant derivative compatible with $g_{\mu\nu}$.

In deriving Eq. (21) we have used the normalization conditions

$$\hat{g}_{\alpha\beta} \hat{u}^\alpha \hat{u}^\beta = -1 = g_{\alpha\beta} u^\alpha u^\beta, \quad (22)$$

or equivalently, $\hat{g}_{\alpha\beta} \hat{p}^\alpha \hat{p}^\beta = -m_2^2 + \mathcal{O}(s^2) = g_{\alpha\beta} p^\alpha p^\beta$. By writing $\hat{u}^\alpha = \frac{d\tau}{d\hat{\tau}} u^\alpha$ in the normalization conditions, one finds

$$\frac{d\tau}{d\hat{\tau}} = \sqrt{1 - h_{\alpha\beta}^R u^\alpha u^\beta}, \quad (23)$$

or equivalently we may write

$$\hat{u}^\alpha = u^\alpha + \delta u^\alpha \quad (24)$$

with

$$\delta u^\alpha = \left(1 - \sqrt{1 - h_{uu}^R}\right) u^\alpha \approx \frac{1}{2} h_{uu}^R u^\alpha, \quad (25)$$

where the indices replaced with u indicate contraction with the background four-velocity.

We emphasise that Eq. (24) does not represent an expansion of the four-velocity around a zeroth-order value. \hat{u}^α and u^α are both tangent to the same self-accelerated curve z^α ; they differ only in the choice of parameter ($\hat{\tau}$ or τ) along that curve.

III. PRECESSION AND NUTATION OF THE SECONDARY SPIN

In this section we present a convenient parametrization of the secondary's spin degrees of freedom. By decomposing the spin vector in a local, Fermi-Walker transported frame along the particle's worldline, we show its components in this frame are exactly constant. The spin's precession then corresponds to the local frame's rotation relative to a second frame.

A. Decomposition in a Fermi-Walker tetrad

Along the particle's worldline, we construct a tetrad $\{\hat{e}_0^\alpha, \hat{e}_A^\alpha\}$ ($A = 1, 2, 3$) that is orthonormal in the effective metric $\hat{g}_{\alpha\beta}$:

$$\hat{g}_{\alpha\beta} \hat{e}_0^\alpha \hat{e}_0^\beta = -1, \quad (26a)$$

$$\hat{g}_{\alpha\beta} \hat{e}_0^\alpha \hat{e}_A^\beta = 0, \quad (26b)$$

$$\hat{g}_{\alpha\beta} \hat{e}_A^\alpha \hat{e}_B^\beta = \delta_{AB}. \quad (26c)$$

We take the zeroth leg of the tetrad to be the four-velocity, $\hat{e}_0^\alpha = \hat{u}^\alpha$, which is propagated along the worldline according to

$$\frac{\hat{D}\hat{e}_0^\alpha}{d\hat{\tau}} = \hat{a}^\alpha. \quad (27)$$

Here $\hat{a}^\alpha \equiv \frac{\hat{D}\hat{u}^\alpha}{d\hat{\tau}}$ is the covariant acceleration in $\hat{g}_{\alpha\beta}$, given by Eq. (15a). We take the spatial triad \hat{e}_A^α to be Fermi-Walker transported along the worldline, meaning [11]

$$\frac{\hat{D}\hat{e}_A^\alpha}{d\hat{\tau}} = \hat{a}_A \hat{u}^\alpha, \quad (28)$$

where $\hat{a}_A \equiv \hat{e}_A^\beta \hat{a}_\beta$. The resulting tetrad $\{\hat{e}_0^\alpha, \hat{e}_A^\alpha\}$ represents the particle's local frame in the effective metric $\hat{g}_{\alpha\beta}$. Decomposed in this frame, the spin vector reads

$$\hat{S}^\alpha = \hat{S}^0 \hat{e}_0^\alpha + \hat{S}^A \hat{e}_A^\alpha. \quad (29)$$

We can show almost immediately that the tetrad components \hat{S}^0 and \hat{S}^A are constant (up to spin-squared terms). The Tulczyjew-Dixon SSC (16) implies

$$\hat{S}^0 = 0. \quad (30)$$

The parallel-transport equation (20) then implies

$$\frac{d\hat{S}^A}{d\hat{\tau}} \hat{e}_A^\alpha + \hat{S}^A \hat{a}_A \hat{u}^\alpha = \mathcal{O}(s^2). \quad (31)$$

Contracting this equation with the four-velocity yields

$$\hat{S}^A \hat{a}_A = \mathcal{O}(s^2), \quad (32)$$

while contracting with a member of the spatial triad yields

$$\frac{d\hat{S}^A}{d\hat{\tau}} = \mathcal{O}(s^2). \quad (33)$$

Equation (32) is trivially satisfied because \hat{a}_α is proportional to the spin. Equation (33) shows that the spin's nonzero degrees of freedom are constant.

Given the trajectory z^α , one can always construct the triad \hat{e}_A^α by finding a triad orthogonal to \hat{u}^α at a single point on the trajectory and then solving the Fermi-Walker transport equation (28). We do so in effect by expressing the Fermi-Walker transported tetrad as a perturbed version of a commonly used background tetrad: the Marck tetrad [112].

B. Marck tetrad

The Marck tetrad $\{e_0^\alpha, e_A^\alpha\}$ forms a basis of solutions to the parallel transport equation along geodesics of Kerr spacetime [51, 52, 112]. As we describe below, it is not Fermi-Walker transported in the background metric $g_{\alpha\beta}$, but the violation is $\mathcal{O}(\varepsilon, s)$, meaning we will be able to write the Fermi-Walker tetrad as a small deformation of the Marck tetrad:

$$\hat{e}_0^\alpha = e_0^\alpha + \delta e_0^\alpha + \mathcal{O}(\varepsilon^2), \quad (34a)$$

$$\hat{e}_A^\alpha = e_A^\alpha + \delta e_A^\alpha + \mathcal{O}(\varepsilon^2). \quad (34b)$$

The Marck tetrad legs on z^μ can be constructed from the four-velocity u^α and the Killing-Yano tensor $Y_{\alpha\beta}$:

$$e_0^\alpha = u^\alpha, \quad (35a)$$

$$e_1^\alpha = \cos \psi_s \sigma_1^\alpha + \sin \psi_s \sigma_2^\alpha, \quad (35b)$$

$$e_2^\alpha = \cos \psi_s \sigma_2^\alpha - \sin \psi_s \sigma_1^\alpha, \quad (35c)$$

$$e_3^\alpha = Y_\beta^\alpha u^\beta / \sqrt{K^{(0)}}, \quad (35d)$$

where ψ_s is a spin-precession phase discussed below,

$$\sigma_0^\alpha = u^\alpha, \quad (36a)$$

$$\sigma_1^\alpha = \epsilon^{\alpha\beta\gamma\delta} \sigma_\beta^0 \sigma_\gamma^2 \sigma_\delta^3, \quad (36b)$$

$$\sigma_2^\alpha = \frac{1}{N} P^{\alpha\beta} K_{\beta\gamma} u^\gamma, \quad (36c)$$

$$\sigma_3^\alpha = e_3^\alpha, \quad (36d)$$

$K_{\mu\nu} = Y_\mu^\rho Y_{\nu\rho}$ is the Killing tensor, $K^{(0)} = K^{\alpha\beta} u_\alpha u_\beta$ and $N \equiv -\sqrt{P^{\alpha\beta} K_{\beta\gamma} u^\gamma P_\alpha^\lambda K_{\delta\lambda} u^\delta}$. One can straightforwardly check that $\{e_0^\alpha, e_A^\alpha\}$ is orthonormal in the background metric, satisfying

$$g_{\alpha\beta} e_0^\alpha e_0^\beta = -1, \quad (37a)$$

$$g_{\alpha\beta} e_0^\alpha e_A^\beta = 0, \quad (37b)$$

$$g_{\alpha\beta} e_A^\alpha e_B^\beta = \delta_{AB} \quad (37c)$$

at all points along z^α .

If z^α were a geodesic of the Kerr background, the Marck tetrad would be parallel propagated (with respect to $g_{\alpha\beta}$) along it. The Fermi-Walker transport equation reduces to the parallel transport equation at zeroth order in acceleration, and the corresponding basis of solutions reduces to the Marck tetrad. Equivalently, we can say the Marck tetrad represents the local frame of an inertial observer (freely falling in the Kerr background) whose trajectory is instantaneously co-moving with z^α . In the remainder of this section, we assess the tetrad's failure to be Fermi-Walker propagated along the accelerated trajectory, and we explain the interpretation of its precession.

Given $e_0^\alpha = u^\alpha$, we first note the tetrad leg e_3^α defined from Eq. (35d) is trivially orthogonal to e_0^α because $Y_{\alpha\beta}$ is antisymmetric. We can loosely interpret the quantity

$$l^\alpha \equiv Y_\beta^\alpha u^\beta \quad (38)$$

as the particle's leading-order specific orbital angular momentum (noting $l^\alpha l_\alpha = K^{(0)}$) [94]. If the spin of either body is co-directional with l^α , it will not exhibit precession. l^α plays a role analogous to the orbital angular momentum vector in post-Newtonian theory, and $\sqrt{K^{(0)}}$ has the units of a specific angular momentum. e_3^α is then a unit vector in the direction of this angular momentum. If we project the Killing-Yano tensor onto the intermediary tetrad $\{\sigma_0^\alpha, \sigma_A^\alpha\}$, we find only three independent components,

$$Y_{\alpha\beta} \sigma_0^\alpha \sigma_3^\beta = -\sqrt{K^{(0)}}, \quad (39a)$$

$$Y_{\alpha\beta} \sigma_1^\alpha \sigma_2^\beta = \frac{\mathcal{Z}}{\sqrt{K^{(0)}}}, \quad (39b)$$

$$Y_{\alpha\beta} \sigma_2^\alpha \sigma_3^\beta = -\frac{N}{\sqrt{K^{(0)}}}, \quad (39c)$$

having introduced the Killing-Yano scalar, which in Kerr spacetime takes the value $\mathcal{Z} = ra \cos \theta$. By definition,

$$\frac{D\sigma_3^\alpha}{d\tau} = \frac{1}{\sqrt{K^{(0)}}} Y_\beta^\alpha a^\beta - \frac{1}{K^{(0)}} \sigma_3^\alpha (K_{\delta\gamma} u^\gamma a^\delta), \quad (40)$$

where a^α is the covariant acceleration in $g_{\alpha\beta}$, given by Eq. (21a). Substituting Eq. (39) yields

$$\begin{aligned} \frac{D\sigma_3^\alpha}{d\tau} &= (a^\beta \sigma_\beta^3) u^\alpha + \frac{\mathcal{Z}}{K^{(0)}} (a^\beta \sigma_\beta^2) \sigma_1^\alpha \\ &\quad - \frac{1}{K^{(0)}} (\mathcal{Z} a^\beta \sigma_\beta^1 + N a^\beta \sigma_\beta^3) \sigma_2^\alpha. \end{aligned} \quad (41)$$

The terms proportional to σ_A^α on the right-hand side represent the failure of σ_3^α (and therefore of e_3^α) to be Fermi-Walker transported along the trajectory. From the formula above, we see that if the trajectory is confined to the equatorial plane, $\theta = \pi/2$ with $a^\theta = 0$, then σ_3^α is automatically Fermi-Walker transported.

At leading order, the secondary spin's precession corresponds to rigid rotation of the remaining two legs, e_1^α and e_2^α , around l^α . To understand this, first consider an arbitrary pair of mutually orthonormal vectors σ_1^α and σ_2^α that are also orthonormal to e_0^α and e_3^α . The triad σ_A^α (with $\sigma_3^\alpha \equiv e_3^\alpha$) necessarily satisfies

$$\frac{D\sigma_A^\alpha}{d\tau} = (\sigma_A^\alpha a_\alpha) u^\alpha + \omega_A^B \sigma_B^\alpha \quad (42)$$

for some antisymmetric $\omega_{AB} = -\omega_{BA}$, which represents the angular velocity of the triad relative to the particle's natural Fermi-Walker frame. Here and elsewhere, triad indices are raised with δ^{AB} and lowered with δ_{AB} . Equation (42) (together with $Du^\alpha/d\tau = a^\alpha$) is the most general form of propagation of any tetrad $\{u^\alpha, \sigma_A^\alpha\}$ along an accelerated worldline; this can be seen by writing $\frac{D\sigma_A^\alpha}{d\tau}$ as a linear combination of u^α and σ_B^α , contracting it with u_α or σ_α^C , and using the identities

$$u_\alpha \frac{D\sigma_A^\alpha}{d\tau} = -a_\alpha \sigma_A^\alpha \quad \text{and} \quad \sigma_\alpha^B \frac{D\sigma_A^\alpha}{d\tau} = -\sigma_A^\alpha \frac{D\sigma_\alpha^B}{d\tau}, \quad (43)$$

which follow from orthonormality. Such a calculation also shows

$$\omega_A^B = \frac{D\sigma_A^\alpha}{d\tau} \sigma_\alpha^B. \quad (44)$$

Any two triads that are orthogonal to u^α must be related to each other by a rigid rotation. Hence, starting from a generic $\sigma_A^\alpha = (\sigma_1^\alpha, \sigma_2^\alpha, e_3^\alpha)$, we can construct the last two Marck tetrad legs using the rotation in Eqs. (35b) and (35c). More compactly, we write the Marck triad as $e_A^\alpha = R_A^B \sigma_B^\alpha$, where

$$R_A^B = \begin{pmatrix} \cos \psi_s & \sin \psi_s & 0 \\ -\sin \psi_s & \cos \psi_s & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \mathbf{R}_z(\psi_s) \quad (45)$$

is the rotation matrix.

The covariant derivative of e_A^α along z^μ is

$$\frac{De_A^\alpha}{d\tau} = \frac{dR_A^B}{d\tau} \sigma_B^\alpha + R_A^B \frac{D\sigma_B^\alpha}{d\tau}. \quad (46)$$

First contracting this with u_α , and using $u_\alpha e_A^\alpha = 0 = u_\alpha \sigma_A^\alpha$, we find

$$u_\alpha \frac{De_A^\alpha}{d\tau} = -a_A = -R_A^B a_\alpha \sigma_B^\alpha. \quad (47)$$

Next contracting Eq. (46) with $\sigma_{C\alpha}$ and using $(R_A^B)^{-1} = R_A^B$, we find

$$R_A^B \frac{dR_{BC}}{d\tau} = -\frac{D\sigma_A^\alpha}{d\tau} \sigma_{C\alpha} = -\omega_{AC} + \mathcal{O}(\varepsilon, s). \quad (48)$$

The left-hand side can be immediately evaluated and the equation reduced to

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{d\psi_s}{d\tau} = -\omega_{AB} + \mathcal{O}(\varepsilon, s), \quad (49)$$

which implies

$$\frac{d\psi_s}{d\tau} = -\omega_{12} + \mathcal{O}(\varepsilon, s) = \omega_{21} + \mathcal{O}(\varepsilon, s). \quad (50)$$

In other words, the dyad $\{\sigma_1^\alpha, \sigma_2^\alpha\}$ precesses around l^α with an angular frequency ω_{21} . Note that Eq. (49) also requires $\omega_{31} = \mathcal{O}(\varepsilon) = \omega_{32}$, but this is satisfied by virtue of Eq. (41).

The spin precession frequency Ω_s will be an appropriate orbit average of ω_{21} . We note this frequency depends on *which* dyad $\{\sigma_1^\alpha, \sigma_2^\alpha\}$ we begin with. Our derivation of Eq. (50) is valid for any choice of this dyad; we have not made use of the specific choice in Eq. (36). If $\{\sigma_1^\alpha, \sigma_2^\alpha\}$ is chosen to have fixed orientation relative to a suitably stationary frame in the Kerr background [113], then the Marck tetrad legs precess around l^α with a frequency ω_{21} when measured in the stationary frame.

C. Fermi-Walker tetrad

Our goal is to find a Fermi-Walker tetrad $\{\hat{e}_0^\alpha, \hat{e}_A^\alpha\}$ such that each of its legs reduces to the corresponding Marck tetrad leg when $h_{\alpha\beta}^R$ and S^α vanish. We achieve this by first constructing a perturbation to the triad σ_A^α and then finding the necessary rotation to eliminate the frame's angular velocity.

Given $\hat{e}_0^\alpha = \hat{u}^\alpha = u^\alpha + \delta u^\alpha$, we construct a triad

$$\hat{\sigma}_A^\alpha = \sigma_A^\alpha + \delta\sigma_A^\alpha + \mathcal{O}(\varepsilon^2), \quad (51)$$

satisfying the same orthonormality conditions (26b) and (26c) as \hat{e}_A^α . Writing $\delta\sigma_A^\alpha$ as a linear combination of u^α and σ_A^α and using the fact that δu^α is parallel to u^α , we immediately find that the orthonormality conditions imply

$$\delta\sigma_A^\alpha = u^\alpha h_{\beta\gamma}^R u^\beta \sigma_A^\gamma - \frac{1}{2} \sigma^{B\alpha} h_{\beta\gamma}^R \sigma_B^\beta \sigma_A^\gamma. \quad (52)$$

This frame has an angular velocity

$$\hat{\omega}_A^B = \frac{\hat{D}\hat{\sigma}_A^\alpha}{d\hat{\tau}} \hat{\sigma}_\alpha^B \quad (53)$$

relative to a Fermi-Walker frame.

From the triad $\hat{\sigma}_A^\alpha$, we now construct our Fermi-Walker triad as

$$\hat{e}_A^\alpha \equiv \hat{R}_A^B \hat{\sigma}_B^\alpha, \quad (54)$$

in analogy with the background triad, where \hat{R}_A^B is a perturbed version of the rotation matrix (45). We can

enforce that \hat{e}_A^α is Fermi-Walker propagated by following the same steps as for the background tetrad, arriving again at Eq. (48) (with hats on all quantities). We rewrite that equation in terms of Euler angles $(\psi_s, \vartheta_s, \varphi_s)$, decomposing the rotation matrix in the form

$$\hat{\mathbf{R}} = \mathbf{R}_z(\varphi_s) \mathbf{R}_x(\vartheta_s) \mathbf{R}_z(\psi_s), \quad (55)$$

where $\mathbf{R}_z(\alpha)$ is a rotation around the ‘z’ axis, as defined in Eq. (45), and

$$\mathbf{R}_x(\vartheta_s) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \vartheta_s & \sin \vartheta_s \\ 0 & -\sin \vartheta_s & \cos \vartheta_s \end{pmatrix} \quad (56)$$

is a rotation around the ‘x’ axis. The angles are defined in the ranges $\psi_s, \varphi_s \in (0, 2\pi)$ and $\vartheta_s \in (0, \pi)$.

In terms of these angles, the hatted version of Eq. (48) reduces to

$$\frac{d\psi_s}{d\hat{\tau}} = \hat{\omega}_{21} - (\hat{\omega}_{31} \cos \psi_s + \hat{\omega}_{32} \sin \psi_s) \cot \vartheta_s, \quad (57a)$$

$$\frac{d\vartheta_s}{d\hat{\tau}} = \hat{\omega}_{32} \cos \psi_s - \hat{\omega}_{31} \sin \psi_s, \quad (57b)$$

$$\frac{d\varphi_s}{d\hat{\tau}} = (\hat{\omega}_{31} \cos \psi_s + \hat{\omega}_{32} \sin \psi_s) \csc \vartheta_s. \quad (57c)$$

These are the standard evolution equations for the Euler angles describing rigid-body motion, given the angular-velocity tensor $\hat{\omega}_{AB}$ [114]. When $\vartheta_s \rightarrow 0$, the rotation matrix in Eq. (56) becomes the identity matrix. While Eq. (57) is singular in that limit, the total rotation in Eq. (55) remains regular as it goes to $\mathbf{R}_z(\psi_s + \varphi_s)$ and the singular terms clearly cancel in the evolution of the sum of the two angles.

The angular velocity $\hat{\omega}_{AB}$ may be expressed in terms of its background equivalent and $h_{\alpha\beta}^R$ as

$$\hat{\omega}_{AB} = \frac{d\tau}{d\hat{\tau}} (\omega_{AB} + \delta\omega_{AB}). \quad (58)$$

Explicit evaluation of Eq. (53) yields

$$\delta\omega_{AB} = -h_{\alpha\beta;\gamma}^R u^\alpha \sigma_A^{[\beta} \sigma_B^{\gamma]} + \mathcal{O}(\varepsilon^2, s). \quad (59)$$

Thus, in Eq. (57) we can write the angular velocity components as

$$\hat{\omega}_{21} = \frac{d\tau}{d\hat{\tau}} [\omega_{21} + \delta\omega_{21} + \mathcal{O}(\varepsilon^2, s)], \quad (60a)$$

$$\hat{\omega}_{3B} = \delta\omega_{3B} + \mathcal{O}(\varepsilon^2, s), \quad (60b)$$

using the fact that $\omega_{31} = \mathcal{O}(s) = \omega_{32}$. We are neglecting the $\mathcal{O}(s)$ contributions to the angular velocity since an $\mathcal{O}(s)$ violation of the Fermi-Walker transport equation does not affect Eq. (33). The secondary’s spin contributions would be straightforward to include perturbatively by folding them into the definition of $\delta\omega_{AB}$.

Given this form of the angular velocity, we seek a solution to Eq. (57) of the form

$$\psi_s = \tilde{\psi}_s - \delta\varphi \cot \vartheta_0 + \mathcal{O}(\varepsilon^2), \quad (61a)$$

$$\vartheta_s = \vartheta_0 + \delta\vartheta + \mathcal{O}(\varepsilon^2), \quad (61b)$$

$$\varphi_s = \delta\varphi \csc \vartheta_0 + \mathcal{O}(\varepsilon^2), \quad (61c)$$

where $\delta\vartheta$ and $\delta\varphi$ are linear in $h_{\alpha\beta}^R$. The $\delta\varphi$ term in Eq. (61a) accounts for the oscillatory terms in Eq. (57a). ϑ_0 is an arbitrary constant designed to avoid the singularity of the Euler angles at $\vartheta_s = 0$; we will ultimately take the limit $\vartheta_0 \rightarrow 0$ to ensure our perturbed tetrad reduces to the unperturbed one in the $\epsilon \rightarrow 0$ limit. To facilitate taking the $\vartheta_0 \rightarrow 0$ limit, we have factored out singular functions of ϑ_0 , allowing us to work with variables $\tilde{\psi}_s$ and $\delta\varphi$ that are smooth at $\vartheta_0 = 0$.

Substituting the ansatz into Eq. (57), we quickly find

$$\frac{d\tilde{\psi}_s}{d\tau} = \omega_{21} + \delta\omega_{21}, \quad (62)$$

$$\frac{d\delta\vartheta}{d\tau} = \delta\omega_{32} \cos \tilde{\psi}_s - \delta\omega_{31} \sin \tilde{\psi}_s, \quad (63)$$

$$\frac{d\delta\varphi}{d\tau} = \delta\omega_{31} \cos \tilde{\psi}_s + \delta\omega_{32} \sin \tilde{\psi}_s. \quad (64)$$

Note that unlike the original variable ψ_s , the new variable $\tilde{\psi}_s$ has a non-oscillatory rate of change.³ It represents the mean precession angle, the secularly growing piece of the original phase ψ_s ; Eq. (61a) can hence be interpreted as a near-identity averaging transformation [79].

The angle $\delta\vartheta$ represents nutation of the Fermi-Walker frame. Given the form of the nutation equation (63), we can factor out its precession dependence with an ansatz

$$\delta\vartheta = \delta\vartheta_c \cos \tilde{\psi}_s + \delta\vartheta_s \sin \tilde{\psi}_s, \quad (65)$$

where $\delta\vartheta_c$ and $\delta\vartheta_s$ are independent of $\tilde{\psi}_s$.⁴ This reduces Eq. (63) to coupled differential equations for the nutation degrees of freedom $\delta\vartheta_c$ and $\delta\vartheta_s$:

$$\frac{d\delta\vartheta_c}{d\tau} - \omega_{12}\delta\vartheta_s = -\delta\omega_{23}, \quad (66)$$

$$\frac{d\delta\vartheta_s}{d\tau} + \omega_{12}\delta\vartheta_c = \delta\omega_{13}. \quad (67)$$

We can also straightforwardly check that Eq. (64) can be satisfied with an ansatz in terms of these same functions $\delta\vartheta_c$ and $\delta\vartheta_s$:

$$\delta\varphi = -\delta\vartheta_s \cos \tilde{\psi}_s + \delta\vartheta_c \cos \tilde{\psi}_s. \quad (68)$$

³ More precisely, $\frac{d\tilde{\psi}_s}{d\tau}$ is independent of $\tilde{\psi}_s$. It can still have oscillatory dependence on the orbital phases.

⁴ This ansatz would need to be modified when including the secondary spin since $\delta\omega_{AB}$ would also depend on $\tilde{\psi}_s$ via precession terms in the spin-curvature coupling force.

The necessary conditions for the parallel transport of the spin vector in the effective metric are finally boiled down to the evolution of the precession phase variable $\tilde{\psi}_s$ and two nutation variables, $\delta\vartheta_c$ and $\delta\vartheta_s$. Their evolution is determined by Eq. (62), Eq. (66), and Eq. (67), respectively. See Appendix A for a sketch of the solution to Eqs. (66) and (67).

D. Summary

We now return to the spin vector itself,

$$\hat{S}^\alpha = \hat{S}^A \hat{e}_A^\alpha. \quad (69)$$

Without loss of generality, we set

$$\hat{S}^1 = \chi_\perp, \quad \hat{S}^2 = 0, \quad \hat{S}^3 = \chi_\parallel. \quad (70)$$

The choice $\hat{S}^2 = 0$ corresponds to a choice of origin for the precession angle $\tilde{\psi}_s$; see Ref. [4] for discussion of analogous freedom in the nonspinning case. By construction, according to Eq. (33), the two spin magnitudes are constants:

$$\frac{d\chi_\perp}{d\tau} = 0 = \frac{d\chi_\parallel}{d\tau}, \quad (71)$$

including during orbital resonances. These spin components are hence freely specifiable constants describing the particle, akin to its mass m_2 .

