1 Introduction

The Kazhdan-Lusztig polynomial $P_M(t)$ is a fundamental invariant associated with any matroid M, as defined by Elias, Proudfoot, and Wakefield in [2]. This polynomial, denoted $P_M(t)$, exhibits formal similarities to the Kazhdan-Lusztig polynomials defined for Coxeter groups. The coefficients of $P_M(t)$ depend only on the lattice of flats $\mathcal{L}(M)$ of the matroid, and in fact, they are integral linear combinations of the flag Whitney numbers counting chains of flats with specified ranks.

In [1], Braden and Vysogorets presented a formula that relates the Kazhdan-Lusztig polynomial of a matroid M to that of the matroid obtained by deleting an element e, denoted $M \setminus e$, as well as various contractions and localizations of M. Specifically, for a simple matroid M where e is not a coloop, their main result, Theorem 2.8, states:

$$P_M(t) = P_{M\setminus e}(t) - tP_{M_e}(t) + \sum_{F\in S} \tau(M_{F\cup e}) \cdot t^{\mathrm{crk}(F)/2} \cdot P_{M^F}(t)$$

where the sum is taken over the set S of all subsets F of $E \setminus e$ such that both F and $F \cup e$ are flats of M, and $\tau(M)$ is the coefficient of $t^{(\mathrm{rk}(M)-1)/2}$ in $P_M(t)$ if $\mathrm{rk}(M)$ is odd, and zero otherwise.

The inverse Kazhdan-Lusztig polynomial $Q_M(x)$ is another important invariant. There is a related polynomial $\hat{Q}_M(x) = (-1)^{\mathrm{rk}(M)}Q_M(x)$ which acts as the inverse of the Kazhdan-Lusztig polynomial $P_M(t)$ within the incidence algebra of the lattice of flats $\mathcal{L}(M)$.

In this paper, we aim to prove the following deletion formula for $\hat{Q}_M(x)$:

Theorem 1 (Deletion Formula for $\hat{Q}_M(x)$)**.** *Let M be a simple matroid and e an element that is not a coloop. Then*

$$\hat{Q}_M(x) = \hat{Q}_{M\setminus e}(x) - (1+x) \cdot \hat{Q}_{M_e}(x) - \sum_{G\in S'} \tau(M_e^G) \cdot x^{\mathrm{rk}(G)/2} \cdot \hat{Q}_{M_G}(x)$$

where $S' = \{F \in \mathcal{L}(M) \mid e \in F$ and $F \setminus e \notin \mathcal{L}(M)\}$.

The proof of this theorem is the main goal of these notes.

2 Perverse elements and the KL basis

Let M be a matroid and $\mathcal{L}(M)$ be its lattice of flats.

- **The Module $\mathcal{H}(M)$**: Let $\mathcal{H} = \mathcal{H}(M)$ be the **free $\mathbb{Z}[t, t^{-1}]$-module with basis indexed by $\mathcal{L}(M)$**. Elements of $\mathcal{H}$ are formal sums of the form

$$\alpha = \sum_{F\in\mathcal{L}(M)} \alpha_F \cdot F, \quad \alpha_F \in \mathbb{Z}[t, t^{-1}].$$

- **The Abelian Subgroup $\mathcal{H}_p$**: $\mathcal{H}_p$ is an **abelian subgroup of $\mathcal{H}$** consisting of all $\alpha \in \mathcal{H}$ such that for every flat $F \in \mathcal{L}(M)$, the following two conditions hold:
  i. $\alpha_F \in \mathbb{Z}[t]$.
  ii. $\sum_{G\geq F} t^{\mathrm{rk}(F)-\mathrm{rk}(G)}\alpha_G \in Pal(0)$, where $Pal(0)$ is the set of Laurent polynomials $f(t)$ such that $f(t) = f(t^{-1})$.
- **The Elements ζ^F**: For any flat $F \in \mathcal{L}(M)$, an element $\zeta^F \in \mathcal{H}$ is defined as

$$\zeta^F = \sum_{G\leq F} t^{\mathrm{rk}(F)-\mathrm{rk}(G)} P_{M_G^F}(t^{-2}) \cdot G.$$

- **Basis of $\mathcal{H}_p$**: Proposition 2.13 of [1] states that the set of elements $\{\zeta^F\}_{F\in\mathcal{L}(M)}$ forms a **$\mathbb{Z}$-basis for** $\mathcal{H}_p$. Any element $\beta \in \mathcal{H}_p$ can be uniquely expressed as a linear combination of the ζ^F with integer coefficients:
$$\beta = \sum_{F\in\mathcal{L}(M)} \beta_F(0)\zeta^F.$$

This algebraic framework, involving the module $\mathcal{H}(M)$ and its subgroup $\mathcal{H}_p$ with the basis $\{\zeta^F\}$, provides a foundation for studying the Kazhdan–Lusztig polynomials of matroids, as demonstrated by its role in the derivation of deletion formulas.

