

# EQUIVARIANT CHOW POLYNOMIALS OF MATROIDS

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## 1. INTRODUCTION

**1.1. Overview.** Given a matroid  $M = (E, \mathcal{F})$ , we can define its Chow ring  $\underline{\text{CH}}$  and augmented Chow ring  $\text{CH}$  for which the bases are given by:

$$\text{FY} = \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} : (\emptyset = F_0) \subset F_1 \subset F_2 \subset \cdots \subset F_k, \text{ and } m_i \leq \text{rk}(F_i) - \text{rk}(F_{i-1}) - 1\},$$

and

$$\widetilde{\text{FY}} = \{x_{F_1}^{a_1} x_{F_2}^{a_2} \cdots x_{F_m}^{a_m} \mid \emptyset \subset F_1 \subset \cdots \subset F_m, 1 \leq a_1 \leq \text{rk}(F_1), a_i \leq \text{rk}(F_i) - \text{rk}(F_{i-1}) - 1 \text{ for } i > 1\}$$

respectively. Here  $\emptyset \subset F_1 \subset \cdots \subset F_m$  is a strictly increasing chain of flats of the matroid  $M$ . The following theorems were proved in [FMSV24]:

**Theorem 1.1** *There is a unique way to assign to each loopless matroid  $M$  a palindromic polynomial  $\underline{H}_M(x) \in \mathbb{Z}[x]$  such that the following properties hold:*

- (i) *If  $\text{rk}(M) = 0$ , then  $\underline{H}_M(x) = 1$ .*
- (ii) *If  $\text{rk}(M) > 0$ , then  $\deg \underline{H}_M(x) = \text{rk}(M) - 1$ .*
- (iii) *For every matroid  $M$ , the polynomial*

$$H_M(x) := \sum_{F \in \mathcal{L}(M)} x^{\text{rk}(F)} \underline{H}_{M/F}(x)$$

*is palindromic.*

**Theorem 1.2** *There is a unique way to assign to each loopless matroid  $M$  a polynomial  $\underline{H}_M(x) \in \mathbb{Z}[x]$  such that the following conditions hold:*

- (i) *If  $\text{rk}(M) = 0$ , then  $\underline{H}_M(x) = 1$ .*
- (ii) *For every matroid  $M$ , the following recursion holds:*

$$\underline{H}_M(x) = \sum_{\substack{F \in \mathcal{L}(M) \\ F \neq \emptyset}} \bar{\chi}_{M|F}(x) \underline{H}_{M/F}(x).$$

It was shown in [FMSV24] that these polynomials  $H_M$  and  $\underline{H}_M$  are the Hilbert-Poincare series for the augmented Chow ring  $\text{CH}$  and the Chow ring  $\underline{\text{CH}}$  respectively. In other words,

$$\underline{H}_M(x) = |\text{FY}^0| + |\text{FY}^1|x + \cdots + |\text{FY}^{r-1}|x^{r-1}$$

where  $|\text{FY}^i|$  denotes the number of fy-monomials of degree  $i$  (which equals the dimension of the degree  $i$  piece of the Chow ring).

## 2. MATROIDS

**2.1. Action.** Given a finite set  $E$  with  $n$  elements, the symmetric group  $\mathfrak{S}_n$  can always act on  $E$  by permutation and this induces an action on the power set  $2^E$ . If  $M = (E, \mathcal{F})$  is a matroid, let  $G$  be the stabilizer subgroup  $\text{Stab}_{\mathfrak{S}_n}(\mathcal{F})$  of  $\mathfrak{S}_n$  that stabilizes the set  $\mathcal{F} \subseteq 2^E$ . Then, we say that the group  $G$  acts on the matroid  $M$ .

The action of  $G$  on  $M$  induces an action on the Chow ring and the augmented Chow ring of  $M$  by permuting the fy-monomials. Since the action doesn't affect the degree of the monomials, it is clear that  $G$  acts on the Chow ring by acting separately on each graded piece of  $\text{CH}$ .

