Equivariant Chow Polynomials of Matroids

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Goal

When the Chow ring $\underline{\mathrm{CH}}(\mathsf{M})$ of a matroid M carries an action of a group G, we study the equivariant Chow polynomial $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) \in \mathrm{VRep}_G[x]$:

$$\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = \underline{\mathrm{CH}}^0(\mathsf{M}) + \underline{\mathrm{CH}}^1(\mathsf{M})x + \cdots + \underline{\mathrm{CH}}^{\mathrm{rk}(\mathsf{M})-1}(\mathsf{M})x^{\mathrm{rk}(\mathsf{M})-1}$$

and describe some of its properties.

Introduction

A matroid M on the ground set E with n elements can be identified with a geometric lattice $\mathcal{L}(\mathsf{M}) \subseteq 2^{[n]}$. The following are lattices corresponding to the braid matroid $\mathsf{M}(K_4)$ (the graphic matroid associated to the complete graph on 4 vertices) and the uniform matroid $U_{3,4}$.

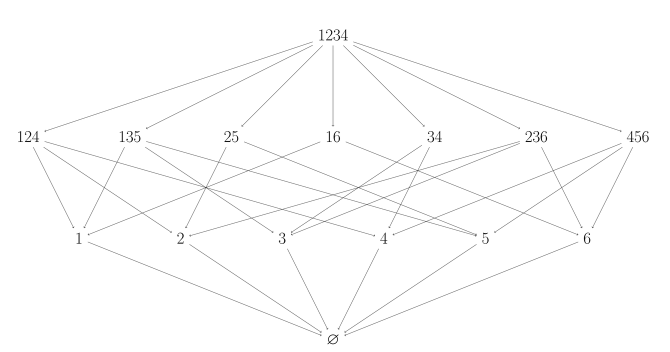

Figure 1: $\mathsf{M}(K_4)$.

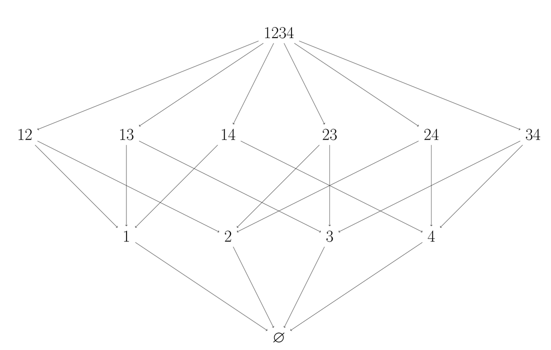

Figure 2: $U_{3,4}$.

Chow rings of matroids

The Chow ring of a matroid was first introduced by Feichtner and Yuzvinsky in [3]. Adiprasito, Huh and Katz [1] use this ring to prove the Heron–Rota–Welsh conjecture: the sequence of absolute values of the coefficients of the characteristic polynomial of a matroid is log-concave. For a loopless matroid M on $[n]$, **the Chow ring** $\underline{\mathrm{CH}}(\mathsf{M})$ is defined as:

$$\underline{\mathrm{CH}}(\mathsf{M}) := \mathbb{Q}\left[\{x_F\}_{F \in \mathcal{L}(\mathsf{M}) \setminus \{\varnothing\}}\right] / (I + J)$$

where I is the ideal $\langle x_F x_G :\ F, G \text{ are incomparable}\rangle$ and J is the ideal $\langle \sum_i x_F :\ F \ni i\rangle$ for $1 \le i \le n$. The ring is graded and has a basis given by the following **FY-monomials**:

$$x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} \ :\ \varnothing = F_0 \subset F_1 \subset \cdots \subset F_k;$$
$$0 \le m_i \le \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1.$$

We denote by FY^i the set of degree i FY-monomials. The restriction on the exponents m_i of x_{F_i} ensures that there are exactly $\mathrm{rk}(\mathsf{M})$ graded pieces. The **(non-equivariant) Chow polynomial** $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ is defined as the Hilbert series of $\underline{\mathrm{CH}}(\mathsf{M})$:

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = a_0 + a_1 x + \cdots + a_{\mathrm{rk}(\mathsf{M})-1} x^{\mathrm{rk}(\mathsf{M})-1},$$

where $a_i = \dim \underline{\mathrm{CH}}^i(\mathsf{M})$.

