Algebra I

Homework 3

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November 1, 2022

(5.4 - 8) Assume that x, y both commute with $[x, y]$. Show that $(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}$ for any positive integer n.

We first note that $yx = xy[y, x] = x[y, x]y = [y, x]xy$. Assume that the statement

$$P(k) : \quad (xy)^k = x^k y^k [y, x]^{\binom{k}{2}}$$

is true for some integer $k > 1$. Then, when $k = 2$

$$P(2) : \quad (xy)^2 = xyxy = xxy[y, x]y = x^2 y^2 [y, x]$$

Now, for $k + 1$,

$$\begin{aligned} P(k+1) : \quad (xy)^{k+1} = xy(xy)^k = xyx^k y^k [y, x]^{\binom{k}{2}} &= xxy[y, x]x^{k-1} y^k [y, x]^{\binom{k}{2}} \\ &= x^2 y[y, x]x^{k-1} y^k [y, x]^{\binom{k}{2}+1} \end{aligned}$$

Repeating the above process k times, we obtain,

$$P(k+1) : \quad (xy)^{k+1} = x^k y^k [y, x]^{\binom{k}{2}+k} = x^k y^k [y, x]^{\binom{k+1}{2}}.$$

Hence, the given statement is true by mathematical induction.

(5.5 - 11) Classify groups of order 28.

If G has order 28 then it contains Sylow 7-subgroups such that $n_7 \equiv 1 \mod 7$ and $n_7 \mid 4$. So $n_7 = 1$ and G contains a unique normal subgroup H of order 7. Let $K \in \mathrm{Syl}_2(G)$ be a subgroup of order 4 in G, so $H \cap K = 1$. Then $G = H \rtimes K$. Now we find the possible semidirect products between H and K.

Since the order of K is 4, $K \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ or $K \simeq \mathbb{Z}_4$. Furthermore $\mathrm{Aut}(H) \simeq \mathbb{Z}_6$.

When $K = \mathbb{Z}_4$ we have two homomorphisms from K to $\text{Aut}(H) = \mathbb{Z}_6$ given by

$$\varphi_1(1) = \overline{0}, \qquad \text{and} \qquad \varphi_2(1) = \overline{3}.$$

Each of these homomorphisms give us 2 distict groups (φ_1 gives us the regular direct product). Now, when $K = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = (ab)^2 = 1 \rangle$, the possible homomorphisms are

$$\begin{aligned} &\varphi_3(a) = 0, \varphi_3(b) = 0, && \varphi_4(a) = 0, \varphi_4(b) = 3, \\ &\varphi_5(a) = 3, \varphi_5(b) = 0 && \varphi_6(a) = 3, \varphi_6(b) = 3. \end{aligned}$$

φ_4 gives the usual direct product as above. The last 3 homomorphisms are isomorphic and hence give another semidirect product $H \rtimes K$.

(6.1 - 6) Show that if $G/Z(G)$ is nilpotent then G is nilpotent.

We consider the upper central series of G and $G/Z(G)$

$$1 = Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq \cdots \trianglelefteq G,$$

$$1 = Z_0(G/Z(G)) \trianglelefteq Z_1(G/Z(G)) \trianglelefteq \cdots \trianglelefteq Z_k(G/Z(G)) = G/Z(G).$$

We know that the subgroups of $G/Z(G)$ are in bijection with the subgroups of G containing $Z(G)$. Then given above upper central series for $G/Z(G)$ we take the corresponding series taken by taking the preimage of each $Z_i(G/Z(G))$,

$$1 = Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq \cdots \trianglelefteq Z_k = G,$$

Then,

$$\begin{aligned} Z_{i+1}(G)/Z_i(G) =& (Z_{i+1}(G)/Z(G))/(Z_i(G)/Z(G)) \\ =& Z_{i+1}(G/Z(G))/Z_i(G/Z(G)) \\ =& Z((G/Z(G))/Z_i(G/Z(G))) \end{aligned}$$

But, $Z_i(G/Z(G)) = Z_i(G)/Z(G)$. Hence, $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$ and G is nilpotent.

(6.3 - 7) Show that the quaternion group $\mathcal{Q}_8$ can be presented by $\langle a, b | \ a^2 = b^2, a^{-1}ba = b^{-1} \rangle$.

Let $G = \langle a, b | a^2 = b^2, a^{-1}ba = b^{-1} \rangle$.

