Smooth Manifolds

Nutan Nepal

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Homework 1

1. Prove that the open disks $D_r(p)$ are open subsets of $\mathbb{R}^n$.

> **Solution:** To prove that the open disks $D_r(p)$ are open subsets of $\mathbb{R}^n$, we show that for every point $x \in D_r(p)$ we have another open disk $D_\epsilon(x)$, $\epsilon > 0$ such that $D_\epsilon(x) \subset D_r(p)$.
>
> For any $x \in D_r(p)$, we have $\delta = d(x,p) < r$, we take $0 < \epsilon < r - \delta$. Then we see that for all $y \in D_\epsilon(x)$
>
> $$d(p,y) \leq d(p,x) + d(x,y) < \delta + \epsilon < \delta + r - \delta = r.$$
>
> Hence, $y \in D_r(p)$ for all $y \in D_\epsilon(x)$ which implies that $D_\epsilon(x) \subset D_r(p)$. So, $D_r(p)$ is an open subset.

2. Prove the second part of Proposition 2.17 (a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous everywhere if and only if for all open subsets V of $\mathbb{R}^m$, the preimage $f^{-1}(V)$ of V under f is open in $\mathbb{R}^n$).

> **Solution:** The Proposition 2.17 is reproduced below:
>
> Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and let $p \in \mathbb{R}^n$. Let $q = f(p)$.
>
> i. f is continuous at p if and only if for any open neighborhood V of q in $\mathbb{R}^m$, the preimage of V under f (i.e. $f^{-1}(V)$) contains an open subset U of $\mathbb{R}^n$ which in turn contains p, i.e. there is some open neighborhood U of $p \in \mathbb{R}^n$ such that f sends U inside V .
>
> ii. f is continuous (everywhere) if and only if for any open subset V of $\mathbb{R}^m$, the preimage of V under f (i.e. $f^{-1}(V)$) is an open subset of $\mathbb{R}^n$.
>
> For $\Leftarrow$: If for all open subsets V of $\mathbb{R}^m$ the preimage $f^{-1}(V)$ of V under f is open in $\mathbb{R}^n$, then f is continuous.
>
> Let $V \subset \mathbb{R}^m$ be an open subset such that $f(x) \in V$. Then we have an open disk $D_\epsilon(f(x)) \subset V$. As the disk $D_\epsilon(f(x))$ is open in $\mathbb{R}^m$, we have the preimage $f^{-1}(D_\epsilon(f(x))) \subset \mathbb{R}^n$ which is open and contains x. Then we can find a $\delta > 0$ such that $D_\delta(x) \subset f^{-1}(D_\epsilon(f(x)))$. That is, for every ϵ-ball around $f(x)$, we can find a δ-ball around x such that
>
> $$y \in D_\delta(x) \implies f(y) \in D_\epsilon(f(x))$$
>
> for some y. Hence f is continuous.

3. Show that a composition of continuous functions $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^k$ is continuous.

Solution: Since g is continuous, we have $g^{-1}(V)$ open for all open set $V \subset \mathbb{R}^k$. Similarly we have f continuous, so $f^{-1}(U)$ is open for all open sets $U \subset \mathbb{R}^m$. Then, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is open for all open sets V in $\mathbb{R}^k$. Hence the composition is continuous.

4. Show that a function $f : X \to Y$ between sets is invertible if and only if it is bijective.

Solution: A function $f : X \to Y$ is invertible if there exists a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

i. f invertible $\implies$ f bijective
Note that since id_Y is surjective, f must be surjective. Now for injectivity, we observe that if $f(x) = f(y)$ then

$$x = (g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y) = y.$$

Hence, f is bijective.

ii. f bijective $\implies$ f invertible
Since f is both injective and surjective, we define a function $g : Y \to X$ by $g(y) = x$ for $y \in Y$ whenever $f(x) = y$. Note that g is well-defined since there exists only one y for each x. Then, for all $x \in X$, $g(f(x)) = g(y) = x \implies g \circ f = \mathrm{id}_X$. Similarly, for all $y \in Y$, $f(g(y)) = f(x) = y \implies f \circ g = \mathrm{id}_Y$. So, f is invertible.

