Equivariant Chow Polynomials of Matroids

Nutan Nepal

North Carolina State University

Goal

Define the equivariant Chow polynomial $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) \in \mathrm{VRep}_G[x]$ of a matroid M :

$$\mathrm{H}^G_{\mathsf{M}}(x) = P(A_0) + P(A_1)x + \cdots P(A_{\mathrm{rk}(M)-1})x^{\mathrm{rk}(M)-1}$$

and describe some of its properties.

Introduction

A matroid M on the ground set E can be identified with a geometric lattice $\mathcal{L}(\mathsf{M}) \subseteq 2^{[n]}$

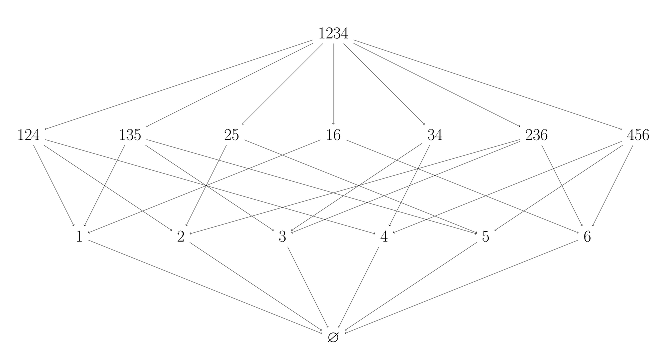

Figure 1:Lattice of flats of the matroid on the complete graph K_4.

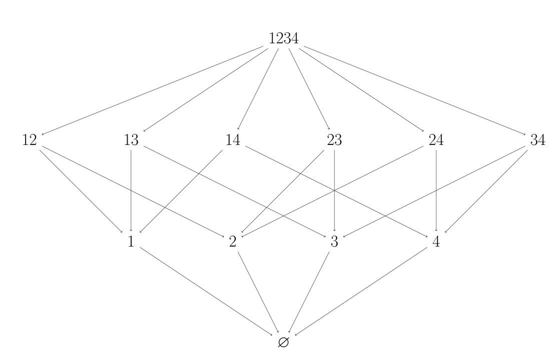

Figure 2: Lattice of flats of the matroid $U(3,4)$.

Chow rings of matroids

For a loopless matroid M on $[n]$ **the Chow ring** $\underline{\mathrm{CH}}_{\mathsf{M}}$ is defined as:

$$\underline{\mathrm{CH}}_{\mathsf{M}} := \mathbb{Z}\left[\{x_F\}_{F \in \mathcal{L}(\mathsf{M}) \setminus \{\varnothing\}}\right]/(I+J)$$

where I is the ideal $\langle x_F x_G : F, G \text{ are incomparable}\rangle$ and J is the ideal $\langle \sum_i x_F : F \ni i\rangle$ for $1 \leq i \leq n$. The ring is graded and has a basis given by the following **FY-monomials**:

$$x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} \ : \varnothing = F_0 \subset F_1 \subset \cdots \subset F_k; \quad 0 \leq m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1.$$

The restriction on the exponents m_i of x_{F_i} ensures that there are exactly $\mathrm{rk}(M)$ graded pieces. The **(non equivariant) Chow polynomial** $\underline{\mathrm{H}}_{\mathsf{M}}$ is defined as the Hilbert series of $\underline{\mathrm{CH}}_{\mathsf{M}}$:

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = a_0 + a_1 x + \cdots a_{\mathrm{rk}(M)-1}x^{\mathrm{rk}(M)-1}$$

where a_i is the rank of degree i piece in $\underline{\mathrm{CH}}_{\mathsf{M}}$.

The Chow ring of $\mathsf{M}(K_4)$ above is
$\mathbb{Z} \oplus \mathbb{Z}(x_{124}, x_{135}, x_{25}, x_{16}, x_{34}, x_{236}, x_{456}, x_{1234}) + \mathbb{Z}(x_{1234})$

Augmented Chow rings of matroids

The **augmented Chow ring** CH_{M} can be defined as:

$$\mathrm{CH}_{\mathsf{M}} := \frac{\mathbb{Z}\left[\{x_F\}_{F \in \mathcal{L}(\mathsf{M}) \setminus [n]} \cup \{y_1, \ldots, y_n\}\right]/(I+J)}{\langle y_i - \sum_{F : i \notin F} x_F\rangle_{i=1,2,\ldots,n}}$$

where I is the ideal $\langle x_F x_G : F, G \text{ are incomparable}\rangle$ and J is the ideal $\langle y_i x_F : i \notin F\rangle$.

Lemma [Eur-Huh-Larson]

The ring CH_{M} has the following monomial basis:

$$x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} \ : \varnothing = F_0 \subset F_1 \subset \cdots \subset F_k; \ m_1 \leq \mathrm{rk}\, F_1, \quad 0 \leq m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1.$$

The augmented Chow ring is itself a Chow ring some other matroid.

Equivariant Chow Polynomial

For a matroid M with an action of a group G, there is an induced action on the Chow ring of M. It can be shown that G acts on each graded piece of $\underline{\mathrm{CH}}_{\mathsf{M}}$ separately by permuting the FY-monomials of that degree. The **equivariant Chow polynomial** $\mathrm{H}^G_{\mathsf{M}}(x) \in \mathrm{VRep}_G[x]$ is defined as:

$$\mathrm{H}^G_{\mathsf{M}}(x) = P(FY^0) + P(FY^1)x + \cdots P(FY^{\mathrm{rk}(M)-1})x^{\mathrm{rk}(M)-1}$$

where $P(FY^i)$ denotes the permutation representation of G on the set FY^i of degree i FY-monomials.

Properties

Theorem [Angarone-Nathanson-Reiner]

Let M be a simple matroid of rank $r+1$ with G a group of automorphisms of M. Then there exist

- G-equivariant bijections $\pi : FY^k \to FY^{r-k}$ for $k \leq r/2$, and
- G-equivariant injections $\lambda : FY^k \to FY^{k+1}$ for $k < r/2$.

Definition of τ_j

Example of CFT

Complete CFT example

Main conjecture (proof in progress)

The bijection T determines $\mathbb{T}$ on simple perverse sheaves; that is, $\mathbb{T}(\mathrm{IC}(\mathcal{O}_\lambda)) = \mathrm{IC}(\mathcal{O}_{\mathsf{T}(\lambda)})$.

References

The bijection T determines $\mathbb{T}$ on simple perverse sheaves; that is, $\mathbb{T}(\mathrm{IC}(\mathcal{O}_\lambda)) = \mathrm{IC}(\mathcal{O}_{\mathsf{T}(\lambda)})$.