We now substitute Eqs. (54), (61), (65), and (68) into Eq. (69) and take the limit $\vartheta_0 \rightarrow 0$. The result is

$$\hat{S}^\alpha = S^\alpha + \varepsilon \delta S^\alpha + \mathcal{O}(\varepsilon^2) \quad (72)$$

where the two terms are parameterized as

$$S^\alpha = \chi_\parallel \sigma_3^\alpha + \chi_\perp \cos \tilde{\psi}_s \sigma_1^\alpha + \chi_\perp \sin \tilde{\psi}_s \sigma_2^\alpha, \quad (73a)$$

$$\begin{aligned} \delta S^\alpha &= \chi_\parallel \delta \sigma_3^\alpha + \chi_\parallel \delta \vartheta_s \sigma_1^\alpha - \chi_\parallel \delta \vartheta_c \sigma_2^\alpha \\ &\quad + \chi_\parallel (\delta \vartheta_c \sin \tilde{\psi}_s - \delta \vartheta_s \cos \tilde{\psi}_s) \sigma_3^\alpha \\ &\quad + \chi_\perp \cos \tilde{\psi}_s \delta \sigma_1^\alpha + \chi_\perp \sin \tilde{\psi}_s \delta \sigma_2^\alpha, \end{aligned} \quad (73b)$$

with $\delta \sigma_A^\alpha$ as defined in Eq. (52). As a consequence of the normalisations of the tetrad legs, we have

$$\hat{S}^\alpha \hat{S}_\alpha = S^\alpha S_\alpha = \chi^2 = \chi_\parallel^2 + \chi_\perp^2. \quad (74)$$

The parameterization (73) has cleanly separated oscillatory and non-oscillatory dependence on the precession phase $\tilde{\psi}_s$. If we average over that phase, we obtain

$$\langle S^\alpha \rangle_{\tilde{\psi}_s} = \chi_\parallel \sigma_3^\alpha, \quad (75a)$$

$$\langle \delta S^\alpha \rangle_{\tilde{\psi}_s} = \chi_\parallel (\delta \sigma_3^\alpha + \delta \vartheta_s \sigma_1^\alpha - \delta \vartheta_c \sigma_2^\alpha). \quad (75b)$$

Finally expressing the spin tensor as $\hat{S}^{\alpha\beta} = S^{\alpha\beta} + \varepsilon \delta S^{\alpha\beta} + \mathcal{O}(\varepsilon^2)$, then by Eq. (19) we have

$$S^{\mu\nu} = -\epsilon^{\mu\nu\alpha\beta} S_\alpha u_\beta, \quad (76a)$$

$$\delta S^{\alpha\beta} = \epsilon^{\alpha\beta}_{\gamma\lambda} u^\gamma \delta S^\lambda + \frac{1}{2} P^{\gamma\lambda} h_{\gamma\lambda}^R S^{\alpha\beta} - 2 h_\lambda^{R[\beta} S^{\alpha]\lambda}. \quad (76b)$$

In the next section we utilize our results here to develop a complete 1PA expansion of the orbital motion and Einstein field equations. The spin magnitudes χ_\parallel and χ_\perp are exactly constant, but the nutation angles $\delta\vartheta_c$ and $\delta\vartheta_s$ will generally evolve. In principle, their evolution contributes to the dissipative self-force at the same order as the leading linear-in-spin energy and angular momentum fluxes and must be taken into account. However, in Sec. IV D and Appendix D, we show that the nutation equations do not need to be explicitly solved to compute the inspiral (and waveform) at 1PA order.

IV. MULTISCALE EXPANSION FOR GENERIC ORBITS IN KERR SPACETIME

In the Introduction we recalled the structure of the multiscale expansion for a nonspinning secondary on a generic orbit in Kerr spacetime. Here, using our results from the previous section, we derive the extension to the case of a spinning secondary. The goal is to solve the Einstein field equations (14) coupled to the equations of motion (21).

Throughout this section, for visual clarity we suppress functional dependence on the constant parameters $m_1^{(0)}$, $\chi_1^{(0)}$, m_2 , χ_\parallel , and χ_\perp .

A. Orbital motion

As summarized in Sec. III D, we have reduced the equation for the secondary spin, Eq. (21b), to evolution equations for a precession angle and two nutation angles, Eqs. (62), (66), and (67). We now wish to similarly reduce the equation of orbital motion (21a) to evolution equations of the form (2) and (3). We adopt Boyer-Lindquist coordinates (t, x^i) , with spatial coordinates $x^i = (r, \theta, \phi)$.

We first rewrite Eq. (21a) in terms of our solution for the spin vector:

$$\begin{aligned} \frac{Du^\mu}{d\tau} &= -\frac{1}{2} P^{\mu\nu} (g_\nu^\lambda - h_\nu^{R(1)\lambda}) (2h_{\lambda\rho;\sigma}^R - h_{\rho\sigma;\lambda}^R) u^\rho u^\sigma \\ &\quad - \frac{m_2}{2} R^\mu_{\alpha\beta\gamma} \left(1 - \frac{1}{2} h_{\rho\sigma}^{R(1)} u^\rho u^\sigma \right) u^\alpha S^{\beta\gamma} \\ &\quad - \frac{m_2}{2} R^\mu_{\alpha\beta\gamma} u^\alpha \delta S^{\beta\gamma} \\ &\quad + \frac{m_2}{2} P^{\mu\nu} (2h_{\nu(\alpha;\beta)\gamma}^R - h_{\alpha\beta;\nu\gamma}^{R(1)}) u^\alpha S^{\beta\gamma} \\ &\quad + \mathcal{O}(\varepsilon^3, s^2), \end{aligned} \quad (77)$$

where $S^{\alpha\beta}$ and $\delta S^{\alpha\beta}$ are given by Eq. (76) with Eq. (73). To mesh with the multiscale expansion of the Einstein equations and the waveform generation scheme, it is convenient to use t , rather than τ as the parameter along the worldline, in which case Eq. (77) becomes

$$\frac{Dz^\mu}{dt} = a^\mu / (u^t)^2 + \kappa z^\mu, \quad (78)$$

where an overdot denotes d/dt , a^μ is the right-hand side of Eq. (77), $\kappa \equiv -\frac{1}{u^t} \frac{du^t}{dt}$, and $g_{\alpha\beta} u^\alpha u^\beta = -1$ implies

$$\dot{u}^t = [-(g_{tt} + 2g_{ti}\dot{z}^i + g_{\alpha\beta}\dot{z}^i\dot{z}^j)]^{-1/2}. \quad (79)$$

Equation (78) can be recast in terms of slow and fast variables following Ref. [76], starting from osculating geodesics [115, 116] and then performing averaging transformations [79]. We do not belabor the details as this type of procedure is thoroughly illustrated elsewhere [79, 96, 117]; see, in particular, Ref. [63], which carried out an analysis very similar to the one we do here but omitted dissipative $\mathcal{O}(\varepsilon^2)$ terms in the equations of motion.

The analysis in Ref. [76] is valid for generic accelerated orbits in Kerr, and we will make extensive use of equations in it. However, our treatment differs from Ref. [76] in that we work with t as our time parameter from the beginning, while Ref. [76] begins with a complete analysis in terms of Mino time λ and then transforms to variables based on t .⁵ Differential equations in terms of the two variables are related by the factor

$$\frac{dt}{d\lambda} \equiv f_t, \quad (80)$$

given in Eq. (205) of Ref. [76].

1. Osculating geodesic equations

As in Ref. [76], we first introduce quasi-Keplerian orbital elements $\pi_i = \{p, e, \iota\}$ and phases $\psi^i = \{\psi^r, \psi^\theta, \phi\}$, with which we parameterize the Boyer-Lindquist trajectory z^i and velocity \dot{z}^i as follows:

$$r(\psi^i, \pi_i) = \frac{pm_1}{1 + e \cos \psi^r}, \quad (81a)$$

$$\cos \theta(\psi^i, \pi_i) = \cos \iota \cos \psi^\theta, \quad (81b)$$

and

$$\dot{z}^i(\psi^j, \pi_j) = \frac{\partial z^i}{\partial \psi^j} \omega_{(0)}^j(\psi^k, \pi_k). \quad (82)$$

Here $\omega_{(0)}^j$ is the expression for $\dot{\psi}^j$ on a Kerr geodesic, given by

$$\omega_{(0)}^j = \frac{d\lambda}{dt} f_j = f_j/f_t, \quad (83)$$

with f_t , f_r , f_θ , and f_ϕ as given in Eqs. (205), (216), (217), and (206) of Ref. [76], respectively.

The parameters p , e , and ι are referred to as osculating orbital elements; they correspond to the eccentricity, semi-latus rectum, and maximum inclination of the geodesic that is instantaneously comoving with the accelerated orbit. If the motion were geodesic, we would have $d\pi_i/dt = 0$ and $d\psi^i/dt = \omega_{(0)}^i$, while for the accelerated motion we have $d\pi_i/dt \neq 0$ and $d\psi^i/dt \neq \omega_{(0)}^i$. We can also simply think of Eqs. (81) and (82) as a coordinate transformation on the particle's orbital phase space, $(z^i, \dot{z}^i) \mapsto (\psi^i, \pi_i)$. The full list of slowly evolving binary parameters we denote with $\varpi_I = \{\pi_i, \delta m_1, \delta \chi_1\}$.

Applying the chain rule $d/dt = \dot{\psi}^i \partial_{\psi^i} + \dot{\varpi}_I \partial_{\varpi_I}$ on the left-hand side of Eq. (82), we obtain the kinematical equation

$$\frac{\partial z^i}{\partial \psi^j} \left(\dot{\psi}^j - \omega_{(0)}^j \right) + \frac{\partial z^i}{\partial \pi_j} \dot{\pi}_j = 0. \quad (84)$$

Substituting Eqs. (79)–(82) into Eq. (78) and appealing to the same chain rule, we obtain the dynamical equation

$$\frac{\partial \dot{z}^i}{\partial \psi^j} \left(\dot{\psi}^j - \omega_{(0)}^j \right) + \frac{\partial \dot{z}^i}{\partial \pi_j} \dot{\pi}_j = a^i/(u^t)^2 + (\kappa - \kappa_{(0)}) \dot{z}^i, \quad (85)$$

where

$$\kappa - \kappa_{(0)} = -\frac{1}{u^t} \left[\frac{\partial u^t}{\partial \psi^i} (\dot{\psi}^i - \omega_{(0)}^i) + \frac{\partial u^t}{\partial \pi_j} \dot{\pi}_j \right]. \quad (86)$$

In obtaining Eq. (85) we have used the Kerr geodesic equation in the form $\frac{\partial \dot{z}^i}{\partial \psi^j} \omega_{(0)}^j + \Gamma_{\beta\gamma}^i \dot{z}^\beta \dot{z}^\gamma = \kappa_{(0)} \dot{z}^i$. Note we have restricted to spatial components of the equation of orbital motion because $z^t = t$ and $\dot{z}^t = 1$ trivially.

We can rearrange Eqs. (84) and (85) to obtain equations for $\dot{\psi}^i$ and $\dot{\pi}_i$, while the evolution of the primary's mass and spin is described by Eq. (4) and Eq. (5) respectively. Given the form of the acceleration (77), these equations take the form

$$\frac{d\psi^i}{dt} = \omega_{(0)}^i(\psi^j, \pi_j) + \varepsilon \omega_{(1)}^i(\psi^j, \varpi_J, \tilde{\psi}_s) + \mathcal{O}(\varepsilon^2), \quad (87)$$

$$\frac{d\pi_i}{dt} = \varepsilon f_i^{(0)}(\psi^j, \varpi_J, \tilde{\psi}_s) + \varepsilon^2 f_i^{(1)}(\psi^j, \varpi_J, \tilde{\psi}_s) + \mathcal{O}(\varepsilon^3). \quad (88)$$

Here the ‘frequency’ corrections $\omega_{(1)}^i$ and forcing functions $f_i^{(n)}$ are linear combinations of the acceleration components, which we write as

$$\omega_{(1)}^i(\psi^i, \varpi_i, \tilde{\psi}_s) = A_{ij}^i(\psi^i, \pi_i) a_{(1)}^j(\psi^i, \varpi_I, \tilde{\psi}_s), \quad (89)$$

$$f_i^{(n)}(\psi^i, \varpi_i, \tilde{\psi}_s) = B_{ij}(\psi^i, \pi_i) a_{(n+1)}^j(\psi^i, \varpi_I, \tilde{\psi}_s), \quad (90)$$

with the right-hand side of Eq. (77) expanded as

$$a^i = \varepsilon a_{(1)}^i(\psi^i, \varpi_I, \tilde{\psi}_s) + \varepsilon^2 a_{(2)}^i(\psi^i, \varpi_I, \tilde{\psi}_s) + \mathcal{O}(\varepsilon^3, s^2). \quad (91)$$

⁵ We also make several changes of notation from Ref. [76]: $p^i \rightarrow \pi_i$, $\varphi_i \rightarrow \dot{\psi}^i$, $p_\varphi^i \rightarrow \dot{\pi}_i$, and $\mathcal{P}^\alpha \rightarrow \varpi_I$, along with various less noteworthy alterations.

The coefficients $A^i{}_j$ and B_{ij} can be read off Eqs. (289)–(291) and (293) in Ref. [76], after dividing those equations by $dt/d\lambda = \ell_t$.

Equations (87) and (88) are coupled to the spin precession phase, whose evolution equation (62) we can write as

$$\frac{d\tilde{\psi}_s}{dt} = \omega_{s(0)}(\psi^j, \pi_j) + \varepsilon\omega_{s(1)}(\psi^j, \varpi_J) + \mathcal{O}(\varepsilon^2), \quad (92)$$

where

$$\omega_{s(0)} = \frac{\omega_{21}}{u^t}, \quad \text{and} \quad \omega_{s(1)} = \frac{\delta\omega_{21}}{u^t} \quad (93)$$

with $\delta\omega_{21}$ given in Eq. (59).

It is also useful to introduce geodesic action angles, which we denote $\dot{\psi}_{(0)}^i$. For geodesics, these satisfy

$$\frac{d\dot{\psi}_{(0)}^i}{dt} = \Omega_{(0)}^i(\pi_i) \quad (\text{geodesic case}), \quad (94)$$

growing exactly linearly in time. The frequencies $\Omega_{(0)}^i$ are an appropriate average of the ‘frequencies’ $\omega_{(0)}^i$ [76], such that $\dot{\psi}_{(0)}^i$ represents the non-oscillatory part of ψ^i . Such angle variables and their frequencies were first derived in Ref. [118]. We review them, and appeal to them in deriving some of our results, in Appendix C.

2. Averaging transformation

We now transform to new variables $\{\dot{\psi}^i, \dot{\varpi}_J, \dot{\psi}_s\}$ that contain no oscillations in their rates of change, meaning they satisfy equations of the form

$$\frac{d\dot{\psi}^i}{dt} = \Omega_{(0)}^i(\dot{\pi}_j) + \varepsilon\Omega_{(1)}^i(\dot{\varpi}_J) + \mathcal{O}(\varepsilon^2), \quad (95a)$$

$$\frac{d\dot{\pi}_i}{dt} = \varepsilon F_i^{(0)}(\dot{\pi}_j) + \varepsilon^2 F_i^{(1)}(\dot{\varpi}_J) + \mathcal{O}(\varepsilon^3), \quad (95b)$$

$$\frac{d\dot{\psi}_s}{dt} = \Omega_{s(0)}(\dot{\pi}_j) + \varepsilon\Omega_{s(1)}(\dot{\varpi}_J) + \mathcal{O}(\varepsilon^2). \quad (95c)$$

The new variables are related to the old ones by an averaging transformation of the form

$$\psi^i = \dot{\psi}^i + \Delta\psi^i(\dot{\psi}^j, \dot{\pi}_j) + \varepsilon\delta\psi^i(\dot{\psi}^j, \dot{\varpi}_J, \dot{\psi}_s) + \mathcal{O}(\varepsilon^2), \quad (96a)$$

$$\pi_i = \dot{\pi}_i + \varepsilon\delta\pi_i(\dot{\psi}^j, \dot{\varpi}_J, \dot{\psi}_s) + \mathcal{O}(\varepsilon^2), \quad (96b)$$

$$\tilde{\psi}_s = \dot{\psi}_s + \Delta\tilde{\psi}_s(\dot{\psi}^j, \dot{\pi}_j) + \varepsilon\delta\psi_s(\dot{\psi}^j, \dot{\varpi}_J) + \mathcal{O}(\varepsilon^2), \quad (96c)$$

where all functions are 2π -periodic in $\dot{\psi}^j$ and $\dot{\psi}_s$.⁶ The zeroth-order terms in this transformation are precisely

⁶ If δm_1 and $\delta\chi_1$ are defined via integrals over the primary’s perturbed horizon, they will generally contain oscillatory phase de-

pendence which may be removed in the averaging transformation by defining the new variables $\delta\dot{m}_1$ and $\delta\dot{\chi}_1$. In practice, when identifying the corrections to the primary’s parameters at the level of the field equations or by integrating the average horizon fluxes in Eqs. (4) and (5), we work directly with the non-oscillatory parameters. Thus in this work it is sufficient to take $\delta\dot{m}_1 = \delta m_1$ and $\delta\dot{\chi}_1 = \delta\chi_1$.

The functions $\Delta\psi^i$, $\delta\psi^i$, $\delta\pi_i$, $\Delta\tilde{\psi}_s$, and $\delta\psi_s$ are chosen to remove oscillations from the equations of motion. They, along with the functions on the right-hand sides of the equations of motion (95), can be determined (up to a residual freedom that we discuss below) by substituting Eq. (96), with Eq. (95), into Eqs. (87), (88), and (92).

After these substitutions, the leading-order terms in Eqs. (87), (88), and (92) become

$$\Omega_{(0)}^i + \Omega_{(0)}^j \frac{\partial\Delta\psi^i}{\partial\dot{\psi}^j} = \omega_{(0)}^i(\psi_{(0)}^j, \dot{\pi}_j), \quad (97a)$$

$$F_i^{(0)} + \Omega_{(0)}^j \frac{\partial\delta\pi_i}{\partial\dot{\psi}^j} + \Omega_{s(0)} \frac{\partial\delta\pi_i}{\partial\dot{\psi}_s} = f_i^{(0)}(\psi_{(0)}^j, \dot{\varpi}_J, \tilde{\psi}_{s(0)}), \quad (97b)$$

$$\Omega_{s(0)} + \Omega_{(0)}^j \frac{\partial\Delta\tilde{\psi}_s}{\partial\dot{\psi}^j} = \omega_{s(0)}(\psi_{(0)}^j, \dot{\pi}_j), \quad (97c)$$

where

$$\psi_{(0)}^i \equiv \dot{\psi}^i + \Delta\psi^i(\dot{\psi}^j, \dot{\pi}_j), \quad (98)$$

$$\tilde{\psi}_{s(0)} \equiv \dot{\psi}_s + \Delta\tilde{\psi}_s(\dot{\psi}^j, \dot{\pi}_j), \quad (99)$$

are the zeroth-order terms in Eq. (96a) and (96c).

Averaging Eq. (97) over angles reveals that the leading term in each of the equations of motion (95) is simply the average of the corresponding term in Eqs. (87), (88), and (92):

$$\Omega_{(0)}^i(\dot{\pi}_j) = \left\langle \omega_{(0)}^i(\psi_{(0)}^j, \dot{\pi}_j) \right\rangle, \quad (100a)$$

$$F_i^{(0)}(\dot{\pi}_j) = \left\langle f_i^{(0)}(\psi_{(0)}^j, \dot{\varpi}_J, \tilde{\psi}_{s(0)}) \right\rangle, \quad (100b)$$

$$\Omega_{s(0)}(\dot{\pi}_j) = \left\langle \omega_{s(0)}(\psi_{(0)}^j, \dot{\pi}_j) \right\rangle, \quad (100c)$$

where

$$\langle \cdot \rangle \equiv \frac{1}{(2\pi)^4} \oint \cdot d^4\dot{\psi} \quad (101)$$

is the average over $\dot{\psi}^i$ and $\dot{\psi}_s$. Note that $\dot{\psi}^\phi$ does not appear in the equations of motion, meaning $\langle \cdot \rangle$ reduces to $\frac{1}{(2\pi)^3} \oint \cdot d\dot{\psi}^r d\dot{\psi}^\theta d\dot{\psi}_s$ in the above expressions. We also note that the 0PA forcing function $F_i^{(0)}$ is independent of χ_2 , δm_1 , and $\delta\chi_1$ despite the fact that $f_i^{(0)}$ depends on these quantities. This is because the first-order MPD force and the linear force due to δm_1 and $\delta\chi_1$ are purely

conservative. We remind the reader of this property in more detail in Appendix B.

Equations (100) eliminate the non-oscillatory parts of Eqs. (87), (88), and (92). The oscillatory parts then determine the functions $\Delta\psi^i$, $\delta\pi_i$, and $\Delta\tilde{\psi}_s$:

$$\Omega_{(0)}^j \frac{\partial \Delta\psi^i}{\partial \dot{\psi}^j} = \omega_{(0)}^i - \langle \omega_{(0)}^i \rangle, \quad (102a)$$

$$\Omega_{(0)}^j \frac{\partial \delta\pi_i}{\partial \dot{\psi}^j} + \Omega_{s(0)} \frac{\partial \delta\pi_i}{\partial \dot{\psi}_s} = f_i^{(0)} - \langle f_i^{(0)} \rangle, \quad (102b)$$

$$\Omega_{(0)}^j \frac{\partial \Delta\tilde{\psi}_s}{\partial \dot{\psi}^j} = \omega_{s(0)} - \langle \omega_{s(0)} \rangle. \quad (102c)$$

At the first subleading order, the averaged part of Eqs. (87), (88), and (92), with Eq. (96) and (95) substituted, yields the first subleading terms in Eqs. (87), (88), and (92):

$$\begin{aligned} \Omega_{(1)}^i &= \langle \omega_{(1)}^i \rangle + \left\langle \delta\pi_j \frac{\partial \omega_{(0)}^i}{\partial \dot{\pi}_j} + \delta\psi^j \frac{\partial \omega_{(0)}^i}{\partial \dot{\psi}_{(0)}^j} \right\rangle \\ &\quad - F_j^{(0)} \frac{\partial \langle \Delta\psi^i \rangle}{\partial \dot{\pi}_j}, \end{aligned} \quad (103a)$$

$$\begin{aligned} F_i^{(1)} &= \langle f_i^{(1)} \rangle + \left\langle \delta\pi_j \frac{\partial f_i^{(0)}}{\partial \dot{\pi}_j} + \delta\psi^j \frac{\partial f_i^{(0)}}{\partial \dot{\psi}_{(0)}^j} \right\rangle \\ &\quad - F_j^{(0)} \frac{\partial \langle \delta\pi_i \rangle}{\partial \dot{\pi}_j}, \end{aligned} \quad (103b)$$

$$\begin{aligned} \Omega_{s(1)} &= \langle \omega_{s(1)} \rangle + \left\langle \delta\pi_j \frac{\partial \omega_{s(0)}}{\partial \dot{\pi}_j} + \delta\psi^j \frac{\partial \omega_{s(0)}}{\partial \dot{\psi}_{(0)}^j} \right\rangle \\ &\quad - F_j^{(0)} \frac{\partial \langle \Delta\tilde{\psi}_s \rangle}{\partial \dot{\pi}_j}, \end{aligned} \quad (103c)$$

where functions of (ψ^i, π_i) on the right-hand side are evaluated at $(\dot{\psi}_{(0)}^i, \dot{\pi}_i)$.

Equations (103) require $\delta\psi^i$, the subleading term in the transformation (96a). This function is determined by the complete (not averaged) $\mathcal{O}(\varepsilon)$ part of Eq. (87):

$$\begin{aligned} \Omega_{(1)}^i + \Omega_{(1)}^j \frac{\partial \Delta\psi^i}{\partial \dot{\psi}^j} + F_j^{(0)} \frac{\partial \Delta\psi^i}{\partial \dot{\pi}_j} + \Omega_{(0)}^j \frac{\partial \delta\psi^i}{\partial \dot{\psi}^j} + \Omega_{s(0)} \frac{\partial \delta\psi^i}{\partial \dot{\psi}_s} \\ = \omega_{(1)}^i + \delta\pi_j \frac{\partial \omega_{(0)}^i}{\partial \dot{\pi}_j} + \delta\psi^j \frac{\partial \omega_{(0)}^i}{\partial \dot{\psi}_{(0)}^j}. \end{aligned} \quad (104)$$

Equation (104) is complicated by the fact that $\Omega_{(1)}^i$ depends on $\delta\psi^i$ through Eq. (103a). However, it admits a series solution for $\delta\psi^i$ of the form (325) in Ref. [76]. Alternatively, we can find $\delta\psi^i$ using the geodesic action angles $\dot{\psi}_{(0)}^i$ as a stepping stone. We present that method in Appendix C.

Finally, the (omitted) order- ε^2 term in the transformation (96b) and the order- ε term in the transformation (96c) are responsible for satisfying the oscillatory

part of Eqs. (88) and (92) at first subleading order. However, these terms in the transformations are not explicitly required at 1PA order because (unlike $\delta\psi^i$) they do not contribute to the evolution equations (95) at the given orders, nor (again, unlike $\delta\psi^i$) do they enter the field equations through second order.

