Analysis I

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September 20, 2022

Homework 1

1. Choose either d_1 or d_2 below and show that it is a metric on $\mathbb{R}^n$.

$$d_1(x,y) = max\{|x_i - y_i|\} \text{ and } d_2(x,y) = \sum_{i=1}^{n} |x_i - y_i| \text{ (taxicab metric).}$$

Solution:

i. Since d_2 is a finite sum of positive numbers, $0 \leq d_2(x,y) < \infty$.

ii. $d_2(x,y) = d_2(y,x)$ since $|x_i - y_i| = |y_i - x_i|$ for all i.

iii. $d_2(x,x) = 0$ since it is the sum of zeros.

iv. Since we have $|x_i - z_i| = |(x_i - y_i) + (y_i - z_i)| \leq |x_i - y_i| + |y_i - z_i|$ (using triangle inequality for each $x_i, y_i, z_i \in \mathbf{R}$), we get

$$d_2(x,z) = \sum_{i=1}^{n} |x_i - z_i| \leq \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) = d_2(x,y) + d_2(y,z).$$

Hence, d_2 also satisfies the triangle inequality for $x, y, z \in \mathbb{R}^n$.

Thus, d_2 is a metric in $\mathbb{R}^n$. □

2. Let $C[0,1]$ be the space of continuous functions on $[0,1]$.
Show that $d(f,g) = \displaystyle\int_0^1 |f(x) - g(x)| dx$ is a metric on $C[0,1]$

Solution: For all $f \in C[0,1]$, f is bounded. Let $|f(x)| < M$ and $|g(x)| < N$ for $x \in [0,1]$.

i. Then $0 \leq \int_0^1 |f(x) - g(x)| dx \leq \int_0^1 |f(x)| + |g(x)| dx \leq (M+N)(1-0) < \infty$. Hence $0 \leq d(f,g) < \infty$.

ii. $d(f,g) = d(g,f)$ since $|f(x) - g(x)| = |g(x) - f(x)|$ for all $x \in [0,1]$.

iii. $d(f,f) = 0$ since it is the integration of zero function.

iv. For each $x \in [0,1]$ and $f, g, h \in C[0,1]$, we have $|f(x) - h(x)| = |(f(x) - g(x)) + (g(x) - h(x))| \leq |f(x) - g(x)| + |g(x) - h(x)|$ (using triangle inequality for real numbers). Then

$$d(f,h) = \int_0^1 |f(x) - h(x)| dx \leq \int_0^1 (|f(x) - g(x)| + |g(x) - h(x)|)\, dx = d(f,g) + d(g,h).$$

Hence, d satisfies the triangle inequality for $f, g, h \in C[0,1]$.

Thus, d is a metric on $C[0,1]$. $\square$

3. Let $x = \{x_n\}_1^\infty$ be a sequence.

   a. True or False: If $x \in l^p$ for some $1 \le p < \infty$, then $x_n \to 0$ as $n \to \infty$. Justify your answer.

   Solution: True. If $x \in l^p$ for some $1 \le p < \infty$, then $\left(\sum_1^\infty |x_i|^p\right)^{1/p} < \infty$. Hence $\sum_i^\infty |x_i|^p$ is a convergent series and $|x_i|^p \to 0$ as $i \to \infty$ which implies that $x_i \to 0$ as $i \to \infty$.

   b. True or False: If $x_n \to 0$ as $n \to \infty$, then $x_n \in l^p$, for some $1 \le p < \infty$. Justify your answer.

   Solution: False. The sequence given by $x = \{x_i = \frac{1}{\log(i+1)}\}_1^\infty \to 0$ as $i \to \infty$ but the sum $\sum_2^\infty |x_i|^p$ does not converge for any $1 \le p < \infty$. So, $x \notin l^p$ for any $1 \le p < \infty$.

4. Let $a, b \ge 0$, and $p \ge 1$. Prove that

$$(a+b)^p \le 2^{p-1}(a^p + b^p).$$

Solution: Let $f(x) = x^p$, $f : [0,\infty) \to \mathbf{R}$ and $p \ge 1$. Since f is a *convex* function, we have

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y) \quad \text{for } \alpha \in [0,1].$$

Taking $\alpha = 1/2$, we get

$$\begin{aligned} & f\left(\frac{a}{2} + \frac{b}{2}\right) \le \frac{f(a)}{2} + \frac{f(b)}{2} \\ \text{or,} \quad & \frac{1}{2^p} f(a+b) \le \frac{1}{2}\left(f(a) + f(b)\right) \\ \text{or,} \quad & (a+b)^p \le 2^{p-1}\left(a^p + b^p\right). \quad \square \end{aligned}$$

5. For $p > 1$, let q be its conjugate, i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Prove the **Young's Inequality**:

$$u \cdot v \le \frac{1}{p} u^p + \frac{1}{q} v^q, \quad \forall u, v \ge 0$$

Solution: If either u or v equals 0, then the inequality follows immediately. Suppose $u > 0$, $v > 0$ and let $f(x) = e^x$. Since f is a *convex* function,

$$\begin{aligned} u \cdot v &= \exp\left(\log u + \log v\right) \\ &= f\left(\frac{1}{p} \log u^p + \frac{1}{q} \log v^q\right) \\ &\le \frac{1}{p} f(\log u^p) + \frac{1}{q} f(\log v^q) \\ &= \frac{u^p}{p} + \frac{v^q}{q}. \quad \square \end{aligned}$$

