EQUIVARIANT CHOW POLYNOMIALS OF MATROIDS

NUTAN NEPAL

ABSTRACT. beeboop beeboop...

1. INTRODUCTION

1.1. **Overview.** Given a matroid $\mathsf{M} = (E, \mathcal{I})$, we can define its Chow ring $\underline{\mathrm{CH}}$ and augmented Chow ring CH for which the bases are given by:

$$\mathrm{FY} = \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} : (\varnothing = F_0) \subset F_1 \subset F_2 \subset \cdots \subset F_k, \text{ and } m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1\},$$

and

$$\widetilde{FY} = \{x_{F_1}^{a_1} x_{F_2}^{a_2} \cdots x_{F_m}^{a_m} \mid \varnothing \subset F_1 \subset \cdots \subset F_m,\ 1 \leq a_1 \leq \mathrm{rk}(F_1),\ a_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 \text{ for } i > 1\}$$

respectively. Here $\varnothing \subset F_1 \subset \cdots \subset F_m$ is a strictly increasing chain of flats of the matroid M. The following theorems were proved in [FMSV24]:.

Theorem 1.1 *There is a unique way to assign to each loopless matroid* M *a palindromic polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following properties hold:*

(i) *If* $\mathrm{rk}(\mathsf{M}) = 0$*, then* $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1$.
(ii) *If* $\mathrm{rk}(\mathsf{M}) > 0$*, then* $\deg \underline{\mathrm{H}}_{\mathsf{M}}(x) = \mathrm{rk}(\mathsf{M}) - 1$.
(iii) *For every matroid* M*, the polynomial*

$$\mathrm{H}_{\mathsf{M}}(x) := \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\mathrm{rk}(F)}\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x)$$

is palindromic.

Theorem 1.2 *There is a unique way to assign to each loopless matroid* M *a polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following conditions hold:*

(i) *If* $\mathrm{rk}(\mathsf{M}) = 0$*, then* $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1$.
(ii) *For every matroid* M*, the following recursion holds:*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \varnothing}} \overline{\chi}_{\mathsf{M}|_F}(x)\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x).$$

It was shown in [FMSV24] that these polynomials H_{M} and $\underline{\mathrm{H}}_{\mathsf{M}}$ are the Hilbert-Poincare series for the augmented Chow ring CH and the Chow ring $\underline{\mathrm{CH}}$ respectively. In other words,

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = |FY^0| + |FY^1|x + \cdots + |FY^{r-1}|x^{r-1}$$

where $|FY^i|$ denotes the number of fy-monomials of degree i (which equals the dimension of the degree i piece of the Chow ring).

2. Matroids

2.1. **Action.** Given a finite set E with n elements, the symmetric group $\mathfrak{S}_n$ can always act on E by permutation and this induces an action on the power set 2^E. If $\mathsf{M} = (E, \mathcal{I})$ is a matroid, let G be the stabilizer subgroup $\mathsf{Stab}_{\mathfrak{S}_n}(\mathcal{I})$ of $\mathfrak{S}_n$ that stabiizes the set $\mathcal{I} \subseteq 2^E$. Then, we say that the group G acts on the matroid M.

The action of G on M induces an action on the Chow ring and the augmented Chow ring of M by permuting the fy-monomials. Since the action doesn't affect the degree of the monomials, it is clear that G acts on the Chow ring by acting separately on each graded piece of CH.

Let V^i denote the permutation representation of G on the set FY^i. Then $\dim(V^i) = |FY^i|$, the dimension of the i-th piece. We define the polynomial $V^0 + V^1 x + \cdots V^{r-1} x^{r-1} \in \mathrm{VRep}_G[x]$ to be the equivariant Chow polynomial of the matroid M. In this paper, we will prove the following theorems:

Theorem 2.1 *Let M be a loopless matroid and $\underline{\mathrm{H}}^G_{\mathsf{M}}$ be its equivariant Chow polynomial. Then $\underline{\mathrm{H}}^G_{\mathsf{M}}$ is given by*

$$\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = \sum_{\emptyset = F_0 \subset F_1 \subset \cdots \subset F_m} \frac{|G_{F_0 \cdots F_m}|}{|G|} \left(\prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1})}{1-x} \right) \mathrm{Ind}^G \left(1_{G_{F_0 \cdots F_m}} \right). \tag{1}$$

Here, $G_{F_0 \cdots F_m} = G_{F_0} \cap \cdots \cap G_{F_m}$ denotes the stabilizer of the chain $(F_0 \subset F_1 \subset \cdots \subset F_m)$ and the sum is taken over all nonempty chains of flats starting at $\emptyset$.