Let  $V^i$  denote the permutation representation of  $G$  on the set  $FY^i$ . Then  $\dim(V^i) = |FY^i|$ , the dimension of the  $i$ -th piece. We define the polynomial  $V^0 + V^1x + \dots + V^{r-1}x^{r-1} \in \text{VRep}_G[x]$  to be the equivariant Chow polynomial of the matroid  $M$ . In this paper, we will prove the following theorems:

**Theorem 2.1** *Let  $M$  be a loopless matroid and  $\underline{H}_M^G$  be its equivariant Chow polynomial. Then  $\underline{H}_M^G$  is given by*

$$(1) \quad \underline{H}_M^G(x) = \sum_{\emptyset = F_0 \subset F_1 \subset \dots \subset F_m} \frac{|G_{F_0 \dots F_m}|}{|G|} \left( \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \text{Ind}^G \left( 1_{G_{F_0 \dots F_m}} \right).$$

Here,  $G_{F_0 \dots F_m} = G_{F_0} \cap \dots \cap G_{F_m}$  denotes the stabilizer of the chain  $(F_0 \subset F_1 \subset \dots \subset F_m)$  and the sum is taken over all nonempty chains of flats starting at  $\emptyset$ .

**Theorem 2.2** *Let  $M$  be a loopless matroid and  $H_M^G$  be its equivariant augmented Chow polynomial. Then  $H_M^G$  is given by*

$$(2) \quad H_M^G(x) = 1_G + \sum_{F_0 \subset F_1 \subset \dots \subset F_m} \frac{|G_{F_0 \dots F_m}|}{|G|} \frac{x(1 - x^{\text{rk}(F_0)})}{1 - x} \left( \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \text{Ind}^G \left( 1_{G_{F_0 \dots F_m}} \right).$$

Here,  $G_{F_0 \dots F_m} = G_{F_0} \cap \dots \cap G_{F_m}$  denotes the stabilizer of the chain  $(F_0 \subset F_1 \subset \dots \subset F_m)$  and the sum is taken over all nonempty chains of flats.

**Theorem 2.3** *There is a unique way to assign to each loopless matroid  $M$  a palindromic polynomial  $\underline{H}_M^G(x) \in \text{VRep}_G[x]$  such that the following properties hold:*

- (i) *If  $\text{rk}(M) = 0$ , then  $\underline{H}_M^G(x) = 1_G$ .*
- (ii) *If  $\text{rk}(M) > 0$ , then  $\deg \underline{H}_M^G(x) = \text{rk}(M) - 1$ .*
- (iii) *For every matroid  $M$ , the polynomial*

$$H_M^G(x) := \sum_{F \in \mathcal{L}(M)} x^{\text{rk}(F)} \frac{|G_F|}{|G|} \text{Ind}^G \left( \underline{H}_{M/F}^{G_F}(x) \right)$$

*is palindromic.*

**Theorem 2.4** *There is a unique way to assign to each loopless matroid  $M$  a polynomial  $\underline{H}_M^G(x) \in \text{VRep}_G[x]$  such that the following conditions hold:*

- (i) *If  $\text{rk}(M) = 0$ , then  $\underline{H}_M^G(x) = 1_G$ .*
- (ii) *For every matroid  $M$ , the following recursion holds:*

$$\underline{H}_M^G(x) = \sum_{\substack{F \in \mathcal{L}(M) \\ F \neq \emptyset}} \frac{|G_F|}{|G|} \text{Ind}^G \left( \overline{\chi}_{M|F}^{G_F}(x) \otimes \underline{H}_{M/F}^{G_F}(x) \right).$$

## 3. SCRATCH

**Proposition 3.1** *Let  $M$  be a loopless matroid. The Hilbert–Poincare series of the Chow ring  $\text{CH}(M)$  is given by*

$$(3) \quad \underline{H}_M(x) = \sum_{\emptyset=F_0 \subset F_1 \subset \dots \subset F_m} \left( \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right).$$

Here, the sum is taken over all nonempty chains of flats starting at  $\emptyset$ .

We notice a few things about the formula in 3.1:

- (i) The chain corresponding to just the empty flat gives an empty product which equals 1.
- (ii) If  $\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1 = 0$  for some  $i$  in the chain  $F_0 \subset F_1 \subset \dots \subset F_m$ , then the product is 0.
- (iii) So, given a chain  $F_0 \subset F_1 \subset \dots \subset F_m$  with  $\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1 > 0$  for all  $i$ , we can write the product as  $a_1x + \dots + a_{r-1}x^{r-1}$  for some positive integers  $a_j$ 's.