6. Prove Holder's Inequality for Sums.

Solution:

Holder's inequality: Let $p, q \geq 1$ be conjugate exponents. Let $x = \{x_i\}_1^\infty \in l^p$ and $y = \{y_i\}_1^\infty \in l^q$. Then

a. $xy = \{x_i y_i\}_1^\infty \in l^1$ and

b. $\sum_1^\infty |x_i y_i| \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} \cdot (\sum_1^\infty |y_i|^q)^{\frac{1}{q}}$.

Let $u_i = \frac{x_i}{(\sum_1^\infty |x_i|^p)^{1/p}}$ and $v_i = \frac{y_i}{(\sum_1^\infty |y_i|^q)^{1/q}}$. Then by Young's inequality,

$$\begin{aligned} u_i \cdot v_i &= \frac{x_i}{(\sum_1^\infty |x_i|^p)^{1/p}} \cdot \frac{y_i}{(\sum_1^\infty |y_i|^q)^{1/q}} \\ &\leq \frac{x_i^p}{p \sum_1^\infty |x_i|^p} + \frac{y_i^q}{q \sum_1^\infty |y_i|^q} \end{aligned}$$

Let $m = (\sum_1^\infty |x_i|^p)^{1/p}$ and $n = (\sum_1^\infty |y_i|^q)^{1/q}$. Then from above we have

$$\begin{aligned} \sum_1^\infty |x_i y_i| = mn \sum_1^\infty |u_i v_i| \leq mn \sum_1^\infty \left| \frac{1}{pm^p} x_i^p + \frac{1}{qn^q} \cdot y_i^q \right| &\leq mn \left(\frac{1}{pm^p} \cdot \sum_1^\infty |x_i^p| + \frac{1}{qn^q} \sum_1^\infty |y_i^q| \right) \\ &= mn \left(\frac{1}{pm^p} \cdot m^p + \frac{1}{qn^q} \cdot n^q \right) = mn \end{aligned}$$

Hence $\sum_1^\infty |x_i y_i| \leq mn = (\sum_1^\infty |x_i|^p)^{1/p} \cdot (\sum_1^\infty |y_i|^q)^{1/q}$ which proves (b). Since $0 \leq \sum_1^\infty |x_i y_i| < \infty$, we also have (a) by definition. $\square$

7. Prove Minkowski's Inequality for Sums.

Solution:

Minkowski's inequality : Let $p \geq 1$ and $x = \{x_i\}_1^\infty \in l^p$ and $y = \{y_i\}_1^\infty \in l^p$. Then

a. $x + y = \{x_i + y_i\}_1^\infty \in l^p$ and

b. $(\sum_1^\infty |x_i + y_i|^p)^{\frac{1}{p}} \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} + (\sum_1^\infty |y_i|^p)^{\frac{1}{p}}$.

First we show that $x + y \in l^p$ by showing that

$$\left(\sum_1^\infty |x_i + y_i|^p \right)^{1/p} < \infty$$

We have,

$$\sum_1^\infty |x_i + y_i|^p \leq \sum_i^\infty (|x_i| + |y_i|)^p \leq 2^{p-1} \left(\sum_i^\infty |x_i|^p + \sum_i^\infty |y_i|^p \right) < \infty.$$

Now, since $x, y \in l^p$, $d_p(x, y) < \infty$. If $p = 1$ then the Minkowski inequality follows from the triangle inequality of real numbers. Let $p > 1$ then

$$\begin{aligned} \sum_1^\infty |x_i + y_i|^p = \sum_1^\infty |x_i + y_1||x_i + y_i|^{p-1} &\leq \sum_1^\infty \left(|x_i||x_i + y_i|^{p-1} + |y_i||x_i + y_i|^{p-1} \right) && (1) \\ &= \sum_1^\infty \left(|x_i||x_i + y_i|^{p-1} \right) + \sum_1^\infty \left(|y_i||x_i + y_i|^{p-1} \right) && (2) \end{aligned}$$

Now let q be the conjugate exponent of p, then we have $\frac{1}{p} + \frac{1}{q} = 1 \iff p + q = pq \iff p = p(q-1)$. Then, at line 2

$$\left(\sum_{1}^{\infty} |x_i + y_i|^{(p-1)q}\right)^{1/q} = \left(\sum_{1}^{\infty} |x_i + y_i|^{p}\right)^{1/q} < \infty$$

which shows that $\{|x_i + y_i|^{p-1}\}_i^{\infty} \in l^q$. Then by Holder's inequality,

$$\begin{aligned} \sum_{1}^{\infty} |x_i||x_i + y_i|^{p-1} &\leq \left(\sum_{1}^{\infty} |x_i|^p\right)^{1/p} \cdot \left(\sum_{1}^{\infty} |x_i + y_i|^{(p-1)q}\right)^{1/q} && (3) \\ &= \left(\sum_{1}^{\infty} |x_i|^p\right)^{1/p} \cdot \left(\sum_{1}^{\infty} |x_i + y_i|^{p}\right)^{1/q} && (4) \end{aligned}$$