Theorem 2.2 *Let M be a loopless matroid and $\mathrm{H}^G_{\mathsf{M}}$ be its equivariant augmented Chow polynomial. Then $\mathrm{H}^G_{\mathsf{M}}$ is given by*
(2)
$$\mathrm{H}^G_{\mathsf{M}}(x) = 1_G + \sum_{F_0 \subset F_1 \subset \cdots \subset F_m} \frac{|G_{F_0 \cdots F_m}|}{|G|} \frac{x(1 - x^{\mathrm{rk}(F_0)})}{1-x} \left(\prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1})}{1-x} \right) \mathrm{Ind}^G \left(1_{G_{F_0 \cdots F_m}} \right).$$

Here, $G_{F_0 \cdots F_m} = G_{F_0} \cap \cdots \cap G_{F_m}$ denotes the stabilizer of the chain $(F_0 \subset F_1 \subset \cdots \subset F_m)$ and the sum is taken over all nonempty chains of flats.

Theorem 2.3 *There is a unique way to assign to each loopless matroid M a palindromic polynomial $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) \in \mathrm{VRep}_G[x]$ such that the following properties hold:*

(i) *If $\mathrm{rk}(\mathsf{M}) = 0$, then $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = 1_G$.*
(ii) *If $\mathrm{rk}(\mathsf{M}) > 0$, then $\deg \underline{\mathrm{H}}^G_{\mathsf{M}}(x) = \mathrm{rk}(\mathsf{M}) - 1$.*
(iii) *For every matroid M, the polynomial*

$$\mathrm{H}^G_{\mathsf{M}}(x) := \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\mathrm{rk}(F)} \frac{|G_F|}{|G|} \mathrm{Ind}^G \left(\underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}(x) \right)$$

is palindromic.

Theorem 2.4 *There is a unique way to assign to each loopless matroid M a polynomial $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) \in \mathrm{VRep}_G[x]$ such that the following conditions hold:*

(i) *If $\mathrm{rk}(\mathsf{M}) = 0$, then $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = 1_G$.*
(ii) *For every matroid M, the following recursion holds:*

$$\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \emptyset}} \frac{|G_F|}{|G|} \mathrm{Ind}^G \left(\overline{\chi}^{G_F}_{\mathsf{M}|_F}(x) \otimes \underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}(x) \right).$$

3. Scratch

Proposition 3.1 *Let* M *be a loopless matroid. The Hilbert–Poincare series of the Chow ring* $\underline{\mathrm{CH}}(M)$ *is given by*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \sum_{\emptyset=F_0\subset F_1\subset\cdots\subset F_m} \left(\prod_{i=1}^{m} \frac{x(1-x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x}\right). \tag{3}$$

Here, the sum is taken over all nonempty chains of flats starting at $\emptyset$.

We notice a few things about the formula in 3.1:

(i) The chain corresponding to just the empty flat gives an empty product which equals 1.
(ii) If $\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1=0$ for some i in the chain $F_0\subset F_1\subset\cdots\subset F_m$, then the product is 0.
(iii) So, given a chain $F_0\subset F_1\subset\cdots\subset F_m$ with $\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1>0$ for all i, we can write the product as $a_1x+\cdots+a_{r-1}x^{r-1}$ for some positive integers a_j's.

