Elementary Number Theory
Homework One

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1

Write out the terms of the expression and then evaluate.

1.
$$\sum_{j=-1}^{2} \left((3j)^2 - 5j\right)$$

Solution:

$$\sum_{j=-1}^{2} \left((3j)^2 - 5j\right) = \left((-3)^2 + 5\right) + \left((0)^2 + 0\right) + \left((3)^2 - 5\right) + \left((3 * 2)^2 - 5 * 2\right)$$
$$= 14 + 0 + 4 + 26 = 44$$

2.
$$\prod_{k=2}^{7} \left(1 - \frac{1}{k^2}\right)$$

Solution:

$$\prod_{k=2}^{7} \left(1 - \frac{1}{k^2}\right) = \left(1 - 1/2^2\right) \cdot \left(1 - 1/3^2\right) \cdot \left(1 - 1/4^2\right) \cdot \left(1 - 1/5^2\right) \cdot \left(1 - 1/6^2\right) \cdot \left(1 - 1/7^2\right)$$
$$= \frac{3}{4} \cdot \frac{8}{9} \cdot \frac{15}{16} \cdot \frac{24}{25} \cdot \frac{35}{36} \cdot \frac{48}{49} = \frac{16}{21}$$

2

1. .

Solution: The n^{th} pentagonal number p_n is given by

$$p_n = p_{n-1} + 3n + 1$$

where the base case is given by $p_0 = 1$.

2. .

Solution: We see that $p_0 = 1$ and using the recursive definition, the few other terms are given by:

$$\begin{aligned}
p_1 &= 1 + 3 * 1 + 1 = 2 + 3 * 1 \\
p_2 &= (1 + 3 * 1 + 1) + 3 * 2 + 1 = 3 + 3 * 3 \\
p_3 &= (3 + 3 * 3) + 3 * 3 + 1 = 4 + 3 * 6 \\
&\vdots \\
p_n &= (n + 1) + 3 * t_n
\end{aligned}$$

where t_n is the nth triangular number.

So, we have

$$p_n = \sum_{i=0}^{n} 1 + 3i$$

3. .

Solution: We get,

$$H_k : p_k = \sum_{i=0}^{k} 1 + 3i = (k + 1) + \frac{3k(k + 1)}{2} = \frac{(k + 1)(3k + 2)}{2} \tag{1}$$

Here, we see that the base case $p_0 = 2/2 = 1$, which is true. Assuming that the statement H_k is true for some k, we have

$$\begin{aligned}
p_{k+1} &= p_k + 3(k + 1) + 1 \\
&= \frac{(k + 1)(3k + 2)}{2} + 3k + 4 \\
&= \frac{3k^2 + 5k + 2 + 6k + 8}{2} \\
&= \frac{3k^2 + 11k + 10}{2} \\
&= \frac{(k + 2)(3k + 5)}{2}
\end{aligned}$$

Then, since $H_k \implies H_{k+1}$, the statement is true for all integers $k \geq 0$.

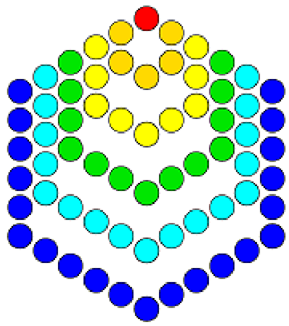
Note: If we start the numbering of terms from 1, we get the recursive formula $p_n = p_{n-1}+3(n-1)+1$ and the closed formula

$$p_n = \frac{n(3n - 1)}{2}$$

3

1. Explain, geometrically, the recursive formula for nth hexagonal number.

Solution: To get nth hexagonal number we add 4 sides each of size $4n$. But since there are 3 common vertices for the added 4 sides, we subtract 3. Hence the recursive definition of the hexagonal



numbers are given by:

$$h_1 = 1, \quad h_n = h_{n-1} + 4n - 3$$

2. .

Solution:

$$h_n = n(2n-1), \quad n \geq 1$$

We see that $h(1) = 1$, which is true. So let the statement

$$H_k : h_k = k(2k-1)$$

be true for some integer $k > 1$. Then,

$$\begin{aligned} h_{k+1} &= h_k + 4(k+1) - 3 \\ &= 2k^2 - k + 4k + 4 - 3 \\ &= 2k^2 + 3k + 1 \\ &= (2k+1)(k+1) \\ &= (k+1)\left(2(k+1) - 1\right) \end{aligned}$$

Then since $H_k \implies H_{k+1}$, the statement is true for all $k \in \mathbf{N}$

4

1. .

Solution: We see that

$$\begin{aligned} p_n + t_{n-1} &= \frac{n(3n-1)}{2} + \frac{n(n-1)}{2} \\ &= \frac{n}{2}\,(3n-1+n-1) \\ &= n(2n-1) = h_n \end{aligned}$$

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5

1. For $m \le k \le n$, prove
$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

Solution: Here

$$\begin{aligned} \binom{n}{k}\binom{k}{m} &= \frac{n!k!}{k!m!(n-k)!(k-m)!} \\ &= \frac{n!(n-m)!}{m!(n-m)!(n-k)!(k-m)!} \\ &= \frac{n!}{m!(n-m)!} \cdot \frac{(n-m)!}{(k-m)!\,((n-m)-(k-m))!} \\ &= \binom{n}{m}\binom{n-m}{k-m} \end{aligned}$$

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