1 Likelihood

- S_t: the t-th sample.
- $S_T^{in} = \bigcup_{t=1}^{T} S_t$: set of sampled individuals up to and including T-th sample.
- S_T^{out}: set of individuals that has not been sampled after T sample draws. ($S_T^{out} \cap S_T^{in} = \emptyset$).
- $\mathcal{P} = S_T^{in} \cup S_T^{out}$: the entire population.
- $d_{i(T)}$: number of samples that contained individual i:

$$d_{i(T)} = \sum_{t=1}^{T} \mathbb{1}\{i \in S_t\}$$

Assuming that samples are drawn independently:

$$\mathbb{E}[d_{i(T)}] = \sum_{t=1}^{T} \pi_i = T\pi_i$$

where π_i denotes inclusion probability of i. This suggests estimating π_i by

$$\hat{\pi}_i = \frac{d_{i(T)}}{T} \quad \text{or} \quad \hat{\pi}_i = \frac{1 + d_{i(T)}}{1 + T} \tag{1}$$

where the latter comes from enforcing the probabilities to be non-zero. Note that in this case $\hat{\pi}_i = \frac{1}{1+T} \forall i \in S_T^{out}$.

Consider the problem from the Bayesian perspective. Assume Beta prior for inclusion probabilities:

$$\pi_i \sim Be(\alpha, \beta)$$

Then

$$\mathbb{E}[\sum_{i \in \mathcal{P}} \pi_i] = \sum_{i \in \mathcal{P}} \frac{\alpha}{\alpha + \beta} = |\mathcal{P}| \frac{\alpha}{\alpha + \beta} = (|S_T^{out}| + |S_T^{in}|) \frac{\alpha}{\alpha + \beta}$$

Let $n_t := |S_t|$ be the sample size. While we assume independent replications of the same sampling scheme, depending on the chosen scheme, it is possible for n_t to be random. However, for simplicity, assume here that $n_t = n$ is fixed.

$$\mathbb{E}[\sum_{i \in \mathcal{P}} \pi_i] = \mathbb{E}[n_t] \Leftrightarrow (|S_T^{out}| + |S_T^{in}|) \frac{\alpha}{\alpha + \beta} = n \tag{2}$$

(in case of random n_t, we can estimate $\mathbb{E}[n_t]$ by $T^{-1} \sum_{t=1}^{T} n_t$).

Condition (2) will be the constraint for our model.

The likelihood would be

$$d_{i(T)} | \pi_i \sim Bin(T, \pi)$$

Then the posterior distribution is

$$\begin{aligned}
f(\pi_i | d_{i(T)} = k) &= \frac{\mathbb{P}(d_{i(T)} = k | \pi_i) f(\pi_i)}{\mathbb{P}(d_{i(T)} = k)} \\
&\propto \pi_i^k (1 - \pi_i)^{T-k} \pi_i^{\alpha-1} (1 - \pi_i)^{\beta-1} \\
&= \pi_i^{\alpha+k-1} (1 - \pi_i)^{\beta+T-k-1} \\
&\Rightarrow \pi_i | d_{i(T)} = k \sim Be(\alpha + k, \beta + T - k) \\
&\Rightarrow \mathbb{E}[\pi_i | d_{i(T)} = k] = \frac{\alpha + k}{\alpha + \beta + T}
\end{aligned}$$

The marginal likelihood is:

$$\mathbb{P}(d_{i(T)} = k) = \int_0^1 f(\pi_i, d_{i(T)})d\pi_i = \int_0^1 \mathbb{P}(d_{i(T)} = k|\pi_i)f(\pi_i)d\pi_i$$
$$= \binom{T}{k}\frac{\mathrm{B}(\alpha + k, \beta + T - k)}{\mathrm{B}(\alpha, \beta)}$$

Note that we never observe $d_{i(T)} = 0$. The observed frequences $d_{i(T)} > 0$ for $i \in S_T^{in}$ follow a truncated distribution:

$$\mathbb{P}(d_{i(T)} = k|d_{i(T)} > 0) = \begin{cases} \frac{\mathbb{P}(d_{i(T)}=k)}{1-\mathbb{P}(d_{i(T)}=0)}, & \text{if } k > 0 \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} \binom{T}{k}\frac{\mathrm{B}(\alpha+k,\beta+T-k)}{\mathrm{B}(\alpha,\beta)-\mathrm{B}(\alpha,\beta+T)}, & \text{if } k > 0 \\ 0, & \text{otherwise} \end{cases}$$

