Algebra I

Homework 2 - All Questions

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(3.3 - 1) Let F be a finite field of order q and let $n \in \mathbb{Z}$. Prove that $|GL_n(F) : SL_n(F)| = q - 1$. [See Exercise 35, Section 1]

(3.3 - 2) Prove all parts of the Lattice Isomorphism Theorem.

(3.3 - 3) Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either

(a) $K \leq H$ or

(b) $G = HK$ and $|K : K \cap H| = p$.

(3.4 - 3) Prove that if G is an abelian simple group then $G \cong \mathbb{Z}_p$ for some prime p (do not assume G is a finite group).

(3.4 - 5) Prove that subgroups and quotient groups of a solvable group are solvable.

(3.5 - 3) Prove that S_n is generated by $\{(i \quad i+1) : 1 \leq i \leq n-1\}$. [Consider conjugates, viz. $(2\ 3)(1\ 2)(2\ 3)^{-1}$.]

(3.5 - 9) Prove that the (unique) subgroup of order 4 in A_4 is normal and is isomorphic to V_4.

(4.1 - 2) Let G be a permutation group on the set A (i.e., $G \leq S_A$), let $\sigma \in G$ and let $a \in A$. Prove that $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$. Deduce that if G acts transitively on A then

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = 1.$$

(4.1 - 10) Let H and K be subgroups of the group G. For each $x \in G$ define the HK double coset of x in G to be the set

$$HxK = \{hxk : h \in H,\ k \in K\}.$$

(a) Prove that HxK is the union of the left cosets $x_1K, \ldots, x_nK$ where $\{x_1K, \ldots, x_nK\}$ is the orbit containing xK of H acting by left multiplication on the set of left cosets of K.

(b) Prove that HxK is a union of right cosets of H.

(c) Show that HxK and HyK are either the same set or are disjoint for all x, $y \in G$. Show that the set of HK double cosets partition G.

(d) Prove that $|HxK| = |K| \cdot |H : H \cap xKx^{-1}|$.

(e) Prove that $|HxK| = |H| \cdot |K : K \cap xHx^{-1}|$

(4.2 - 8) Prove that if H has finite index n then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n!$.

(4.2 - 9) Prove that if p is a prime and G is a group of order p^α for some $\alpha \in \mathbb{Z}^+$, then every subgroup of index p is normal in G. Deduce that every group of order p^2 has a normal subgroup of order p.

(4.3 - 5) If the center of G is of index n, prove that every conjugacy class has at most n elements.

(4.3 - 9) Show that $|C_{S_n}((1\ 2)(3\ 4))| = 8 \cdot (n-4)!$ for all $n \geq 4$. Determine the elements in this centralizer explicitly.