

# Calculus on Manifolds

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## 1 1-1

## 2 1-2

**Theorem 2.1** (Heine-Borel Theorem). *The closed interval  $[a, b]$  is compact.*

*Proof.* If  $\mathcal{O}$  is an open cover of  $[a, b]$ , let

$$A = \{x : a \leq x \leq b \text{ and } [a, x] \text{ is covered by some finite number of open sets in } \mathcal{O}\}.$$

We know that  $a \in A$  since we can choose any open set in  $\mathcal{O}$  containing  $a$ .  $A$  certainly has a least upper bound since  $A$  is bounded above by  $b$ . So, we will show that if some  $\alpha$  is the least upper bound of  $A$ , then  $\alpha \in A$  and  $\alpha = b$ .

Since  $\alpha = \sup A$ , for every  $x \in A$ , there exists an  $\varepsilon$  such that  $\alpha - x < \varepsilon$ . Since  $[a, x]$  is covered by some finite number of open sets, we can choose any open  $\varepsilon$ -neighborhood centered at  $\alpha$ . Hence, we see that  $[a, \alpha]$  is also covered by finitely many open sets. This shows  $\alpha \in A$ .

To show that  $\alpha = b$ , assume that  $\alpha < b$ . Since, we can find some  $x'$  between  $\alpha$  and  $b$ , such that  $x'$  is contained in some open neighborhood around  $\alpha$ , we see that  $[a, \alpha]$  is covered by a single open set. Then certainly,  $x' \in A$ . But this contradicts that  $\alpha = \sup A$ . Hence,  $\alpha = b$ .  $\square$

## 3 1-3

**Definition 3.1.** Let  $X$  and  $Y$  be metric spaces, with metrics  $d_X$  and  $d_Y$ , respectively. Then, we say that a function  $f : X \rightarrow Y$  is **continuous at the point**  $x_0 \in X$  if for each open set  $U \subset Y$ ,  $f(x_0) \in U$ , there is an open set  $V \subset X$ ,  $x_0 \in V$  such that  $f(V) \subset U$ .

**Theorem 3.1.** *If  $A \subset \mathbf{R}^n$ , a function  $f : A \rightarrow \mathbf{R}^m$  is continuous if and only if for every open set  $U \subset \mathbf{R}^m$  there is some open set  $V \subset \mathbf{R}^n$  such that  $f^{-1}(U) = V \cap A$ .*

*Proof.* Suppose  $f$  is continuous. If  $a \in f^{-1}(U)$ , then  $f(a) \in U$ . Since  $U$  is open, there is some open ball  $B \subset U$  such that  $f(a) \in B$ . And since,  $f$  is continuous at  $a$ , we know that  $f(x) \in B$  provided we choose a sufficiently small open ball  $C$  such that  $a \in C$ . If we do this for each  $a \in f^{-1}(U)$  and call their union  $V$  (also an open set), then clearly  $f^{-1}(U) = V \cap A$ .  $\square$

1. [1-23] If  $f : A \rightarrow \mathbf{R}^m$  and  $a \in A$ , show that  $\lim_{x \rightarrow a} f(x) = b$  if and only if  $\lim_{x \rightarrow a} f_i(x) = b_i$  for  $i = 1, \dots, m$ .

**Solution:** If  $\lim_{x \rightarrow a} f(x) = b$ , then for every  $\varepsilon > 0$ , we can find a  $\delta > 0$  such that

$$0 < \|x - a\| < \delta \implies \|f(x) - b\| < \varepsilon.$$

Then we have

$$\begin{aligned} \sum_{i=1}^m (f_i(x) - b_i)^2 &< \varepsilon^2 \\ \implies |f_i(x) - b_i| &< \varepsilon \quad \text{for } i = 1, \dots, m. \end{aligned}$$

This implies that  $\lim_{x \rightarrow a} f_i(x) = b_i$  for  $i = 1, \dots, m$ .

Now, if  $\lim_{x \rightarrow a} f_i(x) = b_i$  for  $i = 1, \dots, m$  then for every  $\varepsilon > 0$ , we can find  $\delta_1, \dots, \delta_m$  such that

$$0 < \|x - a\| < \delta_i \implies \|f_i(x) - b_i\| < \varepsilon \quad \text{for } i = 1, \dots, m.$$

Then, for  $\delta = \min \delta_i$ , for  $i = 1, \dots, m$ , we have,

$$\begin{aligned} 0 < \|x - a\| < \delta &\implies \sum_{i=1}^m \|f_i(x) - b_i\|^2 < m\varepsilon^2 \\ &\implies \|f(x) - b\| < \sqrt{m}\varepsilon. \end{aligned}$$

Hence,  $\lim_{x \rightarrow a} f(x) = b$

2. [1-24] Prove that  $f : A \rightarrow \mathbf{R}^m$  is continuous if and only if each  $f_i$  is.

**Solution:** If  $f : A \rightarrow \mathbf{R}^m$  is continuous then for all  $a \in A$ ,  $\lim_{x \rightarrow a} f(x) = f(a)$ . But this means that for all  $a \in A$ ,  $\lim_{x \rightarrow a} f_i(x) = f_i(a)$ . Hence, each  $f_i(x)$  is continuous.

Similarly, converse follows from [1-23].

3. [1-25] Prove that a linear transformation  $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is continuous.

