Commutative Algebra and Algebraic Geometry

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Notes 0.1. *Basic properties of ideals. (Matsumura 1)*

1. In a surjective ring homomorphism $f : A \to B$, the ideals J of B and the ideals $I = f^{-1}(J)$ are in one-to-one correspondence. When we need to think about ideals of A containing I, we can work on $B = A/I$: if I' is any ideal of A then $f(I')$ is an ideal of B with $f^{-1}(f(I')) = I + I'$, and $f(I') = (I + I')/I$.

2. For $a \in A$, $(a) = (1)$ iff a has an inverse in A. If a is a unit and x is nilpotent, then $a + x$ is a unit.

3. Using Zorn's lemma on a set of proper ideals (ordered by inclusion) containing some ideal I, we see that there exists at least one maximal ideal M containing I. A/M is a field.

4. A proper ideal P is prime if $x, y \notin P \implies xy \notin P$. A/P is an integral domain.

5. If I is an ideal disjoint from a multiplicative set S, then $A - S$ has a maximal ideal containing I which is prime. (prove...)

6. If I is an ideal, then the radical of I
$$\sqrt{I} = \{a \in A : a^n \in I \text{ for some } n > 0\}$$
is also an ideal. If P is a prime ideal containing I then $\sqrt{I} \subset P$. Furthermore, if $x \notin \sqrt{I}$, then we can find a prime ideal containing $\sqrt{I}$ but not x. And hence,
$$\sqrt{I} = \bigcap_{P \supset I} P$$

7. The intersection of all prime ideals is nilradical $\mathrm{nil}(A)$ and the intersection of all maximal ideals is Jacobson radical $\mathrm{rad}(A)$. $x \in \mathrm{rad}(A)$ iff $1 + Ax$ consists entirely of units of A.

8. $II' \subset I \cap I'$. If $I + I' = (1)$, then $II' = I \cap I'$. Also, if $I + I' = (1)$ and $I + I'' = (1)$, then $I + I'I'' = (1)$. Hence, for ideals I_i which are coprime in pairs
$$I_1 I_2 ... I_n = I_1 \cap I_2 \cap ... \cap I_n.$$

9. If $I + I' = (1)$, then $A/II' \simeq A/I \times A/I'$. This can be extended with n ideals like in 8.

10. A maximal ideal $\mathfrak{m}$ of A corresponds with the maximal ideal $\mathbf{m} = \mathfrak{m}B + (X_1, ..., X_n)$ of the ring $B = A[[X_1, ..., X_n]]$. Here, $\mathbf{m} \cap A = \mathfrak{m}$. These properties are not necessarily true in case of polynomial rings.

11. $a \in A$ is called irreducible element if a is not a unit of A and
$$a = bc \implies b \text{ or } c \text{ is a unit of A.}$$
a is irreducible iff aA is maximal among proper principal ideals and a is prime if aA is a prime ideal.

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