Introduction to Manifold Theory

Homework 1

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1. Prove that the open disks $D_r(p)$ are open subsets of $\mathbb{R}^n$.

To prove that the open disks $D_r(p)$ are open subsets of $\mathbb{R}^n$, we show that for every point $x \in D_r(p)$ we have another open disk $D_\epsilon(x)$, $\epsilon > 0$ such that $D_\epsilon(x) \subset D_r(p)$.

For any $x \in D_r(p)$ such that $\delta = d(x,p) < r$, we take $0 < \epsilon < r - \delta$. Then we see that for all $y \in D_\epsilon(x)$

$$d(p,y) \leq d(p,x) + d(x,y) < \delta + \epsilon < \delta + r - \delta = r.$$

Hence, $y \in D_r(p)$ for all $y \in D_\epsilon(x)$ which implies that $D_\epsilon(x) \subset D_r(p)$. So, $D_r(p)$ is an open subset.

2. Prove the second part of Proposition 2.17 (a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous everywhere if and only if for all open subsets V of $\mathbb{R}^m$, the preimage $f^{-1}(V)$ of V under f is open in $\mathbb{R}^n$).

We prove for $\Leftarrow:$ If for all open subsets V of $\mathbb{R}^m$ the preimage $f^{-1}(V)$ of V under f is open in $\mathbb{R}^n$, then f is continuous.

Let $V \subset \mathbb{R}^m$ be an open subset such that $f(x) \in V$. Then we have an open disk $D_\epsilon(f(x)) \subset V$. As the disk $D_\epsilon(f(x))$ is open in $\mathbb{R}^m$, we have an open set $f^{-1}(D_\epsilon(f(x))) \subset \mathbb{R}^n$ which contains x. Then we can find a $\delta > 0$ such that $D_\delta(x) \subset f^{-1}(D_\epsilon(f(x)))$. That is, for every ϵ-ball around $f(x)$, we can find a δ-ball around x such that

$$y \in D_\delta(x) \implies f(y) \in D_\epsilon(f(x))$$

for some y. Hence f is continuous.

3. Show that a composition of continuous functions $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^k$ is continuous.

Since g is continuous, we have $g^{-1}(V)$ open for all open set $V \subset \mathbb{R}^k$. Similarly we have f continuous, so $f^{-1}(U)$ is open for all open sets $U \subset \mathbb{R}^m$. Then, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is open for all open sets V in $\mathbb{R}^k$. Hence the composition is continuous.

4. Show that a function $f : X \to Y$ between sets is invertible if and only if it is bijective.

A function $f : X \to Y$ is invertible if there exists a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

i. f invertible $\implies$ f bijective
   Note that since id_Y is surjective, f must be surjective. Now for injectivity, we observe that if $f(x) = f(y)$ then
   $$x = (g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y) = y.$$
   Hence, f is bijective.
ii. f bijective $\implies$ f invertible
   Since f is both injective and surjective, we define a function $g : Y \to X$ by $g(y) = x$ whenever $f(x) = y$. Note that g is well-defined since there exists only one y for each x. Then, for all $x \in X$, $g(f(x)) = g(y) = x \implies g \circ f = \mathrm{id}_X$. Similarly, for all $y \in Y$, $f(g(y)) = f(x) = y \implies f \circ g = \mathrm{id}_Y$. So, f is invertible.

5. Show that the product topology on a product $X \times Y$ of topological spaces is a valid topology.

X and Y are topological spaces. We define a set $U \subset X \times Y$ to be open if for all $(x, y) \in U$ we have open neighborhoods $U_x \subset X$ and $U_y \subset Y$ such that $U_x \times U_y \subset U$.

i. Clearly the null set ϕ and the whole set $X \times Y$ are open since X is open in X and Y is open in Y.
ii. Arbitrary union $\bigcup U_\alpha$ of open sets is open.

   Let (x, y) be an arbitrary point in $\bigcup U_\alpha$, then $(x, y) \in U_i$ for some i. Then, by definition, there are open neighborhoods $U_x \subset X$ and $U_y \subset Y$ such that $U_x \times U_y \subset U_i \subset \bigcup U_\alpha$.
iii. Finite intersection $U_i \cap U_j$ of open sets is open.

