
Lemma 1. *If 1_H denotes the trivial module for KH, then $\mathrm{Ind}_H^G(1_H)$ affords the permutation representation of G on the right (or left) cosets of H.*

Theorem 2. *Let $X = \{x_1, \ldots, x_n\}$ be a transitive G-set and $G_{x_1} = \mathrm{Stab}_G(x_1)$ denote the stabilizer of $x_1 \in E$ under the action of G. Let Y denote the set of left cosets of G_{x_1}. Then $X \cong Y$ as G-sets.*

Corollary 3. *Let X be a G-set and V be the permutation representation of G on X. Let X/G be the set of orbits under the action of G and G_x be the stabilizer subgroup of $x \in X$. Then*

$$V = \bigoplus_{\overline{x} \in X/G} \mathrm{Ind}_{G_x}^G(1_{G_x}).$$

Corollary 4 ([2][1])**.** *The Chow ring $A(\mathrm{M})$ of a matroid M is free as a $\mathbb{Z}$-module, with $\mathbb{Z}$-basis given by the FY-monomials*

$$FY = \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} :\ (\emptyset = F_0) \subset F_1 \subset F_2 \subset \cdots \subset F_k,\ \textit{and } m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1\}.$$

Proposition 5. *Let M be a loopless matroid. The Hilbert-Poincare series of the Chow ring $A(\mathrm{M})$ is given by*

$$\underline{\mathrm{H}}_{\mathrm{M}} = \sum_{\emptyset = F_0 \subset F_1 \subset \cdots \subset F_m} \prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1})}{1 - x}.$$

Given a flag $F = F_0 \subset F_1 \subset \ldots \subset F_m$ of strictly increasing flats, the formula in proposition 5 essentially computes the number of degree i monomials formed precisely by $x_{F_1} x_{F_2} \cdots x_{F_m}$ as coefficients of x^i and sums over all flags to obtain the Hilbert-Poincare series.

Consider the polynomial ring $2^{FY}[x]$ with addition and multiplication defined as follows:

1. $Sx^i + Tx^i = (S \cup T)x^i$,
2. $Sx^i \cdot Tx^j = (ST)x^{i+j}$,

where S and T are subsets of FY and $ST = \{s \cdot t \mid s \in S,\ t \in T\}$. A slight modification of the formula in proposition 5 is as follows:

$$A = \sum_{\emptyset = F_0 \subset F_1 \subset \cdots \subset F_m} \left(\prod_{i=1}^{m} \{F_i\}x + \{F_i^2\}x^2 + \cdots \{F_i^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1}\} x^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1} \right) \tag{1}$$

assuming that the factor is 0 when $\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) = 1$. This formula explicitly computes the FY-monomials corresponding to the flag $F_0 \subset F_1 \subset \cdots \subset F_m$ and summing over all flags gives us

$$A = FY^0 + FY^1 x + \cdots + FY^{r-1} x^{r-1}$$

for a matroid of rank r.

If a group G acts on the set of atoms, then we have an induced action on the sets Flats and FY. For a fixed flag $P = F_0 \subset F_1 \subset \cdots \subset F_m$, consider the summand of P in A:

$$A_P = \prod_{i=1}^{m} \{F_i\}x + \{F_i^2\}x^2 + \cdots \{F_i^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1}\}x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1} = P^0 + P^1 x + \cdots P^{r-1}x^{r-1}.$$

We have the following facts about the action of G on Flats and FY:

1. For every monomial m in $P^i \subseteq FY^i$, the stabilizer group is $G_P = G_{F_0} \cap G_{F_1} \cap \cdots \cap G_{F_m}$. Equivalently, the orbits of any two elements in P^i under the action of G are disjoint.
2. If Q is in the orbit of the flag P, then Q^i is in the orbit of P^i.
3. The permutation representation of G on the orbit $\overline{m} \in FY^i/G$ is given by $\mathrm{Ind}_{G_P}^{G}(1_{G_P})$ (theorem above).
4. From above, the permutation representation on orbits of P^i equals $|P^i| \cdot \mathrm{Ind}_{G_P}^{G}(1_{G_P})$.
5. Hence, the permutation representation on the orbits of P can be decomposed as

$$\begin{aligned} V(A_{\overline{P}}) =& (|P^0| + |P^1|x + \cdots + |P^{r-1}|x^{r-1}) \cdot \mathrm{Ind}^G(1_{G_P}) \\ =& \left(\prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x} \right) \cdot \mathrm{Ind}^G(1_{G_P}). \end{aligned}$$

From (5), we have the decomposition of the permutation representation on FY as follows:

$$\underline{\mathrm{H}}_{\mathrm{M}}^{G} = \sum_{\overline{P} \in \mathrm{Flags}_0/G} \left(\prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x} \right) \cdot \mathrm{Ind}^G(1_{G_P})$$

where Flags_0 denotes the set of flags starting at $F_0 = \emptyset$.

Some computation then provides us with the following form:

$$\underline{\mathrm{H}}_{\mathrm{M}} = 1_G + \sum_{F \neq \emptyset} \frac{x(1 - x^{\mathrm{rk}(F)-1})}{1-x} \cdot \frac{|G_F|}{|G|} \cdot \mathrm{Ind}^G(\underline{\mathrm{H}}_{\mathrm{M/F}}^{G_F})$$

We have the following FY-monomials set for the augmented Chow ring $\widetilde{A}(M)$ as given in [3] Corollary 3.12.

$\widetilde{FY} = \{x_{F_1}^{a_1} x_{F_2}^{a_2} \cdots x_{F_m}^{a_m} \mid \emptyset \subset F_1 \subset \cdots \subset F_m,\ 1 \leq a_1 \leq \mathrm{rk}(F_1),\ a_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 \text{ for } i > 1\}$.

The computation of equivariant Chow polynomial of $\widetilde{A}_M$ follows exactly the same as that of $A(M)$:

$$\mathrm{H}_{\mathrm{M}}^{G} = 1_G + \sum_{P \in \mathrm{Flags}} \left(\frac{x(1 - x^{\mathrm{rk}(F_0)})}{1-x} \cdot \prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1})}{1-x} \right) \cdot \frac{|G_P|}{|G|} \cdot \mathrm{Ind}^G(1_{G_P})$$

Fixing $F = F_0$ gives us

$$\begin{aligned}
\mathrm{H}_{\mathrm{M}}^{G} &= 1_G + \sum_{P \in \mathrm{Flags}} \left(\frac{x(1 - x^{\mathrm{rk}(F_0)})}{1-x} \cdot \prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1})}{1-x} \right) \cdot \frac{|G_P|}{|G|} \cdot \mathrm{Ind}^{G}(1_{G_P}) \\
&= 1_G + \sum_{F \in \mathrm{Flats}} \frac{x(1 - x^{\mathrm{rk}(F)})}{1-x} \cdot \mathrm{Ind}_{G_F}^{G} \left(\sum_{P \in \mathrm{Flags}/F} \left(\prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1})}{1-x} \right) \cdot \right. \\
&\qquad \left. \frac{|G_P|}{|G_F|} \cdot \mathrm{Ind}^{G}(1_{G_P}) \right)
\end{aligned}$$

References

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[3] Hsin-Chieh Liao. *Stembridge codes and Chow rings*. 2022. arXiv: 2212.05362 [math.CO]. URL: https://arxiv.org/abs/2212.05362.