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**Lemma 1.** If  $1_H$  denotes the trivial module for  $KH$ , then  $\text{Ind}_H^G(1_H)$  affords the permutation representation of  $G$  on the right (or left) cosets of  $H$ .

**Theorem 2.** Let  $X = \{x_1, \dots, x_n\}$  be a transitive  $G$ -set and  $G_{x_1} = \text{Stab}_G(x_1)$  denote the stabilizer of  $x_1 \in X$  under the action of  $G$ . Let  $Y$  denote the set of left cosets of  $G_{x_1}$ . Then  $X \cong Y$  as  $G$ -sets.

**Corollary 3.** Let  $X$  be a  $G$ -set and  $V$  be the permutation representation of  $G$  on  $X$ . Let  $X/G$  be the set of orbits under the action of  $G$  and  $G_x$  be the stabilizer subgroup of  $x \in X$ . Then

$$V = \bigoplus_{\bar{x} \in X/G} \text{Ind}_{G_x}^G(1_{G_x}).$$

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**Corollary 4** ([2][1]). The Chow ring  $A(M)$  of a matroid  $M$  is free as a  $\mathbb{Z}$ -module, with  $\mathbb{Z}$ -basis given by the  $FY$ -monomials

$$FY = \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} : (\emptyset = F_0) \subset F_1 \subset F_2 \subset \cdots \subset F_k, \text{ and } m_i \leq \text{rk}(F_i) - \text{rk}(F_{i-1}) - 1\}.$$

**Proposition 5.** Let  $M$  be a loopless matroid. The Hilbert-Poincare series of the Chow ring  $A(M)$  is given by

$$\underline{H}_M = \sum_{\emptyset = F_0 \subset F_1 \subset \cdots \subset F_m} \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x}.$$

Given a flag  $F = F_0 \subset F_1 \subset \cdots \subset F_m$  of strictly increasing flats, the formula in proposition 5 essentially computes the number of degree  $i$  monomials formed precisely by  $x_{F_1} x_{F_2} \cdots x_{F_m}$  as coefficients of  $x^i$  and sums over all flags to obtain the Hilbert-Poincare series.

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Consider the polynomial ring  $2^{FY}[x]$  with addition and multiplication defined as follows:

1.  $Sx^i + Tx^i = (S \cup T)x^i$ ,
2.  $Sx^i \cdot Tx^j = (ST)x^{i+j}$ ,

where  $S$  and  $T$  are subsets of  $FY$  and  $ST = \{s \cdot t \mid s \in S, t \in T\}$ . A slight modification of the formula in proposition 5 is as follows:

$$A = \sum_{\emptyset = F_0 \subset F_1 \subset \cdots \subset F_m} \left( \prod_{i=1}^m \{F_i\}x + \{F_i^2\}x^2 + \cdots \{F_i^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1}\}x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1} \right) \quad (1)$$

assuming that the factor is 0 when  $\text{rk}(F_i) - \text{rk}(F_{i-1}) = 1$ . This formula explicitly computes the  $FY$ -monomials corresponding to the flag  $F_0 \subset F_1 \subset \cdots \subset F_m$  and summing over all flags gives us

$$A = FY^0 + FY^1x + \cdots + FY^{r-1}x^{r-1}$$

for a matroid of rank  $r$ .

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If a group  $G$  acts on the set of atoms, then we have an induced action on the sets Flats and  $FY$ . For a fixed flag  $P = F_0 \subset F_1 \subset \cdots \subset F_m$ , consider the summand of  $P$  in  $A$ :

$$A_P = \prod_{i=1}^m \{F_i\}x + \{F_i^2\}x^2 + \cdots \{F_i^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1}\}x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1} = P^0 + P^1x + \cdots P^{r-1}x^{r-1}.$$

We have the following facts about the action of  $G$  on Flats and  $FY$ :

