Analysis I

Homework 5

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Pack Pledge: I have neither given nor received unauthorized aid on this test or assignment.

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1. Let $1 < p < \infty$. Show that the dual space of l^p is l^q.

Taking the Schauder basis $(e_k) = (\delta_{kj})$ for the space l^p, we see that every $x \in l^p$ has a unique representation $x = \sum_{i=1}^{\infty} \xi_i e_i$. For the linear and bounded operator $f \in (l^p)'$, we have

$$f(x) = \sum_{k=1}^{\infty} \xi_k \gamma_k \qquad \text{where } \gamma_k = f(e_k).$$

We consider $x_n = (\xi_k^n)$ with

$$\xi_k^n = \begin{cases} |\gamma_k|^q / \gamma_k & \text{if } k \leq n \text{ and } \gamma_k \neq 0 \\ 0 & \text{if } k > n \text{ or } \gamma_k = 0 \end{cases}$$

where q is the conjugate of p. Then we obtain $f(x_n) = \sum_{k=1}^{\infty} \xi_k^n \gamma_k = \sum_{k=1}^{n} |\gamma_k|^q$ and

$$\begin{aligned} f(x_n) \leq \|f\| \|x_n\| &= \|f\| \left(\sum_{k=1}^{\infty} |xi_k^n|^p \right)^{1/p} \\ &= \|f\| \left(\sum_{k=1}^{n} |\gamma_k|^{(q-1)p} \right)^{1/p} \\ &= \|f\| \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1/p}. \end{aligned}$$

So we have, $\sum_{k=1}^{n} |\gamma_k|^q \leq \|f\| \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1/p} \implies \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1/q} \leq \|f\|$. Letting $n \longrightarrow \infty$, we see that $(\gamma_k) = (f(e_k)) \in l^q$ since the infinite sum is bounded. Now, conversely, for any sequence $b = (\beta_k) \in l^q$, we define the correspondinf bounded linear functional $g \in l^p$ by $g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k$

where $x = (\xi_k) \in l^p$. g is linear and bounded by Holder's inequality. Thus $g \in (l^p)'$.

Now, we show that the norm of the functional f is the norm on the space l^q. We have

$$|f(x)| = \left|\sum_{k=1}^{\infty} \xi_k \gamma_k\right| \leq \left(\sum_{k=1}^{\infty} |\xi_k\|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |\gamma_k\|^q\right)^{1/q} = \|x\| \left(\sum_{k=1}^{\infty} |\gamma_k\|^q\right)^{1/q}.$$

Taking sup over all x of norm 1 we obtain $\|f\| \leq (\sum_{k=1}^{\infty} |\gamma_k\|^q)^{1/q}$. But we also have $(\sum_{k=1}^{n} |\gamma_k|^q)^{1/q} \leq \|f\|$ so $(\sum_{k=1}^{n} |\gamma_k|^q)^{1/q} = \|f\|$. Thus we see that the mapping of $(l^p)'$ to l^q is linear and bijective and also norm preserving.

2. Prove the completion theorem for inner product spaces.

Theorem 1 (Completion)**.** *For any inner product space X there exists a Hilbert space H and an isomorphism A from X into a dense subspace $W \subset H$. The space H is unique except for isomorphisms.*

Proof. By completion of Banach spaces, there exists a Banach space H and an isometry A from X onto a dense subspace W of H. For continuity, A preserves sums and scalar multiplications and hence, A is an isomorphism of normed spaces. By continuity of inner product, we can define an inner product on H by $\langle \hat{x}, \hat{y} \rangle = \lim_{n\to\infty} \langle x_n, y_n \rangle$ where $\{x_n\}$ and $\{y_n\}$ are sequences in X converging to x and y respectively and they are representatives of $\hat{x}$ and $\hat{y}$ in H. Since the inner product is continuous, the parallelogram and polarization identities are also preserved and hence, we see that A is an isomorphism of inner product spaces from X onto W.
The space H is unique except for isomorphisms by the completion of Banach spaces theorem. □

3. **Kreyszig p.135 / Problem 4.** If an inner product space X is real, show that the condition $\|x\| = \|y\|$ implies $\langle x + y, x - y \rangle = 0$. What does this mean geometrically if $X = \mathbb{R}^2$? What does the condition imply if X is complex?

