Flat Modules

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1 Introduction

In the study algebraic sets, we can consider a family of varieties as follows: using the coordinate ring R of an affine variety Y over an algebraically closed field k and a collection of polynomials $f_i(x_1, \ldots, x_n; b) \in R[x_1, \ldots, x_n]$, we consider the collection of zero sets

$$V(f_i(x; b)) = \{a \in \mathbb{A}_k^n \mid f_i(a, b) = 0\}, \quad b \in Y$$

to be in **a family parametrized by** Y. The fibers over each $b \in Y$ have covering by affine varieties and can be considered as subschemes of $\mathbb{A}_k^n$. If I is the ideal generated by the polynomials f_i's in the ring $R[x_1, \ldots, x_n]$, then we get the ring morphism $f \colon R \to R[x_1, \ldots, x_n]/I$. This ring morphism induces the morphism of the maximal spectrums (the topological space with maximal ideals as the points)

$$\operatorname{mSpec}(R[x_1, \ldots, x_n]/I) \to \operatorname{mSpec} R = Y$$

which describes the maps between the subschemes $V(I)$ and Y. The fiber over each point $b \in Y$ is then given by

$$\operatorname{mSpec}(R[x_1, \ldots, x_n]/I \otimes R/m_b) = \operatorname{mSpec}(k[x_1, \ldots, x_n]/(f_i(x, b))).$$

The well-behavedness of the fibers over Y depends on the family of the subschemes i.e. on the morphisms of the subschemes.

Example 1. The projection

$$\{(x, y, t) \in \mathbb{A}_k^3 \mid y - xt = 0\} \to \mathbb{A}_k^2$$

defines a family of subvarieties of the $V(y - xt)$. The morphism has fiber over $(0, 0)$ as one-dimensional variety (since any value of t satisfies the polynomial) and the fibers over other points are zero-dimensional.▲

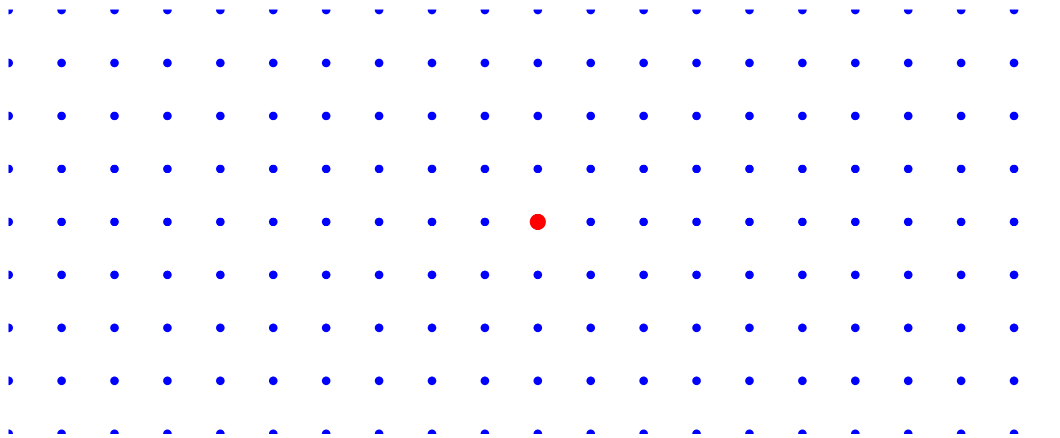

Figure 1: Fibers over integer lattice points are shown in the diagram. The blue points are the zero dimensional fibers and the red is the line (one-dimensional fiber) over the point (0,0).

Hence, a morphism of varieties $f\colon X \to Y$ itself defines a family of subvarieties of the source parametrized by the target. The families where the fibers vary "nicely" are called **flat families** and these families are characterized by **flat morphisms** between the subschemes. In the above example, the fibers of projection had jumps in their dimensions. The notion of flatness was first introduced by Serre for algebraic reasons and Grothendieck later recognized the geometric significance of it.[1]

Similar considerations with homogenous polynomials in $R[x_0, \dots, x_n]$ defines a family of subschemes of $\mathbb{P}^n_k$. Here we describe the notion of flat morphisms between two schemes.