3 The module homomorphism $\mathcal{H}(M) \to \mathcal{H}(M \setminus e)$

Let M be a simple matroid and e be an element of its ground set that is not a coloop. There is a surjective map $\mathcal{L}(M) \to \mathcal{L}(M \setminus e)$ sending a flat $F \in \mathcal{L}(M)$ to $F \setminus e \in \mathcal{L}(M \setminus e)$. There also exists a module homomorphism $\Delta : \mathcal{H}(M) \to \mathcal{H}(M \setminus e)$ which is $\mathbb{Z}[t, t^{-1}]$-linear and is defined on the basis elements $F \in \mathcal{L}(M)$ by

$$\Delta(F) = \begin{cases} F & \text{if } F \not\ni e \\ F \setminus e & \text{if } F \ni e \text{ and } F \setminus e \notin \mathcal{L}(M) \\ t^{-1} \cdot (F \setminus e) & \text{if } F \ni e \text{ and } F \setminus e \in \mathcal{L}(M). \end{cases}$$

This definition is taken from Section 2.6 of Braden and Vysogorets [1]. In [1], the authors also prove the following lemma:

Lemma 2. *Let $F \in \mathcal{L}(M)$ be a flat such that $e \in F$ and $F \setminus e \notin \mathcal{L}(M)$. Then*
$$\Delta(\zeta^F) = \zeta^{F\setminus e} + \sum_{G\in S(M^F)} \tau(M^F_{G\cup e}) \cdot \zeta^G$$
where $S(M^F) = \{G \in \mathcal{L}(M^F) \mid e \notin G \text{ and } G \cup e \in \mathcal{L}(M^F)\}$.

Theorem 3. *Let $F \in \mathcal{L}(M)$ be a flat such that $e \in F$ and $F \setminus e \in \mathcal{L}(M)$, then*
$$\Delta(\zeta^F) = (t + t^{-1})\zeta^{F\setminus e}$$
where $\zeta^{F\setminus e}$ on the right-hand side is the ζ-element in $\mathcal{H}(M \setminus e)$ associated with the flat $F \setminus e \in \mathcal{L}(M \setminus e)$.

Proof. Let $F_0 = F \setminus e$. We are given $F_0 \in \mathcal{L}(M)$. By the definition of $\Delta(G)$:

- If $G \leq F$ and $e \notin G$: Then $G \leq F_0$. Since $G \in \mathcal{L}(M)$ and $e \notin G$, G is also a flat in $M \setminus e$, so $G \in \mathcal{L}(M \setminus e)$. In this case, $\Delta(G) = G$.
- If $G \leq F$ and $e \in G$: Let $G_0 = G \setminus e$. Then $G_0 \leq F_0$. Since $F_0 \in \mathcal{L}(M)$, it follows that $G_0 = G \cap F_0 \in \mathcal{L}(M)$. Thus $G_0 \in \mathcal{L}(M \setminus e)$. In this case, $\Delta(G) = t^{-1}G_0$.

Applying this to $\Delta(\zeta^F)$:

$$\begin{aligned}
\Delta(\zeta^F) &= \Delta\left(\sum_{G\leq F} t^{\mathrm{rk}_M(F)-\mathrm{rk}_M(G)} P_{M^F_G}(t^{-2}) \cdot G\right) \\
&= \sum_{G\leq F, G\not\ni e} t^{\mathrm{rk}_M(F)-\mathrm{rk}_M(G)} P_{M^F_G}(t^{-2}) \cdot \Delta(G) \\
&\quad + \sum_{G\leq F, G\ni e} t^{\mathrm{rk}_M(F)-\mathrm{rk}_M(G)} P_{M^F_G}(t^{-2}) \cdot \Delta(G) \\
&= \sum_{G\leq F_0} t^{\mathrm{rk}_M(F)-\mathrm{rk}_M(G)} P_{M^F_G}(t^{-2}) \cdot G \\
&\quad + \sum_{\substack{G_0\leq F_0 \\ (G=G_0\cup\{e\})}} t^{\mathrm{rk}_M(F)-\mathrm{rk}_M(G_0\cup\{e\})} P_{M^F_{G_0\cup\{e\}}}(t^{-2}) \cdot (t^{-1}G_0).
\end{aligned}$$

