Analysis II

Homework 3

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Pack Pledge: I have neither given nor received unauthorized aid on this test or assignment.

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1. Let $(X, \mathcal{M})$ be a measurable space.

   (a) Let $f : X \to \mathbb{R}$, measurable and bounded. Show that for each $\varepsilon > 0$, there are simple functions φ_ε and ψ_ε on X such that

   $$\varphi_\varepsilon \leq f \leq \psi_\varepsilon \quad \text{and} \quad 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon \text{ on } X.$$

   (b) Show that $f : X \to [-\infty, \infty]$ is measurable if and only if there exists a sequence $\{s_n\}$ of simple, measurable functions on X such that $s_n \to f$ pointwise as $n \to \infty$, and $|s_n| \leq |f|$ on X, for all n.

   (a) Since f is bounded, we note that there exists an open interval $[c, d]$ such that $f(X) \subset [c, d]$. For every $\varepsilon > 0$, we can take the partition of the interval $[c, d]$ such that $y_k - y_{k-1} < \varepsilon$ and $c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$ for some integer n. For each $I_k = [y_{k-1}, y_k)$, we define E_k to be $f^{-1}(I_k)$ which is measurable since f is measurable.

   Now we define the functions φ_ε and ψ_ε by

   $$\varphi_\varepsilon(x) = \sum_{k=1}^{n} y_{k-1} \cdot \chi_{E_k} \quad \text{and} \quad \psi_\varepsilon(x) = \sum_{k=1}^{n} y_k \cdot \chi_{E_k}.$$

   For $x \in E$, there is a unique $k \in \overline{1, \ldots, n}$ such that $x \in E_k$ and we have $y_{k-1} \leq f(x) < y_k$. But $\varphi_\varepsilon(x) = y_{k-1}$ and $\psi_\varepsilon(x) = y_k$ and hence $\varphi_\varepsilon(x) \leq f(x) < \psi_\varepsilon(x)$ with $\psi_\varepsilon - \varphi_\varepsilon < \varepsilon$ on X.

   (b) For a measurable function f, we take the sequence of simple functions defined on **Homework 2 - Problem 11** which we proved to be monotone increasing and pointwise convergent to f.

   Now assume that we have a sequence $\{s_n\}$ of simple, measurable functions that converge to f pointwise and $|s_n| \leq |f|$ on X for all n. We first note that for any number $c \in [-\infty, \infty]$, since $\lim_{n\to\infty} s_n(x) = f(x)$ for each x, we have $f(x) < c$ if and only if there exist n, $k \in \mathbb{N}$ with $f_j(x) < c - 1/n$ for all $j \geq k$. Since each set $E_{j,n} = \{x \in E :$

$f_j(x) < c - 1/n\}$ is measurable (because f_j is measurable), for each k, we have $\bigcap_{j=k}^{\infty} E_{j,n}$ is measurable. Hence

$$\{x \in E : \; f(x) < c\} = \bigcup_{1 \le k, n < \infty} \left(\bigcap_{j=k}^{\infty} E_{j,n} \right)$$

which, in turn, implies that f is a measurable function.

2. Let $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$, and μ is the counting measure. Show that for every function $f : \mathbb{N} \to [0, \infty]$,

$$\int_X f \; d\mu = \sum_{n=1}^{\infty} f(n).$$

We take the sets $E_k = \{1, 2, \ldots, k\}$ and define the sequence $\{f_k\}$ by $f_k = f \cdot \chi_{E_k}$. Then $\{f_k\}$ is a monotone increasing sequence of simple functions that converges pointwise to f and

$$\int_X f_k \; d\mu = \int_{E_k} f_k \; d\mu = \sum_{n=1}^{k} f(n).$$

By monotone convergence theorem we have the required result as $k \longrightarrow \infty$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f : X \to [0, \infty]$ be measurable. Show that $\varphi(A) = \int_A f \; d\mu$ is a positive measure on $\mathcal{M}$.

(a) For any set $A \in \mathcal{M}$, we see that $\varphi(A) = \int_A f \quad d\mu \ge 0$ since f is a non-negative function ($\varphi(\emptyset) = 0$).

(b) For any countable collection of measurable pairwise disjoint sets $\{A_n\}$, we have

$$\varphi\left(\bigcup_{n=1}^{\infty} A_n \right) = \int_{\bigcup_{n=1}^{\infty} A_n} f \quad d\mu = \sum_{n=1}^{\infty} \left(\int_{A_n} f \quad d\mu \right) = \sum_{n=1}^{\infty} \varphi(A_n)$$

where the second equality follows from the additivity property of integrals over the domain of integration.

Thus φ is a positive measure on $\mathcal{M}$.

