Analysis I

Homework 2

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Pack Pledge: I have neither given nor received unauthorized aid on this test or assignment.

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1. Let (X, ρ) and (Y, σ) be metric spaces, and let $f : (X, \rho) \to (Y, \sigma)$ be a map such that $f^{-1}(V)$ is open in X, for all V open in Y. Show that f is continuous on X.

2. (**Continuous mapping**) Show that a mapping $T : X \to Y$ is continuous if and only if the inverse image of any closed set $M \subset Y$ is a closed set in X.

3. Assume that $f : (\mathbb{R}^2, d_1 = \text{ Euclidean metric}) \to \mathbb{R}$ is continuous at $x \in \mathbb{R}^2$. Show that $f : (\mathbb{R}^2, d_2 = \text{taxicab metric}) \to \mathbb{R}$ is also continuous at x.

4. Show that the discrete metric space (X, d) is separable iff X is countable.

5. Show that l^p, with $1 \leq p < \infty$ is separable.

6. Show that l^∞ is not separable.

7. Let $\{x_n\}_{n\geq 1}$ be a sequence in a m.s. (X, d) which converges to x. Show that $\{x_n\}_{n\geq 1}$ is a bounded sequence. Then let $\{y_n\}_{n\geq 1}$ be a sequence in (X, d) which converges to y. Show that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

8. Show that any nonempty set $A \subset (X, d)$ is open if and only if it is a union of open balls.

9. Let (X, ρ) be a metric space, $E \subset X$, and $x \in X$. Prove that the following are equivalent:
   (a) $x \in \overline{E}$
   (b) $B(x, r) \cap E \neq \emptyset$, $\forall r > 0$
   (c) $\exists \{x_n\} \in E$ s.t. $x_n \to x$

10. If d_1 and d_2 are metrics on the same set X and there are positive numbers a and b such that for all x, $y \in X$,
$$ad_1(x, y) \leq d_2(x, y) \leq bd_1(x, y),$$
show that the Cauchy sequences in (X, d_1) and (X, d_2) are the same.

11. Show that l^p, with $1 \leq p < \infty$ is complete.

12. Prove that $(\mathbb{R}, d(x, y) = |x - y|)$ is complete.
Hint: Follow the steps provided in class. You can use Bolzano-Weierstrass Theorem without proving it.

13. Prove that $(\mathbb{Q}, d(x, y) = |x - y|)$ is incomplete.

14. Prove that $\left(\mathrm{C}[-1, 1], d(f, g) = \int_{-1}^{1} |f(t) - g(t)|\, dt\right)$ is incomplete.
Hint: Follow the steps provided in class.

15. Determine whether or not the discrete metric space is complete. Justify your answer.

16. Prove the Completion of a Metric Space Theorem.