Algebra I

Homework 3

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(5.4 - 8) Assume that x, y both commute with $[x, y]$. Show that $(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}$ for any positive integer n.

> We first note that $yx = xy[y, x] = x[y, x]y = [y, x]xy$. Assume that the statement
>
> $$P(k) : \quad (xy)^k = x^k y^k [y, x]^{\binom{k}{2}}$$
>
> is true for some integer $k > 1$. Then,

(5.5 - 11) Classify groups of order 28.

(6.1 - 6) Show that if $G/Z(G)$ is nilpotent then G is nilpotent.

(6.3 - 7) Show that the quaternion group $\mathcal{Q}_8$ can be presented by $\{a, b|\ a^2 = b^2, a^{-1}ba = b^{-1}\}$.

(7.1 - 8) Find the center of the real Hamiltonian Quaternions $\mathbb{H}$. Prove that $\{a + bi|a, b \in R\}$ is a subring of $\mathbb{H}$ which is a field but is not contained in the center of $\mathbb{H}$.

(7.1 - 13) An element $a \in R$ is called nilpotent if $x^m = 0$ for some $m \in \mathbb{Z}^+$.

(a) Show that if $n = a^k b$ for some integers a, b, then ab is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.

(b) If $a \in \mathbb{Z}$, show that the element $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$ iff every prime divisor of n is also a divisor of a. In particular, find all nilpotent elements of $\mathbb{Z}/72\mathbb{Z}$.

(c) Let R be the ring of functions from a nonempty set X to a field F. Prove that R contains no nilpotent elements.

(7.2 - 7) Show that the center of the ring $M_n(R)$ is $R \cdot I$, where $I = diag(1, \ldots, 1)$.

(7.3 - 4) Find all ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}/30\mathbb{Z}$. In each case, describe the kernel and the image.

(7.3 - 18) Prove that the intersection $I \cap J$ of ideals I, J of a ring R is also an ideal of R. Let $\{I_\alpha\}_{\alpha \in S}$ be a collection of ideals of R. Show that $\bigcap_{\alpha \in S} I_\alpha$ is an ideal of R.

(7.3 - 29) Let R be a commutative ring. Prove that the set of nilpotent elements of R form an ideal—called the nilradical $\mathfrak{N}(R)$.