

Calculus on Manifolds

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February 5, 2019

1 1-1

2 1-2

Theorem 2.1 (Heine-Borel Theorem). *The closed interval $[a, b]$ is compact.*

Proof. If \mathcal{O} is an open cover of $[a, b]$, let

$$A = \{x : a \leq x \leq b \text{ and } [a, x] \text{ is covered by some finite number of open sets in } \mathcal{O}\}.$$

We know that $a \in A$ since we can choose any open set in \mathcal{O} containing a . A certainly has a least upper bound since A is bounded above by b . So, we will show that if some α is the least upper bound of A , then $\alpha \in A$ and $\alpha = b$.

Since $\alpha = \sup A$, for every $x \in A$, there exists an ε such that $\alpha - x < \varepsilon$. Since $[a, x]$ is covered by some finite number of open sets, we can choose any open ε -neighborhood centered at α . Hence, we see that $[a, \alpha]$ is also covered by finitely many open sets. This shows $\alpha \in A$.

To show that $\alpha = b$, assume that $\alpha < b$. Since, we can find some x' between α and b , such that x' is contained in some open neighborhood around α , we see that $[a, \alpha]$ is covered by a single open set. Then certainly, $x' \in A$. But this contradicts that $\alpha = \sup A$. Hence, $\alpha = b$. \square

3 1-3

Definition 3.1. Let X and Y be metric spaces, with metrics d_X and d_Y , respectively. Then, we say that a function $f : X \rightarrow Y$ is **continuous at the point** $x_0 \in X$ if for each open set $U \subset Y$, $f(x_0) \in U$, there is an open set $V \subset X$, $x_0 \in V$ such that $f(V) \subset U$.

Theorem 3.1. *If $A \subset \mathbf{R}^n$, a function $f : A \rightarrow \mathbf{R}^m$ is continuous if and only if for every open set $U \subset \mathbf{R}^m$ there is some open set $V \subset \mathbf{R}^n$ such that $f^{-1}(U) = V \cap A$.*

Proof. Suppose f is continuous. If $a \in f^{-1}(U)$, then $f(a) \in U$. Since U is open, there is some open ball $B \subset U$ such that $f(a) \in B$. And since, f is continuous at a , we know that $f(x) \in B$ provided we choose a sufficiently small open ball C such that $a \in C$. If we do this for each $a \in f^{-1}(U)$ and call their union V (also an open set), then clearly $f^{-1}(U) = V \cap A$. \square

1. [1-23] If $f : A \rightarrow \mathbf{R}^m$ and $a \in A$, show that $\lim_{x \rightarrow a} f(x) = b$ if and only if $\lim_{x \rightarrow a} f_i(x) = b_i$ for $i = 1, \dots, m$.

Solution: If $\lim_{x \rightarrow a} f(x) = b$, then for every $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$0 < \|x - a\| < \delta \implies \|f(x) - b\| < \varepsilon.$$

Then we have

$$\begin{aligned} \sum_{i=1}^m (f_i(x) - b_i)^2 &< \varepsilon^2 \\ \implies |f_i(x) - b_i| &< \varepsilon \quad \text{for } i = 1, \dots, m. \end{aligned}$$

This implies that $\lim_{x \rightarrow a} f_i(x) = b_i$ for $i = 1, \dots, m$.

Now, if $\lim_{x \rightarrow a} f_i(x) = b_i$ for $i = 1, \dots, m$ then for every $\varepsilon > 0$, we can find $\delta_1, \dots, \delta_m$ such that

$$0 < \|x - a\| < \delta_i \implies \|f_i(x) - b_i\| < \varepsilon \quad \text{for } i = 1, \dots, m.$$

Then, for $\delta = \min \delta_i$, for $i = 1, \dots, m$, we have,

$$\begin{aligned} 0 < \|x - a\| < \delta &\implies \sum_{i=1}^m \|f_i(x) - b_i\|^2 < m\varepsilon^2 \\ &\implies \|f(x) - b\| < \sqrt{m}\varepsilon. \end{aligned}$$

Hence, $\lim_{x \rightarrow a} f(x) = b$

2. [1-24] Prove that $f : A \rightarrow \mathbf{R}^m$ is continuous if and only if each f_i is.

Solution: If $f : A \rightarrow \mathbf{R}^m$ is continuous then for all $a \in A$, $\lim_{x \rightarrow a} f(x) = f(a)$. But this means that for all $a \in A$, $\lim_{x \rightarrow a} f_i(x) = f_i(a)$. Hence, each $f_i(x)$ is continuous.

Similarly, converse follows from [1-23].

3. [1-25] Prove that a linear transformation $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is continuous.