Using the results from line 2 and line 4 on line 1,

$$\begin{aligned} \sum_{1}^{\infty} |x_i + y_i|^p &\leq \sum_{1}^{\infty} \left(|x_i||x_i + y_i|^{p-1}\right) + \sum_{1}^{\infty} \left(|y_i||x_i + y_i|^{p-1}\right) && (5) \\ \text{or,} \quad \sum_{1}^{\infty} |x_i + y_i|^p &\leq \left(\sum_{1}^{\infty} |x_i + y_i|^{p}\right)^{1/q} \cdot \left(\left(\sum_{1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{1}^{\infty} |y_i|^p\right)^{1/p}\right) && (6) \end{aligned}$$

Dividing both sides by $\left(\sum_1^{\infty} |x_i + y_i|^p\right)^{1/q}$, we get the Minkowski's inequality (since $1 - \frac{1}{q} = \frac{1}{p}$).

□

8. For $1 \leq p < \infty$, let $l^p = \{x = \{x_i\}_1^{\infty} \mid \sum_1^{\infty} |x_i|^p < \infty\}$. For any $x, y \in l^p$, define

$$d_p(x, y) = \Big(\sum_{1}^{\infty} |x_i - y_i|^p\Big)^{1/p}$$

Prove that (l^p, d_p) is a metric space.

Solution:

i. Since $d_p(x, y)$ is the *pth* root of a sum of positive numbers, $d_p \geq 0$. Also from Minkowski inequality (a.), we have $d_p < \infty$.

ii. $d_p(x, y) = d_p(y, x)$ since $|x_i - y_i| = |y_i - x_i|$ for all i.

iii. $d_p(x, x) = 0$ since $|x_i - x_i| = 0$ for all i.

iv. The triangle inequality for d_p follows from the Minkowski inequality (b.)

$$\begin{gathered} \left(\sum_{1}^{\infty} |x_i - z_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{1}^{\infty} |x_i - y_i|^p\right)^{\frac{1}{p}} + \left(\sum_{1}^{\infty} |y_i - z_i|^p\right)^{\frac{1}{p}} \\ \text{or,} \quad d_p(x, z) \leq d_p(x, y) + d_p(y, z) \end{gathered}$$

□

9. Prove Jensen's Inequality for Sums.

Solution:

$$\left(\sum_{i=1}^{\infty}|x_i|^{p_2}\right)^{1/p_2} \leq \left(\sum_{i=1}^{\infty}|x_i|^{p_1}\right)^{1/p_1} \qquad \forall 1 \leq p_1 < p_2 < \infty$$

Let $|y_i| = |x_i|^{p_1}$. Then we need to show that $\left(\sum_{i=1}^{\infty}|y_i|^{p_2/p_1}\right)^{p_1/p_2} \leq \sum_{i=1}^{\infty}|y_i|$. First we show that this is true for a finite sequence $\{x_i\}_1^n$ using induction on n. Then we take the limit as $n \to \infty$ to prove Jensen's inequality.

When $n = 1$, $(|y_1|^{p_2/p_1})^{p_1/p_2} = y_1$. (True)

Let $H(k) : \left(\sum_{i=1}^{k}|y_i|^{p_2/p_1}\right)^{p_1/p_2} \leq \sum_{i=1}^{k}|y_i|$ be true for some integer $k > 1$. Then

$$\begin{aligned}
\left(\sum_{i=1}^{k+1}|y_i|^{p_2/p_1}\right)^{p_1/p_2} &= \left(\sum_{i=1}^{k}|y_i|^{p_2/p_1} + |y_{k+1}|^{p_2/p_1}\right)^{p_1/p_2} \\
&\leq \left(\sum_{i=1}^{k}|y_i|^{p_2/p_1}\right)^{p_1/p_2} + \left(|y_{k+1}|^{p_2/p_1}\right)^{p_1/p_2} && \text{[by Minkowski inequality]} \\
&\leq \sum_{i=1}^{k}|y_i| + |y_{k+1}| && \text{[by induction hypothesis]} \\
&= \sum_{i=1}^{k+1}|y_i|
\end{aligned}$$

Hence $H(k) \implies H(k+1)$ which proves that $H(n)$ is true for all $n \in \mathbb{Z}$. Taking the limit as $n \to \infty$ we get the required Jensen's inequality. □

10. Show that $l^1 \subset l^2$ without using Jensen's inequality. Then show that inclusion is strict, i.e., find an element in l^2 that is not in l^1.