Note that the criterion of removing oscillations only determines the oscillatory parts of the functions $\Delta\psi^i$, $\delta\pi_i$, and $\Delta\tilde{\psi}_s$ (and their higher-order analogues); their averages $\langle \Delta\psi^i \rangle$, $\langle \Delta\pi_i \rangle$, and $\langle \Delta\tilde{\psi}_s \rangle$ do not enter into equations such as (102). Hence, these non-oscillatory pieces of the transformation can be chosen arbitrarily. However, such a choice affects the subleading terms in the equations of motion (87), (88), and (92). Concretely, we see from Eqs. (103) that the non-oscillatory functions have the following contribution to the subleading frequency corrections and forcing functions:

$$\Delta\Omega_{(1)}^i = \langle \delta\pi_j \rangle \frac{\partial \Omega_{(0)}^i}{\partial \dot{\pi}_j} - F_j^{(0)} \frac{\partial \langle \Delta\psi^i \rangle}{\partial \dot{\pi}_j}, \quad (105a)$$

$$\Delta F_i^{(1)} = \langle \delta\pi_j \rangle \frac{\partial F_i^{(0)}}{\partial \dot{\pi}_j} - F_j^{(0)} \frac{\partial \langle \delta\pi_i \rangle}{\partial \dot{\pi}_j}, \quad (105b)$$

$$\Delta\Omega_{s(1)} = \langle \delta\pi_j \rangle \frac{\partial \Omega_{s(0)}}{\partial \dot{\pi}_j} - F_j^{(0)} \frac{\partial \langle \Delta\tilde{\psi}_s \rangle}{\partial \dot{\pi}_j}, \quad (105c)$$

where we have used $\langle \langle \delta\pi_j \rangle \frac{\partial}{\partial \dot{\pi}_j} \cdot \rangle = \langle \delta\pi_j \rangle \frac{\partial}{\partial \dot{\pi}_j} \langle \cdot \rangle$ and Eqs. (100). We return to this residual gauge freedom in later sections.

B. Field equations

In the multiscale expansion of the field equations, all time dependence is encoded in a dependence on the mechanical phase-space variables, such that $h_{\alpha\beta} = h_{\alpha\beta}(\dot{\psi}^i, \dot{\varpi}_I, \dot{\psi}_s, x^i, \varepsilon)$ [76, 119]. The expansion at small ε then becomes

$$h_{\alpha\beta} = \varepsilon \dot{h}_{\alpha\beta}^{(1)}(\dot{\psi}^i, \dot{\varpi}_I, x^i) + \varepsilon^2 \dot{h}_{\alpha\beta}^{(2)}(\dot{\psi}^i, \dot{\varpi}_I, \dot{\psi}_s, x^i) + \dots, \quad (106)$$

where all functions are 2π -periodic in $\dot{\psi}^i$ and $\dot{\psi}_s$. The coefficients here are not identical to those in Eq. (8). Instead, each coefficient in Eq. (8) must be re-expanded in multiscale form,

$$h_{\alpha\beta}^{(n)}(t, x^i, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k h_{\alpha\beta}^{(n,k)}(\dot{\psi}^i(t, \varepsilon), \dot{\varpi}_I(t, \varepsilon), \dot{\psi}_s(t, \varepsilon), x^i), \quad (107)$$

such that

$$\dot{h}_{\alpha\beta}^{(1)}(\dot{\psi}^i, \dot{\varpi}_I, x^i) = h_{\alpha\beta}^{(1,0)}(\dot{\psi}^i, \dot{\varpi}_I, x^i), \quad (108)$$

$$\begin{aligned} \dot{h}_{\alpha\beta}^{(2)}(\dot{\psi}^i, \dot{\varpi}_I, \dot{\psi}_s, x^i) &= h_{\alpha\beta}^{(2,0)}(\dot{\psi}^i, \dot{\varpi}_I, \dot{\psi}_s, x^i) \\ &\quad + h_{\alpha\beta}^{(1,1)}(\dot{\psi}^i, \dot{\varpi}_I, \dot{\psi}_s, x^i), \end{aligned} \quad (109)$$

and so on.⁷

When substituting the expansion (106) into the field equations, we apply the chain rule

$$\frac{\partial}{\partial t} = \frac{d\dot{\psi}^i}{dt} \frac{\partial}{\partial \dot{\psi}^i} + \frac{d\dot{\varpi}_I}{dt} \frac{\partial}{\partial \dot{\varpi}_I} + \frac{d\dot{\psi}_s}{dt} \frac{\partial}{\partial \dot{\psi}_s} \quad (110a)$$

$$\begin{aligned} &= \Omega_{(0)}^i \frac{\partial}{\partial \dot{\psi}^i} + \Omega_{s(0)} \frac{\partial}{\partial \dot{\psi}_s} \\ &\quad + \varepsilon \left(\Omega_{(1)}^i \frac{\partial}{\partial \dot{\psi}^i} + F_I^{(0)} \frac{\partial}{\partial \dot{\varpi}_I} + \Omega_{s(1)} \frac{\partial}{\partial \dot{\psi}_s} \right) + \mathcal{O}(\varepsilon^2), \end{aligned} \quad (110b)$$

having denoted $F_I^{(0)} = \{F_i^{(0)}, \dot{\mathcal{E}}_{\mathcal{H}}^{(1)}, \dot{\mathcal{L}}_{\mathcal{H}}^{(1)}\}$. This implies expansions

$$\delta G_{\alpha\beta}(\dot{h}^{(n)}) = \delta G_{\alpha\beta}^{(0)}(\dot{h}^{(n)}) + \varepsilon \delta G_{\alpha\beta}^{(1)}(\dot{h}^{(n)}) + \mathcal{O}(\varepsilon^2), \quad (111)$$

$$\delta^2 G_{\alpha\beta}(\dot{h}^{(1)}, \dot{h}^{(1)}) = \delta^2 G_{\alpha\beta}^{(0)}(\dot{h}^{(1)}, \dot{h}^{(1)}) + \mathcal{O}(\varepsilon). \quad (112)$$

We can further decompose the field equations into the Fourier domain by expanding the metric perturbations in discrete Fourier series,

$$\dot{h}_{\alpha\beta}^{(1)} = \sum_{\mathbf{k} \in \mathbb{Z}^3} \dot{h}_{\alpha\beta}^{(1,\mathbf{k})} (\dot{\varpi}_I, x^i) e^{-i\dot{\psi}_{\mathbf{k}}}, \quad (113)$$

$$\dot{h}_{\alpha\beta}^{(2)} = \sum_{\mathbf{k} \in \mathbb{Z}^3} \sum_{q=-1}^{+1} \dot{h}_{\alpha\beta}^{(2,\mathbf{k},q)} (\dot{\varpi}_I, x^i) e^{-i\dot{\psi}_{\mathbf{k}} - iq\dot{\psi}_s}, \quad (114)$$

with

$$\mathbf{k} = k_i = (k_r, k_\theta, k_\phi) \quad (115)$$

and

$$\dot{\psi}_{\mathbf{k}} \equiv k_i \dot{\psi}^i. \quad (116)$$

Note the i appearing in the exponential is the imaginary number, not to be confused with a spatial index. The precession phase only enters with mode numbers $q = 0, \pm 1$ because it only appears in the field equations through the sines and cosines in Eq. (73). Given these Fourier expansions, the chain rule (110) becomes

$$\begin{aligned} \left(\frac{\partial}{\partial t} \right)_{\mathbf{k}, q} &= -i(k_i \Omega_{(0)}^i + q \Omega_{s(0)}) \\ &\quad + \varepsilon \left[F_I^{(0)} \frac{\partial}{\partial \dot{\varpi}_I} - i(k_i \Omega_{(1)}^i + q \Omega_{s(1)}) \right] + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (117)$$

⁷ For simplicity we take t to be Boyer-Lindquist time. However, our analysis throughout this paper applies for any choice of time coordinate t ; for times other than Boyer-Lindquist, the only change is the formula for ℓ_t in Eq. (80). As long as the time coordinate reduces to Boyer-Lindquist time along the particle's trajectory, then even Eq. (80) remains unchanged. The multiscale expansion is usually formulated in terms of hyperboloidal time [75, 119] for reasons explained in Refs. [75, 120], and hyperboloidal coordinates also bring other advantages [121].

The operators $\delta^k G_{\alpha\beta}^{(n)}$ correspondingly become operators on individual Fourier coefficients $\dot{h}_{\alpha\beta}^{(1,\mathbf{k})}$ and $\dot{h}_{\alpha\beta}^{(2,\mathbf{k},q)}$:

$$\delta G_{\alpha\beta}^{(n)}(\dot{h}^{(1)}) = \sum_{\mathbf{k} \in \mathbb{Z}^3} \delta G_{\alpha\beta}^{(n,\mathbf{k})}(\dot{h}^{(1,\mathbf{k})}) e^{-i\dot{\psi}_{\mathbf{k}}}, \quad (118)$$

$$\delta G_{\alpha\beta}^{(0)}(\dot{h}^{(2)}) = \sum_{\mathbf{k} \in \mathbb{Z}^3} \sum_{q=-1}^1 \delta G_{\alpha\beta}^{(0,\mathbf{k},q)}(\dot{h}^{(2,\mathbf{k},q)}) e^{-i\dot{\psi}_{\mathbf{k}} - iq\dot{\psi}_s}, \quad (119)$$

and

$$\begin{aligned} &\delta^2 G_{\alpha\beta}^{(0)}(\dot{h}^{(1)}, \dot{h}^{(1)}) \\ &= \sum_{\mathbf{k} \in \mathbb{Z}^3} \sum_{\mathbf{k}' \in \mathbb{Z}^3} \delta^2 G_{\alpha\beta}^{(0,\mathbf{k},\mathbf{k}')}(\dot{h}^{(1,\mathbf{k}')}, \dot{h}^{(1,\mathbf{k}'')}) e^{-i\dot{\psi}_{\mathbf{k}}} \end{aligned} \quad (120)$$

with $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$. The linear operator $\delta G_{\alpha\beta}^{(0,\mathbf{k})}$ is identical to the familiar linearized Einstein tensor in the frequency domain, with frequency $\omega_{\mathbf{k}}^{(0)} = k_i \Omega_{(0)}^i$. We refer to Refs. [75, 119] for more detailed explorations of multiscale expansions of the Einstein field equations.

To fully expand the field equations, we must also expand the stress-energy tensor. We write the monopole term (10a) as

$$T_{\alpha\beta}^{(m)}(\dot{\psi}^i, \dot{\varpi}_I, \dot{\psi}_s, x^i, \varepsilon) = m_2 \frac{\hat{g}_{\alpha\mu} \hat{g}_{\alpha\nu} \dot{z}^\mu \dot{z}^\nu}{\sqrt{-\hat{g}_{\rho\sigma} \dot{z}^\rho \dot{z}^\sigma}} \frac{\delta^3(x^i - z^i)}{\sqrt{-\hat{g}}} \quad (121)$$

after evaluating the integral and using

$$\frac{dt}{d\hat{\tau}} = \frac{1}{\sqrt{-\hat{g}_{\alpha\beta} \dot{z}^\alpha \dot{z}^\beta}}. \quad (122)$$

In Eq. (121), the effective metric, trajectory, and velocity are

$$\hat{g}_{\alpha\beta} = g_{\alpha\beta}(x^i) + \varepsilon \dot{h}_{\alpha\beta}^{R(1)}(\dot{\psi}^i, \dot{\varpi}_I, x^i) + \mathcal{O}(\varepsilon^2), \quad (123)$$

$$z^i = z_{(0)}^i(\psi_{(0)}^i, \dot{\pi}_i) + \varepsilon z_{(1)}^i(\psi_{(0)}^i, \dot{\varpi}_I, \dot{\psi}_s) + \mathcal{O}(\varepsilon^2), \quad (124)$$

$$\dot{z}^i = v_{(0)}^i(\psi_{(0)}^i, \dot{\pi}_i) + \varepsilon v_{(1)}^i(\psi_{(0)}^i, \dot{\varpi}_I, \dot{\psi}_s) + \mathcal{O}(\varepsilon^2), \quad (125)$$

where $\psi_{(0)}^i$ is given in Eq. (98). The coefficients $z_{(n)}^i$ and $v_{(n)}^i$ are obtained from Eqs. (81) and (82) with the expansions (96). Concretely, at leading order,

$$r_{(0)} = \frac{\dot{p} m_1}{1 + \dot{e} \cos \psi_{(0)}^r}, \quad (126a)$$

$$\cos \theta_{(0)} = \cos \dot{i} \cos \psi_{(0)}^\theta, \quad (126b)$$

$\phi_{(0)} = \psi_{(0)}^\phi$, and

$$v_{(0)}^i = \frac{\partial z_{(0)}^i}{\partial \psi_{(0)}^j} \omega_{(0)}^j(\psi_{(0)}^k, \dot{\pi}_k). \quad (127)$$

We emphasize that this leading-order trajectory is not a geodesic of the background spacetime; it would only be

a geodesic if $\dot{\pi}_i$ were constant and $\dot{\psi}^i$ were exactly linear in t . At the next order,

$$z_{(1)}^i = \delta\psi^j \frac{\partial z_{(0)}^i}{\partial\psi_{(0)}^j} + \delta\pi_i \frac{\partial z_{(0)}^i}{\partial\dot{\pi}_j}, \quad (128)$$

$$v_{(1)}^i = \delta\psi^j \frac{\partial v_{(0)}^i}{\partial\psi_{(0)}^j} + \delta\pi_i \frac{\partial v_{(0)}^i}{\partial\dot{\pi}_j}. \quad (129)$$

We also have the trivial identities

$$v_{(0)}^t = 1 \quad \text{and} \quad v_{(n>0)}^t = 0. \quad (130)$$

These expansions of $\hat{g}_{\mu\nu}$, z^i , and \dot{z}^i imply

$$\begin{aligned} T_{\alpha\beta}^{(m)} &= \varepsilon T_{\alpha\beta}^{(m,0)}(\dot{\psi}^i, \dot{\pi}_i, x^i) \\ &\quad + \varepsilon^2 T_{\alpha\beta}^{(m,1)}(\dot{\psi}^i, \dot{\varpi}_I, \dot{\psi}_s, x^i) + \mathcal{O}(\varepsilon^3) \end{aligned} \quad (131)$$

with

$$T_{\alpha\beta}^{(m,0)} = m_2 \frac{g_{\alpha\mu} g_{\alpha\nu} v_{(0)}^\mu v_{(0)}^\nu}{\sqrt{-g_{\rho\sigma} v_{(0)}^\rho v_{(0)}^\sigma}} \frac{\delta^3(x^i - z_{(0)}^i)}{\sqrt{-g}}, \quad (132)$$

$$T_{\alpha\beta}^{(m,1)} = \left(z_{(1)}^i \frac{\partial}{\partial z_{(0)}^i} + v_{(1)}^i \frac{\partial}{\partial v_{(0)}^i} + h_{\mu\nu}^{\text{R}(1)} \frac{\partial}{\partial g_{\mu\nu}} \right) T_{\alpha\beta}^{(m,0)}, \quad (133)$$

where we note $\frac{\partial\sqrt{-g}}{\partial g_{\mu\nu}} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}$.

Similarly, the dipole term (10b) is expanded as

$$T_{\alpha\beta}^{(d)} = \varepsilon^2 T_{\alpha\beta}^{(d,0)}(\dot{\psi}^i, \dot{\pi}_i, \dot{\psi}_s, x^i) + \mathcal{O}(\varepsilon^3) \quad (134)$$

with

$$T_{\alpha\beta}^{(d,0)} = (m_2)^2 g_{\alpha\mu} g_{\beta\nu} \nabla_\rho \left(\frac{\delta^3(x^i - z_{(0)}^i)}{\sqrt{-g}} v_{(0)}^{(\mu} S^{\nu)\rho} \right). \quad (135)$$

The total stress-energy tensor is hence

$$T_{\alpha\beta} = \varepsilon T_{\alpha\beta}^{(1)}(\dot{\psi}^i, \dot{\pi}_i, x^i) + \varepsilon^2 T_{\alpha\beta}^{(2)}(\dot{\psi}^i, \dot{\varpi}_I, \dot{\psi}_s, x^i) + \mathcal{O}(\varepsilon^3), \quad (136)$$

where

$$T_{\alpha\beta}^{(1)} = T_{\alpha\beta}^{(m,0)}, \quad (137)$$

$$T_{\alpha\beta}^{(2)} = T_{\alpha\beta}^{(m,1)} + T_{\alpha\beta}^{(d,0)}. \quad (138)$$

These are decomposed into Fourier modes using

$$T_{\alpha\beta}^{(n,\mathbf{k},q)} = \frac{1}{(2\pi)^4} \oint T_{\alpha\beta}^{(n)} e^{i\psi_{\mathbf{k}} + iq\dot{\psi}_s} d^4\dot{\psi}. \quad (139)$$

Given the expansions of the Einstein tensor and stress-energy tensor, the Einstein equation (14) divides into first- and second-order equations for the Fourier mode coefficients $\dot{h}_{\alpha\beta}^{(1,\mathbf{k})}$ and $\dot{h}_{\alpha\beta}^{(2,\mathbf{k},q)}$:

$$\delta G_{\mu\nu}^{(0,\mathbf{k})}(\dot{h}^{(1,\mathbf{k})}) = 8\pi T_{\mu\nu}^{(1,\mathbf{k})}, \quad (140)$$

$$\begin{aligned} \delta G_{\mu\nu}^{(0,\mathbf{k})}(\dot{h}^{(2,\mathbf{k},q)}) &= 8\pi T_{\mu\nu}^{(2,\mathbf{k},q)} - \delta^2 G_{\mu\nu}^{(0,\mathbf{k})}(\dot{h}^{(1)}, \dot{h}^{(1)}) \\ &\quad - \delta G_{\mu\nu}^{(1,\mathbf{k})}(\dot{h}^{(1,\mathbf{k})}), \end{aligned} \quad (141)$$

where

$$\delta^2 G_{\mu\nu}^{(0,\mathbf{k})}(\dot{h}^{(1)}, \dot{h}^{(1)}) = \sum_{\mathbf{k}' \in \mathbb{Z}^3} \delta^2 G_{\mu\nu}^{(0,\mathbf{k},\mathbf{k}')}(\dot{h}^{(1,\mathbf{k}')}, \dot{h}^{(1,\mathbf{k}-\mathbf{k}')}). \quad (142)$$

The field equations (140) and (141) are (elliptic) partial differential equations in x^i for each Fourier coefficient. They can be converted to Teukolsky equations as described in Refs. [105, 122]. The first-order equation, Eq. (140), is identical to the standard frequency-domain Einstein equation for a point particle on a geodesic orbit (though we once again emphasize that the leading-order trajectory $z_{(0)}^i$ is not a geodesic).

Equations (140) and (141) can be solved on a grid of $\dot{\pi}_i$ (and background spin) values. From the solutions $\dot{h}_{\alpha\beta}^{(1,\mathbf{k})}$ and $\dot{h}_{\alpha\beta}^{(2,\mathbf{k},q)}$, the forcing functions $F_i^{(n)}$ and frequency corrections $\Omega_i^{(n)}$ can be computed, and the waveform amplitudes can be read off the asymptotic values of $\dot{h}_{\alpha\beta}^{(1,\mathbf{k})}$ and $\dot{h}_{\alpha\beta}^{(2,\mathbf{k},q)}$. Using this precomputed data, Eqs. (95) can be solved to evolve the system through parameter space and generate a waveform, as described in the Introduction. We return to the waveform generation scheme in Sec. V.

Finally, we note that the expansion (91) of the acceleration used in Sec. IV A assumes the force is given in the form $a^i(\psi^j, \varpi_I, \dot{\psi}_s, \varepsilon)$. If we directly substitute the multiscale expansion (106) into Eq. (77), we instead obtain the force in the form $a^i(\psi^j, \pi_j, \tilde{\psi}_s, \varepsilon; \dot{\psi}^j, \dot{\varpi}_j, \dot{\psi}_s)$, where the dependence on $(\psi^j, \pi_j, \dot{\psi}_s)$ arises from evaluating $\dot{h}_{\alpha\beta}^{\text{R}(n)}(\dot{\psi}^i, \dot{\varpi}_I, \dot{\psi}_s, x^i)$ (and other fields appearing in a^i) at $x^i = z^i$, as well as from evaluating the four-velocity and spin in terms of the osculating elements and phases. The arguments after the semicolon, on the other hand, arise from the first three arguments in $\dot{h}_{\alpha\beta}^{\text{R}(n)}$.

To accommodate this mixed form of the acceleration, one can treat the expansion (91) as an expansion of $a^i(\psi^j, \pi_j, \tilde{\psi}_s, \varepsilon; \dot{\psi}^j, \dot{\varpi}_j, \dot{\psi}_s)$ holding all arguments (other than ε) fixed. When calculating derivatives such as $\partial f_i^{(0)}/\partial\dot{\pi}_j$ in Eq. (103b), one must note that these terms arise from substituting the expansions (96) into the first three arguments of $a^i(\psi^j, \pi_j, \tilde{\psi}_s, \varepsilon; \dot{\psi}^j, \dot{\varpi}_j, \dot{\psi}_s)$. The derivative $\partial f_i^{(0)}/\partial\dot{\pi}_j$ in Eq. (103b) then involves $\frac{\partial}{\partial\dot{\pi}_j} a^i_{(1)}(\psi_{(0)}^i, \dot{\pi}_i, \tilde{\psi}_{s(0)}; \dot{\psi}^i, \dot{\varpi}_i, \dot{\psi}_s)$, where the derivative only acts on the second argument.

C. Linear-in-spin effects

In Secs. IV A and IV B, we have kept the treatment generic, without highlighting spin contributions or explicitly dropping quadratic-in-spin terms. We now isolate the linear spin terms.

We first define $a_{(n-\chi_2)}^i$, the secondary spin's linear contribution to the coefficients $a_{(n)}^i$ in the expansion (91).

Referring to the complete acceleration (77), we read off

$$a_{(1-\chi_2)}^\mu = -\frac{m_2}{2} R^\mu_{\alpha\beta\gamma} u^\alpha S^{\beta\gamma}, \quad (143)$$

$$\begin{aligned} a_{(2-\chi_2)}^\mu &= -\frac{1}{2} P^{\mu\nu} \left(2\dot{h}_{\nu\rho;\sigma}^{\text{R}(2-\chi_2)} - \dot{h}_{\rho\sigma;\nu}^{\text{R}(2-\chi_2)} \right) u^\rho u^\sigma \\ &\quad + \frac{m_2}{4} R^\mu_{\alpha\beta\gamma} \dot{h}_{\rho\sigma}^{\text{R}(1)} u^\rho u^\sigma u^\alpha S^{\beta\gamma} \\ &\quad - \frac{m_2}{2} R^\mu_{\alpha\beta\gamma} u^\alpha \delta S^{\beta\gamma} \\ &\quad + \frac{m_2}{2} P^{\mu\nu} \left(2\dot{h}_{\nu(\alpha;\beta)\gamma}^{\text{R}(1)} - \dot{h}_{\alpha\beta;\nu\gamma}^{\text{R}(1)} \right) u^\alpha S^{\beta\gamma} \end{aligned} \quad (144)$$

where all fields are evaluated at z^i (rather than $z_{(0)}^i$, for example). Here we have used the ‘mixed’ form of the acceleration described at the end of the previous section.

As discussed below Eq. (101), these accelerations do not contribute to any of the leading terms ($\Omega_{(0)}^i$, $F_i^{(0)}$, and $\Omega_{s(0)}$) in the evolution equations (95a)–(95c). From Eqs. (103a)–(103c), we can read off the linear spin contributions to the subleading terms in the evolution equations:

$$\begin{aligned} \Omega_{(1-\chi_2)}^i &= \left\langle A^i_j a_{(1-\chi_2)}^j \right\rangle \\ &\quad + \left\langle \delta\pi_j^{(\chi_2)} \frac{\partial\omega_{(0)}^i}{\partial\dot{\pi}_j} + \delta\psi_{(\chi_2)}^j \frac{\partial\omega_{(0)}^i}{\partial\psi_{(0)}^j} \right\rangle, \end{aligned} \quad (145)$$

$$\begin{aligned} F_i^{(1-\chi_2)} &= \left\langle B_{ij} a_{(2-\chi_2)}^j \right\rangle - F_j^{(0)} \frac{\partial \left\langle \delta\pi_i^{(\chi_2)} \right\rangle}{\partial\dot{\pi}_j} \\ &\quad + \left\langle \delta\pi_j^{(\chi_2)} \frac{\partial f_i^{(0-1\text{SF})}}{\partial\dot{\pi}_j} + \delta\psi_{(\chi_2)}^j \frac{\partial f_i^{(0-1\text{SF})}}{\partial\psi_{(0)}^j} \right\rangle \\ &\quad + \left\langle \delta\pi_j^{(1\text{SF})} \frac{\partial f_i^{(0-\chi_2)}}{\partial\dot{\pi}_j} + \delta\psi_{(1\text{SF})}^j \frac{\partial f_i^{(0-\chi_2)}}{\partial\psi_{(0)}^j} \right\rangle. \end{aligned} \quad (146)$$

Here we have used Eqs. (89) and (90) to make the forces explicit, and we observed that the linear spin contribution to $\Omega_{s(1)}$ can be neglected as an overall $\mathcal{O}(s^2)$ effect in the dynamics, analogous to the linear spin contribution to $\delta\omega_{AB}$, which we neglected in Eq. (59). The quantity $\delta\pi_i^{(\chi_2)}$ in Eqs. (145) and (146) is extracted from the linear spin terms in Eq. (102b), and $\delta\psi_{(\chi_2)}^i$ can be extracted from the linear spin terms in either Eq. (104) or Eq. (C12). Quantities labeled with ‘1SF’ are calculated from the first-order self-force.

There are several immediate takeaways from these formulas:

1. Since the expressions are linear in the secondary spin, the average over the precession phase entirely eliminates contributions from the orthogonal, precessing components of the spin. In other words, in calculating the right-hand sides of the evolution equations (95), one can replace S^α and δS^α with the precession-averaged spin vectors $\langle S^\alpha \rangle_{\tilde{\psi}_s}$

and $\langle \delta S^\alpha \rangle_{\tilde{\psi}_s}$ given in Eq. (75). This irrelevance of the precession phase in the 1PA orbital dynamics has been pointed out many times previously (e.g., [51, 65]). However, one can still consistently include the precession’s modulation effect on the waveform at this order. We return to this last point at the end of this section and in Sec. V.

