## Algebra II

Homework 1

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1. (10.1 - 5) For any left ideal I of R define

$$IM = \left\{ \sum_{\text{finite}} a_i m_i \mid a_i \in I, \ m_i \in M \right\}$$

to be the collection of all finite sums of elements of the form am where  $a \in I$  and  $m \in M$ . Prove that IM is a submodule of M.

We show that IM is a submodule of M and is closed under the action of the ring elements:

Clearly,  $0 \in IM$ . If  $p = \alpha_1 f_1 + \cdots + \alpha_n f_n$  and  $q = \beta_1 g_1 + \cdots + \alpha_m g_m$  in IM with  $\alpha$ ,  $\beta \in I$  and  $f_i, g_i \in M$ , then  $p + q = \alpha_1 f_1 + \cdots + \alpha_n f_n + \cdots + \beta_1 g_1 + \cdots + \alpha_m g_m$  is a finite sum with coefficients in the ideal and hence belongs to the set IM. For any finite sum  $p = \alpha_1 f_1 + \cdots + \alpha_n f_n \in IM$ , its inverse -p is also a finite sum with coefficients in the ideal and hence belongs to IM. So, IM is a subgroup. Now, for any ring element  $r \in R$  and  $p = \alpha_1 f_1 + \cdots + \alpha_n f_n \in IM$ , we have  $rp = r(\alpha_1 f_1 + \cdots + \alpha_n f_n) = r\alpha_1 f_1 + \cdots + r\alpha_n f_n$ . Since  $rI \subset I$ , we see that IM is closed under the ring action and hence is a submodule of M.

2. (10.1 - 6) Show that the intersection of any nonempty collection of submodules of an R-module is a submodule.

Let  $N = \bigcap_{\lambda \in \Lambda} N_{\lambda}$  be the intersection of the nonempty collection of submodules  $N_{\lambda}$  of M indexed by the set  $\Lambda$ . Clearly,  $0 \in N$ , so N is nonempty. Now, for any  $p, q \in N$ ,  $p+q \in N_{\lambda}$  for every  $\lambda \in \Lambda$  since each  $N_{\lambda}$  is a submodule and thus  $p+q \in N$ . Also,  $p \in N \implies p \in N_{\lambda} \implies -p \in N_{\lambda}$  for all  $\lambda$  which implies that  $-p \in N$ . Thus, N is a subgroup of M. Now, for  $p \in N$  and  $r \in R$ , we note that  $rp \in N_{\lambda}$  for all  $\lambda$ . So  $rp \in N$  and N is closed under the action of ring elements. Thus N is a submodule.

3. (10.2 - 6) Prove that  $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/(n, m)\mathbb{Z}$ .

We first note that if  $\alpha \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$  with  $\alpha(1) = p$ , then  $0 = \alpha(n) = pn$ . Since  $m \mid pn$  we have,  $m/(m,n) \mid p$ . Thus all elements  $\alpha \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$  are given by:

1

$$\alpha(1) = \frac{mx}{(m,n)}$$

for  $x \in \mathbb{Z}/n\mathbb{Z}$ .

Clearly, any map defined as above is a homomorphism in  $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})$ .

We define a map  $\varphi: \mathbb{Z}/(n,m)\mathbb{Z} \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})$  defined by  $\varphi(x) = \alpha_x$  where  $\alpha_x(1) = mx/(m,n)$ . Since these are all the unique homomorphisms in  $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})$ ,  $\varphi$  is surjective. If  $\varphi(x) = \varphi(y)$ , then  $mx/(m,n) = my/(m,n) \implies x = y$ . Thus  $\varphi$  is injective.  $\varphi$  is a homomorphism since  $\varphi(x+y)(1) = m(x+y)/(m,n) = \varphi(x)(1) + \varphi(y)(1)$  and  $\varphi(ry)(1) = mry/(m,n) = r\varphi(y)(1)$ . Thus, this is an isomorphism of  $\mathbb{Z}$ -modules.

