Calculus on Manifolds

Nutan Nepal

October 1, 2019

Chapter 1

Theorem 0.1 (Heine-Borel Theorem)**.** *The closed interval $[a, b]$ is compact.*

Proof. If $\mathscr{O}$ is an open cover of $[a, b]$, let

$$A = \{x : a \leq x \leq b \text{ and } [a, x] \text{ is covered by some finite number of open sets in } \mathscr{O}\}.$$

We know that $a \in A$ since we can choose any open set in $\mathscr{O}$ containing a. A certainly has a least upper bound since A is bounded above by b. So, we will show that if some α is the least upper bound of A, then $\alpha \in A$ and $\alpha = b$.

Since $\alpha = \sup A$, for every $x \in A$, there exists an ε such that $\alpha - x < \varepsilon$. Since $[a, x]$ is covered by some finite number of open sets, we can choose any open ε-neighborhood centered at α. Hence, we see that $[a, \alpha]$ is also covered by finitely many open sets. This shows $\alpha \in A$.

To show that $\alpha = b$, assume that $\alpha < b$. Since, we can find some x' between α and b, such that x' is contained in some open neighborhood around α, we see that $[a, \alpha]$ is covered by a single open set. Then certainly, $x' \in A$. But this contradicts that $\alpha = \sup A$. Hence, $\alpha = b$. $\square$

Definition 0.1. Let X and Y be metric spaces, with metrics d_X and d_Y, respectively. Then, we say that a function $f : X \to Y$ is **continuous at the point** $x_0 \in X$ if for each open set $U \subset Y$, $f(x_0) \in U$, there is an open set $V \subset X$, $x_0 \in V$ such that $f(V) \subset U$.

Theorem 0.2. *If $A \subset \mathbf{R}^n$, a function $f : A \to \mathbf{R}^m$ is continuous if and only if for every open set $U \subset \mathbf{R}^m$ there is some open set $V \subset \mathbf{R}^n$ such that $f^{-1}(U) = V \cap A$.*

Proof. Suppose f is continuous. If $a \in f^{-1}(U)$, then $f(a) \in U$. Since U is open, there is some open ball $B \subset U$ such that $f(a) \in B$. And since, f is continuous at a, we know that $f(x) \in B$ provided we choose a sufficiently small open ball C such that $a \in C$. If we do this for each $a \in f^{-1}(U)$ and call their union V (also an open set), then clearly $f^{-1}(U) = V \cap A$.

$\square$

1. [1-23] If $f : A \to \mathbf{R}^m$ and $a \in A$, show that $\lim_{x \to a} f(x) = b$ if and only if $\lim_{x \to a} f_i(x) = b_i$ for $i = 1, ..., m$.

Solution: If $\lim_{x \to a} f(x) = b$, then for every $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$0 < \|x - a\| < \delta \implies \|f(x) - b\| < \varepsilon.$$

Then we have

$$\sum_{i=1}^{m} \left(f_i(x) - b_i\right)^2 < \varepsilon^2$$
$$\implies |f_i(x) - b_i| < \varepsilon \quad \text{for } i = 1, ..., m.$$

This implies that $\lim_{x \to a} f_i(x) = b_i$ for $i = 1, ..., m$.

Now, if $\lim_{x \to a} f_i(x) = b_i$ for $i = 1, ..., m$ then for every $\varepsilon > 0$, we can find $\delta_1, ..., \delta_m$ such that

$$0 < \|x - a\| < \delta_i \implies \|f_i(x) - b_i\| < \varepsilon \text{ for } i = 1, ..., m.$$

Then, for $\delta = \min \delta_i$, for $i = 1, ..., m$, we have,

$$0 < \|x - a\| < \delta \implies \sum_{i=1}^{m} \|f_i(x) - b_i\|^2 < m\varepsilon^2$$
$$\implies \|f(x) - b\| < \sqrt{m}\varepsilon.$$

Hence, $\lim_{x \to a} f(x) = b$

2. [1-24] Prove that $f : A \to \mathbf{R}^m$ is continuous if and only if each f_i is.

Solution: If $f : A \to \mathbf{R}^m$ is continuous then for all $a \in A$, $\lim_{x \to a} f(x) = f(a)$. But this means that for all $a \in A$, $lim_{x \to a} f_i(x) = f_i(a)$. Hence, each $f_i(x)$ is continuous.

Similarly, converse follows from [1-23].

3. [1-25] Prove that a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous.

Solution:

Proof. We need to show that $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous at all $a \in \mathbf{R}^n$. That is, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that $0 < \|x - a\| < \delta \implies \|T(x) - T(a)\| < \varepsilon$, where $x \in \mathbf{R}^n$. But we have,

$$\|T(x) - T(a)\| = \|T(x - a)\| \leq M\|x - a\|$$

for some $M \in \mathbf{R}$. So for any given $\varepsilon > 0$, we can choose $\delta = \varepsilon/M$. Then certainly, if $0 < \|x - a\| < \delta$, then

$$\|T(x) - T(a)\| \leq M\|x - a\| < M\delta = \varepsilon.$$

So it follows that the linear transformation is continuous. $\square$

4. [1-29] If A is compact, prove that every continuous function $f : A \to \mathbf{R}$ takes on a maximum and a minimum value.

