Mathematische Methoden

Blatt 5

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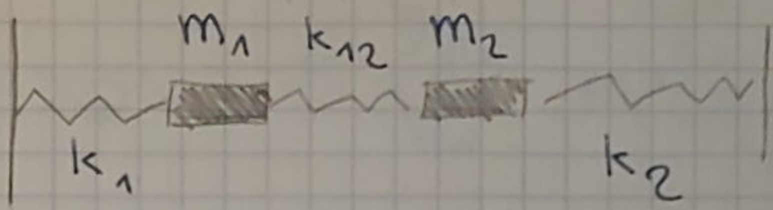


$$\begin{pmatrix} k_1 + k_{12} - m_1 \omega^2 & -k_{12} \\ -k_{12} & k_2 + k_{12} - m_2 \omega^2 \end{pmatrix} = A$$

$$A \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix}$$

$4k^2 - 4km\omega^2 \quad m^2\omega^4 + 2k$

$\det(A) =$

$$(k_1 + k_{12} - m_1 \omega^2)(k_2 + k_{12} - m_2 \omega^2) - ((-k_{12})(-k_{12}))$$

$$= k_1 k_2 + k_1 k_{12} - k_1 m_2 \omega^2 + k_{12} k_2 + (k_{12})^2 - k_{12} m_2 \omega^2 - k_2 m_1 \omega^2 - k_{12} m_1 \omega^2 + m_1 m_2 \omega^4 - k_{12}^2$$

$m_1 = m_2 =: m \qquad k_1 = k_2 =: k$

$$= k^2 + k^2 - km\omega^2 + k^2 + k^2 - km\omega^2 - km\omega^2 - km\omega^2 + m^2\omega^4 - k^2$$

$$= 3k^2 - 4km\omega^2 + m^2\omega^4 \ldots$$

$$3k^2 - 4km\omega^2 + m^2\omega^4 \overset{!}{=} 0$$

$$m^2\omega^4 - 4km\omega^2 + 3k^2 = 0 \quad |:m^2$$

$$\omega^4 - \frac{4k}{m}\omega^2 + \frac{3k^2}{m^2} = 0$$

pq - Formel

$$\omega_{1,2}^2 = -\left(\frac{-4k}{2m}\right) \pm \sqrt{\left(\frac{-4k}{2m}\right)^2 - \left(\frac{3k^2}{m^2}\right)}$$

$$\omega_{1,2}^2 = \frac{2k}{m} \pm \sqrt{\frac{4k^2}{m^2} - \left(\frac{3k^2}{m^2}\right)}$$

$$= \frac{2k}{m} \pm \sqrt{\frac{k^2}{m^2}} = \frac{2k}{m} \pm \frac{k}{m}$$

$$\omega_1^2 = \frac{3k}{m} \qquad \omega_2^2 = \frac{k}{m}$$

$$\omega_1 = \pm\sqrt{\frac{3k}{m}} \qquad \omega_2 = \pm\sqrt{\frac{k}{m}}$$ Eigenwert

Eigenmoden:

$$A = \begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix}$$

$$A\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

$$\Rightarrow \quad 2k\alpha_1 - m\omega^2\alpha_1 - \alpha_2 k = 0$$

$$-\alpha_1 k + 2k\alpha_2 - m\omega\alpha_2$$

Für $\omega = \pm\sqrt{\frac{3k}{m}}$

$$2k\alpha_1 - \cancel{m}\frac{3k}{\cancel{m}}\alpha_1 - \alpha_2 k = 0$$

$$-\alpha_1 k + 2k\alpha_2 - \cancel{m}\frac{3k}{\cancel{m}}\alpha_2 = 0$$

$$\Leftrightarrow \quad -\alpha_1 k - \alpha_2 k = 0$$
$$-\alpha_2 k - \alpha_1 k = 0$$

$$\Leftrightarrow \quad -\alpha_1 \cancel{k} = \alpha_2 \cancel{k} \qquad \Rightarrow \alpha_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}}$$
$$-\alpha_1 \cancel{k} = \alpha_2 \cancel{k}$$

