Calculus on Manifolds

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1 1-1

2 1-2

Theorem 2.1 (Heine-Borel Theorem)**.** *The closed interval $[a, b]$ is compact.*

Proof. If $\mathscr{O}$ is an open cover of $[a, b]$, let

$$A = \{x : a \leq x \leq b \text{ and } [a, x] \text{ is covered by some finite number of open sets in } \mathscr{O}\}.$$

We know that $a \in A$ since we can choose any open set in $\mathscr{O}$ containing a. A certainly has a least upper bound since A is bounded above by b. So, we will show that if some α is the least upper bound of A, then $\alpha \in A$ and $\alpha = b$.

Since $\alpha = \sup A$, for every $x \in A$, there exists an ε such that $\alpha - x < \varepsilon$. Since $[a, x]$ is covered by some finite number of open sets, we can choose any open ε-neighborhood centered at α. Hence, we see that $[a, \alpha]$ is also covered by finitely many open sets. This shows $\alpha \in A$.

To show that $\alpha = b$, assume that $\alpha < b$. Since, we can find some x' between α and b, such that x' is contained in some open neighborhood around α, we see that $[a, \alpha]$ is covered by a single open set. Then certainly, $x' \in A$. But this contradicts that $\alpha = \sup A$. Hence, $\alpha = b$. $\square$

3 1-3

Definition 3.1. Let X and Y be metric spaces, with metrics d_X and d_Y, respectively. Then, we say that a function $f : X \to Y$ is **continuous at the point** $x_0 \in X$ if for each open set $U \subset Y$, $f(x_0) \in U$, there is an open set $V \subset X$, $x_0 \in V$ such that $f(V) \subset U$.

Theorem 3.1. *If $A \subset \mathbf{R}^n$, a function $f : A \to \mathbf{R}^m$ is continuous if and only if for every open set $U \subset \mathbf{R}^m$ there is some open set $V \subset \mathbf{R}^n$ such that $f^{-1}(U) = V \cap A$.*

Proof. Suppose f is continuous. If $a \in f^{-1}(U)$, then $f(a) \in U$. Since U is open, there is some open ball $B \subset U$ such that $f(a) \in B$. And since, f is continuous at a, we know that $f(x) \in B$ provided we choose a sufficiently small open ball C such that $a \in C$. If we do this for each $a \in f^{-1}(U)$ and call their union V (also an open set), then clearly $f^{-1}(U) = V \cap A$.

$\square$

1. [1-23] If $f : A \to \mathbf{R}^m$ and $a \in A$, show that $\lim_{x\to a} f(x) = b$ if and only if $\lim_{x\to a} f_i(x) = b_i$ for $i = 1, ..., m$.

> **Solution:** If $\lim_{x\to a} f(x) = b$, then for every $\varepsilon > 0$, we can find a $\delta > 0$ such that
>
> $$0 < \|x - a\| < \delta \implies \|f(x) - b\| < \varepsilon.$$
>
> Then we have
>
> $$\begin{aligned} &\sum_{i=1}^{m} (f_i(x) - b_i)^2 < \varepsilon^2 \\ \implies &|f_i(x) - b_i| < \varepsilon \quad \text{for } i = 1, ..., m. \end{aligned}$$
>
> This implies that $\lim_{x\to a} f_i(x) = b_i$ for $i = 1, ..., m$.
> Now, if $\lim_{x\to a} f_i(x) = b_i$ for $i = 1, ..., m$ then for every $\varepsilon > 0$, we can find $\delta_1, ..., \delta_m$ such that
>
> $$0 < \|x - a\| < \delta_i \implies \|f_i(x) - b_i\| < \varepsilon \text{ for } i = 1, ..., m.$$
>
> Then, for $\delta = \min \delta_i,$ for $i = 1, ..., m$, we have,
>
> $$\begin{aligned} 0 < \|x - a\| < \delta \implies &\sum_{i=1}^{m} \|f_i(x) - b_i\|^2 < m\varepsilon^2 \\ \implies &\|f(x) - b\| < \sqrt{m}\varepsilon. \end{aligned}$$
>
> Hence, $\lim_{x\to a} f(x) = b$

2. [1-24] Prove that $f : A \to \mathbf{R}^m$ is continuous if and only if each f_i is.

> **Solution:** If $f : A \to \mathbf{R}^m$ is continuous then for all $a \in A$, $\lim_{x\to a} f(x) = f(a)$. But this means that for all $a \in A$, $lim_{x\to a} f_i(x) = f_i(a)$. Hence, each $f_i(x)$ is continuous.
> Similarly, converse follows from [1-23].

3. [1-25] Prove that a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous.