Solution:

Proof. Since A is compact and f is continuous, we know that the image of A under f is compact in $\mathbf{R}$. Hence, it follows from 3.1, that f takes on a maximum and a minimum value. $\square$

Lemma 0.3. *A compact set in $\mathbf{R}$ has a maximum and a minimum value.*

Proof. We know that a compact set is closed and bounded and in $\mathbf{R}$, a compact set is in the form $[a, b]$. And since $a, b \in [a, b]$, all we need to show is that a and b are infimum and supremum, respectively, of the given interval. $\square$

5. [1-30] Let $f : [a, b] \to \mathbf{R}$ be an increasing function. If $x_1, ..., x_n \in [a, b]$ are distinct, show that

$$\sum_{i=1}^{n} o(f, x_i) \leq f(b) - f(a).$$

Solution: Let order be defined in $\{x_1, ..., x_n\}$ such that $x_1 < ... < x_n$, then since f is an increasing function, we get $f(x_1) \leq ... \leq f(x_n)$. We have

$$o(f, x_i) = \lim_{\delta \to 0} \left(M(x_i, f, \delta) - m(x_i, f, \delta) \right)$$

where,

$$M(x_i, f, \delta) = \sup \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\},$$
$$m(x_i, f, \delta) = \inf \{f(x) : x \in [a, b] \text{ and } \|x - x_i\| < \delta\}.$$

If we denote a δ-neighborhood of some $x_i \in [a, b]$ by $N_\delta(x_i)$, then since $\delta \to 0$, we can choose a sufficiently small $\delta > 0$ such that

$$\bigcap_{i=1}^{n} N_\delta(x_i) = \phi.$$

Then for all such δ we have,

$$f(x_{i+1}) \geq M(x_i, f, \delta) \geq f(x_{i-1}), \text{ and}$$
$$f(x_{i+1}) \geq m(x_i, f, \delta) \geq f(x_{i-1}).$$

We simplify the given summation as

$$\sum_{i=1}^{n} o(f, x_i) = \lim_{\delta \to 0} \sum_{i=1}^{n} \left(M(x_i, f, \delta) - m(x_i, f, \delta) \right)$$

$$\leq \lim_{\delta \to 0} \sum_{i=1}^{n} \left(f(x_{i+1}) - f(x_{i-1}) \right)$$

$$= f(x_{n+1}) - f(x_0),$$
$$\leq f(b) - f(a).$$

where the last statement follows from the fact that the max and min $f(x)$ can get is $f(b)$ and $f(a)$. $\square$

Chapter 2

1. [2-1] Prove that if $f : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $a \in \mathbf{R}^n$, then it is continuous at a.

Solution: If f is differentiable at $a \in \mathbf{R}^n$, then

$$\lim_{h \to a} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0.$$

But this means that, for any given $\varepsilon_0 > 0$, we can find a $\delta > 0$ such that

$$0 < \|h\| < \delta \implies \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} < \varepsilon_0$$
$$\implies \|f(a+h) - f(a) - \lambda(h)\| < \varepsilon_0 \|h\|$$
$$\implies \|f(a+h) - f(a)\| < (\varepsilon_0 + M)\|h\|$$

for some $M \in \mathbf{R}$. So, when $h = x - a$ for any given $\varepsilon > 0$, we can choose $0 < \delta < \varepsilon/(\varepsilon_0 + M)$. Then it follows that,

$$0 < \|x - a\| < \delta \implies \|f(x) - f(a)\| < (\varepsilon_0 + M)\delta < \varepsilon.$$

Hence, f is continuous at a.

---

2. [2-2] A function $f : \mathbf{R}^2 \to \mathbf{R}$ is **independent of the second variable** if for each $x \in \mathbf{R}$ we have $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbf{R}$. Show that f is independent of the second variable if and only if there is a function $g : \mathbf{R} \to \mathbf{R}$ such that $f(x, y) = g(x)$. What is $f'(a, b)$ in terms of g'?

Solution: Define $g(x) = f(x, 0)$. Then for all $y \in \mathbf{R}$, if f is independent of the second variable, we have $f(x, y) = f(x, 0) = g(x)$.

Similarly, since g is independent of y, we have $g(x) = f(x, 0) = f(x, y_1) = f(x, y_2)$.

Now let $z = (h, k)$. Then, assuming that f is differentiable at (a, b), we have

$$\lim_{(h,k) \to 0} \frac{\|f(a+h, b+k) - f(a,b) - Df(a,b)(h,k)\|}{\|(h,k)\|} = 0$$
$$\text{or, } \lim_{h \to 0} \frac{\|g(a+h) - g(a) - Df(a,b)(h,k)\|}{|h|} = 0$$

Since $g : \mathbf{R} \to \mathbf{R}$,

$$\lim_{h \to 0} \frac{g(a+h) - g(a)}{|h|} = \lim_{h \to 0} \frac{Df(a,b)(h,k)}{|h|}$$
$$\text{or, } g'(a) = \lim_{h \to 0} \frac{Df(a,b)(h,k)}{|h|}$$

Then we see that

$$Df(a,b)(h,k) = h \cdot g'(a)$$

satisfies the equation. Hence, $f'(a, b) = g'(a)$.