In particular, we can restate the equation 3 as

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1+\sum_{P_\emptyset} a_1(P_\emptyset)x+\cdots+a_{r-1}(P_\emptyset)x^{r-1},$$

where the sum is taken over all chains $P_\emptyset=(F_0\subset F_1\subset\cdots\subset F_m)$ starting at $\emptyset$ with $\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1>0$ for all i and $a_j(P_\emptyset)$ are some integers depending on the chain.

$$S_0+S_1x+\cdots+S_rx^r = \sum_{i\in I}\sum_{j=0}^{r} T_{ij}x^j$$
$$S_k = \sum_{i\in I} T_{ik}$$
$$|S_0|+|S_1|x+\cdots+|S_r|x^r = \sum_{i\in I} p_i(x)$$

3.1. **Continuing without the proof of 2.1 and 2.2.** The transitivity of the Ind functor allows us to carry on analogously to the non-equivariant proof of Proposition 3.7 in [FMSV24]. Using $G_{F_0}=G$, we can rewrite the equation 1 in theorem 2.1 as:

$$\begin{aligned}
\underline{\mathrm{H}}^G_{\mathsf{M}}(x) &= \sum_{\emptyset=F_0\subset F_1\subset\cdots\subset F_m} \frac{|G_{F_0\cdots F_m}|}{|G|}\left(\prod_{i=1}^{m}\frac{x(1-x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x}\right)\mathrm{Ind}^G\left(1_{G_{F_0\cdots F_m}}\right)\\
&= 1_G + \sum_{\emptyset=F_0\subset F_1\subset\cdots\subset F_m} \frac{|G_{F_0F_1}|}{|G|}\frac{x(1-x^{\mathrm{rk}(F_1)-1})}{1-x}\\
&\qquad \mathrm{Ind}^G_{G_{F_0F_1}}\left(\frac{|G_{F_0\cdots F_m}|}{|G_{F_0F_1}|}\prod_{i=2}^{m}\frac{x(1-x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x}\mathrm{Ind}^{G_{F_0F_1}}\left(1_{G_{F_0\cdots F_m}}\right)\right)\\
&= 1_G + \sum_{F\neq\emptyset}\frac{|G_F|}{|G|}\frac{x(1-x^{\mathrm{rk}(F)-1})}{1-x}\\
&\qquad \mathrm{Ind}^G_{G_F}\left(\sum_{F=F_1\subset\cdots\subset F_m}\frac{|G_{F_1\cdots F_m}|}{|G_F|}\left(\prod_{i=2}^{m}\frac{x(1-x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x}\right)\mathrm{Ind}^G\left(1_{G_{F_1\cdots F_m}}\right)\right)\\
&= 1_G + \sum_{F\neq\emptyset}\frac{|G_F|}{|G|}\frac{x(1-x^{\mathrm{rk}(F)-1})}{1-x}\mathrm{Ind}^G_{G_F}\left(\underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}\right)
\end{aligned}$$

Analogously, fixing a flat $F = F_0$ in the formula in theorem 2.2 we get,

$$\begin{aligned}
\mathrm{H}^G_{\mathsf{M}}(x) &= 1_G + \sum_{F\in\mathcal{L}} \frac{|G_F|}{|G|}\frac{x(1-x^{\mathrm{rk}(F)})}{1-x} \\
&\qquad \left(\sum_{F\subset F_1\subset\cdots\subset F_m} \frac{|G_{F_0\cdots F_m}|}{|G_F|}\prod_{i=1}^{m}\frac{x(1-x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x}\right)\mathrm{Ind}^G\left(1_{G_{F_0\cdots F_m}}\right) \\
&= 1_G + \sum_{F\in\mathcal{L}} \frac{|G_F|}{|G|}\frac{x(1-x^{\mathrm{rk}(F)})}{1-x} \\
&\qquad \mathrm{Ind}^G_{G_F}\left(\sum_{F\subset F_1\subset\cdots\subset F_m} \frac{|G_{F_0\cdots F_m}|}{|G_F|}\left(\prod_{i=1}^{m}\frac{x(1-x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x}\right)\mathrm{Ind}^{G_F}\left(1_{G_{F_0\cdots F_m}}\right)\right) \\
&= 1_G + \sum_{F\in\mathcal{L}} \frac{|G_F|}{|G|}\frac{x(1-x^{\mathrm{rk}(F)})}{1-x}\,\mathrm{Ind}^G_{G_F}\left(\underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}\right) \\
&= 1_G + \sum_{F\neq\emptyset} \frac{|G_F|}{|G|}\frac{x(1-x^{\mathrm{rk}(F)})}{1-x}\,\mathrm{Ind}^G_{G_F}\left(\underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}\right).
\end{aligned}$$

