
Lemma 1. *If 1_H denotes the trivial module for KH , then $\text{Ind}_H^G(1_H)$ affords the permutation representation of G on the right (or left) cosets of H .*

Proof. [2] □

Theorem 2. *Let $X = \{x_1, \dots, x_n\}$ be a transitive G -set and $G_{x_1} = \text{Stab}_G(x_1)$ denote the stabilizer of $x_1 \in X$ under the action of G . Let Y denote the set of left cosets of G_{x_1} . Then $X \cong Y$ as G -sets.*

Proof. By Orbit-Stabilizer theorem, we have $[G : H] = n$. Since the action is transitive, there exist $g_1, \dots, g_n \in G$ with $g_i \cdot x_1 = x_i$ such that g_1H, \dots, g_nH are the distinct left cosets of H in G . (Rotman's group theory book, section on G -sets.)

to be continued. (?) □

Corollary 3. *Let X be a G -set and V be the permutation representation of G on X . Let X/G be the set of orbits under the action of G and G_x be the stabilizer subgroup of $x \in X$. Then*

$$V = \bigoplus_{\bar{x} \in X/G} \text{Ind}_{G_x}^G(1_{G_x}).$$

Corollary 4 ([3][1]). *The Chow ring $A(M)$ of a matroid M is free as a \mathbb{Z} -module, with \mathbb{Z} -basis given by the FY -monomials*

$$FY = \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} : (\emptyset = F_0) \subset F_1 \subset F_2 \subset \cdots \subset F_k, \text{ and } m_i \leq \text{rk}(F_i) - \text{rk}(F_{i-1}) - 1\}.$$

Proposition 5. *Let M be a loopless matroid. The Hilbert-Poincare series of the Chow ring $A(M)$ is given by*

$$\underline{H}_M = \sum_{\emptyset = F_0 \subset F_1 \subset \cdots \subset F_m} \prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x}.$$

Given a flag $F = F_0 \subset F_1 \subset \cdots \subset F_m$ of strictly increasing flats, the formula in proposition 5 essentially computes the number of degree i monomials divisible precisely by $x_{F_1} x_{F_2} \cdots x_{F_m}$ as coefficients of x^i and sums over all flags to obtain the Hilbert-Poincare series.

Consider the polynomial ring $2^{FY}[x]$ with addition and multiplication defined as follows:

1. $Sx^i + Tx^i = (S \cup T)x^i$,
2. $Sx^i \cdot Tx^j = (ST)x^{i+j}$,

where S and T are subsets of FY and $ST = \{s \cdot t \mid s \in S, t \in T\}$. A slight modification of the formula in proposition 5 is as follows:

$$A = \sum_{\emptyset = F_0 \subset F_1 \subset \dots \subset F_m} \left(\prod_{i=1}^m \{F_i\}x + \{F_i^2\}x^2 + \dots \{F_i^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1}\}x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1} \right) \quad (1)$$

assuming that the factor is 0 when $\text{rk}(F_i) - \text{rk}(F_{i-1}) = 1$. The new formula explicitly computes the FY-monomials corresponding to the flag $F_0 \subset F_1 \subset \dots \subset F_m$ and summing over all flags gives us

$$A = FY^0 + FY^1x + \dots + FY^{r-1}x^{r-1}$$

for a matroid of rank r .

If a group G acts on the set of atoms, then we have an induced action on the sets Flats and FY . For a fixed flag $P = F_0 \subset F_1 \subset \dots \subset F_m$, consider the summand of P in A :

$$A_P = \prod_{i=1}^m \{F_i\}x + \{F_i^2\}x^2 + \dots \{F_i^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1}\}x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1} = P^0 + P^1x + \dots + P^{r-1}x^{r-1}.$$

We have the following facts about the action of G on Flats and FY :

1. For every monomial m in $P^i \subseteq FY^i$, the stabilizer group is $G_P = G_{F_0} \cap G_{F_1} \cap \dots \cap G_{F_m}$. Equivalently, the orbits of any two elements in P^i under the action of G are disjoint.
2. If Q is in the orbit of the flag P , then $Q^i = P^i$.
3. The permutation representation of G on the orbit $\bar{m} \in FY^i/G$ is given by $\text{Ind}_{G_P}^G(1_{G_P})$ (theorem above).
4. From above, the permutation representation on orbits of P^i equals $|P^i| \cdot \text{Ind}_{G_P}^G(1_{G_P})$.
5. Hence, the permutation representation on the orbits of P can be decomposed as

$$\begin{aligned} V(A_{\bar{P}}) &= (|P^0| + |P^1|x + \dots + |P^{r-1}|x^{r-1}) \cdot \text{Ind}^G(1_{G_P}) \\ &= \left(\prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \text{Ind}^G(1_{G_P}). \end{aligned}$$

From (5), we have the decomposition of the permutation representation on FY as follows:

$$\underline{\mathbb{H}}_M^G = \sum_{\bar{P} \in \text{Flags}_0/G} \left(\prod_{i=1}^m \frac{x(1 - x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \text{Ind}^G(1_{G_P})$$

where Flags_0 denotes the set of flags starting at $F_0 = \emptyset$.

Some computation then provides us with the following form:

$$\underline{\mathbb{H}}_M = 1_G + \sum_{F \neq \emptyset} \frac{x(1 - x^{\text{rk}(F) - 1})}{1 - x} \cdot \frac{|G_F|}{|G|} \cdot \text{Ind}^G(\underline{\mathbb{H}}_{M/F}^{G_F})$$

References

- [1] Robert Angarone, Anastasia Nathanson, and Victor Reiner. *Chow Rings of Matroids as Permutation Representations*. 2024. arXiv: 2309.14312 [math.CO]. URL: <https://arxiv.org/abs/2309.14312>.
- [2] M. J. Collins. *Representations and characters of finite groups*. Cambridge University Press, Mar. 1990. Chap. General Representation Theory, p. 34.
- [3] Eva Maria Feichtner and Sergey Yuzvinsky. “Chow rings of toric varieties defined by atomic lattices”. In: *Inventiones mathematicae* 155.3 (Mar. 2004), pp. 515–536. DOI: 10.1007/s00222-003-0327-2. URL: <https://doi.org/10.1007/s00222-003-0327-2>.