Calculus on Manifolds

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Chapter One

Theorem 0.1 (Heine-Borel Theorem). *The closed interval* $[a, b]$ *is compact.*

Proof. If $\mathscr{O}$ is an open cover of $[a, b]$, let

$$A = \{x : a \leq x \leq b \text{ and } [a, x] \text{ is covered by some finite number of open sets in } \mathscr{O}\}.$$

We know that $a \in A$ since we can choose any open set in $\mathscr{O}$ containing a. A certainly has a least upper bound since A is bounded above by b. So, we will show that if some α is the least upper bound of A, then $\alpha \in A$ and $\alpha = b$.

Since $\alpha = \sup A$, for every $x \in A$, there exists an ε such that $\alpha - x < \varepsilon$. Since $[a, x]$ is covered by some finite number of open sets, we can choose any open ε-neighborhood centered at α. Hence, we see that $[a, \alpha]$ is also covered by finitely many open sets. This shows $\alpha \in A$.

To show that $\alpha = b$, assume that $\alpha < b$. Since, we can find some x' between α and b, such that x' is contained in some open neighborhood around α, we see that $[a, \alpha]$ is covered by a single open set. Then certainly, $x' \in A$. But this contradicts that $\alpha = \sup A$. Hence, $\alpha = b$. $\square$

Definition 0.1. Let X and Y be metric spaces, with metrics d_X and d_Y, respectively. Then, we say that a function $f : X \to Y$ is **continuous at the point** $x_0 \in X$ if for each open set $U \subset Y$, $f(x_0) \in U$, there is an open set $V \subset X$, $x_0 \in V$ such that $f(V) \subset U$.

Theorem 0.2. *If* $A \subset \mathbf{R}^n$*, a function* $f : A \to \mathbf{R}^m$ *is continuous if and only if for every open set* $U \subset \mathbf{R}^m$ *there is some open set* $V \subset \mathbf{R}^n$ *such that* $f^{-1}(U) = V \cap A$.

Proof. Suppose f is continuous. If $a \in f^{-1}(U)$, then $f(a) \in U$. Since U is open, there is some open ball $B \subset U$ such that $f(a) \in B$. And since, f is continuous at a, we know that $f(x) \in B$ provided we choose a sufficiently small open ball C such that $a \in C$. If we do this for each $a \in f^{-1}(U)$ and call their union V (also an open set), then clearly $f^{-1}(U) = V \cap A$.

$\square$

1. [1-23] If $f : A \to \mathbf{R}^m$ and $a \in A$, show that $\lim_{x\to a} f(x) = b$ if and only if $\lim_{x\to a} f_i(x) = b_i$ for $i = 1, ..., m$.

> **Solution:** If $\lim_{x\to a} f(x) = b$, then for every $\varepsilon > 0$, we can find a $\delta > 0$ such that
>
> $$0 < \|x - a\| < \delta \implies \|f(x) - b\| < \varepsilon.$$
>
> Then we have
>
> $$\begin{aligned} &\sum_{i=1}^{m} (f_i(x) - b_i)^2 < \varepsilon^2 \\ \implies &|f_i(x) - b_i| < \varepsilon \quad \text{for } i = 1, ..., m. \end{aligned}$$
>
> This implies that $\lim_{x\to a} f_i(x) = b_i$ for $i = 1, ..., m$.
> Now, if $\lim_{x\to a} f_i(x) = b_i$ for $i = 1, ..., m$ then for every $\varepsilon > 0$, we can find $\delta_1, ..., \delta_m$ such that
>
> $$0 < \|x - a\| < \delta_i \implies \|f_i(x) - b_i\| < \varepsilon \text{ for } i = 1, ..., m.$$
>
> Then, for $\delta = \min \delta_i,$ for $i = 1, ..., m$, we have,
>
> $$\begin{aligned} 0 < \|x - a\| < \delta &\implies \sum_{i=1}^{m} \|f_i(x) - b_i\|^2 < m\varepsilon^2 \\ &\implies \|f(x) - b\| < \sqrt{m}\varepsilon. \end{aligned}$$
>
> Hence, $\lim_{x\to a} f(x) = b$

2. [1-24] Prove that $f : A \to \mathbf{R}^m$ is continuous if and only if each f_i is.

