Problems from "Problems from the Book"

Problems

1. **Problem 1:** Find all triples x, y, z of positive real numbers, solutions to the system:
$$\begin{cases} x^2 + y^2 + z^2 = xyz + 4 \\ xy + yz + zx = 2(x + y + z) \end{cases}$$

2. **Problem 2:** Prove that if $x, y, z > 0$ satisfy $xy + yz + zx + 2xyz = 1$, then
$$xyz \leq \frac{1}{8} \quad \text{and} \quad xy + yz + zx \geq \frac{3}{4}.$$

3. **Problem 3:** Prove that for any positive real numbers a, b, c the following inequality holds:
$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{9}{2}.$$

4. **Problem 4:** Let $a, b, c > 0$ such that $a^2 + b^2 + c^2 + abc = 4$. Prove that
$$\sqrt{\frac{(2-a)(2-b)}{(2+a)(2+b)}} + \sqrt{\frac{(2-b)(2-c)}{(2+b)(2+c)}} + \sqrt{\frac{(2-c)(2-a)}{(2+c)(2+a)}} = 1.$$

5. **Problem 5:** Prove that if $a, b, c \geq 0$ satisfy the condition $|a^2 + b^2 + c^2 - 4| = abc$, then

$$(a-2)(b-2) + (b-2)(c-2) + (c-2)(a-2) \geq 0.$$

6. **Problem 6:** Prove that if $x, y, z > 0$ and $xyz = x + y + z + 2$, then

$$xy + yz + zx \geq 2(x+y+z) \quad \text{and} \quad \sqrt{x} + \sqrt{y} + \sqrt{z} \leq \frac{3}{2}\sqrt{xyz}.$$

7. **Problem 7:** Let $x, y, z > 0$ such that $xy + yz + zx = 2(x+y+z)$. Prove that $xyz \leq x + y + z + 2$.

8. **Problem 8:** Prove that in any triangle ABC the following inequality holds:

$$\cos A + \cos B + \cos C \geq \frac{1}{4}\left(3 + \cos(A-B) + \cos(B-C) + \cos(C-A)\right).$$

9. **Problem 9:** Prove that in every acute-angled triangle ABC,

$$(\cos A + \cos B)^2 + (\cos B + \cos C)^2 + (\cos C + \cos A)^2 \leq 3.$$

10. **Problem 10:** Find all triples (a, b, c) of positive real numbers, solutions to the system:

$$\begin{cases} a^2 + b^2 + c^2 + abc = 4 \\ a + b + c = 3 \end{cases}$$

11. **Problem 11:** Find all triplets of positive integers (k, l, m) with sum 2002 and for which the system

$$\begin{cases} \frac{x}{y} + \frac{y}{x} = k \\ \frac{y}{z} + \frac{z}{y} = l \\ \frac{z}{x} + \frac{x}{z} = m \end{cases}$$

has real solutions.

12. **Problem 12:** Prove that in any triangle the following inequality holds:

$$\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right)^2 \le \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}.$$

13. **Problem 13:** Find all functions $f : (0, \infty) \to (0, \infty)$ with the following properties:
    (a) $f(x) + f(y) + f(z) + f(xyz) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all x, y, z;
    (b) if $1 \le x < y$, then $f(x) < f(y)$.

14. **Problem 14:** Prove that if $a, b, c \ge 2$ satisfy the condition $a^2 + b^2 + c^2 = abc + 4$, then

$$a + b + c + ab + ac + bc \ge 2\sqrt{(a + b + c + 3)(a^2 + b^2 + c^2 - 3)}.$$

15. **Problem 15:** Let $x, y, z > 0$ such that $xy + yz + zx + xyz = 4$. Prove that

$$3\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}}\right)^2 \ge (x + 2)(y + 2)(z + 2).$$

16. **Problem 16:** Prove that in any acute-angled triangle the following inequality holds:

$$\left(\frac{\cos A}{\cos B}\right)^2 + \left(\frac{\cos B}{\cos C}\right)^2 + \left(\frac{\cos C}{\cos A}\right)^2 + 8\cos A\cos B\cos C \geq 4.$$

17. **Problem 17:** Solve in positive integers the equation

$$(x+2)(y+2)(z+2) = (x+y+z+2)^2.$$

18. **Problem 18:** Let $n > 4$ be a given positive integer. Find all pairs of positive integers (x, y) such that

$$xy - \frac{(x+y)^2}{n} = n - 4.$$

19. **Problem 19:** Let the sequence $(a_n)_n \geq 0$, where $a_0 = a_1 = 97$ and

$$a_{n+1} = a_{n-1}a_n + \sqrt{(a_n^2 - 1)(a_{n-1}^2 - 1)}$$

for all $n \geq 1$. Prove that $2 + \sqrt{2 + 2a_n}$ is a perfect square for all $n \geq 0$.

20. **Problem 20:** Prove that if $a, b, c \geq 0$ satisfy $a^2 + b^2 + c^2 + abc = 4$ then

$$0 \leq ab + bc + ca - abc \leq 2.$$