Algebra I

Homework 2 - All Questions

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(3.3 - 3) Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either

(a) $K \leq H$ or

(b) $G = HK$ and $|K : K \cap H| = p$.

Since, H is normal in G, i.e. $N_G(H) = G$, we can apply second isomorphism theorem. So we have, H normal in KH, $K \cap H$ normal in K and

$$KH/H \simeq K/(K \cap H).$$

H is normal, so we have $KH = HK$. We also have that for subgroup B, C of A, $|A : C| = |A : B| \cdot |B : C|$. So, in our case,

$$|G : H| = |G : HK| \cdot |HK : H|$$

Since $|G : H|$ is prime p, $|G : HK|$ is either 1 or p. If $|G : HK| = p$, then $|HK : H| = 1 \implies H = HK \implies K \leq H$.

If $|G : HK| = 1$, then $G = HK$ and

$$|K : K \cap H| = |KH : H| = |G : H| = p.$$

(3.4 - 5) Prove that subgroups and quotient groups of a solvable group are solvable.

Let the chain of normal subgroups of G be

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

with each G_i/G_{i-1} solvable. Then for any subgroup H of G, we take the chain

$$1 = H_0 \leq G_1 \cap H \leq \cdots \leq G_{n-1} \cap H \leq G_n \cap H = H.$$

Relabelling each $G_i \cap H_i$ as H_i, we write,

$$1 = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = H.$$

We first show that H_i is normal in H_{i+1}. Let $x \in H_{i+1} = G_{i+1} \cap H$, then for any $y \in H_i = G_i \cap H$, we have $xyx^{-1} \in G_i$ since G_i is normal in G_{i+1}. Also $xyx^{-1} \in H$ since both $x,\ y \in H$. Hence $xyx^{-1} \in H_i \implies H_i \trianglelefteq H_{i+1}$. Now, to show that H_{i+1}/H_i is abelian, we note that

$$H_{i+1}/H_i = \frac{G_{i+1} \cap H}{G_i \cap H} = \frac{G_{i+1} \cap H}{G_i \cap (G_{i+1} \cap H)}.$$

By second isomorphism theorem, we have

$$H_{i+1}/H_i = \frac{G_{i+1} \cap H}{G_i \cap (G_{i+1} \cap H)} \simeq \frac{(G_{i+1} \cap H)G_i}{G_i} \leq G_{i+1}/G_i.$$

Hence H_{i+1}/H_i is abelian since it is the subgroup of an abelian group and so the subgroup H is solvable.

Now for any normal subgroup N of G, we consider the chain of quotient groups

$$1 = G_0N/N \leq G_1N/N \leq \cdots \leq G_{n-1}N/N \leq G_nN/N = GN/N.$$

We first show that $G_iN \trianglelefteq G_{i+1}N$ which implies that $G_iN/N \trianglelefteq G_{i+1}N/N$. If $x = gn_1 \in G_iN$ and $y = hn_2 \in G_{i+1}N$ with $g \in G_i$, $h \in G_{i+1}$ and $n \in N$, then

$$yxy^{-1} = hn_2gn_1n_2^{-1}h^{-1} = hgn_3h^{-1} = hgh^{-1}n_4 \in G_iN$$

for some n_3 and n_4 in N. The third and fourth equalities come from the facr that N is normal in G and the fourth equality comes from $G_i \trianglelefteq G_{i+1}$. So, $G_iN/N \trianglelefteq G_{i+1}N/N$ by Lattice isomorphism theorem. Now, we note that $(G_iN/N)/(G_{i+1}N/N) \simeq G_iN/G_{i+1}N$. Let $x,\ y \in G_iN/G_{i+1}N$. Then $x = g_1n_1(G_{i+1}N)$ and $y = g_2n_2(G_{i+1}N)$ for some $g_1,\ g_2 \in G_i$ and $n_1,\ n_2 \in N$. So,

$$\begin{aligned}
xyx^{-1}y^{-1} =& g_1n_1g_2n_2n_1^{-1}g_1^{-1}n_2^{-1}g_2^{-1}(G_{i+1}N) \\
=& g_1g_2g_1^{-1}g_2^{-1}n_3(NG_{i+1}) \\
=& g_1g_2g_1^{-1}g_2^{-1}(G_{i+1}N) \\
=& G_{i+1}N
\end{aligned}$$

