1 Introduction

The Kazhdan-Lusztig polynomial $P_M(t)$ is a fundamental invariant associated with any matroid M, as defined by Elias, Proudfoot, and Wakefield in [2]. This polynomial, denoted $P_M(t)$, exhibits formal similarities to the Kazhdan–Lusztig polynomials defined for Coxeter groups. The coefficients of $P_M(t)$ depend only on the lattice of flats $L(M)$ of the matroid, and in fact, they are integral linear combinations of the flag Whitney numbers counting chains of flats with specified ranks.

In [1], Braden and Vysogorets presented a formula that relates the Kazhdan-Lusztig polynomial of a matroid M to that of the matroid obtained by deleting an element e, denoted $M \setminus e$, as well as various contractions and localizations of M. Specifically, for a simple matroid M where e is not a coloop, their main result, Theorem 2.8, states:

$$P_M(t) = P_{M\setminus e}(t) - tP_{M/e}(t) + \sum_{F\in S} \tau(M_{F\cup e}) \cdot t^{\mathrm{crk}(F)/2} \cdot P_{M^F}(t)$$

where the sum is taken over the set S of all subsets F of $E \setminus e$ such that both F and $F \cup e$ are flats of M, and $\tau(M)$ is the coefficient of $t^{(\mathrm{rk}(M)-1)/2}$ in $P_M(t)$ if $\mathrm{rk}(M)$ is odd, and zero otherwise.

The inverse Kazhdan-Lusztig polynomial $Q_M(x)$ is another important invariant. There is a related polynomial $\hat{Q}_M(x) = (-1)^{\mathrm{rk}(M)} Q_M(x)$ which acts as the inverse of the Kazhdan-Lusztig polynomial $P_M(t)$ within the incidence algebra of the lattice of flats under appropriate variable transformation.

In this paper, we aim to prove the following deletion formula for $\hat{Q}_M(x)$:

$$\hat{Q}_M(x) = \hat{Q}_{M\setminus e}(x) - (1 + x) \cdot \hat{Q}_{M/e}(x) - \sum_{G\in S'} \tau(M_e^G) \cdot x^{\mathrm{rk}(G)/2} \cdot \hat{Q}_{M/G}(x)$$

where $S' = \{F \in \mathcal{L}(M) \mid e \in F \text{ and } F \setminus e \notin \mathcal{L}(M)\}$.

2 Perverse elements and the KL basis

Let M be a matroid and $\mathcal{L}(M)$ be its lattice of flats.

- **The Module** $\mathcal{H}(M)$: Let $\mathcal{H} = \mathcal{H}(M)$ be the **free** $\mathbb{Z}[t, t^{-1}]$**-module with basis indexed by** $\mathcal{L}(M)$. Elements of $\mathcal{H}$ are formal sums of the form

$$\alpha = \sum_{F\in\mathcal{L}(M)} \alpha_F \cdot F, \quad \alpha_F \in \mathbb{Z}[t, t^{-1}].$$

- **The Abelian Subgroup** $\mathcal{H}_p$: $\mathcal{H}_p$ is an **abelian subgroup of** $\mathcal{H}$ consisting of all $\alpha \in \mathcal{H}$ such that for every flat $F \in \mathcal{L}(M)$, the following two conditions hold:
  i. $\alpha_F \in \mathbb{Z}[t]$.
  ii. $\sum_{G\geq F} t^{\mathrm{rk}(F)-\mathrm{rk}(G)}\alpha_G \in Pal(0)$, where $Pal(0)$ is the set of Laurent polynomials $f(t)$ such that $f(t) = f(t^{-1})$.
- **The Elements** ζ^F: For any flat $F \in \mathcal{L}(M)$, an element $\zeta^F \in \mathcal{H}$ is defined as

$$\zeta^F = \sum_{G\leq F} \zeta^F_G \cdot G = \sum_{G\leq F} t^{\mathrm{rk}(F)-\mathrm{rk}(G)} P_{M^F_G}(t^{-2}) \cdot G.$$

- **Basis of $\mathcal{H}_p$**: Proposition 2.13 states that the set of elements $\{\zeta^F\}_{F\in\mathcal{L}(M)}$ forms a $\mathbb{Z}$**-basis for** $\mathcal{H}_p$. Any element $\beta \in \mathcal{H}_p$ can be uniquely expressed as a linear combination of the ζ^F with integer coefficients:

$$\beta = \sum_{F\in\mathcal{L}(M)} \beta_F(0)\zeta^F.$$

This algebraic framework, involving the module $\mathcal{H}(M)$ and its subgroup $\mathcal{H}_p$ with the basis $\{\zeta^F\}$, provides a foundation for studying the Kazhdan–Lusztig polynomials of matroids, as demonstrated by its role in the derivation of deletion formulas.

References

[1] Tom Braden and Artem Vysogorets. "Kazhdan–Lusztig polynomials of matroids under deletion". In: *Electron. J. Combin.* 27.1 (2020), P1.17.

[2] Ben Elias, Nicholas Proudfoot, and Max Wakefield. "The Kazhdan-Lusztig polynomial of a matroid". In: *Adv. Math.* 299 (2016), pp. 36–70.