Flat Modules

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1 Introduction

In the study algebraic sets, we can consider a family of varieties as follows: using the coordinate ring R of an affine variety Y over an algebraically closed field k and a collection of polynomials $f_i(x_1, \ldots, x_n; b) \in R[x_1, \ldots, x_n]$, we consider the collection of zero sets

$$V(f_i(x; b)) = \{a \in \mathbb{A}_k^n | \ f_i(a, b) = 0\}, \quad b \in Y$$

to be in **a family parametrized by** Y. The polynomials f_i were not over the field k and the zero sets need not be algebraic sets themselves. However, they have covering by affine varieties and can be considered as subschemes of $\mathbb{A}_k^n$. If I is the ideal generated by the polynomials f_i's in the ring $R[x_1, \ldots, x_n]$, then we get the ring morphism $f : R \to R[x_1, \ldots, x_n]/I$. This ring morphism induces the morphism of the maximal spectrums

$$\operatorname{mSpec}(R[x_1, \ldots, x_n]/I) \to \operatorname{mSpec} R = Y$$

which describes the maps between the subschemes $V(I)$ and Y. The fiber over each point $b \in Y$ is then given by

$$\operatorname{mSpec}(R[x_1, \ldots, x_n]/I \otimes R/m_b) = \operatorname{mSpec}(k[x_1, \ldots, x_n]/(f_i(x, b))).$$

The well-behavedness of the fibers over Y depends on the family of the subschemes i.e. on the morphisms of the subschemes.

Example 1. The projection

$$\{(x, y, t) \in \mathbb{A}_k^3 \mid y - xt = 0\} \to \mathbb{A}_k^2$$

defines a family of subvarieties of the $V(y - xt)$. The morphism has fiber over $(0, 0)$ as one-dimensional variety (since any value of t satisfies the polynomial) and the fibers over other points are zero-dimensional.▲

Hence, a morphism of varieties $f : X \to Y$ itself defines a family of subvarieties of the source parametrized by the target. The families where the fibers vary "nicely" are called **flat families** and these families are characterized by **flat morphisms** between the subschemes. The notion of flatness was first introduced by Serre for algebraic reasons and Grothendieck later recognized the geometric significance of it.

Similar considerations with homogenous polynomials in $R[x_0, \dots, x_n]$ defines a family of subschemes of $\mathbb{P}^n_k$. Here we will try to describe the notion of flat morphisms between two schemes.

2 Flat modules and flat maps

For any $A-$module M, every short exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ of $A-$modules induces the exact sequence $N' \otimes M \longrightarrow N \otimes M \longrightarrow N'' \otimes M \longrightarrow 0$. An $A-$module M is flat if for every short exact sequence $\mathcal{S} : 0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$, the induced sequence $\mathcal{S} \otimes M$:

$$0 \longrightarrow N' \otimes M \longrightarrow N \otimes M \longrightarrow N'' \otimes M \longrightarrow 0$$

is also exact. M is called faithfully flat if we have: $\mathcal{S}$ exact $\iff$ $\mathcal{S} \otimes M$ exact. A ring homomorphism $f : R \to S$ is called flat (resp. faithfully flat) if S is flat (resp. faithfully flat) as an $R-$module.

Examples and facts:

1. Projective modules are flat. Injective modules need not be flat and flat modules need not be projective or injective.

2. $\mathbb{Z}$ is a projective (and hence, flat) $\mathbb{Z}-$module. In general, $- \otimes_A A$ is the identity endofunctor the category of $A-$modules i.e. $\mathcal{S} \otimes_A A = \mathcal{S}$ and so, A is always a flat $A-$module.

3. The localization $S^{-1}N$ of an $A-$module N by the multiplicative set S of A is an exact functor . So $A \to S^{-1}A$ is a flat ring morphism.

4. $\mathbb{Q}$ is a flat $\mathbb{Z}-$module. ($\mathbb{Q}$ is also an injective $\mathbb{Z}-$module)

5. Flat modules are torsion-free. Hence, the $\mathbb{Z}-$modules $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$ (which is injective) are not flat.

6. Finitely generated modules over principal ideal domains are flat if and only if they are torsion-free if and only if they are free.