---

3. [2-4] Let g be a continuous real-valued function on the unit circle $x \in \mathbf{R}^2 : \|x\| = 1$ such that $g(0, 1) =$

$g(1, 0) = 0$ and $g(-x) = -g(x)$. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x) = \begin{cases} \|x\| \cdot g\left(\dfrac{x}{\|x\|}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(a) If $x \in \mathbf{R}^2$ and $h : \mathbf{R} \to \mathbf{R}$ is defined by $h(t) = f(tx)$, show that h is differentiable.

> **Solution:** We need to show that for every $a \in \mathbf{R}$, there exists a $\lambda : \mathbf{R} \to \mathbf{R}$ such that
>
> $$\lim_{t \to 0} \frac{h(a + t) - h(a) - \lambda(t)}{t} = 0. \tag{1}$$
>
> We see that, when $tx \neq 0$,
>
> $$h(t) = f(tx) = \begin{cases} -|t| \cdot \|x\| \cdot g\left(\hat{x}\right) = tf(x) & t < 0, \\ |t| \cdot \|x\| \cdot g\left(\hat{x}\right) = tf(x) & t > 0. \end{cases}$$
>
> Then h is differentiable when the following limit exists for any $a \in \mathbf{R}$:
>
> $$\lim_{a \to 0} \frac{h(t + a) - h(t)}{a}.$$
>
> But we have,
>
> $$\begin{aligned} \lim_{a \to 0} \frac{h(t + a) - h(t)}{a} &= \lim_{a \to 0} \frac{(t + a)f(x) - tf(x)}{a} \\ &= f(x). \end{aligned}$$
>
> The limit always exists and is equal to the derivative of h at t.

(b) Show that f is not differentiable at $(0, 0)$ unless $g = 0$.

> **Solution:**

4. [2-8] Let $f : \mathbf{R} \to \mathbf{R}^2$. Prove that f is differentiable at $a \in \mathbf{R}$ if and only if f_1 and f_2 are, and that in this case

$$f'(a) = \begin{pmatrix} f_1'(a) \\ f_2'(a) \end{pmatrix}.$$

> **Solution:** If $f : \mathbf{R} \to \mathbf{R}^2$ is differentiable at a, then for some linear transformation $\lambda : \mathbf{R} \to \mathbf{R}^2$,
>
> $$\lim_{h \to 0} \frac{\|f(a + h) - f(a) - \lambda(h)\|}{|h|} = 0$$
>
> So, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that
>
> $$0 < |h| < \delta \implies \|f(a + h) - f(a)\| < |h|\varepsilon.$$
>
> But this means that each f_1 and f_2 satisfies $\|f_i(a + h) - f_i(a)\| < |h|\varepsilon$. So each f_i is differentiable. The converse follows similarly.

Then,

$$\frac{\|f(a+h) - f(a) - \lambda(h)\|}{|h|} = \left\| \left(\begin{array}{c} \dfrac{f_1(a+h) - f(a) - \lambda_1(h)}{h} \\ \dfrac{f_2(a+h) - f(a) - \lambda_2(h)}{h} \end{array} \right) \right\|$$

Taking limits on both sides, we see that each of the component of the right hand side must approach to 0. We also have $f_i'(a) = \lambda_i(h)/|h|$. Hence the required expression for $f'(a)$ follows.

$$|h|f'(a) = \lambda(h) = \begin{pmatrix} \lambda_1(h) \\ \lambda_2(h) \end{pmatrix} = \begin{pmatrix} |h|f_1'(a) \\ |h|f_2'(a) \end{pmatrix}.$$

Theorem 0.4. Corollary from the book
If $f, g : \mathbf{R}^n \to \mathbf{R}$ are differentiable at a,

$$D(f + g)(a) = Df(a) + Dg(a)$$
$$D(f \cdot g)(a) = g(a)Df(a) + f(a)Dg(a)$$

If, moreover, $g(a) \neq 0$, then

$$D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{[g(a)]^2}$$

Proof. The first one is done in the text. So we'll do the second and the third one. So, using the notations from the text, since $f \cdot g = p \circ (f, g)$,

$$D(f.g)(a) = Dp(f(a), g(a)) \circ D(f, g)(a)$$
$$= Dp(f(a), g(a))(Df(a), Dg(a))$$
$$= g(a)Df(a) + f(a)Dg(a)$$

The third relation follows from the above product rule and 0.5. $\square$

Lemma 0.5. *If $q : \mathbf{R} \to \mathbf{R}, g : \mathbf{R}^n \to \mathbf{R}$ is defined by $q(x) = \frac{1}{g}(x)$, then*

$$Dq(a) = -\frac{Dg(a)}{[g(a)]^2}.$$

Proof. We have, $q(x) \cdot g(x) = 1$. Then, $D(1) = q(x)Dg(x) + g(x)Dq(x)$. Substituting $q(x) = 1/g(x)$ gives the required result. $\square$

5. [2-12] A function $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^p$ is **bilinear** if for $x, x_1, x_2 \in \mathbf{R}^n$, $y, y_1, y_2 \in \mathbf{R}^m$, and $a \in \mathbf{R}$ we have

$$f(ax, y) = af(x, y) = f(x, ay),$$
$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y),$$
$$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2).$$