> **Solution:** If $f : A \to \mathbf{R}^m$ is continuous then for all $a \in A$, $\lim_{x\to a} f(x) = f(a)$. But this means that for all $a \in A$, $lim_{x\to a} f_i(x) = f_i(a)$. Hence, each $f_i(x)$ is continuous.
> Similarly, converse follows from [1-23].

3. [1-25] Prove that a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous.

> **Solution:**
>
> *Proof.* We need to show that $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous at all $a \in \mathbf{R}^n$. That is, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that $0 < \|x - a\| < \delta \implies \|T(x) - T(a)\| < \varepsilon$, where $x \in \mathbf{R}^n$. But we have,
>
> $$\|T(x) - T(a)\| = \|T(x - a)\| \leq M\|x - a\|$$
>
> for some $M \in \mathbf{R}$. So for any given $\varepsilon > 0$, we can choose $\delta = \varepsilon/M$. Then certainly, if $0 < \|x - a\| < \delta$, then
>
> $$\|T(x) - T(a)\| \leq M\|x - a\| < M\delta = \varepsilon.$$
>
> So it follows that the linear transformation is continuous. $\square$

4. [1-29] If A is compact, prove that every continuous function $f : A \to \mathbf{R}$ takes on a maximum and a minimum value.

Solution:

Proof. Since A is compact and f is continuous, we know that the image of A under f is compact in $\mathbf{R}$. Hence, it follows from 3.1, that f takes on a maximum and a minimum value. □

Lemma 0.3. *A compact set in* $\mathbf{R}$ *has a maximum and a minimum value.*

Proof. We know that a compact set is closed and bounded and in $\mathbf{R}$, a compact set is in the form $[a, b]$. And since $a, b \in [a, b]$, all we need to show is that a and b are infimum and supremum, respectively, of the given interval. □

5. [1-30] Let $f : [a, b] \to \mathbf{R}$ be an increasing function. If $x_1, ..., x_n \in [a, b]$ are distinct, show that

$$\sum_{i=1}^{n} o(f, x_i) \leq f(b) - f(a).$$

Solution: Let order be defined in $\{x_1, ..., x_n\}$ such that $x_1 < ... < x_n$, then since f is an increasing function, we get $f(x_1) \leq ... \leq f(x_n)$. We have

$$o(f, x_i) = \lim_{\delta \to 0} \left(M(x_i, f, \delta) - m(x_i, f, \delta) \right)$$

where,

$$M(x_i, f, \delta) = \sup \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\},$$
$$m(x_i, f, \delta) = \inf \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\}.$$

If we denote a δ-neighborhood of some $x_i \in [a, b]$ by $N_\delta(x_i)$, then since $\delta \to 0$, we can choose a sufficiently small $\delta > 0$ such that

$$\bigcap_{i=1}^{n} N_\delta(x_i) = \phi.$$

Then for all such δ we have,

$$f(x_{i+1}) \geq M(x_i, f, \delta) \geq f(x_{i-1}), \text{ and}$$
$$f(x_{i+1}) \geq m(x_i, f, \delta) \geq f(x_{i-1}).$$

We simplify the given summation as

$$\begin{aligned}
\sum_{i=1}^{n} o(f, x_i) &= \lim_{\delta \to 0} \sum_{i=1}^{n} \left(M(x_i, f, \delta) - m(x_i, f, \delta) \right) \\
&\leq \lim_{\delta \to 0} \sum_{i=1}^{n} \left(f(x_{i+1}) - f(x_{i-1}) \right) \\
&= f(x_{n+1}) - f(x_0), \\
&\leq f(b) - f(a).
\end{aligned}$$

where the last statement follows from the fact that the max and min $f(x)$ can get is $f(b)$ and $f(a)$. □

Chapter 2

1. [2-1] Prove that if $f : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $a \in \mathbf{R}^n$, then it is continuous at a.