Für $\omega = \pm\sqrt{\frac{k}{m}}$

$$2k\alpha_1 - \cancel{m}\frac{k}{\cancel{m}}\alpha_1 - \alpha_2 k = 0$$
$$-\alpha_1 k + 2k\alpha_2 - \cancel{m}\frac{k}{\cancel{m}}\alpha_2 = 0$$

$$\Leftrightarrow \quad k\alpha_1 - \alpha_2 k = 0$$
$$k\alpha_1 - \alpha_2 k = 0$$

$$\Leftrightarrow \quad \cancel{k}\alpha_1 = \alpha_2 \cancel{k} \qquad \Rightarrow \alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$
$$\cancel{k}\alpha_1 = \alpha_2 \cancel{k}$$

Die beiden Eigenmoden sind

$$\alpha_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}} \quad \text{und} \quad \alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

2)

$$\underline{r}(t)\ R\begin{pmatrix}\cos(\omega t)\\ \frac{\sin(\omega t)}{\sqrt{2}}\\ \frac{\sin(\omega t)}{\sqrt{2}}\end{pmatrix}$$

a) Natürliche Parametrisierung $\underline{r}(s)$

$$\dot{\underline{r}}(t)=\begin{pmatrix}-\omega\sin(\omega t)\\ \frac{1}{\sqrt{2}}\omega\cdot\cos(\omega t)\\ \frac{1}{\sqrt{2}}\omega\cdot\cos(\omega t)\end{pmatrix}R$$

$$|\dot{\underline{r}}(t)|=\sqrt{(-\omega\cdot\sin(\omega t))^2+2\left(\frac{1}{\sqrt{2}}\omega\cos(\omega t)\right)^2}\cdot R$$

$$=\sqrt{\omega^2\cdot\sin^2(\omega t)+\cancel{2}\left(\frac{1}{\cancel{2}}\omega^2\cos^2(\omega t)\right)}\cdot R$$

$$=\sqrt{\omega^2\,(\underbrace{\sin^2(\omega t)+\cos^2(\omega t)}_{1})}\cdot R$$

$$=R\omega$$

$$s(t)=\int_0^t R\omega\, d\tilde{t}=R\omega\tilde{t}\Big|_0^t=R\omega t$$

Umlaufzeit:

$$\Rightarrow s(t)=R\omega t \Rightarrow t=\frac{s}{R\omega}$$

$\Rightarrow$ natürliche Parametrisierung:

$$\underline{r}(s)=R\cdot\begin{pmatrix}\cos\left(\omega\cdot\frac{s}{R\omega}\right)\\ \frac{1}{\sqrt{2}}\sin\left(\omega\cdot\frac{s}{R\omega}\right)\\ \frac{1}{\sqrt{2}}\sin\left(\omega\cdot\frac{s}{R\omega}\right)\end{pmatrix}=R\cdot\begin{pmatrix}\cos\left(\frac{s}{R}\right)\\ \frac{1}{\sqrt{2}}\sin\left(\frac{s}{R}\right)\\ \frac{1}{\sqrt{2}}\sin\left(\frac{s}{R}\right)\end{pmatrix}$$

Tangenteneinheitsvektor $\underline{\hat{t}}(s)$

$$\underline{\hat{t}} = \frac{d\underline{r}(s)}{ds} = \cancel{R} \begin{pmatrix} \frac{-\sin(\frac{s}{R})}{R} \\ \frac{\cos(\frac{s}{R})}{\sqrt{2}\ \cancel{R}} \\ \frac{\cos(\frac{s}{R})}{\sqrt{2}\ R} \end{pmatrix} = \begin{pmatrix} -\sin(\frac{s}{R}) \\ \cos(\frac{s}{R})\frac{1}{\sqrt{2}} \\ \cos(\frac{s}{R})\frac{1}{\sqrt{2}} \end{pmatrix}$$

Krümmung κ

$$\kappa = \left| \frac{d\underline{\hat{t}}(s)}{ds} \right| = \left| \begin{pmatrix} \frac{-\cos(\frac{s}{R})}{R} \\ \frac{-\sin(\frac{s}{R})}{\sqrt{2}\ R} \\ \frac{-\sin(\frac{s}{R})}{\sqrt{2}\ R} \end{pmatrix} \right|$$

$$= \frac{1}{R}\sqrt{\left((\cos(\tfrac{s}{R}))^2 + 2\left(\cdot\frac{\sin(\frac{s}{R})}{\sqrt{2}}\right)^2\right.}$$

$$= \frac{1}{R}\sqrt{\cancel{\cos(\tfrac{s}{R})^2 + \sin(\tfrac{s}{R})^2}} = \frac{1}{R}$$

