# EQUIVARIANT CHOW POLYNOMIALS OF MATROIDS

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ABSTRACT. beeboop beeboop. . .

## 1. Introduction

1.1. **Overview.** Given a matroid $\mathsf{M} = (E, \mathcal{I})$, we can define its Chow ring $\underline{\mathrm{CH}}$ and augmented Chow ring CH for which the bases are given by:

$$\mathrm{FY} = \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} : (\varnothing = F_0) \subset F_1 \subset F_2 \subset \cdots \subset F_k, \text{ and } m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1\},$$

and

$$\widetilde{FY} = \{x_{F_1}^{a_1} x_{F_2}^{a_2} \cdots x_{F_m}^{a_m} \mid \varnothing \subset F_1 \subset \cdots \subset F_m,\ 1 \leq a_1 \leq \mathrm{rk}(F_1),\ a_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 \text{ for } i > 1\}$$

respectively. Here $\varnothing \subset F_1 \subset \cdots \subset F_m$ is a strictly increasing chain of flats of the matroid $\mathsf{M}$. The following theorems were proved in [FMSV24]:.

**Theorem 1.1** *There is a unique way to assign to each loopless matroid* $\mathsf{M}$ *a palindromic polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following properties hold:*

(i) *If* $\mathrm{rk}(\mathsf{M}) = 0$*, then* $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1$.
(ii) *If* $\mathrm{rk}(\mathsf{M}) > 0$*, then* $\deg \underline{\mathrm{H}}_{\mathsf{M}}(x) = \mathrm{rk}(\mathsf{M}) - 1$.
(iii) *For every matroid* $\mathsf{M}$*, the polynomial*

$$\mathrm{H}_{\mathsf{M}}(x) := \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\mathrm{rk}(F)}\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x)$$

*is palindromic.*

**Theorem 1.2** *There is a unique way to assign to each loopless matroid* $\mathsf{M}$ *a polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following conditions hold:*

(i) *If* $\mathrm{rk}(\mathsf{M}) = 0$*, then* $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1$.
(ii) *For every matroid* $\mathsf{M}$*, the following recursion holds:*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \varnothing}} \overline{\chi}_{\mathsf{M}|_F}(x)\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x).$$

It was shown in [FMSV24] that these polynomials $\mathrm{H}_{\mathsf{M}}$ and $\underline{\mathrm{H}}_{\mathsf{M}}$ are the Hilbert-Poincare series for the augmented Chow ring CH and the Chow ring $\underline{\mathrm{CH}}$ respectively. In other words,

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = |FY^0| + |FY^1|x + \cdots + |FY^{r-1}|x^{r-1}$$

where $|FY^i|$ denotes the number of fy-monomials of degree $i$ (which equals the dimension of the degree $i$ piece of the Chow ring).

## 2. Matroids

2.1. **Action.** Given a finite set $E$ with $n$ elements, the symmetric group $\mathfrak{S}_n$ can always act on $E$ by permutation and this induces an action on the power set $2^E$. If $\mathsf{M} = (E, \mathcal{I})$ is a matroid, let $G$ be the stabilizer subgroup $\mathsf{Stab}_{\mathfrak{S}_n}(\mathcal{I})$ of $\mathfrak{S}_n$ that stabiizes the set $\mathcal{I} \subseteq 2^E$. Then, we say that the group $G$ acts on the matroid $\mathsf{M}$.

The action of $G$ on $\mathsf{M}$ induces an action on the Chow ring and the augmented Chow ring of $\mathsf{M}$ by permuting the fy-monomials. Since the action doesn't affect the degree of the monomials, it is clear that $G$ acts on the Chow ring by acting separately on each graded piece of CH.