Examples of Chow rings of matroids

- The Chow ring of $\mathsf{M}(K_4)$ has basis given by the FY-monomials $\mathsf{FY}^0 = \{1\}$, $\mathsf{FY}^1 = \{x_{124}, x_{135}, x_{25}, x_{16}, x_{34}, x_{236}, x_{456}, x_{1234}\}$, $\mathsf{FY}^2 = \{x_{1234}^2\}$. Thus, $\dim \underline{\mathrm{CH}}^0(\mathsf{M}) = 1$, $\dim \underline{\mathrm{CH}}^1(\mathsf{M}) = 8$ and $\dim \underline{\mathrm{CH}}^2(\mathsf{M}) = 1$.
- THe Chow ring of $U_{3,4}$ has basis given by $\mathsf{FY}^0 = \{1\}$, $\mathsf{FY}^1 = \{x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, x_{1234}\}$, $\mathsf{FY}^2 = \{x_{1234}^2\}$. Thus, $\dim \underline{\mathrm{CH}}^0(\mathsf{M}) = 1$, $\dim \underline{\mathrm{CH}}^1(\mathsf{M}) = 7$ and $\dim \underline{\mathrm{CH}}^2(\mathsf{M}) = 1$.

Equivariant Chow Polynomial

For a matroid M with an action of a group G, there is an induced action on the Chow ring of M. It can be shown that G acts on each graded piece of $\underline{\mathrm{CH}}(\mathsf{M})$ separately by permuting the FY-monomials of that degree.

Theorem [Angarone–Nathanson–Reiner[2]

] Let M be a simple matroid of rank $r+1$ with G a group of automorphisms of M. Then there exist

- G-equivariant bijections $\pi : \mathsf{FY}^k \to \mathsf{FY}^{r-k}$ for $k \le r/2$, and
- G-equivariant injections $\lambda : \mathsf{FY}^k \to \mathsf{FY}^{k+1}$ for $k < r/2$.

The **equivariant Chow polynomial** $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) \in \mathrm{VRep}_G[x]$ is defined as:

$$\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = P(\mathsf{FY}^0) + P(\mathsf{FY}^1)x + \cdots P(\mathsf{FY}^{\mathrm{rk}(\mathsf{M})-1})x^{\mathrm{rk}(\mathsf{M})-1}$$

where $P(\mathsf{FY}^i)$ denotes the permutation representation of G on the set FY^i of degree i FY-monomials.

Examples of equivariant Chow polynomials

- For $\mathfrak{S}_4 \curvearrowright \mathsf{M}(K_4)$, the equivariant Chow polynomial is:
$$\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = V_{(4)} + (V_{(4)}^{\oplus 3} \oplus V_{(3,1)} \oplus V_{(2,2)})x + V_{(4)}x^2$$
where V_λ denotes the irreducible representation of $\mathfrak{S}_4$ corresponding to the partition λ.
- For $\mathfrak{S}_4 \curvearrowright U_{3,4}$: the polynomial is
$$V_{(4)} + (V_{(4)}^{\oplus 2} \oplus V_{(3,1)} \oplus V_{(2,2)})x + V_{(4)}x^2.$$
- For $\mathfrak{S}_4 \curvearrowright U_{4,4}$: the polynomial is
$$V_{(4)} + (V_{(4)}^{\oplus 3} \oplus V_{(3,1)}^{\oplus 2} \oplus V_{(2,2)})x + (V_{(4)}^{\oplus 3} \oplus V_{(3,1)}^{\oplus 2} \oplus V_{(2,2)})x^2 + V_{(4)}x^3.$$

In these examples, we can see the G-equivariant bijections and injections claimed in the previous theorem.

Theorem [Nepal]

There is a unique way to assign to each loopless matroid M a polynomial $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) \in \mathrm{VRep}_G[x]$ such that the following conditions hold:

1. If $\mathrm{rk}(M) = 0$, then $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1_G$.
2. For every matroid M, the following recursion holds:
$$\underline{\mathrm{H}}^G_{\mathsf{M}}(x) = \sum_{[F] \in \mathcal{L}(M)/G} \mathrm{Ind}^G_{G_F}\left(\overline{\chi}^{G_F}_{\mathsf{M}|_F}(x) \otimes \underline{\mathrm{H}}^{G_F}_{\mathsf{M}/F}(x)\right).$$

This theorem relies on results of Liao [5] and equivariant versions of results in [4]. Future work on equivariant Chow polynomials includes:

- Recover formulas for uniform matroids in [5] using the recursion.
- Find formulas for braid matroids and thagomizer matroids.
- Find similar recursion for other building sets.

Acknowledgements

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