(a) Prove that if f is bilinear, then

$$\lim_{(h,k) \to 0} \frac{\|f(h, k)\|}{\|(h, k)\|} = 0$$

Solution: Let $h = \alpha_1 e_1 + ... + \alpha_n e_n$ and $k = \beta_1 e_1 + ... + \beta_m e_m$. Since f is bilinear, we can write $f(h, k)$ as

$$f(h, k) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j f(e_i, e_j).$$

Let $\alpha = \max\{|\alpha_i| : i = 1, ..., n\}$ and $\beta = \max\{|\beta_i| : i = 1, ..., m\}$. Then $\alpha \le \|h\|$ and $\beta \le \|k\|$. Also let $M = \max\{\|f(e_i, e_j)\| : i = 1, ..., n \text{ and } j = 1, ..., m\}$. So we have,

$$\lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} \le \lim_{(h,k)\to 0} \frac{|mn\alpha\beta M|}{\|(h,k)\|}$$
$$\le \lim_{(h,k)\to 0} \frac{mnM\|h\|\|k\|}{\|(h,k)\|}$$

Now,

$$\|h\|\|k\| \le \begin{cases} \|h\|^2 & \text{if } \|k\| \le \|h\|, \\ \|k\|^2 & \text{if } \|h\| \le \|k\|. \end{cases}$$

Hence $\|h\|\|k\| \le \|h\|^2 + \|k\|^2$ and since, $\|(h, k)\| = \sqrt{\|h\|^2 + \|k\|^2}$ we have

$$\lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} \le \lim_{(h,k)\to 0} \frac{mnM(\|h\|^2 + \|k\|^2)}{\sqrt{\|h\|^2 + \|k\|^2}} = \lim_{(h,k)\to 0} mnM\sqrt{\|h\|^2 + \|k\|^2} = 0.$$

(b) Prove that $Df(a, b)(x, y) = f(a, y) + f(x, b)$.

Solution: Here, for $a, h \in \mathbf{R}^n$ and $b, k \in \mathbf{R}^m$,

$$\lim_{(h,k)\to 0} \frac{\|f(a+h, b+k) - f(a,b) - \lambda(h,k)\|}{\|(h,k)\|}$$
$$\le \lim_{(h,k)\to 0} \frac{\|f(h,k)\|}{\|(h,k)\|} + \lim_{(h,k)\to 0} \frac{\|f(a,b) + f(a,k) + f(h,b) - f(a,b) - \lambda(h,k)\|}{\|(h,k)\|}$$

Then certainly, $\lambda(h, k) = f(a, k) + f(h, b)$ implies,

$$\lim_{(h,k)\to 0} \frac{\|f(a+h, b+k) - f(a,b) - \lambda(h,k)\|}{\|(h,k)\|} = 0.$$

Hence, $Df(a, b)(x, y) = f(a, y) + f(x, b)$.

(c) When $p : \mathbf{R}^2 \to \mathbf{R}$ is defined by $p(x, y) = x \cdot y$, then

$$Dp(a, b)(x, y) = bx + ay.$$

Show that the above formula is a special case of b.

Solution: It's clear that p is a bilinear function. Hence the derivative of p at (a, b) is given by,

$$Dp(a, b)(x, y) = p(a, y) + p(x, b)$$
$$= ay + bx.$$

6. [2-13] Define $IP : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by $IP(x, y) = \langle x, y \rangle$.

(a) Find $D(IP)(a, b)$ and $(IP)'(a, b)$.

> **Solution:** Since inner product is a bilinear function, we have
> $$D(IP)(a, b)(x, y) = IP(a, y) + IP(x, b) = \langle a, y \rangle + \langle x, b \rangle.$$
> And since $\langle a, y \rangle + \langle x, b \rangle = a_1 y_1 + ... + a_n y_n + b_1 x_1 + ... + b_n x_n$, we have
> $$(IP)'(a, b) = (b_1, ..., b_n, a_1, ..., a_n) = (b, a).$$

(b) If $f, g : \mathbf{R} \to \mathbf{R}^n$ are differentiable and $h : \mathbf{R} \to \mathbf{R}$ is defined by $h(t) = \langle f(t), g(t) \rangle$, show that
$$h'(a) = \langle f'(a)^T, g(a) \rangle + \langle f(a), g'(a)^T \rangle.$$

> **Solution:** Here, we can write h as $h = (IP) \circ (f, g)$. Then
> $$\begin{aligned} h'(a) &= (IP)'(f(a), g(a)) \cdot \begin{pmatrix} f'(a) \\ g'(a) \end{pmatrix} \\ &= (g(a), f(a)) \cdot \begin{pmatrix} f'(a) \\ g'(a) \end{pmatrix} \\ &= \sum_{i=1}^{n} g_i(a) \cdot f_i'(a) + \sum_{i=1}^{n} f_i(a) \cdot g_i'(a) \\ &= \langle f'(a)^T, g(a) \rangle + \langle f(a), g'(a)^T \rangle. \end{aligned}$$

(c) If $f : \mathbf{R} \to \mathbf{R}^n$ is differentiable function and $\|f(t)\| = 1$ for all t, show that $\langle f'(t)^T, f(t) \rangle = 0$.

> **Solution:** Define $h : \mathbf{R} \to \mathbf{R}$ by $h = \langle f(t), f(t) \rangle$. Then
> $$h(t) = \|f(t)\|^2 = 1$$
> From b, we have
> $$h'(t) = 2 \langle f'(t)^T, f(t) \rangle.$$
> Since $h(t) = 1$, we have $h'(t) = 0$ and the required result follows.

