An Overview of Stanley Symmetric Functions and Applications

Introduction

Stanley symmetric functions, introduced by Richard Stanley, are a family of symmetric functions that arise in the study of reduced words of permutations. These functions have deep connections to algebraic combinatorics, representation theory, and geometry. They are particularly useful in understanding Schubert polynomials, the geometry of flag varieties, and the combinatorics of reduced words.

1 Definition of Stanley Symmetric Functions

Let w be a permutation in the symmetric group S_n. The Stanley symmetric function F_w is defined as:

$$F_w(x_1, x_2, \ldots) = \sum_{a \in R(w)} x_{a_1} x_{a_2} \cdots x_{a_k},$$

where the sum is over all reduced words $a = (a_1, a_2, \ldots, a_k)$ of w, and $R(w)$ denotes the set of all reduced words of w. A reduced word is a sequence of integers corresponding to the simple transpositions s_i such that their product equals w and the length of the sequence is minimal.

2 Properties of Stanley Symmetric Functions

- **Symmetry:** F_w is symmetric in the variables $x_1, x_2, \ldots$.
- **Stability:** The function F_w is stable under the addition of new variables, i.e., $F_w(x_1, \ldots, x_n, 0) = F_w(x_1, \ldots, x_n)$.
- **Expansion in Schur Functions:** Stanley symmetric functions can be expressed as a positive sum of Schur functions:

  $$F_w = \sum_{\lambda} c_{w,\lambda} s_{\lambda},$$

  where $c_{w,\lambda}$ are non-negative integers.

3 Applications of Stanley Symmetric Functions

3.1 Schubert Polynomials

Stanley symmetric functions are closely related to Schubert polynomials, which represent cohomology classes of Schubert varieties in the flag variety. For a permutation w, the Stanley symmetric function F_w is the stable limit of the Schubert polynomial $\mathfrak{S}_w$:

$$F_w = \lim_{n \to \infty} \mathfrak{S}_w(x_1, \ldots, x_n).$$

3.2 Geometry of Flag Varieties

Stanley symmetric functions encode information about the geometry of flag varieties, particularly the structure of their cohomology rings. They are used to study the intersection theory of Schubert varieties.

3.3 Combinatorics of Reduced Words

The coefficients of Stanley symmetric functions count the number of reduced words of permutations. This has applications in understanding the structure of Coxeter groups and their reduced expressions.

3.4 Representation Theory

Stanley symmetric functions appear in the representation theory of the symmetric group and Hecke algebras. They are used to study the characters of certain representations and their branching rules.

4 Conclusion

Stanley symmetric functions are a powerful tool in algebraic combinatorics, with connections to geometry, representation theory, and beyond. Their rich combinatorial structure and applications make them a central object of study in modern mathematics.

References

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- A. Knutson and T. Tao, *The Honeycomb Model of $GL_n(\mathbb{C})$ Tensor Products I: Proof of the Saturation Conjecture*, Journal of the American Mathematical Society, 1999.