# Analysis II

## Homework 5

Nutan Nepal

March 27, 2023

**Pack Pledge:** I have neither given nor received unauthorized aid on this test or assignment.

---

1. Lebesgue's Criteria for R-integrability: A bounded $f : [a, b] \to \mathbb{R}$ is R-intb. iff $f$ is continuous a.e. In class, we showed "LHS $\Rightarrow$ RHS". Prove the converse.

2. Let $f : [a, \infty) \to \mathbb{R}$ be Riemann integrable on every closed subinterval of $[a, \infty)$. Moreover, assume that $f$ is Lebesgue integrable on $[a, \infty)$. Show that
$\int_a^\infty |f(x)|\, dx \; < \; \infty$, and moreover, $\int_{[a,\infty)} f \; dm = \int_a^\infty f(x)\; dx$.

3. Prove Holder's Inequality: Let $1 \leq p \leq \infty$ and let $q$ be its conjugate exponent. Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$ and

$$\Big| \int_\Omega fg \; d\Omega \Big| \leq \int_\Omega |fg| \; d\Omega \leq \|f\|_p \|g\|_q$$

If $f \neq 0$ and $g \neq 0$, then the inequality is trivial. For $p = 1$, the result follows from the Young's inequality.

4. In any measure space $(X, \mathcal{M}, \mu)$, show that if $1 \leq p \leq r \leq q \leq \infty$, then

$$L^p(\mu) \cap L^q(\mu) \subset L^r(\mu).$$

5. Prove Minkowski's Inequality for Lebesgue spaces.

$$
\begin{aligned}
\|f+g\|_p &= \int_E (f+g)\cdot(f+g)^* \\
&= \int_E f\cdot(f+g)^* + \int_E g\cdot(f+g)^* \\
&= \|f\|_p\cdot\|(f+g)^*\|_q + \|g\|_p\cdot\|(f+g)^*\|_q \\
&= \|f\|_p + \|g\|_p
\end{aligned}
$$

6. Give explicit examples of measure spaces $(X, \mathcal{M}, \mu)$ where each of the following are true:

   (a) $L^p(\mu) \subset L^q(\mu)$, if $0 < p < q < \infty$

   (b) $L^p(\mu) \subset L^q(\mu)$, if $0 < q < p < \infty$

   (c) $L^p(\mu)$ does not contain $L^q(\mu)$ unless $p = q$.

7. Suppose that $\mu(X) = 1$ and let $f$ and $g$ be positive, measurable functions on $X$ s.t. $f, g \in L^1(\mu)$ and $fg \geq 1$. Prove that $(\int_X f\, d\mu)(\int_X g\, d\mu) \geq 1$.

8. **Royden, p. 143/ 13** Show that if $f$ is a bounded function on $E$ that belongs to $L^{p_1}(E)$, then it belongs to $L^{p_2}(E)$ for any $p_2 > p_1$.

   If $f$ is bounded we have $|f(x)| \leq M$ for some constant $M$. Then

$$
f \in L^{p_1}(E) \implies \int_E |f|^{p_1} < \infty
$$

$$
\int_E |f|^{p_2} = \int_E \left(|f|^{p_2/p_1}\right)^{p_1} < \infty
$$

9. **Royden, p. 150/ 25** Assume that $E$ has finite measure and $1 \leq p_1 < p_2 < \infty$. Show that if If $\{f_n\} \to f$ in $L^{p_2}(E)$, then $\{f_n\} \to f$ in $L^{p_1}(E)$.

10. Let $\{f_n\}_{n\geq 1}$ in $L^p(\mu)$. Assume that $f_n \xrightarrow[n\to\infty]{a.e.} f$.

    (a) If $\{f_n\}_{n\geq 1}$ is bounded in $L^p(\mu)$, then $f \in L^p(\mu)$ and $\|f\|_p \leq \liminf_{n\to\inf} \|f_n\|_p$.

(b) If $\exists\, g \in L^p(\mu)$ s.t. $|f_n(x)| \leq g(x)$ a.e. for all $n \geq 1$, then $f_n \xrightarrow[n\to\infty]{L^p} f$.