5. Show that the product topology on a product $X \times Y$ of topological spaces is a valid topology.

Solution: X and Y are topological spaces. We define a set $U \subset X \times Y$ to be open if for all $(x, y) \in U$ we have open neighborhoods $U_x \subset X$ and $U_y \subset Y$ such that $U_x \times U_y \subset U$.

i. Clearly the null set ϕ and the whole set $X \times Y$ are open since X is open in X and Y is open in Y.

ii. Arbitrary union $\bigcup U_\alpha$ of open sets is open.

Let (x, y) be an arbitrary point in $\bigcup U_\alpha$, then $(x, y) \in U_i$ for some i. Then, by definition, there are open neighborhoods $U_x \subset X$ and $U_y \subset Y$ such that $U_x \times U_y \subset U_i \subset \bigcup U_\alpha$.

iii. Finite intersection $U_i \cap U_j$ of open sets is open.

Let (x, y) be an arbitrary point of $U_i \cap U_j$, then $(x, y) \in U_i$ and $(x, y) \in U_j$. Then, by definition, there are open neighborhoods $U_{ix}, U_{jx} \subset X$ and $U_{iy}, U_{jy} \subset Y$ such that $U_{ix} \times U_{iy} \subset U_i$ and $U_{jx} \times U_{jy} \subset U_j$. Then

$$U_i \cap U_j \supset (U_{ix} \times U_{iy}) \cap (U_{jx} \times U_{jy}) = (U_{ix} \cap U_{jx}) \times (U_{iy} \cap U_{jy}) \ni (x, y).$$

Since $(U_{ix} \times U_{jx})$ and $(U_{iy} \times U_{jy})$ are open in X and Y respectively, we see that $U_i \cap U_j$ is open.

6. Verify the three basic properties of closed sets that correspond to the three axioms for open sets.

Solution: We define a set $V \subset X$ to be closed if its complement V^c is open in X.

i. The null set ϕ and the whole set X are closed.

$\phi^c = X$ and $X^c = \phi$ which are open in X.

ii. Arbitrary intersection $\bigcap V_\alpha$ of closed sets is closed.

Here we use the set-theoretic fact that

$$\left(\bigcup U_\beta\right)^c = \bigcap U_\beta^c \tag{1}$$

where $\{U_\beta\}$ is the collection of indexed sets. Since each sets V_α are closed, we write V_α as U_α^c where U_α is an open set of X. Then from 1 we have

$$\bigcap V_\alpha = \bigcap U_\alpha^c = \left(\bigcup U_\alpha\right)^c \tag{2}$$

Hence, since $\bigcup U_\alpha$ is open in X, $\bigcap V_\alpha$ must be closed.

iii. Finite union $V_i \cup V_j$ of closed sets is closed.

We have $V_i \cup V_j = U_i^c \cup U_j^c = (U_i \cap U_j)^c$. Since finite intersection of open sets are open, we observe that $V_i \cup V_j$ is the complement of an open set. Hence $V_i \cup V_j$ is closed.

7. Show that if we have $X'' \subset X' \subset X$, then the "subspace of a subspace" topology on X'' is the same as the "subspace of the biggest space" topology on X''.

