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s-PROCESS STUDIES: EXACT EVALUATION OF AN EXPONENTIAL DISTRIBUTION OF EXPOSURES

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ABSTRACT

We show that the solution of the *s*-process chain can be easily evaluated exactly if the seed nuclei have been irradiated with an exponential distribution of exposures. For the same distribution of exposures, the shape of the $\sigma_k N_k$ curve past the last seed is independent of the distribution of seed nuclei. Evaluation of these results confirms the accuracy of the well-known CFHZ approximation.

Subject headings: nuclear reactions — nucleosynthesis

The set of differential equations of the *s*-process chain of neutron captures is

$$\frac{dN_k}{d\tau} = -\sigma_k N_k + \sigma_{k-1} N_{k-1}\,, \tag{1}$$

where σ_k are the (n, γ) cross-sections, N_k is the abundance of the kth nucleus (taken as unique) on the capture path, and $\tau = \int n_n V_{Th} dt$ is the neutron irradiation (Clayton *et al.* 1961; Clayton 1968). The exact solution for the related function $\psi_k(\tau) \equiv \sigma_k N_k(\tau)/N_1(0)$ subject to the initial condition that $N_k(0) = N_1(0)\delta_{1k}$ is also given by those authors as

$$\psi_k(\tau) = \sum_{i=1}^{k} \left(\prod_{j=1}^{k} \frac{\sigma_j}{\sigma_j - \sigma_i}\right)_{j \neq i} \exp(-\sigma_i \tau)\,. \tag{2}$$

The subscript $j \neq i$ means that the factor with $j = i$ is omitted from the product. In practice this exact solution is difficult to evaluate for large k due to numerical cancellations within the k terms. To circumvent this numerical difficulty, Clayton *et al.* introduced the so-called CFHZ approximation:

$$\psi_k^{\mathrm{CFHZ}} = \frac{\lambda_k^{m_k} \tau^{m_k - 1}}{\Gamma(m_k)} \exp(-\lambda_k \tau)\,, \tag{3}$$

where λ_k and m_k are defined by them in terms of the set of cross-sections σ_k. The size of the error of this approximation had never been successfully evaluated until Clayton and Newman (1974) recently evaluated it with a special chain in which each cross-section assumed one of three values. They showed equation (3) to be adequate for most astrophysical purposes. In this work we confirm that conclusion by a different approach involving a related result of even greater general interest; namely, that although formula (2) cannot in general be evaluated, a superposition of its values for an exponential distribution of neutron irradiations can easily be evaluated exactly.

Again following the terminology of Clayton *et al.*, we let $\rho(\tau)d\tau$ be the number of seed nuclei ($k = 1$) exposed to irradiation τ in the interval $d\tau$. The σN product of this superposition is then $\sigma_k N_k = \int \psi_k(\tau)\rho(\tau)d\tau$. For the special case of the exponential exposure distribution

$$\rho(\tau) = G \exp(-\tau/\tau_0)\,, \tag{4}$$

this integral can be evaluated from equation (2) as

$$\sigma_k N_k = \sum_{i=1}^{k} \left(\prod_{j=1}^{k} \frac{\sigma_j}{\sigma_j - \sigma_i}\right)_{j \neq i} \left[\frac{G}{\sigma_i + (1/\tau_0)}\right]. \tag{5}$$

Although the sum in equation (5) is again difficult to evaluate, its factorization yields the numerically much simpler form

$$\sigma_k N_k = G\left(1 + \frac{1}{\tau_0 \sigma_k}\right)^{-1} \left(1 + \frac{1}{\tau_0 \sigma_{k-1}}\right)^{-1} \cdots \left(1 + \frac{1}{\tau_0 \sigma_1}\right)^{-1}. \tag{6}$$

This somewhat surprising identity is most easily established by noting that the exponential distribution of exposures is formally equivalent to taking the Laplace transform of $\psi_k(\tau)$. Clayton *et al.* had derived this Laplace transform

