Calculus on Manifolds

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1 1-30

1. Let $f : [a, b] \to \mathbf{R}$ be an increasing function. If $x_1, ..., x_n \in [a, b]$ are distinct, show that

$$\sum_{i=1}^{n} o(f, x_i) < f(b) - f(a).$$

Solution: Let order be defined in $\{x_1, ..., x_n\}$ such that $x_1 < ... < x_n$, then since f is an increasing function, we get $f(x_1) \leq ... \leq f(x_n)$. We have

$$o(f, x_i) = \lim_{\delta \to 0} (M(x_i, f, \delta) - m(x_i, f, \delta))$$

where,

$$M(x_i, f, \delta) = \sup \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\},$$
$$m(x_i, f, \delta) = \inf \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\}.$$

If we denote a δ-neighborhood of some $x_i \in [a, b]$ by $N_\delta(x_i)$, then since $\delta \to 0$, we can choose a sufficiently small $\delta > 0$ such that

$$\bigcap_{i=1}^{n} N_\delta(x_i) = \phi.$$

Then for all such δ we have,

$$f(x_{i+1}) \geq M(x_i, f, \delta) \geq f(x_{i-1}), \text{ and}$$
$$f(x_{i+1}) \geq m(x_i, f, \delta) \geq f(x_{i-1}).$$

We simplify the given summation as

$$\begin{aligned}
\sum_{i=1}^{n} o(f, x_i) &= \lim_{\delta \to 0} \sum_{i=1}^{n} (M(x_i, f, \delta) - m(x_i, f, \delta)) \\
&\leq \lim_{\delta \to 0} \sum_{i=1}^{n} (f(x_{i+1}) - f(x_{i-1})) \\
&= f(x_{n+1}) - f(x_0), \\
&\leq f(b) - f(a).
\end{aligned}$$

where the last statement follows from the fact that the max and min $f(x)$ can get is $f(b)$ and $f(a)$. ■

2 1-25

1. Prove that a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous.

Solution:

Proof. We need to show that $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous at all $a \in \mathbf{R}^n$. That is, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that $0 < \|x - a\| < \delta \implies \|T(x) - T(a)\| < \varepsilon$, where $x \in \mathbf{R}^n$. But we have,

$$\|T(x) - T(a)\| = \|T(x - a)\| \leq M\|x - a\|$$

for some $M \in \mathbf{R}$. So for any given $\varepsilon > 0$, we can choose $\delta = \varepsilon/M$. Then certainly, if $0 < \|x-a\| < \delta$, then

$$\|T(x) - T(a)\| \leq M\|x - a\| < M\delta = \varepsilon.$$

So it follows that the linear transformation is continuous. ■

3 1-29

1. If A is compact, prove that every continuous function $f : A \to \mathbf{R}$ takes on a maximum and a minimum value.

Solution:

Proof. Since A is compact and f is continuous, we know that the image of A under f is compact in $\mathbf{R}$. Hence, it follows from 3.1, that f takes on a maximum and a minimum value. ■

Lemma 3.1. *A compact set in* $\mathbf{R}$ *has a maximum and a minimum value.*

Proof. We know that a compact set is closed and bounded and in $\mathbf{R}$, a compact set is in the form $[a, b]$. And since $a, b \in [a, b]$, all we need to show is that a and b are infimum and supremum, respectively, of the given interval. ■

4 2-4

Let g be a continuous real-valued function on the unit circle $x \in \mathbf{R}^2 : \|x\| = 1$ such that $g(0, 1) = g(1, 0) = 0$ and $g(-x) = -g(x)$. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x) = \begin{cases} \|x\| \cdot g\left(\frac{x}{\|x\|}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

1. If $x \in \mathbf{R}^2$ and $h : \mathbf{R} \to \mathbf{R}$ is defined by $h(t) = f(tx)$, show that h is differentiable.

Solution: We need to show that for every $a \in \mathbf{R}$, there exists a $\lambda : \mathbf{R} \to \mathbf{R}$ such that

$$\lim_{t\to 0} \frac{h(a + t) - h(a) - \lambda(t)}{t} = 0.$$

We see that

$$h(a + t) = f(ax + tx) = \begin{cases} \|(a + t)x\| \cdot g\left(\frac{x}{\|x\|}\right) & (a + t)x \neq 0, \\ 0 & (a + t)x = 0. \end{cases}$$

This follows from the fact that $g\left(\frac{(a+t)x}{\|(a+t)x\|}\right) = g\left(\frac{\pm x}{\|x\|}\right) = g\left(\frac{x}{\|x\|}\right)$.

5

Theorem 5.1 (Heine-Borel Theorem). *The closed interval* $[a, b]$ *is compact.*

Proof. If $\mathscr{O}$ is an open cover of $[a, b]$, let

$$A = \{x : a \leq x \leq b \text{ and } [a, x] \text{ is covered by some finite number of open sets in } \mathscr{O}\}$$

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