Let $V^i$ denote the permutation representation of $G$ on the set $FY^i$. Then $\dim(V^i) = |FY^i|$, the dimension of the $i$-th piece. We define the polynomial $V^0 + V^1x + \cdots V^{r-1}x^{r-1} \in \mathrm{VRep}_G[x]$ to be the equivariant Chow polynomial of the matroid $\mathsf{M}$. In this paper, we will prove the following theorems:

**Theorem 2.1** *Let $\mathsf{M}$ be a loopless matroid and $\underline{\mathrm{H}}^G_{\mathsf{M}}$ be its equivariant Chow polynomial. Then $\underline{\mathrm{H}}^G_{\mathsf{M}}$ is given by*

$$\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = \sum_{\varnothing = F_0 \subset F_1 \subset \cdots \subset F_m} \frac{|G_{F_0 \cdots F_m}|}{|G|} \left( \prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1})}{1 - x} \right) \mathrm{Ind}^G \left( 1_{G_{F_0 \cdots F_m}} \right). \tag{1}$$

*Here, $G_{F_0 \cdots F_m} = G_{F_0} \cap \cdots \cap G_{F_m}$ denotes the stabilizer of the chain $(F_0 \subset F_1 \subset \cdots \subset F_m)$ and the sum is taken over all nonempty chains of flats starting at $\varnothing$.*

**Theorem 2.2** *Let $\mathsf{M}$ be a loopless matroid and $\mathrm{H}^G_{\mathsf{M}}$ be its equivariant augmented Chow polynomial. Then $\mathrm{H}^G_{\mathsf{M}}$ is given by*
(2)
$$\mathrm{H}^G_{\mathsf{M}}(x) = 1_G + \sum_{F_0 \subset F_1 \subset \cdots \subset F_m} \frac{|G_{F_0 \cdots F_m}|}{|G|} \frac{x(1 - x^{\mathrm{rk}(F_0)})}{1 - x} \left( \prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1})}{1 - x} \right) \mathrm{Ind}^G \left( 1_{G_{F_0 \cdots F_m}} \right).$$

*Here, $G_{F_0 \cdots F_m} = G_{F_0} \cap \cdots \cap G_{F_m}$ denotes the stabilizer of the chain $(F_0 \subset F_1 \subset \cdots \subset F_m)$ and the sum is taken over all nonempty chains of flats.*

**Theorem 2.3** *There is a unique way to assign to each loopless matroid $\mathsf{M}$ a palindromic polynomial $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) \in \mathrm{VRep}_G[x]$ such that the following properties hold:*

(i) *If $\mathrm{rk}(\mathsf{M}) = 0$, then $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = 1_G$.*
(ii) *If $\mathrm{rk}(\mathsf{M}) > 0$, then $\deg \underline{\mathrm{H}}^G_{\mathsf{M}}(x) = \mathrm{rk}(\mathsf{M}) - 1$.*
(iii) *For every matroid $\mathsf{M}$, the polynomial*

$$\mathrm{H}^G_{\mathsf{M}}(x) := \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\mathrm{rk}(F)} \frac{|G_F|}{|G|} \mathrm{Ind}^G \left( \underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}(x) \right)$$

*is palindromic.*

**Theorem 2.4** *There is a unique way to assign to each loopless matroid $\mathsf{M}$ a polynomial $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) \in \mathrm{VRep}_G[x]$ such that the following conditions hold:*

(i) *If $\mathrm{rk}(\mathsf{M}) = 0$, then $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = 1_G$.*
(ii) *For every matroid $\mathsf{M}$, the following recursion holds:*

$$\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \varnothing}} \frac{|G_F|}{|G|} \mathrm{Ind}^G \left( \overline{\chi}^{G_F}_{\mathsf{M}|_F}(x) \otimes \underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}(x) \right).$$

## 3. Scratch

**Proposition 3.1** *Let* $\mathsf{M}$ *be a loopless matroid. The Hilbert–Poincare series of the Chow ring* $\underline{\mathrm{CH}}(M)$ *is given by*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \sum_{\varnothing = F_0 \subset F_1 \subset \cdots \subset F_m} \left( \prod_{i=1}^{m} \frac{x(1-x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x} \right). \tag{3}$$

*Here, the sum is taken over all nonempty chains of flats starting at* $\varnothing$.