   Let (x, y) be an arbitrary point of $U_i \cap U_j$, then $(x, y) \in U_i$ and $(x, y) \in U_j$. Then, by definition, there are open neighborhoods $U_{ix}, U_{jx} \subset X$ and $U_{iy}, U_{jy} \subset Y$ such that $U_{ix} \times U_{iy} \subset U_i$ and $U_{jx} \times U_{jy} \subset U_j$. Then
   $$U_i \cap U_j \supset (U_{ix} \times U_{iy}) \cap (U_{jx} \times U_{jy}) = (U_{ix} \cap U_{jx}) \times (U_{iy} \cap U_{jy}) \ni (x, y).$$
   Since $(U_{ix} \times U_{jx})$ and $(U_{iy} \times U_{jy})$ are open in X and Y respectively, we see that $U_i \cap U_j$ is open.

6. Verify the three basic properties of closed sets that correspond to the three axioms for open sets.

We define a set $V \subset X$ to be closed if its complement V^c is open in X.

i. The null set ϕ and the whole set X are closed.

   $\phi^c = X$ and $X^c = \phi$ which are open in X.
ii. Arbitrary intersection $\bigcap V_\alpha$ of closed sets is closed.

Here we use the set-theoretic fact that

$$\left(\bigcup U_\beta\right)^c = \bigcap U_\beta^c \tag{1}$$

where $\{U_\beta\}$ is the collection of indexed sets. Since each sets V_α are closed, we write V_α as U_α^c where U_α is an open set of X. Then from 1 we have

$$\bigcap V_\alpha = \bigcap U_\alpha^c = \left(\bigcup U_\alpha\right)^c \tag{2}$$

Hence, since $\bigcup U_\alpha$ is open in X, $\bigcap V_\alpha$ must be closed.

iii. Finite union $V_i \cup V_j$ of closed sets is closed.

We have $V_i \cup V_j = U_i^c \cup U_j^c = (U_i \cap U_j)^c$. Since finite intersection of open sets are open, we observe that $V_i \cup V_j$ is the complement of an open set. Hence $V_i \cup V_j$ is closed.

7. Show that if we have $X'' \subset X' \subset X$, then the "subspace of a subspace" topology on X'' is the same as the "subspace of the biggest space" topology on X''.

Suppose (X, τ) is a topological space. The subspace topology on X' is given by

$$\tau' = \{U' \subset X' : U' = U \cap X' \text{ for some } U \in \tau\}$$

and the subspace topology on X'' induced by τ' is given by

$$\tau'' = \{U'' \subset X'' : U'' = U' \cap X'' \text{ for some } U' \in \tau'\}.$$

We need to show that τ'' is equal to the the subspace topology on X'' induced by τ

$$T = \{U'' \subset X'' : U'' = U \cap X'' \text{ for some } U \in \tau\}.$$

Let $A \in \tau''$, then $A = U' \cap X''$ for some $U' \in \tau'$. Since $U' = U \cap X'$ for some $U \in \tau$ we have, $A = U \cap X' \cap X'' = U \cap X'' \in T$. Hence $\tau'' \subset T$. Similarly, let $B \in T$, then $B = U \cap X''$ for some $U \in \tau$. Since we can write X'' as $X' \cap X''$ we have $B = U \cap X' \cap X'' = U' \cap X'' \in \tau''$ for some U' in τ'. Hence $T \subset \tau''$ which gives $T = \tau''$ ending our proof.

8. Prove the characterization of closed sets of a subspace as intersections with closed sets of the larger space: Let X be a topological space and let $X' \subset X$ be a subspace. Show that a subset $E \subset X'$ is closed if and only if $E = F \cap X'$ for some closed subset F of X.

If $E \subset X'$ is closed, then $X' \setminus E$ is open in X'. Since all open sets of X' are in the form of $U \cap X'$ for some open set U of X we have, $X' \setminus E = U \cap X'$.

$$E = X' \setminus (X' \setminus E) = X' \setminus (U \cap X') = X' \setminus U = X' \cap U^c = X' \cap F$$

for a closed subset F of X.

If $E = F \cap X'$ for some closed subset F of X, then for the open set $U = F^c$ of X, $E =$

$(X\backslash U) \cap X' = (X \cap X')\backslash(U \cap X') = X'\backslash(U \cap X')$. Since $U \cap X'$ is an open set of X', we see that $E = X'\backslash(U \cap X')$ is closed in X'.