computation

February 28, 2025

[33]:
```
import sys
sys.path.append('..')
from db1 import *

R = PolynomialRing(QQ, 't')
S = PolynomialRing(ZZ, 't')
t = R.gen()

def is_paving(M):
    n = M.size()
    r = M.rank()
    return (len(M.independent_r_sets(r-1)) == binomial(n, r-1))

def q_kl(k, h):
    return kazhdan_lusztig_inverse_uniform(k, h+1) -␣
 ↪kazhdan_lusztig_inverse_uniform(k-1, h)

def kl_inverse_fast(M):
    if M.loops(): return R(0)
    k, n = M.rank(), M.size()
    if k == n or k == 0: return R(1)
    if not M.is_connected():
        ans = R(1)
        CC = M.components()
        for N in CC:
            res = M.delete(M.groundset() - N)
            ans = ans * kl_inverse_fast(res)
        return ans

    if is_paving(M):
        return kl_inverse_paving(M)
    if is_paving(M.dual()):
        return kl_inverse_copaving(M)
    """
    if n <= 8 and M.is_connected():
        for i in range(len(mat[n][k])):
            if mat[n][k][i].is_isomorphic(M):
```

```
                return ikl[n][k][i]
    """
    LF = M.lattice_of_flats()
    ans = R(0)
    for F in LF:
        if len(F) != n:
            Res = M.delete(M.groundset() - F)
            Con = M.contract(F)
            chi = characteristic_polynomial(Con)(1/t) * t**(Con.rank())
            PPP = kl_inverse_fast(Res)(t) * (-1)**(Res.rank())
            ans = ans + chi * PPP
    assert (t**k * ans(1/t)).numerator() == -ans(t)
    ans = ans.numerator() * (-1)**(k+1)
    return ans.truncate((k+1)//2)

def kazhdan_lusztig_inverse_uniform(k, n):
    if k == n:
        return R(1)
    d = k
    m = n - d
    ans = 0
    for j in range((d-1)//2 + 1):
        ans = ans + m * (d-2*j)/((m+j) * (m+d-j)) * binomial(d, j) * t**j
    return ans * binomial(m+d, d)

def kl_inverse_paving(M):
    assert is_paving(M)
    n = M.size()
    k = M.rank()
    ans = kazhdan_lusztig_inverse_uniform(k, n)
    for H in M.hyperplanes():
        h = len(H)
        if h >= k:
            ans = ans - q_kl(k, h)
    return ans

def kl_inverse_copaving(M):
    assert is_paving(M.dual())
    n = M.size()
    k = M.rank()
    ans = kazhdan_lusztig_inverse_uniform(k, n)
    for H in M.dual().hyperplanes():
        h = len(H)
        if h >= n-k:
            ans = ans - kli_vtilde_dual(n-k, h, n) +␣
 ↪kazhdan_lusztig_inverse_uniform(h-n+k+1, h) *␣
 ↪kazhdan_lusztig_inverse_uniform(n-h-1, n-h)
```

```
    return ans

def kli_vtilde_dual(k, h, n):
    return helper1(n-k, h, n)

def helper1(k, h, n):
    c = n - h
    ans1 = kazhdan_lusztig_inverse_uniform(k, n)
    ans2 = helper2(c, k, n)
    ans3 = kazhdan_lusztig_inverse_uniform(k-c+1, h) *␣
 ↪kazhdan_lusztig_inverse_uniform(c-1, c)
    return ans1 - ans2 + ans3

def helper2(c, k, n):
    h = n - c
    ans = 0
    for j in range(k-c+1):
        ans = ans + binomial(n-c, j) * (-1)**(c-1+j) *␣
 ↪kazhdan_lusztig_inverse_uniform(c-1, c) * t**(k-c-j+1) * chuly(k-c-j+1,␣
 ↪n-c-j)(1/t)
    for i in range(c-1):
        for j in range(k-i):
            ans = ans + binomial(c, i) * binomial(n-c, j) * (-1)**(i+j) *␣
 ↪t**(k-i-j) * helper4(c, k, n, i, j)(1/t)
    ans = ans.numerator().truncate((k-1)//2 + 1)
    if ans[0] < 0:
        ans = -ans
    return ans

def helper3(c, k, n):
    ans = 0
    for j in range(k-c+1):
        ans = ans + binomial(n-c, j) * kazhdan_lusztig_uniform_matroid(c-1, c)␣
 ↪* (-1)**(k-c-j+1) * kazhdan_lusztig_inverse_uniform(k-c-j+1, n-c-j)
    for i in range(c-1):
        for j in range(k-i):
            ans = ans + binomial(c, i) * binomial(n-c, j) * (-1)**(k-i-j) *␣
 ↪helper2(c-i, k-i-j, n-i-j)
    return -ans

def helper4(c, k, n, i, j):
    ans = 0
    for l in range(c-i-1):
        ans = ans + (-1)**l * (t-1)**(max(n-i-j-l-1, 0))
    for u in range(n-k-1):
        ans = doit_once(ans)
    return ans
```

