Equivariant Chow Polynomials of Matroids

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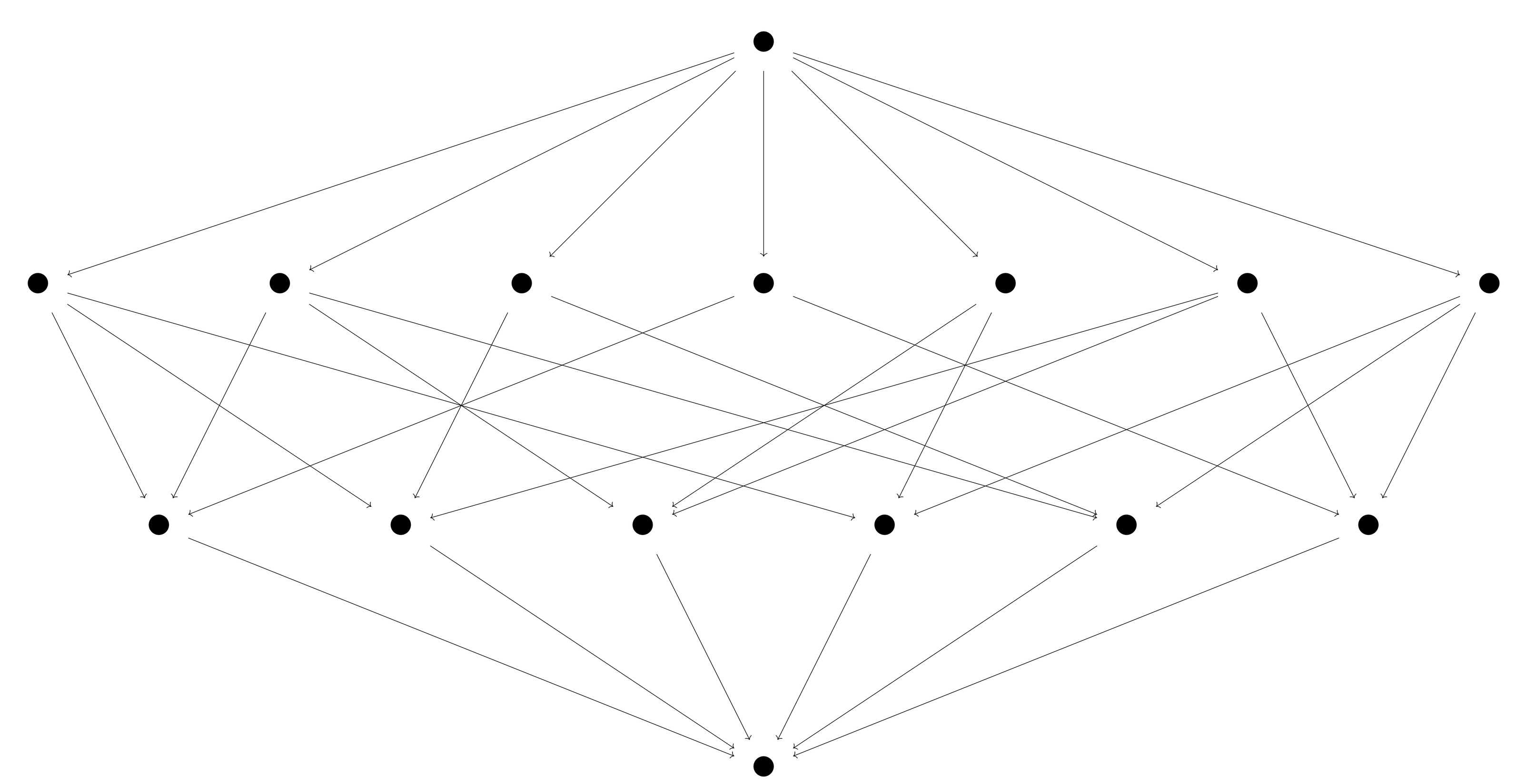
Goal

Define the equivariant Chow polynomial $\underline{\mathrm{H}}^G_{\mathsf{M}}(x) \in \mathrm{VRep}_G[x]$ of a matroid M :

