Commutative Algebra and Algebraic Geometry

Nutan Nepal

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Notes 0.1. *Basic properties of ideals. (Matsumura 1)*

1. In a surjective ring homomorphism $f : A \to A/I$, the ideals J of A/I and the ideals $f^{-1}(J)$ of A are in one-to-one correspondence. (lattice isomorphism theorem) When we need to think about ideals of A containing I, we can work on A/I: if I' is any ideal of A then $f(I')$ is an ideal of A/I with $f^{-1}(f(I')) = I + I'$, and $f(I') = (I + I')/I$.
2. For $a \in A$, $(a) = (1)$ iff a has an inverse in A. If a is a unit and x is nilpotent, then $a + x$ is a unit.
3. Using Zorn's lemma on a set of proper ideals (ordered by inclusion) containing some ideal I, we see that there exists at least one maximal ideal M containing I. A/M is a field.
4. A proper ideal P is prime if $x, y \notin P \implies xy \notin P$. A/P is an integral domain.
5. If I is an ideal disjoint from a multiplicative set S, then $A - S$ has a maximal ideal containing I which is prime. (prove...)
6. If I is an ideal, then the radical of I
$$\sqrt{I} = \{a \in A : a^n \in I \text{ for some } n > 0\}$$
is also an ideal. If P is a prime ideal containing I then $\sqrt{I} \subset P$. Furthermore, if $x \notin \sqrt{I}$, then we can find a prime ideal containing $\sqrt{I}$ but not x. And hence,
$$\sqrt{I} = \bigcap_{P \supset I} P$$
7. The intersection of all prime ideals is nilradical $\mathrm{nil}(A)$ and the intersection of all maximal ideals is Jacobson radical $\mathrm{rad}(A)$. $x \in \mathrm{rad}(A)$ iff $1 + Ax$ consists entirely of units of A.
8. $II' \subset I \cap I'$. If $I + I' = (1)$, then $II' = I \cap I'$. Also, if $I + I' = (1)$ and $I + I'' = (1)$, then $I + I'I'' = (1)$. Hence, for ideals I_i which are coprime in pairs
$$I_1 I_2 ... I_n = I_1 \cap I_2 \cap ... \cap I_n.$$
9. If $I + I' = (1)$, then $A/II' \simeq A/I \times A/I'$. This can be extended with n ideals like in 8.
10. A maximal ideal $\mathfrak{m}$ of A corresponds with the maximal ideal $\mathbf{m} = \mathfrak{m}B + (X_1, ..., X_n)$ of the ring $B = A[[X_1, ..., X_n]]$. Here, $\mathbf{m} \cap A = \mathfrak{m}$. These properties are not necessarily true in case of polynomial rings.
11. $a \in A$ is called irreducible element if a is not a unit of A and
$$a = bc \implies b \text{ or } c \text{ is a unit of A.}$$

a is irreducible iff aA is maximal among proper principal ideals and a is prime if aA is a prime ideal.

12. $A = \mathbb{Z}[\sqrt{-5}] = \mathbb{Z}[X]/(X^2+5)$; then setting $k = \mathbb{Z}/2\mathbb{Z}$ we have

$$A/2A = \mathbb{Z}[X]/(2, X^2+5) = k[X]/(X^2-1) = k[X]/(X-1)^2.$$

Then $P = (2, 1-\sqrt{-5})$ is a maximal ideal of A containing 2.

Problems 0.1.

1. Let A be a ring, and $I \subset$ nil(A) an ideal made up of nilpotent elements. If $a \in A$ maps to a unit of A/I then a is a unit of A.

2. Let $A_1, ..., A_n$ be rings; then the prime ideals of $A = A_1 \times \cdots \times A_n$ are of the form

$$P = A_1 \times \cdots \times A_{i-1} \times P_i \times A_{i+1} \times \cdots \times A_n,$$

where P_i is a prime ideal of A_i.

For $A_1 \times A_2$, let $P = I_1 \times I_2$ be a prime ideal.Then, since the product of integral domains is not itself an integral domain, only one from I_1 or I_2 is prime. $(A_1 \times A_2)/(I_1 \times I_2) \simeq A_1/I_1 \times A_2/I_2$. This can be extended to n ideals in a similar fashion.

3. Let A and B be rings and $f : A \to B$ a surjective homomorphism.

   a. Prove that $f(\text{rad } A) \subset \text{rad } B$, and construct an example where the inclusion is strict.
   If $x \in \text{rad } A$, then $f(1 + Ax) = 1 + Bf(x)$ should be a set of units of B.

   b. Prove that if A is a semilocal ring then $f(\text{rad } A) = \text{rad } B$.

4. Let A be an integral domain. Then A is a UFD iff every irreducible element is prime and the principal ideals of A satisfy the ascending chain condition. (Equivalently, every non-empty family of principal ideals has a maximal element.)

5. Let $\{P_\lambda\}_{\lambda\in\Lambda}$ be a non-empty family of prime ideals, and suppose that the P_λ are totally ordered by inclusion; then $\bigcap P_\lambda$ is a prime ideal. Also, if I is any proper ideal, the set of prime ideals containing I has a minimal element.

6. Let A be a ring, $I, P_1, \ldots, P_r$ ideals of A, and suppose that $P_3, \ldots, P_r$ are prime, and that I is not contained in any of the P_i; then there exists an element $x \in I$ not contained on any P_i.

Notes 0.2. *Story of Commutative Algebra. (Eisenbud 1)*

1. Gauss proved that $\mathbb{Z}[i]$ is a UFD and used this on 1928 paper on biquadratic residues to prove results about ordinary numbers.

2. Euler, Gauss, Dirichlet, and Kummer used this idea for $\mathbb{Z}[\zeta]$, with ζ a nth root of unity, to prove some special cases of Fermat's last theorem.

3. $\mathbb{Z}[\zeta]$ doesn't always have unique factorization. (first example n=23) The search for generalization of unique factorization birthed Dedekind's idea of ideals of a ring.

4. The search culminated in two major theories: Dedekind's unique factorization of ideals into prime ideals (Dedekind domains); and Kronecker's theory of polynomial rings and Lasker's theory of primary decomposition in them.

5. Dedekind represented an element $r \in R$ by the ideal (r) and found conditions under which a ring has unique factorization of ideals into prime ideals. (the ring of all integers in any number field)

6. Kronecker put the notion of adjoining the root of polynomial to a field k on firm footing by introducing the ring $k[x]$, with the desired ring being $k[x]/f(x)$. There is no way to factorize ideals in a polynomial rings but Lasker later showed how to generalize unique factorization into "primary decomposition".