Krümmungsradius

$$\rho = \frac{1}{\kappa} = R$$

Berechnen Sie das begleitende Dreibein

$(\hat{\underline{t}}, \hat{\underline{n}}, \hat{\underline{b}})$

$$\hat{\underline{n}} = \rho \cdot \frac{d\, \hat{\underline{t}}(s)}{ds}$$

$$= R \cdot \frac{d}{ds} \begin{pmatrix} -\sin(\frac{s}{R}) \\ \frac{1}{\sqrt{2}} \cos(\frac{s}{R}) \\ \frac{1}{\sqrt{2}} \cos(\frac{s}{R}) \end{pmatrix} = R \cdot \begin{pmatrix} \frac{1}{R} \cdot -\cos(\frac{s}{R}) \\ \frac{1}{\sqrt{2}R} \cdot -\sin(\frac{s}{R}) \\ \frac{1}{\sqrt{2}R} \cdot -\sin(\frac{s}{R}) \end{pmatrix}$$

$$= - \begin{pmatrix} \cos(\frac{s}{R}) \\ \frac{1}{\sqrt{2}} \sin(\frac{s}{R}) \\ \frac{1}{\sqrt{2}} \sin(\frac{s}{R}) \end{pmatrix}$$

$$\hat{\underline{b}} = \hat{\underline{t}} \times \hat{\underline{n}} = \begin{pmatrix} -\sin(\frac{s}{R}) \\ \frac{1}{\sqrt{2}} \cos(\frac{s}{R}) \\ \frac{1}{\sqrt{2}} \cos(\frac{s}{R}) \end{pmatrix} \begin{pmatrix} -\cos(\frac{s}{R}) \\ -\frac{1}{\sqrt{2}} \sin(\frac{s}{R}) \\ -\frac{1}{\sqrt{2}} \sin(\frac{s}{R}) \end{pmatrix}$$

$$= \begin{pmatrix} (\frac{1}{\sqrt{2}} \cos(\frac{s}{R}))(-\frac{1}{\sqrt{2}} \sin(\frac{s}{R})) - (\frac{1}{\sqrt{2}} \cos(\frac{s}{R}))(-\frac{1}{\sqrt{2}} \sin(\frac{s}{R})) \\ \frac{1}{\sqrt{2}} \cos(\frac{s}{R})(-\cos(\frac{s}{R})) - (-\sin(\frac{s}{R}))(-\frac{1}{\sqrt{2}} \sin(\frac{s}{R})) \\ (-\sin(\frac{s}{R}))(-\frac{1}{\sqrt{2}} \sin(\frac{s}{R})) - (\frac{1}{\sqrt{2}} \cos(\frac{s}{R}))(-\cos(\frac{s}{R})) \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{-\frac{1}{2} \cos(\frac{s}{R}) \sin(\frac{s}{R}) + \frac{1}{2} \cos(\frac{s}{R}) \sin(\frac{s}{R})} = 0 \\ -\frac{1}{\sqrt{2}} \cos(\frac{s}{R})^2 - \frac{1}{\sqrt{2}} \sin(\frac{s}{R})^2 = -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \sin(\frac{s}{R})^2 + \frac{1}{\sqrt{2}} \cos(\frac{s}{R})^2 = \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\hat{\underline{b}} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Das Dreibein ist

$$\hat{\underline{t}} = \begin{pmatrix} -\sin\left(\frac{s}{R}\right) \\ \frac{1}{\sqrt{2}}\cos\left(\frac{s}{R}\right) \\ \frac{1}{\sqrt{2}}\cos\left(\frac{s}{R}\right) \end{pmatrix} \qquad \hat{\underline{n}} = \begin{pmatrix} -\cos\left(\frac{s}{R}\right) \\ -\frac{1}{\sqrt{2}}\sin\left(\frac{s}{R}\right) \\ -\frac{1}{\sqrt{2}}\sin\left(\frac{s}{R}\right) \end{pmatrix} \text{ und } \hat{\underline{b}} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Torsion

$$\frac{d\hat{\underline{b}}}{ds} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \tau \begin{pmatrix} -\cos \\ \dots \\ \dots \end{pmatrix} \Rightarrow \tau = 0$$

