

Commutative Algebra and Algebraic Geometry

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Notes 0.1. *Basic properties of ideals. (Matsumura 1)*

1. In a surjective ring homomorphism $f : A \rightarrow A/I$, the ideals J of A/I and the ideals $f^{-1}(J)$ of A are in one-to-one correspondence. (lattice isomorphism theorem) When we need to think about ideals of A containing I , we can work on A/I : if I' is any ideal of A then $f(I')$ is an ideal of A/I with $f^{-1}(f(I')) = I + I'$, and $f(I') = (I + I')/I$.
2. For $a \in A$, $(a) = (1)$ iff a has an inverse in A . If a is a unit and x is nilpotent, then $a + x$ is a unit.
3. Using Zorn's lemma on a set of proper ideals (ordered by inclusion) containing some ideal I , we see that there exists at least one maximal ideal M containing I . A/M is a field.
4. A proper ideal P is prime if $x, y \notin P \implies xy \notin P$. A/P is an integral domain.
5. If I is an ideal disjoint from a multiplicative set S , then $A - S$ has a maximal ideal containing I which is prime. (prove...)
6. If I is an ideal, then the radical of I

$$\sqrt{I} = \{a \in A : a^n \in I \text{ for some } n > 0\}$$

is also an ideal. If P is a prime ideal containing I then $\sqrt{I} \subset P$. Furthermore, if $x \notin \sqrt{I}$, then we can find a prime ideal containing \sqrt{I} but not x . And hence,

$$\sqrt{I} = \bigcap_{P \supset I} P$$

7. The intersection of all prime ideals is nilradical $\text{nil}(A)$ and the intersection of all maximal ideals is Jacobson radical $\text{rad}(A)$. $x \in \text{rad}(A)$ iff $1 + Ax$ consists entirely of units of A .
8. $II' \subset I \cap I'$. If $I + I' = (1)$, then $II' = I \cap I'$. Also, if $I + I' = (1)$ and $I + I'' = (1)$, then $I + I'I'' = (1)$. Hence, for ideals I_i which are coprime in pairs

$$I_1 I_2 \dots I_n = I_1 \cap I_2 \cap \dots \cap I_n.$$

9. If $I + I' = (1)$, then $A/II' \simeq A/I \times A/I'$. This can be extended with n ideals like in 8.
10. A maximal ideal \mathfrak{m} of A corresponds with the maximal ideal $\mathfrak{m} = \mathfrak{m}B + (X_1, \dots, X_n)$ of the ring $B = A[[X_1, \dots, X_n]]$. Here, $\mathfrak{m} \cap A = \mathfrak{m}$. These properties are not necessarily true in case of polynomial rings.
11. $a \in A$ is called irreducible element if a is not a unit of A and

$$a = bc \implies b \text{ or } c \text{ is a unit of } A.$$

a is irreducible iff aA is maximal among proper principal ideals and a is prime if aA is a prime ideal.

12. $A = \mathbb{Z}[\sqrt{-5}] = \mathbb{Z}[X]/(X^2 + 5)$; then setting $k = \mathbb{Z}/2\mathbb{Z}$ we have

$$A/2A = \mathbb{Z}[X]/(2, X^2 + 5) = k[X]/(X^2 - 1) = k[X]/(X - 1)^2.$$

Then $P = (2, 1 - \sqrt{-5})$ is a maximal ideal of A containing 2.

Problems 0.1.

1. Let A be a ring, and $I \subset \text{nil}(A)$ an ideal made up of nilpotent elements. If $a \in A$ maps to a unit of A/I then a is a unit of A .
2. Let A_1, \dots, A_n be rings; then the prime ideals of $A = A_1 \times \cdots \times A_n$ are of the form

$$P = A_1 \times \cdots \times A_{i-1} \times P_i \times A_{i+1} \times \cdots \times A_n,$$

where P_i is a prime ideal of A_i .

For $A_1 \times A_2$, let $P = I_1 \times I_2$ be a prime ideal. Then, since the product of integral domains is not itself an integral domain, only one from I_1 or I_2 is prime. $(A_1 \times A_2)/(I_1 \times I_2) \simeq A_1/I_1 \times A_2/I_2$. This can be extended to n ideals in a similar fashion.

3. Let A and B be rings and $f : A \rightarrow B$ a surjective homomorphism.
 - a. Prove that $f(\text{rad } A) \subset \text{rad } B$, and construct an example where the inclusion is strict.
If $x \in \text{rad } A$, then $f(1 + Ax) = 1 + Bf(x)$ should be a set of units of B .
 - b. Prove that if A is a semilocal ring then $f(\text{rad } A) = \text{rad } B$.
4. Let A be an integral domain. Then A is a UFD iff every irreducible element is prime and the principal ideals of A satisfy the ascending chain condition. (Equivalently, every non-empty family of principal ideals has a maximal element.)
5. Let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a non-empty family of prime ideals, and suppose that the P_λ are totally ordered by inclusion; then $\bigcap P_\lambda$ is a prime ideal. Also, if I is any proper ideal, the set of prime ideals containing I has a minimal element.
6. Let A be a ring, I, P_1, \dots, P_r ideals of A , and suppose that P_1, \dots, P_r are prime, and that I is not contained in any of the P_i ; then there exists an element $x \in I$ not contained on any P_i .