

Flat Modules

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Introduction

For any A -module M , every short exact sequence $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ of A -modules induces the exact sequence $N' \otimes M \rightarrow N \otimes M \rightarrow N'' \otimes M \rightarrow 0$. An A -module M is flat if for every short exact sequence $\mathcal{S} : 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$, the induced sequence $\mathcal{S} \otimes M$:

$$0 \rightarrow N' \otimes M \rightarrow N \otimes M \rightarrow N'' \otimes M \rightarrow 0$$

is also exact. M is called faithfully flat if we have: \mathcal{S} exact $\iff \mathcal{S} \otimes M$ exact. A ring homomorphism $f : R \rightarrow S$ is called flat (resp. faithfully flat) if S is flat (resp. faithfully flat) as an R -module.

Examples and facts:

1. Projective modules are flat. Injective modules need not be flat and flat modules need not be projective or injective.
2. \mathbb{Z} is a projective (and hence, flat) \mathbb{Z} -module. In general, $- \otimes_A A$ is the identity endofunctor the category of A -modules i.e. $\mathcal{S} \otimes_A A = \mathcal{S}$ and so, A is always a flat A -module.
3. The localization $S^{-1}N$ of an A -module N by the multiplicative set S of A is an exact functor. So $A \rightarrow S^{-1}A$ is a flat ring morphism.
4. \mathbb{Q} is a flat \mathbb{Z} -module. (\mathbb{Q} is also an injective \mathbb{Z} -module)
5. Flat modules are torsion-free. Hence, the \mathbb{Z} -modules $\mathbb{Z}/n\mathbb{Z}$ and \mathbb{Q}/\mathbb{Z} (which is injective) are not flat.
6. The direct sum of flat modules are flat (since the tensor product commutes with arbitrary direct sums). In particular, the \mathbb{Z} -module $\mathbb{Q} \oplus \mathbb{Z}$ (which is neither projective nor injective) is flat.

7. The localization $S^{-1}N$ of an A -module N by the multiplicative set S of A is an exact functor. So $A \rightarrow S^{-1}A$ is a flat ring morphism.
8. Flatness is preserved by change of base ring: If M is a flat B -module and $B \rightarrow A$ is a ring morphism, then $M \otimes_B A$ is a flat A -module.
9. Flatness is preserved by composition: If A is a flat B -algebra and M is a flat A -module, then M is also B -flat.
10. Flatness is a local property: An A -module M is A -flat if and only if $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$ -flat for all prime ideals $\mathfrak{p} \subset A$.

In the study algebraic sets, we can consider a family of varieties as follows: using the coordinate ring R of a variety Y and a collection of polynomials $f_i(x_1, \dots, x_n) \in R[x_1, \dots, x_n]$, we consider the varieties $V(f_i(x, b)) = \{a \in \mathbb{A}_k^n \mid f_i(a, b) = 0\}$ to be a family for each $b \in Y$ (i.e. the families are parametrized by Y).