We have $\langle x + y, x - y \rangle = \langle x, x \rangle - \langle x, y \rangle + \langle y, x \rangle - \langle y, y \rangle = \|x\|^2 - \|y\|^2 - \langle x, y \rangle + \langle x, y \rangle = 0$. If $X = \mathbb{R}^2$, then we see that $x + y$ and $x - y$ are orthogonal for all x and y.

4. **Kreyszig p.135 / Problem 6.** Let $x \neq 0$ and $y \neq 0$.
   (a) If $x \perp y$, show that $\{x, y\}$ is a linearly independent set.
   (b) Extend the result to mutually orthogonal nonzero vectors $x_1, \ldots, x_m$.

(a) If $x \perp y$ then for the equation $\lambda_1 x + \lambda_2 y = 0$ we have,

$$0 = \langle \lambda_1 x + \lambda_2 y, y \rangle = \langle \lambda_1 x, y \rangle + \langle \lambda_2 y, y \rangle = \lambda_2 \|y\|^2 \implies \lambda_2 = 0 \qquad \text{and,}$$

$$0 = \langle \lambda_1 x + \lambda_2 y, x \rangle = \langle \lambda_1 x, x \rangle + \langle \lambda_2 y, x \rangle = \lambda_1 \|x\|^2 \implies \lambda_1 = 0.$$

Thus, x and y are linearly independent.

(b) Using above method, we see that if $\sum_{i=1}^{m} \lambda_i x_i = 0$, then for any $i \in \overline{\{1, \ldots, m\}}$,

$$0 = \left\langle \sum_{i=1}^{m} \lambda_i x_i, x_i \right\rangle = \lambda_i \|x_i\|^2 \implies \lambda_i = 0.$$

Thys the mutually orthogonal vectors $x_1, \ldots, x_m$ are linearly independent when $x_i \neq 0$ for all i.

5. **Kreyszig p.135 / Problem 10.** Let z_1 and z_2 denote complex numbers. Show that $\langle z_1, z_2 \rangle = z_1 \overline{z}_2$ defines an inner product, which yields the usual metric on the complex plane. Under what condition do we have orthogonality?

$\langle z_1, z_2 \rangle = z_1 \overline{z}_2$. We check that this definition satisfies the axioms:
(a) $\langle x + y, z \rangle = (x + y) \cdot \overline{z} = x\overline{z} + y\overline{z} = \langle x, z \rangle + \langle y, z \rangle$,
(b) $\langle \alpha x, y \rangle = (\alpha x)\overline{y} = \alpha \langle x, y \rangle$,
(c) $\langle x, y \rangle = x\overline{y} = \overline{y\overline{x}} = \overline{\langle y, x \rangle}$,
(d) $\langle x, x \rangle = x\overline{x} = \|x\|^2$, where $\|\cdot\|$ denotes the usual metric on the complex plane. So, $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$.

The inner product yields the usual metric by taking $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$. For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where x_j and y_j are real numbers, we have $\langle z_1, z_2 \rangle = (x_1 + iy_1)(x_2 - iy_2) = (x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)$. So we have orthogonality where the two equations $x_1x_2 + y_1y_2 = 0$ and $y_1x_2 - x_1y_2 = 0$ satisfy.

6. **Kreyszig p.141 / Problem 8.** Show that in an inner product space, $x \perp y$ if and only if $\|x + ay\| \geq \|x\|$ for all scalars a.

If $x \perp y$, we have $\|x + ay\|^2 = \langle x + ay, x + ay \rangle = \langle x, x \rangle + \langle x, ay \rangle + \langle ay, x \rangle + \langle ay, ay \rangle$. Thus,

$$\|x + ay\|^2 = \|x\|^2 + 2\mathfrak{Re}(\overline{a}\langle x, y \rangle) + a\overline{a}\|y\|^2 \geq \|x\|^2.$$

Taking square root, we get the required result.

Now, if $\|x + ay\| \geq \|x\|$, we have

$$\|x + ay\|^2 = \|x\|^2 + 2\mathfrak{Re}(\overline{a}\langle x, y \rangle) + a\overline{a}\|y\|^2 \geq \|x\|^2 \implies 2\mathfrak{Re}(\overline{a}\langle x, y \rangle) + |a|^2\|y\|^2 \geq 0.$$

If $\langle x, y \rangle = k \neq 0$, we can choose $a = -k$ to get $-2k^2 + k^2\|y\|^2 \geq 0$. This gives rise to a contradiction when $\|y\| < \sqrt{2}$. Hence $\langle x, y \rangle = 0$.

7. **Kreyszig p.141 / Problem 10. (Zero operator)** Let $T : X \to X$ be a bounded linear operator on a complex inner product space X. If $\langle Tx, x \rangle = 0$ for all $x \in X$, show that $T = 0$.

Show that this does not hold in the case of a real inner product space. Hint: Consider a rotation of the Euclidean plane.

Let $x = \alpha p + q$ for some $p, q \in X$ and a scalar α, then we get

$$\langle Tx, x\rangle = \langle \alpha Tp + Tq, \alpha p + q\rangle = \alpha\overline{\alpha}\,\langle Tp, p\rangle + \alpha\,\langle Tp, q\rangle + \overline{\alpha}\,\langle Tq, p\rangle + \langle Tq, q\rangle = \alpha\,\langle Tp, q\rangle + \overline{\alpha}\,\langle Tq, p\rangle\,.$$

Taking $\alpha = 1$, we get $\langle Tp, q\rangle + \langle Tq, p\rangle = 0$ and taking $\alpha = i$, we get $\langle Tp, q\rangle - \langle Tq, p\rangle = 0$. Solving these two equations we get $\langle Tp, q\rangle = 0$ and $\langle Tq, p\rangle = 0$. Since these are true for all p and q we take $q = Tp$ to get $\|Tp\|^2 = \langle Tp, q\rangle = 0$. Thus T must equal 0.

If we take the linear operator on the real inner product space $\mathbb{R}^2$ defined by the matrix

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

we see that $\langle T(x, y), (x, y)\rangle = \langle(-y, x), (x, y)\rangle = -yx + xy = 0$. But $T \neq 0$.

8. **Kreyszig p.150 / Problem 7.** Let A and $B \supset A$ be nonempty subsets of an inner product space X. Show that

   (a) $A \subset A^{\perp\perp}$

   (b) $B^{\perp} \subset A^{\perp}$

   (c) $A^{\perp\perp\perp} = A^{\perp}$

(a) If $x \in A$, then $\langle x, y\rangle = 0$ for all $y \in A^{\perp}$ and $x \perp A^{\perp}$. So $x \in A^{\perp\perp}$.
(b) If $x \in B^{\perp}$, $x \perp B$. Since $B \supset A$, $x \in A^{\perp}$. Thus $B^{\perp} \subset A^{\perp}$.
(c) From (a), $A^{\perp} \subset A^{\perp\perp\perp}$. Also from (a) and (b), $A^{\perp\perp} \supset A \implies A^{\perp} \supset A^{\perp\perp\perp}$. Thus, $A^{\perp\perp\perp} = A^{\perp}$.

9. **Kreyszig p.150 / Problem 8.** Show that the annihilator $M^{\perp}$ of a set $M \neq \emptyset$ in an inner product space X is a closed subspace of X.

Let $\{x_n\}$ be a sequence in $M^{\perp}$ which converges to some $x \in X$. Then for all n and all $y \in M$, $\langle x_n, y\rangle = 0$. By the continuity of inner product, we have,

$$0 = \lim_{n\to\infty} \langle x_n, y\rangle = \langle x, y\rangle\,.$$

Thus $x \in M^{\perp}$ and $M^{\perp}$ is a closed subspace.

10. **Kreyszig p.150 / Problem 10.** If $M \neq \emptyset$ is any subset of a Hilbert space H, show that $M^{\perp\perp}$ is the smallest closed subspace of H which contains M, that is, $M^{\perp\perp}$ is contained in any closed subspace $Y \subset H$ such that $Y \supset M$.

Let $Y \supset M$ be a closed subspace of H. Then $Y = Y^{\perp\perp}$. By **Kreyszig p.150 / Problem 7**, $Y^{\perp} \subset M^{\perp}$ and $Y^{\perp\perp} \supset M^{\perp\perp}$. Thus, $M^{\perp\perp}$ is contained in the closed subspace Y which contained M.

11. Let Y be a closed subspace of a Hilbert space H. Show that the projection operator $P : H \to Y$ is a bounded linear operator.

The projection operator P maps a point $x \in H = Y \bigoplus Y^{\perp}$ to the point $y \in Y$ such that $x = y + z$ for some $z \in Y^{\perp}$. We note that this representation is unique and hence the map P is well-defined.

If $p = y_1 + z_1$ and $p_2 = y_2 + z_2$ are two points in H written as the sum in $Y \bigoplus Y^{\perp}$ then $\alpha p_1 + p_2 = \alpha y_1 + y_2 + z_1 + z_2$. Since $z_1 + z_2 \in Y^{\perp}$ and $\alpha y_1 + y_2 \in Y$ we see that $P(\alpha p_1 + p_2) = \alpha y_1 + y_2 = \alpha P(p_1) + P(p_2)$. Hence, P is linear.

To show that P is bounded we first note that if $x = y + z$ as above, then

$$\|x\|^2 = \langle y + z, y + z \rangle = \|y\|^2 + \|z\|^2.$$

Then $\|P(x)\|^2 = \|x\|^2 - \|z\|^2 \leq \|x\|^2$. Hence, P is bounded.

12. For any subset $M \neq \emptyset$ of a Hilbert space H, Span(M) is dense in H if and only if $M^{\perp} = \{0\}$.

Assume that Span(M) is dense in H and let $x \in M^{\perp}$. There exists a sequence (x_n) in Span(M) such that $x_n \longrightarrow x$. We have $\langle x_n, x \rangle = 0$ for all n and by continuity of inner product, $\lim_{n\to\infty} \langle x_n, x \rangle = \langle x, x \rangle = 0 \implies x = 0$. Hence $M^{\perp} = \{0\}$.

Now, if $M^{\perp} = \{0\}$, then since $M \oplus M^{\perp} = H$, the subspace Span(M) is dense in H.

13. **Kreyszig p.194 / Problem 6.** Show that Theorem 3.8-1 defines an isometric bijection $T : H \to H'$, $z \mapsto f_z = \langle \cdot, z \rangle$ which is not linear but conjugate linear, that is, $\alpha z + \beta v \mapsto \overline{\alpha} f_z + \overline{\beta} f_v$.

We see that $T(\alpha z + \beta v)(x) = \langle x, \alpha z + \beta v \rangle = \langle x, \alpha z \rangle + \langle x, \beta v \rangle = \overline{\alpha} \langle x, z \rangle + \overline{\beta} \langle x, v \rangle = \overline{\alpha} T(z)(x) + \overline{\beta} T(v)(x)$. Thus $\alpha z + \beta v \mapsto \overline{\alpha} f_z + \overline{\beta} f_v$ and the map is conjugate linear and isometric. The fact that T is a bijection comes from the statement of Theorem 3.8-1.

14. **Kreyszig p.194 / Problem 7.** Show that the dual space H' of a Hilbert space H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_1$ defined by

$$\langle f_z, f_v \rangle_1 = \overline{\langle z, v \rangle} = \langle v, z \rangle,$$

where $f_z(x) = \langle x, z \rangle$, etc.

By Riesz's Theorem, we know that every linear bounded functional on H can be represented as f_z for some $z \in H$. Also, f_x is a linear and bounded functional for all $x \in H$. Here, $f_{x+y}(p) = \langle p, x + y \rangle = f_x(p) + f_y(p)$ and $f_{\alpha x}(p) = \langle p, \alpha x \rangle = \overline{\alpha} f_x(p)$. Now, we show that the given inner product satisfies the axioms:

(a) $\langle f_x + f_y, f_z \rangle_1 = \langle z, x + y \rangle = \langle z, x \rangle + \langle z, y \rangle = \langle f_x, f_z \rangle_1 + \langle f_y, f_z \rangle_1$,
(b) $\langle \alpha f_x, f_y \rangle_1 = \langle f_{\overline{\alpha} x}, f_y \rangle = \langle y, \overline{\alpha} x \rangle = \alpha \langle f_x, f_y \rangle_1$,
(c) $\langle f_x, f_y \rangle_1 = \langle y, x \rangle = \overline{\langle x, y \rangle} = \overline{\langle f_y, f_x \rangle_1}$,
(d) $\langle f_x, f_x \rangle_1 = \langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$ so $\langle f_x, f_x \rangle = 0 \iff f_x = 0$.

Since $x + y \mapsto f_x + f_y$, we also see that the map is isometric.