In particular, we can restate the equation 3 as

$$\underline{H}_M(x) = 1 + \sum_{P_\emptyset} a_1(P_\emptyset)x + \dots + a_{r-1}(P_\emptyset)x^{r-1},$$

where the sum is taken over all chains  $P_\emptyset = (F_0 \subset F_1 \subset \dots \subset F_m)$  starting at  $\emptyset$  with  $\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1 > 0$  for all  $i$  and  $a_j(P_\emptyset)$  are some integers depending on the chain.

$$S_0 + S_1x + \dots + S_rx^r = \sum_{i \in I} \sum_{j=0}^r T_{ij}x^j$$

$$S_k = \sum_{i \in I} T_{ik}$$

$$|S_0| + |S_1|x + \dots + |S_r|x^r = \sum_{i \in I} p_i(x)$$

**3.1. Continuing without the proof of 2.1 and 2.2.** The transitivity of the Ind functor allows us to carry on analogously to the non-equivariant proof of Proposition 3.7 in [FMSV24]. Using  $G_{F_0} = G$ , we can rewrite the equation 1 in theorem 2.1 as:

$$\begin{aligned} \underline{H}_M^G(x) &= \sum_{\emptyset=F_0 \subset F_1 \subset \dots \subset F_m} \frac{|G_{F_0 \dots F_m}|}{|G|} \left( \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \text{Ind}^G \left( 1_{G_{F_0 \dots F_m}} \right) \\ &= 1_G + \sum_{\emptyset=F_0 \subset F_1 \subset \dots \subset F_m} \frac{|G_{F_0 F_1}|}{|G|} \frac{x(1 - x^{\text{rk}(F_1) - 1})}{1 - x} \\ &\quad \text{Ind}_{G_{F_0 F_1}}^G \left( \frac{|G_{F_0 \dots F_m}|}{|G_{F_0 F_1}|} \prod_{i=2}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \text{Ind}_{G_{F_0 F_1}}^G \left( 1_{G_{F_0 \dots F_m}} \right) \right) \\ &= 1_G + \sum_{F \neq \emptyset} \frac{|G_F|}{|G|} \frac{x(1 - x^{\text{rk}(F) - 1})}{1 - x} \\ &\quad \text{Ind}_{G_F}^G \left( \sum_{F=F_1 \subset \dots \subset F_m} \frac{|G_{F_1 \dots F_m}|}{|G_F|} \left( \prod_{i=2}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \text{Ind}^G \left( 1_{G_{F_1 \dots F_m}} \right) \right) \\ &= 1_G + \sum_{F \neq \emptyset} \frac{|G_F|}{|G|} \frac{x(1 - x^{\text{rk}(F) - 1})}{1 - x} \text{Ind}_{G_F}^G \left( \underline{H}_{M/F}^G \right) \end{aligned}$$

Analogously, fixing a flat  $F = F_0$  in the formula in theorem 2.2 we get,

$$\begin{aligned}
H_M^G(x) &= 1_G + \sum_{F \in \mathcal{L}} \frac{|G_F|}{|G|} \frac{x(1 - x^{\text{rk}(F)})}{1 - x} \\
&\quad \left( \sum_{F \subset F_1 \subset \dots \subset F_m} \frac{|G_{F_0 \dots F_m}|}{|G_F|} \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \text{Ind}^G \left( 1_{G_{F_0 \dots F_m}} \right) \\
&= 1_G + \sum_{F \in \mathcal{L}} \frac{|G_F|}{|G|} \frac{x(1 - x^{\text{rk}(F)})}{1 - x} \\
&\quad \text{Ind}_{G_F}^G \left( \sum_{F \subset F_1 \subset \dots \subset F_m} \frac{|G_{F_0 \dots F_m}|}{|G_F|} \left( \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \text{Ind}^{G_F} \left( 1_{G_{F_0 \dots F_m}} \right) \right) \\
&= 1_G + \sum_{F \in \mathcal{L}} \frac{|G_F|}{|G|} \frac{x(1 - x^{\text{rk}(F)})}{1 - x} \text{Ind}_{G_F}^G \left( \underline{H}_{M/F}^{G_F} \right) \\
&= 1_G + \sum_{F \neq \emptyset} \frac{|G_F|}{|G|} \frac{x(1 - x^{\text{rk}(F)})}{1 - x} \text{Ind}_{G_F}^G \left( \underline{H}_{M/F}^{G_F} \right).
\end{aligned}$$

From these two equations we get:

$$H_M(x) = \sum_{F \in \mathcal{L}} x^{\text{rk}(F)} \frac{|G_F|}{|G|} \text{Ind}^G \left( \underline{H}_{M/F}^{G_F}(x) \right).$$

We now restate the following lemma from [FMSV24] that will help us prove theorem 2.3.

**Lemma 3.2** *Let  $p(x)$  be a polynomial of degree  $d$ . There exist unique polynomials  $a(x)$  of degree  $d$  and  $b(x)$  of degree at most  $d - 1$  with the properties that  $a(x) = x^d a(x^{-1})$  and  $b(x) = x^{d-1} b(x^{-1})$ , and that satisfy*

$$p(x) = a(x) + b(x).$$

We note that this lemma is true for polynomials over any ring. In particular, it is true for polynomials over  $\text{VRep}_G$ .