(d) Exhibit a differentiable function $f : \mathbf{R} \to \mathbf{R}$ such that the function $\|f\|$ defined by $\|f\|(t) = \|f(t)\|$ is not differentiable.

> **Solution:** $f(t) = t$ is differentiable. $\|f\|(t) = \|f(t)\| = |t|$ is not differentiable.

7. [2-14] Let $E_i, i = 1, ..., k$ be Euclidean spaces of various dimensions. A function $f : E_1 \times ... \times E_k \to \mathbf{R}^p$ is called **multilinear** if for each choice of $x_j \in E_j, j \neq i$ the function $g : E_i \to \mathbf{R}^p$ defined by $g(x) = f(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_k)$ is a linear transformation.

(a) If f is multilinear and $i \neq j$, show that for $h = (h_1, ..., h_k)$, with $h_l \in E_l$, we have
$$\lim_{h \to 0} \frac{\|f(a_1, ..., h_i, ..., h_j, ..., a_k)\|}{\|h\|} = 0$$

Solution: Define $g : E_i \times E_j \to \mathbf{R}^p$ by $g(x, y) = f(a_1, ..., x, ..., y, ...a_k)$. Then we see that g is bilinear since

$$\begin{aligned}
g(x_1 + x_2, y) &= f(a_1, ..., x_1 + x_2, ..., y, ..., a_k) \\
&= f(a_1, ..., x_1, ..., y, ..., a_k) + f(a_1, ..., x_2, ..., y, ..., a_k) \\
&= g(x_1, y) + g(x_2, y)
\end{aligned}$$

and similarly,

$$\begin{aligned}
g(kx, y) &= f(a_1, ..., kx, ..., y, ..., a_k) \\
&= kf(a_1, ..., x, ..., y, ..., a_k) = kg(x, y).
\end{aligned}$$

(b) Prove that

$$Df(a_1, ..., a_k)(x_1, ..., x_k) = \sum_{i=1}^{k} f(a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_k).$$

Solution:

8. [2-24] Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq 0, \\ 0 & (x, y) = 0. \end{cases}$$

(a) Show that $D_2 f(x, 0) = x$ for all x and $D_1 f(0, y) = -y$ for all y.

Solution: Here,

$$\begin{aligned}
D_2 f(x, 0) &= \lim_{h \to 0} \frac{f(x, h) - f(x, 0)}{h} \\
&= \lim_{h \to 0} \frac{xh\dfrac{x^2 - h^2}{x^2 + h^2} - 0}{h} \\
&= \lim_{h \to 0} \frac{x(x^2 - h^2)}{x^2 + h^2} \\
&= x.
\end{aligned}$$

Similarly, $D_2 f(0, y) = -y$.

(b) Show that $D_{1,2} f(0, 0) \neq D_{2,1} f(0, 0)$.

Solution: We have,

$$\begin{aligned}
D_{1,2} f(0, 0) &= D_2(D_1)(f(0, 0)) \\
&= \lim_{h \to 0} \frac{D_1(0, h) - D_1(0, 0)}{h} \\
&= \lim_{h \to 0} \frac{-h}{h} = -1.
\end{aligned}$$

Similarly, we get $D_{2,1} f(0, 0) = 1$.

9. [1-25] Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} e^{-x^{-2}} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that f is a C^∞ function and $f^{(i)}(0) = 0$ for all i.

Solution: We know that,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and equals $\dfrac{2}{x^3} e^{x^{-2}}$ when $x \neq 0$. The first derivative is given by

$$f'(x) = \frac{2}{x^3} f(x).$$

The subsequent derivatives $f^n(x)$ can be then found by the quotient rule, which guarantees that there won't be more zeroes in the denominator of the derivatives except $x = 0$. Hence if the first derivative is continuous at $x = 0$, the derivatives of all order exist. Now, we'll prove the existence of derivative at $x = 0$. So when $x = 0$,

$$f'(x) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{1/h}{e^{h^{-2}}}.$$

Then by L'Hospital rule we have,

$$f'(0) = \lim_{h \to 0} \frac{-1/h^2}{(2/h^3) f(h)} = \lim_{h \to 0} \frac{-h}{2f(h)} = 0.$$

The derivative clearly exists at $x = 0$. To show the continuity, we observe that,

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \frac{2}{x^3} e^{-x^{-2}} = 0 = f'(0)$$

follows from the L'Hospital rule.

10. [2-26] Let

$$f(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & x \in (-1, 1), \\ 0 & x \notin (-1, 1). \end{cases}$$

(a) Show that $f : \mathbf{R} \to \mathbf{R}$ is a C^∞ function which is positive on $(-1, 1)$ and 0 elsewhere.