Solution: Let $x \in l^1$ then $0 \leq \sum_1^{\infty}|x_1| < \infty$ which implies that the sequence x converges to 0. Let $N \in \mathbb{Z}$ such that $x_i < 1$ for all $i > N$. Then for $i > N$, we have $|x_i|^2 < |x_i|$. Hence,

$$0 \leq \sum_{i=1}^{\infty}|x_i|^2 \leq \sum_{i=1}^{N}|x_i|^2 + \sum_{i=N+1}^{\infty}|x_i|^2 < \sum_{i=1}^{N}|x_i|^2 + \sum_{i=N+1}^{\infty}|x_i| < \infty.$$

So, $l^1 \subset l^2$.

The harmonic series given by the sequence $x = \{x_i = \frac{1}{i}\}_1^{\infty}$ does not converge. However

$$\sum_{i=1}^{\infty}\frac{1}{i^2} = \frac{\pi^2}{6} < \infty.$$

Here we see that $x \notin l^1$ but $x \in l^2$. Hence, the inclusion is strict. □

Homework 2

1. Let (X, ρ) and (Y, σ) be metric spaces, and let $f : (X, \rho) \to (Y, \sigma)$ be a map such that $f^{-1}(V)$ is open in X, for all V open in Y. Show that f is continuous on X.

Solution: Let V be the open set around the image $f(x)$ of a point $x \in X$. Then there exists an open set $f^{-1}(V) \in X$ such that $x \in f^{-1}(V)$. Let V be the open ball $B_{x,\varepsilon}(f(x))$ around the point $f(x)$ for some given $\varepsilon > 0$. Then $f^{-1}(B_{x,\varepsilon}(f(x)))$ is also open in X. Since x is in the open set $f^{-1}(B_{x,\varepsilon}(f(x)))$, we can find a $\delta > 0$ such that $x \in B_\delta(x) \subset f^{-1}(B_{x,\varepsilon}(f(x)))$. This means that for all $z \in X$ such that $d(z,x) < \delta$ we have $d(f(z), f(x)) < \varepsilon$. Hence f is continuous a4 x. Since we can do this at every point $x \in X$, we see that f is continuous on X.

2. (**Continuous mapping**) Show that a mapping $T : X \to Y$ is continuous if and only if the inverse image of any closed set $M \subset Y$ is a closed set in X.

Solution: We first note that for any set $A \subset Y$, $f^{-1}(A) = \{x \in X : \ f(x) \in A\}$ and $f^{-1}(Y \backslash A) = \{x \in X : f(x) \notin A\}$. Then clearly, $f^{-1}(A)$ and $f^{-1}(Y \backslash A)$ are disjoint sets of X. Particularly,

$$f^{-1}(Y \backslash A) = X \backslash f^{-1}(A).$$

($\Longleftarrow$) Let V be a closed set of Y. Then there exists a closed set $f^{-1}(V)$ in X. Since all open sets can be written as a complement of closed set, we see that for any open set $Y \backslash V$ in Y there exists an open set $X \backslash f^{-1}(V)$ in X. Since $X \backslash f^{-1}(V) = f^{-1}(Y \backslash A)$, the preimage of any open set of Y is open in X. Hence f is continuous.

($\Longrightarrow$)

3. Assume that $f : (\mathbb{R}^2, d_1 = \text{ Euclidean metric}) \to \mathbb{R}$ is continuous at $x \in \mathbb{R}^2$. Show that $f : (\mathbb{R}^2, d_2 = \text{ taxicab metric}) \to \mathbb{R}$ is also continuous at x.

Solution: Since $f : (\mathbb{R}^2, d_1 = \text{ Euclidean metric}) \to \mathbb{R}$ is continuous, we know that $f^{-1}(V)$ is open in $\mathbb{R}^2$, for all V open in $\mathbb{R}$. Our proof will be complete if we can show that every open set U in $(\mathbb{R}^2, d_1)$ is also open with respect to the taxicab metric.

Let V be open in $(\mathbb{R}^2, d_1)$. Then for every point $(p,q) \in X$ there exists an open ball $B_\delta((p,q)) \subset V$ containing (p,q).

$$B_\delta((p,q)) = \{(x,y) \in \mathbb{R}^2 : \ (x-p)^2 + (y-q)^2 < \delta^2\}$$

4. Show that the discrete metric space (X,d) is separable iff X is countable.

Solution: If X is countable, then X is the countable dense subset in X. So, it is separable.

Now, we show that if (X,d) is separable, then it is countable. Let $Y \subset X$ be the countable dense subset of X. Then every open set of X must contain a point from Y. We note that all the singleton sets of X are open in X and so Y must contain all the points from the singleton sets. Hence we get $X \subset Y \implies X = Y$. So, X is countable.

5. Show that l^p, with $1 \leq p < \infty$ is separable.

Solution: We will show that the set M of sequences in l^p with finite non-zero rational terms is the countable dense subset of l^p with $1 \leq p < \infty$. That is M has all the sequences x of the form

$$(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$$

with each $x_i \in \mathbb{Q}$. First, we show that M is countable. Note that for a fixed n, the set of sequences in l^p with rational terms and all but the first n terms zero is countable. Then M is the countable union of countable sets and, hence, is countable.

