Homework 1 Nutan Nepal
Number Theory February 7, 2019

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1. Write out the terms of the expression and then evaluate.

   (a) $\sum_{j=-1}^{2} \left((3j)^2 - 5j\right)$

   Solution:

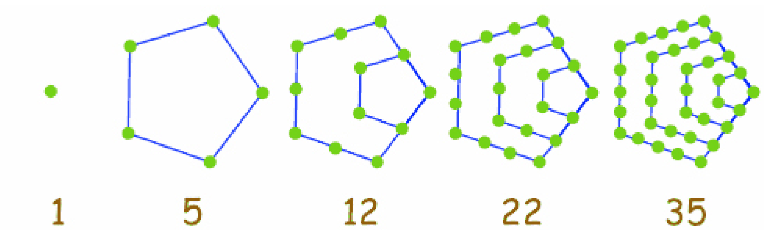
   $$\sum_{j=-1}^{2} \left((3j)^2 - 5j\right) = \left((-3)^2 + 5\right) + \left((0)^2 + 0\right) + \left((3)^2 - 5\right) + \left((3*2)^2 - 5*2\right)$$
   $$= 14 + 0 + 4 + 26 = 44$$

   (b) $\prod_{k=2}^{7} \left(1 - \frac{1}{k^2}\right)$

   Solution:

   $$\prod_{k=2}^{7} \left(1 - \frac{1}{k^2}\right) = \left(1 - 1/2^2\right) \cdot \left(1 - 1/3^2\right) \cdot \left(1 - 1/4^2\right) \cdot \left(1 - 1/5^2\right) \cdot \left(1 - 1/6^2\right) \cdot \left(1 - 1/7^2\right)$$
   $$= \frac{3}{4} \cdot \frac{8}{9} \cdot \frac{15}{16} \cdot \frac{24}{25} \cdot \frac{35}{36} \cdot \frac{48}{49} = \frac{16}{21}$$

2. The pentagonal numbers are the integers that count the number of "coins" that can be arranged to form pentagons. Some examples are shown below.



   (a) Determine a recursion equation for the pentagonal numbers and explain how this can be understood from the geometry. For example, we could see that in the triangular numbers that we were simply adding a new row with n numbers to see that $t_n = t_{n-1} + n$.

   Solution: Each time we add 3 sides with n points each and subtract 2 points. So, the n^{th} pentagonal number p_n is given by

   $$p_n = p_{n-1} + 3n - 2$$

   where the base case is given by $p_1 = 1$.

   (b) Use the recursion equation in the previous part to write the equation as a sum.
   For example, we saw that $t_n = t_{n-1} + n$ and then that $t_n = \sum_{i=1}^{n} i$.
   We also showed that the square numbers $s_n = s_{n-1} + 2n - 1$ and then that $s_n = \sum_{i=1}^{n}(2i - 1)$.

Solution: Using the recursive definition, the nth term is given by:

$$p_n = \sum_{i=1}^{n} 3i - 2$$

Note: $p_n = s_n + t_{n-1}$, where t_n is the nth triangular number and s_n is the nth square number.

(c) Determine a closed-form solution for the pentagonal numbers. Use induction to show that this is equivalent to the recursion equation stated above. For example, we showed that $\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ using induction in class.

Solution: Let,

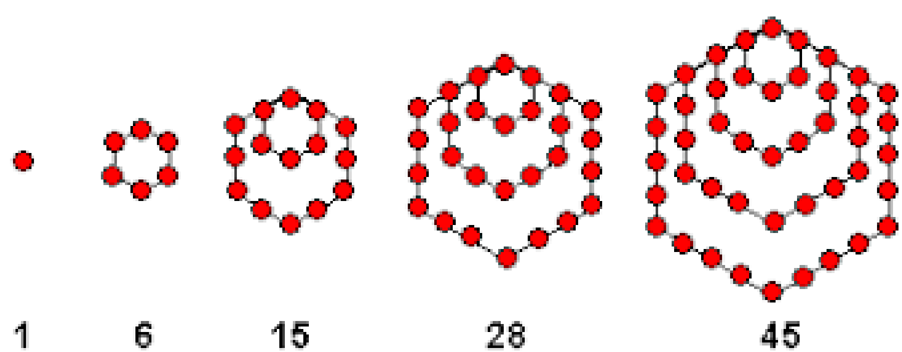
$$H_k : p_k = \frac{k(3k-1)}{2}. \tag{1}$$

