Flat Modules

Nutan Nepal

March 28, 2024

Introduction

For any $A-$module M, every short exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ of $A-$modules induces the exact sequence $N' \otimes M \longrightarrow N \otimes M \longrightarrow N'' \otimes M \longrightarrow 0$. An $A-$module M is flat if for every short exact sequence $\mathcal{S} : 0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$, the induced sequence $\mathcal{S} \otimes M$:

$$0 \longrightarrow N' \otimes M \longrightarrow N \otimes M \longrightarrow N'' \otimes M \longrightarrow 0$$

is also exact. M is called faithfully flat if we have: $\mathcal{S}$ exact $\iff$ $\mathcal{S} \otimes M$ exact. A ring homomorphism $f : R \to S$ is called flat (resp. faithfully flat) if S is flat (resp. faithfully flat) as an $R-$module.

Examples and facts:

1. Projective modules are flat. Injective modules need not be flat and flat modules need not be projective or injective.

2. $\mathbb{Z}$ is a projective (and hence, flat) $\mathbb{Z}-$module. In general, $- \otimes_A A$ is the identity endofunctor the category of $A-$modules i.e. $\mathcal{S} \otimes_A A = \mathcal{S}$ and so, A is always a flat $A-$module.

3. The localization $S^{-1}N$ of an $A-$module N by the multiplicative set S of A is an exact functor . So $A \to S^{-1}A$ is a flat ring morphism.

4. $\mathbb{Q}$ is a flat $\mathbb{Z}-$module. ($\mathbb{Q}$ is also an injective $\mathbb{Z}-$module)

5. Flat modules are torsion-free. Hence, the $\mathbb{Z}-$modules $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$ (which is injective) are not flat.

6. The direct sum of flat modules are flat (since the tensor product commutes with arbitrary direct sums). In particular, the $\mathbb{Z}-$module $\mathbb{Q} \oplus \mathbb{Z}$ (which is neither projective nor injective) is flat.

7. The localization $S^{-1}N$ of an $A-$module N by the multiplicative set S of A is an exact functor . So $A \to S^{-1}A$ is a flat ring morphism.

8. Flatness is preserved by change of base ring: If M is a flat $B-$module and $B \to A$ is a ring morphism, then $M \otimes_B A$ is a flat $A-$module.

9. Flatness is preserved by composition: If A is a flat $B-$algebra and M is a flat $A-$module, then M is also $B-$flat.

10. Flatness is a local property: An $A-$module M is $A-$flat if and only if $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}-$flat for all prime ideals $\mathfrak{p} \subset A$.

In the study algebraic sets, we can consider a family of varieties as follows: using the coordinate ring R of a variety Y and a collection of polynomials $f_i(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$, we consider the varieties $V(f_i(x, b)) = \{a \in \mathbb{A}_k^n | \ f_i(a, b) = 0\}$ to be a family for each $b \in Y$ (i.e. the families are parametrized by Y).