> **Solution:**
>
> *Proof.* We need to show that $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous at all $a \in \mathbf{R}^n$. That is, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that $0 < \|x - a\| < \delta \implies \|T(x) - T(a)\| < \varepsilon$, where $x \in \mathbf{R}^n$. But we have,
>
> $$\|T(x) - T(a)\| = \|T(x - a)\| \leq M\|x - a\|$$
>
> for some $M \in \mathbf{R}$. So for any given $\varepsilon > 0$, we can choose $\delta = \varepsilon/M$. Then certainly, if $0 < \|x-a\| < \delta$, then
>
> $$\|T(x) - T(a)\| \leq M\|x - a\| < M\delta = \varepsilon.$$
>
> So it follows that the linear transformation is continuous. $\square$

4. [1-29] If A is compact, prove that every continuous function $f : A \to \mathbf{R}$ takes on a maximum and a minimum value.

Solution:

Proof. Since A is compact and f is continuous, we know that the image of A under f is compact in $\mathbf{R}$. Hence, it follows from 3.1, that f takes on a maximum and a minimum value. $\square$

Lemma 3.2. *A compact set in* $\mathbf{R}$ *has a maximum and a minimum value.*

Proof. We know that a compact set is closed and bounded and in $\mathbf{R}$, a compact set is in the form $[a, b]$. And since $a, b \in [a, b]$, all we need to show is that a and b are infimum and supremum, respectively, of the given interval. $\square$

5. [1-30] Let $f : [a, b] \to \mathbf{R}$ be an increasing function. If $x_1, ..., x_n \in [a, b]$ are distinct, show that

$$\sum_{i=1}^{n} o(f, x_i) \leq f(b) - f(a).$$

Solution: Let order be defined in $\{x_1, ..., x_n\}$ such that $x_1 < ... < x_n$, then since f is an increasing function, we get $f(x_1) \leq ... \leq f(x_n)$. We have

$$o(f, x_i) = \lim_{\delta \to 0} \left(M(x_i, f, \delta) - m(x_i, f, \delta)\right)$$

where,

$$\begin{aligned} M(x_i, f, \delta) &= \sup\{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\}, \\ m(x_i, f, \delta) &= \inf\{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\}. \end{aligned}$$

If we denote a δ-neighborhood of some $x_i \in [a, b]$ by $N_\delta(x_i)$, then since $\delta \to 0$, we can choose a sufficiently small $\delta > 0$ such that

$$\bigcap_{i=1}^{n} N_\delta(x_i) = \phi.$$

Then for all such δ we have,

$$\begin{aligned} f(x_{i+1}) \geq M(x_i, f, \delta) \geq f(x_{i-1}), &\text{ and} \\ f(x_{i+1}) \geq m(x_i, f, \delta) \geq f(x_{i-1}). \end{aligned}$$

We simplify the given summation as

$$\begin{aligned} \sum_{i=1}^{n} o(f, x_i) &= \lim_{\delta \to 0} \sum_{i=1}^{n} \left(M(x_i, f, \delta) - m(x_i, f, \delta)\right) \\ &\leq \lim_{\delta \to 0} \sum_{i=1}^{n} \left(f(x_{i+1}) - f(x_{i-1})\right) \\ &= f(x_{n+1}) - f(x_0), \\ &\leq f(b) - f(a). \end{aligned}$$

where the last statement follows from the fact that the max and min $f(x)$ can get is $f(b)$ and $f(a)$. $\square$

4 2-1

1. [2-1] Prove that if $f : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $a \in \mathbf{R}^n$, then it is continuous at a.

> **Solution:** If f is differentiable at $a \in \mathbf{R}^n$, then
>
> $$\lim_{h \to a} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0.$$
>
> But this means that, for any given $\varepsilon_0 > 0$, we can find a $\delta > 0$ such that
>
> $$\begin{aligned} 0 < \|h\| < \delta \implies & \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} < \varepsilon_0 \\ \implies & \|f(a+h) - f(a) - \lambda(h)\| < \varepsilon_0 \|h\| \\ \implies & \|f(a+h) - f(a)\| < (\varepsilon_0 + M)\|h\| \end{aligned}$$
>
> for some $M \in \mathbf{R}$. So, when $h = x - a$ for any given $\varepsilon > 0$, we can choose $0 < \delta < \varepsilon/(\varepsilon_0 + M)$. Then it follows that,
>
> $$0 < \|x - a\| < \delta \implies \|f(x) - f(a)\| < (\varepsilon_0 + M)\delta < \varepsilon.$$
>
> Hence, f is continuous at a.

2. [2-2] A function $f : \mathbf{R}^2 \to \mathbf{R}$ is **independent of the second variable** if for each $x \in \mathbf{R}$ we have $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbf{R}$. Show that f is independent of the second variable if and only if there is a function $g : \mathbf{R} \to \mathbf{R}$ such that $f(x, y) = g(x)$. What is $f'(a, b)$ in terms of g'?