2 Flat modules and flat maps

For any A-module M, every short exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ of $A-$modules induces the exact sequence $N' \otimes M \longrightarrow N \otimes M \longrightarrow N'' \otimes M \longrightarrow 0$. An A-module M is flat if for every short exact sequence $\mathcal{S} : 0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$, the induced sequence $\mathcal{S} \otimes M$:

$$0 \longrightarrow N' \otimes M \longrightarrow N \otimes M \longrightarrow N'' \otimes M \longrightarrow 0$$

is also exact. M is called faithfully flat if we have: $\mathcal{S}$ exact $\iff$ $\mathcal{S} \otimes M$ exact. A ring homomorphism $f\colon R \to S$ is called flat (resp. faithfully flat) if S is flat (resp. faithfully flat) as an R-module.

Example 2. ▲

While the definition looks quite unmotivating, the following list provides plenty of examples and general facts about flat modules.

Examples and facts:

1. Projective modules are flat. Injective modules need not be flat and flat modules need not be projective or injective. In particular, free modules are flat.
2. $\mathbb{Z}$ is a projective (and hence, flat) $\mathbb{Z}$-module. In general, $- \otimes_A A$ is the identity endofunctor in the category of A-modules i.e. $\mathcal{S} \otimes_A A = \mathcal{S}$ and so, A is always a flat A-module.
3. $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. ($\mathbb{Q}$ is also an injective $\mathbb{Z}$-module)
4. Flat modules are torsion-free. Hence, the $\mathbb{Z}$-modules $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$ are not flat. The latter is an example of injective module that is not flat.
5. Finitely generated modules over principal ideal domains are flat if and only if they are torsion-free if and only if they are free.
6. The direct sum of flat modules are flat (since the tensor product commutes with arbitrary direct sums). In particular, the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}$ (which is neither projective nor injective) is flat.
7. The localization $S^{-1}N$ of an A-module N by the multiplicative set S of A is an exact functor. So $A \to S^{-1}A$ is a flat ring morphism.
8. Flatness is preserved by change of base ring: If M is a flat B-module and $B \to A$ is a ring morphism, then $M \otimes_B A$ is a flat A-module.
9. Flatness is preserved by composition: If A is a flat B-algebra and M is a flat A-module, then M is also $B-$flat.
10. Flatness is a local property: An A-module M is A-flat if and only if $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-flat for all prime ideals $\mathfrak{p} \subset A$.
11. An A-module M is A-flat if and only if for all ideals I of A, we have $I \otimes M \cong IM$.

3 Quasicoherent sheaves and flat morphisms

Given a ring A and an A-module M, we can form a sheaf of abelian groups $\widetilde{M}$ on $X = \operatorname{Spec} R$ by taking $\widetilde{M}(D(f)) = M \otimes f^{-1}A = f^{-1}M$, which is the localization of M by the multiplicative set $\{1, f, f^2, \ldots\}$, and then extending to all the open sets. The sheaf $\widetilde{M}$ has the structure of an $\mathcal{O}_X$-module: each $\widetilde{M}(U)$ is an $A(U)$-module and the module action commutes with the restriction maps. In other words, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r, "action"] \arrow[d, "\mathrm{res}_{V,U} \times \mathrm{res}_{V,U}"'] & \mathcal{F}(V) \arrow[d, "\mathrm{res}_{V,U}"'] \\
\mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "action"] & \mathcal{F}(U)
\end{tikzcd}$$

1. A sheaf $\mathcal{F}$ on X is called **quasicoherent** if for each affine open subset Spec A of X, the sheaf $\mathcal{F}|_{\operatorname{Spec} A}$ is isomorphic to $\widetilde{M}$ for some A-module M. The category of quasicoherent sheaves over an affine scheme $\operatorname{Spec} A$ is equivalent to the category of A-modules: $\mathcal{QC}oh_{\operatorname{Spec} A} \overset{\sim}{\longleftrightarrow} \mathfrak{mod}_A$.