We notice a few things about the formula in 3.1:

(i) The chain corresponding to just the empty flat gives an empty product which equals 1.
(ii) If $\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 = 0$ for some $i$ in the chain $F_0 \subset F_1 \subset \cdots \subset F_m$, then the product is 0.
(iii) So, given a chain $F_0 \subset F_1 \subset \cdots \subset F_m$ with $\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 > 0$ for all $i$, we can write the product as $a_1 x + \cdots + a_{r-1}x^{r-1}$ for some positive integers $a_j$'s.

In particular, we can restate the equation 3 as

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1 + \sum_{P_\varnothing} a_1(P_\varnothing)x + \cdots + a_{r-1}(P_\varnothing)x^{r-1},$$

where the sum is taken over all chains $P_\varnothing = (F_0 \subset F_1 \subset \cdots \subset F_m)$ starting at $\varnothing$ with $\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 > 0$ for all $i$ and $a_j(P_\varnothing)$ are some integers depending on the chain.

$$S_0 + S_1 x + \cdots + S_r x^r = \sum_{i \in I} \sum_{j=0}^{r} T_{ij} x^j$$

$$S_k = \sum_{i \in I} T_{ik}$$

$$|S_0| + |S_1| x + \cdots + |S_r| x^r = \sum_{i \in I} p_i(x)$$

3.1. **Continuing without the proof of 2.1 and 2.2.** The transitivity of the Ind functor allows us to carry on analogously to the non-equivariant proof of Proposition 3.7 in [FMSV24]. Using $G_{F_0} = G$, we can rewrite the equation 1 in theorem 2.1 as:

$$\begin{aligned}
\underline{\mathrm{H}}^G_{\mathsf{M}}(x) &= \sum_{\varnothing = F_0 \subset F_1 \subset \cdots \subset F_m} \frac{|G_{F_0 \cdots F_m}|}{|G|} \left( \prod_{i=1}^{m} \frac{x(1-x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x} \right) \mathrm{Ind}^G \left( 1_{G_{F_0 \cdots F_m}} \right) \\
&= 1_G + \sum_{\varnothing = F_0 \subset F_1 \subset \cdots \subset F_m} \frac{|G_{F_0 F_1}|}{|G|} \frac{x(1-x^{\mathrm{rk}(F_1)-1})}{1-x} \\
&\qquad \mathrm{Ind}^G_{G_{F_0 F_1}} \left( \frac{|G_{F_0 \cdots F_m}|}{|G_{F_0 F_1}|} \prod_{i=2}^{m} \frac{x(1-x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x} \mathrm{Ind}^{G_{F_0 F_1}} \left( 1_{G_{F_0 \cdots F_m}} \right) \right) \\
&= 1_G + \sum_{F \neq \varnothing} \frac{|G_F|}{|G|} \frac{x(1-x^{\mathrm{rk}(F)-1})}{1-x} \\
&\qquad \mathrm{Ind}^G_{G_F} \left( \sum_{F = F_1 \subset \cdots \subset F_m} \frac{|G_{F_1 \cdots F_m}|}{|G_F|} \left( \prod_{i=2}^{m} \frac{x(1-x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x} \right) \mathrm{Ind}^G \left( 1_{G_{F_1 \cdots F_m}} \right) \right) \\
&= 1_G + \sum_{F \neq \varnothing} \frac{|G_F|}{|G|} \frac{x(1-x^{\mathrm{rk}(F)-1})}{1-x} \mathrm{Ind}^G_{G_F} \left( \underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F} \right)
\end{aligned}$$