Solution: If f is differentiable at $a \in \mathbf{R}^n$, then

$$\lim_{h \to a} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0.$$

But this means that, for any given $\varepsilon_0 > 0$, we can find a $\delta > 0$ such that

$$\begin{aligned} 0 < \|h\| < \delta &\implies \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} < \varepsilon_0 \\ &\implies \|f(a+h) - f(a) - \lambda(h)\| < \varepsilon_0 \|h\| \\ &\implies \|f(a+h) - f(a)\| < (\varepsilon_0 + M)\|h\| \end{aligned}$$

for some $M \in \mathbf{R}$. So, when $h = x - a$ for any given $\varepsilon > 0$, we can choose $0 < \delta < \varepsilon/(\varepsilon_0 + M)$. Then it follows that,

$$0 < \|x - a\| < \delta \implies \|f(x) - f(a)\| < (\varepsilon_0 + M)\delta < \varepsilon.$$

Hence, f is continuous at a.

2. [2-2] A function $f : \mathbf{R}^2 \to \mathbf{R}$ is **independent of the second variable** if for each $x \in \mathbf{R}$ we have $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbf{R}$. Show that f is independent of the second variable if and only if there is a function $g : \mathbf{R} \to \mathbf{R}$ such that $f(x, y) = g(x)$. What is $f'(a, b)$ in terms of g'?

Solution: Define $g(x) = f(x, 0)$. Then for all $y \in \mathbf{R}$, if f is independent of the second variable, we have $f(x, y) = f(x, 0) = g(x)$.

Similarly, since g is independent of y, we have $g(x) = f(x, 0) = f(x, y_1) = f(x, y_2)$.

Now let $z = (h, k)$. Then, assuming that f is differentiable at (a, b), we have

$$\lim_{(h,k) \to 0} \frac{\|f(a+h, b+k) - f(a, b) - Df(a, b)(h, k)\|}{\|(h, k)\|} = 0$$

$$\text{or,} \ \lim_{h \to 0} \frac{\|g(a+h) - g(a) - Df(a, b)(h, k)\|}{|h|} = 0$$

Since $g : \mathbf{R} \to \mathbf{R}$,

$$\lim_{h \to 0} \frac{g(a+h) - g(a)}{|h|} = \lim_{h \to 0} \frac{Df(a, b)(h, k)}{|h|}$$

$$\text{or,} \ g'(a) = \lim_{h \to 0} \frac{Df(a, b)(h, k)}{|h|}$$

Then we see that

$$Df(a, b)(h, k) = h \cdot g'(a)$$

satisfies the equation. Hence, $f'(a, b) = g'(a)$.

3. [2-4] Let g be a continuous real-valued function on the unit circle $x \in \mathbf{R}^2 : \|x\| = 1$ such that $g(0, 1) =$

$g(1,0)=0$ and $g(-x)=-g(x)$. Define $f:\mathbf{R}^2\to\mathbf{R}$ by

$$f(x)=\begin{cases}\|x\|\cdot g\left(\dfrac{x}{\|x\|}\right) & x\neq 0,\\ 0 & x=0.\end{cases}$$

(a) If $x\in\mathbf{R}^2$ and $h:\mathbf{R}\to\mathbf{R}$ is defined by $h(t)=f(tx)$, show that h is differentiable.