Frenetsche Formeln für $\underline{r}(s)$

$$\begin{pmatrix} \hat{\underline{t}}' \\ \hat{\underline{n}}' \\ \hat{\underline{b}}' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{R} & 0 \\ -\frac{1}{R} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\underline{t}} \\ \hat{\underline{n}} \\ \hat{\underline{b}} \end{pmatrix}$$

Berechnen Sie $\underline{a}(t)$ Geben sie Tangential und Radialteil an.

$$a(t) = \ddot{\underline{r}}(t) = \begin{pmatrix} -\omega^2\cos(\omega t) \\ -\frac{\omega^2}{\sqrt{2}}\sin(\omega t) \\ -\frac{\omega^2}{\sqrt{2}}\sin(\omega t) \end{pmatrix}$$

$$a_n(t) = \frac{v^2}{\rho} = \frac{|\dot{r}(t)|}{R} =$$

$$\frac{(|\dot{\underline{r}}(t)|)^2}{R} = \left| \begin{matrix} -\omega \sin(\omega t) \\ \frac{1}{\sqrt{2}}\, \omega \cdot \cos(\omega t) \\ \frac{1}{\sqrt{2}}\, \omega \cdot \cos(\omega t) \end{matrix} \right|^2 \cdot \frac{1}{R}$$

$$= \sqrt{(-\omega \sin(\omega t))^2 + 2\left(\frac{1}{\sqrt{2}}\, \omega \cos(\omega t)\right)^2}^{\,2} \cdot \frac{1}{R}$$

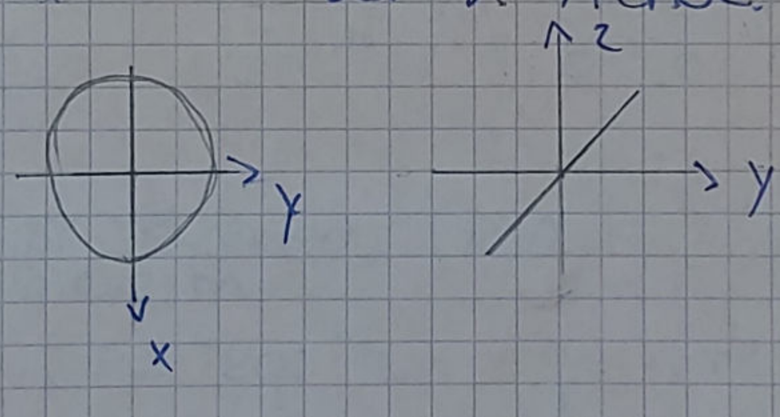
$$= \omega^2 \left(\cancel{\sin(\omega t)^2 + \cos(\omega t)^2}\right) \cdot \frac{1}{R} = \frac{\omega^2}{R} = a_n$$

$$a_t = \dot{v} = \frac{d}{dt}\, \omega^2 = 0$$

Was für eine Kurve?

Die Kurve ist ein Kreis im Drei-dimensionalen Raum

Er ist an der x-Achse um 45° gekippt.



3) Wegintegral 1. Art

$$f(x, y, z) = (6x + 2y)z = 6xz + 2yz$$

$$\underline{r}(t) = \begin{pmatrix} t-1 \\ 2t+3 \\ \frac{1}{2} t^2 \end{pmatrix}, \quad 1 \le t \le 2$$

$$\int_r f = \int_1^2 f(r(t)) \left| \frac{d\, r(t)}{dt} \right| dt$$

$$\frac{dr}{dt} = \begin{pmatrix} 1 \\ 2 \\ t \end{pmatrix} \qquad \left| \frac{dr}{dt} \right| = \sqrt{1^2 + 2^2 + t^2}$$

$$= \sqrt{5 + t^2}$$

$$\int_1^2 \left(6(t-1)\left(\frac{1}{2}t^2\right) + 2(2t+3)\left(\frac{1}{2}t^2\right)\right) \sqrt{5+t^2}\, dt$$