Analogously, fixing a flat $F = F_0$ in the formula in theorem 2.2 we get,

$$\begin{aligned}
\mathrm{H}^G_{\mathsf{M}}(x) &= 1_G + \sum_{F\in\mathcal{L}} \frac{|G_F|}{|G|}\frac{x(1-x^{\mathrm{rk}(F)})}{1-x} \\
&\quad \left(\sum_{F\subset F_1\subset\cdots\subset F_m} \frac{|G_{F_0\cdots F_m}|}{|G_F|}\prod_{i=1}^m \frac{x(1-x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x}\right)\mathrm{Ind}^G\left(1_{G_{F_0\cdots F_m}}\right) \\
&= 1_G + \sum_{F\in\mathcal{L}} \frac{|G_F|}{|G|}\frac{x(1-x^{\mathrm{rk}(F)})}{1-x} \\
&\quad \mathrm{Ind}^G_{G_F}\left(\sum_{F\subset F_1\subset\cdots\subset F_m} \frac{|G_{F_0\cdots F_m}|}{|G_F|}\left(\prod_{i=1}^m \frac{x(1-x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x}\right)\mathrm{Ind}^{G_F}\left(1_{G_{F_0\cdots F_m}}\right)\right) \\
&= 1_G + \sum_{F\in\mathcal{L}} \frac{|G_F|}{|G|}\frac{x(1-x^{\mathrm{rk}(F)})}{1-x}\mathrm{Ind}^G_{G_F}\left(\underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}\right) \\
&= 1_G + \sum_{F\neq\varnothing} \frac{|G_F|}{|G|}\frac{x(1-x^{\mathrm{rk}(F)})}{1-x}\mathrm{Ind}^G_{G_F}\left(\underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}\right).
\end{aligned}$$

From these two equations we get:

$$\mathrm{H}_{\mathsf{M}}(x) = \sum_{F\in\mathcal{L}} x^{\mathrm{rk}(F)}\frac{|G_F|}{|G|}\mathrm{Ind}^G\left(\underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}(x)\right).$$

We now restate the following lemma from [FMSV24] that will help us prove theorem 2.3.

**Lemma 3.2** *Let $p(x)$ be a polynomial of degree $d$. There exist unique polynomials $a(x)$ of degree $d$ and $b(x)$ of degree at most $d-1$ with the properties that $a(x) = x^d a(x^{-1})$ and $b(x) = x^{d-1}b(x^{-1})$, and that satisfy*

$$p(x) = a(x) + b(x).$$

We note that this lemma is true for polynomials over any ring. In particular, it is true for polynomials over $\mathrm{VRep}_G$.

**Theorem 3.3** *There is a unique way to assign to each loopless matroid $\mathsf{M}$ a palindromic polynomial $\underline{\widetilde{\mathrm{H}}}^G_{\mathsf{M}}(x) \in \mathrm{VRep}_G[x]$ such that the following properties hold:*

(i) *If* $\mathrm{rk}(\mathsf{M}) = 0$, *then* $\underline{\widetilde{\mathrm{H}}}^G_{\mathsf{M}}(x) = 1_G$.

(ii) *If* $\mathrm{rk}(\mathsf{M}) > 0$, *then* $\deg \underline{\widetilde{\mathrm{H}}}^G_{\mathsf{M}}(x) = \mathrm{rk}(\mathsf{M}) - 1$.

(iii) *For every matroid* $\mathsf{M}$, *the polynomial*

$$\widetilde{\mathrm{H}}^G_{\mathsf{M}}(x) := \sum_{F\in\mathcal{L}(\mathsf{M})} x^{\mathrm{rk}(F)}\frac{|G_F|}{|G|}\mathrm{Ind}^G\left(\underline{\widetilde{\mathrm{H}}}^{G_F}_{\mathsf{M}/F}(x)\right)$$