Solution: We need to show that for every $a\in\mathbf{R}$, there exists a $\lambda:\mathbf{R}\to\mathbf{R}$ such that

$$\lim_{t\to 0}\frac{h(a+t)-h(a)-\lambda(t)}{t}=0. \tag{1}$$

We see that, when $tx\neq 0$,

$$h(t)=f(tx)=\begin{cases}-|t|\cdot\|x\|\cdot g(\hat{x})=tf(x) & t<0,\\ |t|\cdot\|x\|\cdot g(\hat{x})=tf(x) & t>0.\end{cases}$$

Then h is differentiable when the following limit exists for any $a\in\mathbf{R}$:

$$\lim_{a\to 0}\frac{h(t+a)-h(t)}{a}.$$

But we have,

$$\begin{aligned}\lim_{a\to 0}\frac{h(t+a)-h(t)}{a}&=\lim_{a\to 0}\frac{(t+a)f(x)-tf(x)}{a}\\&=f(x).\end{aligned}$$

The limit always exists and is equal to the derivative of h at t.

(b) Show that f is not differentiable at $(0,0)$ unless $g=0$.

Solution:

4. [2-8] Let $f:\mathbf{R}\to\mathbf{R}^2$. Prove that f is differentiable at $a\in\mathbf{R}$ if and only if f_1 and f_2 are, and that in this case

$$f'(a)=\begin{pmatrix}f_1'(a)\\ f_2'(a)\end{pmatrix}.$$

Solution: If $f:\mathbf{R}\to\mathbf{R}^2$ is differentiable at a, then for some linear transformation $\lambda:\mathbf{R}\to\mathbf{R}^2$,

$$\lim_{h\to 0}\frac{\|f(a+h)-f(a)-\lambda(h)\|}{|h|}=0$$

So, for every $\varepsilon>0$, there exists a $\delta>0$ such that

$$0<|h|<\delta\implies\|f(a+h)-f(a)\|<|h|\varepsilon.$$

But this means that each f_1 and f_2 satisfies $\|f_i(a+h)-f_i(a)\|<|h|\varepsilon$. So each f_i is differentiable. The converse follows similarly.

Then,

$$\frac{\|f(a+h)-f(a)-\lambda(h)\|}{|h|} = \left\| \begin{pmatrix} \dfrac{f_1(a+h)-f(a)-\lambda_1(h)}{h} \\ \dfrac{f_2(a+h)-f(a)-\lambda_2(h)}{h} \end{pmatrix} \right\|$$

Taking limits on both sides, we see that each of the component of the right hand side must approach to 0. We also have $f_i'(a) = \lambda_i(h)/|h|$. Hence the required expression for $f'(a)$ follows.

$$|h|f'(a) = \lambda(h) = \begin{pmatrix} \lambda_1(h) \\ \lambda_2(h) \end{pmatrix} = \begin{pmatrix} |h|f_1'(a) \\ |h|f_2'(a) \end{pmatrix}.$$

Theorem 0.4. Corollary from the book
If $f, g : \mathbf{R}^n \to \mathbf{R}$ are differentiable at a,

$$D(f+g)(a) = Df(a) + Dg(a)$$
$$D(f \cdot g)(a) = g(a)Df(a) + f(a)Dg(a)$$

If, moreover, $g(a) \neq 0$, then

$$D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{[g(a)]^2}$$

Proof. The first one is done in the text. So we'll do the second and the third one.
So, using the notations from the text, since $f \cdot g = p \circ (f, g)$,

$$\begin{aligned} D(f.g)(a) &= Dp(f(a), g(a)) \circ D(f, g)(a) \\ &= Dp(f(a), g(a))(Df(a), Dg(a)) \\ &= g(a)Df(a) + f(a)Dg(a) \end{aligned}$$

The third relation follows from the above product rule and 0.5. □

Lemma 0.5. *If $q : \mathbf{R} \to \mathbf{R}, g : \mathbf{R}^n \to \mathbf{R}$ is defined by $q(x) = \frac{1}{g}(x)$, then*

$$Dq(a) = -\frac{Dg(a)}{[g(a)]^2}.$$

Proof. We have, $q(x) \cdot g(x) = 1$. Then, $D(1) = q(x)Dg(x) + g(x)Dq(x)$. Substituting $q(x) = 1/g(x)$ gives the required result. □