**Theorem 3.3** *There is a unique way to assign to each loopless matroid  $M$  a palindromic polynomial  $\tilde{H}_M^G(x) \in \text{VRep}_G[x]$  such that the following properties hold:*

- (i) *If  $\text{rk}(M) = 0$ , then  $\tilde{H}_M^G(x) = 1_G$ .*
- (ii) *If  $\text{rk}(M) > 0$ , then  $\deg \tilde{H}_M^G(x) = \text{rk}(M) - 1$ .*
- (iii) *For every matroid  $M$ , the polynomial*

$$\tilde{H}_M^G(x) := \sum_{F \in \mathcal{L}(M)} x^{\text{rk}(F)} \frac{|G_F|}{|G|} \text{Ind}^G \left( \tilde{H}_{M/F}^{G_F}(x) \right)$$

*is palindromic.*

*Proof.* We prove the theorem by induction on the size of the ground set of  $M$ . When the ground set has cardinality 0, the polynomial  $\tilde{H}_M^G(x)$  is unique and equals  $1_G$  by the first property. We now assume that the uniqueness has been established for all the matroids with cardinality less than  $n$  and consider the matroid of cardinality  $n$ . The polynomial

$$S_M^G(x) = \sum_{F \neq \emptyset} x^{\text{rk}(F)} \frac{|G_F|}{|G|} \text{Ind}^G \left( \tilde{H}_{M/F}^{G_F}(x) \right)$$

is uniquely determined since  $\widetilde{\underline{H}}$  is unique for all the matroids  $M/F$  by the inductive step.  $S_M^G(x)$  has a summand of degree  $\text{rk}(M)$  which we get when  $F = E$ . For all other flats, the summands have degree  $\text{rk}(F) + \text{rk}(M/F) - 1 = \text{rk}(M) - 1$ .

Hence the degree of  $S_M^G(x)$  is  $\text{rk}(M)$ , and we can decompose it into unique polynomials  $a(x)$  and  $b(x) \in \text{VRep}_G[x]$  such that  $a(x)$  is palindromic of degree  $\text{rk}(M)$  and  $b(x)$  has degree at most  $\text{rk}(M) - 1$ . Taking  $\widetilde{\underline{H}}_M^G(x) = -b(x)$  we get,  $\widetilde{H}_M^G(x) = S_M^G(x) + \widetilde{\underline{H}}_M^G(x) = a(x) + b(x) + \widetilde{\underline{H}}_M^G(x) = a(x)$  which is palindromic. The uniqueness of  $\widetilde{\underline{H}}$  also implies the uniqueness of  $\widetilde{H}$ . Finally, since  $\underline{H}$  and  $H$  satisfy these conditions, they must be the polynomials  $\underline{H}$  and  $H$  respectively.  $\square$

**Theorem 3.4** *Let  $M$  be a loopless matroid. The equivariant Hilbert series of the Chow ring of  $M$  satisfies:*

$$\underline{H}_M^G(x) = \sum_{\substack{F \in \mathcal{L}(M) \\ F \neq \emptyset}} \frac{|G_F|}{|G|} \text{Ind}^G \left( \overline{\chi}_{M|_F}^{G_F}(x) \otimes \underline{H}_{M/F}^{G_F}(x) \right).$$

*Proof.* To prove this, we define the polynomials

$$\widetilde{\underline{H}}_M^G(x) := \begin{cases} 1_G & \text{if } M \text{ is empty,} \\ \sum_{\substack{F \in \mathcal{L}(M) \\ F \neq \emptyset}} \frac{|G_F|}{|G|} \text{Ind}^G \left( \overline{\chi}_{M|_F}^{G_F}(x) \otimes \underline{H}_{M/F}^{G_F}(x) \right) & \text{if } M \text{ is nonempty} \end{cases}$$

satisfy all the properties of that statement. The first two conditions are immediate to check, and for the last, it will suffice to verify that  $\widetilde{\underline{H}}_M^G(x)$  satisfies the recursion of Remark . . . . We have a

chain of equalities:

$$\begin{aligned}
& 1_G + x \sum_{\substack{F \in \mathcal{L}(\mathbf{M}) \\ F \neq E}} \frac{|G_F|}{|G|} \text{Ind}^G \left( \widetilde{\mathbf{H}}_{\mathbf{M}/F}^{G_F}(x) \right) \\
&= 1_G + x \sum_{\substack{F \in \mathcal{L}(\mathbf{M}) \\ F \neq E}} \frac{|G_F|}{|G|} \text{Ind}^G \left( \sum_{\substack{F' \in \mathcal{L}(\mathbf{M}/F) \\ F' \neq \emptyset}} \frac{|G_{FF'}|}{|G_F|} \text{Ind}_{G_{FF'}}^{G_F} \left( \overline{\chi}_{(\mathbf{M}/F)|_{F'}}^{G_{FF'}}(x) \otimes \mathbf{H}_{\mathbf{M}/F/F'}^{G_{FF'}}(x) \right) \right) \\
&= 1_G + x \sum_{\substack{F \in \mathcal{L}(\mathbf{M}) \\ F \neq E}} \sum_{\substack{F' \in \mathcal{L}(\mathbf{M}) \\ F' \supset F}} \frac{|G_{FF'}|}{|G|} \text{Ind}_{G_{FF'}}^G \left( \overline{\chi}_{(\mathbf{M}|_{F'})/F}^{G_{FF'}}(x) \otimes \mathbf{H}_{\mathbf{M}/F'}^{G_{FF'}}(x) \right) \\
&= 1_G + x \sum_{\substack{F' \in \mathcal{L}(\mathbf{M}) \\ F' \neq \emptyset}} \sum_{\substack{F \in \mathcal{L}(\mathbf{M}) \\ F \subset F'}} \frac{|G_{FF'}|}{|G|} \text{Ind}_{G_{FF'}}^G \left( \text{Ind}_{G_{FF'}}^{G_{F'}} \left( \overline{\chi}_{(\mathbf{M}|_{F'})/F}^{G_{FF'}}(x) \otimes \mathbf{H}_{\mathbf{M}/F'}^{G_{FF'}}(x) \right) \right) \\
&= 1_G + x \sum_{\substack{F' \in \mathcal{L}(\mathbf{M}) \\ F' \neq \emptyset}} \text{Ind}_{G_{F'}}^G \left( \sum_{\substack{F \in \mathcal{L}(\mathbf{M}) \\ F \subset F'}} \frac{|G_{FF'}|}{|G|} \text{Ind}_{G_{FF'}}^{G_{F'}} \left( \overline{\chi}_{(\mathbf{M}|_{F'})/F}^{G_{FF'}}(x) \right) \otimes \mathbf{H}_{\mathbf{M}/F'}^{G_{F'}}(x) \right) \\
&= 1_G + x \sum_{\substack{F' \in \mathcal{L}(\mathbf{M}) \\ F' \neq \emptyset}} \frac{|G_{F'}|}{|G|} \text{Ind}_{G_{F'}}^G \left( \mathbf{H}_{\mathbf{M}/F'}^{G_{F'}}(x) \otimes \sum_{\substack{F \in \mathcal{L}(\mathbf{M}) \\ F \subset F'}} \frac{|G_{FF'}|}{|G_{F'}|} \text{Ind}_{G_{FF'}}^{G_{F'}} \left( \overline{\chi}_{(\mathbf{M}|_{F'})/F}^{G_{FF'}}(x) \right) \right) \\
(4) \quad &= 1_G + x \sum_{\substack{F' \in \mathcal{L}(\mathbf{M}) \\ F' \neq \emptyset}} \frac{|G_{F'}|}{|G|} \text{Ind}_{G_{F'}}^G \left( \mathbf{H}_{\mathbf{M}/F'}^{G_{F'}}(x) \otimes \left( 1_{F'} + 1_{F'X} + \cdots + 1_{F'X^{\text{rk}(F')-1}} \right) \right) \\
(5) \quad &= 1_G + x \sum_{\substack{F' \in \mathcal{L}(\mathbf{M}) \\ F' \neq \emptyset}} \frac{|G_{F'}|}{|G|} \text{Ind}_{G_{F'}}^G \left( \mathbf{H}_{\mathbf{M}/F'}^{G_{F'}}(x) \right) \cdot \frac{1 - x^{\text{rk}(\mathbf{M}|_{F'})}}{1 - x} \\
&= \mathbf{H}_{\mathbf{M}}^G(x),
\end{aligned}$$

□

#### 4. GOING ON A TANGENT

#### REFERENCES

- [FMSV24] Ferroni, Luis and Matherne, Jacob P. and Stevens, Matthew and Vecchi, Lorenzo, *Hilbert–Poincaré series of matroid Chow rings and intersection cohomology*, *Advances in Mathematics*. **449** (2024) 1, 3, 4

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