MA 515 - Analysis I **Homework 1**

Pack Pledge: I have neither given nor received unauthorized aid on this test or assignment.
Name: Nutan Nepal

---

1. Choose either d_1 or d_2 below and show that it is a metric on $\mathbb{R}^n$.

$$d_1(x,y) = max\{|x_i - y_i|\} \text{ and } d_2(x,y) = \sum_{i=1}^{n} |x_i - y_i| \text{ (taxicab metric).}$$

   i. Since d_2 is a finite sum of positive numbers, $0 \le d_2(x,y) < \infty$.
   ii. $d_2(x,y) = d_2(y,x)$ since $|x_i - y_i| = |y_i - x_i|$ for all i.
   iii. $d_2(x,x) = 0$ since it is the sum of zeros.
   iv. Since we have $|x_i - z_i| = |(x_i - y_i) + (y_i - z_i)| \le |x_i - y_i| + |y_i - z_i|$ (using triangle inequality for each $x_i, y_i, z_i \in \mathbf{R}$), we get

$$d_2(x,z) = \sum_{i=1}^{n} |x_i - z_i| \le \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) = d_2(x,y) + d_2(y,z).$$

   Hence, d_2 also satisfies the triangle inequality for $x, y, z \in \mathbb{R}^n$.

   Thus, d_2 is a metric in $\mathbb{R}^n$. ■

2. Let $C[0,1]$ be the space of continuous functions on $[0,1]$.
   Show that $d(f,g) = \displaystyle\int_0^1 |f(x) - g(x)|dx$ is a metric on $C[0,1]$

   For all $f \in C[0,1]$, f is bounded. Let $|f(x)| < M$ and $|g(x)| < N$ for $x \in [0,1]$.
   i. Then $0 \le \int_0^1 |f(x) - g(x)|dx \le \int_0^1 |f(x)| + |g(x)|dx \le (M+N)(1-0) < \infty$. Hence $0 \le d(f,g) < \infty$.
   ii. $d(f,g) = d(g,f)$ since $|f(x) - g(x)| = |g(x) - f(x)|$ for all $x \in [0,1]$.
   iii. $d(f,f) = 0$ since it is the integration of zero function.
   iv. For each $x \in [0,1]$ and $f, g, h \in C[0,1]$, we have $|f(x) - h(x)| = |(f(x) - g(x)) + (g(x) - h(x))| \le |f(x) - g(x)| + |g(x) - h(x)|$ (using triangle inequality for real numbers). Then

$$d(f,h) = \int_0^1 |f(x) - h(x)|dx \le \int_0^1 (|f(x) - g(x)| + |g(x) - h(x)|)\, dx = d(f,g) + d(g,h).$$

   Hence, d satisfies the triangle inequality for $f, g, h \in C[0,1]$.

   Thus, d is a metric on $C[0,1]$. ■

3. Let $x = \{x_n\}_1^\infty$ be a sequence.

   (a) True or False: If $x \in l^p$ for some $1 \le p < \infty$, then $x_n \to 0$ as $n \to \infty$. Justify your answer.

   True. If $x \in l^p$ for some $1 \le p < \infty$, then $\left(\sum_1^\infty |x_i|^p\right)^{1/p} < \infty$. Hence $\sum_i^\infty |x_i|^p$ is a convergent series and $|x_i|^p \to 0$ as $i \to \infty$ which implies that $x_i \to 0$ as $i \to \infty$.

(b) True or False: If $x_n \to 0$ as $n \to \infty$, then $x_n \in l^p$, for some $1 \leq p < \infty$. Justify your answer.

False. The sequence given by $x = \{x_i = \frac{1}{\log(i+1)}\}_1^\infty \to 0$ as $i \to \infty$ but the sum $\sum_2^\infty |x_i|^p$ does not converge for any $1 \leq p < \infty$. So, $x \notin l^p$ for any $1 \leq p < \infty$.

4. Let $a, b \geq 0$, and $p \geq 1$. Prove that

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

Use the hints from class.