7. The direct sum of flat modules are flat (since the tensor product commutes with arbitrary direct sums). In particular, the $\mathbb{Z}-$module $\mathbb{Q} \oplus \mathbb{Z}$ (which is neither projective nor injective) is flat.

8. The localization $S^{-1}N$ of an $A-$module N by the multiplicative set S of A is an exact functor. So $A \to S^{-1}A$ is a flat ring morphism.

9. Flatness is preserved by change of base ring: If M is a flat $B-$module and $B \to A$ is a ring morphism, then $M \otimes_B A$ is a flat $A-$module.

10. Flatness is preserved by composition: If A is a flat $B-$algebra and M is a flat $A-$module, then M is also $B-$flat.

11. Flatness is a local property: An $A-$module M is $A-$flat if and only if $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}-$flat for all prime ideals $\mathfrak{p} \subset A$.

12. An $A-$module M is $A-$flat if and only if for all ideals I of A, we have $I \otimes M \cong IM$.

3 Quasicoherent sheaves and flat morphisms

Given a ring R and an $R-$module M, we can form a sheaf of abelian groups $\widetilde{M}$ on $X = \operatorname{Spec} R$ by taking $\widetilde{M}(D(f)) = M \otimes f^{-1}R = f^{-1}M$, which is the localization of M by the multiplicative set $\{1, f, f^2, \dots\}$, and then extending to all the open sets. The sheaf $\widetilde{M}$ has the structure of an $\mathcal{O}_X-$module.

1. A sheaf $\mathcal{F}$ on X is called **quasicoherent** if for each affine open subset Spec A of X, the sheaf $\mathcal{F}\mid_{\operatorname{Spec} A}$ is isomorphic to $\widetilde{M}$ for some $A-$module M. The category of quasicoherent sheaves over an affine scheme $\operatorname{Spec} A$ is equivalent to the category of $A-$modules: $\mathcal{QC}oh_{\operatorname{Spec} A} \overset{\sim}{\longleftrightarrow} \mathfrak{mod}_A$.

2. A **quasicoherent sheaf** $\mathcal{F}$ on X is said to be **flat** at $p \in X$ if $\mathcal{F}_p$ is a flat $\mathcal{O}_{X,p}-$module. A **quasicoherent sheaf** $\mathcal{F}$ on X is said to be **flat** over X if for every point $p \in X$, $\mathcal{F}_p$ is a flat $\mathcal{O}_{X,p}-$module.

3. A morphism $f : X \to Y$ of schemes is said to be **flat** at $p \in X$ if $\mathcal{O}_{X,p}$ is a flat $\mathcal{O}_{Y,f(p)}-$module. A morphism $f : X \to Y$ of schemes is called a **flat morphism** if for every $p \in X$, $\mathcal{O}_{X,p}$ is a flat $\mathcal{O}_{Y,f(p)}-$module. A morphism is called **faithfully flat** if it is both flat and surjective.

   In particular, if $Y = \operatorname{mSpec} A$, a morphism (or family) of schemes $f : X \to Y$ is flat if $\mathcal{O}(U)$ is a flat $A-$module for every open set $U \subset X$.

Examples and facts

1. Given a ring map $B \to A$, A is faithfully flat over B if and only if $\operatorname{Spec} A \to \operatorname{Spec} B$ is a faithfully flat morphism.

2. Open embeddings are flat. $f : X \to Y$ is an **open embedding** if X is isomorphic to an open set of Y. For a morphism $f : D(y - x^2) \to \mathbb{A}^2_k$ which is clearly an open embedding we see that

3. A morphism of rings $A \to B$ is flat if and only if the corresponding morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$. More generally, if $B \to A$ is a ring homomorphism and M is an $A-$module, then M is $B-$flat if and only if $\widetilde{M}$ is flat over $\operatorname{Spec} B$.

4. The fibers of a flat morphism of varieties $f : X \to Y$ all all have the same dimension dim $X-$ dim Y.

5. If $f : X \to Y$ is a surjective morphism of a variety to a non-singular curve, then it is flat.

6. The above two statements imply: The fibers of a surjective morphism of a variety to a non-singular curve have dimension equal to dim $X - 1$.

References

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