Following empirical Bayes approach, maximise the marginal likelihood to obtain hyperparameters α and β.

$$\begin{aligned} L(\alpha, \beta) &:= L(\alpha, \beta; T, d_{i(T)} = k_i \forall i \in S_T^{in}) \\ &\overset{indep}{=} \prod_{i \in S_T^{in}} \binom{T}{k_i}\frac{\Gamma(\alpha + k_i)\Gamma(\beta + T - k_i)}{\Gamma(\alpha + \beta + T)} \cdot \frac{1}{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} - \frac{\Gamma(\alpha)\Gamma(\beta+T)}{\Gamma(\alpha+\beta+T)}} \\ &= \prod_{i \in S_T^{in}} \binom{T}{k_i}\frac{\Gamma(\alpha + k_i)\Gamma(\beta + T - k_i)\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + T) - \Gamma(\alpha)\Gamma(\beta + T)\Gamma(\alpha + \beta)} \end{aligned} \tag{3}$$

Using the recursive formula of gamma function $\Gamma(x) = (x - 1)\Gamma(x - 1)$ and the fact that $T, k_i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ allows us to rewrite the marginal likelihood as:

$$\begin{aligned} L(\alpha, \beta) &\propto \prod_{i \in S_T^{in}} \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta)} \\ &\quad \times \frac{\prod_{j=1}^{k_i}(\alpha + k_i - j)\prod_{j=1}^{T-k_i}(\beta + T - k_i - j)}{\prod_{j=1}^{T}(\alpha + \beta + T - j) - \prod_{j=1}^{T}(\beta + T - j)} \\ &= \prod_{i \in S_T^{in}} \frac{\prod_{j=1}^{k_i}(\alpha + k_i - j)\prod_{j=1}^{T-k_i}(\beta + T - k_i - j)}{\prod_{j=1}^{T}(\alpha + \beta + T - j) - \prod_{j=1}^{T}(\beta + T - j)} \end{aligned}$$

The parameters α and β must satisfy (2), so equation (3) is to be maximised subject to the constraint. The optimisation problem can be simplified by solving the constraint for β and plugging into the objective function:

$$\max_{\alpha,\beta} L(\alpha, \beta) \quad \text{s.t.} \quad |\mathcal{P}|\frac{\alpha}{\alpha + \beta} = n$$
$$\Leftrightarrow \beta = \left(\frac{|\mathcal{P}|}{n} - 1\right)\alpha := q\alpha$$
$$\Rightarrow \max_{\alpha,q} L(\alpha, q)$$

More explicitly,

$$\begin{aligned}\max_{\alpha,\beta} L(\alpha,\beta) \quad \text{s.t.} \quad (|S_T^{in}| + |S_T^{out}|)\frac{\alpha}{\alpha+\beta} = n \\ \Leftrightarrow \beta = n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| - \alpha \\ \Rightarrow \max_{\alpha, |S_T^{out}|} L(\alpha, |S_T^{out}|)\end{aligned}$$

1.1 Variant 1

$$L(\alpha,q) \propto \prod_{i\in S_T^{in}} \frac{\prod_{j=1}^{k_i}(\alpha+k_i-j)\prod_{j=1}^{T-k_i}(q\alpha+T-k_i-j)}{\prod_{j=1}^{T}(\alpha+q\alpha+T-j)-\prod_{j=1}^{T}(q\alpha+T-j)}$$

$$\begin{aligned}l(\alpha,q) &:= \log L(\alpha,q) \\ &= const - |S_T^{in}| \log\left[\prod_{j=1}^{T}(\alpha+q\alpha+T-j) - \prod_{j=1}^{T}(q\alpha+T-j)\right] \\ &+ \sum_{i\in S_T^{in}}\sum_{j=1}^{k_i}\log(\alpha+k_i-j) + \sum_{j=1}^{T-k_i}\log(q\alpha+T-k_i-j)\end{aligned} \tag{4}$$

