Topics:

- Outer measures, Measurability criteria; Lebesgue outer measure: infimum of the length of enveloping intervals; measures: Pushforward, Counting, Point mass
- Measurable sets and their “limits”: ascending chain, descending chain
- Measurable functions and their pointwise limits
- Every non-negative measurable function induces another measure
- Fundamental approximation theorem: approximation by simple functions
- Integration of non-negative simple functions: monotonicity and linearity
- Integration of non-negative functions: approx. by simple functions
- Convergence Theorems, Monotone Convergence Theorem, Fubini’s Theorem
- (requirement of monotonicity in MCT: two reasons)
- Fatou’s Lemma; (requirement of non-negativity in Fatou’s Lemma)
- Integration of measurable functions by decomposition into positive and negative parts
- Lebesgue integrability: finiteness of absolute integrability
- Absolute continuity of Lebesgue integral : for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta \implies \|f\| < \varepsilon$
- linearity and monotonicity of Lebesgue integration
- Dominated Convergence Theorem
- Countable additivity of the integrals as a consequence of LDCT
- Almost everywhere extensions of the convergence theorems
- Borel-Cantelli Lemma
- Complete measure space: completion theorem
- Caratheodory Extension: Restriction of Lebesgue outer measure to ”measurable” sets
- Non-measurable sets: Vitali’s Theorem
- Cantor Dust: uncountable set with measure zero; Nested set theorem
- Lebesgue integration coincides with Riemann integration of Riemann integrable functions
- Lebesgue-Vitali Theorem (criteria for Riemann integrability)
- Improper Riemann-integration and its relation to Lebesgue integrability
- Lebesgue spaces L^p : equivalence classes of p-integrable functions
- Finite essential upper bound in L^∞
- Normed spaces L^p : respective norms
- Young’s inequality, Holder’s inequality

- Extension of Holder's inequality for k different functions
- Corollary for the extension: $f \in L^r$ for every r in between ...
- Minkowski's inequality: triangle inequality for L^p
- In a set of finite measure, every power $> p$ integrable
- Convergence in L^p: convergence of norms
- Riesz-Fischer Theorem: L^p are Banach spaces
- Rapidly Cauchy subsequence; Convergence in L^p implies pointwise a.e. convergence of a subsequence
- Modes of convergence
- Convergence in measure : if the functions differ only in a set of measure zero
- Uniform convergence implies L^p convergence in a set of finite measure
- L^p convergence implies convergence in measure
- Convergence in measure implies pointwise a.e. convergence <u>of a subsequence</u>
- Pointwise convergence a.e. (finite a.e.) implies convergence in measure in a set of finite measure
- Chebyshev's inequality: attaching a constant to every integrable function
- Reiteration: convergence on finite measure space
- Egoroff's Theorem: pointwise a.e. convergence in fms implies almost uniform convergence
- Almost uniform convergence implies convergence in measure
- Lusin's Theorem: measurable functions have continuous approximations on a complement of arbitralily small closed set
- Littlewood's three principles
- Approximations in L^p spaces
- Simple functions L^p_s in L^p are dense
- Simple approximation lemma : bounded measurable functions are bounded above and below by simple functions
- Step functions in L^p are dense in L^p_s : use of approximation of measurable sets by finite intervals
- Continuous functions with compact support C_c are dense in L^p; $p \neq \infty$
- Lusin's property
- Definition of L^p spaces as completion of C_c ($p \neq \infty$) or L^p_s
- C_c not dense in L^∞; C_c^∞ dense in L^p ($p \neq \infty$)
- L^∞ completion of C_c is C_0, continuous functions that "vanish at infinity"
- Weierstrass approximation theorem : uniform approximation of continuous function on $[a, b]$ by polynomials