Hence $xyx^{-1}y^{-1} = 1 \implies xy = yx$. So $G_iN/G_{i+1}N$ is abelian and the quotient group G/N is solvable.

(3.5 - 3) Prove that S_n is generated by $\{(i \ \ i+1) : 1 \leq i \leq n-1\}$. [Consider conjugates, viz. $(2\ 3)(1\ 2)(2\ 3)^{-1}$.]

Let $G = \{(i \ \ i+1) : 1 \leq i \leq n-1\}$. We first note that

$(i+1 \ \ i+2)(i \ \ i+1)(i+1 \ \ i+2)^{-1} = (i+1 \ \ i+2)(i \ \ i+1)(i+1 \ \ i+2) = (i \ \ i+2).$

Then for any transposition $(i \ \ i+k)$, we can write it as

$$(i+k-1 \ \ i+k)\cdots(i+1 \ \ i+2)(i \ \ i+1)(i+1 \ \ i+2)\cdots(i+k-1 \ \ i+k).$$

Hence, any transposition in S_n can be generated by the consecutive transposition and is in G. Since every element of S_n can be written as the product of transposition, we see that every element is generated by the transpositions $(i \quad i+1)$.

(4.1 - 2) Let G be a permutation group on the set A (i.e., $G \leq S_A$), let $\sigma \in G$ and let $a \in A$. Prove that $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$. Deduce that if G acts transitively on A then

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = 1.$$

We first note that

$$\sigma G_a \sigma^{-1} = \{\sigma g \sigma^{-1} : \; g \in G, \; g(a) = a\} \text{ and,}$$

$$G_{\sigma(a)} = \{g \in G : \; g(\sigma(a)) = \sigma(a)\}.$$

If $f \in \sigma G_a \sigma^{-1}$, then $f = \sigma g \sigma^{-1}$ for some $g \in G$ and $f(a) = \sigma g \sigma^{-1}(a) = a$. So, $f(\sigma(a)) = \sigma g \sigma^{-1}(\sigma(a)) = \sigma g(a) = \sigma(a) \implies f \in G_{\sigma(a)}$.

Now if $f \in G_{\sigma(a)}$, then $f(\sigma(a)) = \sigma(a) \implies \sigma g \sigma^{-1}(\sigma(a)) = \sigma(a)$ for some $g = \sigma f \sigma^{-1}$. Hence, $f \in \sigma G_a \sigma^{-1}$. So, $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$.

Now, if G acts transitively on A, then

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = \bigcap_{\sigma \in G} G_{\sigma(a)}$$

But since $\sigma(A) = A$, $\bigcap_{\sigma \in G} G_{\sigma(a)}$ contains elements of G that fixes all $a \in A$ which is just the identity element. Hence

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = \bigcap_{\sigma \in G} G_{\sigma(a)} = 1.$$

(4.2 - 8) Prove that if H has finite index n then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n!$.

Let P be the set of left cosets of H in G. Then we define a map $\varphi : G \to P$ by $\varphi(g) = gxH$ for some $xH \in P$. We know that this defines a homomorphism from G to the symmetry of n elements of P. So we have a homomorphism $\alpha : G \to S_n$, whose kernel K is normal in G. Furthermore, $|G| = |K| \cdot |S_n| \implies |G : K| \leq n!$.

(4.3 - 5) If the center of G is of index n, prove that every conjugacy class has at most n elements.