Derivatives:

$$\begin{aligned}\frac{\partial l(\alpha,q)}{\partial\alpha} =& -|S_T^{in}|\frac{\sum_{j=1}^{T}(1+q)\prod_{1\le m\ne j\le T}(\alpha+q\alpha+T-m) - \sum_{j=1}^{T} q\prod_{1\le m\ne j\le T}(q\alpha+T-m)}{\prod_{j=1}^{T}(\alpha+q\alpha+T-j)-\prod_{j=1}^{T}(q\alpha+T-j)} \\ &+ \sum_{i\in S_T^{in}}\sum_{j=1}^{k_i}(\alpha+k_i-j)^{-1} + q\sum_{j=1}^{T-k_i}(q\alpha+T-k_i-j)^{-1} \\ =& -(1+q)|S_T^{in}|\frac{\prod_{j=1}^{T}(\alpha+q\alpha+T-j)\sum_{j=1}^{T}(\alpha+q\alpha+T-j)^{-1}}{\prod_{j=1}^{T}(\alpha+q\alpha+T-j)-\prod_{j=1}^{T}(q\alpha+T-j)} \\ &+ q|S_T^{in}|\frac{\prod_{j=1}^{T}(q\alpha+T-j)\sum_{j=1}^{T}(q\alpha+T-j)^{-1}}{\prod_{j=1}^{T}(\alpha+q\alpha+T-j)-\prod_{j=1}^{T}(q\alpha+T-j)} \\ &+ \sum_{i\in S_T^{in}}\sum_{j=1}^{k_i}(\alpha+k_i-j)^{-1} + q\sum_{j=1}^{T-k_i}(q\alpha+T-k_i-j)^{-1} \\ =& -(1+q)|S_T^{in}|\frac{A_p(\alpha,q)A_s(\alpha,q)}{A_p(\alpha,q)-B_p(\alpha,q)} + q|S_T^{in}|\frac{B_p(\alpha,q)B_s(\alpha,q)}{A_p(\alpha,q)-B_p(\alpha,q)} \\ &+ \sum_{i\in S_T^{in}}\sum_{j=1}^{k_i}(\alpha+k_i-j)^{-1} + q\sum_{j=1}^{T-k_i}(q\alpha+T-k_i-j)^{-1}\end{aligned} \tag{5}$$

$$
\begin{aligned}
\frac{\partial l(\alpha,q)}{\partial q} =& -|S_T^{in}|\frac{\sum_{j=1}^{T}\alpha\prod_{1\le m\neq j\le T}(\alpha+q\alpha+T-m)-\sum_{j=1}^{T}\alpha\prod_{1\le m\neq j\le T}(q\alpha+T-m)}{\prod_{j=1}^{T}(\alpha+q\alpha+T-j)-\prod_{j=1}^{T}(q\alpha+T-j)} \\
&+\sum_{i\in S_T^{in}}\alpha\sum_{j=1}^{T-k_i}(q\alpha+T-k_i-j)^{-1} \\
=& -\alpha|S_T^{in}|\frac{\prod_{j=1}^{T}(\alpha+q\alpha+T-j)\sum_{j=1}^{T}(\alpha+q\alpha+T-j)^{-1}}{\prod_{j=1}^{T}(\alpha+q\alpha+T-j)-\prod_{j=1}^{T}(q\alpha+T-j)} \\
&+\alpha|S_T^{in}|\frac{\prod_{j=1}^{T}(q\alpha+T-j)\sum_{j=1}^{T}(q\alpha+T-j)^{-1}}{\prod_{j=1}^{T}(\alpha+q\alpha+T-j)-\prod_{j=1}^{T}(q\alpha+T-j)} \\
&+\alpha\sum_{i\in S_T^{in}}\sum_{j=1}^{T-k_i}(q\alpha+T-k_i-j)^{-1} \\
=& -\alpha|S_T^{in}|\frac{A_p(\alpha,q)A_s(\alpha,q)-B_p(\alpha,q)B_s(\alpha,q)}{A_p(\alpha,q)-B_p(\alpha,q)}+\alpha\sum_{i\in S_T^{in}}\sum_{j=1}^{T-k_i}(q\alpha+T-k_i-j)^{-1}
\end{aligned} \tag{6}
$$

where

$$
A_p(\alpha,q):=\prod_{j=1}^{T}(\alpha+q\alpha+T-j),\quad A_s(\alpha,q):=\sum_{j=1}^{T}(\alpha+q\alpha+T-j)^{-1}
$$
$$
B_p(\alpha,q):=\prod_{j=1}^{T}(q\alpha+T-j),\quad B_s(\alpha,q):=\sum_{j=1}^{T}(q\alpha+T-j)^{-1}
$$