1. [2-12] A function $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^p$ is **bilinear** if for $x, x_1, x_2 \in \mathbf{R}^n$, $y, y_1, y_2 \in \mathbf{R}^m$, and $a \in \mathbf{R}$ we have

$$\begin{aligned} f(ax, y) &= af(x, y) = f(x, ay), \\ f(x_1 + x_2, y) &= f(x_1, y) + f(x_2, y), \\ f(x, y_1 + y_2) &= f(x, y_1) + f(x, y_2). \end{aligned}$$

(a) Prove that if f is bilinear, then

$$\lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} = 0$$

Solution: Let $h = \alpha_1 e_1 + ... + \alpha_n e_n$ and $k = \beta_1 e_1 + ... + \beta_m e_m$. Since f is bilinear, we can write $f(h,k)$ as

$$f(h,k) = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \beta_j f(e_i, e_j).$$

Let $\alpha = \max\{|\alpha_i| : i = 1, ..., n\}$ and $\beta = \max\{|\beta_i| : i = 1, ..., m\}$. Then $\alpha \le \|h\|$ and $\beta \le \|k\|$. Also let $M = \max\{\|f(e_i, e_j)\| : i = 1, ..., n \text{ and } j = 1, ..., m\}$. So we have,

$$\begin{aligned}\lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} &\le \lim_{(h,k)\to 0} \frac{|mn\alpha\beta M|}{\|(h,k)\|} \\ &\le \lim_{(h,k)\to 0} \frac{mnM\|h\|\|k\|}{\|(h,k)\|}\end{aligned}$$

Now,

$$\|h\|\|k\| \le \begin{cases} \|h\|^2 & \text{if } \|k\| \le \|h\|, \\ \|k\|^2 & \text{if } \|h\| \le \|k\|. \end{cases}$$

Hence $\|h\|\|k\| \le \|h\|^2 + \|k\|^2$ and since, $\|(h,k)\| = \sqrt{\|h\|^2 + \|k\|^2}$ we have

$$\lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} \le \lim_{(h,k)\to 0} \frac{mnM(\|h\|^2 + \|k\|^2)}{\sqrt{\|h\|^2 + \|k\|^2}} = \lim_{(h,k)\to 0} mnM\sqrt{\|h\|^2 + \|k\|^2} = 0.$$

(b) Prove that $Df(a,b)(x,y) = f(a,y) + f(x,b)$.

Solution: Here, for $a, h \in \mathbf{R}^n$ and $b, k \in \mathbf{R}^m$,

$$\begin{aligned}&\lim_{(h,k)\to 0} \frac{\|f(a+h, b+k) - f(a,b) - \lambda(h,k)\|}{\|(h,k)\|} \\ &\le \lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} + \lim_{(h,k)\to 0} \frac{\|f(a,b) + f(a,k) + f(h,b) - f(a,b) - \lambda(h,k)\|}{\|(h,k)\|}\end{aligned}$$

Then certainly, $\lambda(h,k) = f(a,k) + f(h,b)$ implies,

$$\lim_{(h,k)\to 0} \frac{\|f(a+h, b+k) - f(a,b) - \lambda(h,k)\|}{\|(h,k)\|} = 0.$$

Hence, $Df(a,b)(x,y) = f(a,y) + f(x,b)$.

(c) When $p : \mathbf{R}^2 \to \mathbf{R}$ is defined by $p(x,y) = x \cdot y$, then

$$Dp(a,b)(x,y) = bx + ay.$$

Show that the above formula is a special case of b.

Solution: It's clear that p is a bilinear function. Hence the derivative of p at (a, b) is given by,

$$\begin{aligned} Dp(a,b)(x,y) &= p(a,y) + p(x,b) \\ &= ay + bx. \end{aligned}$$

2. [2-13] Define $IP : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by $IP(x, y) = \langle x, y \rangle$.

(a) Find $D(IP)(a, b)$ and $(IP)'(a, b)$.