By Proposition 6 (Chapter 4), we note that the number of conjugates of an element s equals the index of the centralizer of s in G. So for every conjugacy class H in G, if $s \in H$, then $|H| = |G : C_G(s|$. We know that $Z(G) \subset C_G(s)$ for all elements s, so $|H| = |G : C_G(s| \leq |G : Z(G)| = n$. Hence, each conjugacy class has at most

n elements.

(4.4 - 2) Prove that if G is abelian and of order pq, where $p \neq q$ are primes. Show that G is cyclic.

Since G is an abelian group of order pq, it has two distinct elements x, y with $x^p = y^q = 1$. We note that the order of xy divides pq and if $(xy)^n = 1$ then

$$1 = (xy)^n = x^n \cdot y^n.$$

Since the order of x is p and the order of y is q, p and q both must divide $n \implies pq|n$. So the order of the element xy is pq and $\langle xy \rangle = G$. Hence, G is cyclic.

(4.4 - 12) Let G be a group of order 3825. Prove that if H is a normal subgroup of order 17 in G then $H \leq Z(G)$.

$3825 = 3^2 \cdot 5^2 \cdot 17$. If H is normal in G, then G acts on H by conjugation as automorphisms and we have the permutation representation $\varphi : G \to Aut(H)$ which has $C_G(H)$ as its kernel. Then for some subgroup K of $Aut(H)$, we have

$$G/C_G(H) \simeq K.$$

For the normal subgroup H of G of order 17, since H is cyclic, we have $|Aut(H)| = \varphi(17) = 16$. Then, since K is the subgroup of $Aut(H)$, $|K|$ must divide 16. So $|K|$ must be 1, 2, 4, 8 or 16. But we also have $|G| = |C_G(H)| \cdot |K|$, so $|K| = 1 \implies K = 1$.

Now, since $G/C_G(H) \simeq K = \{1\}$, we have $G = C_G(H) \implies H \leq Z(G)$.

(4.5 - 13) Prove that a group of order 56 has a normal Sylow p-subgroup for some prime dividing its order.

Let G be a group of order $56 = 2^3 \cdot 7$. Then G has at least one subgroup of order 8 and 7 each. Then the number of Sylow p-groups given by n_p for each 2 and 7 satisfy

$$\begin{aligned} n_2 &\equiv 1 \mod 2,\ n_2|7 \implies n_2 = 1 \text{ or } 7 \\ n_7 &\equiv 1 \mod 7,\ n_7|8 \implies n_7 = 1 \text{ or } 8. \end{aligned}$$

If $n_p = 1$ then the unique subgroup is normal Sylow p-subgroup with prime dividing the order. But if $n_7 = 8$, then the remaining 8 elements must form the unique Sylow 2-subgroup of order 8. Since this group is unique, it must be normal. Hence, the group G has a normal Sylow p-subgroup for some prime dividing its order.

(4.5 - 22) Prove that if $|G| = 132$ then G is not simple.

Since $|G| = 132 = 2^2 \cdot 3 \cdot 11$, the number of Sylow p-groups given by n_p for each p satisfy

$$\begin{aligned} n_2 \equiv 1 \mod 2,\ n_2|33 &\implies n_2 = 1, 3 \text{ } or \text{ } 11 \\ n_3 \equiv 1 \mod 3,\ n_3|44 &\implies n_3 = 1, 4 \text{ } or \text{ } 22 \\ n_{11} \equiv 1 \mod 11,\ n_{11}|12 &\implies n_{11} = 1 \text{ } or \text{ } 12. \end{aligned}$$

Assume that G is not simple so $n_p \neq 1$ for any p. Then $n_{11} = 12 \implies 120$ unique elements in G have order 11. If $n_3 = 4$, then 8 unique elements in G have order 3. We have 132-120-8=4 elements remaining which must be inside the unique Sylow 2-subgroup of order 4. This subgroup is normal since it's unique and hence we have a contradiction. So, G is not simple.