The product terms $A_p(\alpha,q)$ and $B_p(\alpha,q)$ are computationally expensive to calculate, as even small values of T will lead yield extremely large quantities. For practical purposes, one can factor out $q\alpha+T$ from both expressions. Then the log-likelihood and the derivatives can be re-written as

$$
\begin{aligned}
l(\alpha,q) = &\, const - T|S_T^{in}|\log(q\alpha+T)-|S_T^{in}|\log\left[\widetilde{A_p}(\alpha,q)-\widetilde{B_p}(\alpha,q)\right] \\
&+\sum_{i\in S_T^{in}}\sum_{j=1}^{k_i}\log(\alpha+k_i-j)+\sum_{j=1}^{T-k_i}\log(q\alpha+T-k_i-j)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l(\alpha,q)}{\partial \alpha} =& -(1+q)|S_T^{in}|\frac{\widetilde{A_p}(\alpha,q)A_s(\alpha,q)}{\widetilde{A_p}(\alpha,q)-\widetilde{B_p}(\alpha,q)}+q|S_T^{in}|\frac{\widetilde{B_p}(\alpha,q)B_s(\alpha,q)}{\widetilde{A_p}(\alpha,q)-\widetilde{B_p}(\alpha,q)} \\
&+\sum_{i\in S_T^{in}}\sum_{j=1}^{k_i}(\alpha+k_i-j)^{-1}+q\sum_{j=1}^{T-k_i}(q\alpha+T-k_i-j)^{-1}
\end{aligned}
$$

$$
\frac{\partial l(\alpha,q)}{\partial q} = -\alpha|S_T^{in}|\frac{\widetilde{A_p}(\alpha,q)A_s(\alpha,q)-\widetilde{B_p}(\alpha,q)B_s(\alpha,q)}{\widetilde{A_p}(\alpha,q)-\widetilde{B_p}(\alpha,q)}+\alpha\sum_{i\in S_T^{in}}\sum_{j=1}^{T-k_i}(q\alpha+T-k_i-j)^{-1}
$$

where

$$\widetilde{A_p}(\alpha, q) := \prod_{j=1}^{T} \left(1 - \frac{j-\alpha}{q\alpha + T}\right), \quad \widetilde{B_p}(\alpha, q) := \prod_{j=1}^{T} \left(1 - \frac{j}{q\alpha + T}\right)$$

1.2 Variant 2

$$L(\alpha, |S_T^{out}|) \propto \prod_{i \in S_T^{in}} \frac{\prod_{j=1}^{k_i}(\alpha + k_i - j) \prod_{j=1}^{T-k_i}(n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| - \alpha + T - k_i - j)}{\prod_{j=1}^{T}(n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| + T - j) - \prod_{j=1}^{T}(n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| - \alpha + T - j)}$$

$$\begin{aligned}
l(\alpha, |S_T^{out}|) &:= \log L(\alpha, |S_T^{out}|) \\
&= const - |S_T^{in}| \log\left[A_p(\alpha, |S_T^{out}|) - B_p(\alpha, |S_T^{out}|)\right] \\
&+ \sum_{i \in S_T^{in}} \sum_{j=1}^{k_i} \log(\alpha + k_i - j) + \sum_{j=1}^{T-k_i} \log(n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| - \alpha + T - k_i - j) \\
&= const - T|S_T^{in}| \log(n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| + T) - |S_T^{in}| \log\left[\widetilde{A_p}(\alpha, |S_T^{out}|) - \widetilde{B_p}(\alpha, |S_T^{out}|)\right] \\
&+ \sum_{i \in S_T^{in}} \sum_{j=1}^{k_i} \log(\alpha + k_i - j) + \sum_{j=1}^{T-k_i} \log(n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| - \alpha + T - k_i - j)
\end{aligned} \tag{7}$$

