

Calculus on Manifolds

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1 1-1

2 1-2

Theorem 2.1 (Heine-Borel Theorem). *The closed interval $[a, b]$ is compact.*

Proof. If \mathcal{O} is an open cover of $[a, b]$, let

$$A = \{x : a \leq x \leq b \text{ and } [a, x] \text{ is covered by some finite number of open sets in } \mathcal{O}\}.$$

We know that $a \in A$ since we can choose any open set in \mathcal{O} containing a . A certainly has a least upper bound since A is bounded above by b . So, we will show that if some α is the least upper bound of A , then $\alpha \in A$ and $\alpha = b$.

Since $\alpha = \sup A$, for every $x \in A$, there exists an ε such that $\alpha - x < \varepsilon$. Since $[a, x]$ is covered by some finite number of open sets, we can choose any open ε -neighborhood centered at α . Hence, we see that $[a, \alpha]$ is also covered by finitely many open sets. This shows $\alpha \in A$.

To show that $\alpha = b$, assume that $\alpha < b$. Since, we can find some x' between α and b , such that x' is contained in some open neighborhood around α , we see that $[a, \alpha]$ is covered by a single open set. Then certainly, $x' \in A$. But this contradicts that $\alpha = \sup A$. Hence, $\alpha = b$. \square

3 1-3

Definition 3.1. Let X and Y be metric spaces, with metrics d_X and d_Y , respectively. Then, we say that a function $f : X \rightarrow Y$ is **continuous at the point** $x_0 \in X$ if for each open set $U \subset Y$, $f(x_0) \in U$, there is an open set $V \subset X$, $x_0 \in V$ such that $f(V) \subset U$.

Theorem 3.1. *If $A \subset \mathbf{R}^n$, a function $f : A \rightarrow \mathbf{R}^m$ is continuous if and only if for every open set $U \subset \mathbf{R}^m$ there is some open set $V \subset \mathbf{R}^n$ such that $f^{-1}(U) = V \cap A$.*

Proof. Suppose f is continuous. If $a \in f^{-1}(U)$, then $f(a) \in U$. Since U is open, there is some open ball $B \subset U$ such that $f(a) \in B$. And since, f is continuous at a , we know that $f(x) \in B$ provided we choose a sufficiently small open ball C such that $a \in C$. If we do this for each $a \in f^{-1}(U)$ and call their union V (also an open set), then clearly $f^{-1}(U) = V \cap A$. \square

1. [1-23] If $f : A \rightarrow \mathbf{R}^m$ and $a \in A$, show that $\lim_{x \rightarrow a} f(x) = b$ if and only if $\lim_{x \rightarrow a} f_i(x) = b_i$ for $i = 1, \dots, m$.

Solution: If $\lim_{x \rightarrow a} f(x) = b$, then for every $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$0 < \|x - a\| < \delta \implies \|f(x) - b\| < \varepsilon.$$

Then we have

$$\begin{aligned} \sum_{i=1}^m (f_i(x) - b_i)^2 &< \varepsilon^2 \\ \implies |f_i(x) - b_i| &< \varepsilon \quad \text{for } i = 1, \dots, m. \end{aligned}$$

This implies that $\lim_{x \rightarrow a} f_i(x) = b_i$ for $i = 1, \dots, m$.

Now, if $\lim_{x \rightarrow a} f_i(x) = b_i$ for $i = 1, \dots, m$ then for every $\varepsilon > 0$, we can find $\delta_1, \dots, \delta_m$ such that

$$0 < \|x - a\| < \delta_i \implies \|f_i(x) - b_i\| < \varepsilon \quad \text{for } i = 1, \dots, m.$$

Then, for $\delta = \min \delta_i$, for $i = 1, \dots, m$, we have,

$$\begin{aligned} 0 < \|x - a\| < \delta &\implies \sum_{i=1}^m \|f_i(x) - b_i\|^2 < m\varepsilon^2 \\ &\implies \|f(x) - b\| < \sqrt{m}\varepsilon. \end{aligned}$$

Hence, $\lim_{x \rightarrow a} f(x) = b$

2. [1-24] Prove that $f : A \rightarrow \mathbf{R}^m$ is continuous if and only if each f_i is.

Solution: If $f : A \rightarrow \mathbf{R}^m$ is continuous then for all $a \in A$, $\lim_{x \rightarrow a} f(x) = f(a)$. But this means that for all $a \in A$, $\lim_{x \rightarrow a} f_i(x) = f_i(a)$. Hence, each $f_i(x)$ is continuous.