Solution:

Proof. We need to show that $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is continuous at all $a \in \mathbf{R}^n$. That is, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that $0 < \|x - a\| < \delta \implies \|T(x) - T(a)\| < \varepsilon$, where $x \in \mathbf{R}^n$. But we have,

$$\|T(x) - T(a)\| = \|T(x - a)\| \leq M\|x - a\|$$

for some $M \in \mathbf{R}$. So for any given $\varepsilon > 0$, we can choose $\delta = \varepsilon/M$. Then certainly, if $0 < \|x - a\| < \delta$, then

$$\|T(x) - T(a)\| \leq M\|x - a\| < M\delta = \varepsilon.$$

So it follows that the linear transformation is continuous. □

4. [1-29] If A is compact, prove that every continuous function $f : A \rightarrow \mathbf{R}$ takes on a maximum and a minimum value.

Solution:

Proof. Since A is compact and f is continuous, we know that the image of A under f is compact in \mathbf{R} . Hence, it follows from 3.1, that f takes on a maximum and a minimum value. \square

Lemma 3.2. *A compact set in \mathbf{R} has a maximum and a minimum value.*

Proof. We know that a compact set is closed and bounded and in \mathbf{R} , a compact set is in the form $[a, b]$. And since $a, b \in [a, b]$, all we need to show is that a and b are infimum and supremum, respectively, of the given interval. \square

5. [1-30] Let $f : [a, b] \rightarrow \mathbf{R}$ be an increasing function. If $x_1, \dots, x_n \in [a, b]$ are distinct, show that

$$\sum_{i=1}^n o(f, x_i) \leq f(b) - f(a).$$

Solution: Let order be defined in $\{x_1, \dots, x_n\}$ such that $x_1 < \dots < x_n$, then since f is an increasing function, we get $f(x_1) \leq \dots \leq f(x_n)$. We have

$$o(f, x_i) = \lim_{\delta \rightarrow 0} (M(x_i, f, \delta) - m(x_i, f, \delta))$$

where,

$$\begin{aligned} M(x_i, f, \delta) &= \sup \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\}, \\ m(x_i, f, \delta) &= \inf \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\}. \end{aligned}$$

If we denote a δ -neighborhood of some $x_i \in [a, b]$ by $N_\delta(x_i)$, then since $\delta \rightarrow 0$, we can choose a sufficiently small $\delta > 0$ such that

$$\bigcap_{i=1}^n N_\delta(x_i) = \emptyset.$$

Then for all such δ we have,

$$\begin{aligned} f(x_{i+1}) &\geq M(x_i, f, \delta) \geq f(x_{i-1}), \text{ and} \\ f(x_{i+1}) &\geq m(x_i, f, \delta) \geq f(x_{i-1}). \end{aligned}$$

We simplify the given summation as

$$\begin{aligned} \sum_{i=1}^n o(f, x_i) &= \lim_{\delta \rightarrow 0} \sum_{i=1}^n (M(x_i, f, \delta) - m(x_i, f, \delta)) \\ &\leq \lim_{\delta \rightarrow 0} \sum_{i=1}^n (f(x_{i+1}) - f(x_{i-1})) \\ &= f(x_{n+1}) - f(x_0), \\ &\leq f(b) - f(a). \end{aligned}$$

where the last statement follows from the fact that the max and min $f(x)$ can get is $f(b)$ and $f(a)$. \square

4 2-1

1. [2-1] Prove that if $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is differentiable at $a \in \mathbf{R}^n$, then it is continuous at a .

Solution: Hi.

2. [2-4] Let g be a continuous real-valued function on the unit circle $x \in \mathbf{R}^2 : \|x\| = 1$ such that $g(0, 1) = g(1, 0) = 0$ and $g(-x) = -g(x)$. Define $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ by

$$f(x) = \begin{cases} \|x\| \cdot g\left(\frac{x}{\|x\|}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

- (a) If $x \in \mathbf{R}^2$ and $h : \mathbf{R} \rightarrow \mathbf{R}$ is defined by $h(t) = f(tx)$, show that h is differentiable.

Solution: We need to show that for every $a \in \mathbf{R}$, there exists a $\lambda : \mathbf{R} \rightarrow \mathbf{R}$ such that

$$\lim_{t \rightarrow 0} \frac{h(a+t) - h(a) - \lambda(t)}{t} = 0. \quad (1)$$

We see that, when $tx \neq 0$,

$$h(t) = f(tx) = \begin{cases} -|t| \cdot \|(x)\| \cdot g(\hat{x}) = tf(x) & t < 0, \\ |t| \cdot \|(x)\| \cdot g(\hat{x}) = tf(x) & t > 0. \end{cases}$$

Then h is differentiable when the following limit exists for any $a \in \mathbf{R}$:

$$\lim_{a \rightarrow 0} \frac{h(t+a) - h(t)}{a}.$$

But we have,

$$\begin{aligned} \lim_{a \rightarrow 0} \frac{h(t+a) - h(t)}{a} &= \lim_{a \rightarrow 0} \frac{(t+a)f(x) - tf(x)}{a} \\ &= f(x). \end{aligned}$$

The limit always exists and is equal to the derivative of h at t .

- (b) Show that f is not differentiable at $(0, 0)$ unless $g = 0$.

Solution: