Introduction to Manifold Theory

Homework 4

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1. Precisely specify a function f and an element c of the codomain of f such that the level set of f at level c is the ellipsoid in $\mathbb{R}^3$ defined by the equation

$$x^2 + 2y^2 + 3z^2 = 4.$$

The given equation is the level set of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x, y, z) = x^2 + 2y^2 + 3z^2$$

at level $c = 4$.

2. Precisely specify a function f and an element c of the codomain of f such that the level set of f at level c is the graph of

$$g : \mathbb{R}^2 \to \mathbb{R}$$
$$g(x, y) = x^3 - y^4.$$

The graph of g is given by the set

$$\{(x, y, z) \in \mathbb{R}^3 : \ z = x^3 - y^4\}.$$

Then, this graph of g is the level set of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x, y, z) = x^3 - y^4 - z$$

at level $c = 0$.

3. Precisely specify a function f whose image is the line in $\mathbb{R}^3$ defined by the system of equations

$$\begin{cases} y = 2; \\ x - 3z = 5. \end{cases}$$

In the given line, the second coordinate is always 2 and the first coordinate can be written as a function of the third coordinate as $x = 5 + 3z$. Then, the function $f : \mathbb{R} \to \mathbb{R}^3$ defined as

$$f(x) = (5 + 3x, 2, x)$$

has the given line as its image.

4. Do Exercise 3.3: check that the total derivative T of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point p of $\mathbb{R}^n$ (if it exists) is unique.

Let T and T' both be the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point p. Then for the total derivative T,

$$\lim_{q \to p} \frac{f(q) - f(p) - T(q-p)}{\|q-p\|} = 0.$$

Let $h = q - p$, so we have,

$$\lim_{h \to 0} \frac{f(p+h) - f(p) - T(h)}{\|h\|} = 0.$$

Now, for T and T',

$$\begin{aligned} 0 \le \lim_{h \to 0} \frac{\|T(h) - T'(h)\|}{\|h\|} &= \lim_{h \to 0} \frac{\|f(p+h) - f(p) - T'(h) - (f(p+h) - f(p) - T(h))\|}{\|h\|} \\ &\le \lim_{h \to 0} \frac{\|f(p+h) - f(p) - T'(h)\| + \|(f(p+h) - f(p) - T(h))\|}{\|h\|} \\ &= 0 \end{aligned}$$

Then, since $tx \to 0$ as $t \to 0$, we can say that, for $x \neq 0$ and $h = tx$ we have, (by linearity of T and T')

$$0 = \lim_{h \to 0} \frac{\|T(h) - T'(h)\|}{\|h\|} = \lim_{t \to 0} \frac{\|T(tx) - T'(tx)\|}{\|tx\|} = \lim_{t \to 0} \frac{|t|\|T(x) - T'(x)\|}{\|tx\|} = \frac{\|T(x) - T'(x)\|}{\|x\|}.$$

Hence $\|T(x) - T'(x)\| \implies T = T'$. So, the total derivative T is unique.

5. Do Exercise 3.4: establish the given formula for the Jacobian of the "matrix multiplication" map

$$\mu : \mathbb{R}^{km} \times \mathbb{R}^{mn} \to \mathbb{R}^{kn}.$$

Let T be the Jacobian of the map $\mu : \mathbb{R}^{km} \times \mathbb{R}^{mn} \to \mathbb{R}^{kn}$ at (a, b). Then T is given by

$$\lim_{(A,B) \to 0} \frac{\mu(a + A, b + B) - \mu(a, b) - T(A, B)}{\|(A, B)\|} = 0.$$

Here μ is the matrix multiplication map that takes $k \times m$ matrix and $m \times n$ matrix.

Furthermore,

$$\begin{aligned} &\lim_{(A,B)\to 0} \frac{\mu(a+A, b+B) - \mu(a,b) - T(A,B)}{\|(A,B)\|} = 0 \\ \implies &\lim_{(A,B)\to 0} \frac{\|\mu(a+A, b+B) - \mu(a,b) - T(A,B)\|}{\|(A,B)\|} = 0. \end{aligned}$$

Since $tA \to 0$ as $t \to 0$ and $tB \to 0$ as $t \to 0$, when $(A,B) \neq 0$, we can write the above limit as

$$\begin{aligned} \lim_{t\to 0} \frac{\|\mu(a+tA, b+tB) - \mu(a,b) - T(tA,tB)\|}{\|(tA,tB)\|} &= \lim_{t\to 0} \frac{\|ab + tAb + taB + t^2AB - ab - tT(A,B)\|}{\|(tA,tB)\|} \\ &= \lim_{t\to 0} \frac{|t|\|Ab + aB + tAB - T(A,B)\|}{|t|\|(A,B)\|} \\ &= \lim_{t\to 0} \frac{\|Ab + aB + tAB - T(A,B)\|}{\|(A,B)\|} \\ &= \frac{\|Ab + aB - T(A,B)\|}{\|(A,B)\|} \end{aligned}$$

Since above limit equals 0, we see that the Jacobian $T(A,B)$ evaluated at (a,b) equals $Ab + aB$.