Solution:

11. [2-28 d] $F(x, y) = f(x, g(x), h(x, y))$

Solution: Define the functions k and $\bar{g}$ by

$$k(x, y) = x,$$
$$\bar{g}(x, y) = g(x).$$

Then $F(x, y) = f(k(x, y), \bar{g}(x, y), h(x, y))$. We also have,

$$D_1 k(x, y) = 1, \qquad\qquad\qquad D_2 k(x, y) = 0,$$
$$D_1 \bar{g}(x, y) = g'(x), \qquad\qquad D_2 \bar{g}(x, y) = 0.$$

Hence by theorem 2-9, letting $a = (x, g(x), h(x, y))$,

$$D_1 F(x, y) = D_1 f(a) \cdot D_1 k(x, y) + D_2 f(a) \cdot D_1 \bar{g}(x, y) + D_3 f(a) \cdot D_1 h(x, y)$$
$$= D_1 f(a) + D_2 f(a) \cdot g'(x) + D_3 f(a) \cdot D_1 h(x, y)$$
$$D_2 F(x, y) = D_1 f(a) \cdot D_2 k(x, y) + D_2 f(a) \cdot D_2 \bar{g}(x, y) + D_3 f(a) \cdot D_2 h(x, y)$$
$$= D_3 f(a) \cdot D_2 h(x, y).$$

12. [2-29] Let $f : \mathbf{R}^n \to \mathbf{R}$. For $x \in \mathbf{R}^n$, the limit

$$\lim_{t \to 0} \frac{f(a + tx) - f(a)}{t},$$

if it exists, is denoted $D_x f(a)$, and is called the **directional** derivarive of f at a, in the direction x.

(a) Show that $D_{e_i} f(a) = D_i f(a)$.

Solution: Here,

$$D_{e_i} f(a) = \lim_{t \to 0} \frac{f((a_1, ..., a_i, ..., a_n) + (0, ..., t, ..., 0)) - f(a_1, ..., a_i, ..., a_n)}{t}$$
$$= \lim_{t \to 0} \frac{f(a_1, ..., a_i + t, ..., a_n) - f(a_1, ..., a_i, ..., a_n)}{t} = D_i f(a).$$

(b) Show that $D_{tx} f(a) = t D_x f(a)$.

Solution: Since $th \to 0$ as $h \to 0$, for $t, h \in \mathbf{R}$, we have

$$D_{tx} f(a) = t \lim_{h \to 0} \frac{f(a + thx) - f(a)}{th} = t D_x f(a).$$

(c) If f is differentiable at a, show that $D_x f(a) = Df(a)(x)$, and therefore $D_{x+y} f(a) = D_x f(a) + D_y f(a)$.

Solution: Here

$$Df(a)(x) = \|x\| \lim_{x \to 0} \frac{f(a + x) - f(a)}{\|x\|}$$

and if $y \in \mathbf{R}^n$, $ty \to 0$ as $t \to 0$. So for $y \neq 0$,

$$= \|x\| \lim_{t \to 0} \frac{f(a + ty) - f(a)}{\|ty\|}$$
$$= \frac{\|x\|}{\|y\|} D_y f(a)$$
$$= t \cdot D_{x/t} f(a) = D_x f(a).$$

Since $Df(a)$ is linear, the remaining results immediately follow.

13. [2-30] Let f be defined as in (3). Show that $D_x f(0,0)$ exists for all x, but if $g \neq 0$, then $D_{x+y} f(0,0) = D_x f(0,0) + D_y f(0,0)$ is not true for all x and y.

In the referred exercise: $f : \mathbf{R}^2 \to \mathbf{R}$ is defined by

$$f(x) = \begin{cases} \|x\| \cdot g\left(\dfrac{x}{\|x\|}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

where g is a continuous real-valued function on a unit circle centered at origin in $\mathbf{R}^2$ such that $g(0,1) = g(1,0) = 0$ and $g(-x) = -g(x)$.

Solution: For $x \in \mathbf{R}^2$,

$$D_x f(0,0) = \lim_{t \to 0} \frac{f(tx) - f(0,0)}{t} = \lim_{t \to 0} \frac{t f(x)}{t} = f(x)$$

exists. But

$$D_{x+y} f(0,0) = f(x+y) = \|x + y\| \cdot g\left(\frac{x+y}{\|x+y\|}\right)$$

, and in order for the above equation to be equal to

$$D_x f(0,0) + D_y f(0,0) = f(x) + f(y) = \|x\| \cdot g\left(\frac{x}{\|x\|}\right) + \|y\| \cdot g\left(\frac{y}{\|y\|}\right).$$

---

14. [2-32]

(a) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that f is differentiable at 0 but f' is not continuous at 0.

Solution: At $x = 0$,

$$Df(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

But,

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

doesn't exist at $x = 0$, and hence f' is not continuous.

(b) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin \dfrac{1}{\sqrt{x^2 + y^2}} & (x, y) \neq 0, \\ 0 & (x, y) = 0. \end{cases}$$

Show that f is differentiable at $(0,0)$ but $D_i f$ is not continuous at $(0,0)$.

Solution: If $h = \sqrt{x^2 + y^2}$, define $g : \mathbf{R} \to \mathbf{R}$ as $g(h) = f(x, y)$. Then by (a), $g(h) = f(x, y)$ is differentiable at $(0, 0)$ but not continuous at that point.

15. [2-33.5] If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^n$, find $Df(x)$.