We see that $a^{-1}ba = b^{-1} \implies a^{-1}b^2a = b^{-2}$. Hence, $a^2 = b^{-2}$ and $a^4 = b^4 = 1$. Thus we see that G has at least 2 subgroups of order 4 and 1 subgroup $\{1, a^2\}$ of order 2. The quaternion group $\mathcal{Q}_8$ is $\{1, i, j, k, -1, -i, -j, -k\}$ and satisfies the relation $i^2 = j^2 = -1$ and $i^{-1}ji = j^{-1}$. Thus we have a homomorphism $\varphi : G \to \mathcal{Q}_8$ defined as $\varphi(a) = i$ and $\varphi(b) = j$. But, since $a^4 = 1$ and $a^2 = b^2$, the free group generated by a and b with the given relations have at most 8 elements. Thus, the group G is actually the quaternion group itself.

(7.1 - 8) Find the center of the real Hamiltonian Quaternions $\mathbb{H}$. Prove that $\{a+bi|a,b\in\mathbb{R}\}$ is a subring of $\mathbb{H}$ which is a field but is not contained in the center of $\mathbb{H}$.

An element $y = a+bi+cj+dk \in \mathbb{H}$ is in the center iff $xy - yx$ is the identity element 0 for all $x \in \mathbb{H}$. When $x = i$,

$$xy - yx = ia + ibi + icj + idk - ai - bii - cji - dki = 2ck - 2dj.$$

Then $xy = yx \iff c = d = 0$. Similarly, when $x = j$,

$$xy - yx = ja + jbi + jcj + jdk - aj - bij - cjj - dkj = -2bk + 2di.$$

So, $xy = yx \iff b = d = 0$. We see that when $b = c = d = 0$, $y \in \mathbb{R}$ and hence the center $Z(\mathbb{H}) \subset \mathbb{R}$. But every real number commutes with quaternions. Hence the center is the set $\mathbb{R}$.

For two elements $x, y \in R = \{a+bi|a,b\in\mathbb{R}\}$, we have $x\cdot y = (a_1+b_1i)(a_2+b_2i) = (a_1a_2 - b_1b_2) + (a_1b_2 + b_1a_2)i \in R$. R is clearly a group under addition and hence it is a subring of $\mathbb{H}$. We also see that for a and b not equal to zero, the inverse of an element $x = a+bi$ for the product operation is given by $\frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}$. Hence R is a field but is not contained in the center.

(7.1 - 13) An element $a \in R$ is called nilpotent if $x^m = 0$ for some $m \in \mathbb{Z}^+$.

(a) Show that if $n = a^k b$ for some integers a, b, then ab is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.

(b) If $a \in \mathbb{Z}$, show that the element $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$ is nilpotent iff every prime divisor of n is also a divisor of a. In particular, find all nilpotent elements of $\mathbb{Z}/72\mathbb{Z}$.

(c) Let R be the ring of functions from a nonempty set X to a field F . Prove that R contains no nonzero nilpotent elements.

(a) We have, $(ab)^k = a^k b^k = nb^{k-1} \equiv 0$. Hence ab is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.
(b) If a is nilpotent, then $a^k \equiv 0$ for some $k \in \mathbb{Z}$. Let $p_1, \ldots, p_m$ be the prime divisors of n. But, we know that $n \mid a^k$ and thus each p_i must divide a^k. This implies that $p_i \mid a$.

Now, if every prime divisors $p_1, \ldots, p_m$ of n divides a, then $a = q \cdot p_1^{i_1} \cdots p_m^{i_m}$ for some integer q. And we have, $n = p_1^{j_1} \cdots p_m^{j_m}$ and $a^k = q^k \cdot p_1^{ki_1} \cdots p_m^{ki_m}$. We see that, for an integer k such that $ki_\alpha \geq j_i$ for each α, $n \mid a^k$. Hence, a is nilpotent.
(c) If R did contain a nilpotent element $f \neq 0$, then f^k is a 0 function for some integer k.

(7.2 - 7) Show that the center of the ring $M_n(R)$ is $R \cdot I$, where $I = diag(1, \ldots, 1)$.

Clealy $R \cdot I$ is contained in the center since for $A \in M_n(R)$ and $B = pI \in R \cdot I$, we have $AB = BA = pA$. We now prove that the center is contained in $R \cdot I$.

