Introduction to Manifold Theory

Homework 3

Nutan Nepal

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1. Do Exercise 3.1 (show that if $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, then a function $f : U \to V$ is smooth if and only if each of its component functions $f^i : U \to R$ are smooth).

If $f(x_1, \ldots, x_n) = (f^1(x_1, \ldots, x_n), \ldots, f^m(x_1, \ldots, x_n)))$ then for $i \in \{1, 2, \ldots, n\}$, the first-order partial derivative at p is given by the limit

$$\lim_{t \to 0} \frac{f(p + te_i) - f(p)}{t} = \lim_{t \to 0} \frac{(0, \ldots, 0, f^i(p_1, \ldots, p_i + t, \ldots, p_n) - f^i(p_1, \ldots, p_n), 0, \ldots, 0)}{t} \tag{1}$$

Then for each $i \in \{1, 2, \ldots, n\}$, the partial derivative exists at $p \in U$ iff the limit

$$\lim_{t \to 0} \frac{f^i(p_1, \ldots, p_i + t, \ldots, p_n) - f^i(p_1, \ldots, p_n)}{t} \tag{2}$$

exists at p. But the limit on equation (2) is the derivative of the component function f^i. Hence, the derivative of f exists at p iff each of its component functions are differentiable. The partial derivative at a point p is a function $g : U \to \mathbb{R}^m$. Then, as above, we see that the partial derivatives of g exist iff each of its component functions are differentiable.

If $f : U \to V$ is smooth then all k^{th}-order partial derivatives exist on U for all k. Then, inductively, from above, all k^{th}-order partial derivatives of each component functions also exist on U for all k. Similarly, if all k^{th}-order partial derivatives of each component functions exist on U for all k, then f is also smooth.

2. Check that Definition 3.6 gives an equivalence relation (a binary relation that is reflexive, symmetric, and transitive) on the set of smooth atlases on a given topological manifold X.

Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ and $\mathcal{B} = \{(V_\beta, \psi_\beta) : \beta \in B\}$ be smooth atlases on the topological manifold X for some indexed set A and B. We say that $\mathcal{A} \sim \mathcal{B}$ if their union is a smooth atlas on X. The reflexive ($\mathcal{A} \sim \mathcal{A}$) and symmetric ($\mathcal{A} \sim \mathcal{B} \implies \mathcal{B} \sim \mathcal{A}$) properties are obvious.

We now prove for the transitivity of the relation $\sim$.

If $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$, with $\mathcal{C} = \{(W_\gamma, \zeta_\gamma) : \gamma \in C\}$ for some indexed set C, then for all $\alpha \in A$ and $\beta \in B$ such that $U_\alpha \cap V_\beta$ is non-empty, the map

$$\psi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap V_\beta) \to \psi_\beta(U_\alpha \cap V_\beta) \tag{3}$$

is smooth, and for all $\gamma \in C$ and $\beta \in B$ such that $W_\gamma \cap V_\beta$ is non-empty, the map

$$\zeta_\gamma \circ \psi_\beta^{-1} : \psi_\beta(W_\gamma \cap V_\beta) \to \zeta_\gamma(W_\gamma \cap V_\beta) \tag{4}$$

is smooth. Then, we take all $\alpha \in A$ and $\gamma \in C$ such that $U_\alpha \cap W_\gamma$ is non-empty. For each $x \in U_\alpha \cap W_\gamma$, we take a chart $(V, \psi) \in \mathcal{B}$ that contains $x \in X$. Then from (3) and (4), we get the composition of smooth maps

$$\zeta_\gamma \circ \psi^{-1} \circ \psi \circ \varphi_\alpha^{-1} = \zeta_\gamma \circ \varphi_\alpha^{-1}$$

from $\varphi_\alpha(U_\alpha \cap W_\gamma) \to \zeta_\gamma(U_\alpha \cap W_\gamma)$ which is smooth. Analogously, we can show that the inverse map

$$\varphi_\alpha \circ \zeta_\gamma^{-1} : \zeta_\gamma(U_\alpha \cap W_\gamma) \to \varphi_\alpha(U_\alpha \cap W_\gamma)$$

is also smooth. Hence, this proves transitivity and that $\sim$ is an equivalence relation.