4. (10.2 - 10) Let R be a commutative ring. Prove that  $Hom_R(R, R)$  and R are isomorphic as rings.

We first note that if  $\varphi \in \operatorname{Hom}_R(R,R)$  then for any  $x \in R$ ,  $\varphi(x) = \varphi(x \cdot 1) = r \cdot \varphi(1)$ . Since,  $\varphi(1)$  can be any element of R we have  $\varphi(x) = x \cdot r$  for some  $r \in R$ . Hence  $\varphi$  is completely determined by its value on the identity of R.

Now, if  $\alpha: R \to R$  is any map such that  $\alpha(x) = x \cdot r$  for some  $r \in R$  then we have  $\alpha(px + qy) = (px + qy) \cdot r = p\alpha(x) + q\alpha(y)$ . Thus,  $\alpha \in \operatorname{Hom}_R(R, R)$  and we see that every  $\varphi \in \operatorname{Hom}_R(R, R)$  is of this form. Hence we can define a map  $\psi: \operatorname{Hom}_R(R, R) \to R$  by  $\psi(\varphi) = \varphi(1)$  which is surjective as we saw above. Furthermore, if  $\psi(\varphi) = \varphi(1) = 0$  then  $\varphi$  is a 0 map and hence  $\psi$  is injective. We now show that  $\psi$  is a ring homomorphism:

- (a)  $\psi(\alpha + \beta) = (\alpha + \beta)(1) = \psi(\alpha) + \psi(\beta)$
- (b)  $\psi(\alpha \circ \beta) = (\alpha \circ \beta)(1) = \alpha(\beta(1)) = \alpha(1 \cdot \beta(1)) = \alpha(1) \cdot \beta(1) = \psi(\alpha) \cdot \psi(\beta)$
- (c)  $\psi(e) = e(1) = 1$

where  $e \in \operatorname{Hom}_R(R,R)$  is the identity map. Thus,  $\psi$  is a ring isomorphism.

5. (10.3 - 4) An R-module M is called a torsion module if for each  $m \in M$  there is a nonzero element  $r \in R$  such that rm = 0, where r may depend on m (i.e., M = Tor(M) in the notation of Exercise 8 of Section 1). Prove that every finite abelian group is a torsion  $\mathbb{Z}$ -module. Give an example of an infinite abelian group that is a torsion  $\mathbb{Z}$ -module.

If  $p \neq 0$  is the order of the any given finite abelian group then  $p \cdot m = 0$  for every element  $m \in M$ . Thus every finite abelian group is a torsion  $\mathbb{Z}$ -module.

We take  $M = \mathbb{F}_2[x]$ , the group of polynomials over the finite field  $\mathbb{F}_2$  to be the our abelian group. Clearly, it's infinite as it has elements of every degree. Considered as a  $\mathbb{Z}$ -module, we see that  $2 \cdot f(x) = 0$  for all  $f(x) \in M$ . Thus it is an infinite abelian torsion  $\mathbb{Z}$ -module.

6. (10.3 - 9) An R-module M is called *irreducible* if  $M \neq 0$  and if 0 and M are the only submodules of M. Show that M is irreducible if and only if  $M \neq 0$  and M is a cyclic module with any nonzero element as generator. Determine all the irreducible  $\mathbb{Z}$ -modules.

We first prove that if M is irreducible then  $M \neq 0$  and M is a cyclic module with any nonzero element as generator. Assume that  $M \neq 0$  is irreducible and  $m \in M$  is any element. Then

since  $Rm \subset M$  is a submodule of M and M contains no non-trivial submodule, Rm = M and m is the nonzero generator. Now, assume that  $M \neq 0$  is a cyclic module with any nonzero generator. Then if  $m \neq 0$ ,  $Rm \subset M$ . But M is generated by any nonzero element of M so  $M \subset Rm$ . Thus M = Rm, meaning that M and 0 are the only submodules of M. Hence M is irreducible.

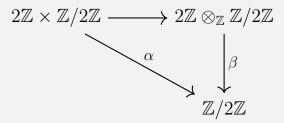
From above, we see that the only irreducible  $\mathbb{Z}$ -modules are cyclic groups of prime order.

7. (10.4 - 2) Show that the element " $2 \otimes 1$ " is 0 in  $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$  but is nonzero in  $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ .

We note that in  $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ ,  $2 \otimes 1 = 2 \cdot 1 \otimes 1 = 1 \otimes 2 \cdot 1$ . But since  $2 \equiv 0$  in  $\mathbb{Z}/2\mathbb{Z}$ , we have,  $2 \otimes 1 = 1 \otimes 2 = 1 \otimes 0 = 0$ .

To show that it is nonzero in  $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ , we consider the map  $\alpha : 2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$  defined by  $\alpha(2m, n) = mn \mod 2$  for  $2m \in 2\mathbb{Z}$  and  $n \in \mathbb{Z}/2\mathbb{Z}$ . Clearly, the following holds:

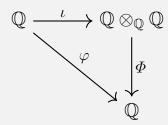
- (a)  $\alpha(r_1 \cdot 2m + r_2 \cdot 2p, n) = r_1 \cdot \alpha(2m, n) + r_2 \cdot \alpha(2p, n) = r_1 mn + r_2 pn \mod 2$ .
- (b)  $\alpha(2m, r_1 \cdot n + r_2 \cdot q) = r_1 \cdot \alpha(2m, n) + r_2 \cdot \alpha(2m, q) = r_1 m n + r_2 m q \mod 2$ . for  $m, p \in 2\mathbb{Z}$  and  $n, q \in \mathbb{Z}/2\mathbb{Z}$ . Then,  $\alpha$  is a  $\mathbb{Z}$ -bilinear map and induces a  $\mathbb{Z}$ -linear map  $\beta : 2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$  such that  $\beta(2m \otimes n) = mn \mod 2$ .



Since  $\beta(2 \otimes 1) = 1 \cdot 1 \mod 2 = 1 \neq 0$  in  $\mathbb{Z}/2\mathbb{Z}$ , we see that  $2 \otimes 1 \neq 0$  in  $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$  since  $\beta$  is an R-linear map.

8. (10.4 - 4) Show that  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$  are isomorphic left  $\mathbb{Q}$ -modules. [Show they are both 1-dimensional vector spaces over  $\mathbb{Q}$ .]

By Theorem 8, letting  $\iota : \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$  be the ring homomorphism defined by  $\iota(q) = 1 \otimes q$ , we have the commutative diagram:



If we let  $\varphi$  to be the identity map then we get  $\Phi \circ \iota$  to be the identity map. Then  $\iota$  is an isomorphism of  $\mathbb{Q}$ -modules. To show that  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$  we define a map  $\beta : \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$  by

 $\beta(q) = 1 \otimes q$ . We can prove that this is a  $\mathbb{Q}$ -linear map by

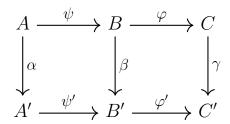
$$\beta\left(\frac{a}{b}q\right) = 1 \otimes \frac{a}{b}q = a \otimes \frac{q}{b} = \frac{ab}{b} \otimes \frac{q}{b} = \frac{a}{b} \otimes q = \frac{a}{b}(1 \otimes q) = \frac{a}{b}\beta(q).$$

Now, we can define a map  $\alpha : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$  by  $\alpha(p \otimes q) = pq$ . It is a well-defined map and we have  $\alpha \circ \beta(q) = q$  and

$$\beta \circ \alpha(p \otimes q) = \beta(pq) = 1 \otimes pq = p \otimes q.$$

Thus, since the maps are inverse of each other, we see that  $\beta$  is an isomorphism of  $\mathbb{Q}$ -modules. Since, both  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$  are isomorphic to  $\mathbb{Q}$  as a bimodule, we see that they are isomorphic to each other as a left  $\mathbb{Q}$ -module.

9. (10.5 - 1) Suppose that



is a commutative diagram of groups and that the rows are exact. Prove that

(a) if  $\varphi$  and  $\alpha$  are surjective, and  $\beta$  is injective then  $\gamma$  is injective. [If  $c \in \ker(\gamma)$ , show there is a  $b \in B$  with  $\varphi(b) = c$ . Show that  $\varphi'(\beta(b)) = 0$  and deduce that  $\beta(b) = \psi'(a')$  for some  $a' \in A'$ . Show there is an  $a \in A$  with  $\alpha(a) = a'$  and that  $\beta(\psi(a)) = \beta(b)$ . Conclude that  $b = \psi(a)$  and hence  $c = \varphi(b) = 0$ .]

Let  $c \in \ker(\gamma)$ . Since  $\varphi$  is surjective, there exists  $b \in B$  with  $\varphi(b) = c$ . The given diagram is commutative and hence we have  $\varphi' \circ \beta(b) = \gamma \circ \varphi(b) = 0$ . Hence  $\beta(b) \in \ker(\varphi') = \operatorname{im}(\psi')$  and we have  $\beta(b) = \psi'(a')$  for some  $a' \in A'$ . Since  $\alpha$  is surjective, there exists  $a \in A$  such that  $\alpha(a) = a'$ . Thus  $\psi'(\alpha(a)) = \beta(\psi(a)) = \beta(b) \implies b = \psi(a)$ . Hence  $c = \varphi(b) = \varphi \circ \psi(a) = 0$ . So,  $\gamma$  is injective.

(b) if  $\psi'$ ,  $\alpha$ , and  $\gamma$  are injective, then  $\beta$  is injective,

Since  $\psi' \circ \alpha$  is injective, we see that  $\beta \circ \psi$  is injective and so  $\psi$  is injective. Let  $b \in \ker(\beta)$ . Then  $\varphi' \circ \beta(b) = 0 = \gamma \circ \varphi(b)$ . Since,  $\gamma$  is injective and  $\varphi(b) \in \ker(\gamma)$ ,  $\varphi(b) = 0 \implies b \in \operatorname{im}(\psi)$ . Thus there exists  $a \in A$  with  $b = \psi(a)$ . But  $\psi$  is injective and  $\beta(b) = \beta \circ \psi(a) = 0$ . So a = 0 and  $b = \psi(a) = 0$ .  $\beta$  is injective.

4

(c) if  $\varphi$ ,  $\alpha$ , and  $\gamma$  are surjective, then  $\beta$  is surjective,

Let  $b' \in B'$  be any element. We want to show that there exists a  $p \in B$  with  $\beta(p) = b'$ . Since  $\gamma \circ \varphi$  is surjective, there exists a  $b \in B$  such that  $\gamma \circ \varphi(b) = \varphi'(b') = \varphi' \circ \beta(b)$ . Then  $\varphi'(\beta(b) - b') = 0 \implies b' - \beta(b) \in \operatorname{im}(\psi')$ . So there exists  $a' \in A'$  with  $\psi'(a') = \beta(b) - b'$ . Since  $\alpha$  is surjective, we have  $a \in A$  such that  $\psi' \circ \alpha(a) = \beta(b) - b'$ . So,  $\beta \circ \psi(a) = \psi' \circ \alpha(a) = \beta(b) - b' \implies \varphi'(\beta \circ \psi(a)) = \varphi'(\beta(b) - b')$ . If we let  $p = \psi(a) + b$ ,

$$\beta(\psi(a) + b) = \beta \circ \psi(a) + \beta(b) = b' = \beta(b) + \beta(b) = b'.$$

So,  $\beta$  is surjective.

(d) if  $\beta$  is injective,  $\alpha$  and  $\varphi$  are surjective, then  $\gamma$  is injective,

Let  $c \in \ker(\gamma)$ . Then since  $\varphi$  is surjective, there exists  $b \in B$  with  $c = \varphi(b)$ . Then  $\varphi' \circ \beta(b) = \gamma \circ \varphi(b) = 0$ . Then  $b' = \beta(b) \in \ker(\varphi') = \operatorname{im}(\psi')$ . So we have,  $a' \in A$  with  $\psi'(a) = b'$ . Since  $\alpha$  is surjective, we have  $a \in A$  with  $\alpha(a) = a'$  and  $\psi' \circ \alpha(a) = \beta \circ \psi(a) = b' = \beta(b)$ . Since  $\beta$  is injective, we have  $\psi(a) = b \implies b \in \operatorname{im}(\psi) \implies c = \varphi(b) = 0$ . Thus  $\gamma$  is injective.

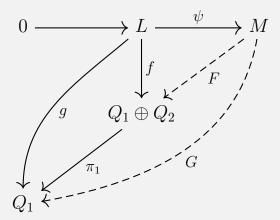
(e) if  $\beta$  is surjective,  $\gamma$  and  $\psi'$  are injective, then  $\alpha$  is surjective.

Let  $a' \in A'$  be any element. We want to show that there exists a  $a \in A$  with  $\alpha(a) = a'$ . Since  $\beta$  is surjective, there exists  $b \in B$  with  $\beta(b) = \psi'(a') = b'$  where  $b' \in B'$  is some element of B'. Then  $\gamma \circ \varphi(b) = \varphi' \circ \beta(b) = \varphi' \circ \psi'(a') = 0$  follows from the exactness of the lower row. Since  $\gamma$  is injective,  $\gamma \circ \varphi(b) = 0 \implies \varphi(b) = 0 \implies b \in \operatorname{im}(\psi)$ . So there exists  $a \in A$  such that  $b = \psi(a)$ . Then  $\beta \circ \psi(a) = \psi' \circ \alpha(a) = \psi'(a')$ . But since  $\psi'$  is injective, we have  $\alpha(a) = a'$  as required.

10. (10.5 - 4) Let  $Q_1$  and  $Q_2$  be R-modules. Prove that  $Q_1 \oplus Q_2$  is an injective R-module if and only if both  $Q_1$  and  $Q_2$  are injective.

If  $Q_1 \oplus Q_2$  is injective, then for any R-modules L, M, if  $0 \to L \xrightarrow{\psi} M$  is exact, then every R-module homomorphism f from L to  $Q_1 \oplus Q_2$  lifts to an R-module homomorphism F of M into  $Q_1 \oplus Q_2$ . Now, if  $g: L \to Q_1$  is any R-module homomorphism, then g factors through  $Q_1 \oplus Q_2$  with  $g = \pi_1 \circ f$  for some  $f \in \operatorname{Hom}_R(L, Q_1 \oplus Q_2)$ . If F is the lift of f, then  $\pi_1 \circ F \in \operatorname{Hom}_R(M, Q_1)$  is the lift of g. Hence,  $Q_1$  is injective. We see that  $Q_2$  is also

injective similarly.



Now, if  $Q_1$  and  $Q_2$  are injective modules and  $f: L \to Q_1 \oplus Q_2$  is any module homomorphism, then  $\pi_1 \circ f$  and  $\pi_2 \circ f$  are homomorphisms from L to  $Q_1$  and  $Q_2$  respectively. If  $F_1$  and  $F_2$  are the lift of these homomorphisms to  $\operatorname{Hom}_R(M,Q_1)$  and  $\operatorname{Hom}_R(M,Q_2)$  respectively, then we define  $F: M \to Q_1 \oplus Q_2$  by  $F(m) = (F_1(m),F_2(m))$ . Then this is a lift of the map f and hence the direct sum is injective.