4. Let $f_n : X \to [0, \infty]$ be a monotone decreasing sequence of functions with $f_n \searrow f$ pointwise.

(a) Show by counterexample that $\lim\limits_{n \to \infty} \displaystyle\int_X f_n \; d\mu$ is not necessarily $\int_X f \; d\mu$.

(b) Find an additional assumption that would make the statement true.

(a) We define a sequence $\{f_n\}$ of functions $f_n : X = [0,\infty] \to [0,\infty]$ as follows:

$$f_n(x) = \begin{cases} n & x \in [n,\infty] \\ 0 & x \in [0,n). \end{cases}$$

We see that $\{f_n\}$ is clearly a decreasing sequence of functions and $\int_X f_n \quad d\mu = \infty$ for all n, but $\int_X \lim_{n\to\infty} f_n \quad d\mu = \int_X 0 \quad d\mu = 0$.

(b) If $f_1 \in L^1(\mu)$ then the statement is true. We first note that if $f_1 \geq f_2 \geq \cdots \geq f \geq 0$, then $-f_1 \leq -f_2 \leq \cdots \leq -f \leq 0$ is a monotone increasing sequence. We define $g_n = f_1 - f_n$ and take the sequence $\{g_n\}$ which is monotone increasing and converges to $f_1 - f$. f is measurable since it is a pointwise limit of measurable functions and by monotone convergence theorem for increasing sequence we have

$$\lim_{n\to\infty} \int_X f_1 - f_n \quad d\mu = \int_X f_1 - f \quad d\mu \implies \lim_{n\to\infty} \int_X f_n \quad d\mu = \int_X f \quad d\mu.$$

5. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. Suppose $E_1, E_2, ..., E_n$ are a finite number of measurable sets in X, such that each point in X belongs to at least M of these sets (where M is a positive integer with $M \leq n$). Show that there exists k such that $\mu(E_k) \geq \frac{M}{n}$.

Define a function $g : X \to \mathbb{N}$ by $g(x) = \sum_{k=1}^{n} \chi_{E_k}$. Then $g(x) \geq M$ for all $x \in X$. We have

$$M = M \cdot \mu(X) \leq \int_X g(x)\ d\mu \leq \sum_{k=1}^{n} \int_{E_k} g(x)\ d\mu.$$

For each E_k, $\int_{E_k} g(x)\ d\mu$ is at most $\mu(E_k)$. Thus, $M \leq n \cdot \mu(E_k)$ for some k and hence we have the required result.

6. Prove an analogous result to Fatou's Lemma for $\limsup\limits_{n\to\infty}$.

Theorem 1 (Fatou's Lemma, reverse)**.** *Let $\{f_n\}$ be a sequence of non-negative bounded measurable functions on X and $f_n \to f$ pointwise, then*

$$\limsup_{n\to\infty} \int_X f_n \quad d\mu \leq \int_X f \quad d\mu.$$

Proof. Since $\{f_n\}$ is a bounded sequence and hence there exists a function g such that $f_n(x) \leq g(x)$ for all n. Then $\{g - f_n\}$ is a sequence of non-negative functions that converge to $g - f$. By Fatou's Lemma, we have

$$\int_X g - f \quad d\mu \leq \liminf_{n\to\infty} \int_X g - f_n \quad d\mu.$$

Using linearity and multiplying by -1, we have

$$\int_X f \quad d\mu \geq -\liminf_{n\to\infty} \int_X -f_n \quad d\mu = \limsup_{n\to\infty} \int_X f_n \quad d\mu$$

as required. $\square$

7. Give an example where we have strict inequality in Fatou's Lemma. Then illustrate by example that the assumption "f_n are non-negative" is necessary in Fatou's Lemma.

We define a sequence of functions $\{f_n\}$ by

$$f_n(x) = \begin{cases} 1/n & x \in [0, n] \\ 0 & \text{otherwise.} \end{cases}$$

The sequence $\{f_n\}$ converges to the 0 function which has integral 0. However, each function f_n has integral 1. Thus we have the strict inequality.

To see the importance of the non-negativity, we define a similar functions as above by

$$f_n(x) = \begin{cases} -1/n & x \in [0, n] \\ 0 & \text{otherwise.} \end{cases}$$

Each functions f_n has integral -1 but the limit is the 0 function which has integral 0 which is more than the lim inf of the integral.

