Group Actions on Matroids

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Any matroid M can be thought of as a geometric lattice (i.e. atomic and semimodular). An action of S_n on a matroid M over n-elements ground set $E = \{x_1, \ldots, x_n\}$ is defined as the natural extension of its action on E such that flats are mapped to flats. We note that such an action has the following properties on the lattice:

1. $g \cdot (a \vee b) = g \cdot a \vee g \cdot b$.
2. $g \cdot (a \wedge b) = g \cdot a \wedge g \cdot b$.
3. $\text{rk } (g \cdot a) = \text{rk } a$.
4. $A \subseteq B \iff g \cdot A \subseteq g \cdot B$.

Let $\mathfrak{F} = \{F_1, \ldots, F_k\}$ be the collection of non empty proper flats and $E = \{1, \ldots, n\}$ be the set of atoms of the matroid $M = (E, \mathfrak{F})$. The action $S_n \times E \to E$ of S_n on E extends to an action $S_n \times M \to M$.

For the graded ring

$$R = k \oplus k(x_{F_1}, \ldots, x_{F_k}) \oplus k(x_{F_i} x_{F_j} \mid 0 < i \leq j \leq k) \oplus \cdots ,$$

let I be the ideal generated by the elements $x_{F_i} \cdot x_{F_j}$ for incomparable flats F_i and F_j. Let J be the ideal of R generated by the elements $\alpha_i = \sum_{F_j \ni i} x_{F_j}$. Then the Chow ring $A(M)$ of M is given as the quotient $R/(I + J)$.

Proposition 1.1. *The action of S_n on the matroid M over $E = \{x_1, \ldots, x_n\}$ induces an action on R which stabilizes the ideals I and J.*

Proof. Given an action $k \mapsto \sigma \cdot k$ for $k \in E$, we define the induced action of on R as the linear extension of $x_F \mapsto x_{\sigma \cdot F}$ where $\sigma \cdot F = \sigma \cdot \{k_1, \ldots, k_r\} = \{\sigma \cdot k_1, \ldots, \sigma \cdot k_r\}$. We note that the action as the automorphism of the matroid induces an isomorphism of $k-$modules.

We now show that the action stabilizes the ideals I and J: F_j contains F_i if and only if $\sigma \cdot F_j$ contains $\sigma \cdot F_i$. Hence

$$\sigma \cdot \alpha_k = \sum_{\sigma \cdot F_j \ni \sigma \cdot k} \sigma \cdot F_j = \alpha_{\sigma \cdot k} \in J.$$

Similarly, since F_i and F_j are incomparable precisely when $F_i \nsubseteq F_j$ and $F_j \nsubseteq F_i$, we have F_i and F_j incomparable if and only if $\sigma \cdot F_i$ and $\sigma \cdot F_j$ incomparable. Thus, $x_{F_i} \cdot x_{F_j} \in I$ if and only if $x_{\sigma \cdot F_i} \cdot x_{\sigma \cdot F_j} \in I$. $\square$

The ideal $I + J$ has a monic Grobner basis $\{g_1, \dots, g_t\}$ with respect to a monomial ordering. Thus, by [CLO15-2.5-5.3], $R/(I + J)$ is a free $\mathbb{Z}-$module and has a basis given by standard monomials. The Chow ring has the monomial $\mathbb{Z}-$basis

$$FY := \{x_{F_1}^{m_1} x_{F_2}^{m_2} \cdots x_{F_k}^{m_k} : (\emptyset =: F_0) \subsetneq F_1 \subsetneq F_2 \cdots \subsetneq F_k, \text{ and } m_i \leq rk(F_i) - rk(F_{i-1}) - 1\}$$

given by the $FY-$monomials.

The total degree of the monomials

$$\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} (rk(F_i) - rk(F_{i-1}) - 1) = r$$

when the rank of the matroid is $r + 1$. Thus $A(M) = \oplus_{i=1}^{r} A^k(M)$.

The group S_n acts on each graded piece of degree k as

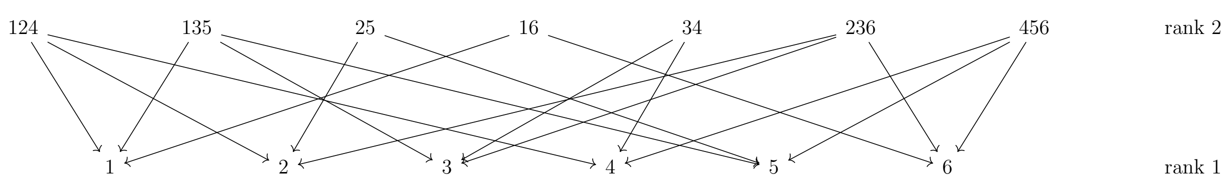
$$x_{F_1}^{m_1} \cdots x_{F_k}^{m_k} \mapsto x_{\sigma \cdot F_1}^{m_1} \cdots x_{\sigma \cdot F_k}^{m_k}$$

since the incomparability relations are preserved under group actions.

Braid−3:

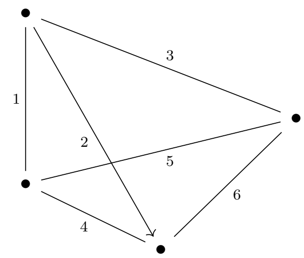
The non-empty proper flats have the following lattice structure. The $FY-$monomials are given as:

1. degree 0: 1,
2. degree 1: x_E; x_F for all rank 2 flats F
3. degree 2: x_E^2.



In particular, for any rank 3 matroid, the analogous result about the $FY-$monomials holds.

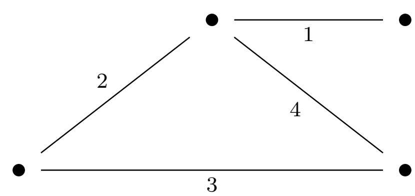
Not all elements $\sigma \in S_6$ give an action $x_F \mapsto x_{\sigma \cdot F}$. For example, (1 2) doesn't give an action on M since (1 6) $\mapsto$ (2 6) and (2 6) is not a flat of M; so it doesn't give an action on the Chow ring either. The braid matroid can be realized as the following graph.



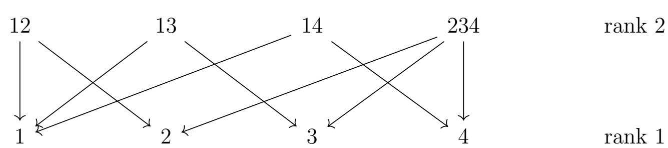
The subgroup S_4 in S_6 corresponding to the rigid symmetry of the tetrahedron acts on the matroid $B-3$. For example, (1 2 4)(3 6 5) is a rotation of the tetrahedron keeping the right vertex fixed. The corresponding action on the FY^1 monomials is given below:

1. $x_{124} \mapsto x_{124},\ x_{135} \mapsto x_{236},\ x_{236} \mapsto x_{456},\ x_{456} \mapsto x_{135};$
2. $x_{25} \mapsto x_{34},\ x_{34} \mapsto x_{16},\ x_{16} \mapsto x_{25};$
3. $x_E \mapsto x_E.$

Similarly, the reflection (1 3)(4 6) also gives an action.



Another lattice of flats of a different matroid arising from the graph above is given below. The degree 1 $FY-$monomials are x_{12}, x_{13}, x_{14}, x_{234} and x_E.



The symmetries of the triangle are the automorphisms of the matroid.