> **Solution:** Define $g(x) = f(x, 0)$. Then for all $y \in \mathbf{R}$, if f is independent of the second variable, we have $f(x, y) = f(x, 0) = g(x)$.
>
> Similarly, since g is independent of y, we have $g(x) = f(x, 0) = f(x, y_1) = f(x, y_2)$.
>
> Now let $z = (h, k)$. Then, assuming that f is differentiable at (a, b), we have
>
> $$\begin{aligned} \lim_{(h,k) \to 0} & \frac{\|f(a+h, b+k) - f(a,b) - Df(a,b)(h,k)\|}{\|(h,k)\|} = 0 \\ \text{or, } \lim_{h \to 0} & \frac{\|g(a+h) - g(a) - Df(a,b)(h,k)\|}{|h|} = 0 \end{aligned}$$
>
> Since $g : \mathbf{R} \to \mathbf{R}$,
>
> $$\begin{aligned} \lim_{h \to 0} \frac{g(a+h) - g(a)}{|h|} &= \lim_{h \to 0} \frac{Df(a,b)(h,k)}{|h|} \\ \text{or, } g'(a) &= \lim_{h \to 0} \frac{Df(a,b)(h,k)}{|h|} \end{aligned}$$
>
> Then we see that
>
> $$Df(a,b)(h,k) = h \cdot g'(a)$$
>
> satisfies the equation. Hence, $f'(a, b) = g'(a)$.

3. [2-4] Let g be a continuous real-valued function on the unit circle $x \in \mathbf{R}^2 : \|x\| = 1$ such that $g(0, 1) = g(1, 0) = 0$ and $g(-x) = -g(x)$. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x) = \begin{cases} \|x\| \cdot g\left(\frac{x}{\|x\|}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(a) If $x \in \mathbf{R}^2$ and $h : \mathbf{R} \to \mathbf{R}$ is defined by $h(t) = f(tx)$, show that h is differentiable.

Solution: We need to show that for every $a \in \mathbf{R}$, there exists a $\lambda : \mathbf{R} \to \mathbf{R}$ such that

$$\lim_{t \to 0} \frac{h(a+t) - h(a) - \lambda(t)}{t} = 0. \tag{1}$$

We see that, when $tx \neq 0$,

$$h(t) = f(tx) = \begin{cases} -|t| \cdot \|(x\| \cdot g(\hat{x}) = tf(x) & t < 0, \\ |t| \cdot \|(x\| \cdot g(\hat{x}) = tf(x) & t > 0. \end{cases}$$

Then h is differentiable when the following limit exists for any $a \in \mathbf{R}$:

$$\lim_{a \to 0} \frac{h(t+a) - h(t)}{a}.$$

But we have,

$$\begin{aligned} \lim_{a \to 0} \frac{h(t+a) - h(t)}{a} &= \lim_{a \to 0} \frac{(t+a)f(x) - tf(x)}{a} \\ &= f(x). \end{aligned}$$

The limit always exists and is equal to the derivative of h at t.

(b) Show that f is not differentiable at $(0, 0)$ unless $g = 0$.

Solution:

4. [2-8] Let $f : \mathbf{R} \to \mathbf{R}^2$. Prove that f is differentiable at $a \in \mathbf{R}$ if and only if f_1 and f_2 are, and that in this case

$$f'(a) = \begin{pmatrix} f_1'(a) \\ f_2'(a) \end{pmatrix}.$$

Solution: If $f : \mathbf{R} \to \mathbf{R}^2$ is differentiable at a, then for some linear transformation $\lambda : \mathbf{R} \to \mathbf{R}^2$,

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{|h|} = 0$$

So, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |h| < \delta \implies \|f(a+h) - f(a)\| < |h|\varepsilon.$$

But this means that each f_1 and f_2 satisfies $\|f_i(a+h) - f_i(a)\| < |h|\varepsilon$. So each f_i is differentiable. The converse follows similarly.

Then,

$$\frac{\|f(a+h) - f(a) - \lambda(h)\|}{|h|} = \left\| \begin{pmatrix} \frac{f_1(a+h) - f(a) - \lambda_1(h)}{h} \\ \frac{f_2(a+h) - f(a) - \lambda_2(h)}{h} \end{pmatrix} \right\|$$

We see that each of the component of the right hand side must be 0. We also have $f_i'(a) = \lambda_i(h)/|h|$. Hence the required expression for $f'(a)$ follows.

$$|h|f'(a) = \lambda(h) = \begin{pmatrix} \lambda_1(h) \\ \lambda_2(h) \end{pmatrix} = \begin{pmatrix} |h|f_1'(a) \\ |h|f_2'(a) \end{pmatrix}.$$