Let $f(x) = x^p$, $f : [0, \infty) \to \mathbf{R}$ and $p \geq 1$. Since f is a *convex* function, we have

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \qquad \text{for } \alpha \in [0,1].$$

Taking $\alpha = 1/2$, we get

$$\begin{aligned} f\left(\frac{a}{2} + \frac{b}{2}\right) &\leq \frac{f(a)}{2} + \frac{f(b)}{2} \\ \text{or,} \quad \frac{1}{2^p} f(a+b) &\leq \frac{1}{2}(f(a) + f(b)) \\ \text{or,} \quad (a+b)^p &\leq 2^{p-1}(a^p + b^p). \end{aligned}$$

■

5. For $p > 1$, let q be its conjugate, i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Prove the following inequality:

$$u \cdot v \leq \frac{1}{p} u^p + \frac{1}{q} v^q, \quad \forall u, v \geq 0$$

Use the hints from class.

If either u or v equals 0, then the inequality follows immediately. Suppose $u > 0$, $v > 0$ and let $f(x) = e^x$. Since f is a *convex* function,

$$\begin{aligned} u \cdot v &= \exp(\log u + \log v) \\ &= f\left(\frac{1}{p}\log u^p + \frac{1}{q}\log v^q\right) \\ &\leq \frac{1}{p} f(\log u^p) + \frac{1}{q} f(\log v^q) \\ &= \frac{u^p}{p} + \frac{v^q}{q}. \end{aligned}$$

■

6. Prove Holder's Inequality for Sums. Use the hints from class.

Holder's inequality: Let $p, q \geq 1$ be conjugate exponents. Let $x = \{x_i\}_1^\infty \in l^p$ and $y = \{y_i\}_1^\infty \in l^q$. Then

a. $xy = \{x_i y_i\}_1^\infty \in l^1$ and

b. $\sum_1^\infty |x_i y_i| \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} \cdot (\sum_1^\infty |y_i|^q)^{\frac{1}{q}}$.

Let $u_i = \frac{x_i}{\left(\sum_1^\infty |x_i|^p\right)^{1/p}}$ and $v_i = \frac{y_i}{\left(\sum_1^\infty |y_i|^q\right)^{1/q}}$. Then by Young's inequality,

$$\begin{aligned} u_i \cdot v_i &= \frac{x_i}{\left(\sum_1^\infty |x_i|^p\right)^{1/p}} \cdot \frac{y_i}{\left(\sum_1^\infty |y_i|^q\right)^{1/q}} \\ &\leq \frac{x_i^p}{p\sum_1^\infty |x_i|^p} + \frac{y_i^q}{q\sum_1^\infty |y_i|^q} \end{aligned}$$

Let $m = \left(\sum_1^\infty |x_i|^p\right)^{1/p}$ and $n = \left(\sum_1^\infty |y_i|^q\right)^{1/q}$. Then from above we have

$$\begin{aligned} \sum_1^\infty |x_i y_i| = mn \sum_1^\infty |u_i v_i| \leq mn \sum_1^\infty \left| \frac{1}{pm^p} x_i^p + \frac{1}{qn^q} \cdot y_i^q \right| &\leq mn \left(\frac{1}{pm^p} \cdot \sum_1^\infty |x_i^p| + \frac{1}{qn^q} \sum_1^\infty |y_i^q| \right) \\ &= mn \left(\frac{1}{pm^p} \cdot m^p + \frac{1}{qn^q} \cdot n^q \right) = mn \end{aligned}$$

Hence $\sum_1^\infty |x_i y_i| \leq mn = \left(\sum_1^\infty |x_i|^p\right)^{1/p} \cdot \left(\sum_1^\infty |y_i|^q\right)^{1/q}$ which proves (b). Since $0 \leq \sum_1^\infty |x_i y_i| < \infty$, we also have (a) by definition. ■

7. Prove Minkowski's Inequality for Sums. Use the hints from class.

Minkowski's inequality : Let $p \geq 1$ and $x = \{x_i\}_1^\infty \in l^p$ and $y = \{y_i\}_1^\infty \in l^p$. Then

a. $x + y = \{x_i + y_i\}_1^\infty \in l^p$ and

b. $(\sum_1^\infty |x_i + y_i|^p)^{\frac{1}{p}} \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} + (\sum_1^\infty |y_i|^p)^{\frac{1}{p}}$.