Now, we show that the set M is dense in l^p with $1 \le p < \infty$. Let $x = \{x_i\}_1^\infty \in l^p$. Then $\sum_1^\infty |x_i|^p$ is convergent and so for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that,

$$\sum_{N+1}^{\infty} |x_i|^p < \varepsilon^p/2.$$

Then, for the first N terms, since rational numbers are dense in $\mathbb{R}$, we can choose rational numbers y_i such that $|x_i - y_i|^p < \varepsilon^p/2N$. So $(y_1, \dots, y_N, 0, 0, \dots)$ is a point in M and we see that

$$d(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} = \left(\sum_{i=1}^{N} |x_i - y_i|^p + \sum_{i=N+1}^{\infty} |x_i - y_i|^p\right)^{1/p} < (\varepsilon^p/2 + \varepsilon^p/2)^{1/p} = \varepsilon$$

So, for every $\varepsilon > 0$, we can find a point of M in the ε-neighborhood of every point $x \in l^p$. Hence l^p with $1 \le p < \infty$ is separable.

6. Show that l^∞ is not separable.

Solution: Let $x = \{x_i\}_{i=0}^\infty$ be the sequence of zeros and ones. Then since the sequence is bounded, it is in l^∞. Now we will show that the set of such sequences x are uncountably many and have disjoint open neighborhoods for some radius.

For each sequence x we associate a real number y whose binary representation is

$$\sum_{1}^{\infty} \frac{x_i}{2^i}.$$

Then for each $y \in [0,1]$, there exists a unique sequence of zeros and ones as each y has a unique binary representation. Since there are uncountably many y, the sequences associated with them are also uncountable. The metric on l^∞ given by

$$d(x,y) = \sup_{i \in \mathbb{N}} |x_i - y_i|$$

implies that any two distict binary sequences $\{x_i\}_{i=0}^\infty$ must be 1 distance apart. Then we can take $r = 1/2$ to get the disjoint neighborhoods in l^∞ associated with each sequence $\{x_i\}_{i=0}^\infty$.

Now, if M is any dense set in l^∞, then every open set in l^∞ must contain a point of M. So, for each disjoint open neighborhood constructed above, M contains a point in the neighborhood. Hence, any dense set M is uncountable in l^∞ and l^∞ is not separable.

7. Let $\{x_n\}_{n\ge1}$ be a sequence in a m.s. (X,d) which converges to x. Show that $\{x_n\}_{n\ge1}$ is a bounded sequence. Then let $\{y_n\}_{n\ge1}$ be a sequence in (X,d) which converges to y. Show that $\lim_{n\to\infty} d(x_n, y_n) = d(x,y)$.

Solution: Since the sequence $\{x_n\}_{n\ge1}$ is converges to x, for every $\varepsilon > 0$, we can find a N such that $d(x_n, x) < \varepsilon$ for all $n \ge N$. If we take $\varepsilon = 1$, we see $x_n \le |x| + 1$ for all $n \ge N$. Let $M = \max\{x_1, \dots, x_{N-1}\}$. Then

$$x_n \le \max\{M, |x|+1\} \quad \text{for all } n \in \mathbb{Z}.$$

Hence, $\{x_n\}_{n\ge1}$ is bounded.

If $\{x_n\}_{n\ge1}$ converges to x and $\{y_n\}_{n\ge1}$ converges to y, then for every $\varepsilon > 0$, there exists N_1 and N_2 such that

$d(x, x_n) < \varepsilon/2$ for $n > N_1$ and $d(y, y_n) < \varepsilon/2$ for $n > N_2$. Then for $n > \max\{N_1, N_2\}$,

$$d(x_n, y_n) \leq d(x, x_n) + d(x, y_n) \leq d(x, x_n) + d(y, y_n) + d(x, y) \leq \varepsilon + d(x, y).$$

As $n \to \infty$, $d(x_n, y_n) \to d(x, y)$.

8. Show that any nonempty set $A \subset (X, d)$ is open if and only if it is a union of open balls.

Solution: If A is open in X then for all $x \in A$ we exists an open ball $B_{\delta_x}(x)$ such that $x \in B_{\delta_x}(x) \subset A$. Let

$$B = \bigcup_{x \in A} B_{\delta_x}(x).$$

Clearly, $B \subset A$ since each $B_{\delta_x}(x)$ is contained in A. Also, since B contains all the points of A, we have $A = B$. So, A is a union of open balls.

Now, if $A = \bigcup B_\alpha$ is a union of open balls B_α then for each $x \in A$, we have $x \in B_\alpha$ for some B_α. That means that for every $x \in A$ we have some open ball B_α such that $x \in B_\alpha \subset A$. Hence A is open in X.

9. Let (X, ρ) be a metric space, $E \subset X$, and $x \in X$. Prove that the following are equivalent:
   (a) $x \in \overline{E}$
   (b) $B(x, r) \cap E \neq \emptyset$, $\forall r > 0$
   (c) $\exists \{x_n\} \in E$ s.t. $x_n \to x$

Solution: (a) $\implies$ (b). Let $x \in \overline{E} = X \backslash (X \backslash E)^\circ$. Then $x \notin (X \backslash E)^\circ$. Negating this statement, we see that for all $r > 0$, $B(x, r) \cap E \neq \emptyset$.