$$= \int_1^2 (3t^3 - \cancel{3t^2} + 2t^3 + \cancel{3t^2}) \cdot \sqrt{5+t^2}\, dt$$

$$= \int_1^2 5t^3 \cdot \sqrt{5+t^2}\, dt = 5 \int_1^2 t^3 \sqrt{5+t^2}\, dt$$

$$u = t^2 \qquad \frac{du}{dt} = 2t$$

$$\Rightarrow 5 \int_1^2 \cancel{t} \cdot u \cdot \sqrt{5+u} \cdot \frac{1}{2\cancel{t}}\, du = \frac{5}{2} \int_1^4 u \cdot \sqrt{5+u}\, du$$

$$s = 5 + u \qquad \frac{ds}{du} = 1$$

$$\Rightarrow \frac{5}{2} \int_6^9 (s-5) \cdot \sqrt{s}\, ds = \frac{5}{2} \int_6^9 s^{\frac{3}{2}}\, ds - \frac{25}{2} \int_6^9 \sqrt{s}\, ds$$

$$= s^{\frac{5}{2}} \Big|_6^9 - \frac{25}{2} \frac{2}{3} s^{\frac{3}{2}} \Big|_6^9$$

$$= 243 - 36\sqrt{6} + 50\sqrt{6} - 225 = 18 + 14\sqrt{6}$$

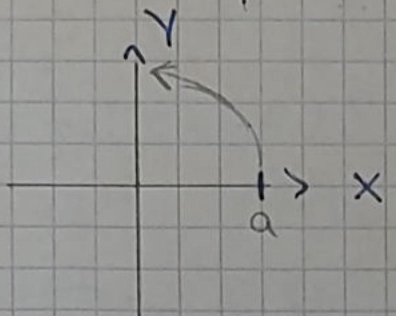
4. Parametrisieren

Aufwärtsschraube

Radius α

Höhengewinn h /Umlauf

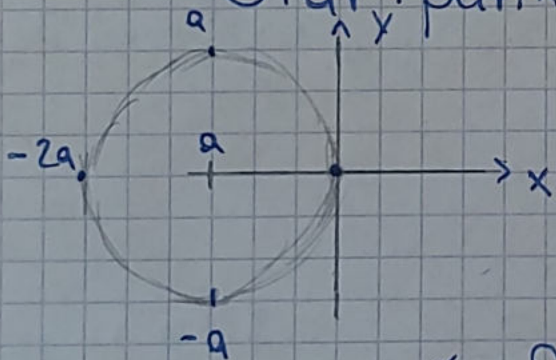
a) Startpunkt $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$ in positive y-Richtung



$$r(t) = \begin{pmatrix} a \cos(t) \\ a \sin(t) \\ h \frac{t}{2\pi} \end{pmatrix} \begin{matrix} \Rightarrow \text{nimmt ab} \\ \Rightarrow \text{nimmt zu} \end{matrix} \Bigg\} \text{ in positive y-Richtung}$$

$\Rightarrow$ bei einer Umderehung $2\pi \Rightarrow h \frac{2\pi}{2\pi} = h$

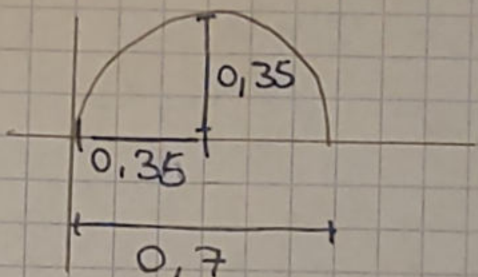
b) Startpunkt $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in positive y-Richtung