Similarly, converse follows from [1-23].

3. [1-25] Prove that a linear transformation $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is continuous.

Solution:

Proof. We need to show that $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is continuous at all $a \in \mathbf{R}^n$. That is, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that $0 < \|x - a\| < \delta \implies \|T(x) - T(a)\| < \varepsilon$, where $x \in \mathbf{R}^n$. But we have,

$$\|T(x) - T(a)\| = \|T(x - a)\| \leq M\|x - a\|$$

for some $M \in \mathbf{R}$. So for any given $\varepsilon > 0$, we can choose $\delta = \varepsilon/M$. Then certainly, if $0 < \|x - a\| < \delta$, then

$$\|T(x) - T(a)\| \leq M\|x - a\| < M\delta = \varepsilon.$$

So it follows that the linear transformation is continuous. □

4. [1-29] If A is compact, prove that every continuous function $f : A \rightarrow \mathbf{R}$ takes on a maximum and a minimum value.

Solution:

Proof. Since A is compact and f is continuous, we know that the image of A under f is compact in \mathbf{R} . Hence, it follows from 3.1, that f takes on a maximum and a minimum value. \square

Lemma 3.2. *A compact set in \mathbf{R} has a maximum and a minimum value.*

Proof. We know that a compact set is closed and bounded and in \mathbf{R} , a compact set is in the form $[a, b]$. And since $a, b \in [a, b]$, all we need to show is that a and b are infimum and supremum, respectively, of the given interval. \square

5. [1-30] Let $f : [a, b] \rightarrow \mathbf{R}$ be an increasing function. If $x_1, \dots, x_n \in [a, b]$ are distinct, show that

$$\sum_{i=1}^n o(f, x_i) \leq f(b) - f(a).$$

Solution: Let order be defined in $\{x_1, \dots, x_n\}$ such that $x_1 < \dots < x_n$, then since f is an increasing function, we get $f(x_1) \leq \dots \leq f(x_n)$. We have

$$o(f, x_i) = \lim_{\delta \rightarrow 0} (M(x_i, f, \delta) - m(x_i, f, \delta))$$

where,

$$\begin{aligned} M(x_i, f, \delta) &= \sup \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\}, \\ m(x_i, f, \delta) &= \inf \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\}. \end{aligned}$$

If we denote a δ -neighborhood of some $x_i \in [a, b]$ by $N_\delta(x_i)$, then since $\delta \rightarrow 0$, we can choose a sufficiently small $\delta > 0$ such that

$$\bigcap_{i=1}^n N_\delta(x_i) = \emptyset.$$

Then for all such δ we have,

$$\begin{aligned} f(x_{i+1}) &\geq M(x_i, f, \delta) \geq f(x_{i-1}), \text{ and} \\ f(x_{i+1}) &\geq m(x_i, f, \delta) \geq f(x_{i-1}). \end{aligned}$$

We simplify the given summation as

$$\begin{aligned} \sum_{i=1}^n o(f, x_i) &= \lim_{\delta \rightarrow 0} \sum_{i=1}^n (M(x_i, f, \delta) - m(x_i, f, \delta)) \\ &\leq \lim_{\delta \rightarrow 0} \sum_{i=1}^n (f(x_{i+1}) - f(x_{i-1})) \\ &= f(x_{n+1}) - f(x_0), \\ &\leq f(b) - f(a). \end{aligned}$$

where the last statement follows from the fact that the max and min $f(x)$ can get is $f(b)$ and $f(a)$. \square

4 2-1

1. [2-1] Prove that if $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is differentiable at $a \in \mathbf{R}^n$, then it is continuous at a .

Solution: If f is differentiable at $a \in \mathbf{R}^n$, then

$$\lim_{h \rightarrow a} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0.$$

But this means that, for any given $\varepsilon_0 > 0$, we can find a $\delta > 0$ such that

$$\begin{aligned} 0 < \|h\| < \delta &\implies \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} < \varepsilon_0 \\ &\implies \|f(a+h) - f(a) - \lambda(h)\| < \varepsilon_0 \|h\| \\ &\implies \|f(a+h) - f(a)\| < (\varepsilon_0 + M)\|h\| \end{aligned}$$

for some $M \in \mathbf{R}$. So, when $h = x - a$ for any given $\varepsilon > 0$, we can choose $0 < \delta < \varepsilon/(\varepsilon_0 + M)$. Then it follows that,

$$0 < \|x - a\| < \delta \implies \|f(x) - f(a)\| < (\varepsilon_0 + M)\delta < \varepsilon.$$

Hence, f is continuous at a .

2. [2-2] A function $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ is **independent of the second variable** if for each $x \in \mathbf{R}$ we have $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbf{R}$. Show that f is independent of the second variable if and only if there is a function $g : \mathbf{R} \rightarrow \mathbf{R}$ such that $f(x, y) = g(x)$. What is $f'(a, b)$ in terms of g' ?

Solution: Define $g(x) = f(x, 0)$. Then for all $y \in \mathbf{R}$, if f is independent of the second variable, we have $f(x, y) = f(x, 0) = g(x)$.

Similarly, since g is independent of y , we have $g(x) = f(x, 0) = f(x, y_1) = f(x, y_2)$.

Now let $z = (h, k)$. Then, assuming that f is differentiable at (a, b) , we have

$$\begin{aligned} \lim_{(h,k) \rightarrow 0} \frac{\|f(a+h, b+k) - f(a, b) - Df(a, b)(h, k)\|}{\|(h, k)\|} &= 0 \\ \text{or, } \lim_{h \rightarrow 0} \frac{\|g(a+h) - g(a) - Df(a, b)(h, k)\|}{|h|} &= 0 \end{aligned}$$

Since $g : \mathbf{R} \rightarrow \mathbf{R}$,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{|h|} &= \lim_{h \rightarrow 0} \frac{Df(a, b)(h, k)}{|h|} \\ \text{or, } g'(a) &= \lim_{h \rightarrow 0} \frac{Df(a, b)(h, k)}{|h|} \end{aligned}$$

Then we see that

$$Df(a, b)(h, k) = h \cdot g'(a)$$

satisfies the equation. Hence, $f'(a, b) = g'(a)$.

3. [2-4] Let g be a continuous real-valued function on the unit circle $x \in \mathbf{R}^2 : \|x\| = 1$ such that $g(0, 1) = g(1, 0) = 0$ and $g(-x) = -g(x)$. Define $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ by

$$f(x) = \begin{cases} \|x\| \cdot g\left(\frac{x}{\|x\|}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

- (a) If $x \in \mathbf{R}^2$ and $h : \mathbf{R} \rightarrow \mathbf{R}$ is defined by $h(t) = f(tx)$, show that h is differentiable.

Solution: We need to show that for every $a \in \mathbf{R}$, there exists a $\lambda : \mathbf{R} \rightarrow \mathbf{R}$ such that

$$\lim_{t \rightarrow 0} \frac{h(a+t) - h(a) - \lambda(t)}{t} = 0. \quad (1)$$

We see that, when $tx \neq 0$,

$$h(t) = f(tx) = \begin{cases} -|t| \cdot \|x\| \cdot g(\hat{x}) = tf(x) & t < 0, \\ |t| \cdot \|x\| \cdot g(\hat{x}) = tf(x) & t > 0. \end{cases}$$

Then h is differentiable when the following limit exists for any $a \in \mathbf{R}$:

$$\lim_{a \rightarrow 0} \frac{h(t+a) - h(t)}{a}.$$

But we have,

$$\begin{aligned} \lim_{a \rightarrow 0} \frac{h(t+a) - h(t)}{a} &= \lim_{a \rightarrow 0} \frac{(t+a)f(x) - tf(x)}{a} \\ &= f(x). \end{aligned}$$

The limit always exists and is equal to the derivative of h at t .

- (b) Show that f is not differentiable at $(0, 0)$ unless $g = 0$.

Solution:

4. [2-8] Let $f : \mathbf{R} \rightarrow \mathbf{R}^2$. Prove that f is differentiable at $a \in \mathbf{R}$ if and only if f_1 and f_2 are, and that in this case

$$f'(a) = \begin{pmatrix} f'_1(a) \\ f'_2(a) \end{pmatrix}.$$

Solution: If $f : \mathbf{R} \rightarrow \mathbf{R}^2$ is differentiable at a , then for some linear transformation $\lambda : \mathbf{R} \rightarrow \mathbf{R}^2$,

$$\lim_{h \rightarrow 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{|h|} = 0$$

So, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |h| < \delta \implies \|f(a+h) - f(a)\| < |h|\varepsilon.$$

But this means that each f_1 and f_2 satisfies $\|f_i(a+h) - f_i(a)\| < |h|\varepsilon$. So each f_i is differentiable. The converse follows similarly.

Then,

$$\frac{\|f(a+h) - f(a) - \lambda(h)\|}{|h|} = \left\| \begin{pmatrix} \frac{f_1(a+h) - f_1(a) - \lambda_1(h)}{h} \\ \frac{f_2(a+h) - f_2(a) - \lambda_2(h)}{h} \end{pmatrix} \right\|$$

Taking limits on both sides, we see that each of the component of the right hand side must approach to 0. We also have $f'_i(a) = \lambda_i(h)/|h|$. Hence the required expression for $f'(a)$ follows.

$$|h|f'(a) = \lambda(h) = \begin{pmatrix} \lambda_1(h) \\ \lambda_2(h) \end{pmatrix} = \begin{pmatrix} |h|f'_1(a) \\ |h|f'_2(a) \end{pmatrix}.$$

5 2-2

Theorem 5.1. Corollary from the book

If $f, g : \mathbf{R}^n \rightarrow \mathbf{R}$ are differentiable at a ,

$$\begin{aligned} D(f+g)(a) &= Df(a) + Dg(a) \\ D(f \cdot g)(a) &= g(a)Df(a) + f(a)Dg(a) \end{aligned}$$

If, moreover, $g(a) \neq 0$, then

$$D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{[g(a)]^2}$$

Proof. The first one is done in the text. So we'll do the second and the third one.

So, using the notations from the text, since $f \cdot g = p \circ (f, g)$,

$$\begin{aligned} D(f \cdot g)(a) &= Dp(f(a), g(a)) \circ D(f, g)(a) \\ &= Dp(f(a), g(a))(Df(a), Dg(a)) \\ &= g(a)Df(a) + f(a)Dg(a) \end{aligned}$$

The third relation follows from the above product rule and 5.2. □

Lemma 5.2. If $q : \mathbf{R} \rightarrow \mathbf{R}, g : \mathbf{R}^n \rightarrow \mathbf{R}$ is defined by $q(x) = \frac{1}{g}(x)$, then

$$Dq(a) = -\frac{Dg(a)}{[g(a)]^2}.$$

Proof. We have, $q(x) \cdot g(x) = 1$. Then, $D(1) = q(x)Dg(x) + g(x)Dq(x)$. Substituting $q(x) = 1/g(x)$ gives the required result. □

1. [2-12] A function $f : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^p$ is **bilinear** if for $x, x_1, x_2 \in \mathbf{R}^n$, $y, y_1, y_2 \in \mathbf{R}^m$, and $a \in \mathbf{R}$ we have

$$\begin{aligned} f(ax, y) &= af(x, y) = f(x, ay), \\ f(x_1 + x_2, y) &= f(x_1, y) + f(x_2, y), \\ f(x, y_1 + y_2) &= f(x, y_1) + f(x, y_2). \end{aligned}$$

(a) Prove that if f is bilinear, then

$$\lim_{(h,k) \rightarrow 0} \frac{\|f(h, k)\|}{\|(h, k)\|} = 0$$

Solution: Let $h = \alpha_1 e_1 + \dots + \alpha_n e_n$ and $k = \beta_1 e_1 + \dots + \beta_m e_m$. Since f is bilinear, we can write $f(h, k)$ as

$$f(h, k) = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j f(e_i, e_j).$$

Let $\alpha = \max\{|\alpha_i| : i = 1, \dots, n\}$ and $\beta = \max\{|\beta_i| : i = 1, \dots, m\}$. Then $\alpha \leq \|h\|$ and $\beta \leq \|k\|$. Also let $M = \max\{\|f(e_i, e_j)\| : i = 1, \dots, n \text{ and } j = 1, \dots, m\}$. So we have,

$$\begin{aligned} \lim_{(h,k) \rightarrow 0} \frac{\|f(h, k)\|}{\|(h, k)\|} &\leq \lim_{(h,k) \rightarrow 0} \frac{|mn\alpha\beta M|}{\|(h, k)\|} \\ &\leq \lim_{(h,k) \rightarrow 0} \frac{mnM\|h\|\|k\|}{\|(h, k)\|} \end{aligned}$$

Now,

$$\|h\|\|k\| \leq \begin{cases} \|h\|^2 & \text{if } \|k\| \leq \|h\|, \\ \|k\|^2 & \text{if } \|h\| \leq \|k\|. \end{cases}$$

Hence $\|h\|\|k\| \leq \|h\|^2 + \|k\|^2$ and since, $\|(h, k)\| = \sqrt{\|h\|^2 + \|k\|^2}$ we have

$$\lim_{(h,k) \rightarrow 0} \frac{\|f(h, k)\|}{\|(h, k)\|} \leq \lim_{(h,k) \rightarrow 0} \frac{mnM(\|h\|^2 + \|k\|^2)}{\sqrt{\|h\|^2 + \|k\|^2}} = \lim_{(h,k) \rightarrow 0} mnM\sqrt{\|h\|^2 + \|k\|^2} = 0.$$

(b) Prove that $Df(a, b)(x, y) = f(a, y) + f(x, b)$.