Solution: Since inner product is a bilinear function, we have

$$D(IP)(a,b)(x,y) = IP(a,y) + IP(x,b) = \langle a, y\rangle + \langle x, b\rangle.$$

And since $\langle a, y\rangle + \langle x, b\rangle = a_1y_1 + ... + a_ny_n + b_1x_1 + ... + b_nx_n$, we have

$$(IP)'(a,b) = (b_1, ..., b_n, a_1, ..., a_n) = (b, a).$$

(b) If $f, g : \mathbf{R} \to \mathbf{R}^n$ are differentiable and $h : \mathbf{R} \to \mathbf{R}$ is defined by $h(t) = \langle f(t), g(t)\rangle$, show that

$$h'(a) = \langle f'(a)^T, g(a)\rangle + \langle f(a), g'(a)^T\rangle.$$

Solution: Here, we can write h as $h = (IP) \circ (f, g)$. Then

$$\begin{aligned} h'(a) &= (IP)'(f(a), g(a)) \cdot \begin{pmatrix} f'(a) \\ g'(a) \end{pmatrix} \\ &= (g(a), f(a)) \cdot \begin{pmatrix} f'(a) \\ g'(a) \end{pmatrix} \\ &= \sum_{i=1}^{n} g_i(a) \cdot f_i'(a) + \sum_{i=1}^{n} f_i(a) \cdot g_i'(a) \\ &= \langle f'(a)^T, g(a)\rangle + \langle f(a), g'(a)^T\rangle. \end{aligned}$$

(c) If $f : \mathbf{R} \to \mathbf{R}^n$ is differentiable function and $\|f(t)\| = 1$ for all t, show that $\langle f'(t)^T, f(t)\rangle = 0$.

Solution: Define $h : \mathbf{R} \to \mathbf{R}$ by $h = \langle f(t), f(t)\rangle$. Then

$$h(t) = \|f(t)\|^2 = 1$$

From b, we have

$$h'(t) = 2\langle f'(t)^T, f(t)\rangle.$$

Since $h(t) = 1$, we have $h'(t) = 0$ and the required result follows.

(d) Exhibit a differentiable function $f : \mathbf{R} \to \mathbf{R}$ such that the function $\|f\|$ defined by $\|f\|(t) = \|f(t)\|$ is not differentiable.

Solution: $f(t) = t$ is differentiable. $\|f\|(t) = \|f(t)\| = |t|$ is not differentiable.

3. [2-14] Let $E_i, i = 1, ..., k$ be Euclidean spaces of various dimensions. A function $f : E_1 \times ... \times E_k \to \mathbf{R}^p$ is called **multilinear** if for each choice of $x_j \in E_j, j \neq i$ the function $g : E_i \to \mathbf{R}^p$ defined by $g(x) = f(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_k)$ is a linear transformation.

(a) If f is multilinear and $i \neq j$, show that for $h = (h_1, ..., h_k)$, with $h_l \in E_l$, we have

$$\lim_{h \to 0} \frac{\|f(a_1, ..., h_i, ..., h_j, ..., a_k)\|}{\|h\|} = 0$$

Solution: Define $g : E_i \times E_j \to \mathbf{R}^p$ by $g(x, y) = f(a_1, ..., x, ..., y, ...a_k)$. Then we see that g is bilinear since

$$\begin{aligned} g(x_1 + x_2, y) &= f(a_1, ..., x_1 + x_2, ..., y, ..., a_k) \\ &= f(a_1, ..., x_1, ..., y, ..., a_k) + f(a_1, ..., x_2, ..., y, ..., a_k) \\ &= g(x_1, y) + g(x_2, y) \end{aligned}$$

and similarly,

$$\begin{aligned} g(kx, y) &= f(a_1, ..., kx, ..., y, ..., a_k) \\ &= kf(a_1, ..., x, ..., y, ..., a_k) = kg(x, y). \end{aligned}$$

(b) Prove that

$$Df(a_1, ..., a_k)(x_1, ..., x_k) = \sum_{i=1}^{k} f(a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_k).$$

Solution:

1. [2-24] Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq 0, \\ 0 & (x, y) = 0. \end{cases}$$

(a) Show that $D_2 f(x, 0) = x$ for all x and $D_1 f(0, y) = -y$ for all y.