We use the rank relations:

- $\mathrm{rk}_M(F) = \mathrm{rk}_{M\setminus e}(F_0) + 1$.
- For $G \leq F_0$: $\mathrm{rk}_M(G) = \mathrm{rk}_{M\setminus e}(G)$.
- For $G_0 \leq F_0$: $\mathrm{rk}_M(G_0 \cup \{e\}) = \mathrm{rk}_{M\setminus e}(G_0) + 1$.

And the Kazhdan-Lusztig polynomial identities under these conditions:

- For $G \leq F_0$: $M_G^F \cong (M \setminus e)_G^{F_0} \oplus U_{1,1}(\{e\})$, so $P_{M_G^F}(t^{-2}) = P_{(M\setminus e)_G^{F_0}}(t^{-2})$.
- For $G_0 \leq F_0$: $M_{G_0\cup\{e\}}^F \cong (M \setminus e)_{G_0}^{F_0}$, so $P_{M_{G_0\cup\{e\}}^F}(t^{-2}) = P_{(M\setminus e)_{G_0}^{F_0}}(t^{-2})$.

Substituting these into the sums:

$$\begin{aligned}
\Delta(\zeta^F) = & \sum_{G\leq F_0} t^{(\mathrm{rk}_{M\setminus e}(F_0)+1)-\mathrm{rk}_{M\setminus e}(G)} P_{(M\setminus e)_G^{F_0}}(t^{-2}) \cdot G \\
& + t^{-1} \sum_{G_0\leq F_0} t^{(\mathrm{rk}_{M\setminus e}(F_0)+1)-(\mathrm{rk}_{M\setminus e}(G_0)+1)} P_{(M\setminus e)_{G_0}^{F_0}}(t^{-2}) \cdot G_0 \\
= & \, t \sum_{G\leq F_0, G\in\mathcal{L}(M\setminus e)} t^{\mathrm{rk}_{M\setminus e}(F_0)-\mathrm{rk}_{M\setminus e}(G)} P_{(M\setminus e)_G^{F_0}}(t^{-2}) \cdot G \\
& + t^{-1} \sum_{G_0\leq F_0, G_0\in\mathcal{L}(M\setminus e)} t^{\mathrm{rk}_{M\setminus e}(F_0)-\mathrm{rk}_{M\setminus e}(G_0)} P_{(M\setminus e)_{G_0}^{F_0}}(t^{-2}) \cdot G_0.
\end{aligned}$$

Using the definition of $\zeta_{M\setminus e}^{F_0}$:

$$\begin{aligned}
\Delta(\zeta^F) &= t \cdot \zeta_{M\setminus e}^{F_0} + t^{-1} \cdot \zeta_{M\setminus e}^{F_0} \\
&= (t + t^{-1})\zeta_{M\setminus e}^{F_0}.
\end{aligned}$$

□

4 The Deletion Formula for $\hat{Q}_M(x)$

Lemma 4. *The standard basis $\{F\}_{F\in\mathcal{L}(M)}$ of $\mathcal{H}(M)$ satisfies*

$$F = \sum_{G\leq F} t^{\mathrm{rk}(F)-\mathrm{rk}(G)} \hat{Q}_{M_G^F}(t^{-2}) \cdot \zeta^G. \tag{1}$$

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. □

Proof of 1. Applying the homomorphism Δ to equation 1 when $F = E$, we get

$$[E \setminus e] = \sum_{G\leq E} t^{\mathrm{rk}(E)-\mathrm{rk}(G)} \hat{Q}_{M_G^E}(t^{-2}) \cdot \Delta(\zeta^G).$$

Using lemma 4, the coefficient of $\zeta^{\emptyset}$ in the left-hand side is $t^{\mathrm{rk}(E\setminus e)-\mathrm{rk}(\emptyset)} \cdot \hat{Q}_{M_{\emptyset}^{E\setminus e}}(t^{-2}) = t^{\mathrm{rk}(M)} \cdot \hat{Q}_{M\setminus e}(t^{-2})$.