3. Do Exercise 3.2 (Let X and Y be topological manifolds equipped with smooth atlases $\mathcal{A}$ and $\mathcal{B}$ respectively. Show that $\{U \times V : U \in \mathcal{A}, V \in \mathcal{B}\}$ is a smooth atlas on the topological manifold $X \times Y$).

Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ and $\mathcal{B} = \{(V_\beta, \psi_\beta) : \beta \in B\}$ be smooth atlases on the topological manifolds X and Y respectively for some indexed set A and B. Then the product of the smooth atlases is defined by

$$\mathcal{A} \times \mathcal{B} = \{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta) : \alpha \in A,\ \beta \in B\}.$$

(a) If X is m-manifold and Y is n-manifold, then $\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^{m+n}$ is given by

$$(\varphi_\alpha \times \psi_\beta)(x, y) = (\varphi_\alpha^1(x), \dots, \varphi_\alpha^m(x), \psi_\beta^1(y), \dots, \psi_\beta^n(y))$$

for all $x \in U_\alpha$ and $y \in V_\beta$. Since, each component functions are smooth, we see that $\varphi_\alpha \times \psi_\beta$ is a smooth function on the product topology.

(b) Each φ_α is a homeomorphism from U_α to an open disk $D_\alpha \subset \mathbb{R}^m$ and ψ_β is a homeomorphism from V_β to an open disk $D_\beta \subset \mathbb{R}^n$. Clearly, $D_\alpha \times D_\beta$ is an open disk in $\mathbb{R}^{m+n}$. We define $\varphi_\alpha^{-1} \times \psi_\beta^{-1} : D_\alpha \times D_\beta \to U_\alpha \times V_\beta$ by

$$(\varphi_\alpha^{-1} \times \psi_\beta^{-1})(z_1, \dots, z_m, z_{m+1}, \dots, z_{m+n}) = (\varphi^{-1}(z_1, \dots, z_m, z_{m+1}), \psi_\beta^{-1}(z_{m+1}, \dots, z_{m+n}))$$

Since each component functions are continuous, we see that $\varphi_\alpha^{-1} \times \psi_\beta^{-1}$ is the continuous inverse of the map $\varphi_\alpha \times \psi_\beta$. Hence $\varphi_\alpha \times \psi_\beta$ is a homeomorphism from $U_\alpha \times V_\beta$ to an open disk in $\mathbb{R}^{m+n}$.

(c) Since every point of X is in at least one U_α and every point of Y is in V_β, every point of $X \times Y$ is in some $U_\alpha \times V_\beta$ (by defnition of the product topology).

(d) If $(U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'})$ is non-empty, then the transition map $(\varphi_{\alpha'} \times \psi_{\beta'}) \circ (\varphi_\alpha \times \psi_\beta)^{-1}$: $(\varphi_\alpha \times \psi_\beta)((U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'})) \to (\varphi_{\alpha'} \times \psi_{\beta'})((U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'}))$ is given by $(\varphi_{\alpha'} \times \psi_{\beta'}) \circ (\varphi_\alpha \times \psi_\beta)^{-1}(z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+n}) =$

$$(\varphi_{\alpha'} \circ \varphi_\alpha^{-1}(z_1), \ldots, \varphi_{\alpha'} \circ \varphi_\alpha^{-1}(z_m), \psi_{\beta'} \circ \psi_\beta^{-1}(z_{m+1}), \ldots \psi_{\beta'} \circ \psi_\beta^{-1}(z_{m+n}))$$

Since each component functions are smooth, we see that the transition map is smooth. Hence the product of atlases is an atlas in the product of topological manifolds.

4. Define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f(u,v) = \left(\cos(u^2 v) - e^{u-v},\ \frac{u^2-3}{u^2+v^2},\ e^{e^{uv}}\right)$$

Compute the Jacobian matrix of f.

$$(Jf)_{(u,v)} = \begin{bmatrix} -2uv\sin(u^2v) - e^{u-v} & u^2\sin(u^2v) + e^{u-v} \\ \frac{2uv^2-6u}{(u^2+v^2)^2} & \frac{2v(3-u^2)}{u^2+v^2} \\ ve^{e^{uv}+uv} & ue^{e^{uv}+uv} \end{bmatrix}.$$