2. The correction $\Omega_{(1-\chi_2)}^i$ to the orbital frequency is solely due to the precession-averaged MPD force,

$$\langle a_{(1-\chi_2)}^\mu \rangle_{\tilde{\psi}_s} = -\frac{m_2}{2} R^\mu_{\alpha\beta\gamma} u^\alpha \langle S^{\beta\gamma} \rangle_{\tilde{\psi}_s}, \quad (147)$$

and to gauge freedom. Using Eqs. (C8) and (C14), we can also write this frequency correction as

$$\begin{aligned} \Omega_{(1-\chi_2)}^i &= \left\langle \delta\pi_j^{(\chi_2)} \right\rangle \frac{\partial\Omega_{(0)}^i}{\partial\dot{\pi}_j} \\ &\quad + \left\langle \frac{\partial\dot{\psi}_{(0)}^i}{\partial\psi_{(0)}^j} A^j_k a_{(1-\chi_2)}^k \right\rangle \\ &\quad + \left\langle \frac{\partial\dot{\psi}_{(0)}^i}{\partial\dot{\pi}_j} B_{jk} a_{(1-\chi_2)}^k \right\rangle, \end{aligned} \quad (148)$$

where we recall that $\langle \delta\pi_j^{(\chi_2)} \rangle$ is freely specifiable. This frequency correction was first computed in Ref. [51]. It is given in closed, analytical form (for a specific gauge choice) in Ref. [72] in agreement with the numerical calculations of Ref. [60]. We return to this frequency correction in Secs. IV D, VB, and VI.

3. The spin nutation only contributes to $F_i^{(1-\chi_2)}$, through the force

$$a_{(2-\delta S)}^\mu = -\frac{m_2}{2} R^\mu_{\alpha\beta\gamma} u^\alpha \langle \delta S^{\beta\gamma} \rangle_{\tilde{\psi}_s} \quad (149)$$

that enters in the first term on the right-hand side of Eq. (146). In Appendix D, we show that in fact, the effects of this force in the 1PA orbital dynamics can be computed without solving the nutation equations.

4. Each term in $F_i^{(1-\chi_2)}$ arises from an interaction between the secondary spin and the first-order regular field at the particle. However, as alluded to in the Introduction, none of these local terms need to be evaluated to calculate $F_i^{(1-\chi_2)}$ in practice; thanks to recent results in Refs. [70, 72], Eq. (146) can be replaced with an expression in terms of asymptotic fluxes. We summarize this in Sec. VI.

To complete the summary of first-order spin effects, we now turn to the field equations. The spin only enters the field equations (140) and (141) in two simple ways: (i) through the spin-dipole stress-energy tensor $T_{\alpha\beta}^{(d,0)}$ in Eq. (135); (ii) through the spin’s contribution to $T_{\alpha\beta}^{(m,1)}$

in Eq. (133), which arises from the MPD spin force's contribution to $z_{(1)}^i$ and $v_{(1)}^i$ in Eqs. (128) and (129); and (iii) through contributions $\Omega_{(1-\chi_2)}^i$ and $\Omega_{s(1-\chi_2)}^i$ to Eq. (110), which contributes a term $\delta G_{\mu\nu}^{(1-\chi_2, \mathbf{k})}(\mathring{h}^{(1, \mathbf{k})})$ to Eq. (141).

We write the spin's total contribution to the stress-energy tensor as

$$T_{\alpha\beta}^{(2-\chi_2)} = T_{\alpha\beta}^{(d,0)} + T_{\alpha\beta}^{(m,1-\chi_2)} \quad (150)$$

with

$$T_{\alpha\beta}^{(m,1-\chi_2)} = \left(z_{(1-\chi_2)}^i \frac{\partial}{\partial z_{(0)}^i} + v_{(1-\chi_2)}^i \frac{\partial}{\partial v_{(0)}^i} \right) T_{\alpha\beta}^{(m,0)}. \quad (151)$$

The only term involving χ_2 in the field equation is thus

$$\begin{aligned} \delta G_{\mu\nu}^{(0, \mathbf{k})}(\mathring{h}^{(2-\chi_2, \mathbf{k}, q)}) &= 8\pi T_{\mu\nu}^{(2-\chi_2, \mathbf{k}, q)} \\ &\quad - \delta G_{\mu\nu}^{(1-\chi_2, \mathbf{k})}(\mathring{h}^{(1, \mathbf{k})}), \end{aligned} \quad (152)$$

where $h_{\alpha\beta}^{(2-\chi_2)}$ is the metric perturbation sourced by the spin. The first source term on the right-hand side is confined to the libration region containing the particle's orbit, while the second source term is distributed over the entire spacetime. As highlighted in Ref. [44] and discussed in the next section, there is considerable advantage in choosing a (phase-space) gauge that eliminates this non-compact term.

The solution to Eq. (152) enters the asymptotic waveform in two ways. First, it contributes to the first term on the right-hand side of the local force (144), thereby contributing to the first term on the right-hand side of the 1PA forcing function (146). Second, it contributes directly to the second-order waveform mode amplitudes. We return to these two contributions in Secs. V and VI.

D. Gauge choices

As noted in Sec. IV A, our multiscale expansion admits a residual gauge freedom on the orbital phase space, corresponding to the choice of non-oscillatory terms $\langle \Delta\psi^i \rangle$, $\langle \delta\pi_i \rangle$, and $\langle \Delta\psi_s \rangle$ in the transformations (96). Under different choices of these functions, subleading terms in the evolution equations change according to Eq. (105).

Different choices of these functions correspond to different choices of what we hold fixed when we vary ε . In this section, we describe three convenient gauge choices. Previous discussions along these lines can be found in Refs. [60, 63, 71, 79], for example.

In Sec. V B, we explain how the final 1PA waveform is invariant under such choices. We also emphasize that these choices can nevertheless affect the accuracy of the waveform model.

1. Fixed frequencies and fixed turning points

We first consider the gauge choice adopted in Ref. [76] for generic accelerated orbits. This gauge choice is defined by the properties

1. $\dot{\psi}^i$ vanishes at turning points (or what would be turning points if $\dot{\pi}_i$ were fixed). More generally, the phases $\dot{\psi}^i$ vanish at the same locations as ψ^i .
2. $\dot{\pi}_i$ is geodesically related to the physical frequencies

$$\Omega^i \equiv \frac{d\dot{\psi}^i}{dt}. \quad (153)$$

In other words, in this gauge, $\Omega^i = \Omega_{(0)}^i(\dot{\pi}_i)$.

3. either $\dot{\pi}_i$ is geodesically related to $\Omega_s \equiv \frac{d\dot{\psi}_s}{dt}$, making $\Omega_s = \Omega_{s(0)}(\dot{\pi}_i)$, or $\dot{\psi}_s$ vanishes where $\dot{\psi}_s$ vanishes.

We stress that our condition of ‘fixed turning points’ is unrelated to the condition with the same name in Ref. [60]; ours is a condition on the values of the radial and polar phases $\dot{\psi}^r$ and $\dot{\psi}^\theta$ at turning points, while Ref. [60]’s is a condition on the values of Boyer-Lindquist radius r and polar angle θ at turning points.

The first condition represents a choice of origin for the phases on the tori of constant $\dot{\pi}_i$. Recalling Eq. (98), we see the condition implies

$$\Delta\psi^i(0, \dot{\pi}_i) = 0. \quad (154)$$

To turn that into a condition on $\langle \Delta\psi^i \rangle$, we divide $\Delta\psi^i$ into its average $\langle \Delta\psi^i \rangle$ and a purely oscillatory part, $\Delta\psi_{\text{osc}}^i \equiv \Delta\psi^i - \langle \Delta\psi^i \rangle$. Equation (154) then implies

$$\langle \Delta\psi^i \rangle = -\Delta\psi_{\text{osc}}^i(0, \dot{\pi}_i). \quad (155)$$

Next, to enforce the condition $\Omega^i = \Omega_{(0)}^i(\dot{\pi}_i)$, we choose $\langle \delta\pi_i \rangle$ to eliminate the frequency corrections $\Omega_{(1)}^i$. From Eq. (103a), this requires

$$\begin{aligned} \langle \delta\pi_j \rangle \frac{\partial \Omega_{(0)}^i}{\partial \dot{\pi}_j} &= -\langle \omega_{(1)}^i \rangle + F_j^{(0)} \frac{\partial \langle \Delta\psi^i \rangle}{\partial \dot{\pi}_j} \\ &\quad - \left\langle \delta\pi_j^{\text{osc}} \frac{\partial \omega_{(0)}^i}{\partial \dot{\pi}_j} + \delta\psi_{\text{osc}}^j \frac{\partial \omega_{(0)}^i}{\partial \psi_{(0)}^j} \right\rangle, \end{aligned} \quad (156)$$

where we have split $\delta\pi_i$ and $\delta\psi^i$ into averaged and oscillatory pieces, noted $\langle \omega_{(0)}^i \rangle = \Omega_{(0)}^i$, and further noted that the product of an oscillatory function with a non-oscillatory one averages to zero. The linear spin term in Eq. (156) is

$$\begin{aligned} \left\langle \delta\pi_j^{(\chi_2)} \right\rangle \frac{\partial \Omega_{(0)}^i}{\partial \dot{\pi}_j} &= -\left\langle \omega_{(1-\chi_2)}^i \right\rangle \\ &\quad - \left\langle \delta\pi_j^{(\chi_2)} \frac{\partial \omega_{(0)}^i}{\partial \dot{\pi}_j} + \delta\psi_{(x_2)}^j \frac{\partial \omega_{(0)}^i}{\partial \psi_{(0)}^j} \right\rangle. \end{aligned} \quad (157)$$

Equation (156) has a unique solution so long as the matrix $\frac{\partial \Omega_{(0)}^i}{\partial \dot{\pi}_j}$ is invertible. This means the fixed-frequencies gauge breaks down at (measure-zero) degenerate surfaces in parameter space where the frequencies fail to be good coordinates [123].

Having fixed $\langle \delta\pi_i \rangle$ and $\langle \tilde{\Delta}\psi^i \rangle$, we are only left with $\langle \Delta\psi^i \rangle$. This means we do not have the freedom to simultaneously enforce $\Omega_{s(1)} = 0$ and $\Delta\tilde{\psi}_s(0, \dot{\pi}_i) = 0$ (unless the oscillatory part of $\Delta\tilde{\psi}_s$ is an odd function of ψ^i , in which case it automatically vanishes at $\psi^i = 0$). If we choose to eliminate $\Omega_{s(1)}$, then from Eq. (103c) we see that $\langle \Delta\tilde{\psi}_s \rangle$ must satisfy the following partial differential equation:

$$\begin{aligned} F_j^{(0)} \frac{\partial \langle \Delta\tilde{\psi}_s \rangle}{\partial \dot{\pi}_j} &= \left\langle \omega_{s(1)}^{(1SF)} \right\rangle + \left\langle \delta\pi_j^{(1SF)} \frac{\partial \omega_{s(0)}}{\partial \dot{\pi}_j} \right\rangle \\ &\quad + \left\langle \delta\psi_{(1SF)}^j \frac{\partial \omega_{s(0)}}{\partial \dot{\psi}^j} \right\rangle. \end{aligned} \quad (158)$$

Here we have emphasized that these terms come solely from the first-order regular field because we neglect linear spin corrections to the spin precession.

Assuming we eliminate $\Omega_{s(1)}$, the evolution equations away from the degenerate surfaces become

$$\frac{d\dot{\psi}^i}{dt} = \Omega_{(0)}^i(\dot{\pi}_j), \quad (159a)$$

$$\frac{d\dot{\pi}_i}{dt} = \varepsilon F_i^{(0)}(\dot{\pi}_j) + \varepsilon^2 F_i^{(1)}(\dot{\varpi}_J) + \mathcal{O}(\varepsilon^3), \quad (159b)$$

$$\frac{d\dot{\psi}_s}{dt} = \Omega_{s(0)}(\dot{\pi}_j). \quad (159c)$$

The forcing function $F_i^{(1)}$ is now

$$\begin{aligned} F_i^{(1)} &= \left\langle f_i^{(1)} \right\rangle + \langle \delta\pi_j \rangle \frac{\partial F_i^{(0)}}{\partial \dot{\pi}_j} - F_j^{(0)} \frac{\partial \langle \delta\pi_i \rangle}{\partial \dot{\pi}_j} \\ &\quad + \left\langle \delta\pi_j^{\text{osc}} \frac{\partial f_i^{(0)}}{\partial \dot{\pi}_j} + \delta\psi_{\text{osc}}^j \frac{\partial f_i^{(0)}}{\partial \dot{\psi}^j} \right\rangle, \end{aligned} \quad (160)$$

with $\langle \delta\pi_j \rangle$ given by the solution to Eq. (156).

This is the gauge used in the Introduction, where we used it to replace $\Omega_{(0)}^i$ with Ω^i . It has several advantages:

1. It eliminates $\Omega_{(1)}^i$ terms from the second-order field equations (141) because Eq. (117) reduces to

$$\begin{aligned} \left(\frac{\partial}{\partial t} \right)_{k,q} &= -i(k_i \Omega_{(0)}^i + q \Omega_{s(0)}) \\ &\quad + \varepsilon F_I^{(0)} \frac{\partial}{\partial \dot{\varpi}_I} + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (161)$$

As highlighted in Ref. [44], this is particularly advantageous in the case of the secondary-spin contribution to the field equations because it eliminates

the noncompact source term proportional to $\Omega_{(1)}^i$ in Eq. (152). On the other hand, the correction to $\Omega_{s(0)}$ would only enter the *third*-order field equations, since the precession phase first enters the field equations at second order.

2. It naturally yields observable quantities as functions of the physical, observable frequencies. Such functional relationships have historically been the basis for translating invariant information between different approaches to the two-body problem [2, 124].
3. There is some indication that it might yield more accurate waveforms than other gauge choices, though this evidence is limited to the quasicircular case [5], where the degenerate surfaces do not exist.

This gauge also has a disadvantage due to the degenerate surfaces. At these surfaces, $\delta\pi_i$ diverges, causing the forcing function (160) to diverge. A complete evolution scheme using this gauge would require a method of evolving across these singular surfaces. It is not yet clear how much of an obstacle this represents, particularly since the surfaces are deep in the strong field, near the separatrix where the multiscale expansion breaks down and the particle transitions into a plunge [125–128]. It is also possible to avoid this problem by only eliminating corrections to some, but not all of the frequencies.

2. Fixed frequencies and fixed emissions

We next consider a gauge that eliminates all post-adiabatic terms in the evolution equations. This requires eliminating the forcing functions $F_i^{(1)}$ as well as the frequency corrections $\Omega_{(1)}^i$ and $\Omega_{s(1)}$. The elimination of post-adiabatic forcing functions ('fixed emissions') has been considered previously in Ref. [79], for example.

We see from Eq. (103b) that the forcing function $F_i^{(1)}$ can be eliminated with a choice of $\langle \delta\pi_j \rangle$ satisfying the partial differential equation

$$\begin{aligned} \langle \delta\pi_j \rangle \frac{\partial F_i^{(0)}}{\partial \dot{\pi}_j} - F_j^{(0)} \frac{\partial \langle \delta\pi_i \rangle}{\partial \dot{\pi}_j} &= -\left\langle f_i^{(1)} \right\rangle - \left\langle \delta\pi_j^{\text{osc}} \frac{\partial f_i^{(0)}}{\partial \dot{\pi}_j} \right\rangle \\ &\quad + \left\langle \delta\psi_{\text{osc}}^j \frac{\partial f_i^{(0)}}{\partial \dot{\psi}^j} \right\rangle. \end{aligned} \quad (162)$$

Similarly, we can see from Eq. (103a) that $\Omega_{(1)}^i$ can be eliminated with a choice of $\langle \Delta\psi^i \rangle$ satisfying

$$F_j^{(0)} \frac{\partial \langle \Delta\psi^i \rangle}{\partial \dot{\pi}_j} = \left\langle \omega_{(1)}^i \right\rangle + \left\langle \delta\pi_j \frac{\partial \omega_{(0)}^i}{\partial \dot{\pi}_j} + \delta\psi^j \frac{\partial \omega_{(0)}^i}{\partial \dot{\psi}^j} \right\rangle. \quad (163)$$

The evolution equations in this gauge are simply the 0PA ones to all orders:

$$\frac{d\dot{\psi}^i}{dt} = \Omega_{(0)}^i(\dot{\pi}_j), \quad (164a)$$

$$\frac{d\dot{\pi}_i}{dt} = \varepsilon F_i^{(0)}(\dot{\pi}_j), \quad (164b)$$

$$\frac{d\dot{\psi}_s}{dt} = \Omega_{s(0)}(\dot{\pi}_j). \quad (164c)$$

This gauge might appear to be impossibly advantageous in that it superficially avoids the need to calculate subleading terms when generating waveforms. However, this appearance is misleading because the term $\langle f_i^{(1)} \rangle$ in Eq. (162) is proportional to the second-order force $a_{(2)}^i$. Calculating $a_{(2)}^i$ requires the solution to the second-order field equation (141), which in turn involves $\delta\pi_i$ in the source term (133). Therefore Eq. (162) actually represents an extremely complicated integro-differential equation for $\delta\pi_i$, which might not even have a solution.

We can consider a more practical alternative that eliminates all contributions to $F_i^{(1)}$ *except* those coming from $a_{(2)}^i$. Equation (162) is then replaced with the condition

$$\begin{aligned} \langle \delta\pi_j \rangle \frac{\partial F_i^{(0)}}{\partial \dot{\pi}_j} - F_j^{(0)} \frac{\partial \langle \delta\pi_i \rangle}{\partial \dot{\pi}_j} &= - \left\langle \delta\pi_j^{\text{osc}} \frac{\partial f_i^{(0)}}{\partial \dot{\pi}_j} \right\rangle \\ &\quad + \left\langle \delta\psi_{\text{osc}}^j \frac{\partial f_i^{(0)}}{\partial \dot{\psi}^j} \right\rangle. \end{aligned} \quad (165)$$

The evolution equation for $\dot{\pi}_i$ then becomes

$$\frac{d\dot{\pi}_i}{dt} = \varepsilon \langle B_{ij} a_{(1)}^j \rangle + \varepsilon^2 \langle B_{ij} a_{(2)}^j \rangle + \mathcal{O}(\varepsilon^3), \quad (166)$$

where we have used Eq. (90). However, one should note that this choice can complicate the second-order Einstein equations because the solution to Eq. (165) contributes to the source term (133).

3. Fixed constants of motion

As a final option, we consider a gauge in which the constants of motion take fixed values as we vary ε . This is the gauge used in Ref. [72], a fact that will play an important role in Sec. VI.

To understand this gauge choice, we must first recall the constants of motion for spinning test particles (i.e., spinning particles that do not source a metric perturbation or experience a self-force). In addition to the particle's mass m_2 and spin components χ_{\parallel} and χ_{\perp} , the conserved quantities are spin-corrected versions of the geodesic energy, angular momentum, and Carter constant: $P_i = (E, L_z, K)$.⁸ In any spacetime with a Killing

vector ξ^α , the quantity

$$\Xi_\xi = \xi^\alpha u_\alpha + \frac{m_2}{2} S^{\alpha\beta} \nabla_\alpha \xi_\beta \quad (167)$$

is conserved along solutions to the MPD equations [129]. The energy and angular momentum are the quantities Ξ_ξ associated with Kerr's timelike and axial Killing vectors:

$$E \equiv -\Xi_t \quad \text{and} \quad L_z \equiv \Xi_\phi. \quad (168)$$

The spin-corrected Carter constant, introduced in Ref. [93], is

$$K = K^{\alpha\beta} u_\alpha u_\beta + m_2 L_{\alpha\beta\gamma} S^{\alpha\beta} u^\gamma, \quad (169)$$

in which we have defined [51, 93]

$$L_{\alpha\beta\gamma} \equiv -2 (Y^\delta{}_\beta \nabla_\delta Y_{\gamma\alpha} - Y_\gamma{}^\delta \nabla_\delta Y_{\alpha\beta}). \quad (170)$$

We refer to Ref. [57], for example, for a review of the construction of these conserved quantities.

Each of the three conserved quantities take the form of a geodesic term plus a correction proportional to the spin. Hence, in terms of our osculating elements π_i and phases ψ^i , we can write them as

$$P_i = P_i^{(0)}(\pi_j) + \varepsilon \delta P_i^{(\chi_2)}(\psi^j, \pi_j, S^{\alpha\beta}). \quad (171)$$

Here

$$P_i^{(0)} = (E^{(0)}, L_z^{(0)}, K^{(0)}) = (-u_t, u_\phi, K^{\alpha\beta} u_\alpha u_\beta) \quad (172)$$

are the geodesic conserved quantities, which are given as functions of π_i in Eqs. (222)–(224) of Ref. [76], where the quantity $Q^{(0)}$ in Eq. (224) of Ref. [76] is related to $K^{(0)}$ by $Q^{(0)} = K^{(0)} - (L_z^{(0)} - m_1^{(0)} \chi_1^{(0)} E^{(0)})^2$. Along the accelerated orbit, the geodesic term varies with time, regardless of whether we consider test-particle orbits or self-accelerated ones. Upon substitution of the expansions (96a) and (96b), Eq. (171) becomes

$$\begin{aligned} P_i &= P_i^{(0)}(\dot{\pi}_j) - \varepsilon \delta \pi_j \frac{\partial P_i^{(0)}}{\partial \dot{\pi}_j} \\ &\quad + \varepsilon \delta P_i^{(\chi_2)}(\psi_{(0)}^j, \dot{\pi}_j, S^{\alpha\beta}) + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (173)$$

It will also be necessary to involve the angle-averaged version,

$$\begin{aligned} \langle P_i \rangle &= P_i^{(0)}(\dot{\pi}_j) - \varepsilon \langle \delta \pi_j \rangle \frac{\partial P_i^{(0)}}{\partial \dot{\pi}_j} \\ &\quad + \varepsilon \left\langle \delta P_i^{(\chi_2)}(\psi_{(0)}^j, \dot{\pi}_j, S^{\alpha\beta}) \right\rangle + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (174)$$

⁸ The conserved quantities are more often described as P_i together

with the mass, spin magnitude, and Rüdiger's constant $C_Y = l_\alpha S^\alpha$; see, e.g., Refs. [57, 65, 68]. Here l_α is the 'orbital angular momentum' vector defined in Eq. (38). In the test-particle limit, C_Y is equivalent (up to a factor) to our χ_{\parallel} .

The gauge condition we consider now is

$$\langle P_i \rangle = P_i^{(0)}(\dot{\pi}_j). \quad (175)$$

In other words, we choose $\dot{\pi}_j$ to be geodesically related to the (averaged) spinning-particle constants of motion. From Eq. (174), we see this condition is enforced with the choice

$$\langle \delta\pi_i \rangle = \frac{\partial P_i^{(0)}}{\partial \dot{\pi}_j} \langle \delta P_i \rangle. \quad (176)$$

This gauge is convenient because it enables us to immediately find the evolution of $\dot{\pi}_i$ from the evolution of $\langle P_i \rangle$:

$$\frac{d\dot{\pi}_i}{dt} = \frac{\partial \dot{\pi}_i}{\partial P_k^{(0)}} \frac{d\langle P_k \rangle}{dt}, \quad (177)$$

where $\partial \dot{\pi}_i / \partial P_k^{(0)}$ denotes the inverse of the (geodesic) Jacobian $\partial P_k^{(0)} / \partial \dot{\pi}_i$. This formula holds even for the spin-independent part of the dynamics. However, since we do not have a useful formula for the 1PA spin-independent piece of $d\langle P_k \rangle / dt$, it is not (at the moment) particularly useful for computing that piece of the 1PA dynamics. Instead, the formula becomes especially useful, as we explain in Sec. VI, when calculating the linear-in-spin 1PA piece of $d\dot{\pi}_i / dt$.

Note that in this gauge we are still left with the freedom to choose $\langle \Delta\psi^i \rangle$ and $\langle \Delta\tilde{\psi}_s \rangle$.

V. WAVEFORM

We now summarize the waveform-generation scheme that results from the multiscale expansion. We then explain how the waveform is invariant under the gauge freedom discussed in the preceding section. Finally, we discuss the impact that the secondary spin's precession has on the waveform.

A. Waveform generation scheme

The gravitational waveform is extracted from the metric perturbation at future null infinity,

$$h = h_+ - ih_\times = \lim_{r \rightarrow \infty} (rh_{\alpha\beta}\bar{m}^\alpha\bar{m}^\beta), \quad (178)$$

where the limit is taken at fixed retarded time u , and \bar{m}^α is the standard Newman-Penrose complex basis vector on the celestial sphere [76]. Given the form of the metric perturbation (113)–(114), the waveform through second order in ε can be written as

$$\begin{aligned} h = \sum_{\mathbf{k} \in \mathbb{Z}^3} & \left[\varepsilon \mathring{h}_{\mathbf{k}}^{(1)}(\dot{\pi}_i, \theta, \phi) + \varepsilon^2 \mathring{h}_{\mathbf{k}}^{(2-\chi_2)}(\mathring{\varpi}_i, \theta, \phi) \right. \\ & + \varepsilon^2 \chi_{\parallel} \mathring{h}_{\mathbf{k}}^{(2-\chi_{\parallel})}(\dot{\pi}_i, \theta, \phi) \\ & \left. + \varepsilon^2 \chi_{\perp} \sum_{q=\pm 1} \mathring{h}_{\mathbf{k}q}^{(2-\chi_{\perp})}(\dot{\pi}_i, \theta, \phi) e^{-iq\tilde{\psi}_s} \right] e^{-i\tilde{\psi}_{\mathbf{k}}}, \end{aligned} \quad (179)$$

with $\mathring{h}_{\mathbf{k}}^{(1)} = \lim_{r \rightarrow \infty} (rh_{\alpha\beta}^{(1,\mathbf{k})}\bar{m}^\alpha\bar{m}^\beta)$, for example. Equation (179) extends Eq. (1) to include the secondary spin. Here $\mathring{h}_{\alpha\beta}^{(2-\chi_2)}$ denotes the χ_2 -independent part of the second-order metric perturbation, identical to the second-order term in Eq. (1). We have also divided the linear spin contribution—the solution to Eq. (152)—into pieces proportional to χ_{\parallel} and χ_{\perp} , respectively.