Overview



The Chow Ring

For a matroid M with flats $F_1, \ldots, F_m$, **the Chow ring** $\underline{\mathrm{CH}}_{\mathsf{M}}$ can be defined as a graded $\mathbb{Z}$-module generated by the following monomials:

$$x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} \mid \varnothing \subset F_1 \subset \cdots \subset F_k,\ 0 \le m_i \le \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1.$$

The restriction on the exponents m_i of x_{F_i} ensures that there are exactly $\mathrm{rk}(M)$ graded pieces. The **(non equivariant) Chow polynomial** $\underline{\mathrm{H}}_{\mathsf{M}}$ is defined as:

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = a_0 + a_1 x + \cdots a_{\mathrm{rk}(M)-1} x^{\mathrm{rk}(M)-1}$$

where a_i is the rank of degree i piece in $\underline{\mathrm{CH}}_{\mathsf{M}}$.

For the braid matroid K_4 depicted above, the Chow polynomial is $1 + 8x + x^2$.

The set of triangular arrays P(w)

Define the set $\mathbf{P}(\mathbf{w})$ of triangular arrays of nonnegative integers such that:

- $\forall j$, the entries in the j^{th} chute sum to w_j.
- Ladders are weakly decreasing.

Theorem [Achar–Kulkarni–M.]

There is a bijection

$$\{G(\mathbf{w})\text{-orbits in } E(\mathbf{w})\} \xleftrightarrow{1-1} \mathbf{P}(\mathbf{w}) = \{\text{certain tri. arrays}\}.$$

Example of bijection

$$\mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C}^2 \qquad \begin{array}{ccc} 1 & & \\ & 0 & \\ 1 & & 0 \\ & 0 & \\ 2 & & \end{array}$$

$$\mathbb{C} \xrightarrow{\text{rank } 1} \mathbb{C} \xrightarrow{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} \mathbb{C}^2 \qquad \begin{array}{ccc} 0 & & \\ & 1 & \\ 1 & & 0 \\ & 0 & \\ 2 & & \end{array}$$

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Theorem (Combinatorial Fourier transform) [Achar–Kulkarni–M.]

There is a bijection

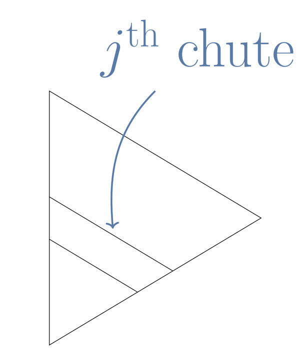
$$\mathbf{P}(\mathbf{w}) \xrightarrow{\mathsf{T}} \mathbf{P}(\mathbf{w}^*)$$

defined inductively by

$$\mathsf{T}\left(\begin{array}{c} Y' \quad y_{1,n} \\ y_{n,1} \cdots \end{array} \right) = \tau_n^{y_{1,n}} \tau_{n-1}^{y_{2,n-1}-y_{1,n}} \cdots \tau_1^{y_{n,1}-y_{n-1,2}} \left(\begin{array}{c} 0 \cdots 0 \\ \mathsf{T}(Y') \end{array} \right)$$

where $\mathsf{T}(a) = a$.

Definition of τ_j



Define $\tau_j : \mathbf{P}(\mathbf{w}) \to \mathbf{P}(\mathbf{w} + \mathbf{e}_1 + \ldots + \mathbf{e}_j)$ by:

- Add 1 as far down the j^{th} chute as possible, drawing an impassable vertical line there.
- Repeat for chutes $j-1, \ldots, 1$ not crossing lines.