Solution: Here, for $a, h \in \mathbf{R}^n$ and $b, k \in \mathbf{R}^m$,

$$\begin{aligned} & \lim_{(h,k) \rightarrow 0} \frac{\|f(a+h, b+k) - f(a, b) - \lambda(h, k)\|}{\|(h, k)\|} \\ & \leq \lim_{(h,k) \rightarrow 0} \frac{\|f(h, k)\|}{\|(h, k)\|} + \lim_{(h,k) \rightarrow 0} \frac{\|f(a, b) + f(a, k) + f(h, b) - f(a, b) - \lambda(h, k)\|}{\|(h, k)\|} \end{aligned}$$

Then certainly, $\lambda(h, k) = f(a, k) + f(h, b)$ implies,

$$\lim_{(h,k) \rightarrow 0} \frac{\|f(a+h, b+k) - f(a, b) - \lambda(h, k)\|}{\|(h, k)\|} = 0.$$

Hence, $Df(a, b)(x, y) = f(a, y) + f(x, b)$.

(c) When $p : \mathbf{R}^2 \rightarrow \mathbf{R}$ is defined by $p(x, y) = x \cdot y$, then

$$Dp(a, b)(x, y) = bx + ay.$$

Show that the above formula is a special case of b.

Solution: It's clear that p is a bilinear function. Hence the derivative of p at (a, b) is given by,

$$\begin{aligned} Dp(a, b)(x, y) &= p(a, y) + p(x, b) \\ &= ay + bx. \end{aligned}$$

2. [2-13] Define $IP : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$ by $IP(x, y) = \langle x, y \rangle$.

(a) Find $D(IP)(a, b)$ and $(IP)'(a, b)$.

Solution: Since inner product is a bilinear function, we have

$$D(IP)(a, b)(x, y) = IP(a, y) + IP(x, b) = \langle a, y \rangle + \langle x, b \rangle.$$

And since $\langle a, y \rangle + \langle x, b \rangle = a_1 y_1 + \dots + a_n y_n + b_1 x_1 + \dots + b_n x_n$, we have

$$(IP)'(a, b) = (b_1, \dots, b_n, a_1, \dots, a_n) = (b, a).$$

(b) If $f, g : \mathbf{R} \rightarrow \mathbf{R}^n$ are differentiable and $h : \mathbf{R} \rightarrow \mathbf{R}$ is defined by $h(t) = \langle f(t), g(t) \rangle$, show that

$$h'(a) = \langle f'(a)^T, g(a) \rangle + \langle f(a), g'(a)^T \rangle.$$

Solution: Here, we can write h as $h = (IP) \circ (f, g)$. Then

$$\begin{aligned} h'(a) &= (IP)'(f(a), g(a)) \cdot \begin{pmatrix} f'(a) \\ g'(a) \end{pmatrix} \\ &= (g(a), f(a)) \cdot \begin{pmatrix} f'(a) \\ g'(a) \end{pmatrix} \\ &= \sum_{i=1}^n g_i(a) \cdot f'_i(a) + \sum_{i=1}^n f_i(a) \cdot g'_i(a) \\ &= \langle f'(a)^T, g(a) \rangle + \langle f(a), g'(a)^T \rangle. \end{aligned}$$

- (c) If $f : \mathbf{R} \rightarrow \mathbf{R}^n$ is differentiable function and $|f(t)| = 1$ for all t , show that $\langle f'(t)^T, f(t) \rangle = 0$.

Solution: Define $h : \mathbf{R} \rightarrow \mathbf{R}$ by $h = \langle f(t), f(t) \rangle$. Then

$$h(t) = \|f(t)\|^2 = 1$$

From b, we have

$$h'(t) = 2\langle f'(t)^T, f(t) \rangle.$$

Since $h(t) = 1$, we have $h'(t) = 0$ and the required result follows.

- (d) Exhibit a differentiable function $f : \mathbf{R} \rightarrow \mathbf{R}$ such that the function $|f|$ defined by $|f|(t) = |f(t)|$ is not differentiable.

Solution: $f(t) = t$ is differentiable. $|f|(t) = |f(t)| = |t|$ is not differentiable.