The waveform's time dependence is governed by Eqs. (95), where t is re-interpreted as retarded time u along future null infinity. This identification between time along the particle's worldline and time at future null infinity is achieved using hyperboloidal slicing, for example [75, 119]. Even with such slicing, there is considerable subtlety in extracting the second-order waveform, as detailed in Ref. [120]. However, this complexity only arises from nonlinearity in the field equations. It does not affect the first-order and linear-in-spin pieces of the waveform, which can be extracted from the Fourier amplitudes $\mathring{h}_{\alpha\beta}^{(1,\mathbf{k})}$ and $\mathring{h}_{\alpha\beta}^{(2-\chi_2,\mathbf{k})}$ as standard in linear perturbation theory [76].

B. Gauge invariance

The residual gauge freedom discussed in Sec. IV A and IV D comprises transformations

$$\dot{\pi}_i \rightarrow \dot{\pi}'_i = \dot{\pi}_i - \varepsilon \langle \delta\pi_i \rangle, \quad (180a)$$

$$\dot{\psi}^i \rightarrow \dot{\psi}'^i = \dot{\psi}^i - \langle \Delta\psi^i \rangle, \quad (180b)$$

$$\dot{\psi}_s \rightarrow \dot{\psi}'_s = \dot{\psi}_s - \langle \Delta\tilde{\psi}_s \rangle. \quad (180c)$$

To understand the minus sign, note that Eq. (96) expresses old variables in terms of new ones, while Eq. (180) represents the inverse: new in terms of old.

Under the phase-space gauge transformation (180), the orbital evolution equations (95) change according to Eqs. (105). We now show that the amplitudes in the waveform (179) change under this transformation in a way that precisely compensates the changes in the orbital evolution equations, leaving the waveform invariant.

The metric perturbation (106) in terms of the new phase-space variables is

$$h_{\alpha\beta} = \varepsilon \mathring{h}_{\alpha\beta}^{(1)}(\dot{\psi}'^i, \mathring{\varpi}'_I, x^i) + \varepsilon^2 \mathring{h}_{\alpha\beta}^{(2)}(\dot{\psi}'^i, \mathring{\varpi}'_I, \dot{\psi}'_s, x^i) + \dots, \quad (181)$$

where $(\dot{\psi}'^i, \dot{\pi}'_i)$ are determined from the ordinary differential equations

$$\frac{d\dot{\psi}'^i}{dt} = \Omega_{(0)}^i(\dot{\pi}'_j) + \varepsilon \Omega_{(1)}^i(\mathring{\varpi}'_J) + \mathcal{O}(\varepsilon^2), \quad (182a)$$

$$\frac{d\dot{\pi}'_i}{dt} = \varepsilon F_i^{(0)}(\dot{\pi}'_j) + \varepsilon^2 F_i^{(1)}(\mathring{\varpi}'_J) + \mathcal{O}(\varepsilon^3), \quad (182b)$$

$$\frac{d\dot{\psi}'_s}{dt} = \Omega_{s(0)}(\dot{\pi}'_j) + \varepsilon \Omega'_{s(1)}(\mathring{\varpi}'_J) + \mathcal{O}(\varepsilon^2). \quad (182c)$$

For simplicity, we assume $\delta m'_1 = \delta m_1$ and $\delta\chi'_1 = \delta\chi_1$, and we exclude transformations involving the perturbations of the primary's mass and spin.

We first find the relationship between $\dot{h}_{\alpha\beta}^{(n)}$ and $\dot{h}_{\alpha\beta}^{(n)}$. To do so, we start from the metric perturbation (106) in the unprimed gauge. After expressing $(\dot{\psi}^i, \dot{\pi}_i)$ in terms of $(\dot{\psi}'^i, \dot{\pi}'_i)$ using Eq. (180), we then re-expand Eq. (106) in powers of ε at fixed $(\dot{\psi}'^i, \dot{\pi}'_i)$. Since the result and Eq. (181) both represent expansions of $h_{\alpha\beta}$ at fixed phase-space coordinates $(\dot{\psi}'^i, \dot{\pi}'_i)$, the coefficients must agree at each order in ε . Making that identification between coefficients in the two expressions, we find

$$\dot{h}_{\alpha\beta}^{(1)}(\dot{\psi}'^i, \dot{\varpi}'_I) = \dot{h}_{\alpha\beta}^{(1)}(\dot{\psi}'^i + \langle \Delta\psi^i \rangle, \dot{\varpi}'_I), \quad (183)$$

$$\begin{aligned} \dot{h}_{\alpha\beta}^{(2)}(\dot{\psi}'^i, \dot{\varpi}'_I, \dot{\psi}'_s) &= \dot{h}_{\alpha\beta}^{(2)}(\dot{\psi}'^i + \langle \Delta\psi^i \rangle, \dot{\varpi}'_I, \dot{\psi}'_s + \langle \Delta\tilde{\psi}_s \rangle) \\ &\quad + \langle \delta\pi_i \rangle \frac{\partial}{\partial \dot{\pi}'_i} \dot{h}_{\alpha\beta}^{(1)}(\dot{\psi}'^i + \langle \Delta\psi^i \rangle, \dot{\pi}'_i), \end{aligned} \quad (184)$$

where we have suppressed the dependence on x^i . Note that the $\dot{\pi}'_i$ derivative in the last line acts only on the second argument of $\dot{h}_{\alpha\beta}^{(1)}$, not on the $\langle \Delta\psi^i \rangle (\dot{\pi}'_i)$ appearing in the first argument.

Since the waveform (179) is expressed in terms of Fourier modes, we next Fourier expand the left- and right-hand sides of Eqs. (183) and (184) with respect to the functions' first and third arguments. This immediately yields relationships between mode coefficients,

$$\dot{h}_{\alpha\beta}^{(1,\mathbf{k})} = \dot{h}_{\alpha\beta}^{(1,\mathbf{k})} e^{-i\langle \Delta\psi_{\mathbf{k}} \rangle}, \quad (185)$$

$$\dot{h}_{\alpha\beta}^{(2-\chi_2,\mathbf{k})} = \left[\dot{h}_{\alpha\beta}^{(2-\chi_2)} + \langle \delta\pi_i^{(\chi_2)} \rangle \frac{\partial}{\partial \dot{\pi}'_i} \dot{h}_{\alpha\beta}^{(1)} \right] e^{-i\langle \Delta\psi_{\mathbf{k}} \rangle}, \quad (186)$$

$$\dot{h}_{\alpha\beta}^{(2-\chi_{\parallel},\mathbf{k})} = \left[\dot{h}_{\alpha\beta}^{(2-\chi_{\parallel})} + \langle \delta\pi_i^{(\chi_{\parallel})} \rangle \frac{\partial}{\partial \dot{\pi}'_i} \dot{h}_{\alpha\beta}^{(1)} \right] e^{-i\langle \Delta\psi_{\mathbf{k}} \rangle}, \quad (187)$$

$$\dot{h}_{\alpha\beta}^{(2-\chi_{\perp},\mathbf{k},q)} = \dot{h}_{\alpha\beta}^{(2-\chi_{\perp},\mathbf{k},q)} e^{-i\langle \Delta\psi_{\mathbf{k}} \rangle - iq\langle \Delta\tilde{\psi}_s \rangle}. \quad (188)$$

Here we have divided $\langle \delta\pi_i \rangle$ into pieces independent of χ_2 and proportional to χ_{\parallel} . We exclude a term proportional to χ_{\perp} in $\langle \delta\pi_i \rangle$ because such a term would introduce a $\dot{\psi}_s$ -independent contribution to $\dot{h}_{\alpha\beta}^{(2-\chi_{\perp})}$.

The relationships between the Fourier mode amplitudes in the two gauges imply that the waveform in the primed gauge can be written as follows in terms of the amplitudes in the unprimed gauge:

$$\begin{aligned} h &= \sum_{\mathbf{k} \in \mathbb{Z}^3} \left\{ \varepsilon \dot{h}_{\mathbf{k}}^{(1)}(\dot{\pi}'_i) + \varepsilon^2 \left[\dot{h}_{\mathbf{k}}^{(2-\chi_2)}(\dot{\varpi}'_I) \right. \right. \\ &\quad + \chi_{\parallel} \dot{h}_{\mathbf{k}}^{(2-\chi_{\parallel})}(\dot{\pi}'_i) + \langle \delta\pi_i \rangle \frac{\partial}{\partial \dot{\pi}'_i} \dot{h}_{\mathbf{k}}^{(1)}(\dot{\pi}'_i) \\ &\quad \left. \left. + \chi_{\perp} \sum_{q=\pm 1} \dot{h}_{\mathbf{k}q}^{(2-\chi_{\perp})}(\dot{\pi}'_i) e^{-iq(\dot{\psi}'_s + \langle \Delta\tilde{\psi}_s \rangle)} \right] \right\} e^{-i(\dot{\psi}'_{\mathbf{k}} + \langle \Delta\psi_{\mathbf{k}} \rangle)}. \end{aligned} \quad (189)$$

We now appeal to the equations of motion (182) to show that the above waveform agrees with the one in

the unprimed gauge, Eq. (179). Examining Eqs. (179) and (189), we see that the two expressions become equivalent, up to $\mathcal{O}(\varepsilon^3)$ differences, if the equations of motion imply the relationships (180). Since the transformation laws (105) for the equations of motion were derived from those same relationships, it is clear that the equations of motion must be compatible with them. But doing the converse, obtaining the desired relationships from the equations of motion, is a worthwhile exercise.

Consider the equation of motion (182a) for the orbital phases. Appealing to Eq. (105), we write it as

$$\begin{aligned} \frac{d\dot{\psi}'^i}{dt} &= \Omega_{(0)}^i(\dot{\pi}'_j) + \varepsilon \left[\Omega_{(1)}^i(\dot{\varpi}'_J) + \langle \delta\pi_j \rangle \frac{\partial \Omega_{(0)}^i}{\partial \dot{\pi}'_j} \right. \\ &\quad \left. - F_j^{(0)} \frac{\partial \langle \Delta\psi^i \rangle}{\partial \dot{\pi}'_j} \right] + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (190)$$

We can immediately rewrite this as

$$\begin{aligned} \frac{d\dot{\psi}'^i}{dt} &= \Omega_{(0)}^i(\dot{\pi}'_j + \varepsilon \langle \delta\pi_j \rangle) + \varepsilon \Omega_{(1)}^i(\dot{\varpi}'_J + \varepsilon \langle \delta\varpi_J \rangle) \\ &\quad - \frac{d\langle \Delta\psi^i \rangle}{dt} + \mathcal{O}(\varepsilon^2), \end{aligned} \quad (191a)$$

$$= \frac{d\dot{\psi}^i}{dt} - \frac{d\langle \Delta\psi^i \rangle}{dt} + \mathcal{O}(\varepsilon^2), \quad (191b)$$

defining $\delta\varpi_J = (\delta\pi_i, 0, 0)$. Hence, we find the required relationship $\dot{\psi}'^i = \dot{\psi}^i - \Delta\psi^i + \mathcal{O}(\varepsilon)$ so long as their initial conditions agree. The difference is of order ε , rather than the ε^2 in Eq. (191), because the integration is over a time scale of order $1/\varepsilon$.

Similarly, the equation of motion (182a), given Eq. (105), can be written as

$$\begin{aligned} \frac{d\dot{\pi}'_i}{dt} &= \varepsilon F_i^{(0)}(\dot{\pi}'_j) + \varepsilon^2 \left[F_i^{(1)}(\dot{\varpi}'_J) + \langle \delta\pi_j \rangle \frac{\partial F_{(0)}^i}{\partial \dot{\pi}'_j} \right. \\ &\quad \left. - F_j^{(0)} \frac{\partial \langle \delta\pi_i \rangle}{\partial \dot{\pi}'_j} \right] + \mathcal{O}(\varepsilon^3). \end{aligned} \quad (192)$$

This implies

$$\begin{aligned} \frac{d\dot{\pi}'_i}{dt} &= \varepsilon F_i^{(0)}(\dot{\pi}'_j + \langle \delta\pi_j \rangle) + \varepsilon^2 F_i^{(1)}(\dot{\varpi}'_J + \langle \delta\varpi_J \rangle) \\ &\quad - \varepsilon \frac{d\langle \delta\pi_i \rangle}{dt} + \mathcal{O}(\varepsilon^3), \end{aligned} \quad (193a)$$

$$= \frac{d\dot{\pi}_i}{dt} - \varepsilon \frac{d\langle \delta\pi_i \rangle}{dt} + \mathcal{O}(\varepsilon^3), \quad (193b)$$

and we find the required relationship $\dot{\pi}'_i = \dot{\pi}_i - \varepsilon \delta\pi_i + \mathcal{O}(\varepsilon^2)$ (again, so long as their initial conditions agree).

Therefore, as promised, the transformation of the evolution equations counterbalances the transformation of the waveform amplitudes, such that the waveform is invariant under the residual gauge freedom we consider. This invariance might appear trivial when shown in this way. However, its significance becomes clearer when we

consider the many manifestations of the choice of gauge in solving the Einstein equation and calculating forcing functions:

1. The choice of $\langle \Delta\psi^i \rangle$ affects the Fourier mode decomposition of the stress-energy tensor and of all the Einstein equations.
2. The choice of $\langle \delta\pi_i \rangle$ affects the first-order terms $z_{(1)}^i$ and $v_{(1)}^i$ in the parametrization of the particle's Boyer-Lindquist trajectory; see Eqs. (128) and (129).
3. The changes in $z_{(1)}^i$ and $v_{(1)}^i$ affect the term $T_{\alpha\beta}^{(m,1)}$ in the stress-energy tensor; see Eq. (133).
4. The change in $T_{\alpha\beta}^{(m,1)}$ affects the piece of $h_{\alpha\beta}^{(2)}$ that $T_{\alpha\beta}^{(m,1)}$ sources; see Eq. (141).
5. The choice of $\langle \Delta\psi^i \rangle$ and $\langle \delta\pi_i \rangle$ affect the value of $\Omega_{(1)}^i$ and therefore the source term $\delta G_{\mu\nu}^{(1,k)}$ in the field equation (141).
6. The change in $\delta G_{\mu\nu}^{(1,k)}$ affects the piece of $h_{\alpha\beta}^{(2)}$ that $\delta G_{\mu\nu}^{(1,k)}$ sources.
7. The changes in $h_{\alpha\beta}^{(2)}$ affect the second-order force $a_{(2)}^i$.

Since the final waveform is invariant, one can choose whichever gauge is deemed most convenient for these calculations. As we discuss in Sec. VI, one can also choose a gauge that is convenient for solving the field equations and then transform the outputs into a gauge that is convenient for calculating forcing functions.

Before moving on, we emphasize two important facts. First, even though the waveform is invariant under the gauge transformations we consider, that does not mean waveforms computed in two different gauges will be identical (given identical initial conditions). The invariance is a limiting statement, in that the two waveforms agree up to nonzero $\mathcal{O}(\varepsilon^3)$ differences, and only in the sense that they agree when one is re-expanded at fixed values of the variables used in the other (i.e., when they are compared in precisely the *same* limit $\epsilon \rightarrow 0$). In other words, they will decidedly *not* be numerically identical functions of time for a given, finite value of ϵ . Different gauges move information between different terms in the expansion, and different gauges will change the magnitude of omitted higher-order terms. This in turn can mean that one choice of gauge can yield more accurate waveforms than another choice.

The second important point is that the 1PA frequency correction $\Omega_{(1)}^i$, forcing function $F_i^{(1)}$, and leading-order amplitudes $\dot{h}_{\mathbf{k}}^{(1)}$ are all affected by the choice of $\langle \Delta\psi^i \rangle$. These three quantities work together in unison, and all three must be computed with a single, consistent choice of gauge—or else transformed into a common gauge as

a post-processing step before using them as inputs in a 1PA waveform model. Recognizing this is especially important because different choices for $\langle \Delta\psi^i \rangle$ are already in use in 0PA waveform generation, as previously discussed in the supplemental material of Ref. [4].

C. Relevance of the secondary spin's precession

So far, we have not specifically highlighted the impact of the secondary spin's precession, except to reaffirm that it does not contribute to the 1PA orbital evolution. We now investigate its role in more detail.

To make the assessment, we write the waveform (179) as

$$h = \sum_{\mathbf{k} \in \mathbb{Z}^3} h_{\mathbf{k}}(\dot{\psi}_{\mathbf{k}}, \dot{\pi}_i, \theta, \phi, \varepsilon) \quad (194a)$$

$$= \sum_{\mathbf{k} \in \mathbb{Z}^3} \sum_{n \geq 0} \varepsilon^n h_{\mathbf{k}}^{(n)}(\dot{\psi}_{\mathbf{k}}, \dot{\pi}_i, \theta, \phi). \quad (194b)$$

We then decompose each mode into a real amplitude and a complex phase factor,

$$h_{\mathbf{k}} = A_{\mathbf{k}} e^{-i\Phi_{\mathbf{k}}}, \quad (195)$$

where $A_{\mathbf{k}} = |h_{\mathbf{k}}|$ and $\Phi_{\mathbf{k}} = -\arg(h_{\mathbf{k}})$. The phase $\Phi_{\mathbf{k}}$ characterizes the total waveform's phase evolution in an invariant way. This contrasts with phases such as $\dot{\psi}_{\mathbf{k}}$, which are associated with complex-valued amplitudes such as $\dot{h}_{\mathbf{k}}^{(1)}$. In a product such as $\dot{h}_{\mathbf{k}}^{(1)} e^{-i\dot{\psi}_{\mathbf{k}}}$, we can freely move phases between the complex amplitude and the complex exponential, making the phase non-unique, as we saw when considering gauge transformations in the previous section. The decomposition (195) avoids this ambiguity.

Decomposing Eq. (179) into the form (195), we find

$$\Phi_{\mathbf{k}} = \dot{\psi}_{\mathbf{k}} - \varepsilon^0 \arg(h_{\mathbf{k}}^{(1)}) + \frac{\varepsilon}{|h_{\mathbf{k}}^{(1)}|^2} \left\{ d_{\mathbf{k}}^0 + \chi_{\perp} d_{\mathbf{k}}^s \sin(\dot{\psi}_s) + \chi_{\perp} d_{\mathbf{k}}^c \cos(\dot{\psi}_s) \right\} + \mathcal{O}(\varepsilon^2), \quad (196)$$

with

$$d_{\mathbf{k}}^0 = -\text{Re } h_{\mathbf{k}}^{(1)} \text{Im } h_{\mathbf{k},0}^{(2)} + \text{Im } h_{\mathbf{k}}^{(1)} \text{Re } h_{\mathbf{k},0}^{(2)}, \quad (197a)$$

$$d_{\mathbf{k}}^s = \left(\text{Im } h_{\mathbf{k},+1}^{(2-\chi_{\perp})} - \text{Im } h_{\mathbf{k},-1}^{(2-\chi_{\perp})} \right) \text{Im } h_{\mathbf{k}}^{(1)} + \left(\text{Re } h_{\mathbf{k},+1}^{(2-\chi_{\perp})} - \text{Re } h_{\mathbf{k},-1}^{(2-\chi_{\perp})} \right) \text{Re } h_{\mathbf{k}}^{(1)}, \quad (197b)$$

$$d_{\mathbf{k}}^c = \left(\text{Re } h_{\mathbf{k},+1}^{(2-\chi_{\perp})} + \text{Re } h_{\mathbf{k},-1}^{(2-\chi_{\perp})} \right) \text{Im } h_{\mathbf{k}}^{(1)} - \left(\text{Im } h_{\mathbf{k},+1}^{(2-\chi_{\perp})} + \text{Im } h_{\mathbf{k},-1}^{(2-\chi_{\perp})} \right) \text{Re } h_{\mathbf{k}}^{(1)}. \quad (197c)$$

Here $h_{\mathbf{k},0}^{(2)} = (h_{\mathbf{k}}^{(2-\chi_{\perp})} + \chi_{\parallel} h_{\mathbf{k}}^{(2-\chi_{\parallel})}) e^{-i\psi_{\mathbf{k}}}$ is the total second-order \mathbf{k} mode excluding the secondary spin precession.

Similarly,

$$A_{\mathbf{k}} = \varepsilon |h_{\mathbf{k}}^{(1)}| + \frac{\varepsilon^2}{|h_{\mathbf{k}}^{(1)}|} \left\{ a_{\mathbf{k}}^0 + \chi_{\perp} a_{\mathbf{k}}^s \sin(\dot{\psi}_s) + \chi_{\perp} a_{\mathbf{k}}^c \cos(\dot{\psi}_s) \right\} + \mathcal{O}(\varepsilon^3), \quad (198)$$

with

$$a_{\mathbf{k}}^0 = \operatorname{Re} h_{\mathbf{k}}^{(1)} \operatorname{Re} h_{\mathbf{k},0}^{(2)} + \operatorname{Im} h_{\mathbf{k}}^{(1)} \operatorname{Im} h_{\mathbf{k},0}^{(2)}, \quad (199a)$$

$$\begin{aligned} a_{\mathbf{k}}^s &= -\left(\operatorname{Re} h_{\mathbf{k},+1}^{(2-\chi_{\perp})} - \operatorname{Re} h_{\mathbf{k},-1}^{(2-\chi_{\perp})}\right) \operatorname{Im} h_{\mathbf{k}}^{(1)} \\ &\quad + \left(\operatorname{Im} h_{\mathbf{k},+1}^{(2-\chi_{\perp})} - \operatorname{Im} h_{\mathbf{k},-1}^{(2-\chi_{\perp})}\right) \operatorname{Re} h_{\mathbf{k}}^{(1)}, \end{aligned} \quad (199b)$$

$$\begin{aligned} a_{\mathbf{k}}^c &= \left(\operatorname{Re} h_{\mathbf{k},+1}^{(2-\chi_{\perp})} + \operatorname{Re} h_{\mathbf{k},-1}^{(2-\chi_{\perp})}\right) \operatorname{Re} h_{\mathbf{k}}^{(1)} \\ &\quad + \left(\operatorname{Im} h_{\mathbf{k},+1}^{(2-\chi_{\perp})} + \operatorname{Im} h_{\mathbf{k},-1}^{(2-\chi_{\perp})}\right) \operatorname{Im} h_{\mathbf{k}}^{(1)}. \end{aligned} \quad (199c)$$

Now, to compare these terms to the usual n PA counting, we note that the form of the evolution equations (95a)–(95c) immediately implies that $\dot{\psi}_{\mathbf{k}}$ can be expanded as

$$\dot{\psi}_{\mathbf{k}}(\varepsilon t, \varepsilon) = \frac{1}{\varepsilon} \left[\dot{\psi}_{\mathbf{k}}^{(0)}(\varepsilon t) + \varepsilon \dot{\psi}_{\mathbf{k}}^{(1)}(\varepsilon t) + \mathcal{O}(\varepsilon^2) \right] \quad (200)$$

in terms of the ‘slow time’ εt ; cf. Eq. (6). Here $\dot{\psi}_{\mathbf{k}}^{(0)}/\varepsilon$ is the 0PA term and subsequent orders are n PA. Comparing this to Eq. (196), we see that the precession of the secondary spin contributes to the waveform phase at the same order in ε as a 2PA effect.

One might hastily conclude two things: (i) if we are justified in neglecting 2PA terms when modeling some class of binaries (such as EMRIs), then we are equally well justified in neglecting the secondary spin precession; and (ii) if we think we should not include 1PA first-order conservative SF effects in a waveform model until we have also included 1PA second-order dissipative SF effects, then we should likewise not include the orthogonal component of the secondary spin unless we include all 2PA terms.

However, both of these conclusions might be *too* hasty. About the first, we note that the effect of the spin precession is qualitatively different than the effect of a 2PA forcing function: unlike a 2PA term, the precession terms (in both the phase and the amplitude) oscillate on a fast time scale rather than only varying on the slow, radiation-reaction time scale of the inspiral. About the second conclusion, we note that first-order conservative and second-order dissipative effects have the same qualitative effect on the waveform, and the division between them is gauge dependent. There is hence no reason to expect that including just one of them will improve the fidelity of a waveform model. This contrasts with the secondary spin precession, which has a qualitatively different signature than other effects that contribute to the waveform phase at the same order in ε . Ultimately, data analysis studies should assess the relevance of the secondary spin precession for gravitational-wave science. In

Ref. [90], we highlight that $a_{\mathbf{k}}^c$, $d_{\mathbf{k}}^c$, $a_{\mathbf{k}}^s$ and $d_{\mathbf{k}}^s$ are numerically small compared to $a_{\mathbf{k}}^0$ and $d_{\mathbf{k}}^0$ for quasicircular, approximately equatorial inspirals into slowly spinning primaries. Their relative contribution to the waveform for generic configurations warrants further investigation.

VI. EVOLUTION WITH ASYMPTOTIC FLUXES

As we have presented, and as summarized in the Introduction, the secondary spin contributes to the 1PA waveform in three ways: (i) through a correction $h_{\mathbf{k}q}^{(2-\chi_2)}$ to the waveform amplitudes, obtained by solving Eq. (152), (ii) through a correction $\Omega_{(1-\chi_2)}^i$ to the orbital frequencies, given analytically in Eq. (145) in terms of the MPD force (and an arbitrary gauge choice), and (iii) through a correction $F_i^{(1-\chi_2)}$ to the 1PA forcing function.

The first two of these ingredients can be obtained independently from all other 1PA calculations; they do not require the solution to the first-order field equation (140)⁹ or any local calculations of regular fields at the particle. However, in the form (146) the forcing function $F_i^{(1-\chi_2)}$ requires as inputs the local force generated by the regular part of the solution $h_{\alpha\beta}^{(2-\chi_2,\mathbf{k},0)}$ to Eq. (152); the regular field and first-order self-force extracted from the solution to the first-order field equation (140); and the spin perturbation $\langle \delta S^\alpha \rangle_{\tilde{\psi}_s}$ in Eq. (75), which in turn seems to require the solution to the nutation equations (66) and (67) (though Appendix D shows that the nutation equations do not need to be explicitly solved even when constructing $F_i^{(1-\chi_2)}$ from local forces and torques).