If $B \in Z(M_n(R))$, then $AB - BA = 0$ for all elements A of $M_n(R)$. We take A to be the matrices of the form E_{ij} whose ij entry is 1 and all other entries are 0. Then the ith row of $E_{ij}B$ is the jth row of B and the jth column of BE_{ij} is the ith column row of B and all other entries are 0. Then $E_{ij}B = BE_{ij}$ implies that $b_{ii} = b_{jj}$ for all i and $j < n$. Also, $E_{ii}B = BE_{ii}$ implies that the matrix B is diagonal. Hence the center is contained in $R \cdot I$.

(7.3 - 4) Find all ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}/30\mathbb{Z}$. In each case, describe the kernel and the image.

Since $30 = 2 \cdot 3 \cdot 5$, $\mathbb{Z}/30\mathbb{Z}$ has elements of order 2, 3, 5, 10, 15 and 30. For each case we have homomorphisms $\varphi_i : \mathbb{Z} \to \mathbb{Z}/30\mathbb{Z}$ given by

$$\varphi_1(1) = \overline{15}, \qquad \mathrm{Im}(\varphi_1) = \overline{15\mathbb{Z}}, \qquad \ker(\varphi_1) = 2\mathbb{Z} \tag{1}$$
$$\varphi_2(1) = \overline{10}, \qquad \mathrm{Im}(\varphi_2) = \overline{10\mathbb{Z}}, \qquad \ker(\varphi_1) = 3\mathbb{Z} \tag{2}$$
$$\varphi_3(1) = \overline{3}, \qquad \mathrm{Im}(\varphi_1) = \overline{3\mathbb{Z}}, \qquad \ker(\varphi_1) = 10\mathbb{Z} \tag{3}$$
$$\varphi_4(1) = \overline{2}, \qquad \mathrm{Im}(\varphi_1) = \overline{2\mathbb{Z}}, \qquad \ker(\varphi_1) = 15\mathbb{Z} \tag{4}$$
$$\varphi_5(1) = \overline{1}, \qquad \mathrm{Im}(\varphi_1) = \overline{\mathbb{Z}}, \qquad \ker(\varphi_1) = 30\mathbb{Z} \tag{5}$$

(7.3 - 18) Prove that the intersection $I \cap J$ of ideals I, J of a ring R is also an ideal of R. Let $\{I_\alpha\}_{\alpha \in S}$ be a collection of ideals of R. Show that $\bigcap_{\alpha \in S} I_\alpha$ is an ideal of R.

Clearly $0 \in I \cap J$ and hence it is non-empty. $I \cap J$ is also closed under addition and contains the additive inverse of elements. For $x, y \in I \cap J$, we see that $xy \in I$ and $xy \in J$ and so $I \cap J$ is a subring of R. Now, if $x \in I \cap J$ and $y \in R$, we see that $xy \in I$ and $xy \in J$. Hence, $I \cap J$ is an ideal of R.

If $\{I_\alpha\}_{\alpha \in S}$ be a collection of ideals of R, then $0 \in J = \bigcap_{\alpha \in S} I_\alpha$. So, J is nonempty. And as above, we see that J is closed under addition, contains additive inverses and is closed under multiplication. Now, if $x \in J$, then $x \in I_\alpha$ for each α. If $y \in R$ is any element then $xy \in I_\alpha$ for all α. Hence, $xy \in J$ and J is an ideal.

(7.3 - 29) Let R be a commutative ring. Prove that the set of nilpotent elements of R form an ideal—called the nilradical $\mathfrak{N}(R)$.

For $x, y \in \mathfrak{N}(R)$, if $x^p = y^q = 0$ for some integers p and q, then clearly $(-x)^p = 0$ and

$$(x+y)^{p+q} = \sum_{i=0}^{p+q} \binom{p+q}{i} x^i \cdot y^{p+q-i} = 0$$

since each term in the sum is a multiple of x^p or y^q. Hence $x + y \in \mathfrak{N}(R)$ and

$-x \in \mathfrak{N}(R)$. Similarly, $(xy)^{pq} = x^{pq}y^{pq} = 0$. Thus, $\mathfrak{N}(R)$ forms a subring of R. Now we show that for $x \in \mathfrak{N}(R)$ and any $y \in R$, $xy \in \mathfrak{N}(R)$. We have

$$(xy)^p = x^p y^p = 0 \cdot y^p = 0.$$

Hence $\mathfrak{N}(R)$ is an ideal of R.