EQUIVARIANT CHOW POLYNOMIALS OF MATROIDS

NUTAN NEPAL

Abstract. beeboop beeboop. . .

1. Introduction

1.1. **Overview.** Given a matroid $\mathsf{M} = (E, \mathcal{I})$, we can define its Chow ring $\underline{\mathrm{CH}}$ and augmented Chow ring CH for which the bases are given by:

$$\mathrm{FY} = \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} \,:\, (\varnothing = F_0) \subset F_1 \subset F_2 \subset \cdots \subset F_k, \text{ and } m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1\},$$

and

$$\widetilde{FY} = \{x_{F_1}^{a_1} x_{F_2}^{a_2} \cdots x_{F_m}^{a_m} \mid \varnothing \subset F_1 \subset \cdots \subset F_m,\ 1 \leq a_1 \leq \mathrm{rk}(F_1),\ a_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 \text{ for } i > 1\}$$

respectively. Here $\varnothing \subset F_1 \subset \cdots \subset F_m$ is a strictly increasing chain of flats of the matroid M. The following theorems were proved in [FMSV24]:.

Theorem 1.1 *There is a unique way to assign to each loopless matroid* M *a palindromic polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following properties hold:*

(i) *If* $\mathrm{rk}(\mathsf{M}) = 0$, *then* $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1$.
(ii) *If* $\mathrm{rk}(\mathsf{M}) > 0$, *then* $\deg \underline{\mathrm{H}}_{\mathsf{M}}(x) = \mathrm{rk}(\mathsf{M}) - 1$.
(iii) *For every matroid* M, *the polynomial*

$$\mathrm{H}_{\mathsf{M}}(x) := \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\mathrm{rk}(F)}\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x)$$

is palindromic.

Theorem 1.2 *There is a unique way to assign to each loopless matroid* M *a polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following conditions hold:*

(i) *If* $\mathrm{rk}(\mathsf{M}) = 0$, *then* $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1$.
(ii) *For every matroid* M, *the following recursion holds:*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \varnothing}} \overline{\chi}_{\mathsf{M}|_F}(x)\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x).$$

It was shown in [FMSV24] that these polynomials H_{M} and $\underline{\mathrm{H}}_{\mathsf{M}}$ are the Hilbert-Poincare series for the augmented Chow ring CH and the Chow ring $\underline{\mathrm{CH}}$ respectively. In other words,

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = |FY^0| + |FY^1|x + \cdots + |FY^{r-1}|x^{r-1}$$

where $|FY^i|$ denotes the number of fy-monomials of degree i (which equals the dimension of the degree i piece of the Chow ring).

2. Matroids

2.1. **Action.** Given a finite set E with n elements, the symmetric group $\mathfrak{S}_n$ can always act on E by permutation and this induces an action on the power set 2^E. If $\mathsf{M} = (E, \mathcal{I})$ is a matroid, let G be the stabilizer subgroup $\mathsf{Stab}_{\mathfrak{S}_n}(\mathcal{I})$ of $\mathfrak{S}_n$ that stabiizes the set $\mathcal{I} \subseteq 2^E$. Then, we say that the group G acts on the matroid M.

The action of G on M induces an action on the Chow ring and the augmented Chow ring of M by permuting the fy-monomials. Since the action doesn't affect the degree of the monomials, it is clear that G acts on the Chow ring by acting separately on each graded piece of CH.

Let V^i denote the permutation representation of G on the set FY^i. Then $\dim(V^i) = |FY^i|$, the dimension of the i-th piece. We define the polynomial $V^0 + V^1x + \cdots V^{r-1}x^{r-1} \in \mathrm{VRep}_G[x]$ to be the equivariant Chow polynomial of the matroid M. In this paper, we will prove the following theorems:

Theorem 2.1 *Let* M *be a loopless matroid and* $\underline{\mathrm{H}}_{\mathsf{M}}^G$ *be its equivariant Chow polynomial. Then* $\underline{\mathrm{H}}_{\mathsf{M}}^G$ *is given by*

$$\underline{\mathrm{H}}_{\mathsf{M}}^G(x) = \sum_{\varnothing = F_0 \subset F_1 \subset \cdots \subset F_m} \frac{|G_{F_0\cdots F_m}|}{|G|} \left(\prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1})}{1 - x} \right) \mathrm{Ind}^G(1_{G_{F_0\cdots F_m}}). \tag{1}$$

Here, $G_{F_0\cdots F_m} = G_{F_0} \cap \cdots \cap G_{F_m}$ *denotes the stabilizer of the chain* $(F_0 \subset F_1 \subset \cdots \subset F_m)$ *and the sum is taken over all nonempty chains of flats starting at* $\varnothing$.