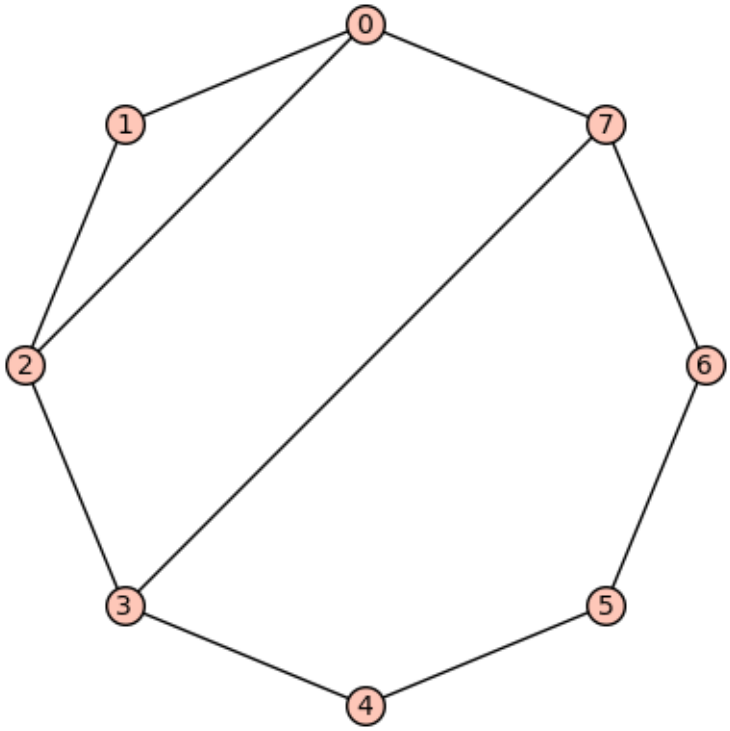
```
def chuly(a, b):
    ans = (t-1)**b
    for i in range(b-a):
        ans = doit_once(ans)
    return ans

def doit_once(p):
    p = p // t**2
    p = p * t
    p = p - p(1)
    return p

def lorenzo(k, h, n):
    c = n - h
    ans1 = kazhdan_lusztig_uniform_matroid(k, n) +␣
 ↪kazhdan_lusztig_uniform_matroid(k-c+1, h) *␣
 ↪kazhdan_lusztig_uniform_matroid(c-1, c)
    ans2 = helper3(c, k, n)
    return ans1 - ans2
```

[34]:
```
n = 8

G = graphs.CycleGraph(n)
G.add_edges({(0, 2), (3, 7)})
e1, e2 = frozenset({(0, 2)}), frozenset({(3,7)})
G.show()
```



[35]:
```
M = Matroid(G)
flats = list(M.lattice_of_flats())

def the_set_S(flats, e):
    return set(F for F in flats if set(e).issubset(set(F)) and set(F) - set(e)␣
 ↪not in flats)

the_set_S(flats, e1)
```

[35]:
```
{frozenset({(0, 1), (0, 2), (1, 2)}),
 frozenset({(0, 1), (0, 2), (1, 2), (2, 3), (3, 4), (5, 6), (6, 7)}),
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            (0, 7),
            (1, 2),
            (2, 3),
            (3, 4),
            (3, 7),
            (4, 5),
            (5, 6),
            (6, 7)})}
```

[36]:
```
def leading_term(p):
    return p.leading_coefficient() * t ** p.degree()

def lt(M):
    r = M.rank()
    if r % 2 == 1:
        return leading_term(kl_inverse_fast(M))
    else:
        return 0

def extra_sum(M, S, e):
    return sum(kl_inverse_fast(M.contract(F)) * lt(M.minor(contractions=e,␣
 ↪deletions=M.groundset() - F)) for F in S)

kl_inverse_fast(M)
```

[36]:
```
7*t^3 + 35*t^2 + 52*t + 24
```

[37]:
```
kl_inverse_fast(M.delete(e1)) - (t+1)*kl_inverse_fast(M.contract(e1)) +␣
  ↪extra_sum(M, the_set_S(flats, e1), e1)
```

[37]:
```
7*t^3 + 24*t^2 + 64*t + 114
```