2. A **quasicoherent sheaf** $\mathcal{F}$ on X is said to be **flat** at $p \in X$ if $\mathcal{F}_p$ is a flat $\mathcal{O}_{X,p}$-module. A **quasicoherent sheaf** $\mathcal{F}$ on X is said to be **flat** over X if for every point $p \in X$, $\mathcal{F}_p$ is a flat $\mathcal{O}_{X,p}$-module.

3. A morphism $f\colon X \to Y$ of schemes is said to be **flat** at $p \in X$ if $\mathcal{O}_{X,p}$ is a flat $\mathcal{O}_{Y,f(p)}$-module. A morphism $f\colon X \to Y$ of schemes is called a **flat morphism** if for every $p \in X$, $\mathcal{O}_{X,p}$ is a flat $\mathcal{O}_{Y,f(p)}$-module. A morphism is called **faithfully flat** if it is both flat and surjective.

   In particular, if $Y = \operatorname{mSpec} A$, a morphism (or family) of schemes $f\colon X \to Y$ is flat if $\mathcal{O}(U)$ is a flat A-module for every open set $U \subset X$.

Examples and facts

1. Given a ring map $B \to A$, A is faithfully flat over B if and only if $\operatorname{Spec} A \to \operatorname{Spec} B$ is a faithfully flat morphism.

2. Open embeddings are flat. $f\colon X \to Y$ is an **open embedding** if X is isomorphic to an open set of Y. For a morphism $f\colon D(y - x^2) \to \mathbb{A}^2_k$ which is clearly an open embedding we see that

3. A morphism of rings $A \to B$ is flat if and only if the corresponding morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$. More generally, if $B \to A$ is a ring homomorphism and M is an A-module, then M is $B-$flat if and only if $\widetilde{M}$ is flat over $\operatorname{Spec} B$.

4. The fibers of a flat morphism of varieties $f\colon X \to Y$ all all have the same dimension $\dim X - \dim Y$.

5. If $f\colon X \to Y$ is a surjective morphism of a variety to a non-singular curve, then it is flat.

6. The above two statements imply: The fibers of a surjective morphism of a variety to a non-singular curve have dimension equal to $\dim X - 1$.

4 Flatness through Tor

$\operatorname{Tor}_i^A(M, \cdot)$ is the derived functor $\mathfrak{mod}_A \to \mathfrak{mod}_A$ of the right-exact covariant functor $M \otimes \cdot$ (meaning $\operatorname{Tor}_0^A(M, N) = M \otimes N$ for all A-modules N). For an A-module M, the following are equivalent:

i. M is a flat $A-$ module.

ii. $\mathrm{Tor}_i^A(M, N) = 0$ for all $i > 0$ and A-module N.

iii. $\mathrm{Tor}_1^A(M, N) = 0$ for all A-module N.

Tor stands for torsion as seen in the following example:

Example 3. If $x \in A$ is not a zero divisor then

$$\mathrm{Tor}_i^A(M, A/(x)) = \begin{cases} M/xM & \text{if } i = 0; \\ (M : x) & \text{if } i = 1; \\ 0 & \text{if } i > 1, \end{cases}$$

where $(M : x) = \{m \in M \mid x \cdot m = 0\}$ is the set of elements annihilated by x. Hence a module over a domain is flat if and only if it is torsion-free.▲

4.1 Examples and facts

1. (**ideal-theoretic criterion for flatness**) An A-module M is flat if and only if $\mathrm{Tor}_1^A(M, A/I) = 0$ for any ideal I.

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A lot of the nice properties that we find in flat morphism turn out to be cohomological in nature. This makes sense since flatness itself is characterized by cohomology: it is equivalent to the vanishing of the Tor functor. For example, if $f\colon X \to Y$ is a projective flat morphism with Y connected and Noetherian, then the numerical invariants of the fibers like the dimension, Hilbert polynomial, degree, the arithmetic genus and other numbers interpretable in terms of an Euler characteristic are constant [1].

References

[1] Ravi Vakil. *Foundations of Algebraic Geometry*. 2023.