- Separability of L^p spaces, $(p \neq \infty)$
- Parallelogram law in Hilbert spaces; strict convexity
- Milman-Pettis Theorem: uniform convexity of norms in Banach spaces implies reflexivity
- Some reflexive spaces admit no uniformly convex norms
- L^p-norm $(1 < p < \infty)$ is uniformly convex
- Clarkson's first $(2 \leq p < \infty)$ and second $(1 < p < 2)$ inequality
- Operator in the dual of L^q induced by an element in L^p : bijective isometry
- Norm of this operator equal to the L^p-norm of the element
- Closed subspace of a reflexive Banach space is reflexive
- Banach space is reflexive iff the space of its bounded linear functionals is reflexive
- Riesz Representation Theorem for L^p $(1 \leq p < \infty)$ spaces
- Hahn-Banach Theorem and its corollaries
- Relations between $(L^1)'$ and L^∞
- Weak convergence : uniqueness of weak limit; boundedness of weak sequence
- Uniform Boundedness Principle
- Riesz Lemma: space is finite dimensional iff the unit ball is compact
- Bessel's inequality
- Bolzano-Weierstrass Theorem and its weak analogue for infinite dimensional
- Equivalent definition of weak convergence: boundedness and convergence of $f(x_n)$ for f in a total subset of dual
- Particular applications to the L^p spaces
- Weak convergence in L^p $(p \neq \infty)$ space equivalent to convergence in every measurable subset
- Weak convergence in L^p $(1 < p < \infty)$ space equivalent to convergence of "indefinite integral" in every closed interval
- Riemann-Lebesgue Lemma
- Pointwise convergence implies weak convergence for $1 < p < \infty$
- Weak convergence in $1 < p < \infty$ implies pointwise convergence iff there is convergence of L^p-norm

Techniques:

- Triangle inequality : $|x - x_n| \leq |x - x_m| + |x_m - x_n|$
- Lebesgue to Riemann integral transitions: if $f = g$ a.e. on X then

$$\int_X f = \int_X g$$

- Weakly convergent sequences are bounded and the limit is unique
- Boundedness and the linear functionals in a total subset of the dual is enough to characterize weakness of a sequence
- $$\text{bounded } f_n \longrightarrow f \iff \int_X g \cdot f_n \longrightarrow \int_X g \cdot f \quad \forall g \in M \text{ total}$$
- Step functions and simple functions are total in L^p for $1 \leq p < \infty$
- If $|f| < 1$ on a set of infinite measure and $f \in L^p$ then $f \in L^q$ for all $q > p$
- If $|f| \geq 1$ on a set of infinite measure and $f \in L^q$ then $f \in L^p$ for all $p < q$
- Usual examples and counter examples: shrinking box, box marching to infinity, shrinking box marching in a circle, flattening box
- Countinuous analogue of the above functions
- Convexity of functions $f(tx + (1-t)y) \leq t \cdot f(x) + (1-t) \cdot f(y)$
- Fatou's lemma : integral of lim inf $\leq$ lim inf of integral; flattening box shows strict inequality
- Defining the sequence $f_n(x) = n$ for $f(x) \geq n$ and $f_n(x) = f(x)$ otherwise helps prove absolute continuity of Lebesgue integral
- For positive f this gives a monotone increasing sequence that converges to f
- $|f| = \text{sgn}(f) \cdot f$ helps define functions that can be multiplied with f and get L^p norm by integrating the product
- Cauchy-Schwartz inequality for inequalities with product of integrals and integral of products
- $L^p \ni u \mapsto Tu \in (L^q)'$ with $(Tu)(f) = \displaystyle\int_X u \cdot f$ for $f \in L^q$
- Functions defined like $\text{sgn}(f) \cdot |f|^{p-2}$ also helps find operator norms
- Defining a complete Lebesgue measure poses a challenge of well definition of the measure: If $A \subset E \subset B$ with $\mu(B \setminus A) = 0$ and $A' \subset E \subset B'$ with $\mu(B' \setminus A') = 0$, then how do we define $\mu(E)$?
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