*is palindromic.*

*Proof.* We prove the theorem by induction on the size of the ground set of $\mathsf{M}$. When the ground set has cardinality 0, the polynomial $\underline{\widetilde{\mathrm{H}}}^G_{\mathsf{M}}(x)$ is unique and equals $1_G$ by the first property. We now assume that the uniqueness has been established for all the matroids with cardinality less than $n$ and consider the matroid of cardinality $n$. The polynomial

$$S^G_{\mathsf{M}}(x) = \sum_{F\neq\varnothing} x^{\mathrm{rk}(F)}\frac{|G_F|}{|G|}\mathrm{Ind}^G\left(\underline{\widetilde{\mathrm{H}}}^{G_F}_{\mathsf{M}/F}(x)\right)$$

is uniquely determined since $\widetilde{\underline{\mathrm{H}}}$ is unique for all the matroids $\mathsf{M}/F$ by the inductive step. $S^G_{\mathsf{M}}(x)$ has a summand of degree $\mathrm{rk}(M)$ which we get when $F = E$. For all other flats, the summands have degree $\mathrm{rk}(F) + \mathrm{rk}(\mathsf{M}/F) - 1 = \mathrm{rk}(\mathsf{M}) - 1$.

Hence the degree of $S^G_{\mathsf{M}}(x)$ is $\mathrm{rk}(\mathsf{M})$, and we can decompose it into unique polynomials $a(x)$ and $b(x) \in \mathrm{VRep}_G[x]$ such that $a(x)$ is palindromic of degree $\mathrm{rk}(M)$ and $b(x)$ has degree at most $\mathrm{rk}(\mathsf{M})-1$. Taking $\widetilde{\underline{\mathrm{H}}}^G_{\mathsf{M}}(x) = -b(x)$ we get, $\widetilde{\mathrm{H}}^G_{\mathsf{M}}(x) = S^G_{\mathsf{M}}(x)+\widetilde{\underline{\mathrm{H}}}^G_{\mathsf{M}}(x) = a(x)+b(x)+\widetilde{\underline{\mathrm{H}}}^G_{\mathsf{M}}(x) = a(x)$ which is palindromic. The uniqueness of $\widetilde{\underline{\mathrm{H}}}$ also implies the uniqueness of $\widetilde{\mathrm{H}}$. Finally, since $\underline{\mathrm{H}}$ and $\mathrm{H}$ satisfy these conditions, they must be the polynomials $\widetilde{\underline{\mathrm{H}}}$ and $\widetilde{\mathrm{H}}$ respectively. ◻

**Theorem 3.4** *Let* $\mathsf{M}$ *be a loopless matroid. The equivariant Hilbert series of the Chow ring of* $\mathsf{M}$ *satisfies:*

$$\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = \sum_{\substack{F\in\mathscr{L}(\mathsf{M})\\ F\neq\emptyset}} \frac{|G_F|}{|G|}\,\mathrm{Ind}^G\left(\overline{\chi}^{G_F}_{\mathsf{M}|_F}(x)\otimes \underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}(x)\right).$$

*Proof.* To prove this, we define the polynomials

$$\widetilde{\underline{\mathrm{H}}}^G_{\mathsf{M}}(x) := \begin{cases} 1_G & \text{if } \mathsf{M} \text{ is empty,}\\ \displaystyle\sum_{\substack{F\in\mathscr{L}(\mathsf{M})\\ F\neq\emptyset}} \frac{|G_F|}{|G|}\,\mathrm{Ind}^G\left(\overline{\chi}^{G_F}_{\mathsf{M}|_F}(x)\otimes \underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}(x)\right) & \text{if } \mathsf{M} \text{ is nonempty}\end{cases}$$

satisfy all the properties of that statement. The first two conditions are immediate to check, and for the last, it will suffice to verify that $\widetilde{\underline{\mathrm{H}}}_{\mathsf{M}}(x)$ satisfies the recursion of Remark .... We have a chain of equalities:

$$
\begin{aligned}
& 1_G + x \sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \neq E}} \frac{|G_F|}{|G|} \operatorname{Ind}^G \left( \underline{\widetilde{\mathrm{H}}}^{G_F}_{\mathsf{M}/F}(x) \right) \\
= \; & 1_G + x \sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \neq E}} \frac{|G_F|}{|G|} \operatorname{Ind}^G \left( \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}/F) \\ F' \neq \varnothing}} \frac{|G_{FF'}|}{|G_F|} \operatorname{Ind}^{G_F}_{G_{FF'}} \left( \overline{\chi}^{G_{FF'}}_{(\mathsf{M}/F)|_{F'}}(x) \otimes \underline{\mathrm{H}}^{G_{FF'}}_{\mathsf{M}/F/F'}(x) \right) \right) \\
= \; & 1_G + x \sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \neq E}} \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}) \\ F' \supset F}} \frac{|G_{FF'}|}{|G|} \operatorname{Ind}^G_{G_{FF'}} \left( \overline{\chi}^{G_{FF'}}_{(\mathsf{M}|_{F'})/F}(x) \otimes \underline{\mathrm{H}}^{G_{FF'}}_{\mathsf{M}/F'}(x) \right) \\
= \; & 1_G + x \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}) \\ F' \neq \varnothing}} \sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \subset F'}} \frac{|G_{FF'}|}{|G|} \operatorname{Ind}^G_{G_{F'}} \left( \operatorname{Ind}^{G_{F'}}_{G_{FF'}} \left( \overline{\chi}^{G_{FF'}}_{(\mathsf{M}|_{F'})/F}(x) \otimes \underline{\mathrm{H}}^{G_{FF'}}_{\mathsf{M}/F'}(x) \right) \right) \\
= \; & 1_G + x \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}) \\ F' \neq \varnothing}} \operatorname{Ind}^G_{G_{F'}} \left( \sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \subset F'}} \frac{|G_{FF'}|}{|G|} \operatorname{Ind}^{G_{F'}}_{G_{FF'}} \left( \overline{\chi}^{G_{FF'}}_{(\mathsf{M}|_{F'})/F}(x) \right) \otimes \underline{\mathrm{H}}^{G_{F'}}_{\mathsf{M}/F'}(x) \right) \\
= \; & 1_G + x \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}) \\ F' \neq \varnothing}} \frac{|G_{F'}|}{|G|} \operatorname{Ind}^G_{G_{F'}} \left( \underline{\mathrm{H}}^{G_{F'}}_{\mathsf{M}/F'}(x) \otimes \sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \subset F'}} \frac{|G_{FF'}|}{|G_{F'}|} \operatorname{Ind}^{G_{F'}}_{G_{FF'}} \left( \overline{\chi}^{G_{FF'}}_{(\mathsf{M}|_{F'})/F}(x) \right) \right) \\
= \; & 1_G + x \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}) \\ F' \neq \varnothing}} \frac{|G_{F'}|}{|G|} \operatorname{Ind}^G_{G_{F'}} \left( \underline{\mathrm{H}}^{G_{F'}}_{\mathsf{M}/F'}(x) \otimes \left( 1_{F'} + 1_{F'} x + \cdots + 1_{F'} x^{\operatorname{rk}(F')-1} \right) \right) \qquad (4) \\
= \; & 1_G + x \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}) \\ F' \neq \varnothing}} \frac{|G_{F'}|}{|G|} \operatorname{Ind}^G_{G_{F'}} \left( \underline{\mathrm{H}}^{G_{F'}}_{\mathsf{M}/F'}(x) \right) \cdot \frac{1 - x^{\operatorname{rk}(\mathsf{M}|_{F'})}}{1 - x} \qquad (5) \\
= \; & \mathrm{H}^G_{\mathsf{M}}(x),
\end{aligned}
$$

□

## 4. Going on a tangent

## References


[FMSV24] Ferroni, Luis and Matherne, Jacob P. and Stevens, Matthew and Vecchi, Lorenzo, *Hilbert–Poincaré series of matroid Chow rings and intersection cohomology*, Advances in Mathematics. **449** (2024) 1, 3, 4



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