On a Recurrence Relation for Matroid Polynomials

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1 Introduction

Let M be a matroid of rank r on a finite ground set E. Let $P_M(t)$ and $Q(M)$ denote the Kazhdan-Lusztig polynomial and the inverse Kazhdan-Lusztig polynomial, respectively.

We also define

$$\hat{Q}_M(t) = (-1)^{\mathrm{rk}(M)} Q_M(t),$$
$$A(M)(t) = t^{\mathrm{rk}(M)} \hat{Q}_M(t^{-2}),$$

and

$$B(M)(t) = t^{\mathrm{rk}(M)} P_M(t^{-2}).$$

Let $[F]$ and ζ^F, indexed by $\mathcal{L}(M)$, be the standard basis and the Kazhdan-Lusztig basis of the module $\mathcal{H}(M)$, respectively. We have the following relation between the two bases:

$$\zeta^F = \sum_{G \le F} B(M_G^F) \cdot [G],$$

and

$$[F] = \sum_{G \le F} A(M_G^F) \cdot \zeta^G.$$

where M_G^F is the matroid obtained by restricting M to the flat F and then contracting the flat G.

2 The Braden-Vysogorets Approach

The proof technique in the Braden-Vysogorets paper [BV20] centers around an algebraic framework using Hecke algebra modules and a specific homomorphism relating the modules for M and its deletion $M \setminus e$.

Let $\mathcal{H}(M)$ be the Hecke algebra module associated with the lattice of flats $L(M)$, equipped with the standard basis $\{[F] \mid F \in L(M)\}$ and the Kazhdan-Lusztig (KL) basis $\{\zeta^F \mid F \in L(M)\}$.

BV introduce a $\mathbb{Z}[t, t^{-1}]$-linear homomorphism $\Delta : \mathcal{H}(M) \to \mathcal{H}(M \setminus e)$. This map is defined by its action on the standard basis elements:

$$\Delta([F]) = \begin{cases} t^{-1} \cdot [F \setminus e] & \text{if } e \in \text{coloops}(M|_F) \\ [F \setminus e] & \text{otherwise.} \end{cases}$$

where $F \setminus e$ is interpreted as the corresponding flat in $L(M \setminus e)$.

We then have:

$$\begin{aligned} \Delta(\zeta^F) &= \sum_{G \le F} B(M_G^F) \cdot \Delta[G] \\ &= \sum_{H \le F;\ e \notin coloops(H)} B(M_G^F) \cdot [H \setminus e] + \sum_{H \le F;\ e \in coloops(H)} B(M_G^F) \cdot t^{-1}[H \setminus e] \\ &= \zeta^{F \setminus e} + \sum_{H \le F;\ e \in coloops(H)} B(M_G^F) \cdot t^{-1}[H \setminus e] \end{aligned}$$

This leads to the expansion (Equation (12) in [BV20]):

$$\Delta(\zeta^E) = \zeta^{E\setminus e} + \sum_{F' \in S} \tau(M_{F' \cup e})\zeta^{F'}$$

Further steps in the proof involve applying Δ to another specific element, $Z_M(t)$, related to the identity element of the algebra, and manipulating the resulting expressions using properties of the KL basis and the $P_M(t)$ polynomial to arrive at their main recurrence relation for $P_M(t)$ (Theorem 2.8).

(Note: The recurrence stated in Theorem 2.8 of the BV paper relates P_M, $P_{M\setminus e}$, and terms involving $P_{M_{F'\cup e}}$, which differs from the formula involving contraction $P_{M/e}$ that you included in your LaTeX outline. The formula you stated might be from a different source or context).