Here, we see that the base case $p_1 = 2/2 = 1$, which is true. Assuming that the statement H_k is true for some integer $k > 1$, we have

$$\begin{aligned} p_{k+1} &= p_k + 3(k+1) - 2 \\ &= \frac{k(3k-1)}{2} + 3k + 1 \\ &= \frac{3k^2 - k + 6k + 2}{2} \\ &= \frac{3k^2 - 5k + 2}{2} \\ &= \frac{(k+1)(3(k+1)-1)}{2} \end{aligned}$$

Then, since $H_k \implies H_{k+1}$, the statement is true for all integers $k \geq 0$.

3. (a) The hexagonal numbers satisfy the recursive equation $h_n = h_{n-1} + 4n - 3, h_1 = 1$. Explain geometrically why this describes these numbers. A picture of the hexagonal numbers is given below.



Solution: To get nth hexagonal number we add 4 sides each of size $4n$. But since there are 3 common vertices for the added 4 sides, we subtract 3. Hence the recursive definition of the hexagonal numbers are given by:

$$h_1 = 1, \quad h_n = h_{n-1} + 4n - 3$$

(b) Use induction to show that the closed form $h_n = n(2n-1), n \geq 1$ also evaluates to the hexagonal numbers.

Solution:

$$h_n = n(2n-1), \quad n \geq 1$$

We see that $h(1) = 1$, which is true. So let the statement

$$H_k : h_k = k(2k-1)$$

be true for some integer $k > 1$. Then,

$$\begin{aligned} h_{k+1} &= h_k + 4(k+1) - 3 \\ &= 2k^2 - k + 4k + 4 - 3 \\ &= 2k^2 + 3k + 1 \\ &= (2k+1)(k+1) \\ &= (k+1)\left(2(k+1) - 1\right) \end{aligned}$$

Then since $H_k \implies H_{k+1}$, the statement is true for all $k \in \mathbf{N}$

4. In class, Craig noticed that a square number is just the sum of two triangular numbers; $s_n = t_n + t_{n-1}$. A similar pattern is true for other polygonal numbers. Show that the hexagonal numbers $h_n = p_n + t_{n-1}$ is the sum of a pentagonal number and a triagular number. You may give a geometric or algebraic proof.

Solution: We see that

$$\begin{aligned} p_n + t_{n-1} &= \frac{n(3n-1)}{2} + \frac{n(n-1)}{2} \\ &= \frac{n}{2}(3n - 1 + n - 1) \\ &= n(2n-1) = h_n \end{aligned}$$

5. For $m \leq k \leq n$, prove the identity

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

Solution: Here

$$\begin{aligned} \binom{n}{k}\binom{k}{m} &= \frac{n!k!}{k!m!(n-k)!(k-m)!} \\ &= \frac{n!(n-m)!}{m!(n-m)!(n-k)!(k-m)!} \\ &= \frac{n!}{m!(n-m)!} \cdot \frac{(n-m)!}{(k-m)!\left((n-m)-(k-m)\right)!} \\ &= \binom{n}{m}\binom{n-m}{k-m} \end{aligned}$$

6. Bonus: Create a general formula for the nth polygonal number with sides of length k. For example $k = 3$ for triangular numbers, $k = 4$ for square numbers, $k = 5$ for pentagonal numbers, etc...

   Use the notation $p(k, n)$ to mean the nth number on the list of the k-ogonal numbers. For example, $p(3, 4) = 10$ because the 4th triagular number is 10. While $p(4, 3) = 9$ since the 3rd square number is 9.

Solution: Let $p(k, n)$ be the nth number on the list of k-ogonal numbers. Then for all $k \geq 3, k \in \mathbb{N}$, we have $p(k, 1) = 1$.

To find the $(n+1)^{th}$ k-ogonal number, we add $(k-2)$ sides, each of size $(n+1)$ and then subtract $k-3$ (the number of common vertices) from the resultant. So we have

$$p(k, n+1) = p(k, n) + (n+1)(k-2) + 3 - k$$

Using this recursive definition, we find

$$\begin{aligned} p(k, n) &= \sum_{i=1}^{n} (i(k-2) + 3 - k) \\ &= \frac{n(n+1)}{2}(k-2) + n(3-k) \end{aligned}$$

Note: We can also see that $p(k, n) = kt_{n-1} - s_{n-1} + 1$.