8. Suppose that $\mu(X) < \infty$, and $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

We know that each f_n is bounded since f_n converges uniformly to f and thus f is also bounded. Since each f_n are measurable, f is also measurable. Now, for each $\varepsilon > 0$, we know that there exists $N \in \mathbb{N}$ such that $|f - f_n| < \varepsilon/\mu(X)$ for all $n > N$. Then

$$\left| \int_E f - \int_E f_n \right| = \left| \int_E f - f_n \right| \leq \int_E |f - f_n| \leq \frac{\varepsilon}{\mu(X)} \cdot \mu(X) = \varepsilon.$$

Thus, $\lim_{n\to\infty} \int_E f_n = \int_E f$.

9. Assume that $f \in L^1(\mu)$ and $\left| \int_X f \, d\mu \right| = \int_X |f| \, d\mu$. Then there exists $\alpha \in \mathbb{C}$, with $|\alpha| = 1$ such that $\alpha f = |f|$ a.e. on X.

We define $\beta = \left(\int_X f \, d\mu \right) \Big/ \left| \int_X f \, d\mu \right|$. Clearly, $|\beta| = 1$ and we have $\int_X f \, d\mu = \beta \left| \int_X f \, d\mu \right| = \beta \int_X |f| \, d\mu$. Since the integrals are equal, we conclude that $f = \beta|f|$ a.e. on X. Taking $\alpha = 1/\beta$ gives the required equality.

10. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose f is a non-negative measurable function on X. If $\int_X f \ d\mu = 0$, show that $\mu(\{x \in X \mid f(x) \neq 0\}) = 0$ (i.e. $f = 0$ a.e. on X).

Assume that $\mu(\{x \in X \mid f(x) \neq 0\}) > 0$ instead. We take the sets $A_n = \{x \mid f(x) > 1/n\}$ and note that $\mu(A_k) > 0$ for some k. Then,

$$\int_X f \quad d\mu \geq \int_{A_k} f \quad d\mu > \int_{A_k} \frac{1}{k} \quad d\mu = \frac{1}{k} \cdot \mu(A_k) > 0$$

which is a contradiction. Thus $\mu(\{x \in X \mid f(x) \neq 0\}) = 0$.

11. (A small extension of the LDCT)

Let $\{f_n\}$ be a sequence of either complex-valued or extended real-valued functions such that $f_n(x) \to f(x)$ a.e. on X and suppose there is $g \in L^1(\mu)$ such that $|f_n(x)| \leq g(x)$ for a.e. X. Show that

$$\lim_{n\to\infty} \int_X f_n(x) \quad d\mu = \int_X f(x) \quad d\mu$$

Let N be the set where $|f_n(x)| > |g(x)|$ or where f_n doesn't converge to f. This set is a union of two sets that have measure 0 and hence itself has measure 0. Then by Lebesgue Dominated Convergence theorem,

$$\lim_{n\to\infty} \int_{X-N} f_n(x) \ d\mu = \int_{X-N} f(x) \quad d\mu.$$

But, for each n, we have $\int_X f_n(x) \quad d\mu = \int_{X-N} f_n(x) \quad d\mu + \int_N f_n(x) \quad d\mu$. We note that $\int_N f_n(x) \quad d\mu \leq \int_N g(x) \quad d\mu \leq \sup_{x \in N} g(x) \cdot \mu(N) = 0$. Using this result for each n and also for f itself, we get the required result.

12. (Absolute Continuity of the Integral) Let f be a non-negative measurable function in $L^1(\mu)$. Show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every measurable set A with $\mu(A) < \delta$, we have $\int_A f \quad d\mu < \varepsilon$.

Hint: Argue by contradiction: If not, then there is some ε_0 and a sequence of measurable sets A_n with $\mu(A_n) < 2^{-n}$ and

$$\int_{A_n} f \ d\mu > \varepsilon_0.$$

Consider the sequence $g_n = f \cdot \chi_{A_n}$. Show that g_n converges to 0 except at points x in infinitely many of the sets A_n. What is the measure of this "exceptional" set? Now apply a convergence result to the sequence $f_n = f - g_n$ to get a contradiction.

We prove the given statement by contradiction. If the statement is false then there is some $\varepsilon_0 > 0$ and a sequence of measurable sets A_n with $\mu(A_n) < 2^{-n}$ and $\int_{A_n} f \ d\mu > \varepsilon_0$. We define a sequence $\{g_n\}$ of functions by $g_n = f \cdot \chi_{A_n}$. For $x \notin A_n$, we have $g_n(x) = 0$ and for $x \in A_n$,

$g_n(x) = f$ with $\mu(A_n) = 2^{-n}$. Then $f_n = f - g_n$ is 0 in the set A_n and is f otherwise.

$$0 = \int_{A_n} f - g_n \; d\mu = \int_{A_n} f \; d\mu - \int_{A_n} g_n \; d\mu.$$

Not able to continue. I cannot see how I can make progress on this question.