By Theorem 3, the coefficient $\zeta^\emptyset$ on the right-hand side is non-zero only when $G \in \{\emptyset, e\} \cup S'$. In each of these cases, we have the following:

- For $G = \emptyset$, we have $\Delta(\zeta^\emptyset) = \zeta^\emptyset$.
- For $G = e$, we have $\Delta(\zeta^e) = (t + t^{-1})\zeta^\emptyset$.
- For $G \in S'$, we have $\Delta(\zeta^G) = \tau(M_e^G) \cdot \zeta^\emptyset +$ terms not including $\zeta^\emptyset$.

Collecting the coefficients of $\zeta^\emptyset$ on the right-hand side, we have:

$$\begin{aligned} &t^{\mathrm{rk}(E)-\mathrm{rk}(\emptyset)}\hat{Q}_{M^E_\emptyset}(t^{-2}) + t^{\mathrm{rk}(E)-\mathrm{rk}(e)}(t+t^{-1})\hat{Q}_{M^E_e}(t^{-2}) + \sum_{G \in S'} t^{\mathrm{rk}(E)-\mathrm{rk}(G)}\hat{Q}_{M^E_G}(t^{-2}) \cdot \tau(M_e^G) \\ &= t^{\mathrm{rk}(M)}\hat{Q}_M(t^{-2}) + t^{\mathrm{rk}(M)-1}(t+t^{-1})\hat{Q}_{M_e}(t^{-2}) + \sum_{G \in S'} t^{\mathrm{rk}(M)-\mathrm{rk}(G)}\hat{Q}_{M_G}(t^{-2}) \cdot \tau(M_e^G). \end{aligned}$$

Equating the coefficients of $\zeta^\emptyset$ on both sides, we obtain:

$$\hat{Q}_{M\setminus e}(t^{-2}) = \hat{Q}_M(t^{-2}) + (1+t^{-2})\hat{Q}_{M_e}(t^{-2}) + \sum_{G \in S'} t^{-\mathrm{rk}(G)}\hat{Q}_{M_G}(t^{-2}) \cdot \tau(M_e^G).$$

Finally, taking $x = t^{-2}$ and rearranging yields the desired statement:

$$\hat{Q}_M(x) = \hat{Q}_{M\setminus e}(x) - (1+x) \cdot \hat{Q}_{M_e}(x) - \sum_{G \in S'} \tau(M_e^G) \cdot x^{\mathrm{rk}(G)/2} \cdot \hat{Q}_{M_G}(x).$$

□

The inverse Kazhdan-Lusztig polynomial $Q_M(x) = (-1)^{\mathrm{rk}(M)}\hat{Q}_M(x)$ then satisfies the following:

Corollary 5. *Let M be a simple matroid and e an element that is not a coloop. Then*

$$Q_M(x) = Q_{M\setminus e}(x) + (1+x) \cdot Q_{M_e}(x) - \sum_{G \in S'} \tau(M_e^G) \cdot x^{\mathrm{rk}(G)/2} \cdot Q_{M_G}(x)$$

where $S' = \{F \in \mathcal{L}(M) \mid e \in F$ and $F \setminus e \notin \mathcal{L}(M)\}$.

Corollary 6. *Let $M := U_{m,d}$ be the uniform matroid of rank d on $m+d$ elements with $d \geq 1$. Then*

$$Q_{U_{m,d}}(x) = Q_{U_{m-1,d}}(x) + (1+x) \cdot Q_{U_{m,d-1}}(x) - \tau(U_{m,d-1}) \cdot x^{d/2}.$$

Proof. As none of the elements in the ground set are coloops, we may apply the deletion formula for a generic element e in the ground set. The set S' then consists only of the top flat E and the sum over S' reduces to

$$\tau(M_e) \cdot x^{\mathrm{rk}(M)/2} = \tau(U_{m,d-1}) \cdot x^{d/2}.$$

□

♣ Jacob: [For the previous corollary, can you now use induction to get a formula for $Q_{m,d}$ for any m and d that does not have any Q's in it? (Perhaps starting with the fact the inverse KL polynomials of Boolean matroids are equal to 1?)]

References

[1] Tom Braden and Artem Vysogorets. "Kazhdan–Lusztig polynomials of matroids under deletion". In: *Electron. J. Combin.* 27.1 (2020), P1.17.

[2] Ben Elias, Nicholas Proudfoot, and Max Wakefield. "The Kazhdan-Lusztig polynomial of a matroid". In: *Adv. Math.* 299 (2016), pp. 36–70.