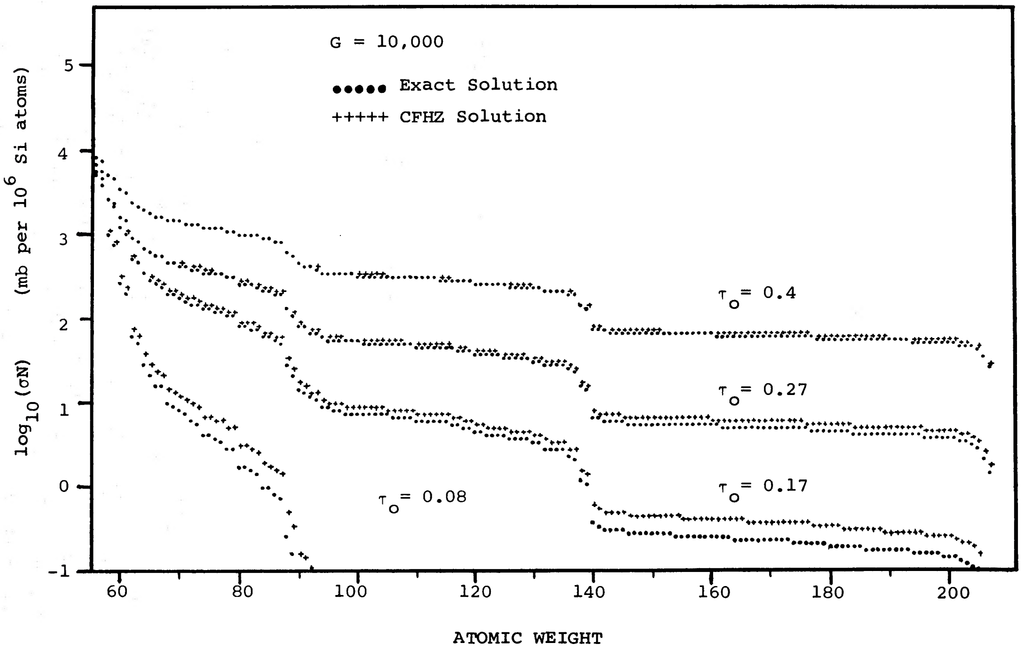

FIG. 1.—Comparison of the exact solution with the CFHZ approximation for an exponential distribution of exposures. The curve for $\tau_0 = 0.269$ millibarns^{-1} resembles the solar abundance distribution.

in the form (6) from the set of differential equations themselves. Therefore, equation (6) is an easily evaluated exact solution of the chain with arbitrary cross-sections for an exponential distribution of exposures. It is immediately evident, for example, that superposition (6) has no numerical difficulty even if two cross-sections are exactly equal to each other, in contrast to the difficulty with equations (2) and (5). Equation (6) was derived by Ulrich (1973) from his assumption of steady-state abundances in a mixing model characterized by repeated infinitesimal homogeneous shell flashes. Our result links his astrophysical model more directly to the basic mathematics of the *s*-process.

Seeger, Fowler, and Clayton (1965) showed that the exponential $\rho(\tau)$ gave a good characterization of the solar σN curve by using the CFHZ approximation, leading to

$$(\sigma_k N_k)^{\text{CFHZ}} = G\left(1 + \frac{1}{\tau_0 \lambda_k}\right)^{-m_k} . \tag{7}$$

Now that the exact result (6) has been discovered, the CFHZ approximation can be compared with it. We show this comparison in figure 1 for four different values of τ_0. The values of σ_k and of the parameters λ_k and m_k that are calculable from σ_k have been taken from the review by Allen, Gibbons, and Macklin (1971). The CFHZ approximation is seen to be quite good except for the smallest values of τ_0. These results are shown as percentage error ϵ in figure 2, defined as

$$\epsilon = 100 \frac{(\sigma N)^{\text{CFHZ}} - (\sigma N)^{\text{exact}}}{(\sigma N)^{\text{exact}}} . \tag{8}$$

These results confirm again the usefulness of the CFHZ approximation. For τ_0 near the value characterizing the solar distribution (0.3–0.4), the CFHZ superposition is an overestimate of only a few percent for $A < 140$ and of about 10 percent for $A > 140$. Because the percentage error changes sizably in passing through $A \approx 140$, it is clear that the exact superposition is preferable when evaluating the shape of the drop in σN near $A = 140$.

We have also noted the interesting result that, because the CFHZ approximation matches the first three moments of its Laplace transform to that of the exact solution for $\psi_k(\tau)$, the superpositions in the exponential-distribution form (4) likewise match the first three terms of the series expansions in $1/\tau_0$. That is, the power-series expansions of equations (6) and (7) first differ in the coefficient of $(1/\tau_0)^3$.