**Solution:**

*Proof.* We need to show that  $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is continuous at all  $a \in \mathbf{R}^n$ . That is, for every  $\varepsilon > 0$  we can find a  $\delta > 0$  such that  $0 < \|x - a\| < \delta \implies \|T(x) - T(a)\| < \varepsilon$ , where  $x \in \mathbf{R}^n$ . But we have,

$$\|T(x) - T(a)\| = \|T(x - a)\| \leq M\|x - a\|$$

for some  $M \in \mathbf{R}$ . So for any given  $\varepsilon > 0$ , we can choose  $\delta = \varepsilon/M$ . Then certainly, if  $0 < \|x - a\| < \delta$ , then

$$\|T(x) - T(a)\| \leq M\|x - a\| < M\delta = \varepsilon.$$

So it follows that the linear transformation is continuous. □

4. [1-29] If  $A$  is compact, prove that every continuous function  $f : A \rightarrow \mathbf{R}$  takes on a maximum and a minimum value.

**Solution:**

*Proof.* Since  $A$  is compact and  $f$  is continuous, we know that the image of  $A$  under  $f$  is compact in  $\mathbf{R}$ . Hence, it follows from 3.1, that  $f$  takes on a maximum and a minimum value.  $\square$

**Lemma 3.2.** *A compact set in  $\mathbf{R}$  has a maximum and a minimum value.*

*Proof.* We know that a compact set is closed and bounded and in  $\mathbf{R}$ , a compact set is in the form  $[a, b]$ . And since  $a, b \in [a, b]$ , all we need to show is that  $a$  and  $b$  are infimum and supremum, respectively, of the given interval.  $\square$

5. [1-30] Let  $f : [a, b] \rightarrow \mathbf{R}$  be an increasing function. If  $x_1, \dots, x_n \in [a, b]$  are distinct, show that

$$\sum_{i=1}^n o(f, x_i) \leq f(b) - f(a).$$

**Solution:** Let order be defined in  $\{x_1, \dots, x_n\}$  such that  $x_1 < \dots < x_n$ , then since  $f$  is an increasing function, we get  $f(x_1) \leq \dots \leq f(x_n)$ . We have

$$o(f, x_i) = \lim_{\delta \rightarrow 0} (M(x_i, f, \delta) - m(x_i, f, \delta))$$

where,

$$\begin{aligned} M(x_i, f, \delta) &= \sup \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\}, \\ m(x_i, f, \delta) &= \inf \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\}. \end{aligned}$$

If we denote a  $\delta$ -neighborhood of some  $x_i \in [a, b]$  by  $N_\delta(x_i)$ , then since  $\delta \rightarrow 0$ , we can choose a sufficiently small  $\delta > 0$  such that

$$\bigcap_{i=1}^n N_\delta(x_i) = \phi.$$

Then for all such  $\delta$  we have,

$$\begin{aligned} f(x_{i+1}) &\geq M(x_i, f, \delta) \geq f(x_{i-1}), \text{ and} \\ f(x_{i+1}) &\geq m(x_i, f, \delta) \geq f(x_{i-1}). \end{aligned}$$

We simplify the given summation as

$$\begin{aligned} \sum_{i=1}^n o(f, x_i) &= \lim_{\delta \rightarrow 0} \sum_{i=1}^n (M(x_i, f, \delta) - m(x_i, f, \delta)) \\ &\leq \lim_{\delta \rightarrow 0} \sum_{i=1}^n (f(x_{i+1}) - f(x_{i-1})) \\ &= f(x_{n+1}) - f(x_0), \\ &\leq f(b) - f(a). \end{aligned}$$

where the last statement follows from the fact that the max and min  $f(x)$  can get is  $f(b)$  and  $f(a)$ .  $\square$

## 4 2-1

1. [2-1] Prove that if  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is differentiable at  $a \in \mathbf{R}^n$ , then it is continuous at  $a$ .

**Solution:** Hi.

2. [2-4] Let  $g$  be a continuous real-valued function on the unit circle  $x \in \mathbf{R}^2 : \|x\| = 1$  such that  $g(0, 1) = g(1, 0) = 0$  and  $g(-x) = -g(x)$ . Define  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  by

$$f(x) = \begin{cases} \|x\| \cdot g\left(\frac{x}{\|x\|}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

- (a) If  $x \in \mathbf{R}^2$  and  $h : \mathbf{R} \rightarrow \mathbf{R}$  is defined by  $h(t) = f(tx)$ , show that  $h$  is differentiable.

**Solution:** We need to show that for every  $a \in \mathbf{R}$ , there exists a  $\lambda : \mathbf{R} \rightarrow \mathbf{R}$  such that

$$\lim_{t \rightarrow 0} \frac{h(a+t) - h(a) - \lambda(t)}{t} = 0. \quad (1)$$

We see that

$$h(a+t) = f(ax + tx) = \begin{cases} \|(a+t)x\| \cdot g\left(\frac{x}{\|x\|}\right) & (a+t)x \neq 0, \\ 0 & (a+t)x = 0. \end{cases}$$

This follows from the fact that  $g\left(\frac{(a+t)x}{\|(a+t)x\|}\right) = g\left(\frac{\pm x}{\|x\|}\right) = g\left(\frac{x}{\|x\|}\right)$ . We have,

$$\begin{aligned} \frac{h(a+t) - h(a) - \lambda(t)}{t} &= \frac{\|(a+t)x\| \cdot g(\hat{x}) - \|a\|g(\hat{x}) - \lambda(t)}{t} \\ &= \end{aligned}$$