# Calculus on Manifolds

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## 1 1-1

## 2 1-2

**Theorem 2.1** (Heine-Borel Theorem)**.** *The closed interval* $[a, b]$ *is compact.*

*Proof.* If $\mathscr{O}$ is an open cover of $[a, b]$, let

$$A = \{x : a \leq x \leq b \text{ and } [a, x] \text{ is covered by some finite number of open sets in } \mathscr{O}\}.$$

We know that $a \in A$ since we can choose any open set in $\mathscr{O}$ containing $a$. $A$ certainly has a least upper bound since $A$ is bounded above by $b$. So, we will show that if some $\alpha$ is the least upper bound of $A$, then $\alpha \in A$ and $\alpha = b$.

Since $\alpha = \sup A$, for every $x \in A$, there exists an $\varepsilon$ such that $\alpha - x < \varepsilon$. Since $[a, x]$ is covered by some finite number of open sets, we can choose any open $\varepsilon$-neighborhood centered at $\alpha$. Hence, we see that $[a, \alpha]$ is also covered by finitely many open sets. This shows $\alpha \in A$.

To show that $\alpha = b$, assume that $\alpha < b$. Since, we can find some $x'$ between $\alpha$ and $b$, such that $x'$ is contained in some open neighborhood around $\alpha$, we see that $[a, \alpha]$ is covered by a single open set. Then certainly, $x' \in A$. But this contradicts that $\alpha = \sup A$. Hence, $\alpha = b$. $\square$

## 3 1-3

**Definition 3.1.** Let $X$ and $Y$ be metric spaces, with metrics $d_X$ and $d_Y$, respectively. Then, we say that a function $f : X \to Y$ is **continuous at the point** $x_0 \in X$ if for each open set $U \subset Y$, $f(x_0) \in U$, there is an open set $V \subset X$, $x_0 \in V$ such that $f(V) \subset U$.

**Theorem 3.1.** *If* $A \subset \mathbf{R}^n$*, a function* $f : A \to \mathbf{R}^m$ *is continuous if and only if for every open set* $U \subset \mathbf{R}^m$ *there is some open set* $V \subset \mathbf{R}^n$ *such that* $f^{-1}(U) = V \cap A$.

*Proof.* Suppose $f$ is continuous. If $a \in f^{-1}(U)$, then $f(a) \in U$. Since $U$ is open, there is some open ball $B \subset U$ such that $f(a) \in B$. And since, $f$ is continuous at $a$, we know that $f(x) \in B$ provided we choose a sufficiently small open ball $C$ such that $a \in C$. If we do this for each $a \in f^{-1}(U)$ and call their union $V$ (also an open set), then clearly $f^{-1}(U) = V \cap A$.

$\square$

1. [1-23] If $f : A \to \mathbf{R}^m$ and $a \in A$, show that $\lim_{x\to a} f(x) = b$ if and only if $\lim_{x\to a} f_i(x) = b_i$ for $i = 1, ..., m$.

**Solution:** If $\lim_{x\to a} f(x) = b$, then for every $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$0 < \|x - a\| < \delta \implies \|f(x) - b\| < \varepsilon.$$

Then we have

$$\begin{aligned} &\sum_{i=1}^{m} (f_i(x) - b_i)^2 < \varepsilon^2 \\ \implies &|f_i(x) - b_i| < \varepsilon \quad \text{for } i = 1, ..., m. \end{aligned}$$

This implies that $\lim_{x\to a} f_i(x) = b_i$ for $i = 1, ..., m$.
Now, if $\lim_{x\to a} f_i(x) = b_i$ for $i = 1, ..., m$ then for every $\varepsilon > 0$, we can find $\delta_1, ..., \delta_m$ such that

$$0 < \|x - a\| < \delta_i \implies \|f_i(x) - b_i\| < \varepsilon \text{ for } i = 1, ..., m.$$

Then, for $\delta = \min \delta_i,$ for $i = 1, ..., m$, we have,

$$\begin{aligned} 0 < \|x - a\| < \delta &\implies \sum_{i=1}^{m} \|f_i(x) - b_i\|^2 < m\varepsilon^2 \\ &\implies \|f(x) - b\| < \sqrt{m}\varepsilon. \end{aligned}$$

Hence, $\lim_{x\to a} f(x) = b$

2. [1-24] Prove that $f : A \to \mathbf{R}^m$ is continuous if and only if each $f_i$ is.

**Solution:** If $f : A \to \mathbf{R}^m$ is continuous then for all $a \in A$, $\lim_{x\to a} f(x) = f(a)$. But this means that for all $a \in A$, $lim_{x\to a} f_i(x) = f_i(a)$. Hence, each $f_i(x)$ is continuous.
Similarly, converse follows from [1-23].

3. [1-25] Prove that a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous.

**Solution:**

*Proof.* We need to show that $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous at all $a \in \mathbf{R}^n$. That is, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that $0 < \|x - a\| < \delta \implies \|T(x) - T(a)\| < \varepsilon$, where $x \in \mathbf{R}^n$. But we have,

$$\|T(x) - T(a)\| = \|T(x - a)\| \leq M\|x - a\|$$

for some $M \in \mathbf{R}$. So for any given $\varepsilon > 0$, we can choose $\delta = \varepsilon/M$. Then certainly, if $0 < \|x-a\| < \delta$, then

$$\|T(x) - T(a)\| \leq M\|x - a\| < M\delta = \varepsilon.$$

So it follows that the linear transformation is continuous. $\square$

4. [1-29] If $A$ is compact, prove that every continuous function $f : A \to \mathbf{R}$ takes on a maximum and a minimum value.

**Solution:**

*Proof.* Since $A$ is compact and $f$ is continuous, we know that the image of $A$ under $f$ is compact in $\mathbf{R}$. Hence, it follows from 3.1, that $f$ takes on a maximum and a minimum value. $\square$

**Lemma 3.2.** *A compact set in* $\mathbf{R}$ *has a maximum and a minimum value.*

*Proof.* We know that a compact set is closed and bounded and in $\mathbf{R}$, a compact set is in the form $[a, b]$. And since $a, b \in [a, b]$, all we need to show is that $a$ and $b$ are infimum and supremum, respectively, of the given interval. $\square$

5. [1-30] Let $f : [a, b] \to \mathbf{R}$ be an increasing function. If $x_1, ..., x_n \in [a, b]$ are distinct, show that

$$\sum_{i=1}^{n} o(f, x_i) \leq f(b) - f(a).$$

**Solution:** Let order be defined in $\{x_1, ..., x_n\}$ such that $x_1 < ... < x_n$, then since $f$ is an increasing function, we get $f(x_1) \leq ... \leq f(x_n)$. We have

$$o(f, x_i) = \lim_{\delta \to 0} (M(x_i, f, \delta) - m(x_i, f, \delta))$$

where,

$$M(x_i, f, \delta) = \sup \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\},$$
$$m(x_i, f, \delta) = \inf \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\}.$$

If we denote a $\delta$-neighborhood of some $x_i \in [a, b]$ by $N_\delta(x_i)$, then since $\delta \to 0$, we can choose a sufficiently small $\delta > 0$ such that

$$\bigcap_{i=1}^{n} N_\delta(x_i) = \phi.$$

Then for all such $\delta$ we have,

$$f(x_{i+1}) \geq M(x_i, f, \delta) \geq f(x_{i-1}), \text{ and}$$
$$f(x_{i+1}) \geq m(x_i, f, \delta) \geq f(x_{i-1}).$$

We simplify the given summation as

$$\begin{aligned}
\sum_{i=1}^{n} o(f, x_i) &= \lim_{\delta \to 0} \sum_{i=1}^{n} (M(x_i, f, \delta) - m(x_i, f, \delta)) \\
&\leq \lim_{\delta \to 0} \sum_{i=1}^{n} (f(x_{i+1}) - f(x_{i-1})) \\
&= f(x_{n+1}) - f(x_0), \\
&\leq f(b) - f(a).
\end{aligned}$$

where the last statement follows from the fact that the max and min $f(x)$ can get is $f(b)$ and $f(a)$. $\square$

## 4 2-1

1. [2-1] Prove that if $f : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $a \in \mathbf{R}^n$, then it is continuous at a.

> **Solution:** If $f$ is differentiable at $a \in \mathbf{R}^n$, then
>
> $$\lim_{h \to a} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0.$$
>
> But this means that, for any given $\varepsilon_0 > 0$, we can find a $\delta > 0$ such that
>
> $$\begin{aligned} 0 < \|h\| < \delta &\implies \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} < \varepsilon_0 \\ &\implies \|f(a+h) - f(a) - \lambda(h)\| < \varepsilon_0 \|h\| \\ &\implies \|f(a+h) - f(a)\| < (\varepsilon_0 + M)\|h\| \end{aligned}$$
>
> for some $M \in \mathbf{R}$. So, when $h = x - a$ for any given $\varepsilon > 0$, we can choose $0 < \delta < \varepsilon/(\varepsilon_0 + M)$. Then it follows that,
>
> $$0 < \|x - a\| < \delta \implies \|f(x) - f(a)\| < (\varepsilon_0 + M)\delta < \varepsilon.$$
>
> Hence, $f$ is continuous at $a$.

2. [2-2] A function $f : \mathbf{R}^2 \to \mathbf{R}$ is **independent of the second variable** if for each $x \in \mathbf{R}$ we have $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbf{R}$. Show that $f$ is independent of the second variable if and only if there is a function $g : \mathbf{R} \to \mathbf{R}$ such that $f(x, y) = g(x)$. What is $f'(a, b)$ in terms of $g'$?

> **Solution:** Define $g(x) = f(x, 0)$. Then for all $y \in \mathbf{R}$, if $f$ is independent of the second variable, we have $f(x, y) = f(x, 0) = g(x)$.
>
> Similarly, since $g$ is independent of y, we have $g(x) = f(x, 0) = f(x, y_1) = f(x, y_2)$.
>
> Now let $z = (h, k)$. Then, assuming that $f$ is differentiable at $(a, b)$, we have
>
> $$\begin{aligned} \lim_{(h,k) \to 0} &\frac{\|f(a+h, b+k) - f(a, b) - Df(a, b)(h, k)\|}{\|(h, k)\|} = 0 \\ \text{or, } \lim_{h \to 0} &\frac{\|g(a+h) - g(a) - Df(a, b)(h, k)\|}{|h|} = 0 \end{aligned}$$
>
> Since $g : \mathbf{R} \to \mathbf{R}$,
>
> $$\begin{aligned} \lim_{h \to 0} \frac{g(a+h) - g(a)}{|h|} &= \lim_{h \to 0} \frac{Df(a, b)(h, k)}{|h|} \\ \text{or, } g'(a) &= \lim_{h \to 0} \frac{Df(a, b)(h, k)}{|h|} \end{aligned}$$
>
> Then we see that
>
> $$Df(a, b)(h, k) = h \cdot g'(a)$$
>
> satisfies the equation. Hence, $f'(a, b) = g'(a)$.

3. [2-4] Let $g$ be a continuous real-valued function on the unit circle $x \in \mathbf{R}^2 : \|x\| = 1$ such that $g(0, 1) = g(1, 0) = 0$ and $g(-x) = -g(x)$. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x) = \begin{cases} \|x\| \cdot g\left(\frac{x}{\|x\|}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(a) If $x \in \mathbf{R}^2$ and $h : \mathbf{R} \to \mathbf{R}$ is defined by $h(t) = f(tx)$, show that $h$ is differentiable.

**Solution:** We need to show that for every $a \in \mathbf{R}$, there exists a $\lambda : \mathbf{R} \to \mathbf{R}$ such that

$$\lim_{t \to 0} \frac{h(a+t) - h(a) - \lambda(t)}{t} = 0. \tag{1}$$

We see that, when $tx \neq 0$,

$$h(t) = f(tx) = \begin{cases} -|t| \cdot \|(x\| \cdot g\,(\hat{x}) = tf(x) & t < 0, \\ |t| \cdot \|(x\| \cdot g\,(\hat{x}) = tf(x) & t > 0. \end{cases}$$

Then $h$ is differentiable when the following limit exists for any $a \in \mathbf{R}$:

$$\lim_{a \to 0} \frac{h(t+a) - h(t)}{a}.$$

But we have,

$$\begin{aligned} \lim_{a \to 0} \frac{h(t+a) - h(t)}{a} &= \lim_{a \to 0} \frac{(t+a)f(x) - tf(x)}{a} \\ &= f(x). \end{aligned}$$

The limit always exists and is equal to the derivative of $h$ at $t$.

(b) Show that $f$ is not differentiable at $(0,0)$ unless $g = 0$.

**Solution:**

4. [2-8] Let $f : \mathbf{R} \to \mathbf{R}^2$. Prove that $f$ is differentiable at $a \in \mathbf{R}$ if and only if $f_1$ and $f_2$ are, and that in this case

$$f'(a) = \begin{pmatrix} f_1'(a) \\ f_2'(a) \end{pmatrix}.$$

**Solution:** If $f : \mathbf{R} \to \mathbf{R}^2$ is differentiable at $a$, then for some linear transformation $\lambda : \mathbf{R} \to \mathbf{R}^2$,

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{|h|} = 0$$

So, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |h| < \delta \implies \|f(a+h) - f(a)\| < |h|\varepsilon.$$

But this means that each $f_1$ and $f_2$ satisfies $\|f_i(a+h) - f_i(a)\| < |h|\varepsilon$. So each $f_i$ is differentiable. The converse follows similarly.

Then,

$$\frac{\|f(a+h) - f(a) - \lambda(h)\|}{|h|} = \left\| \begin{pmatrix} \frac{f_1(a+h) - f(a) - \lambda_1(h)}{h} \\ \frac{f_2(a+h) - f(a) - \lambda_2(h)}{h} \end{pmatrix} \right\|$$

Taking limits on both sides, we see that each of the component of the right hand side must be 0. We also have $f_i'(a) = \lambda_i(h)/|h|$. Hence the required expression for $f'(a)$ follows.

$$|h|f'(a) = \lambda(h) = \begin{pmatrix} \lambda_1(h) \\ \lambda_2(h) \end{pmatrix} = \begin{pmatrix} |h|f_1'(a) \\ |h|f_2'(a) \end{pmatrix}.$$

# 5 2-2

**Theorem 5.1.** Corollary from the book
*If* $f, g : \mathbf{R}^n \to \mathbf{R}$ *are differentiable at* $a$,

$$D(f+g)(a) = Df(a) + Dg(a)$$
$$D(f \cdot g)(a) = g(a)Df(a) + f(a)Dg(a)$$

*If, moreover,* $g(a) \neq 0$, *then*

$$D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{[g(a)]^2}$$

*Proof.* The first one is done in the text. So we'll do the second and the third one.
So, using the notations from the text, since $f \cdot g = p \circ (f, g)$,

$$\begin{aligned} D(f.g)(a) &= Dp(f(a), g(a)) \circ D(f, g)(a) \\ &= Dp(f(a), g(a))(Df(a), Dg(a)) \\ &= g(a)Df(a) + f(a)Dg(a) \end{aligned}$$

The third relation follows from the above product rule and 5.2. □

**Lemma 5.2.** *If* $q : \mathbf{R} \to \mathbf{R}, g : \mathbf{R}^n \to \mathbf{R}$ *is defined by* $q(x) = \frac{1}{g}(x)$, *then*

$$Dq(a) = -\frac{Dg(a)}{[g(a)]^2}.$$

*Proof.* We have, $q(x) \cdot g(x) = 1$. Then, $D(1) = q(x)Dg(x) + g(x)Dq(x)$. Substituting $q(x) = 1/g(x)$ gives the required result. □