Lemma 1. If 1_H denotes the trivial module for KH, then $\operatorname{Ind}_H^G(1_H)$ affords the permutation representation of G on the right (or left) cosets of H.

Theorem 2. Let $X = \{x_1, \ldots, x_n\}$ be a transitive G-set and $G_{x_1} = \operatorname{Stab}_G(x_1)$ denote the stabilizer of $x_1 \in E$ under the action of G. Let Y denote the set of left cosets of G_{x_1} . Then $X \cong Y$ as G-sets.

Corollary 3. Let X be a G-set and V be the permutation representation of G on X. Let X/G be the set of orbits under the action of G and G_x be the stabilizer subgroup of $x \in X$. Then

$$V = \bigoplus_{\overline{x} \in X/G} \operatorname{Ind}_{G_x}^G(1_{G_x}).$$

Corollary 4 ([2][1]). The Chow ring A(M) of a matroid M is free as a \mathbb{Z} -module, with \mathbb{Z} -basis given by the FY-monomials

$$FY = \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} : (\emptyset = F_0) \subset F_1 \subset F_2 \subset \cdots \subset F_k, \text{ and } m_i \leq \operatorname{rk}(F_i) - \operatorname{rk}(F_{i-1}) - 1\}.$$

Proposition 5. Let M be a loopless matroid. The Hilbert-Poincare series of the Chow ring A(M) is given by

$$\underline{\mathbf{H}}_{\mathbf{M}} = \sum_{\emptyset = F_0 \subset F_1 \subset \cdots \subset F_m} \prod_{i=1}^m \frac{x(1 - x^{\operatorname{rk}(F_i) - \operatorname{rk}(F_{i-1}) - 1})}{1 - x}.$$

Given a flag $F = F_0 \subset F_1 \subset \ldots \subset F_m$ of strictly increasing flats, the formula in proposition 5 essentially computes the number of degree i monomials formed precisely by $x_{F_1}x_{F_2}\cdots x_{F_m}$ as coefficients of x^i and sums over all flags to obtain the Hilbert-Poincare series.

Consider the polynomial ring $2^{FY}[x]$ with addition and multiplication defined as follows:

- 1. $Sx^{i} + Tx^{i} = (S \cup T)x^{i}$.
- 2. $Sx^i \cdot Tx^j = (ST)x^{i+j}$.

where S and T are subsets of FY and $ST = \{s \cdot t \mid s \in S, t \in T\}$. A slight modification of the formula in proposition 5 is as follows:

$$A = \sum_{\emptyset = F_0 \subset F_1 \subset \dots \subset F_m} \left(\prod_{i=1}^m \{F_i\} x + \{F_i^2\} x^2 + \dots \{F_i^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1}\} x^{\text{rk}(F_i) - \text{rk}(F_{i-1}) - 1} \right)$$
(1)

assuming that the factor is 0 when $\operatorname{rk}(F_i) - \operatorname{rk}(F_{i-1}) = 1$. This formula explicitly computes the FY-monomials corresponding to the flag $F_0 \subset F_1 \subset \cdots \subset F_m$ and summing over all flags gives us

$$A = FY^{0} + FY^{1}x + \dots + FY^{r-1}x^{r-1}$$

for a matroid of rank r.

If a group G acts on the set of atoms, then we have an induced action on the sets Flats and FY. For a fixed flag $P = F_0 \subset F_1 \subset \cdots \subset F_m$, consider the summand of P in A:

$$A_P = \prod_{i=1}^m \{F_i\} x + \{F_i^2\} x^2 + \dots \{F_i^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1}\} x^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1} = P^0 + P^1 x + \dots + P^{r-1} x^{r-1}.$$

We have the following facts about the action of G on Flats and FY:

- 1. For every monomial m in $P^i \subseteq FY^i$, the stabilizer group is $G_P = G_{F_0} \cap G_{F_1} \cap \cdots \cap G_{F_m}$. Equivalently, the orbits of any two elements in P^i under the action of G are disjoint.
- 2. If Q is in the orbit of the flag P, then Q^i is in the orbit of P^i .
- 3. The permutation representation of G on the orbit $\overline{m} \in FY^i/G$ is given by $\operatorname{Ind}_{G_P}^G(1_{G_P})$ (theorem above).
- 4. From above, the permutation representation on orbits of P^i equals $|P^i| \cdot \operatorname{Ind}_{G_P}^G(1_{G_P})$.
- 5. Hence, the permutation representation on the orbits of P can be decomposed as

$$V(A_{\overline{P}}) = (|P^{0}| + |P^{1}|x + \dots + |P^{r-1}|x^{r-1}) \cdot \operatorname{Ind}^{G}(1_{G_{P}})$$
$$= \left(\prod_{i=1}^{m} \frac{x(1 - x^{\operatorname{rk}(F_{i}) - \operatorname{rk}(F_{i-1}) - 1})}{1 - x}\right) \cdot \operatorname{Ind}^{G}(1_{G_{P}}).$$