First we show that $x + y \in l^p$ by showing that

$$\left(\sum_1^\infty |x_i + y_i|^p \right)^{1/p} < \infty$$

We have,

$$\sum_1^\infty |x_i + y_i|^p \leq \sum_i^\infty (|x_i| + |y_i|)^p \leq 2^{p-1} \left(\sum_i^\infty |x_i|^p + \sum_i^\infty |y_i|^p \right) < \infty.$$

Now, since $x, y \in l^p$, $d_p(x, y) < \infty$. If $p = 1$ then the Minkowski inequality follows from the triangle inequality of real numbers. Let $p > 1$ then

$$\begin{aligned} \sum_1^\infty |x_i + y_i|^p = \sum_1^\infty |x_i + y_1||x_i + y_i|^{p-1} &\leq \sum_1^\infty \left(|x_i||x_i + y_i|^{p-1} + |y_i||x_i + y_i|^{p-1} \right) && (1) \\ &= \sum_1^\infty \left(|x_i||x_i + y_i|^{p-1} \right) + \sum_1^\infty \left(|y_i||x_i + y_i|^{p-1} \right) && (2) \end{aligned}$$

Now let q be the conjugate exponent of p, then we have $\frac{1}{p}+\frac{1}{q}=1 \iff p+q=pq \iff p=p(q-1)$. Then, at line 2

$$\left(\sum_1^\infty |x_i+y_i|^{(p-1)q}\right)^{1/q} = \left(\sum_1^\infty |x_i+y_i|^p\right)^{1/q} < \infty$$

which shows that $\{|x_i+y_i|^{p-1}\}_i^\infty \in l^q$. Then by Holder's inequality,

$$\begin{aligned} \sum_1^\infty |x_i||x_i+y_i|^{p-1} &\leq \left(\sum_1^\infty |x_i|^p\right)^{1/p} \cdot \left(\sum_1^\infty |x_i+y_i|^{(p-1)q}\right)^{1/q} && (3) \\ &= \left(\sum_1^\infty |x_i|^p\right)^{1/p} \cdot \left(\sum_1^\infty |x_i+y_i|^p\right)^{1/q} && (4) \end{aligned}$$

Using the results from line 2 and line 4 on line 1,

$$\begin{aligned} \sum_1^\infty |x_i+y_i|^p &\leq \sum_1^\infty \left(|x_i||x_i+y_i|^{p-1}\right) + \sum_1^\infty \left(|y_i||x_i+y_i|^{p-1}\right) && (5) \\ \text{or,} \quad \sum_1^\infty |x_i+y_i|^p &\leq \left(\sum_1^\infty |x_i+y_i|^p\right)^{1/q} \cdot \left(\left(\sum_1^\infty |x_i|^p\right)^{1/p} + \left(\sum_1^\infty |y_i|^p\right)^{1/p}\right) && (6) \end{aligned}$$

Dividing both sides by $(\sum_1^\infty |x_i+y_i|^p)^{1/q}$, we get the Minkowski's inequality (since $1-\frac{1}{q}=\frac{1}{p}$). ■

8. For $1 \leq p < \infty$, let $l^p = \{x = \{x_i\}_1^\infty \mid \sum_1^\infty |x_i|^p < \infty\}$. For any $x, y \in l^p$, define

$$d_p(x,y) = \left(\sum_1^\infty |x_i-y_i|^p\right)^{1/p}$$

Prove that (l^p, d_p) is a metric space.