Solution: Here

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{(x + h - x)((x+h)^{n-1} + x(x+h)^{n-2} + \dots + x^{n-1})}{h} \\
&= \lim_{h \to 0} \left[(x+h)^{n-1} + x(x+h)^{n-2} + \dots + x^{n-1} \right] \\
&= nx^{n-1}.
\end{aligned}
$$

16. [2-34] A function $f : \mathbf{R}^n \to \mathbf{R}$ is **homogenous** of degree m if $f(tx) = t^m f(x)$ for all x. If f is also differentiable, show that

$$
\sum_{i=1}^{n} x_i D_i f(x) = m f(x).
$$

Solution: Let a function $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(t) = f(tx) = t^m f(x)$. Then

$$
m t^{m-1} f(x) = g'(t) = f'(tx).
$$

So when $t = 1$,

$$
m f(x) = g'(1) = f'(x) = \sum_{i=1}^{n} x_i D_i f(x).
$$

$\square$

17. [2-36] Let $A \subset \mathbf{R}^n$ be an open set and $f : A \to \mathbf{R}^n$ a continuously differentiable 1-1 function such that $\det f'(x) \neq 0$ for all x. Show that $f(A)$ is an open set and $f^{-1} : f(A) \to A$ is differentiable. Show also that $f(B)$ is open for any open set in A.

Solution: The inverse function theorem: Suppose $f : \mathbf{R}^n \to \mathbf{R}^n$ is a continuously differentiable in an open set containing a, and $\det f'(a) \neq 0$. Then there is an open set V containing a and an open set W containing $f(a)$ such that $f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ which is differentiable.

Then since $f : A \to \mathbf{R}^n$ is continuously differentiable in the open set A with $\det f'(x) \neq 0$ for all $x \in A$ there exists open sets $V_x \subset A$ for every $x \in A$ with $W_x \subset \mathbf{R}^n$ such that each $f : V_x \to W_x$ has continuous and differentiable inverse $f^{-1} : W_x \to V_x$. We note that $A \subset \bigcup_{x \in A} V_x$ and since each

$V_x \subset A$, we have $A = \bigcup_{x \in A} V_x$. Also since f is 1-1 and invertible on each W_x, $f(V_x) = W_x$. Hence,

$$f(A) = f\left(\bigcup_{x \in A} V_x\right) = \bigcup_{x \in A} W_x$$

which is open. Similarly

$$f^{-1}(W_x) = V_x = A.$$

And since f^{-1} is continuous, every open subset B of A has an open subset $f(B) \subset f(A)$.

Chapter 3

1. [3-1] Let $f : [0,1] \times [0,1] \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } 0 \leq x \leq \dfrac{1}{2}, \\ 1 & \text{if } \dfrac{1}{2} \leq x \leq 1. \end{cases}$$

Show that f is integrable and $\int_{[0,1] \times [0,1]} f = \frac{1}{2}$.

Solution: For subrectangles $S_1 = \left[0, \frac{1}{2}\right]$ and $S_2 = \left[\frac{1}{2}, 1\right]$, $m_{S_1} = M_{S_1} = 0$ and $m_{S_2} = M_{S_2} = 1$. Then $L(f,P) = \sum_{k=1}^{2} m_{S_k} \cdot v(S_k) = 0 \cdot v(S_1) + 1 \cdot v(S_2) = v(S_2)$. Similarly, $U(f,P) = v(S_2)$. And hence, the integration of f on $[0,1] \times [0,1] = v(S_2) = 1/2$.

2. Let $f : A \to \mathbf{R}$ be integrable and let $g = f$ except at finitely many points. Show that g is integrable and $\int_A f = \int_A g$.

Solution: Let P be a partition of A such that the function f differs from g at finitely many points contained in the closed subrectangles $S_1, S_2, ..., S_n$ where each subrectangle has a side $\varepsilon > 0$. Then

$$(U(f,P) - L(f,P)) - (U(g,P) - L(g,P))$$

$$= \sum_{k=1}^{n} (M_f(S_k) - m_f(S_k)) \cdot v(S_k) - \sum_{k=1}^{n} (M_g(S_k) - m_g(S_k)) \cdot v(S_k)$$

since the f and g only differ in S_i. Also, the volume of each S_i is given by some $\varepsilon' > 0$. Since ε' is arbitrary, we see that

$$(U(f,P) - L(f,P)) - (U(g,P) - L(g,P)) < p\varepsilon'$$

$$\text{or, } \int_A f - (U(g,P) - L(g,P)) < p\varepsilon'$$

for some $p \in \mathbf{R}$. Also since f is integrable, $(U(g,P) - L(g,P)) < \varepsilon' \cdot \varepsilon''$ for another arbitrary $\varepsilon'' > 0$.

3. [3-8] Prove that $[a_1, b_1] \times \cdots \times [a_n, b_n]$ does not have content 0 if $a_i \leq b_i$ for each i.

Solution: The given rectangle is compact in $\mathbf{R}^n$. Let $\{U_1, \ldots, U_m\}$ be the finite cover of $[a_1, b_1] \times \cdots \times [a_n, b_n]$ by closed rectangles where we assume that each $U_j \subset [a_1, b_1] \times \cdots \times [a_n, b_n]$. And let P_i be the sequence $t_{i,0}, t_{i,1}, \ldots, t_{i,k}$ where $a_i = t_{i,0} < t_{i,1} < \ldots < t_{i,k} = b_i$. Then the partition $P = (P_1, \ldots, P_n)$ divides the rectangle into k^n subrectangles. If we let $a_i = t_{i,0} < t_{i,1} < \ldots < t_{i,k} = b_i$ be all the corners of all U_j, then each U_j is the union of finite subrectangles in P. And hence, each $v(U_j)$ is the sum of volume of certain subrectangles of the partition. Also since each subrectangle lies in at least one U_j,

$$\text{sum of volume of all the closed covers } U_i$$
$$\geq \text{sum of volumes of each subrectangles}$$
$$= v([a_1, b_1] \times \cdots \times [a_n, b_n])$$

4. [3-9]

Show that an unbounded set cannot have content 0.