Derivatives:

$$\begin{aligned}
\frac{\partial l(\alpha, |S_T^{out}|)}{\partial \alpha} =& - |S_T^{in}|(n^{-1}|S_T^{in}| + n^{-1}|S_T^{out}|) \frac{A_p(\alpha, |S_T^{out}|) A_s(\alpha, |S_T^{out}|)}{A_p(\alpha, |S_T^{out}|) - B_p(\alpha, |S_T^{out}|)} \\
&+ |S_T^{in}|(n^{-1}|S_T^{in}| + n^{-1}|S_T^{out}| - 1) \frac{B_p(\alpha, |S_T^{out}|) B_s(\alpha, |S_T^{out}|)}{A_p(\alpha, |S_T^{out}|) - B_p(\alpha, |S_T^{out}|)} \\
&+ \sum_{i \in S_T^{in}} \sum_{j=1}^{k_i} (\alpha + k_i - j)^{-1} + (\frac{|S_T^{in}|}{n} + \frac{|S_T^{out}|}{n} - 1) \sum_{j=1}^{T-k_i} (\alpha \frac{|S_T^{in}|}{n} + \alpha \frac{|S_T^{out}|}{n} - \alpha + T - k_i - j)^{-1} \\
=& - |S_T^{in}|(n^{-1}|S_T^{in}| + n^{-1}|S_T^{out}|) \frac{\widetilde{A_p}(\alpha, |S_T^{out}|) A_s(\alpha, |S_T^{out}|)}{\widetilde{A_p}(\alpha, |S_T^{out}|) - \widetilde{B_p}(\alpha, |S_T^{out}|)} \\
&+ |S_T^{in}|(n^{-1}|S_T^{in}| + n^{-1}|S_T^{out}| - 1) \frac{\widetilde{B_p}(\alpha, |S_T^{out}|) B_s(\alpha, |S_T^{out}|)}{\widetilde{A_p}(\alpha, |S_T^{out}|) - \widetilde{B_p}(\alpha, |S_T^{out}|)} \\
&+ \sum_{i \in S_T^{in}} \sum_{j=1}^{k_i} (\alpha + k_i - j)^{-1} + (\frac{|S_T^{in}|}{n} + \frac{|S_T^{out}|}{n} - 1) \sum_{j=1}^{T-k_i} (\alpha \frac{|S_T^{in}|}{n} + \alpha \frac{|S_T^{out}|}{n} - \alpha + T - k_i - j)^{-1}
\end{aligned} \tag{8}$$

$$
\begin{aligned}
\frac{\partial l(\alpha, |S_T^{out}|)}{\partial |S_T^{out}|} =& - n^{-1}\alpha|S_T^{in}| \frac{A_p(\alpha, |S_T^{out}|)A_s(\alpha, |S_T^{out}|) - B_p(\alpha, |S_T^{out}|)B_s(\alpha, |S_T^{out}|)}{A_p(\alpha, |S_T^{out}|) - B_p(\alpha, |S_T^{out}|)} \\
&+ n^{-1}\alpha \sum_{i \in S_T^{in}} \sum_{j=1}^{T-k_i} (\alpha\frac{|S_T^{in}|}{n} + \alpha\frac{|S_T^{out}|}{n} + T - k_i - j)^{-1} \\
=& - n^{-1}\alpha|S_T^{in}| \frac{\widetilde{A_p}(\alpha, |S_T^{out}|)A_s(\alpha, |S_T^{out}|) - \widetilde{B_p}(\alpha, |S_T^{out}|)B_s(\alpha, |S_T^{out}|)}{\widetilde{A_p}(\alpha, |S_T^{out}|) - \widetilde{B_p}(\alpha, |S_T^{out}|)} \\
&+ n^{-1}\alpha \sum_{i \in S_T^{in}} \sum_{j=1}^{T-k_i} (\alpha\frac{|S_T^{in}|}{n} + \alpha\frac{|S_T^{out}|}{n} + T - k_i - j)^{-1}
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
A_p(\alpha, |S_T^{out}|) &:= \prod_{j=1}^{T} (n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| + T - j) \\
A_s(\alpha, |S_T^{out}|) &:= \sum_{j=1}^{T} (n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| + T - j)^{-1} \\
B_p(\alpha, |S_T^{out}|) &:= \prod_{j=1}^{T} (n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| - \alpha + T - j) \\
B_s(\alpha, |S_T^{out}|) &:= \sum_{j=1}^{T} (n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| - \alpha + T - j)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{A_p}(\alpha, |S_T^{out}|) &:= \prod_{j=1}^{T} \left(1 - \frac{j}{n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| + T}\right) \\
\widetilde{B_p}(\alpha, |S_T^{out}|) &:= \prod_{j=1}^{T} \left(1 - \frac{j+\alpha}{n^{-1}\alpha|S_T^{in}| + n^{-1}\alpha|S_T^{out}| + T}\right)
\end{aligned}
$$