Solution: Here,

$$\begin{aligned} D_2 f(x, 0) &= \lim_{h \to 0} \frac{f(x, h) - f(x, 0)}{h} \\ &= \lim_{h \to 0} \frac{xh\dfrac{x^2 - h^2}{x^2 + h^2} - 0}{h} \\ &= \lim_{h \to 0} \frac{x(x^2 - h^2)}{x^2 + h^2} \\ &= x. \end{aligned}$$

Similarly, $D_2 f(0, y) = -y$.

(b) Show that $D_{1,2}f(0,0) \neq D_{2,1}f(0,0)$.

> **Solution:** We have,
>
> $$\begin{aligned} D_{1,2}f(0,0) &= D_2(D_1)(f(0,0)) \\ &= \lim_{h\to 0} \frac{D_1(0,h) - D_1(0,0)}{h} \\ &= \lim_{h\to 0} \frac{-h}{h} = -1. \end{aligned}$$
>
> Similarly, we get $D_{2,1}f(0,0) = 1$.

2. [1-25] Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} e^{-x^{-2}} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that f is a C^∞ function and $f^{(i)}(0) = 0$ for all i.

> **Solution:** We know that,
>
> $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
>
> exists and equals $\frac{2}{x^3}e^{x^{-2}}$ when $x \neq 0$. The first derivative is given by
>
> $$f'(x) = \frac{2}{x^3}f(x).$$
>
> The subsequent derivatives $f^n(x)$ can be then found by the quotient rule, which guarantees that there won't be more zeroes in the denominator of the derivatives except $x = 0$. Hence if the first derivative is continuous at $x = 0$, the derivatives of all order exist. Now, we'll prove the existence of derivative at $x = 0$. So when $x = 0$,
>
> $$f'(x) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{1/h}{e^{h^{-2}}}.$$
>
> Then by L'Hospital rule we have,
>
> $$f'(0) = \lim_{h\to 0} \frac{-1/h^2}{(2/h^3)f(h)} = \lim_{h\to 0} \frac{-h}{2f(h)} = 0.$$
>
> The derivative clearly exists at $x = 0$. To show the continuity, we observe that,
>
> $$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{2}{x^3}e^{-x^{-2}} = 0 = f'(0)$$
>
> follows from the L'Hospital rule.

3. [2-26] Let

$$f(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & x \in (-1,1), \\ 0 & x \notin (-1,1). \end{cases}$$

(a) Show that $f : \mathbf{R} \to \mathbf{R}$ is a C^∞ function which is positive on $(-1,1)$ and 0 elsewhere.

Solution:

1. [2-28 d] $F(x, y) = f(x, g(x), h(x, y))$

Solution: Define the functions k and $\bar{g}$ by

$$\begin{aligned} k(x, y) &= x, \\ \bar{g}(x, y) &= g(x). \end{aligned}$$

Then $F(x, y) = f(k(x, y), \bar{g}(x, y), h(x, y))$. We also have,

$$\begin{aligned} D_1 k(x, y) &= 1, & D_2 k(x, y) &= 0, \\ D_1 \bar{g}(x, y) &= g'(x), & D_2 \bar{g}(x, y) &= 0. \end{aligned}$$

Hence by theorem 2-9, letting $a = (x, g(x), h(x, y))$,

$$\begin{aligned} D_1 F(x, y) &= D_1 f(a) \cdot D_1 k(x, y) + D_2 f(a) \cdot D_1 \bar{g}(x, y) + D_3 f(a) \cdot D_1 h(x, y) \\ &= D_1 f(a) + D_2 f(a) \cdot g'(x) + D_3 f(a) \cdot D_1 h(x, y) \\ D_2 F(x, y) &= D_1 f(a) \cdot D_2 k(x, y) + D_2 f(a) \cdot D_2 \bar{g}(x, y) + D_3 f(a) \cdot D_2 h(x, y) \\ &= D_3 f(a) \cdot D_2 h(x, y). \end{aligned}$$

2. [2-29] Let $f : \mathbf{R}^n \to \mathbf{R}$. For $x \in \mathbf{R}^n$, the limit

$$\lim_{t \to 0} \frac{f(a + tx) - f(a)}{t},$$

if it exists, is denoted $D_x f(a)$, and is called the **directional** derivarive of f at a, in the direction x.