(b) $\implies$ (c). We first define the radii $r_n = 1/n$ of the open balls around the point $x \in X$. Since $B(x, r) \cap E \neq \emptyset$, $\forall r > 0$, we can take a sequences of points $\{x_i\}_1^\infty$ in E such that $x_i \in B(x, r_i)$ for each i. Then this is the sequence in E which converges to the point x.

(c) $\implies$ (a). Since $x_n \to x$, for every $\varepsilon > 0$ we can find a N such that $d(x, x_n) < \varepsilon$ all $n > N$. That is, every open ball around x contains a point of E. So $x \notin (X \backslash E)^\circ \implies x \in X \backslash (X \backslash E)^\circ$. Thus x is in the closure $\overline{E}$.

10. If d_1 and d_2 are metrics on the same set X and there are positive numbers a and b such that for all x, $y \in X$,

$$ad_1(x, y) \leq d_2(x, y) \leq bd_1(x, y),$$

show that the Cauchy sequences in (X, d_1) and (X, d_2) are the same.

Solution: Let $\{x_n\}_1^\infty$ be a Cauchy sequence in (X, d_1), then for every $\varepsilon' > 0$, we can find an integer N such that $d_1(x_m, x_n) < \varepsilon'$ for all m, $n > N$. Then taking $\varepsilon = \varepsilon'/b$ we have $d_2(x_m, x_n) \leq \varepsilon$. Hence the sequence $\{x_n\}_1^\infty$ is Cauchy in (X, d_2).

Now, let $\{y_n\}_1^\infty$ be a Cauchy sequence in (X, d_2), then for every $\varepsilon' > 0$, we can find an integer N such that $d_2(y_m, y_n) < \varepsilon'$ for all m, $n > N$. Then taking $\varepsilon = a\varepsilon'$ we have $ad_1(y_m, y_n) \leq a\varepsilon \implies d_1(y_m, y_n) \leq \varepsilon$. Hence the sequence $\{y_n\}_1^\infty$ is Cauchy in (X, d_1).

11. Show that l^p, with $1 \leq p < \infty$ is complete.

Solution: Let $\{\{x_i^n\}_{i=1}^{\infty}\}_{n=1}^{\infty}$ be a Cauchy sequence in the space l^p where for each n, $\{x_i^n\}_{i=1}^{\infty}$ is a convergent sequence in $\mathbb{R}$. Then for every $\varepsilon > 0$, there exists an N such that

$$d(x_i^m, x_i^n) = \left(\sum_{i=1}^{\infty} |x_i^m - x_i^n|^p \right)^{1/p} < \varepsilon$$

for all m, $n > N$. Then for each fixed i, the term $|x_i^m - x_i^n|^p < \varepsilon^p \implies |x_i^m - x_i^n| < \varepsilon$. So $\{x_i^j\}_1^{\infty}$ is a Cauchy sequence of real numbers for each fixed i and it converges to some number that we can call x_i. We define $x = \{x_i\}_{i=1}^{\infty}$ and show that this is the limit of our sequence $\{\{x_i^n\}_{i=1}^{\infty}\}_{n=1}^{\infty}$ in l^p.

From above, we have

$$\sum_{i=1}^{k} |x_i^m - x_i^n|^p < \varepsilon^p$$

for all m, $n > N$. Then as $n \to \infty$, we have (by definition) for $m > N$

$$\sum_{i=1}^{k} |x_i^m - x_i|^p \leq \varepsilon^p.$$

Then as $k \to \infty$, we have

$$\sum_{i=1}^{\infty} |x_i^m - x_i^n|^p \leq \varepsilon^p$$

12. Prove that $(\mathbb{R}, d(x, y) = |x - y|)$ is complete.

Solution: First, we show that every Cauchy sequence in $(\mathbb{R}, d)$ is bounded. Let $\{x_n\}_1^{\infty}$ be a Cauchy sequence in $(\mathbb{R}, d)$, then for every $\varepsilon > 0$, we can find an integer N such that $d(x_m, x_n) = |x_m - x_n| < \varepsilon$ for all m, $n > N$. Then when $m = N + 1$ and $\varepsilon = 1$, we see that $|x_m| - |x_{N+1}| \leq |x_m - x_{N+1}| < \varepsilon$. So $|x_m| \leq |x_{N+1}| + 1$ for all $m > N$. Hence the sequence $\{x_n\}_1^{\infty}$ is bounded by M where $M = \max\{|x_1|, \ldots, |x_N|, |x_{N+1}| + 1\}$. Then, since $\{x_n\}_1^{\infty}$ is bounded, by Bolzano-Weierstrass Theorem, we know that $\{x_n\}_1^{\infty}$ has a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$. Let $\{x_{n_i}\}_{i=1}^{\infty} \to x$ in $\mathbb{R}$.

