Group Actions on Matroids

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An action of S_n on a matroid M over n-elements ground set $E = \{x_1, \ldots, x_n\}$ is defined as the natural extension of its action on E such that flats are mapped to flats. We note that such an action has the following properties on its flats-lattice structure:

1. $g \cdot (a \vee b) = g \cdot a \vee g \cdot b$.
2. $g \cdot (a \wedge b) = g \cdot a \wedge g \cdot b$.
3. $\text{rk}\ (g \cdot a) = \text{rk}\ a$.
4. $A \subseteq B \iff g \cdot A \subseteq g \cdot B$.

Let $\mathfrak{F} = \{F_1, \ldots, F_k\}$ be the collection of non empty proper flats and $E = \{1, \ldots, n\}$ be the set of atoms of the matroid $M = (E, \mathfrak{F})$. The action $S_n \times E \to E$ of S_n on E extends to an action $S_n \times M \to M$.

For the graded ring

$$R = k \oplus k(x_{F_1}, \ldots, x_{F_k}) \oplus k(x_{F_i} x_{F_j} \mid 0 < i \leq j \leq k) \oplus \ldots \oplus k(x_{F_1} \cdots x_{F_k}),$$

let I be the ideal generated by the elements $x_{F_i} \cdot x_{F_j}$ for incomparable flats F_i and F_j. Let J be the ideal of R generated by the elements $\alpha_i = \sum_{F_j \ni i} x_{F_j}$. Then the Chow ring $A(M)$ of M is given as the quotient $R/(I + J)$.

Proposition 1.1. *The action of S_n on the matroid M over $E = \{x_1, \ldots, x_n\}$ induces an action on R which stabilizes the ideals I and J.*

Proof. Given an action $k \mapsto \sigma \cdot k$ for $k \in E$, we define the induced action of on R as the linear extension of $x_F \mapsto x_{\sigma \cdot F}$ where $\sigma \cdot F = \sigma \cdot \{k_1, \ldots, k_r\} = \{\sigma \cdot k_1, \ldots, \sigma \cdot k_r\}$. We note that the action as the automorphism of the matroid induces an isomorphism of $k-$modules.

We now show that the action stabilizes the ideals I and J: F_j contains F_i if and only if $\sigma \cdot F_j$ contains $\sigma \cdot F_i$. Hence

$$\sigma \cdot \alpha_k = \sum_{\sigma \cdot F_j \ni \sigma \cdot k} \sigma \cdot F_j = \alpha_{\sigma \cdot k} \in J.$$

Similarly, since F_i and F_j are incomparable precisely when $F_i \nsubseteq F_j$ and $F_j \nsubseteq F_i$, we have F_i and F_j incomparable if and only if $\sigma \cdot F_i$ and $\sigma \cdot F_j$ incomparable. Thus, $x_{F_i} \cdot x_{F_j} \in I$ if and only if $x_{\sigma \cdot F_i} \cdot x_{\sigma \cdot F_j} \in I$. $\square$

The ideal $I + J$ has a monic Grobner basis $\{g_1, \ldots, g_t\}$ with respect to a monomial ordering. Thus, by [CLO15-2.5-5.3], $R/(I + J)$ is a free $\mathbb{Z}-$module and has a basis given by standard monomials. The Chow ring has the monomial $\mathbb{Z}-$basis

$$FY := \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} : (\emptyset =: F_0) \subsetneq F_1 \subsetneq F_2 \cdots \subsetneq F_k, \text{ and } m_i \leq rk(F_i) - rk(F_{i-1}) - 1\}$$

given by the $FY-$monomials.

The total degree of the monomials

$$\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} (rk(F_i) - rk(F_{i-1}) - 1) = r.$$