Example of CFT

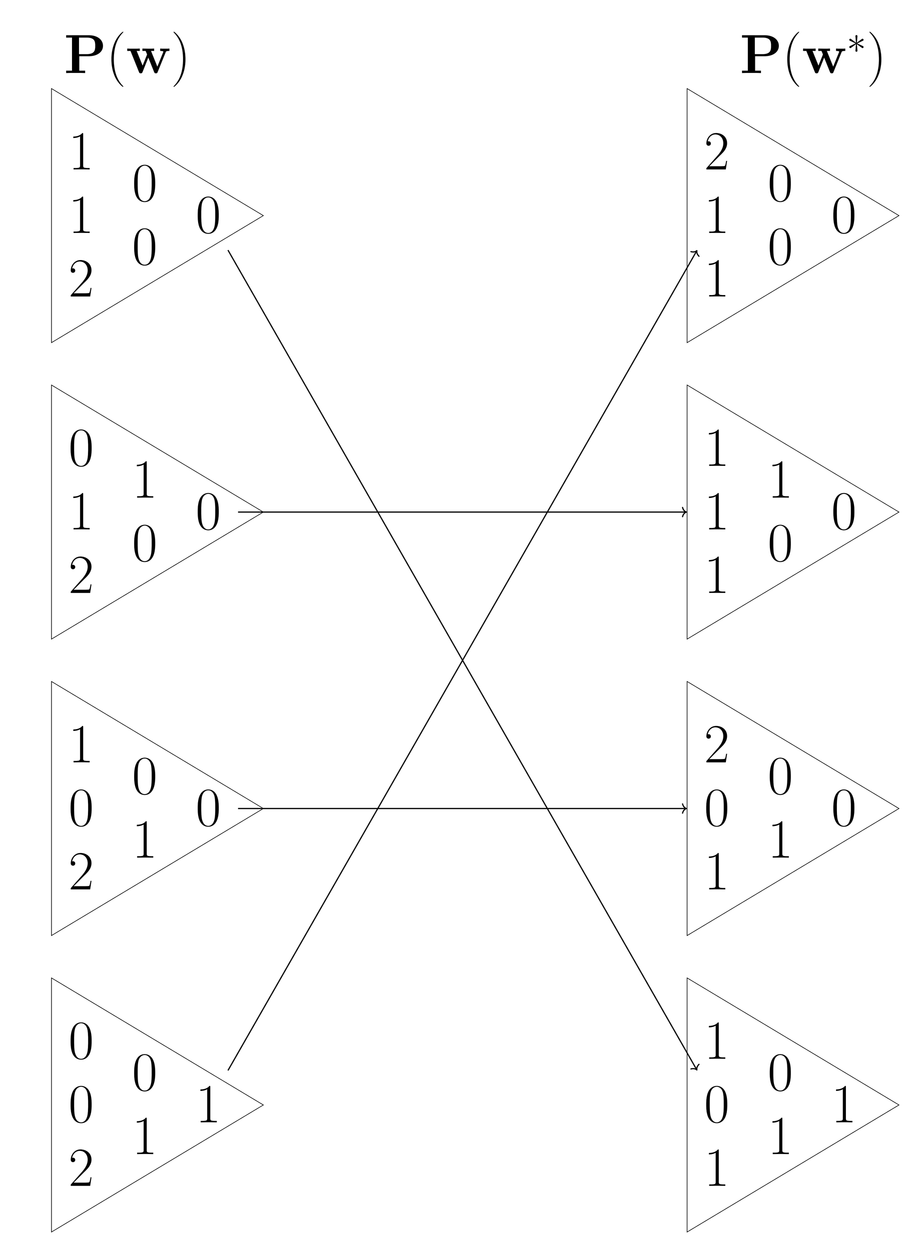
$$\begin{array}{ccc} 1 & & \\ & 0 & \\ 0 & & 0 \\ & 1 & \\ 2 & & \end{array}$$

$$\mathsf{T}\left(1 \right) = 1$$

$$\mathsf{T}\left(\begin{array}{cc} 1 & \\ & 0 \\ 0 & \end{array} \right) = \begin{array}{cc} 0 & \\ & 0 \\ 1 & \end{array}$$

$$\mathsf{T}\left(\begin{array}{ccc} 1 & & \\ & 0 & \\ 0 & & 0 \\ & 1 & \\ 2 & & \end{array} \right) = \tau_2\tau_1 \left(\begin{array}{ccc} 0 & & \\ & 0 & \\ 0 & & 0 \\ & 0 & \\ 1 & & \end{array} \right) = \begin{array}{ccc} 2 & & \\ & 0 & \\ 0 & & 0 \\ & 1 & \\ 1 & & \end{array}$$

Complete CFT example



Main conjecture (proof in progress)

The bijection T determines $\mathbb{T}$ on simple perverse sheaves; that is, $\mathbb{T}(\mathrm{IC}(\mathcal{O}_\lambda)) = \mathrm{IC}(\mathcal{O}_{\mathsf{T}(\lambda)})$.