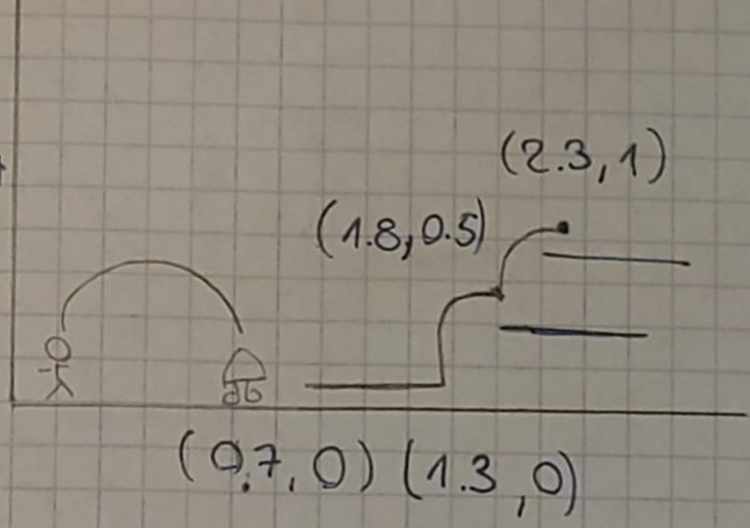


$$r(t) = \begin{pmatrix} a \cos(t) - a \\ a \sin(t) \\ h \frac{t}{2\pi} \end{pmatrix} \begin{matrix} \Rightarrow \in [-2a, 0] \\ \in [a, a] \end{matrix}$$

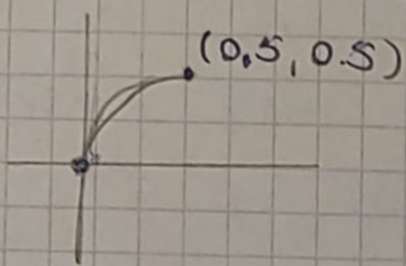
5)

Mit Halbkreis 0,7 weit





Mit Viertelkreis 0,5 hoch



$\Rightarrow (1.3, 0) + (0.5, 0.5) = (1.8, 0.5)$

$(1.8, 0.5) + (0.5, 0.5) = (2.3, 1)$

1. Teil Halbkreis

$$s_1(t) = \begin{pmatrix} -0{,}35\cos(t\pi) + 0{,}35 \\ 0{,}35\sin(t\pi) \end{pmatrix} \quad 0 \le t \le 1$$

2. Teil Gerade

$$s_2(t) = \begin{pmatrix} 0{,}7 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{(t-1)}{2} \\ 0 \end{pmatrix} \quad 1 \le t \le 2$$

3. Teil Viertelkreis

$$s_3(t) = \begin{pmatrix} 1{,}3 \\ 0 \end{pmatrix} + \begin{pmatrix} -0{,}5\cos(\frac{\pi}{2}(t-2)) + 0{,}5 \\ -0{,}5\sin(\frac{\pi}{2}(t-2)) \end{pmatrix} \quad 2 \le t \le 3$$

4. Teil

$$s_4(t) = \begin{pmatrix} 1{,}8 \\ 0{,}5 \end{pmatrix} + \begin{pmatrix} -0{,}5\cos(\frac{\pi}{2}(t-3)) + 0{,}5 \\ -0{,}5\sin(\frac{\pi}{2}(t-3)) \end{pmatrix}$$

$$3 \le t \le 4$$

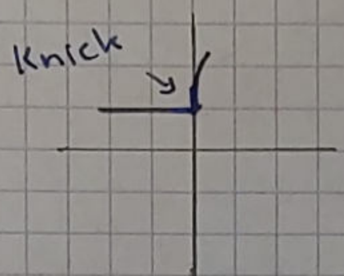
Mariokurve:

$$r(t) = \begin{cases} s_1(t) & \text{für } 0 \leq t \leq 1 \\ s_2(t) & \text{für } 1 \leq t \leq 2 \\ s_3(t) & \text{für } 2 \leq t \leq 3 \\ s_4(t) & \text{für } 3 \leq t \leq 4 \end{cases}$$

mit $0 \leq t \leq 4$

Die resultierende Kurve ist nur stückweise glatt, sie hat Knicke z. B. zwisch s_2 und s_3

$\Rightarrow$



Der Parameter kann als Zeit interpretiert werden, nur sind Marios Bewegungen keine echten Sprünge, da sich von Sprung 1 zu 2 die Schwerkraft ändern müsste. Außerdem beschleunigt er unendlich schnell, was in der Realität ebenfalls nicht möglich ist.

Als Zeitablauf einer "virtuellen" Bahn eignet sich die Parametrisierung aber.