# Introduction to Manifold Theory

Homework 5

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1. Do Exercise 3.6: for part 1, for each of the six types of quadric surfaces listed in the problem, say which you expect to be regular level sets and thus have a smooth manifold structure. For part 2, give an equation defining an example of each of these types of quadric surfaces, view the defining equation as the level set of a function

$$f : \mathbb{R}^3 \to \mathbb{R},$$

and determine algebraically whether the level set is regular.

We expect ellipsoids, elliptic paraboloids, hyperbolic paraboloids, hyperboloids of one sheet and hyperboloids of two sheets to be regular level sets and thus a smooth manifolds.

(a) Ellipsoid:

$$\frac{x^2}{4} + \frac{y^2}{1} + \frac{z^2}{9} = 1$$

This is an example of an ellipsoid in $\mathbb{R}^3$ and we can view this as a level set of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = \frac{x^2}{4} + \frac{y^2}{1} + \frac{z^2}{9} - 1$$

at level 0. The Jacobian given by

$$\begin{bmatrix} 2x/4 & 2y & 2z/9 \end{bmatrix}$$

is 0 only when $(x, y, z) = (0, 0, 0)$ but this point is not our set. So, the Jacobian always has rank $\geq 1$ and hence, every point is regular and the ellipsoid is has a smooth manifold structure.

(b) Elliptic paraboloids:

$$z = x^2 + y^2$$

is a level set of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = x^2 + y^2 - z$$

at level 0. The Jacobian given by

$$\begin{bmatrix} 2x & 2y & 1 \end{bmatrix}$$

always more than 0. Hence this is a regular level set.

(c) Hyperbolic paraboloids:

$$z = x^2 - y^2$$

is a level set of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = x^2 - y^2 - z$$

at level 0. The Jacobian given by

$$\begin{bmatrix} 2x & -2y & -1 \end{bmatrix}$$

always more than 0. Hence this is a regular level set.

(d) Hyperboloids of one sheet:

$$1 = x^2 + y^2 - z^2$$

is a level set of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = x^2 + y^2 - z^2 - 1$$

at level 0. The Jacobian given by

$$\begin{bmatrix} 2x & 2y & -2z \end{bmatrix}$$

always more than 0 since the critical point $(0, 0, 0)$ is not in our level set. Hence this is a regular level set.

(e) Hyperboloids of two sheet:

$$1 = -x^2 - y^2 + z^2$$

is a level set of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = -x^2 - y^2 + z^2 - 1$$

at level 0. The Jacobian given by

$$\begin{bmatrix} -2x & -2y & 2z \end{bmatrix}$$

always more than 0 since the critical point $(0, 0, 0)$ is not in our level set. Hence this is a regular level set.

(f) Double cone:

$$0 = x^2 + y^2 - z^2$$

is a level set of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = x^2 + y^2 - z^2$$

at level 0. The Jacobian given by

$$\begin{bmatrix} 2x & 2y & -2z \end{bmatrix}$$

is 0 when $(x, y, z) = (0, 0, 0)$. This point is in our set and hence the Jacobian has rank 0 at $(0, 0, 0)$. So, this is not a regular level set.

2. Do Exercise 3.5: let $\xi$ be the function from $n \times n$-matrix-space $\mathbb{R}^{n^2}$ to itself sending a matrix $A$ to $AA^T$. Show that the Jacobian of $\xi$ at a point $A \in \mathbb{R}^{n^2}$, viewed as a linear transformation from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{n^2}$, sends a matrix $a \in \mathbb{R}^{n^2}$ to

$$(J\xi)_A(a) = aA^T + Aa^T.$$

Exercise 3.4 may be useful.

We first note that the transpose of the sum of matrices is the sum of transposes of the matrices. That is

$$(a + A)^T = a^T + A^T.$$

Now, if $\mu : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ is the function taking a matrix to its transpose, then denoting the Jacobian at a point $a \in rl^{n^2}$ by $T$ we have,

$$\begin{aligned}
0 = \lim_{t\to 0} \frac{\|\mu(a+tA) - \mu(a) - T(tA)\|}{\|(tA)\|} &= \lim_{t\to 0} \frac{\|(a+tA)(a^T + tA^T) - aa^T - tT(A)\|}{\|(tA, tB)\|} \\
&= \lim_{t\to 0} \frac{\|aa^T + tAa^T + taA^T + t^2 AA^T - aa^T - tT(A)\|}{\|(tA, tB)\|} \\
&= \lim_{t\to 0} \frac{|t|\|Aa^T + aA^T + tAA^T - T(A)\|}{|t|\|A\|} \\
&= \lim_{t\to 0} \frac{\|Aa^T + aA^T + tAA^T - T(A)\|}{\|A\|} \\
&= \frac{\|Aa^T + aA^T - T(A)\|}{\|A\|}
\end{aligned}$$

The above limit is 0 and hence the numerator

$$\|Aa^T + aA^T - T(A)\| = 0 \implies Aa^T + aA^T = T(A)$$

So $T(A)$ evaluated at the point $a$ is given by

$$(J\xi)_A(a) = aA^T + Aa^T.$$