From these two equations we get:

$$\mathrm{H}_{\mathsf{M}}(x) = \sum_{F\in\mathcal{L}} x^{\mathrm{rk}(F)}\frac{|G_F|}{|G|}\mathrm{Ind}^G\left(\underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}(x)\right).$$

We now restate the following lemma from [FMSV24] that will help us prove theorem 2.3.

Lemma 3.2 *Let $p(x)$ be a polynomial of degree d. There exist unique polynomials $a(x)$ of degree d and $b(x)$ of degree at most $d-1$ with the properties that $a(x) = x^d a(x^{-1})$ and $b(x) = x^{d-1}b(x^{-1})$, and that satisfy*

$$p(x) = a(x) + b(x).$$

We note that this lemma is true for polynomials over any ring. In particular, it is true for polynomials over VRep_G.

Proof of Theorem 2.2. We prove the theorem by induction on the size of the ground set of M. When the ground set has cardinality 0, $\mathrm{rk}(\mathsf{M}) = 0$ and the polynomial $\underline{\mathrm{H}}^G_{\mathsf{M}}(x)$ is unique and equals 1_G by the first property. We now assume that the uniqueness has been established for all the matroids with cardinality less than n and consider the matroid of cardinality n. The polynomial

$$S^G_{\mathsf{M}}(x) = \sum_{F\neq\emptyset} x^{\mathrm{rk}(F)}\frac{|G_F|}{|G|}\mathrm{Ind}^G\left(\underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}(x)\right)$$

is uniquely determined since $\underline{\mathrm{H}}$ is unique for all the matroids M/F by the inductive step. We have a summand of degree $\mathrm{rk}(M)$ which we get when $F = E$. For all other flats, the summands have degree $\mathrm{rk}(F) + \mathrm{rk}(\mathsf{M}/F) - 1 = \mathrm{rk}(\mathsf{M}) - 1$.

Hence the degree of $S^G_{\mathsf{M}}(x)$ is $\mathrm{rk}(\mathsf{M})$ and we can decompose it into unique polynomials $a(x)$ and $b(x) \in \mathrm{VRep}_G[x]$ such that a is palindromic and b has degree at most $\mathrm{rk}(\mathsf{M}) - 1$. □

Theorem 3.3 *Let M be a loopless matroid. The equivariannt Hilbert series of the Chow ring of M satisfies:*

$$\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq\emptyset}} \frac{|G_F|}{|G|}\mathrm{Ind}^G\left(\overline{\chi}^{G_F}_{\mathsf{M}|_F}(x)\otimes\underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}(x)\right).$$

Proof. To prove this, we define the polynomials

$$\underline{\widetilde{\mathrm{H}}}^{G}_{\mathsf{M}}(x) := \begin{cases} 1_G & \text{if } \mathsf{M} \text{ is empty,} \\ \displaystyle\sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \neq \varnothing}} \frac{|G_F|}{|G|} \operatorname{Ind}^G \left(\overline{\chi}^{G_F}_{\mathsf{M}|_F}(x) \otimes \underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}(x) \right) & \text{if } \mathsf{M} \text{ is nonempty} \end{cases}$$

satisfy all the properties of that statement. The first two conditions are immediate to check, and for the last, it will suffice to verify that $\underline{\widetilde{\mathrm{H}}}_{\mathsf{M}}(x)$ satisfies the recursion of Remark .... We have a chain of equalities:

$$\begin{aligned}
& 1_G + x \sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \neq E}} \frac{|G_F|}{|G|} \operatorname{Ind}^G \left(\underline{\widetilde{\mathrm{H}}}^{G_F}_{\mathsf{M}/F}(x) \right) \\
= \; & 1_G + x \sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \neq E}} \frac{|G_F|}{|G|} \operatorname{Ind}^G \left(\sum_{\substack{F' \in \mathscr{L}(\mathsf{M}/F) \\ F' \neq \varnothing}} \frac{|G_{FF'}|}{|G_F|} \operatorname{Ind}^{G_F}_{G_{FF'}} \left(\overline{\chi}^{G_{FF'}}_{(\mathsf{M}/F)|_{F'}}(x) \otimes \underline{\mathrm{H}}^{G_{FF'}}_{\mathsf{M}/F/F'}(x) \right) \right) \\
= \; & 1_G + x \sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \neq E}} \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}) \\ F' \supset F}} \frac{|G_{FF'}|}{|G|} \operatorname{Ind}^{G}_{G_{FF'}} \left(\overline{\chi}^{G_{FF'}}_{(\mathsf{M}|_{F'})/F}(x) \otimes \underline{\mathrm{H}}^{G_{FF'}}_{\mathsf{M}/F'}(x) \right) \\
= \; & 1_G + x \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}) \\ F' \neq \varnothing}} \sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \subset F'}} \frac{|G_{FF'}|}{|G|} \operatorname{Ind}^{G}_{G_{F'}} \left(\operatorname{Ind}^{G_{F'}}_{G_{FF'}} \left(\overline{\chi}^{G_{FF'}}_{(\mathsf{M}|_{F'})/F}(x) \otimes \underline{\mathrm{H}}^{G_{FF'}}_{\mathsf{M}/F'}(x) \right) \right) \\
= \; & 1_G + x \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}) \\ F' \neq \varnothing}} \operatorname{Ind}^{G}_{G_{F'}} \left(\sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \subset F'}} \frac{|G_{FF'}|}{|G|} \operatorname{Ind}^{G_{F'}}_{G_{FF'}} \left(\overline{\chi}^{G_{FF'}}_{(\mathsf{M}|_{F'})/F}(x) \right) \otimes \underline{\mathrm{H}}^{G_{F'}}_{\mathsf{M}/F'}(x) \right) \\
= \; & 1_G + x \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}) \\ F' \neq \varnothing}} \frac{|G_{F'}|}{|G|} \operatorname{Ind}^{G}_{G_{F'}} \left(\underline{\mathrm{H}}^{G_{F'}}_{\mathsf{M}/F'}(x) \otimes \sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \subset F'}} \frac{|G_{FF'}|}{|G_{F'}|} \operatorname{Ind}^{G_{F'}}_{G_{FF'}} \left(\overline{\chi}^{G_{FF'}}_{(\mathsf{M}|_{F'})/F}(x) \right) \right) \\
\text{(4)} \quad = \; & 1_G + x \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}) \\ F' \neq \varnothing}} \frac{|G_{F'}|}{|G|} \operatorname{Ind}^{G}_{G_{F'}} \left(\underline{\mathrm{H}}^{G_{F'}}_{\mathsf{M}/F'}(x) \otimes \left(1_{F'} + 1_{F'} x + \cdots + 1_{F'} x^{\operatorname{rk}(F')-1} \right) \right) \\
\text{(5)} \quad = \; & 1_G + x \sum_{\substack{F' \in \mathscr{L}(\mathsf{M}) \\ F' \neq \varnothing}} \frac{|G_{F'}|}{|G|} \operatorname{Ind}^{G}_{G_{F'}} \left(\underline{\mathrm{H}}^{G_{F'}}_{\mathsf{M}/F'}(x) \right) \cdot \frac{1 - x^{\operatorname{rk}(\mathsf{M}|_{F'})}}{1 - x} \\
= \; & \mathrm{H}^{G}_{\mathsf{M}}(x),
\end{aligned}$$

□

4. Going on a tangent

References

[FMSV24] Ferroni, Luis and Matherne, Jacob P. and Stevens, Matthew and Vecchi, Lorenzo, *Hilbert–Poincaré series of matroid Chow rings and intersection cohomology*, Advances in Mathematics. **449** (2024) 1, 3, 4

(N. Nepal) North Carolina State University
Email address: nnepal2@ncsu.edu