2 Simulation results

We simulate capture-recapture procedure to produce $d_{i(T)}$ as follows:

1. Set the true population size to $|\mathcal{P}|$, sample size at each capture to n and number of captures to T. This results in setting $q = (\frac{\mathcal{P}}{n} - 1)$.
2. Set the parameter α.
3. Draw $|\mathcal{P}|$ values from $\pi \sim Beta(\alpha, q\alpha)$.
4. Normalise π_i by dividing by n.
5. At each replication $t = 1, \ldots, T$:

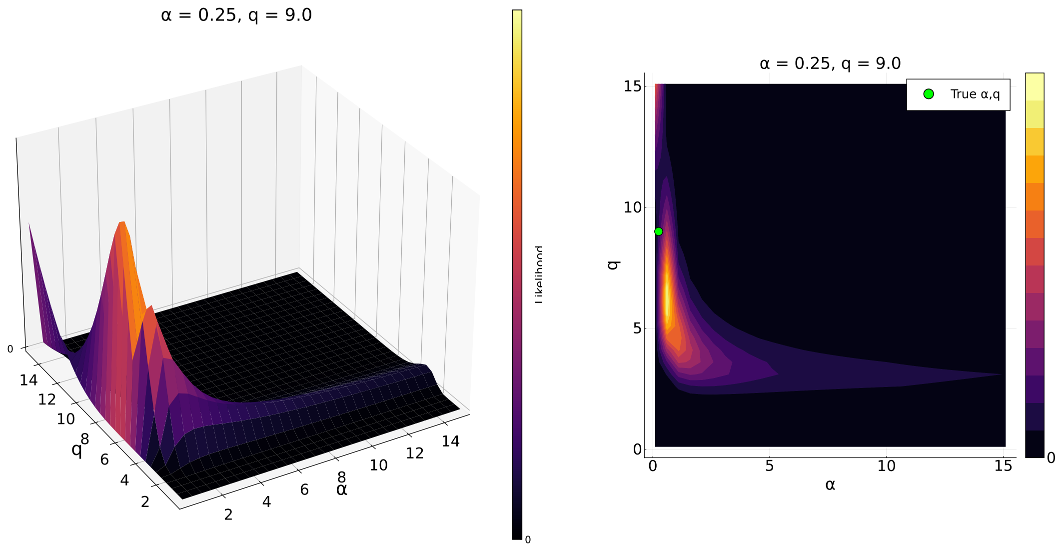

Figure 1: Likelihood function for data simulated by $\pi \sim Beta(0.25, 9 \cdot 0.25), |\mathcal{P}| = 100, n = 10, T = 5$