Remarkably, a flux-balance law recently derived by Grant [70] allows us to entirely bypass this complexity and instead calculate $F_i^{(1-\chi_2)}$ solely from the values of $h_{\alpha\beta}^{(1,\mathbf{k})}$ and $h_{\alpha\beta}^{(2-\chi_2,\mathbf{k},0)}$ (or equivalent Teukolsky mode amplitudes) at future null infinity and at the primary’s future horizon. Working in the pseudo-Hamiltonian framework of Ref. [130], Grant considered small corrections to spinning test-body motion due to the linear metric perturbation that the body produces, also ultimately linearizing in the spin. Such a framework is conceptually different than our own, but since the spin effects we are interested in are linear in the perturbation produced by the body, we can directly import Grant’s result for those effects.

⁹ The exception to this statement is if the fixed-frequencies gauge is not used. In that case, the source term $\delta G_{\alpha\beta}^{(1-\chi_2,\mathbf{k})}$ in Eq. (152) requires the solution to the first-order field equation as input. The analogous statement applies if solving the analogous Teukolsky equation.

We state Grant's flux-balance law as¹⁰

$$\left\langle \frac{dJ_\alpha}{d\tau} \right\rangle_\tau = -\mathcal{F}[h, \partial_{\vartheta_0^\alpha} h] + \mathcal{O}(\varepsilon^2 s^0, \varepsilon^3). \quad (201)$$

Here $(\vartheta^\alpha, J_\alpha)$ are action-angle variables for the linearised MPD system in Kerr spacetime, and ϑ_0^α is the angle value at the reference time where we wish to compute the average. On the left, $\langle \cdot \rangle_\tau$ is an average over proper time around the reference time. On the right, the bilinear operator $\mathcal{F}[h, \partial_{\vartheta_0^\alpha} h]$ is an average flux to future null infinity and down the black hole horizon, constructed from the linear perturbation $h_{\alpha\beta}$ due to a spinning particle. We use $\mathcal{O}(\varepsilon^2 s^0)$ to denote spin-independent 1PA terms.

Later in this section, we describe how Eq. (201) can be recast as

$$\left\langle \frac{dJ_\alpha}{dt} \right\rangle = -\mathcal{F}_\alpha + \mathcal{O}(\varepsilon^2 s^0, \varepsilon^3), \quad (202)$$

where $\langle \cdot \rangle$ is our angle average, and (adapting the notation of Ref. [88]) the flux is¹¹

$$\mathcal{F}_\alpha \equiv \sum_{\ell\mathbf{k}} \frac{\varepsilon_\alpha}{4\pi\omega_{\ell\mathbf{k}}^3} \left(|\mathcal{Z}_{\ell\mathbf{k}}^{\text{out}}|^2 + \frac{\omega_{\mathbf{k}}}{\mathcal{P}_{\mathbf{k}}} |\mathcal{Z}_{\ell\mathbf{k}}^{\text{down}}|^2 \right). \quad (203)$$

Here

$$\omega_{\mathbf{k}} \equiv k_i \Omega^i = k_i \left(\Omega_{(0)}^i + \varepsilon \Omega_{(1-\chi_2)}^i \right), \quad (204)$$

$\varepsilon_\alpha \equiv (-\omega_{\mathbf{k}}, k_r, k_\theta, k_\phi)$, $\mathcal{P}_{\mathbf{k}} \equiv \omega_{\mathbf{k}} - k_\phi \chi_1^{(0)} / (2r_+)$, and r_+ is the usual outer horizon radius. The quantities

$$\mathcal{Z}_{\ell\mathbf{k}}^{\text{out/down}} = \mathcal{Z}_{\ell\mathbf{k}}^{(1)\text{out/down}} + \varepsilon \chi_{\parallel} \mathcal{Z}_{\ell\mathbf{k}}^{(2-\chi_{\parallel})\text{out/down}} \quad (205)$$

are the usual Teukolsky mode amplitudes at the horizon ('down') and at future null infinity ('out'), which here are constructed from the solution to the (linear) Teukolsky equation with a spinning-particle source, corresponding to the sum of the solutions $h_{\alpha\beta}^{(1,\mathbf{k})}$ and $h_{\alpha\beta}^{(2-\chi_2,\mathbf{k},0)}$ to Eqs. (140) and (152). As in the previous sections, there is no contribution from χ_\perp at this order.

Equation (202) also implies evolution equations for the spin-corrected constants of motion $P_i = (E, L_z, K)$ defined in Eqs. (168) and (169):

$$\left\langle \frac{dP_i}{dt} \right\rangle = -\frac{\partial P_i}{\partial J_A} \mathcal{F}_\alpha + \mathcal{O}(\varepsilon^2 s^0, \varepsilon^3), \quad (206)$$

where it is understood that the Jacobian $\partial P_i / \partial J_\alpha$ is calculated for a test particle to linear order in spin. The

form (206) requires the relationships $P_i(J_\alpha)$, which were recently calculated in closed form by Witzany and collaborators [72]. In particular, the linear spin corrections to the Jacobian $\partial P_i / \partial J_\alpha$ can be obtained analytically from Eqs. (B.9) and (B.15) of Ref. [72].

Equation (206) represents a complete, practical description of dissipation at 0PA and of the secondary spin's contribution to dissipation at 1PA order. At 0PA it reduces to the standard flux-driven evolution equations [88, 95]. At 1PA, it recovers the spinning-body energy and angular momentum flux balance formulae of Ref. [53] (also see Appendix D), and it provides a formula for dK/dt that completes the description.

In Sec. VI A below, we describe how Eq. (201) is translated into the forms (202) and (206). In Sec. VI B we outline how to use Eq. (206) to compute the forcing functions $F_i^{(1-\chi_2)}$ in the 1PA waveform-generation scheme of Sec. V. Readers uninterested in the technical details can skip directly to Sec. VI B.

A. Importing the results of Grant, Witzany et al., and Isoyama et al.

We consider each of the three ingredients in Eq. (206) in turn: actions, averages, and fluxes.

1. Actions

Grant's result (201) is valid for any set of phase-space coordinates (ϑ^A, J_A) that behave as action-angle variables in the test-body limit, by which we mean $d\vartheta^A/d\tau = \partial H/\partial J_A = \nu_A(J_B)$ and $dJ_A/d\tau = 0$ when $h_{\alpha\beta}^R \rightarrow 0$. Here calligraphic indices denote coordinates on the 10D phase space for the test-body MPD dynamics at linear order in spin, and H is a suitable Hamiltonian for the MPD dynamics. To relate this setting to ours, note the 10D phase space has (noncanonical) coordinates (x^A, p_A) with $x^A = (x^\alpha, \psi_s)$ and $p_A = (p_\alpha, p_{\psi_s})$, where $p_{\psi_s} = \chi_{\parallel} - \chi_2$ [51]. The bulk of our paper works instead on the physical 7D submanifold defined by the mass-shell condition $\sqrt{-g^{\alpha\beta}} p_\alpha p_\beta = m_2$ and $p_{\psi_s} = \text{constant}$, reducing t to a parameter rather than a coordinate.

Fortunately, Witzany et al. [51, 72] have recently provided action-angle coordinates that are appropriate for use in Eq. (201), based on a Hamilton-Jacobi formulation of the test-body MPD equations (through linear order in s and assuming the Tulczyjew-Dixon SSC). The actions are given by

$$J_t = -E, \quad J_\phi = L_z, \quad J_{\psi_s} = \chi_{\parallel} - \chi_2, \quad (207)$$

and

$$J_y(P_B) \equiv \frac{1}{2\pi m_2} \oint_{\gamma_y} \Pi_A dx^A, \quad (208)$$

for $y = r, \theta$. Here γ_y are any two homotopically inequivalent closed radial and polar contours on the torus of

¹⁰ See Eq. (5.41) in Ref. [70]. Note that our $\mathcal{F}[h, \partial_{\vartheta^\alpha} h]$ corresponds to $(\partial_{\vartheta^\beta})^A \mathcal{F}_A$ in that equation, and we have used Eq. (5.9) therein with P_β replaced by J_β . Equation (5.41) is directly for the constants of motion P_β , while we find it conceptually clearer to start with balance law for the actions.

¹¹ For comparison with Ref. [88], we note that the mode numbers there are written as $k_i = (n, k, m)$.

constant $P_A = (m_2, E, L_z, K, \chi_{\parallel})$ (and any constant t), and we have introduced a factor $1/m_2$ to work with actions that are m_2 -independent at leading order. The quantities $\Pi_B = (\Pi_\mu, \Pi_{\psi_s})$ are the momenta conjugate to $x^A = \{x^\alpha, \psi_s\}$ [50, 72], related to p_A by $\Pi_{\psi_s} = p_{\psi_s}$ and

$$\Pi_\mu \equiv p_\mu + \frac{1}{2} m_2^2 \bar{\omega}_{AB\mu} S^{AB}, \quad (209)$$

with $\bar{\omega}_{AB\mu} \equiv (\nabla_\mu \sigma_A^\alpha) \sigma_{B\alpha}$ such that $u^\mu \bar{\omega}_{AB\mu} = \omega_{AB}$ and S^{AB} are the triad components of the (dimensionless) spin tensor.¹² Equation (208) also applies for $y = \phi, \psi_s$ with appropriate closed contours γ_y , but in those cases it immediately reduces to Eq. (207) because $\Pi_\phi = L_z$ and $\Pi_{\psi_s} = \chi_{\parallel} - \chi_2$ are constant on the torus.

In Ref. [72], Witzany and collaborators derived closed-form analytical expressions for these action variables in terms of the test-body conserved quantities P_A by performing the loop integrals with Hadamard finite-part integration. The results are linearized in spin in the form

$$J_\alpha = J_\alpha^{(0)}(P_i) + \varepsilon \chi_{\parallel} J_\alpha^{(1-\chi_2)}(P_i) + \mathcal{O}(s^2), \quad (210)$$

with no contribution from χ_{\perp} at linear order. They also provided the linear spin corrections to the Jacobian $\partial J_\alpha / \partial P_i$, meaning $\partial J_\alpha^{(1-\chi_2)} / \partial P_i$, from which we can obtain the linear spin correction to $\partial P_i / \partial J_\alpha$ in Eq. (206). Many elements of the Jacobian are trivial by virtue of the relations (207), and (as we explain in Sec. VI B) ultimately the only elements required are $\partial J_y / \partial P_i$ for either $y = r$ or $y = \theta$.

These results are conveniently available in a Mathematica notebook in the supplemental material of Ref. [72]. That notebook also contains closed-form expressions for the frequency corrections due to the spin, in the form

$$\Omega^i = \Omega_{(0)}^i(P_j) + \varepsilon \chi_{\parallel} \Omega_{(1-\chi_2)}^i(P_j) + \mathcal{O}(s^2); \quad (211)$$

recall that χ_{\perp} cannot contribute because the frequency involves an average over the precession phase. Both this and Eq. (210) are expansions in ε at fixed P . Since $P_i = \langle P_i \rangle$ for a spinning test particle, we can replace P_i with $\langle P_i \rangle$ in the above expressions. The expressions then also hold true in the presence of self-force in the fixed-constants-of-motion gauge discussed in Sec. IV D; otherwise, they omit $\mathcal{O}(\varepsilon s^0)$ self-force terms.

2. Averages

Grant's derivation is based on regular perturbation theory rather than a multiscale expansion. Specifically,

¹² To define the derivative $\nabla_\mu \sigma_A^\alpha$, we must promote the tetrad σ_A^α to a field in a neighborhood of the worldline. Since the tetrad is defined along a geodesic (or an accelerated curve as in Sec. III), it is immediately promoted to a field by considering a congruence of such curves [51, 72].

he works with a spinning particle, considers the linear perturbation it sources, and finds the effect of that linear perturbation on its motion. This type of approach is well known to be ill-behaved on large time and space scales [131], but incorporating the results of such an approach into well-behaved schemes is also standard SF lore.

We consider a spacetime described by our multiscale expansion (106). To put it in the form assumed by Grant, we expand our spacetime metric (and particle trajectory) in an ordinary power series in ε near an arbitrary time t_0 . We then define an average with respect to a time λ as

$$\langle \cdot \rangle_\lambda \equiv \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{\lambda(t_0)-T}^{\lambda(t_0)+T} \cdot d\lambda. \quad (212)$$

This is the average in Eq. (206), with $\lambda = \tau$.

Before mapping quantities in the regular expansion onto those in the multiscale expansion, we first convert Eq. (201) to an average over t . At the same time, we effectively convert to the formalism in the body of this paper: a 3+1 split in which t is a parameter along trajectories rather than a coordinate. This is achieved by observing that if J_α is conjugate to ϑ^α for a Hamiltonian H that generates proper-time evolution [i.e., $d\vartheta^\alpha/dt = \partial H/\partial J_\alpha = \nu^\alpha(J_\beta)$], then J_α is also conjugate to $\dot{\psi}^\alpha$ for a Hamiltonian \mathcal{H} that generates coordinate-time evolution [i.e., $d\dot{\psi}^i/dt = \partial \mathcal{H}/\partial J_i = \Omega^i(J_k)$ and $\dot{\psi}^t = t$] [132–134]. Using \mathcal{H} in place of H in Grant's derivation yields

$$\left\langle \frac{dJ_\alpha}{dt} \right\rangle_t = -\mathcal{F}[h, \partial_{\dot{\psi}_0^\alpha} h] \equiv -\mathcal{F}_\alpha, \quad (213)$$

where $\dot{\psi}_0^t \equiv t_0$ and $\dot{\psi}_0^i \equiv \dot{\psi}^i(t_0)$.¹³ Alternatively, we can obtain this result directly from Eq. (201) using the general relation $\langle df/dt \rangle_\tau = \langle df/dt \rangle_t \langle d\tau/dt \rangle_t$ [77, 88] and the Jacobian $\partial \dot{\psi}^\beta / \partial \vartheta^\alpha$.

To relate $\langle dJ_\alpha/dt \rangle_t$ to $\langle dJ_\alpha/dt \rangle$, we now examine the regular expansion around t_0 . Inspecting Eqs. (95) reveals that $(\dot{\psi}^i, \dot{\pi}_i)$ can be written as functions $\dot{\psi}^i(\varepsilon t, \varepsilon)$ and $\dot{\pi}_i(\varepsilon t, \varepsilon)$, where $\dot{\psi}^i(\varepsilon t, \varepsilon)$ has the expansion (6), and $\dot{\pi}_i(\varepsilon t, \varepsilon)$ has an expansion

$$\dot{\pi}_i(\varepsilon t, \varepsilon) = \dot{\pi}_i^{(0)}(\varepsilon t) + \varepsilon \dot{\pi}_i^{(1)}(\varepsilon t) + \mathcal{O}(\varepsilon^2). \quad (214)$$

We expand these around their values at t_0 using $\varepsilon t =$

¹³ When using t as a time parameter, Hamilton's equations only apply for $\alpha = i$. As a consequence, Grant's derivation only yields Eq. (213) for $\alpha = i$. However, the equation for $\alpha = t$ follows from the others as an equation for the Hamiltonian itself: $\left\langle \frac{d\mathcal{H}}{dt} \right\rangle = \left\langle \frac{\partial \mathcal{H}}{\partial J_i} \frac{dJ_i}{dt} \right\rangle = \left\langle \Omega^i \frac{dJ_i}{dt} \right\rangle = -\mathcal{F}[h, \Omega^i \partial_{\dot{\psi}_0^\alpha} h]$. This agrees with the $\alpha = t$ component of Eq. (213) because $J_t = -\mathcal{H}$ and, by virtue of Eq. (215), $h_{\alpha\beta}$ depends on $\dot{\psi}_0^\alpha$ in the combination $(\dot{\psi}_0^i - \Omega^i t_0)$.

$\tilde{t}_0 + \varepsilon \Delta t$ with $\Delta t \equiv (t - t_0)$:

$$\begin{aligned}\mathring{\psi}^i(\varepsilon t, \varepsilon) &= \mathring{\psi}^i(\tilde{t}_0, \varepsilon) + \Delta t \Omega^i(\tilde{t}_0, \varepsilon) + \mathcal{O}(\varepsilon \Delta t^2), \\ \mathring{\pi}_i(\varepsilon t, \varepsilon) &= \mathring{\pi}_i(\tilde{t}_0, \varepsilon) + \varepsilon \Delta t \frac{d}{d\tilde{t}_0} \mathring{\pi}_i(\tilde{t}_0, \varepsilon) + \mathcal{O}(\varepsilon^2 \Delta t^2).\end{aligned}\quad (216)$$

The first term in Eq. (216) represents the constant parameters of a test-particle orbit (plus self-force contributions),

$$\mathring{\pi}_i(\tilde{t}_0, \varepsilon) = \mathring{\pi}_i^{(0)}(\tilde{t}_0) + \varepsilon \mathring{\pi}_i^{(1-\chi_2)}(\tilde{t}_0) + \mathcal{O}(\varepsilon s^0); \quad (217)$$

and the first two terms in Eq. (215) represent the test-particle orbital phases (plus self-force terms), with arbitrary initial values $\mathring{\psi}^i(\tilde{t}_0, \varepsilon)$ and constant frequencies

$$\Omega^i(\tilde{t}_0, \varepsilon) = \Omega^i(\tilde{t}_0) + \varepsilon \Omega_{(1-\chi_2)}^i(\tilde{t}_0) + \mathcal{O}(\varepsilon s^0). \quad (218)$$

Unlike $\mathring{\pi}_i$, the test-body constants of motion P_i , defined in Eqs. (168) and (169), contain oscillatory contributions due to the self-force, and the actions $J_\alpha(P_i)$ inherit those oscillations. We divide them into averaged and oscillatory terms,

$$J_\alpha = \langle J_\alpha \rangle + J_\alpha^{\text{osc}}, \quad (219)$$

as defined generically in Eq. (C16). $\langle J_\alpha \rangle$ is a function of $\mathring{\pi}_i$ and therefore has an expansion of the form (216),

$$\langle J_\alpha \rangle = \langle J_\alpha \rangle(\tilde{t}_0, \varepsilon) + \varepsilon \Delta t \frac{d}{d\tilde{t}_0} \langle J_\alpha \rangle(\tilde{t}_0, \varepsilon) + \mathcal{O}(\varepsilon^2 \Delta t^2). \quad (220)$$

On the other hand, J_α^{osc} is an oscillatory function of the phases and hence has an expansion

$$J_\alpha^{\text{osc}} = \varepsilon \sum_{\mathbf{k} \neq 0} J_\alpha^{(1,\mathbf{k})}(\tilde{t}_0, \varepsilon) e^{-i(\mathring{\psi}_\mathbf{k}^0 + \omega_\mathbf{k} \Delta t)} + \mathcal{O}(\varepsilon^2 \Delta t^2) \quad (221)$$

with $\mathring{\psi}_\mathbf{k}^0 = k_i \mathring{\psi}^i(\tilde{t}_0, \varepsilon)$ and $\omega_\mathbf{k} = k_i \Omega^i(\tilde{t}_0, \varepsilon)$.

By substituting Eq. (219) into the time average (212), we immediately find

$$\left\langle \frac{dJ_\alpha}{dt} \right\rangle_t = \varepsilon \frac{d\langle J_\alpha \rangle(\tilde{t}_0, \varepsilon)}{d\tilde{t}_0} + \mathcal{O}(\varepsilon^2). \quad (222)$$

We also see that the t average is ill defined beyond order ε because of the terms quadratic (and higher order) in Δt in Eqs. (220) and (221). We hence define the average to apply only at leading order, setting subleading terms to zero before averaging.

Since the equality (222) applies for any \tilde{t}_0 , we can rewrite it as

$$\left\langle \frac{dJ_\alpha}{dt} \right\rangle_t = \frac{d\langle J_\alpha \rangle}{dt}, \quad (223)$$

discarding subleading terms as explained above. We also have the trivial identity

$$\left\langle \frac{df}{dt} \right\rangle = \frac{d\langle f \rangle}{dt} \quad (224)$$

for any f since $\langle df^{\text{osc}}/dt \rangle = 0$. Combining these, we obtain our desired identity:

$$\left\langle \frac{dJ_\alpha}{dt} \right\rangle_t = \left\langle \frac{dJ_\alpha}{dt} \right\rangle. \quad (225)$$

We similarly consider the flux \mathcal{F}_α on the right-hand side of Eq. (213), which is an integral of products of $h_{\alpha\beta}$ and $\partial_{\mathring{\psi}_0^\alpha} h_{\alpha\beta}$ over cuts of the horizon and future null infinity, averaged over all time along those surfaces. Here $h_{\alpha\beta}$ is the linear perturbation sourced by a spinning particle on an orbit with constant parameters (217) and corresponding constant frequencies (218), discarding the $\mathcal{O}(\varepsilon s^0)$ terms in those equations. We can write this metric perturbation as

$$h_{\alpha\beta} = \sum_{\mathbf{k} \in \mathbb{Z}^3} h_{\alpha\beta}^{(\mathbf{k})} e^{-i(\mathring{\psi}_\mathbf{k}^0 + \omega_\mathbf{k} \Delta t)} + \mathcal{O}(\varepsilon^2 s^0, \varepsilon^3, \chi_\perp) \quad (226)$$

with

$$h_{\alpha\beta}^{(\mathbf{k})} = \varepsilon \mathring{h}_{\alpha\beta}^{(1,\mathbf{k})}(\mathring{\pi}_i, x^i) + \varepsilon^2 \chi_{\parallel} \mathring{h}_{\alpha\beta}^{(2-\chi_\parallel, \mathbf{k})}(\mathring{\pi}_i, x^i), \quad (227)$$

in the notation of previous sections, where $\mathring{\pi}_i$ is evaluated at t_0 and we omit the $\mathcal{O}(\varepsilon s^0)$ terms in Eqs. (217) and (218). We also omit the term proportional to χ_\perp in the metric perturbation, which can only contribute to the flux at $\mathcal{O}(s^2)$ due to its oscillatory dependence on the precession phase (just as it could only contribute to the local dynamics at that order). Given Eq. (226), the derivative in $\partial_{\mathring{\psi}_0^\alpha} h_{\alpha\beta}$ can be replaced by

$$\frac{\partial}{\partial \mathring{\psi}_0^\alpha} = -i\varepsilon_\alpha, \quad (228)$$

where ε_α is defined below Eq. (204) and we used $\partial_{t_0} \Delta t = -1$.

Equation (226) is precisely what one would obtain for the metric perturbation by substituting the expansions (215), (217), and (218) into the multiscale expansion (106) with Eqs. (113) and (114). The average over time along the horizon and future null infinity also yields precisely the same expression in terms of Fourier mode amplitudes and frequencies as one would obtain by averaging over angles in the multiscale expansion, noting we can neglect δm_1 and $\delta \chi_1$ when calculating linear-in-spin IPA terms.

3. Fluxes

Finally, we convert the fluxes \mathcal{F}_α into the form (203). These fluxes, as mentioned above, are defined from time averages of symplectic currents, which are products of the retarded metric perturbation evaluated on the horizon and at future infinity. They can be expressed in terms of Teukolsky amplitudes following standard methods of metric reconstruction [135]. However, we can skip that

calculation by appealing to the results of Isoyama et al. for $\langle dJ_\alpha/dt \rangle$ in the case of a nonspinning particle [88].

The essential point is that $\mathcal{F}_\alpha \equiv \mathcal{F}[h, \partial_{\dot{\psi}_0^\alpha} h]$ is an identical function of $h_{\mu\nu}$ and $\partial_{\dot{\psi}_0^\alpha} h_{\mu\nu}$ regardless of whether $h_{\mu\nu}$ is sourced by a spinning or a nonspinning particle. More concretely, it is an identical function of the mode coefficients $h_{\alpha\beta}^k$ and frequencies ω_k in Eq. (226) regardless of their values. The conversion into Teukolsky amplitudes is likewise independent of the values of $h_{\alpha\beta}^k$ and frequencies ω_k . Therefore the expression for \mathcal{F}_α , and hence for $\langle dJ_\alpha/dt \rangle$, in terms of Teukolsky amplitudes and mode frequencies, is functionally the same for a spinning body as for a non-spinning body. It follows that \mathcal{F}_α for a spinning particle must be the same function of Teukolsky amplitudes and mode frequencies as Isoyama et al.'s result for $\langle dJ_\alpha/dt \rangle$ for a nonspinning particle. This is the result reproduced in Eq. (203).

B. Pragmatic summary

To incorporate Eq. (206) into the 1PA waveform generation scheme of Sec. V, we only need to extract its linear-in-spin term and convert it into an expression for $\dot{\pi}_i/dt$.

We first write Eq. (206) more explicitly. Noting $J_t = -E$ and $J_\phi = L_z$ along with Eq. (224), we have immediately

$$\frac{d\langle E \rangle}{dt} = \mathcal{F}_t + \mathcal{O}(\varepsilon^2 s^0), \quad (229a)$$

$$\frac{d\langle L_z \rangle}{dt} = -\mathcal{F}_\phi + \mathcal{O}(\varepsilon^2 s^0). \quad (229b)$$

Next, to obtain the evolution formula for the Carter constant, rather than expanding the Jacobian in Eq. (206), we simply rearrange

$$\frac{dJ_r}{dt} = \frac{\partial J_r}{\partial P_j} \frac{dP_j}{dt} \quad (230)$$

to obtain

$$\frac{dK}{dt} = \left(\frac{\partial J_r}{\partial K} \right)^{-1} \left(\frac{dJ_r}{dt} - \frac{\partial J_r}{\partial E} \frac{dE}{dt} - \frac{\partial J_r}{\partial L_z} \frac{dL_z}{dt} \right). \quad (231)$$

Taking the average yields

$$\frac{d\langle K \rangle}{dt} = \left(\frac{\partial J_r}{\partial K} \right)^{-1} \left(-\mathcal{F}_r - \frac{\partial J_r}{\partial E} \mathcal{F}_t + \frac{\partial J_r}{\partial L_z} \mathcal{F}_\phi \right), \quad (232)$$

where it is understood that in $\partial J_r/\partial P_i$ we only include the linear spin correction (neglecting the 1SF correction, which contains oscillations) and replace P_i with $\langle P_i \rangle$. The same expression holds with J_r replaced by J_θ .

Equations (229) and (232) can be straightforwardly linearized in χ_{\parallel} (noting χ_{\perp} does not appear). This involves substituting the frequencies (204) and amplitudes (205) into the fluxes (203) and substituting the radial or polar action (210).