Theorem 2.2 *Let* M *be a loopless matroid and* $\mathrm{H}_{\mathsf{M}}^G$ *be its equivariant augmented Chow polynomial. Then* $\mathrm{H}_{\mathsf{M}}^G$ *is given by*

$$\mathrm{H}_{\mathsf{M}}^G(x) = 1_G + \sum_{F_0 \subset F_1 \subset \cdots \subset F_m} \frac{|G_{F_0\cdots F_m}|}{|G|} \frac{x(1 - x^{\mathrm{rk}(F_0)})}{1 - x} \left(\prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1})}{1 - x} \right) \mathrm{Ind}^G(1_{G_{F_0\cdots F_m}}). \tag{2}$$

Here, $G_{F_0\cdots F_m} = G_{F_0} \cap \cdots \cap G_{F_m}$ *denotes the stabilizer of the chain* $(F_0 \subset F_1 \subset \cdots \subset F_m)$ *and the sum is taken over all nonempty chains of flats.*

Theorem 2.3 *There is a unique way to assign to each loopless matroid* M *a palindromic polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}^G(x) \in \mathrm{VRep}_G[x]$ *such that the following properties hold:*

(i) *If* $\mathrm{rk}(\mathsf{M}) = 0$*, then* $\underline{\mathrm{H}}_{\mathsf{M}}^G(x) = 1_G$.
(ii) *If* $\mathrm{rk}(\mathsf{M}) > 0$*, then* $\deg \underline{\mathrm{H}}_{\mathsf{M}}^G(x) = \mathrm{rk}(\mathsf{M}) - 1$.
(iii) *For every matroid* M*, the polynomial*

$$\mathrm{H}_{\mathsf{M}}^G(x) := \sum_{F \in \mathscr{L}(\mathsf{M})} x^{\mathrm{rk}(F)} \frac{|G_F|}{|G|} \mathrm{Ind}^G \left(\underline{\mathrm{H}}_{\mathsf{M}/F}^{G_F}(x) \right)$$

is palindromic.

Theorem 2.4 *There is a unique way to assign to each loopless matroid* M *a polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}^G(x) \in \mathrm{VRep}_G[x]$ *such that the following conditions hold:*

(i) *If* $\mathrm{rk}(\mathsf{M}) = 0$*, then* $\underline{\mathrm{H}}_{\mathsf{M}}^G(x) = 1_G$.
(ii) *For every matroid* M*, the following recursion holds:*

$$\underline{\mathrm{H}}_{\mathsf{M}}^G(x) = \sum_{\substack{F \in \mathscr{L}(\mathsf{M}) \\ F \neq \varnothing}} \frac{|G_F|}{|G|} \mathrm{Ind}^G \left(\overline{\chi}_{\mathsf{M}|_F}^{G_F}(x) \otimes \underline{\mathrm{H}}_{\mathsf{M}/F}^{G_F}(x) \right).$$

3. Scratch

Proposition 3.1 *Let* M *be a loopless matroid. The Hilbert–Poincare series of the Chow ring* $\underline{\mathrm{CH}}(M)$ *is given by*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \sum_{\varnothing = F_0 \subset F_1 \subset \cdots \subset F_m} \left(\prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1})}{1 - x} \right). \tag{3}$$

Here, the sum is taken over all nonempty chains of flats starting at $\varnothing$.

We notice a few things about the formula in 3.1:

(i) The chain corresponding to just the empty flat gives an empty product which equals 1.

(ii) If $\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 = 0$ for some i in the chain $F_0 \subset F_1 \subset \cdots \subset F_m$, then the product is 0.
(iii) So, given a chain $F_0 \subset F_1 \subset \cdots \subset F_m$ with $\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 > 0$ for all i, we can write the product as $a_1 x + \cdots + a_{r-1}x^{r-1}$ for some positive integers a_j's.

In particular, we can restate the equation 3 as

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1 + \sum_{P_\varnothing} a_1(P_\varnothing)x + \cdots + a_{r-1}(P_\varnothing)x^{r-1},$$

where the sum is taken over all chains $P_\varnothing = (F_0 \subset F_1 \subset \cdots \subset F_m)$ starting at $\varnothing$ with $\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 > 0$ for all i and $a_j(P_\varnothing)$ are some integers depending on the chain.

$$S_0 + S_1 x + \cdots + S_r x^r = \sum_{i \in I} \sum_{j=0}^{r} T_{ij} x^j$$
$$S_k = \sum_{i \in I} T_{ik}$$
$$|S_0| + |S_1|x + \cdots + |S_r|x^r = \sum_{i \in I} p_i(x)$$

4. Going on a tangent

References

[FMSV24] Ferroni, Luis and Matherne, Jacob P. and Stevens, Matthew and Vecchi, Lorenzo, *Hilbert–Poincaré series of matroid Chow rings and intersection cohomology*, Advances in Mathematics. **449** (2024) 1

(N. Nepal) North Carolina State University
Email address: nnepal2@ncsu.edu