From (5), we have the decomposition of the permutation representation on FY as follows:

$$\underline{\mathbf{H}}_{\mathbf{M}}^{G} = \sum_{\overline{P} \in \mathrm{Flags}_{0}/G} \left(\prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_{i}) - \mathrm{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \mathrm{Ind}^{G}(1_{G_{P}})$$

where Flags₀ denotes the set of flags starting at $F_0 = \emptyset$.

For $G_P = \operatorname{Stab}(F_0 \cdots F_m) = \operatorname{Stab}(F_0) \cap \cdots \cap \operatorname{Stab}(F_m) = \operatorname{Stab}(F_1) \cap \cdots \cap \operatorname{Stab}(F_m)$,

$$\underline{\mathbf{H}}_{\mathbf{M}}^{G} = \sum_{\emptyset = F_{0} \subset F_{1} \subset \cdots \subset F_{m}} \left(\prod_{i=1}^{m} \frac{x(1 - x^{\operatorname{rk}(F_{i}) - \operatorname{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \frac{|G_{P}|}{|G|} \cdot \operatorname{Ind}^{G}(1_{G_{P}})$$

$$= 1_{G} + \sum_{\emptyset \neq F} \frac{x(1 - x^{\operatorname{rk}(F) - 1})}{1 - x} \frac{|G_{F}|}{|G|}$$

$$\operatorname{Ind}^{G} \left(\sum_{F = F_{1} \subset \cdots \subset F_{m}} \left(\prod_{i=1}^{m} \frac{x(1 - x^{\operatorname{rk}(F_{i}) - \operatorname{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \frac{|G_{P}|}{|G_{F}|} \cdot \operatorname{Ind}^{G_{F}}(1_{G_{P}}) \right)$$

$$= 1_{G} + \sum_{\emptyset \neq F} \frac{x(1 - x^{\operatorname{rk}(F) - 1})}{1 - x} \frac{|G_{F}|}{|G|} \cdot \operatorname{Ind}^{G}(\underbrace{\mathbf{H}}_{\mathbf{M}/F}^{G_{F}}).$$

We have the following FY-monomials set for the augmented Chow ring $\widetilde{A}(M)$ as given in [3] Corollary 3.12. $\widetilde{FY} = \{x_{F_1}^{a_1} x_{F_2}^{a_2} \cdots x_{F_m}^{a_m} \mid \emptyset \subset F_1 \subset \cdots \subset F_m, \ 1 \leq a_1 \leq \operatorname{rk}(F_1), \ a_i \leq \operatorname{rk}(F_i) - \operatorname{rk}(F_{i-1}) - 1 \text{ for } i > 1\}.$

The computation of equivariant Chow polynomial of \widetilde{A}_M follows exactly the same as that of A(M):

$$\mathbf{H}_{\mathbf{M}}^{G} = \mathbf{1}_{G} + \sum_{P \in \mathbf{Flags}} \left(\frac{x(1 - x^{\mathrm{rk}(F_{0})})}{1 - x} \cdot \prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_{i}) - \mathrm{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \frac{|G_{P}|}{|G|} \cdot \mathrm{Ind}^{G}(\mathbf{1}_{G_{P}})$$

Fixing $F = F_0$ gives us

$$\begin{split} \mathbf{H}_{\mathbf{M}}^{G} &= \mathbf{1}_{G} + \sum_{P \in \mathrm{Flags}} \left(\frac{x(1 - x^{\mathrm{rk}(F_{0})})}{1 - x} \cdot \prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_{i}) - \mathrm{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \frac{|G_{P}|}{|G|} \cdot \mathrm{Ind}^{G}(\mathbf{1}_{G_{P}}) \\ &= \mathbf{1}_{G} + \sum_{F \in \mathrm{Flats}} \frac{x(1 - x^{\mathrm{rk}(F)})}{1 - x} \cdot \mathrm{Ind}_{G_{F}}^{G} \left(\sum_{P \in \mathrm{Flags}/F} \left(\prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_{i}) - \mathrm{rk}(F_{i-1}) - 1})}{1 - x} \right) \cdot \frac{|G_{P}|}{|G_{F}|} \cdot \mathrm{Ind}^{G}(\mathbf{1}_{G_{P}}) \right) \end{split}$$

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