Now, we show that our sequence $\{x_n\}_1^{\infty}$ itself converges to the point $x \in \mathbb{R}$. In the Cauchy sequence $\{x_n\}_1^{\infty}$, we have, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon/2$ for all m, $n > N$. Similarly, in the convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$, there exists $N' \in \mathbb{N}$ such that $|x_{n_i} - x| < \varepsilon/2$ for all $i > N'$. Then for all $\varepsilon > 0$, there exists $N, N' \in \mathbb{N}$ such that, for all $n > N$ and $i > N'$ with $n_i > N$, we have

$$|x_n - x| \leq |x_n - x_{n_i}| + |x_{n_i} - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $\{x_n\}_{n=1}^{\infty}$ converges in $(\mathbb{R}, d)$ and $\mathbb{R}$ is complete.

13. Prove that $(\mathbb{Q}, d(x, y) = |x - y|)$ is incomplete.

Solution: We take the sequence $\{x_n\}_1^{\infty}$ of rational numbers given by $x_n = \left(1 + \frac{1}{n}\right)^n$. Note that this is a convergent sequence in $\mathbb{R}$ with the same metric and hence it is Cauchy in $\mathbb{Q}$ too. However, it does not converge to any number in $\mathbb{Q}$. So, $\mathbb{Q}$ is not complete.

14. Prove that $\left(\mathrm{C}[-1, 1], d(f, g) = \int_{-1}^{1} |f(t) - g(t)|\ dt\right)$ is incomplete.

Solution: We take the sequence $\{f_n\}_1^\infty$ of piecewise defined functions in C$[-1,1]$ such that

$$f_n(x) = \begin{cases} 1, & x > \dfrac{1}{n} \\ nx, & -\dfrac{1}{n} \leq x \leq \dfrac{1}{n} \\ -1, & x < \dfrac{1}{n} \end{cases}$$

We first observe that $d(f_m, f_n) = 2\times$ area of the triangle with base $\left|\frac{1}{m} - \frac{1}{n}\right|$ and height 1 $=|1/m - 1/n|$. Then, the sequence is Cauchy since for every $\varepsilon > 0$ we have an integer $N > 1/\varepsilon$ such that for all $m, n > N$, $d(f_m, f_n) < \varepsilon$.

Now, for every $g \in C[0,1]$, we have

$$d(f_n, g) = \int_{-1}^{1} |f_n(t) - g(t)|dt = \int_{-1}^{-\frac{1}{n}} |-1 - g(t)|dt + \int_{-\frac{1}{n}}^{\frac{1}{n}} |f_n(t) - g(t)|dt + \int_{\frac{1}{n}}^{1} |1 - g(t)|dt$$

Since $d \geq 0$ for all $g \in C[0,1]$, $d(f_n, g) \to 0$ implies that each integral on the right should also approach 0. Then we should have

$$g(t) = -1 \quad \text{for } t \in [-1, 0) \quad \text{and} \quad g(t) = 1 \quad \text{for } t \in (0, 1].$$

But then g cannot be continuous so we have a contradiction. So, the Cauchy sequence $\{f_n\}_1^\infty$ does not converge in $C[0,1]$ and the space is not complete.

15. Determine whether or not the discrete metric space is complete. Justify your answer.

Solution: Let $\{x_i\}_{i=0}^\infty$ be a Cauchy sequence in the discrete metric space. Then for every $\varepsilon > 0$ there exists an N such that for all $m, n > N$, we have

$$d(x_m, x_n) < \varepsilon.$$

If we take $\varepsilon < 1$ then we see that $x_m = x_n = x$ for all $m, n > N$ for some N. So the sequence $\{x_i\}_{i=0}^\infty$ converges to x and the space is complete.

16. Prove the Completion of a Metric Space Theorem.

Solution:

Theorem 1 (Completion of a Metric Space)**.** *For a metric space $X = (X, d)$ there exists a complete metric space $X' = (X', d')$ which has a subspace W that is isometric with X and is dense in X'. This space X' is unique except for isometries, that is, if $\tilde{X}$ is any complete metric space having a dense subspace $\tilde{W}$ isometric with X, then X' and $\tilde{X}$ are isometric.*

Proof. We prove the theorem in the following steps.

(a) We first define a relation $\sim$ on Cauchy sequences of X and show that it is a well-defined equivalence relation. For the Cauchy sequences $x = \{x_i\}_1^\infty$, $y = \{y_i\}_1^\infty$ in X, we say that $x \sim y$ if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Clearly, this relation is symmetric and reflexive. For transitivity we see that if $x \sim y$ and $y \sim z$ then

$$0 \leq \lim_{n\to\infty} d(x_n, z_n) \leq \lim_{n\to\infty} d(x_n, z_n) + d(y_n, z_n) = 0.$$

Hence $\sim$ is an equivalence relation on the Cauchy sequences of X. Now, let X' be the set of all equivalence classes of Cauchy sequences on X and define the function $d' : X' \to \mathbb{R}$ by

$$d'(x', y') = \lim_{n\to\infty} d(x_n, y_n)$$

where x' and y' are the equivalence classes of x and y respectively. We show that this limit exists and the function d' is well defined. We have,

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_n, y_m) \\ \text{or,} \quad |d(x_n, y_n) - d(x_m, y_m)| &\leq d(x_n, x_m) + d(y_n, y_m). \end{aligned}$$

Taking limit as m, n go to ∞ on both sides we obtain,

$$\lim_{n\to\infty} |d(x_n, y_n) - d(x_m, y_m)| \leq \lim_{n\to\infty} [d(x_n, x_m) + d(y_n, y_m)] = 0$$

Hence the limit $d'(x', y') = \lim_{n\to\infty} d(x_n, y_n)$ exists.