(a) Show that $D_{e_i} f(a) = D_i f(a)$.

Solution: Here,

$$\begin{aligned} D_{e_i} f(a) &= \lim_{t \to 0} \frac{f((a_1, ..., a_i, ..., a_n) + (0, ..., t, ..., 0)) - f(a_1, ..., a_i, ..., a_n)}{t} \\ &= \lim_{t \to 0} \frac{f(a_1, ..., a_i + t, ..., a_n) - f(a_1, ..., a_i, ..., a_n)}{t} = D_i f(a). \end{aligned}$$

(b) Show that $D_{tx} f(a) = t D_x f(a)$.

Solution: Since $th \to 0$ as $h \to 0$, for $t, h \in \mathbf{R}$, we have

$$D_{tx} f(a) = t \lim_{h \to 0} \frac{f(a + thx) - f(a)}{th} = t D_x f(a).$$

(c) If f is differentiable at a, show that $D_x f(a) = Df(a)(x)$, and therefore $D_{x+y}f(a) = D_x f(a) + D_y f(a)$.

> **Solution:** Here, if $x = \lambda_1 e_1 + ... + \lambda_n e_n$, then

3. [2-32]

(a) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that f is differentiable at 0 but f' is not continuous at 0.

> **Solution:** At $x = 0$,
> $$Df(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = h \sin \frac{1}{h}.$$
> But,
> $$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$
> doesn't exist at $x = 0$, and hence f' is not continuous.

(b) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin \dfrac{1}{\sqrt{x^2 + y^2}} & (x, y) \neq 0, \\ 0 & (x, y) = 0. \end{cases}$$

Show that f is differentiable at $(0, 0)$ but $D_i f$ is not continuous at $(0, 0)$.

> **Solution:** If $h = \sqrt{x^2 + y^2}$, define $g : \mathbf{R} \to \mathbf{R}$ as $g(h) = f(x, y)$. Then by (a), $g(h) = f(x, y)$ is differentiable at $(0, 0)$ but not continuous at that point.

4. [2-33.5] If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^n$, find $Df(x)$.

> **Solution:** Here
> $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{(x+h-x)((x+h)^{n-1} + x(x+h)^{n-2} + ... + x^{n-1})}{h} \\ &= \lim_{h\to 0} \left[(x+h)^{n-1} + x(x+h)^{n-2} + ... + x^{n-1}\right] \\ &= nx^{n-1}. \end{aligned}$$

5. [2-34] A function $f : \mathbf{R}^n \to \mathbf{R}$ is **homogenous** of degree m if $f(tx) = t^m f(x)$ for all x. If f is also differentiable, show that

$$\sum_{i=1}^{n} x_i D_i f(x) = mf(x).$$

Solution: Let a function $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(t) = f(tx) = t^m f(x)$. Then

$$mt^{m-1} f(x) = g'(t) = f'(tx).$$

So when $t = 1$,

$$mf(x) = g'(1) = f'(x) = \sum_{i=1}^{n} x_i D_i f(x).$$

$\square$

1. Let $A \subset \mathbf{R}^n$ be an open set and $f : A \to \mathbf{R}^n$ a continuously differentiable 1-1 function such that $\det f'(x) \neq 0$ for all x. Show that $f(A)$ is an open set and $f^{-1}(A) : f(A) \to A$ is differentiable. Show also that $f(B)$ is open for any open set in A.

Solution: