Calculus on Manifolds

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February 3, 2019

1 1-1

2 1-2

Theorem 2.1 (Heine-Borel Theorem). *The closed interval $[a, b]$ is compact.*

Proof. If $\mathscr{O}$ is an open cover of $[a, b]$, let

$$A = \{x : a \leq x \leq b \text{ and } [a, x] \text{ is covered by some finite number of open sets in } \mathscr{O}\}.$$

We know that $a \in A$ since we can choose any open set in $\mathscr{O}$ containing a. A certainly has a least upper bound since A is bounded above by b. So, we will show that if some α is the least upper bound of A, then $\alpha \in A$ and $\alpha = b$.

Since $\alpha = \sup A$, for every $x \in A$, there exists an ε such that $\alpha - x < \varepsilon$. Since $[a, x]$ is covered by some finite number of open sets, we can choose any open ε-neighborhood centered at α. Hence, we see that $[a, \alpha]$ is also covered by finitely many open sets. This shows $\alpha \in A$.

To show that $\alpha = b$, assume that $\alpha < b$. Since, we can find some x' between α and b, such that x' is contained in some open neighborhood around α, we see that $[a, \alpha]$ is covered by a single open set. Then certainly, $x' \in A$. But this contradicts that $\alpha = \sup A$. Hence, $\alpha = b$. ■

3 1-3

Theorem 3.1. *If $A \subset \mathbf{R}^n$, a function $f : A \to \mathbf{R}^m$ is continuous if and only if for every open set $U \subset \mathbf{R}^m$ there is some open set $V \subset \mathbf{R}^n$ such that $f^{-1}(U) = V \cap A$.*

Proof. ■

1. [1-23] If $f : A \to \mathbf{R}^m$ and $a \in A$, show that $\lim_{x \to a} f(x) = b$ if and only if $\lim_{x \to a} f_i(x) = b_i$ for $i = 1, ..., m$.

   Solution: If $\lim_{x \to a} f(x) = b$, then for every $\varepsilon > 0$, we can find a $\delta > 0$ such that

   $$0 < \|x - a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon.$$

Then we have

$$\sum_{i=1}^{m} (f_i(x) - b_i)^2 < \varepsilon$$
$$\implies |f_i(x) - b_i| < \sqrt{\varepsilon} \quad \text{for } i = 1, ..., m$$

This implies that $lim_{x \to a} f(x) = b_i$ for $i = 1, ..., m$.

2. [1-25]Prove that a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous.

Solution:

Proof. We need to show that $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous at all $a \in \mathbf{R}^n$. That is, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that $0 < \|x - a\| < \delta \implies \|T(x) - T(a)\| < \varepsilon$, where $x \in \mathbf{R}^n$. But we have,

$$\|T(x) - T(a)\| = \|T(x - a)\| \leq M\|x - a\|$$

for some $M \in \mathbf{R}$. So for any given $\varepsilon > 0$, we can choose $\delta = \varepsilon/M$. Then certainly, if $0 < \|x - a\| < \delta$, then

$$\|T(x) - T(a)\| \leq M\|x - a\| < M\delta = \varepsilon.$$

So it follows that the linear transformation is continuous. ∎

3. [1-29]If A is compact, prove that every continuous function $f : A \to \mathbf{R}$ takes on a maximum and a minimum value.

Solution:

Proof. Since A is compact and f is continuous, we know that the image of A under f is compact in $\mathbf{R}$. Hence, it follows from 3.1, that f takes on a maximum and a minimum value. ∎

Lemma 3.2. *A compact set in* $\mathbf{R}$ *has a maximum and a minimum value.*

Proof. We know that a compact set is closed and bounded and in $\mathbf{R}$, a compact set is in the form $[a, b]$. And since $a, b \in [a, b]$, all we need to show is that a and b are infimum and supremum, respectively, of the given interval. ∎

4. [1-30] Let $f : [a, b] \to \mathbf{R}$ be an increasing function. If $x_1, ..., x_n \in [a, b]$ are distinct, show that

$$\sum_{i=1}^{n} o(f, x_i) \leq f(b) - f(a).$$

Solution: Let order be defined in $\{x_1, ..., x_n\}$ such that $x_1 < ... < x_n$, then since f is an increasing function, we get $f(x_1) \leq ... \leq f(x_n)$. We have

$$o(f, x_i) = \lim_{\delta \to 0} (M(x_i, f, \delta) - m(x_i, f, \delta))$$

where,

$$M(x_i, f, \delta) = \sup\{f(x) : x \in [a,b] \text{ and } \|x - x_i\| < \delta\},$$
$$m(x_i, f, \delta) = \inf\{f(x) : x \in [a,b] \text{ and } \|x - x_i\| < \delta\}.$$

If we denote a δ-neighborhood of some $x_i \in [a,b]$ by $N_\delta(x_i)$, then since $\delta \to 0$, we can choose a sufficiently small $\delta > 0$ such that

$$\bigcap_{i=1}^{n} N_\delta(x_i) = \phi.$$

Then for all such δ we have,

$$f(x_{i+1}) \geq M(x_i, f, \delta) \geq f(x_{i-1}), \text{ and}$$
$$f(x_{i+1}) \geq m(x_i, f, \delta) \geq f(x_{i-1}).$$

We simplify the given summation as

$$\begin{aligned}\sum_{i=1}^{n} o(f, x_i) &= \lim_{\delta \to 0} \sum_{i=1}^{n} (M(x_i, f, \delta) - m(x_i, f, \delta)) \\ &\leq \lim_{\delta \to 0} \sum_{i=1}^{n} (f(x_{i+1}) - f(x_{i-1})) \\ &= f(x_{n+1}) - f(x_0), \\ &\leq f(b) - f(a).\end{aligned}$$

where the last statement follows from the fact that the max and min $f(x)$ can get is $f(b)$ and $f(a)$.
■

4 2-1

1. [2-1]Prove that if $f : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $a \in \mathbf{R}^n$, then it is continuous at a.

Solution: Hi.

2. [2-4]Let g be a continuous real-valued function on the unit circle $x \in \mathbf{R}^2 : \|x\| = 1$ such that $g(0,1) = g(1,0) = 0$ and $g(-x) = -g(x)$. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x) = \begin{cases} \|x\| \cdot g\left(\frac{x}{\|x\|}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(a) If $x \in \mathbf{R}^2$ and $h : \mathbf{R} \to \mathbf{R}$ is defined by $h(t) = f(tx)$, show that h is differentiable.

Solution: We need to show that for every $a \in \mathbf{R}$, there exists a $\lambda : \mathbf{R} \to \mathbf{R}$ such that

$$\lim_{t \to 0} \frac{h(a+t) - h(a) - \lambda(t)}{t} = 0. \tag{1}$$

We see that

$$h(a+t) = f(ax + tx) = \begin{cases} \|(a+t)x\| \cdot g\left(\frac{x}{\|x\|}\right) & (a+t)x \neq 0, \\ 0 & (a+t)x = 0. \end{cases}$$

This follows from the fact that $g\left(\frac{(a+t)x}{\|(a+t)x\|}\right) = g\left(\frac{\pm x}{\|x\|}\right) = g\left(\frac{x}{\|x\|}\right)$. We have,

$$\begin{aligned}\frac{h(a+t) - h(a) - \lambda(t)}{t} &= \frac{\|(a+t)x\| \cdot g(\hat{x}) - \|a\| g(\hat{x}) - \lambda(t)}{t} \\ &=\end{aligned}$$