i. Since $d_p(x,y)$ is the *pth* root of a sum of positive numbers, $d_p \geq 0$. Also from Minkowski inequality (a.), we have $d_p < \infty$.
ii. $d_p(x,y) = d_p(y,x)$ since $|x_i - y_i| = |y_i - x_i|$ for all i.
iii. $d_p(x,x) = 0$ since $|x_i - x_i| = 0$ for all i.
iv. The triangle inequality for d_p follows from the Minkowski inequality (b.)

$$\begin{aligned} \left(\sum_1^\infty |x_i-z_i|^p\right)^{\frac{1}{p}} &\leq \left(\sum_1^\infty |x_i-y_i|^p\right)^{\frac{1}{p}} + \left(\sum_1^\infty |y_i-z_i|^p\right)^{\frac{1}{p}} \\ \text{or,} \quad d_p(x,z) &\leq d_p(x,y) + d_p(y,z) \end{aligned}$$

■

9. Prove Jensen's Inequality for Sums. Use the hints from class.

$$\left(\sum_{i=1}^{\infty}|x_i|^{p_2}\right)^{1/p_2} \leq \left(\sum_{i=1}^{\infty}|x_i|^{p_1}\right)^{1/p_1} \qquad \forall 1 \leq p_1 < p_2 < \infty$$

Let $|y_i| = |x_i|^{p_1}$. Then we need to show that $\left(\sum_{i=1}^{\infty}|y_i|^{p_2/p_1}\right)^{p_1/p_2} \leq \sum_{i=1}^{\infty}|y_i|$. First we show that this is true for a finite sequence $\{x_i\}_1^n$ using induction on n. Then we take the limit as $n \to \infty$ to prove Jensen's inequality.

When $n = 1$, $(|y_1|^{p_2/p_1})^{p_1/p_2} = y_1$. (True)

Let $H(k) : \left(\sum_{i=1}^{k}|y_i|^{p_2/p_1}\right)^{p_1/p_2} \leq \sum_{i=1}^{k}|y_i|$ be true for some integer $k > 1$. Then

$$\begin{aligned}
\left(\sum_{i=1}^{k+1}|y_i|^{p_2/p_1}\right)^{p_1/p_2} &= \left(\sum_{i=1}^{k}|y_i|^{p_2/p_1} + |y_{k+1}|^{p_2/p_1}\right)^{p_1/p_2} \\
&\leq \left(\sum_{i=1}^{k}|y_i|^{p_2/p_1}\right)^{p_1/p_2} + \left(|y_{k+1}|^{p_2/p_1}\right)^{p_1/p_2} && \text{[by Minkowski inequality]} \\
&\leq \sum_{i=1}^{k}|y_i| + |y_{k+1}| && \text{[by induction hypothesis]} \\
&= \sum_{i=1}^{k+1}|y_i|
\end{aligned}$$

Hence $H(k) \implies H(k+1)$ which proves that $H(n)$ is true for all $n \in \mathbb{Z}$. Taking the limit as $n \to \infty$ we get the required Jensen's inequality. ∎

10. Show that $l^1 \subset l^2$ without using Jensen's inequality. Then show that inclusion is strict, i.e., find an element in l^2 that is not in l^1.

Let $x \in l^1$ then $0 \leq \sum_1^{\infty}|x_1| < \infty$ which implies that the sequence x converges to 0. Let $N \in \mathbb{Z}$ such that $x_i < 1$ for all $i > N$. Then for $i > N$, we have $|x_i|^2 < |x_i|$. Hence,

$$0 \leq \sum_{i=1}^{\infty}|x_i|^2 \leq \sum_{i=1}^{N}|x_i|^2 + \sum_{i=N+1}^{\infty}|x_i|^2 < \sum_{i=1}^{N}|x_i|^2 + \sum_{i=N+1}^{\infty}|x_i| < \infty.$$

So, $l^1 \subset l^2$.

The harmonic series given by the sequence $x = \{x_i = \frac{1}{i}\}_1^{\infty}$ does not converge. However

$$\sum_{i=1}^{\infty}\frac{1}{i^2} = \frac{\pi^2}{6} < \infty.$$

Here we see that $x \notin l^1$ but $x \in l^2$. Hence, the inclusion is strict. ∎