Give an example of a closed set of measure 0 which does not have content 0.

Solution:

(a) Unbounded sets cannot have finite cover and hence do not have content 0.

(b) $\mathbb{Z}$ has measure 0 but not content 0.

5. [3-10]

(a) If C is a set of content 0, show that the boundary of C has content 0.

(b) Give an example of a bounded set C of measure 0 such that the boundary of C does not have measure 0.

Solution:

(a) Let A be the set of content 0. Then for every $\varepsilon > 0$, there is a finite cover $\{U_1, \ldots, U_n\}$ of A by closed intervals such that $\sum_{i=1}^{n} v(U_i) < \varepsilon$. Also since $A \subset \cup_{i=1}^{n} U_i$, where each U_i are closed, boundary of A are also contained in the finite union and the same U_i suffices to conclude that the boundary of A has content 0.

(b) A=$\{x \in [0, 1] : x \in \mathbb{Q}\}$ has measure 0. But the boundary of A is all of $[0, 1]$ which doesn't have measure 0.

6. [3-11] Let A be the set of Problem 1-18. If $\sum_{i=1}^{\infty} (b_i - a_i) < 1$, show that the boundary of A does not have measure 0.

[1-18] If $A \subset [0, 1]$ is the union of open intervals (a_i, b_i) such that each rational number in $[0, 1]$ is contained in some (a_i, b_i), show that boundary $A = [0, 1] - A$.

Solution: Since each (a_i, b_i) contains a rational number, the collection $\mathscr{O}$ of (a_i, b_i) is countable. Now, assume that the boundary of A doesnt not have measure zero. Then there exists a countable cover of the boundary by closed rectangles U_i such that $\sum_{i=0}^{\infty} v(U_i) < \varepsilon$ for every $\varepsilon > 0$ Then there is a countable collection of closed rectangles V_j that cover all of $[0, 1]$ such that $\sum_{j=1}^{\infty} v(V_j) < 1$. But, we know that $\sum_{j=1}^{\infty} v(V_j)$ should be at least $1 - 0 = 1$. Hence, we have a contradiction.

7. [3-12] Let $f : [a,b] \to \mathbf{R}$ be an increasing function. Show that $\{x : f$ is discontinuous at $x\}$ has measure 0.

Hint: Use Problem 1-30 to show that $\{x : o(f,x) > 1/n\}$ is finite, for each integer n.

Solution: Let $B = \{x : f$ is discontinuous at $x\}$. Then we know that

$$\sum_{x \in B} o(f,x) < f(b) - f(a).$$

Let A_n be the set $\{x : o(f,x) > 1/n\}$ for each n. Then for an increasing function

$$f(b) - f(a) > \sum_{x \in A_n} o(f,x) > \sum_{x \in A} \frac{1}{n}.$$

However, since $f(b) - f(a)$ is finite ($f([a,b])$ is bounded and closed), there should be finitely many $x \in A_n$ for each n.

---

8. [3-13]

(a) Show that the collection of all rectangles $[a_1, b_1] \times \ldots \times [a_n, b_n]$ with all a_i and b_i rational can be arranged in a sequence.
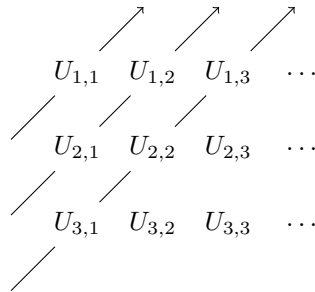
(b) If $A \subset \mathbf{R}^n$ is any set and $\mathscr{O}$ is an open cover of A, show that there is a sequence $U_1, U_2, U_3, \ldots$ of members of $\mathscr{O}$ which also cover A.

Hint: For each $x \in A$ there is a rectangle $B = [a_1, b_1] \times \ldots \times [a_n, b_n]$ with all a_i and b_i rational such that $x \in B \subset U$ for some $U \in$).

Solution:

(a) (a) We will prove this by induction. For base case, we will show that $[a_1, b_1] \times [a_2, b_2]$ is countable if each a_i and b_i are rational.

We know that rational numbers are countable. Let the rational numbers between a_1 and b_1 be arranged in the sequence $a_{1,1}, a_{1,2}, a_{1,3} \ldots$. Then let each intervals $[a_{1,i}, a_{1,i+1}] \subset [a_1, b_1]$ be denoted V_i. We do the same for $[a_2, b_2]$ to get the sequence of countable intervals W_j. Now let $U_{i,j}$ denote $V_i \times W_j$ for each V_i and W_j. It is clear that each corners of $U_{i,j}$ have rational coordinates. Then we can see that

$$
\begin{array}{cccc}
U_{1,1} & U_{1,2} & U_{1,3} & \cdots \\
U_{2,1} & U_{2,2} & U_{2,3} & \cdots \\
U_{3,1} & U_{3,2} & U_{3,3} & \cdots
\end{array}
$$

all the $U_{i,j}$ can be arranged in a sequence.

Now, assuming that we can do the same for $[a_1, b_1] \times \ldots \times [a_n, b_n]$, we get a sequence of closed rectangles $V_i \subset \mathbf{R}^n$ with each a_i and b_i rational. We can also get the sequence of intervals W_j

---