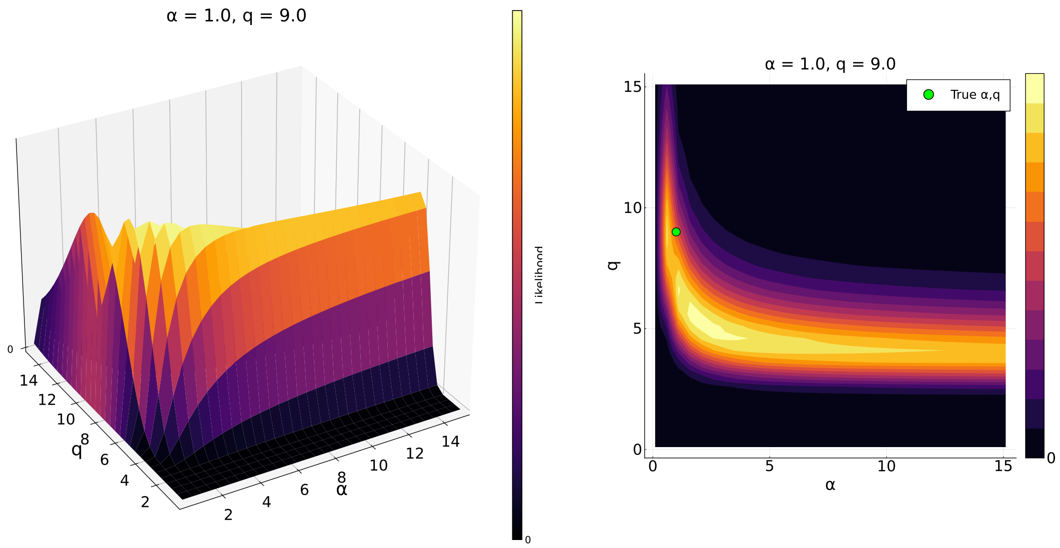

Figure 2: Likelihood function for data simulated by $\pi \sim Beta(1, 9 \cdot 1), |\mathcal{P}| = 100, n = 10, T = 5$

(a) Given $\pi_i \forall\ i = 1, \ldots, |\mathcal{P}|$, draw a sample of n numbers from $\{1, \ldots, |\mathcal{P}|\}$ using Sampford's method. Denote the sample with S_t.

(b) For each $i \in S_t$, increment $d_{i(T)}$ by 1. If $d_{i(T)}$ was never recorded before set $d_{i(T)} = 1$.

Once we acquire $d_{i(T)}$, it is possible to calculate the likelihood function from section 1.1. Figures 1-3 show surface and contour plots of the likelihood for data simulated using different true α at parameter values $\{0.1, 0.6, \ldots, 15.1\}$. The population size was set to 100, 5 samples of size 10 were drawn.

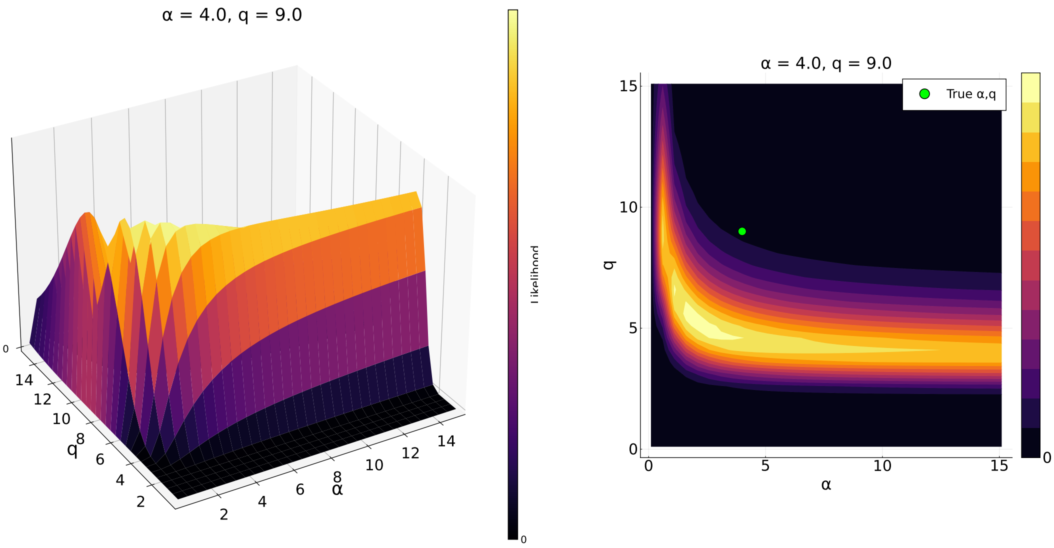

Figure 3: Likelihood function for data simulated by $\pi \sim Beta(4, 9 \cdot 4), |\mathcal{P}| = 100, n = 10, T = 5$