Solution: Suppose (X, τ) is a topological space. The subspace topology on X' is given by

$$\tau' = \{U' \subset X' : U' = U \cap X' \text{ for some } U \in \tau\}$$

and the subspace topology on X'' induced by τ' is given by

$$\tau'' = \{U'' \subset X'' : U'' = U' \cap X'' \text{ for some } U' \in \tau'\}.$$

We need to show that τ'' is equal to the the subspace topology on X'' induced by τ

$$T = \{U'' \subset X'' : U'' = U \cap X'' \text{ for some } U \in \tau\}.$$

Let $A \in \tau''$, then $A = U' \cap X''$ for some $U' \in \tau'$. Since $U' = U \cap X'$ for some $U \in \tau$ we have, $A = U \cap X' \cap X'' = U \cap X'' \in T$. Hence $\tau'' \subset T$. Similarly, let $B \in T$, then $B = U \cap X''$ for some $U \in \tau$. Since we can write X'' as $X' \cap X''$ we have $B = U \cap X' \cap X'' = U' \cap X'' \in \tau''$ for some U' in τ'. Hence $T \subset \tau''$ which gives $T = \tau''$ ending our proof.

Homework 2

1. Do Exercise 2.6 (show that for topological spaces X, Y, Z, the "rearrange-the-parentheses" map from $(X \times Y) \times Z$ to $X \times (Y \times Z)$ is a homeomorphism).

Solution: Let the function $f : (X \times Y) \times Z \to X \times (Y \times Z)$ be defined as

$$f((x, y), z) = f(x, (y, z))$$

where x, y and z are respective points of the topological spaces. We see that the map is clearly bijective and hence invertible.

Now, for each open set $U_x \times (U_y \times U_z)$, the preimage of f is given by $(U_x \times U_y) \times U_z$ which is open in $(X \times Y) \times Z$. Similarly, for each open set $(U_x \times U_y) \times U_z$, the preimage of f^{-1} is given by $U_x \times (U_y \times U_z)$ which is open in $X \times (Y \times Z)$. Hence f and f^{-1} are both continuous and the "rearrange the parentheses" map is homeomorphism.

2. Do Exercise 2.7 (show that the product topology and the usual topology on $\mathbb{R}^n$ agree).

Solution: Suppose $\mathscr{P}$ be the product topology and $\mathscr{T}$ be the usual topology in $\mathbb{R}^n$. Let U be open in the product topology, then for all $x = (x_1, \ldots, x_n) \in U$ there exists open neighborhoods $U_i \in \mathbb{R}$ such that $x_i \in U_i$ and $U_1 \times \cdots \times U_n \subset U$. Then for all x_i, there exists an open interval $(x_i - \delta_i, x_i + \delta_i)$ for some $\delta_i > 0$. Let $\delta = \min\{\delta_i\}$ taken over all i from 1 to n. Clearly, $\delta > 0$ and $x \in B_\delta(x) \subset U_1 \times \cdots \times U_n \subset U$. This shows that $\mathscr{P} \subset \mathscr{T}$.

Now let U be open with respect to the usual topology. Then for all $x \in U$, there exists an open ball $B_\delta(x)$ containing x such that $B_\delta(x) \subset U$ for some $\delta > 0$. Let $\delta_i = \delta/\sqrt{2}$. Then each x_i is contained in the interval $U_i = (x_i - \delta_i, x_i + \delta_i)$ and we see that $B_\delta(x) \supset U_1 \times \cdots \times U_n$. Then $x \in U_1 \times \cdots \times U_n \subset B_\delta(x) \subset U$. Hence U is open in the product topology and $\mathscr{P} \supset \mathscr{T}$. So we see that the two topologies agree.

3. The following exercises are about the "line with two origins" of Example 2.44, which we will call X.
   (a) Show that the construction in Example 2.44 defines a topology on X.
   (b) Show that with this topology, X is locally homeomorphic to R.
   (c) Show that X is not Hausdorff.

Solution: The construction in Example 2.44 is reproduced below:

Let $\mathscr{B}$ be the set of subsets of X that have one of the following two forms:

i. open intervals $(a, b) \subset \mathbb{R}$ (with a and b finite and $a < b$);

ii. sets of the form $((a, b)\backslash 0) \cup \bar{0}$ whenever $a < 0 < b$.

(a) We declare a subset U of X to be open if, for all $x \in U$, there exists a subset B of $\mathscr{B}$ with $x \in B$ and $B \subset U$.

Let $\mathscr{T}$ be the collection of open sets as defined above. We now show that it is a topology.

a. Clearly, $\phi \in \mathscr{T}$ and also $X \in \mathscr{T}$.

b. Let $A = \bigcup_i U_i$ be the union of arbitrary collection of indexed open sets. For all $x \in A$ then there exists a U_i such that $x \in U_i$. So, there exists a subset B of $\mathscr{B}$ with $x \in B$ and $B \subset U_i \subset A$. Hence, A is open.

c. Let $A = U_1 \cap U_2$ be the finite intersection of open sets of X. For any $x \in A$ we see that $x \in U_1$ and $x \in U_2$. Then there exists a subset B_1 of $\mathscr{B}$ with $x \in B_1$ and $B_1 \subset U_1$ and there exists a subset B_2 of $\mathscr{B}$ with $x \in B_2$ and $B_2 \subset U_2$. If $x \neq \bar{0}$ then the problem reduces to $\mathbb{R}$ which implies that A is open. If $x = \bar{0}$ then we see that $B_1 \cap B_2$ is the intersection of open intervals and $\bar{0}$ which is again open in X.

Thus X is a topological space with the topology $\mathscr{T}$.

(b) For any point $x \neq \bar{0}$ in X, we observe that there is an open ball $(x - \delta, x + \delta)$ around x for some $\delta > 0$. Since any open intervals of $\mathbb{R}$ are homeomorphic to $\mathbb{R}$ itself, we see that X is locally homeomorphic $\mathbb{R}$ for every point $x \neq \bar{0}$.

Now, when $x = \bar{0}$ we take $Y = (-\delta, 0) \cup (0, \delta) \cup \{\bar{0}\}$ and define a function $f : Y \to \mathbb{R}$ by $f(\bar{0}) = 0$ and $f(y) = \tan(\pi y/2\delta)$. We see that f is invertible, continuous and has a continuous inverse and hence is a homeomorphism. Thus, X is locally homeomorphic to $\mathbb{R}$.

(c) For every $\epsilon > 0$, the neighborhood $N_\epsilon(0)$ of the point 0 intersects with the neighborhood around the point $\bar{0}$ non-trivially. So, X is not Hausdorff.

Homework 3

1. Do Exercise 3.1 (show that if $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, then a function $f : U \to V$ is smooth if and only if each of its component functions $f^i : U \to R$ are smooth).

Solution: If $f(x_1, \ldots, x_n) = (f^1(x_1, \ldots, x_n), \ldots, f^m(x_1, \ldots, x_n))$ then for $i \in \{1, 2, \ldots, n\}$, the first-order partial derivative at p is given by the limit

$$\lim_{t \to 0} \frac{f(p + te_i) - f(p)}{t} = \lim_{t \to 0} \frac{(f^1(p + te_i) - f^1(p), \ldots, f^j(p + te_i) - f^j(p), \ldots, f^m(p + te_i) - f^m(p))}{t} \tag{3}$$

Then for each $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$, the partial derivative exists at $p \in U$ iff the limit

$$\lim_{t \to 0} \frac{f^j(p_1, \ldots, p_i + t, \ldots, p_n) - f^j(p_1, \ldots, p_n)}{t} \tag{4}$$

exists at p. But the limit on equation (2) is the partial derivative of the component function f^j at x_i. Hence, the derivative of f exists at p iff each of its component functions are differentiable. The partial derivative at a point p, again, is a function $g : U \to \mathbb{R}^m$. Then, as above, we see that the partial derivatives of g exist iff each of its component functions are differentiable.

If $f : U \to V$ is smooth then all k^{th}-order partial derivatives exist on U for all k. Then, inductively, from above, all k^{th}-order partial derivatives of each component functions also exist on U for all k. Similarly, if all k^{th}-order partial derivatives of each component functions exist on U for all k, then f is also smooth.

2. Check that Definition 3.6 gives an equivalence relation (a binary relation that is reflexive, symmetric, and transitive) on the set of smooth atlases on a given topological manifold X.

Solution: Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ and $\mathcal{B} = \{(V_\beta, \psi_\beta) : \beta \in B\}$ be smooth atlases on the topological manifold X for some indexed set A and B. We say that $\mathcal{A} \sim \mathcal{B}$ if their union is a smooth atlas on X. The reflexive ($\mathcal{A} \sim \mathcal{A}$) and symmetric ($\mathcal{A} \sim \mathcal{B} \implies \mathcal{B} \sim \mathcal{A}$) properties are obvious. We now prove for the transitivity of the relation $\sim$.

If $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$, with $\mathcal{C} = \{(W_\gamma, \zeta_\gamma) : \gamma \in C\}$ for some indexed set C, then for all $\alpha \in A$ and $\beta \in B$ such that $U_\alpha \cap V_\beta$ is non-empty, the map

$$\psi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap V_\beta) \to \psi_\beta(U_\alpha \cap V_\beta) \tag{5}$$

is smooth, and for all $\gamma \in C$ and $\beta \in B$ such that $W_\gamma \cap V_\beta$ is non-empty, the map

$$\zeta_\gamma \circ \psi_\beta^{-1} : \psi_\beta(W_\gamma \cap V_\beta) \to \zeta_\gamma(W_\gamma \cap V_\beta) \tag{6}$$

is smooth. Then, we take all $\alpha \in A$ and $\gamma \in C$ such that $U_\alpha \cap W_\gamma$ is non-empty. For each $x \in U_\alpha \cap W_\gamma$, we take a chart $(V, \psi) \in \mathcal{B}$ that contains $x \in X$. Then from (3) and (4), we get the composition of smooth maps

$$\zeta_\gamma \circ \psi^{-1} \circ \psi \circ \varphi_\alpha^{-1} = \zeta_\gamma \circ \varphi_\alpha^{-1}$$

from $\varphi_\alpha(U_\alpha \cap W_\gamma) \to \zeta_\gamma(U_\alpha \cap W_\gamma)$ which is smooth. Analogously, we can show that the inverse map

$$\varphi_\alpha \circ \zeta_\gamma^{-1} : \zeta_\gamma(U_\alpha \cap W_\gamma) \to \varphi_\alpha(U_\alpha \cap W_\gamma)$$

is also smooth. Hence, this proves transitivity and that $\sim$ is an equivalence relation.

3. Do Exercise 3.2 (Let X and Y be topological manifolds equipped with smooth atlases $\mathcal{A}$ and $\mathcal{B}$ respectively. Show that $\{U \times V : U \in \mathcal{A}, V \in \mathcal{B}\}$ is a smooth atlas on the topological manifold $X \times Y$).

Solution: Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ and $\mathcal{B} = \{(V_\beta, \psi_\beta) : \beta \in B\}$ be smooth atlases on the topological manifolds X and Y respectively for some indexed set A and B. Then the product of the smooth atlases is defined by

$$\mathcal{A} \times \mathcal{B} = \{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta) : \alpha \in A,\ \beta \in B\}.$$

1. If X is m-manifold and Y is n-manifold, then $\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^{m+n}$ is given by

   $$(\varphi_\alpha \times \psi_\beta)(x, y) = (\varphi_\alpha^1(x), \ldots, \varphi_\alpha^m(x), \psi_\beta^1(y), \ldots, \psi_\beta^n(y))$$

   for all $x \in U_\alpha$ and $y \in V_\beta$. Since, each component functions are smooth, we see that $\varphi_\alpha \times \psi_\beta$ is a smooth function on the product topology.

2. Each φ_α is a homeomorphism from U_α to an open disk $D_\alpha \subset \mathbb{R}^m$ and ψ_β is a homeomorphism from V_β to an open disk $D_\beta \subset \mathbb{R}^n$. Clearly, $D_\alpha \times D_\beta$ is an open disk in $\mathbb{R}^{m+n}$. We define $\varphi_\alpha^{-1} \times \psi_\beta^{-1} : D_\alpha \times D_\beta \to U_\alpha \times V_\beta$ by

   $$(\varphi_\alpha^{-1} \times \psi_\beta^{-1})(z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+n}) = (\varphi^{-1}(z_1, \ldots, z_m, z_{m+1}), \psi_\beta^{-1}(z_{m+1}, \ldots, z_{m+n}))$$

   Since each component functions are continuous, we see that $\varphi_\alpha^{-1} \times \psi_\beta^{-1}$ is the continuous inverse of the map $\varphi_\alpha \times \psi_\beta$. Hence $\varphi_\alpha \times \psi_\beta$ is a homeomorphism from $U_\alpha \times V_\beta$ to an open disk in $\mathbb{R}^{m+n}$.

3. Since every point of X is in at least one U_α and every point of Y is in V_β, every point of $X \times Y$ is in some $U_\alpha \times V_\beta$ (by defnition of the product topology).

4. If $(U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'})$ is non-empty, then the transition map $(\varphi_{\alpha'} \times \psi_{\beta'}) \circ (\varphi_\alpha \times \psi_\beta)^{-1} : (\varphi_\alpha \times \psi_\beta)((U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'})) \to (\varphi_{\alpha'} \times \psi_{\beta'})((U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'}))$ is given by $(\varphi_{\alpha'} \times \psi_{\beta'}) \circ (\varphi_\alpha \times \psi_\beta)^{-1}(z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+n}) =$

   $$(\varphi_{\alpha'} \circ \varphi_\alpha^{-1}(z_1), \ldots, \varphi_{\alpha'} \circ \varphi_\alpha^{-1}(z_m), \psi_{\beta'} \circ \psi_\beta^{-1}(z_{m+1}), \ldots \psi_{\beta'} \circ \psi_\beta^{-1}(z_{m+n}))$$

   Since each component functions are smooth, we see that the transition map is smooth.

Hence the product of atlases is an atlas in the product of topological manifolds.

4. Define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

   $$f(u, v) = \left(\cos(u^2 v) - e^{u-v},\ \frac{u^2 - 3}{u^2 + v^2},\ e^{e^{uv}}\right)$$

   Compute the Jacobian matrix of f.

Solution:

$$(Jf)_{(u,v)} = \begin{bmatrix} -2uv\sin(u^2 v) - e^{u-v} & u^2 \sin(u^2 v) + e^{u-v} \\ \frac{2uv^2 - 6u}{(u^2+v^2)^2} & \frac{2v(3-u^2)}{u^2+v^2} \\ ve^{e^{uv}+uv} & ue^{e^{uv}+uv} \end{bmatrix}.$$

Homework 4

1. Precisely specify a function f and an element c of the codomain of f such that the level set of f at level c is the ellipsoid in $\mathbb{R}^3$ defined by the equation

   $$x^2 + 2y^2 + 3z^2 = 4.$$

Solution: The given equation is the level set of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x, y, z) = x^2 + 2y^2 + 3z^2$$

at level $c = 4$.

2. Precisely specify a function f and an element c of the codomain of f such that the level set of f at level c is the graph of

$$g : \mathbb{R}^2 \to \mathbb{R}$$
$$g(x, y) = x^3 - y^4.$$

Solution: The graph of g is given by the set

$$\{(x, y, z) \in \mathbb{R}^3 : \ z = x^3 - y^4\}.$$

Then, this graph of g is the level set of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x, y, z) = x^3 - y^4 - z$$

at level $c = 0$.

3. Precisely specify a function f whose image is the line in $\mathbb{R}^3$ defined by the system of equations

$$\begin{cases} y = 2; \\ x - 3z = 5. \end{cases}$$

Solution: In the given line, the second coordinate is always 2 and the first coordinate can be written as a function of the third coordinate as $x = 5 + 3z$. Then, the function $f : \mathbb{R} \to \mathbb{R}^3$ defined as

$$f(x) = (5 + 3x, 2, x)$$

has the given line as its image.

4. Do Exercise 3.3: check that the total derivative T of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point p of $\mathbb{R}^n$ (if it exists) is unique.

Solution: Let T and T' both be the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point p. Then for the total derivative T,

$$\lim_{q \to p} \frac{f(q) - f(p) - T(q - p)}{\|q - p\|} = 0.$$

Let $h = q - p$, so we have,

$$\lim_{h \to 0} \frac{f(p + h) - f(p) - T(h)}{\|h\|} = 0.$$

Now, for T and T',

$$\begin{aligned} 0 \leq \lim_{h \to 0} \frac{\|T(h) - T'(h)\|}{\|h\|} &= \lim_{h \to 0} \frac{\|f(p + h) - f(p) - T'(h) - (f(p + h) - f(p) - T(h))\|}{\|h\|} \\ &\leq \lim_{h \to 0} \frac{\|f(p + h) - f(p) - T'(h)\| + \|(f(p + h) - f(p) - T(h))\|}{\|h\|} \\ &= 0 \end{aligned}$$

Then, since $tx \to 0$ as $t \to 0$, we can say that, for $x \neq 0$ and $h = tx$ we have, (by linearity of T and T')

$$0 = \lim_{h \to 0} \frac{\|T(h) - T'(h)\|}{\|h\|} = \lim_{t \to 0} \frac{\|T(tx) - T'(tx)\|}{\|tx\|} = \lim_{t \to 0} \frac{|t|\|T(x) - T'(x)\|}{\|tx\|} = \frac{\|T(x) - T'(x)\|}{\|x\|}.$$

Hence $\|T(x) - T'(x)\| \implies T = T'$. So, the total derivative T is unique.

5. Do Exercise 3.4: establish the given formula for the Jacobian of the "matrix multiplication" map

$$\mu : \mathbb{R}^{km} \times \mathbb{R}^{mn} \to \mathbb{R}^{kn}.$$

Solution: Let T be the Jacobian of the map $\mu : \mathbb{R}^{km} \times \mathbb{R}^{mn} \to \mathbb{R}^{kn}$ at (a, b). Then T is given by

$$\lim_{(A,B) \to 0} \frac{\mu(a + A, b + B) - \mu(a, b) - T(A, B)}{\|(A, B)\|} = 0.$$

Here μ is the matrix multiplication map that takes $k \times m$ matrix and $m \times n$ matrix.

Furthermore,

$$\begin{aligned} &\lim_{(A,B) \to 0} \frac{\mu(a + A, b + B) - \mu(a, b) - T(A, B)}{\|(A, B)\|} = 0 \\ \implies &\lim_{(A,B) \to 0} \frac{\|\mu(a + A, b + B) - \mu(a, b) - T(A, B)\|}{\|(A, B)\|} = 0. \end{aligned}$$

Since $tA \to 0$ as $t \to 0$ and $tB \to 0$ as $t \to 0$, when $(A, B) \neq 0$, we can write the above limit as

$$\begin{aligned} \lim_{t \to 0} \frac{\|\mu(a + tA, b + tB) - \mu(a, b) - T(tA, tB)\|}{\|(tA, tB)\|} &= \lim_{t \to 0} \frac{\|ab + tAb + taB + t^2 AB - ab - tT(A, B)\|}{\|(tA, tB)\|} \\ &= \lim_{t \to 0} \frac{|t|\|Ab + aB + tAB - T(A, B)\|}{|t|\|(A, B)\|} \\ &= \lim_{t \to 0} \frac{\|Ab + aB + tAB - T(A, B)\|}{\|(A, B)\|} \\ &= \frac{\|Ab + aB - T(A, B)\|}{\|(A, B)\|} \end{aligned}$$

Since above limit equals 0, we see that the Jacobian $T(A, B)$ evaluated at (a, b) equals $Ab + aB$.