1. For every monomial  $m$  in  $P^i \subseteq FY^i$ , the stabilizer group is  $G_P = G_{F_0} \cap G_{F_1} \cap \cdots \cap G_{F_m}$ . Equivalently, the orbits of any two elements in  $P^i$  under the action of  $G$  are disjoint.
2. If  $Q$  is in the orbit of the flag  $P$ , then  $Q^i$  is in the orbit of  $P^i$ .
3. The permutation representation of  $G$  on the orbit  $\overline{m} \in FY^i/G$  is given by  $\text{Ind}_{G_P}^G(1_{G_P})$  (theorem above).
4. From above, the permutation representation on orbits of  $P^i$  equals  $|P^i| \cdot \text{Ind}_{G_P}^G(1_{G_P})$ .
5. Hence, the permutation representation on the orbits of  $P$  can be decomposed as

$$\begin{aligned} V(A_{\overline{P}}) &= (|P^0| + |P^1|x + \cdots + |P^{r-1}|x^{r-1}) \cdot \text{Ind}^G(1_{G_P}) \\ &= \left( \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \text{Ind}^G(1_{G_P}). \end{aligned}$$

From (5), we have the decomposition of the permutation representation on  $FY$  as follows:

$$\underline{H}_M^G = \sum_{\overline{P} \in \text{Flags}_0/G} \left( \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \text{Ind}^G(1_{G_P})$$

where  $\text{Flags}_0$  denotes the set of flags starting at  $F_0 = \emptyset$ .

For  $G_P = \text{Stab}(F_0 \cdots F_m) = \text{Stab}(F_0) \cap \cdots \cap \text{Stab}(F_m) = \text{Stab}(F_1) \cap \cdots \cap \text{Stab}(F_m)$ ,

$$\begin{aligned} \underline{H}_M^G &= \sum_{\emptyset = F_0 \subset F_1 \subset \cdots \subset F_m} \left( \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \frac{|G_P|}{|G|} \cdot \text{Ind}^G(1_{G_P}) \\ &= 1_G + \sum_{\emptyset \neq F} \frac{x(1 - x^{\text{rk}(F) - 1})}{1 - x} \frac{|G_F|}{|G|} \\ &\quad \text{Ind}^G \left( \sum_{F=F_1 \subset \cdots \subset F_m} \left( \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \frac{|G_P|}{|G_F|} \cdot \text{Ind}^{G_F}(1_{G_P}) \right) \\ &= 1_G + \sum_{\emptyset \neq F} \frac{x(1 - x^{\text{rk}(F) - 1})}{1 - x} \frac{|G_F|}{|G|} \cdot \text{Ind}^G(\underline{H}_{M/F}^{G_F}). \end{aligned}$$

We have the following  $FY$ -monomials set for the augmented Chow ring  $\tilde{A}(M)$  as given in [3] Corollary 3.12.

$$\widetilde{FY} = \{x_{F_1}^{a_1} x_{F_2}^{a_2} \cdots x_{F_m}^{a_m} \mid \emptyset \subset F_1 \subset \cdots \subset F_m, 1 \leq a_1 \leq \text{rk}(F_1), a_i \leq \text{rk}(F_i) - \text{rk}(F_{i-1}) - 1 \text{ for } i > 1\}.$$

The computation of equivariant Chow polynomial of  $\tilde{A}_M$  follows exactly the same as that of  $A(M)$ :

$$\underline{H}_M^G = 1_G + \sum_{P \in \text{Flags}} \left( \frac{x(1 - x^{\text{rk}(F_0)})}{1 - x} \cdot \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \frac{|G_P|}{|G|} \cdot \text{Ind}^G(1_{G_P})$$

Fixing  $F = F_0$  gives us

$$\begin{aligned}
H_M^G &= 1_G + \sum_{P \in \text{Flags}} \left( \frac{x(1 - x^{\text{rk}(F_0)})}{1 - x} \cdot \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \frac{|G_P|}{|G|} \cdot \text{Ind}^G(1_{G_P}) \\
&= 1_G + \sum_{F \in \text{Flats}} \frac{x(1 - x^{\text{rk}(F)})}{1 - x} \cdot \text{Ind}_{G_F}^G \left( \sum_{P \in \text{Flags}/F} \left( \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \frac{|G_P|}{|G_F|} \cdot \text{Ind}^G(1_{G_P}) \right)
\end{aligned}$$

## References

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