The explicit form of the result depends on the choice of phase space gauge. In principle, the evolution equations (229) and (232) apply in any gauge. However, the actions (210) and frequencies (211) are expressed in the fixed-constants-of-motion gauge discussed in Sec. IV D. Meanwhile, the field equation for the mode amplitudes $\mathcal{Z}_{\ell k}^{(2-\chi_{\parallel})\text{out/down}}$ is simplest in the fixed-frequencies gauge. In the remainder of this section, we summarize the prescription in these two gauges, which we follow Ref. [71] in labeling 'FF' (fixed frequencies) and 'FC' (fixed constants).

We first derive the transformation between the two gauges. We assume the gauge freedom $\langle \Delta\psi^i \rangle$ is specified in the same way in both cases. We also note that the gauge imposed on 1SF effects is independent of the gauge imposed on linear spin effects, and we only consider the latter here.

The frequencies in the FC gauge are given by Eq. (211), which we rewrite as

$$\Omega^i = \Omega_{(0)}^i(\dot{\pi}_j^{\text{FC}}) + \varepsilon\chi_{\parallel}\Omega_{(1-\text{FC})}^i(\dot{\pi}_j^{\text{FC}}). \quad (233)$$

Here $\dot{\pi}_i$ are geodesically related to $\langle P_i \rangle$, or equivalently, $\langle P_i \rangle(\dot{\pi}_j^{\text{FC}}) = P_i^{(0)}(\dot{\pi}_j^{\text{FC}})$. To transform to the FF gauge, we write

$$\dot{\pi}_i^{\text{FC}} = \dot{\pi}_i^{\text{FF}} + \varepsilon\chi_{\parallel}\delta\pi_i^{\text{FF}}. \quad (234)$$

We then substitute this into Eq. (233) and enforce the fixed-frequencies condition

$$\Omega^i = \Omega_{(0)}^i(\dot{\pi}_j^{\text{FF}}), \quad (235)$$

which yields

$$\delta\pi_j^{\text{FF}} = -\frac{\partial\dot{\pi}_j}{\partial\Omega_{(0)}^i}\Omega_{(1-\text{FC})}^i. \quad (236)$$

Here $\partial\dot{\pi}_j/\partial\Omega_{(0)}^i$ represents the inverse of the geodesic Jacobian $\partial\Omega_{(0)}^i/\partial\dot{\pi}_j$.

With Eq. (236) in hand, we now summarize the prescription in the two gauges:

1. In the FC gauge, we can read off the linear spin contribution to the forcing function in the final evolution equation (95) from Eq. (177):

$$F_i^{(1-\text{FC})} = \frac{\partial\dot{\pi}_i}{\partial P_k^{(0)}} \left(\frac{d\langle P_k \rangle}{dt} \right)^{(1-\text{FC})}, \quad (237)$$

where $\partial\dot{\pi}_i/\partial P_k^{(0)}$ is the inverse of the geodesic Jacobian $\partial P_k^{(0)}/\partial\dot{\pi}_i$. $\left(\frac{d\langle P_k \rangle}{dt} \right)^{(1-\text{FC})}$ is the linear spin term in Eqs. (229) and (232), extracted using the expansions of the frequencies (204), amplitudes (205), and radial action (210). The mode amplitudes $\mathcal{Z}_{\ell k}^{(2-\chi_{\parallel})\text{out/down}}$ are calculated from the Teukolsky analog of Eq. (152), accounting for

$\Omega_{(1-\text{FC})}^i$ terms in the noncompact source term. Alternatively, $\mathcal{Z}_{\ell k}^{(2-\chi_{\parallel})\text{out/down}}$ in the FC gauge can be calculated by first solving the Teukolsky analog of Eq. (152) in the FF gauge (with no noncompact source term), and then using

$$\mathcal{Z}_{\ell k}^{(2-\text{FC})} = \mathcal{Z}_{\ell k}^{(2-\text{FF})} - \chi_{\parallel} \delta \pi_i^{\text{FF}} \partial_{\pi_i} \mathcal{Z}_{\ell k}^{(1)}. \quad (238)$$

2. In the FF gauge, we have

$$\frac{d\dot{\pi}_i}{dt} = \frac{\partial \dot{\pi}_i}{\partial \Omega_{(0)}^j} \left(\frac{\partial \Omega_{(0)}^j}{\partial P_k} + \varepsilon \chi_{\parallel} \frac{\partial \Omega_{(1-\text{FC})}^j}{\partial P_k} \right) \frac{d\langle P_k \rangle}{dt}, \quad (239)$$

where $\partial \dot{\pi}_i / \partial \Omega_{(0)}^j$ is the inverse of the geodesic Jacobian. From this, we read off the linear spin term:

$$F_i^{(1-\text{FF})} = \frac{\partial \dot{\pi}_i}{\partial P_k^{(0)}} \left(\frac{d\langle P_k \rangle}{dt} \right)^{(1-\text{FF})} + \chi_{\parallel} \frac{\partial \dot{\pi}_i}{\partial \Omega_{(0)}^j} \frac{\partial \Omega_{(1-\text{FC})}^j}{\partial P_k} \left(\frac{d\langle P_k \rangle}{dt} \right)^{(0)}, \quad (240)$$

where $\partial \dot{\pi}_i / \partial P_k^{(0)}$ is the inverse of the geodesic Jacobian. In calculating $\left(\frac{d\langle P_k \rangle}{dt} \right)^{(1-\text{FF})}$, we set $\Omega_{(1-\chi_2)}^i$ to zero everywhere it appears. Specifically, it is set to zero in the fluxes (203), and there is no noncompact source term in the Teukolsky equation for the amplitudes $\mathcal{Z}_{\ell k}^{(2-\chi_{\parallel})\text{out/down}}$. In Eq. (232) we require J_r in the FF gauge. This is straightforwardly obtained from its value in the FC gauge:

$$J_{\alpha} = J_{\alpha}^{(0)}(\dot{\pi}_i^{\text{FF}}) + \varepsilon \chi_{\parallel} J_{\alpha}^{(1-\text{FC})}(\dot{\pi}_i^{\text{FF}}) + \varepsilon \chi_{\parallel} \delta \pi_i^{\text{FF}} \frac{\partial P_j^{(0)}}{\partial \dot{\pi}_i} \frac{\partial J_{\alpha}^{(0)}}{\partial P_j}. \quad (241)$$

VII. DISCUSSION AND CONCLUSIONS

We conclude with a summary of this work and of progress toward generic 1PA waveform models.

A. This work

In this paper, we have extended the 1PA multiscale waveform-generation framework of Ref. [76] to include a generic secondary spin. The framework accounts for all 1PA effects for generic orbits of a spinning secondary around a Kerr black hole, including, in particular, all 1PA spin-precession effects (though excluding orbital resonances). The scheme is summarized in Sec. V, and the contribution of the secondary spin is further outlined and streamlined in Sec. VI.

Our analysis began in Sec. III with a detailed study of the spin degrees of freedom for a spinning, gravitating secondary body with an arbitrary, self-accelerated orbital configuration and precessing spin orientation in Kerr spacetime. We have characterized both the precession and the nutation of the secondary's spin with simple parameters and corresponding evolution formulae. Our formulation has elucidated that the conserved quantity related to Rüdiger's constant (χ_{\parallel}) is exactly constant in the MPD-Harte system, at linear order in spin, as a simple consequence of the existence of an orthonormal Fermi-Walker-transported basis in the effective metric.

Having suitably parameterized the secondary spin, we incorporated it into the multiscale framework in Secs. IV and V. In doing so, we recovered the known result that the secondary spin's precession decouples from the 1PA orbital evolution (at least away from resonances). This decoupling was easily anticipated [63–65, 71] because any precession effect at linear order in the spin is purely oscillatory, meaning it cannot survive averaging over the precession period. However, we also showed the less obvious result that the nutation equations do not need to be solved at 1PA order. This required a more careful analysis because the nutation generates a force that *does* survive precession-averaging and does contribute to the 1PA orbital dynamics.

Despite the fact that the precession does not enter the 1PA orbital evolution, we advocated in Sec. V that its direct contribution to the waveform, through an additional $\mathcal{O}(\varepsilon^2)$ oscillatory amplitude, is worth calculating. Since it represents a qualitatively new feature in the waveform, modulating the waveform phase and amplitude, it is potentially relevant for data analysis in some regions of parameter space. On the other hand, nutation will only make a direct contribution to the waveform at a still higher order in the mass ratio, in an order- ε^3 term sourced by the δS^{α} contribution to the dipole stress-energy tensor $T_{\alpha\beta}^{(d)}$. This makes nutation's direct contribution exceedingly unlikely to be relevant (unlike its indirect contribution through its impact on the 1PA orbital dynamics).

In addition to incorporating secondary spin, we have also illuminated broader aspects of the multiscale framework, specifically streamlining the derivation of the final orbital evolution equations in Sec. IVA and analysing the framework's gauge freedom in Secs. IVD and VBD. We described various gauge choices in the 1PA waveform generation scheme. We then formalized the invariance of the resulting waveform under such choices, while emphasizing that, despite this invariance, the choices *can have differing implications on waveform accuracy*.

Finally, we have shown how to combine the results of Grant [70], Witzany et al. [72] and Isoyama et al. [88] into an evolution formula for the spin-corrected Carter constant in terms of Teukolsky amplitudes. As explained in Sec. VI, this formula, alongside the energy and angular momentum flux balance formulae of Ref. [53], can be readily incorporated into our multiscale framework,

where it enables calculations of the secondary spin's complete contribution to 1PA waveforms while avoiding any evaluation of local self-forces and regular fields at the particle. Computing the secondary spin's contribution to the 1PA forcing function $F_i^{(1-\chi_2)}$ across the parameter space hence only requires the Teukolsky amplitudes for a spinning secondary as input (and only the contribution from the nonprecessing component of the spin). These amplitudes are now available from Refs. [65, 71].

In Appendix D, we have also presented a simplified local expression for the averaged rate of change of the Carter constant in terms of the effective metric at the particle, which may be used as a consistency check of the Teukolsky flux formula in future work.

B. Path to a complete 1PA waveform model

The first 1PA waveform model was limited to nonspinning, quasicircular binaries [5]. The most general 1PA waveform models at the time of writing are presented in a series of upcoming companion papers that make immediate use of the multiscale framework we presented in this paper ('Paper I'). In Ref. [90], hereafter 'Paper II', the 1PA waveform model of Ref. [5] is extended to include a generic precessing secondary spin and a slowly spinning primary whose spin axis has a small misalignment with the orbital angular momentum. In Ref. [91], hereafter 'Paper III', the model of Ref. [5] is extended to allow for a rapidly spinning primary whose spin is (anti-)aligned with the orbital angular momentum; this is achieved by hybridizing existing SF information (0PA fluxes and 1SF conservative 1PA effects) with known post-Newtonian results for the 2SF energy fluxes (and terms beyond 1PA order). The results of Papers II and III are combined and extended in Ref. [92] to describe quasi-circular binaries with generic (anti-)aligned spins on both bodies. Reference [92] also demonstrates how the parameter-space coverage of 1PA waveforms can be pushed to high spins and comparable masses by leveraging the resummations in Papers II and III.

These models remain limited in their coverage of spin precession (and eccentricity). There are two major computational hurdles on the path to fully generic 1PA models: (i) the significant effort required in computing the dissipative effects of $h_{\alpha\beta}^{(2)}$ and (ii) the large intrinsic parameter space over which SF calculations must be performed. We stress that these computations are performed offline, and (when optimized) the online waveform generation is computationally inexpensive and rapid. For generic 1PA waveforms relying solely on strong-field SF results, the computational frameworks necessary to compute $h_{\alpha\beta}^{(2)}$ [75, 105, 119, 120, 136–138] are still being extended to include eccentricity and a rapidly spinning primary [121, 122, 139–148]. More study is required to assess whether post-Newtonian results for 2SF dissipative effects (as used in Paper III) will be sufficiently accurate

across the full parameter space of realistic asymmetric binaries.

At 0PA order, modeling efforts have been aided by a substantial simplification: the forcing functions $F_i^{(0)}$ in the evolution equation (3) can be written in terms of asymptotic Teukolsky mode amplitudes at infinity and the primary black hole's horizon [95]. This means all necessary inputs for a 0PA waveform model ($F_i^{(0)}$) and the waveform amplitudes $h_k^{(1)}$) can be determined directly from the solution to the Teukolsky equation with a point-mass source, avoiding the need to reconstruct the complete first-order metric perturbation or to calculate the complete self-force it exerts [2]. As we have highlighted, the same shortcut is now possible for the secondary spin's contribution to 1PA waveform models. We expect the same to also hold for the χ_2 -independent sector of the 1PA dynamics, but such a result has not yet been established. As 1PA waveform models are extended to include spin precession and eccentricity, transient orbital resonances between Ω_r and Ω_θ will also require careful treatment [76, 149]. There has been considerable progress to that end [4, 150–152].

Finally, most of the development of SF waveforms to date has focused on the inspiral stage of the waveform. Current multiscale methods employed in SF models break down as the binary transitions to the merger-ringdown regime. While these stages likely contribute very little to the total signal-to-noise ratio of EMRI signals detected by LISA, for example, they become increasingly important in the intermediate-mass-ratio regime and for asymmetric-mass sources observable by ground-based detectors. Thus, including the merger and ringdown in SF waveform models will be a critical step toward their direct use in these cases. Fortunately, there is significant progress toward this goal [125–127, 153–156], building on the pioneering work of Refs. [157, 158].

Finally we remark that, even before accurate 1PA SF models are extended across the entire parameter space (and into the merger-ringdown regime), their intermediate results may be used in calibrating effective models with more extensive coverage. Such calibration has a long history [2], with Refs. [24, 25, 159] standing as recent examples. The results presented in Refs. [90–92] could be used to further calibrate these models.

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Appendix A: Solution to the nutation equations

The secondary spin vector's nutation is governed by Eqs. (66) and (67), which we restate here for convenience:

$$\frac{d\delta\vartheta_c}{d\tau} - \omega_{12}\delta\vartheta_s = -\delta\omega_{23}, \quad (\text{A1})$$

$$\frac{d\delta\vartheta_s}{d\tau} + \omega_{12}\delta\vartheta_c = \delta\omega_{13}. \quad (\text{A2})$$

In this appendix we describe how to solve these equations.

We first decouple the first-order coupled system in favor of two independent second-order differential equations:

$$\frac{d}{d\tau} \left(\frac{1}{\omega_{12}} \frac{d\delta\vartheta_c}{d\tau} \right) + \omega_{12}\delta\vartheta_c = -\frac{d}{d\tau} \left(\frac{\delta\omega_{23}}{\omega_{12}} \right) + \delta\omega_{13}, \quad (\text{A3})$$

$$\frac{d}{d\tau} \left(\frac{1}{\omega_{12}} \frac{d\delta\vartheta_s}{d\tau} \right) + \omega_{12}\delta\vartheta_s = \frac{d}{d\tau} \left(\frac{\delta\omega_{13}}{\omega_{12}} \right) + \delta\omega_{23}, \quad (\text{A4})$$

or switching time variable,

$$\frac{d}{dt} \left(\frac{u^t}{\omega_{12}} \frac{d\delta\vartheta_c}{dt} \right) + \frac{\omega_{12}}{u^t} \delta\vartheta_c = -\frac{d}{dt} \left(\frac{\delta\omega_{23}}{\omega_{12}} \right) + \frac{\delta\omega_{13}}{u^t}, \quad (\text{A5})$$

$$\frac{d}{dt} \left(\frac{u^t}{\omega_{12}} \frac{d\delta\vartheta_s}{dt} \right) + \frac{\omega_{12}}{u^t} \delta\vartheta_s = \frac{d}{dt} \left(\frac{\delta\omega_{13}}{\omega_{12}} \right) + \frac{\delta\omega_{23}}{u^t}. \quad (\text{A6})$$

Recall that $\frac{\omega_{12}}{u^t} = -\omega_{s(0)}$; see Eq. (93).

Next we note that in the multiscale analysis, ω_{AB} , u^t , and $\delta\omega_{AB}$ are all functions of the form $\omega_{12} = \omega_{12}^{(0)}(\dot{\psi}^i, \dot{\pi}_i) + \mathcal{O}(\varepsilon, s)$, $\delta\omega_{AB} = \delta\omega_{AB}(\dot{\psi}^i, \dot{\varpi}_i) + \mathcal{O}(\varepsilon^2, s)$, etc. We therefore adopt the ansatz $\delta\vartheta_c = \delta\vartheta_{c(0)}(\dot{\psi}^i, \dot{\varpi}_I) + \mathcal{O}(\varepsilon)$ and analogously for $\delta\vartheta_s$. At leading order, the equations (A5) and (A6) depend only on time derivatives of the phases $\dot{\psi}^i$ since time derivatives of $\dot{\varpi}_i$ are subleading order. The first equation hence becomes

$$\Omega_{(0)}^i \Omega_{(0)}^j \frac{\partial}{\partial \dot{\psi}^i} \left(\frac{1}{\omega_{s(0)}} \frac{\partial \delta\vartheta_{c(0)}}{\partial \dot{\psi}^j} \right) + \omega_{s(0)} \delta\vartheta_{c(0)} = \Omega_{(0)}^i \frac{\partial}{\partial \dot{\psi}^i} \left(\frac{\delta\omega_{23}^{(0)}}{\omega_{12}^{(0)}} \right) - \frac{\delta\omega_{13}^{(0)}}{u_{(0)}^t}, \quad (\text{A7})$$

with a similar expression for the second equation. By substituting the Fourier series anzatz

$$\delta\vartheta^{(0)} = \sum_{\mathbf{k}} \delta\vartheta_{\mathbf{k}}^{(0)}(\dot{\pi}_i) e^{-i\dot{\psi}_{\mathbf{k}}}, \quad (\text{A8})$$

for both $\delta\vartheta_{c(0)}$ and $\delta\vartheta_{s(0)}$ along with the Fourier series for the four-velocity, $\omega_{12}^{(0)}$ and $\delta\omega_{AB}^{(0)}$, one obtains an algebraic equation with coupled modes.

Appendix B: Conservative and dissipative forces

The analysis of orbital motion in Sec. IV A makes use of the fact that certain forces are purely conservative and therefore cannot contribute to the 0PA forcing function (100b). Here we explain this fact by recalling some basic features of the problem.

We first define dissipative and conservative forces according to their behavior under time reversal $(t, \psi^i, \tilde{\psi}_s) \mapsto (-t, -\psi^i, -\tilde{\psi}_s)$:

$$a_{\text{diss}}^{\alpha}(\psi^i, \pi_i, \tilde{\psi}_s) \equiv \frac{1}{2} a^{\alpha}(\psi^i, \pi_i, \tilde{\psi}_s) - \frac{1}{2} \varepsilon^{\alpha} a^{\alpha}(-\psi^i, \pi_i, -\tilde{\psi}_s), \quad (\text{B1})$$

$$a_{\text{cons}}^{\alpha}(\psi^i, \pi_i, \tilde{\psi}_s) \equiv \frac{1}{2} a^{\alpha}(\psi^i, \pi_i, \tilde{\psi}_s) + \frac{1}{2} \varepsilon^{\alpha} a^{\alpha}(-\psi^i, \pi_i, -\tilde{\psi}_s), \quad (\text{B2})$$

where $\varepsilon^{\alpha} = (-1, 1, 1, -1)$ in Boyer-Lindquist coordinates and there is no summation over α . Under this definition, the first-order MPD force is purely conservative, as is the linear force due to the mass and spin perturbations δm_1 and $\delta\chi_1$. This fact can be straightforwardly verified by explicit computation.

Next, we note the key symmetry properties of the matrices $A^i_j(\psi^i, \pi_i)$ and $B_{ij}(\psi^i, \pi_i)$ appearing in Eqs. (89) and (90):

$$A^i_y(-\psi^i, \pi_i) = A^i_y(\psi^i, \pi_i), \quad (\text{B3})$$

$$A^i_{\phi}(-\psi^i, \pi_i) = -A^i_{\phi}(\psi^i, \pi_i), \quad (\text{B4})$$

$$B_{iy}(-\psi^i, \pi_i) = -B_{iy}(\psi^i, \pi_i), \quad (\text{B5})$$

$$B_{i\phi}(-\psi^i, \pi_i) = B_{i\phi}(\psi^i, \pi_i), \quad (\text{B6})$$

where y denotes either r or θ . These properties are easily verified by inspection of Eqs. (289)–(295) in Ref. [76].

Finally, we note that $\langle f \rangle = 0$ for any odd function of $(\psi^i, \tilde{\psi}_s)$. This is trivial if we average over $(\psi^i, \tilde{\psi}_s)$ but slightly less obvious for our average over $(\dot{\psi}^i, \dot{\tilde{\psi}}_s)$. To see that it holds for the average over $(\dot{\psi}^i, \dot{\tilde{\psi}}_s)$, consider that the ringed variables are odd functions of the unringed ones (and vice versa) if we choose them to have the same origin, implying the $\dot{\psi}$ -average vanishes for an odd function of the unringed phases. Since the functions of interest here are 2π -periodic in both sets of phases, it follows that the $\dot{\psi}$ -average also vanishes even if the ringed and unringed phases do not have a common origin.

Combining all the above properties, we find

$$\langle A^i_j a_{\text{diss}}^j \rangle = 0, \quad (\text{B7})$$

$$\langle B_{ij} a_{\text{cons}}^j \rangle = 0. \quad (\text{B8})$$

In other words, under an average, A^i_j projects out dissipative pieces of the force, and B_{ij} projects out conservative contributions. The promised conclusion follows: the first-order MPD force and the linear forces due to δm_1 and $\delta\chi_1$ cannot contribute to the 0PA forcing function (100b).

Appendix C: Osculating action angles

In the body of the paper we formulate the orbital motion in terms of quasi-Keplerian phase-space coordinates (ψ^i, π_i) . It is also possible to begin with geodesic action angles $\dot{\psi}_{(0)}^i$ in place of the quasi-Keplerian phases ψ^i . Here we outline that approach and also show how $\dot{\psi}_{(0)}^i$ serves as a helper function for the approach in the body of the paper.

The geodesic action angles $\dot{\psi}_{(0)}^i$ satisfy Eq. (94) in the case of a geodesic motion. They can be defined from a type-2 canonical transformation, but here it will be more useful to define them (for generic, accelerated or geodesic orbits) as the solution to the leading-order part of the transformation (96a):

$$\psi^i = \dot{\psi}_{(0)}^i + \Delta\psi^i(\dot{\psi}_{(0)}^j, \pi_i). \quad (\text{C1})$$

As neatly summarized in Ref. [152], the solution $\dot{\psi}_{(0)}^i(\psi^j, \pi_j)$ can be written analytically as

$$\dot{\psi}_{(0)}^i(\psi^j, \pi_j) = q^i(\psi^j, \pi_j) + \Omega_{(0)}^i(\pi_j)\delta t(\psi^j, \pi_j) \quad (\text{C2})$$

with

$$\delta t(\psi^j, \pi_j) = t^r(q^r(\psi^r, \pi_i), \pi_i) + t^\theta(q^\theta(\psi^\theta, \pi_i), \pi_i). \quad (\text{C3})$$

The functions $q^r(\psi^r, \pi_i)$ and $q^\theta(\psi^\theta, \pi_i)$ are given in Eqs. (20b) and (21b) of Ref. [152]. The functions $t_r(q_r, \pi_i)$ and $t_\theta(q_\theta, \pi_i)$ are given in Eqs. (28) and (39) of Ref. [85], with the quantities ‘ $\lambda^{(r)}$ ’ and ‘ $\lambda^{(\theta)}$ ’, therein replaced by $q^r/\Upsilon^r(\pi_i)$ and $q^\theta/\Upsilon^\theta(\pi_i)$, respectively. Here Υ^α are the ‘Mino time’ orbital frequencies [76, 85].

If we apply the osculating-geodesics approach of Sec. IV A, then in place of Eq. (87) we arrive at equations of the form

$$\frac{d\dot{\psi}_{(0)}^i}{dt} = \Omega_{(0)}^i(\pi_j) + \varepsilon\delta\Omega_{(0)}^i(\dot{\psi}_{(0)}^j, \varpi_j, \tilde{\psi}_s) + \mathcal{O}(\varepsilon^2), \quad (\text{C4})$$

$$\frac{d\pi_i}{dt} = \varepsilon g_i^{(0)}(\dot{\psi}_{(0)}^j, \varpi_j, \tilde{\psi}_s) \quad (\text{C5})$$

$$+ \varepsilon^2 g_i^{(1)}(\dot{\psi}_{(0)}^j, \varpi_j, \tilde{\psi}_s) + \mathcal{O}(\varepsilon^3). \quad (\text{C6})$$

The functions $g_i^{(n)}$ are trivially related to the functions $f_i^{(n)}$ in Eq. (88) by

$$g_i^{(n)}(\dot{\psi}_{(0)}^j) = f_i^{(n)}(\dot{\psi}_{(0)}^j + \Delta\psi^j(\dot{\psi}_{(0)}^k, \pi_k)), \quad (\text{C7})$$

suppressing the arguments ϖ_j and $\tilde{\psi}_s$. The function $\delta\Omega_{(0)}^i$ can be related to those in Eqs. (87) and (88) by differentiating $\dot{\psi}_{(0)}^i(\psi^j, \pi_j)$ with respect to t , substituting Eqs. (87) and (88), and comparing to Eq. (C4). The result is

$$\delta\Omega_{(0)}^i = \frac{\partial\dot{\psi}_{(0)}^i}{\partial\psi^j}\omega_{(1)}^j + \frac{\partial\dot{\psi}_{(0)}^i}{\partial\pi_j}f_j^{(0)} \quad (\text{C8})$$

as well as the geodesic expression

$$\Omega_{(0)}^i(\pi_j) = \frac{\partial\dot{\psi}_{(0)}^i}{\partial\psi^j}\omega_{(0)}^j(\psi^k, \pi_k). \quad (\text{C9})$$

We can transform to the variables $(\dot{\psi}^i, \varpi_I, \dot{\psi}_s)$ used in the multiscale expansion using a near-identity averaging transformation,

$$\dot{\psi}_{(0)}^i = \dot{\psi}^i + \varepsilon\delta\dot{\psi}_{(0)}^i(\dot{\psi}^j, \varpi_J, \dot{\psi}_s) + \mathcal{O}(\varepsilon^2). \quad (\text{C10})$$

Following the same steps as in Sec. IV A, one finds the forcing functions $F_i^{(n)}$, frequency correction $\Omega_{(1)}^i$, and the quantities $\delta\pi_i$ and $\delta\dot{\psi}_{(0)}^i$ in the transformations (C10) and (96b). This process is substantially simplified by the fact that the leading term in Eq. (C4) is non-oscillatory, and hence no $\mathcal{O}(\varepsilon^0)$ oscillatory term is needed in the transformation (C10).