Now, if $x \sim x'$ and $y \sim y'$, then

$$d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y_n, y'_n)$$

And as before,

$$0 \leq \lim_{n\to\infty} |d(x_n, y_n) - d(x'_n, y'_n)| \leq \lim_{n\to\infty} [d(x_n, x'_n) + (y_n, y'_n)] = 0$$

which implies that $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n)$. Hence d' is a well-defined function on X'.

We now show that d' is a metric on X'. Clearly $0 \leq d' < \infty$ since the limit exists and $d'(x', x') = 0$. Furthermore,

$$d'(x', y') = 0 \implies x \sim y \implies x' \sim y'.$$

And,

$$d'(x', z') = \lim_{n\to\infty} d(x_n, z_n) \leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) = d'(x', y') + d'(y', z').$$

So, d' satisfies the definition of a metric.

(b) Now, we construct an isometry $T : X \to W$ where W is a dense subset of X'. Let T be a function that takes each element to the equivalence class x' in X' of the Cauchy sequence $\{x\}_1^\infty = (x, x, x, \ldots)$ associated with that element. Then T is an isometry since for each x, $y \in X$,

$$d'(T(x), T(y)) = \lim_{n\to\infty} d(x_n, y_n) = d(x, y).$$

We note that isometry is injective map and if $W = T(X)$, then $T : X \to W$ is surjective. So, W and X are isometric.

We need to show that W is dense in X'. Let $x' \in X'$ be the equivalence class of the Cauchy sequence $\{y_i\}_1^\infty$. Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(y_n, y_m) < \varepsilon$ for all m, $n > N$. Let $z = y_{N+1}$. Then, if z' is the image of z under T,

$$d'(y', z') = \lim_{n\to\infty} d(y_n - z) < \varepsilon.$$

Thus we see that every open neighborhood around the point x' in X' contains a point z' of W and so, W is dense in X'.

(c) We now show that X' is a complete metric space. Let $\{x'_i\}_1^\infty$ be a Cauchy sequence in X'. Since W is dense in X', every open neighborhood in X' contains a point of W. So, for each i, there exists $z'_i \in W$ such that

$$d'(x'_i, z'_i) < 1/n$$

Then

$$d'(z'_j, z'_i) \leq d'(z'_j, x'_j) + d'(x'_j, x'_i) + d'(x'_i, z'_i) < 1/j + 1/i + d'(x'_j, x'_i)$$

So, since $\{x'_i\}_1^\infty$ is Cauchy, as i, $j \to \infty$, $d'(z'_j, z'_i) \to 0$. So the sequence $\{z'_i\}_1^\infty$ is Cauchy in X'. Since T is an isometry, we see that the sequence $T^{-1}(\{z'_i\}_1^\infty) = (T^{-1}(z'_1), T^{-1}(z'_2), T^{-1}(z'_3), \ldots) = (z_1, z_2, z_3, \ldots)$ is

Cauchy in X. Let x' be the equivalence class of the Cauchy sequence $(z_1, z_2, z_3, \ldots)$. We now show that x' is the limit of our Cauchy sequence $\{x'_i\}_1^\infty$ in X'. We have

$$d'(x'_i, x') \leq d'(x'_i, z'_i) + d'(x', z'_i) < 1/n + d'(x', z'_i)$$

for $z'_i \in W$. Then, since x' is the equivalence class of the Cauchy sequence $(z_1, z_2, z_3, \ldots)$ and z'_i is the equivalence class of $(z_i, z_i, z_i, \ldots)$, $d'(x', z'_i) = \lim_{n\to\infty} d(z_n, z_i)$. Then

$$d'(x'_i, x') < 1/n + \lim_{n\to\infty} d(z_n, z_i).$$

The right hand side goes to zero as $n, i \to \infty$. So the Cauchy sequence $\{x'_i\}_1^\infty$ is convergent in X' and X' is complete.

(d) Now, we show that the space X' is unique upto isometry. If $(\tilde{X}, \tilde{d})$ is another space that contains a dense subset $\tilde{W}$ isometric to X, then for any $\tilde{x}$, $\tilde{y} \in \tilde{X}$, we have sequences $\{\tilde{x}_n\}_1^\infty$ and $\{\tilde{y}_n\}_1^\infty$ in $\tilde{W}$ such that $\tilde{x}_n \to \tilde{x}$ and $\tilde{y}_n \to \tilde{y}$ with

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} \tilde{d}(\tilde{x}_n, \tilde{y}_n).$$

Since W and $\tilde{W}$ are isometric and the closure of W in X' is X' itself, X' and $\tilde{X}$ must be isometric.

$\square$