This exact solution has greater significance than that of a test of the CFHZ approximation, moreover. It is especially useful to have an exact evaluation of σN for the exponential exposure distribution, because the exponential distribution can be obtained from certain mixing models for the *s*-process such as that of Ulrich (1973). For such interesting cases we need no longer rely at all on the CFHZ approximation, because equation (6) is easily evaluated.

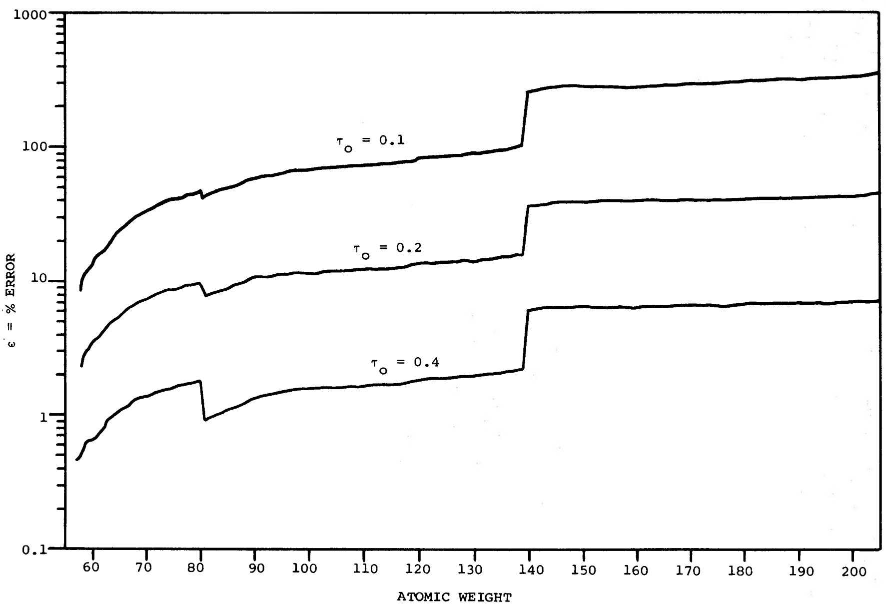

FIG. 2.—The percent error of the CFHZ approximation as a function of atomic weight for an exponential distribution of exposures. The values of τ_0 are in units of 10^{27} per cm^2.

Equation (6) contains another result of importance in applications of s-process theory: *with an exponential distribution of exposures, the relative values of $\sigma_k N_k$ for any two nuclei beyond the last seed nucleus are independent of the distribution of seed nuclei.* Let all significant seed nuclei lie between $k = 1$ and $k = s$, and let each seed nucleus s' be exposed to the same exponential distribution of exposures $\rho_{s'}(\tau) = G_{s'} \exp(-\tau/\tau_0)$. Then the resulting $\sigma_k N_k$ is a sum of terms having the form (6). If $k \geq s$, that is, lying at or beyond the last seed nucleus, then the sum factors into the following form:

$$\sigma_k N_k = \prod_{j=s}^{k} \left(1 + \frac{1}{\tau_0 \sigma_j}\right)^{-1} \left[G_1 \prod_{i=1}^{s-1} \left(1 + \frac{1}{\tau_0 \sigma_i}\right)^{-1} + G_2 \prod_{i=2}^{s-1} \left(1 + \frac{1}{\tau_0 \sigma_i}\right)^{-1} + \cdots + G_s\right] . \tag{9}$$

The entire term in brackets is a constant for all nuclei $k \geq s$, so for any two values of $k' > k \geq s$ we have

$$\frac{\sigma_k N_k}{\sigma_{k'} N_{k'}} = \left(1 + \frac{1}{\tau_0 \sigma_{k'}}\right)\left(1 + \frac{1}{\tau_0 \sigma_{k'-1}}\right) \cdots \left(1 + \frac{1}{\tau_0 \sigma_{k+1}}\right) . \tag{10}$$

This is the same ratio as that obtained with a single seed nucleus, confirming the original allegation. The proof utilized a unique s-process path, consistent with the assumption of rapid beta decays. In a later paper dealing with branching between neutron capture and β-decay, we will show the same result to be valid. These conclusions are true only for the exponential distribution of exposures; but considering the importance of that distribution, the result is noteworthy. It prevents one from altering the shape of $\sigma_k N_k$ by altering conditions upstream.

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