Introduction to Schemes

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Spectrum of a ring

Definition: The spectrum of a ring $\mathrm{Spec}(R)$ is the set of all prime ideals of the commutative ring R. For example:

- For a field k, $\mathrm{Spec}(k)$ is a one point set (0).
- $\mathrm{Spec}(\mathbb{Z})$ is the set $\{(0), (p)\}$ for all primes p.

We can define the Zariski topology on $\mathrm{Spec}(R)$ by defining closed sets to be of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \ \mathfrak{p} \supset \mathfrak{a}\}.$$

With this, $\mathrm{Spec}(R)$ becomes a topological space. The open sets of the form $D(f) = \mathrm{Spec}(R) - V((f)); f \in R$ are called distinguished open sets.

Presheaf

Definition: A presheaf $\mathcal{F}$ on a topological space X consists of following data:

1. For each open set $\mathcal{U} \subset X$ we have a set $\mathcal{F}(\mathcal{U})$. Elements of $\mathcal{F}(\mathcal{U})$ are called sections of $\mathcal{F}$ over $\mathcal{U}$.
2. For each inclusion $\mathcal{U} \hookrightarrow \mathcal{V}$, we have the restriction

$$\mathrm{res}_{\mathcal{V},\mathcal{U}} : \mathcal{F}(\mathcal{V}) \longrightarrow \mathcal{F}(\mathcal{U})$$

   that satisfies:

   - $\mathrm{res}_{\mathcal{U},\mathcal{U}} = \mathrm{id}_{\mathcal{F}(\mathcal{U})}$, and
   - if $\mathcal{U} \hookrightarrow \mathcal{V} \hookrightarrow \mathcal{W}$ is are inclusions of open sets we have

$$\mathrm{res}_{\mathcal{V},\mathcal{U}} \circ \mathrm{res}_{\mathcal{W},\mathcal{V}} = \mathrm{res}_{\mathcal{W},\mathcal{U}}.$$

In other words, a presheaf is a contravariant functor on the poset of open sets of X.

Sheaf

Definition: A sheaf $\mathcal{F}$ on a topological space X is a presheaf that satisfies:

1. Glubability axiom: If $\{\mathcal{U}_i\}$ is an open cover of $\mathcal{U}$, then given $f_i \in \mathcal{F}(\mathcal{U}_i)$ such that
$$\mathrm{res}_{\mathcal{U}_i,\, \mathcal{U}_i \cap \mathcal{U}_j}(f_i) = \mathrm{res}_{\mathcal{U}_j,\, \mathcal{U}_i \cap \mathcal{U}_j}(f_j)$$
there exists $f \in \mathcal{U}$ that satisfies $\mathrm{res}_{\mathcal{U},\mathcal{U}_i}(f) = f_i$.
2. Identity axiom: If $\{\mathcal{U}_i\}$ is an open cover of $\mathcal{U}$, then given $f_1, f_2 \in \mathcal{F}(\mathcal{U})$ such that
$$\mathrm{res}_{\mathcal{U},\mathcal{U}_i}(f_1) = \mathrm{res}_{\mathcal{U},\mathcal{U}_j}(f_2)$$
for all i, j then $f_1 = f_2$.

Here, gluability axiom says that there is at least one way to glue compatible sections and identity axiom says that there is at most one such gluing.

Examples

Functions on a topological space

Given a topological space X, we can take $\mathcal{F}(X)$ to be the continuous, smooth, real or complex valued functions on X and res to be the usual restriction map between sets. Then $\mathcal{F}$ is a sheaf on X.

Skyscraper Sheaf

If $p \in X$ is a point and S is any set, we define

$$\mathcal{F}(U) = \begin{cases} S & p \in U, \\ \{e\} & p \notin U. \end{cases}$$

where $\{e\}$ is any one point set.

Stalks

Definition: The stalk of a presheaf $\mathcal{F}$ at a point $p \in X$ is the set of all germs at p. That is, it is the set

$$\{(f; U) : p \in U; f \in \mathcal{F}(U)\}$$

modulo the relation that $(f; U) \sim (g; V)$ if there is some open set $W \subset U, V$ where $p \in W$ and $\text{res}_{U;W}(f) = \text{res}_{V;W}(g)$.

Ringed Spaces and Locally Ringed Spaces

Definition: Given a sheaf of rings $\mathcal{O}_X$ on X, $(X, \mathcal{O}_X)$ is called a ringed space.
A locally ringed space $(X, \mathcal{O}_X)$ is a ringed space if the stalk of $\mathcal{O}_X$ at every point $p \in X$ is a local ring.

- A morphism of locally ringed spaces $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ constitutes of the continuous map of topological spaces $f : X \to Y$ and morphisms of the structure sheaves.

For example: manifolds with sheaves of smooth functions are locally ringed spaces.
Given the topological space $\mathrm{Spec}(R)$, there is a construction where we consider $\mathcal{O}_{\mathrm{Spec}R}(D(f)) = R_f$, the localization of the ring R at the multiplicative set $\{1, f, f^2, \ldots\}$. With this, we obtain another locally ringed space $(R, \mathcal{O}_{\mathrm{Spec}R})$.

Schemes

Affine Schemes: A locally ringed space which is isomorphic to $(R, \mathcal{O}_{\mathrm{Spec}R})$ for some commutative ring R is called an affine scheme.

Scheme: A locally ringed space is called a scheme if it has an open covering by affine schemes.

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