A disadvantage of this approach is the complicated relationship between $\dot{\psi}_{(0)}^i$ and the Boyer-Lindquist trajectory. This makes the expansion of the stress-energy tensor in Sec. IV B more complicated, for example. However, its advantages can outweigh this disadvantage.

As alluded to above, this approach also provides useful helper functions for the approach taken in the body of the paper. In particular, we can use it to find the function $\delta\psi^i$ in the transformation (96a) without directly solving Eq. (104). To achieve this, we substitute the expansions (96) into $\dot{\psi}_{(0)}^i(\psi^j, \pi_j)$, yielding

$$\begin{aligned} \dot{\psi}_{(0)}^i(\psi^j, \pi_j) &= \dot{\psi}_{(0)}^i(\psi_{(0)}^j, \dot{\pi}_j) + \varepsilon \left(\delta\psi^j \frac{\partial\dot{\psi}_{(0)}^i}{\partial\psi_{(0)}^j} + \delta\pi_j \frac{\partial\dot{\psi}_{(0)}^i}{\partial\dot{\pi}_j} \right) \\ &\quad + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (\text{C11})$$

Identifying this with Eq. (C10) and rearranging, we find $\dot{\psi}_{(0)}^i(\psi_{(0)}^j, \dot{\pi}_j) = \dot{\psi}^i$ and

$$\delta\psi^j \frac{\partial\dot{\psi}_{(0)}^i}{\partial\psi_{(0)}^j} = \delta\dot{\psi}_{(0)}^i - \delta\pi_j \frac{\partial\dot{\psi}_{(0)}^i}{\partial\dot{\pi}_j}. \quad (\text{C12})$$

Since the matrix $\partial\dot{\psi}_{(0)}^i/\partial\psi_{(0)}^j$ is invertible, this determines $\delta\psi^j$ in terms of $\delta\dot{\psi}_{(0)}^i$.

$\delta\dot{\psi}_{(0)}^i$, in turn, is determined (up to an arbitrary non-oscillatory part) by the analogue of Eq. (104). That analogue is the $\mathcal{O}(\varepsilon)$ term in Eq. (C4) after substituting the transformations (C10) and (96b):

$$\Omega_{(1)}^i + \Omega_{(0)}^j \frac{\partial\delta\dot{\psi}_{(0)}^i}{\partial\psi^j} + \Omega_{s(0)} \frac{\partial\delta\dot{\psi}_{(0)}^i}{\partial\tilde{\psi}_s} = \delta\pi_j \frac{\partial\Omega_{(0)}^i}{\partial\dot{\pi}_j} + \delta\Omega_{(0)}^i. \quad (\text{C13})$$

Averaging this equation yields an alternative expression for $\Omega_{(1)}^i$,

$$\Omega_{(1)}^i = \langle\delta\pi_j\rangle \frac{\partial\Omega_{(0)}^i}{\partial\dot{\pi}_j} + \langle\delta\Omega_{(0)}^i\rangle. \quad (\text{C14})$$

Substituting this formula for $\Omega_{(1)}^i$ back into Eq. (C13), we obtain

$$\Omega_{(0)}^j \frac{\partial \delta\dot{\psi}_{(0)}^i}{\partial \dot{\psi}^j} + \Omega_{s(0)} \frac{\partial \delta\dot{\psi}_{(0)}^i}{\partial \dot{\psi}_s} = \delta\pi_j^{\text{osc}} \frac{\partial \Omega_{(0)}^i}{\partial \dot{\pi}_j} + \delta\Omega_{(0)\text{osc}}^i, \quad (\text{C15})$$

where the oscillatory part of a function f is defined as

$$f_{\text{osc}} \equiv f - \langle f \rangle. \quad (\text{C16})$$

The simple equation (C15) contrasts with the complicated equation (104) for $\delta\psi^i$. Equation (C15) can be immediately solved for $\delta\dot{\psi}_{(0)}^i$ by, for example, expanding $\delta\dot{\psi}_{(0)}^i$, $\delta\pi_j^{\text{osc}}$, and $\delta\Omega_{(0)\text{osc}}^i$ in Fourier series. Substituting the solution for $\delta\dot{\psi}_{(0)}^i$ into Eq. (C12) then determines $\delta\psi^i$.

Appendix D: Local expressions for 1PA forcing functions

Here we derive local expressions for $\langle dP_i/dt \rangle$ in terms of $S^{\alpha\beta}$ and $h_{\alpha\beta}^{\text{R}(1)}$. One important outcome is that the correction term δS^α in Eq. (73) does not need to be explicitly computed in order to calculate these rates of change. Hence, one need not calculate the nutation angles at 1PA order even if one uses local evolution equations.

Our overarching approach is to express $\langle dP_i/dt \rangle$ in terms of quantities that would manifestly vanish if the Killing vectors and tensor of Kerr were also Killing in the effective metric.

1. Evolution of energy and angular momentum

Reference [53] derived convenient local expressions for the averaged rates of change of E and L_z . However, that derivation involved errors that compensated earlier errors in Ref. [53]'s expressions for the local forces and torques; see Ref. [44]. Here for completeness we provide a corrected and streamlined derivation of Ref. [53]'s formula.

For a generic vector \hat{v}^α (whose indices are raised and lowered with the effective metric), we introduce the quantities

$$\hat{\Sigma} = \hat{g}_{\alpha\beta} \hat{v}^\alpha \hat{u}^\beta, \quad (\text{D1a})$$

$$\delta\hat{\Sigma} = \hat{S}^{\beta\gamma} \hat{\nabla}_\beta \hat{v}_\gamma, \quad (\text{D1b})$$

with derivatives

$$\frac{d\hat{\Sigma}}{d\hat{\tau}} = \frac{1}{2} \mathcal{L}_{\hat{v}} \hat{g}_{\alpha\beta} \hat{u}^\alpha \hat{u}^\beta + v_\alpha \hat{a}^\alpha + \mathcal{O}(s^2), \quad (\text{D2a})$$

$$\begin{aligned} \frac{d\delta\hat{\Sigma}}{d\hat{\tau}} &= \hat{S}^{\beta\gamma} \hat{u}^\lambda \hat{\nabla}_\lambda \hat{\nabla}_\beta \hat{v}_\gamma + \mathcal{O}(s^2) \\ &= -2\hat{v}_\alpha \hat{u}^\alpha + \hat{S}^{\beta\gamma} \hat{u}^\lambda \left(\hat{\nabla}_\beta \hat{\nabla}_\lambda \hat{v}_\gamma - 2\hat{\nabla}_{[\gamma} \hat{\nabla}_{\lambda]} \hat{v}_\beta \right) \\ &\quad + \mathcal{O}(s^2). \end{aligned} \quad (\text{D2b})$$

In going from the second line to the last line, we have used the Riemann symmetries inside \hat{a}^α and the definition of the Riemann tensor. Next we split up the symmetric and anti-symmetric terms in $\gamma \leftrightarrow \lambda$, using

$$\hat{\nabla}_\lambda \hat{v}_\gamma = \hat{\nabla}_{[\lambda} \hat{v}_{\gamma]} + \hat{\nabla}_{(\lambda} \hat{v}_{\gamma)} = \hat{\nabla}_{[\lambda} \hat{v}_{\gamma]} + \frac{1}{2} \mathcal{L}_{\hat{v}} \hat{g}_{\lambda\gamma}, \quad (\text{D3})$$

and the antisymmetry of the spin tensor. We find Eq. (D2b) reduces to

$$\frac{d\delta\hat{\Sigma}}{d\hat{\tau}} = \hat{S}^{\beta\gamma} \hat{u}^\lambda \hat{\nabla}_\beta \mathcal{L}_{\hat{v}} \hat{g}_{\lambda\gamma} - 2\hat{v}_\alpha \hat{u}^\alpha + \mathcal{O}(s^2), \quad (\text{D4})$$

where we have eliminated terms proportional to $\hat{\nabla}_{[\lambda} \hat{\nabla}_{\gamma]} v_\beta$ by virtue of the first Bianchi identity. Putting these two terms together we have

$$\begin{aligned} \frac{d\hat{\Sigma}}{d\hat{\tau}} + \frac{m_2}{2} \frac{d\delta\hat{\Sigma}}{d\hat{\tau}} &= \frac{1}{2} \mathcal{L}_{\hat{v}} \hat{g}_{\alpha\beta} \hat{u}^\alpha \hat{u}^\beta \\ &\quad + \hat{S}^{\beta\gamma} \hat{u}^\lambda \hat{\nabla}_\beta \mathcal{L}_{\hat{v}} \hat{g}_{\lambda\gamma} + \mathcal{O}(s^2). \end{aligned} \quad (\text{D5})$$

Now consider the quantity

$$\hat{\Xi} = \hat{g}_{\alpha\beta} \xi^\alpha \hat{u}^\beta + \frac{m_2}{2} \hat{S}^{\gamma\delta} \hat{g}_{\delta\beta} \hat{\nabla}_\gamma \xi^\beta, \quad (\text{D6})$$

which corresponds to $\hat{\Sigma} + \frac{m_2}{2} \delta\hat{\Sigma}$ with $\hat{v}^\alpha = \xi^\alpha$. If ξ^α satisfied Killing's equation in the effective metric, then this quantity would be conserved [93], which we have made obvious in Eq. (D5). We can still use the fact that $\mathcal{L}_\xi g_{\alpha\beta} = 0$, which immediately tells us, with Eq. (D5), that

$$\frac{d\hat{\Xi}}{dt} = \frac{1}{2} \dot{z}^\alpha \hat{u}^\beta \mathcal{L}_\xi h_{\alpha\beta}^{\text{R}} - \frac{m_2}{2} \hat{S}^{\gamma\delta} \dot{z}^\lambda \hat{\nabla}_\delta \mathcal{L}_\xi h_{\gamma\lambda}^{\text{R}} + \mathcal{O}(s^2), \quad (\text{D7})$$

where we used $\frac{d\hat{\tau}}{dt} \hat{u}^\alpha = \dot{z}^\alpha$. Consider now the test-body quantity, Ξ , which we can relate to the hatted quantity via $\hat{\Xi} = \Xi + \delta\Xi$. We have

$$\left\langle \frac{d\Xi}{dt} \right\rangle = \left\langle \frac{d\hat{\Xi}}{dt} \right\rangle - \left\langle \frac{d\delta\Xi}{dt} \right\rangle. \quad (\text{D8})$$

Using $d/dt = \Omega_{(0)}^i(\dot{\pi}_j) \partial/\partial \dot{\psi}^i + \mathcal{O}(\varepsilon)$, we see the latter term is too high order to consider:

$$\left\langle \frac{d\delta\Xi}{dt} \right\rangle = \mathcal{O}(\varepsilon^3). \quad (\text{D9})$$

Therefore the average rate of change of the test-body conserved quantity Ξ is

$$\begin{aligned} \left\langle \frac{d\Xi}{dt} \right\rangle &= \frac{1}{2} \left\langle \dot{z}^\alpha \hat{u}^\beta \mathcal{L}_\xi h_{\alpha\beta}^{\text{R}} - m_2 S^{\alpha\beta} \dot{z}^\gamma \hat{\nabla}_\beta \mathcal{L}_\xi h_{\alpha\gamma}^{\text{R}} \right\rangle \\ &\quad + \mathcal{O}(\varepsilon^3, s^2). \end{aligned} \quad (\text{D10})$$

If we neglect terms quadratic in $h_{\alpha\beta}^{\text{R}}$ and introduce a proper-time average [77, 88]

$$\langle \cdot \rangle_\tau \equiv \frac{\langle \frac{d\tau}{dt} \cdot \rangle}{\langle \frac{d\tau}{dt} \rangle}, \quad (\text{D11})$$

then this result takes the more symmetrical form

$$\left\langle \frac{d\Xi}{d\tau} \right\rangle_\tau = \frac{1}{2} \left\langle u^\alpha u^\beta \mathcal{L}_\xi h_{\alpha\beta}^R - m_2 S^{\alpha\beta} u^\gamma \nabla_\beta \mathcal{L}_\xi h_{\alpha\gamma}^R \right\rangle_\tau + \mathcal{O}(\varepsilon^2, s^2). \quad (\text{D12})$$

Reference [53] showed that the above local evolution satisfies a balance law with the asymptotic flux of energy (angular momentum), taking $\xi^\alpha = -t^\alpha(\phi^\alpha)$ and using the approach of Gal'tsov [163] and Mino [82].

As foreshadowed at the beginning of this appendix, the result for $\langle \frac{d\Xi}{dt} \rangle$ does not depend on the spin correction δS^α . The underlying reason for this could perhaps be lost in our streamlined derivation, but we can explain it with a more pedestrian approach. If we consider

$$\Xi = g_{\alpha\beta} \xi^\alpha u^\beta + \frac{m_2}{2} S^{\gamma\delta} g_{\delta\beta} \nabla_\gamma \xi^\beta, \quad (\text{D13})$$

then we first comment that replacing $S^{\gamma\delta}$ with $\hat{S}^{\gamma\delta}$ only changes Ξ at order ε^2 , hence changing $\langle dP_i/dt \rangle$ at order ε^3 , meaning 2PA. The direct contribution of δS^α is therefore not relevant to the averaged rate of change at 1PA order. Next consider

$$\frac{d\Xi}{d\tau} = \xi_\beta a^\beta + \frac{m_2}{2} u^\alpha \nabla_\alpha (S^{\gamma\delta} \nabla_\gamma \xi_\delta). \quad (\text{D14})$$

δS^α contributes to the first term through the acceleration (144). Appealing to the identity $\xi^\mu R_{\mu\alpha\beta\gamma} = \nabla_\alpha \nabla_\beta \xi_\gamma$ for a Killing vector, we can write that contribution to Eq. (D14) as

$$\xi_\beta a^\beta_{(\delta S)} = -\frac{m_2}{2} \xi^\mu R_{\mu\alpha\beta\gamma} u^\alpha \delta S^{\beta\gamma} \quad (\text{D15a})$$

$$= -\frac{d}{d\tau} \left(\frac{m_2}{2} \nabla_\beta \xi_\gamma \delta S^{\beta\gamma} \right) + \frac{m_2}{2} \nabla_\beta \xi_\gamma \frac{D\delta S^{\beta\gamma}}{d\tau}. \quad (\text{D15b})$$

The first term can be neglected under averaging because it is a total derivative of an order- ε^2 quantity. The second term does contribute to $\langle d\Xi/dt \rangle$, but only through the derivative of $\delta S^{\alpha\beta}$ rather than through $\delta S^{\alpha\beta}$ on its own. That derivative can be written directly in terms of the first-order regular field using

$$\frac{D\delta S^{\alpha\beta}}{d\tau} = \frac{D\hat{S}^{\alpha\beta}}{d\tau} - \frac{DS^{\alpha\beta}}{d\tau}, \quad (\text{D16})$$

with Eqs. (21b), (75a) (after precession averaging for simplicity), and (41). Hence, although $\delta S^{\alpha\beta}$ does contribute to the 1PA orbital evolution, it and the nutation angles it involves do not need to be explicitly computed.

2. Evolution of the spin-corrected Carter constant

Next we consider the quantity

$$\hat{K} = K_{\alpha\beta} \hat{u}^\alpha \hat{u}^\beta + m_2 L_{\alpha\beta\gamma} \hat{S}^{\alpha\beta} \hat{u}^\gamma, \quad (\text{D17})$$

in which we have defined

$$L_{\alpha\beta\gamma} \equiv -2g^{\delta\lambda} (Y_{\lambda\beta} \nabla_\delta Y_{\gamma\alpha} - Y_{\gamma\lambda} \nabla_\delta Y_{\alpha\beta}). \quad (\text{D18})$$

The quantity \hat{K} reduces to the spin-corrected Carter constant in the test-body limit. Its rate of change is

$$\frac{d\hat{K}}{d\hat{\tau}} = m_2 \hat{V}_{\alpha\beta\gamma\delta} \hat{S}^{\alpha\beta} \hat{u}^\gamma \hat{u}^\delta + \hat{u}^\alpha \hat{u}^\beta \hat{u}^\gamma \hat{U}_{\alpha\beta\gamma} + \mathcal{O}(s^2), \quad (\text{D19})$$

where we have defined

$$\hat{U}_{\alpha\beta\gamma} \equiv \hat{\nabla}_\gamma K_{\alpha\beta}, \quad (\text{D20a})$$

$$\hat{V}_{\alpha\beta\gamma\delta} \equiv \hat{\nabla}_\delta L_{\alpha\beta\gamma} - K_{\gamma\rho} \hat{R}^\rho_{\delta\alpha\beta}. \quad (\text{D20b})$$

Using the fact that $\hat{S}^{\alpha\beta} = -\hat{\epsilon}^{\alpha\beta\mu\nu} \hat{S}_\mu \hat{u}_\nu$, we can write this as

$$\begin{aligned} \frac{d\hat{K}}{d\hat{\tau}} &= -2m_2 {}^* \hat{V}_{\alpha(\beta\gamma\delta)} \hat{S}^\alpha \hat{u}^\beta \hat{u}^\gamma \hat{u}^\delta \\ &\quad + \hat{u}^\alpha \hat{u}^\beta \hat{u}^\gamma \hat{U}_{(\alpha\beta\gamma)} + \mathcal{O}(s^2), \end{aligned} \quad (\text{D21})$$

having used the notation for the left-handed dual

$${}^* \hat{V}_{\alpha\beta\gamma\delta} = \frac{1}{2} \hat{\epsilon}_{\alpha\beta}^{\mu\nu} \hat{V}_{\mu\nu\gamma\delta}. \quad (\text{D22})$$

Now, to utilize the background Killing and Killing-Yano symmetries, we expand out the dependence on the effective metric such that

$${}^* \hat{V}_{\alpha\beta\gamma\delta} = {}^* V_{\alpha\beta\gamma\delta} + \delta {}^* V_{\alpha\beta\gamma\delta}, \quad (\text{D23a})$$

$$\hat{U}_{\alpha\beta\gamma} = U_{\alpha\beta\gamma} + \delta U_{\alpha\beta\gamma}. \quad (\text{D23b})$$

The background quantities are obtained by removing all the hats in the definitions (D20) and (D22), and the δ terms are linear in the regular field $h_{\alpha\beta}^R$.

The background terms $U_{\alpha\beta\gamma}$ and $V_{\alpha\beta\gamma\delta}$ drop out of Eq. (D21) by virtue of the identities

$$U_{(\alpha\beta\gamma)} = 0, \quad (\text{D24a})$$

$${}^* V_{\alpha(\beta\gamma\delta)} = 0, \quad (\text{D24b})$$

the former of which is simply the statement that $K_{\alpha\beta}$ is a Killing tensor of the background metric, and the latter of which is proved in Ref. [57, 94]. In addition to using these identities, we also allow ourselves to discard terms quadratic in the regular field, since we are interested specifically in terms that are linear in spin.

Combining these simplifications, we reduce the evolution equation (D21) to

$$\frac{d\hat{K}}{d\tau} = V_1 + V_2 + \mathcal{O}(\varepsilon^2 s^0, \varepsilon^3, s^2), \quad (\text{D25})$$

having defined the ‘symmetry violation’ terms

$$V_1 \equiv u^\alpha u^\beta u^\gamma \delta U_{(\alpha\beta\gamma)}, \quad (\text{D26a})$$

$$V_2 \equiv -2m_2 \delta {}^* V_{\alpha(\beta\gamma\delta)} S^\alpha u^\beta u^\gamma u^\delta \quad (\text{D26b})$$

and used $\mathcal{O}(\varepsilon^2 s^0)$ to denote spin-independent $\mathcal{O}(\varepsilon^2)$ terms. At this stage we can see the evolution equation has no explicit dependence on δS^α because it would contribute $\mathcal{O}(\varepsilon^3)$ to V_2 . Like in the analysis of the previous section, the effect of the force generated by δS^α has been put in a form that does not require an explicit calculation of the nutation angles.

Finally, as in the previous subsection, the spin-corrected Carter constant will evolve according to

$$\begin{aligned} \left\langle \frac{dK}{d\tau} \right\rangle_\tau &= \left\langle \frac{d\hat{K}}{d\tau} \right\rangle_\tau + \mathcal{O}(\varepsilon^3) \\ &= \langle V_1 + V_2 \rangle_\tau + \mathcal{O}(\varepsilon^2 s^0, \varepsilon^3, s^2), \end{aligned} \quad (\text{D27})$$

since the difference between the two is a total derivative that can be neglected on average. We also stress that the average immediately eliminates the precessing component of the spin, such that S^α can be replaced with the parallel component (75a). In the remainder of the section, we derive more explicit expressions for the quantities V_1 and V_2 .

The first violation term is straightforward to compute explicitly in terms of the metric perturbation:

$$\begin{aligned} V_1 &= -2u^\alpha u^\beta u^\gamma \delta\Gamma^\delta_{\gamma\alpha} K_{\delta\beta} \\ &= -u^\alpha u^\gamma (2h_{\delta\alpha;\gamma}^R - h_{\alpha\gamma;\delta}^R) n^\delta + \mathcal{O}(\varepsilon^2 s^0), \end{aligned} \quad (\text{D28})$$

where we have defined $n^\alpha \equiv K^\alpha_{\beta\gamma} u^\beta$ and introduced the perturbation to the Christoffel symbol,

$$\delta\Gamma^\alpha_{\beta\gamma} \equiv \frac{1}{2}g^{\alpha\delta} (2h_{\delta(\beta;\gamma)}^R - h_{\beta\gamma;\delta}^R) + \mathcal{O}(\varepsilon^2 s^0). \quad (\text{D29})$$

To compute V_2 we use

$$\delta^* V_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta}^{\rho\sigma} \delta V_{\rho\sigma\gamma\delta} + \frac{1}{2}\delta\epsilon_{\alpha\beta}^{\rho\sigma} V_{\rho\sigma\gamma\delta} + \mathcal{O}(\varepsilon^2 s^0) \quad (\text{D30})$$

with

$$\begin{aligned} \delta V_{\alpha\beta\gamma\delta} &= -\delta\Gamma^\rho_{\delta\alpha} L_{\rho\beta\gamma} - \delta\Gamma^\rho_{\delta\beta} L_{\alpha\rho\gamma} \\ &\quad - \delta\Gamma^\rho_{\delta\gamma} L_{\alpha\beta\rho} - 2K_{\gamma\rho} \delta\Gamma^\rho_{\delta[\beta;\alpha]} + \mathcal{O}(\varepsilon^2 s^0), \end{aligned} \quad (\text{D31})$$

and

$$\delta\epsilon_{\alpha\beta}^{\rho\sigma} = \frac{1}{2}\epsilon_{\alpha\beta}^{\rho\sigma} g^{\mu\nu} h_{\mu\nu}^R + 2\epsilon_{\alpha\beta}^{\mu[\rho} h_{\mu\nu}^{R\sigma]} + \mathcal{O}(\varepsilon^2 s^0). \quad (\text{D32})$$

Substituting Eq. (D30) into Eq. (D26b), we obtain

$$\begin{aligned} V_2 &= -m_2 \left(4\epsilon_{\alpha\beta}^{\mu[\rho} h_{\mu\nu}^{R\sigma]} V_{\rho\sigma\gamma\delta} \right. \\ &\quad \left. + \epsilon_{\alpha\beta}^{\rho\sigma} \delta V_{\rho\sigma\gamma\delta} \right) S^\alpha u^\beta u^\gamma u^\delta + \mathcal{O}(\varepsilon^2 s^0), \end{aligned} \quad (\text{D33})$$

having eliminated a term with Eq. (D24b).

We can now extract the linear-in-spin 1PA terms in Eq. (D27). First,

$$\begin{aligned} V_1^{(1-\chi_2)} &= -u_{(0)}^\alpha u_{(0)}^\gamma \left(2\dot{h}_{\delta\alpha;\gamma}^R - \dot{h}_{\alpha\gamma;\delta}^R \right) n_{(0)}^\delta \\ &\quad + \left(\delta\psi_{(\chi_2)}^i \frac{\partial}{\partial\psi_{(0)}^i} + \delta\pi_i^{(\chi_2)} \frac{\partial}{\partial\dot{\pi}_i} \right) V_1^{(0)}, \end{aligned} \quad (\text{D34})$$

where $V_1^{(0)} = -u_{(0)}^\alpha u_{(0)}^\gamma \left(2\dot{h}_{\delta\alpha;\gamma}^R - \dot{h}_{\alpha\gamma;\delta}^R \right) n_{(0)}^\delta$ and we follow the notation of Sec. IV B. In V_2 , we simply replace $h_{\alpha\beta}^R$ and u^α with $\dot{h}_{\alpha\beta}^R$ and $u_{(0)}^\alpha$ in Eq. (D33).

In the case of a nonspinning secondary (such that $V_2 = 0$), and at leading order in ε , Sago et al. showed that Eq. (D27) can be rewritten as a practical flux-like formula in terms of Teukolsky mode amplitudes [95]. (See also Refs. [83, 88]. Note that the formula requires additional information from the local dynamics at/near orbital resonances, as shown in Ref. [88], for example.) Their derivation began from Mino's result that $\langle dK/d\tau \rangle_\tau$, at 0PA order, only depends on the self-force generated by the radiative part of the metric perturbation [82]. They then followed Gal'tsov's method [163] of reconstructing the radiative metric perturbation in terms of Teukolsky mode amplitudes. It might be possible to extend this analysis to Eq. (D27) in full. As explained in Sec. VI, Grant's result [70] bypasses the need for such an analysis, but it would provide a healthy consistency check.

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