Introduction to Manifold Theory

Homework 8

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1. Do Exercise 5.3 Let X and Y be smooth manifolds. Show that the identity map sending $f \longmapsto f$ is a bijection from the set of smooth maps $f : X \to Y$ to the set of sections of the trivial bundle $(X \times Y, X, \text{proj}_1, Y)$ with base X and fiber Y.(loosely: show that the definition of "section of a fiber bundle" can be viewed as a generalization of the definition of "smooth function between smooth manifolds").

   Let $\mathcal{A}$ be the set of all smooth maps from X to Y and $\mathcal{B}$ be the set of all sections of the trivial bundle $(X \times Y, X, \text{proj}_1, Y)$ with base X and fiber Y. We note that the set $\mathcal{B}$ consists of the smooth functions $f : X \to X \times Y$ such that $\text{proj}_1 \circ f = \text{id}_B$. Since $\text{proj}_1(x, y) = x$ for the points $(x, y) \in X \times Y$, every $f \in \mathcal{B}$ fixes all the points of $x \in X$ meaning that

   $$f(x) = (x, f'(x))$$

   for some $f' : X \to Y$. Since, f is smooth, f' is a smooth map from X to Y. Then we can define the identity map $\text{id} : \mathcal{A} \to \mathcal{B}$ that takes each smooth function $f : X \to Y$ such that $f(x) = y$ to the section $f : X \to X \times Y$ such that $f(x) = (x, f'(x))$. Clearly, this is an injective map since

   $$\text{id}(f)(x) = \text{id}(g)(x) \implies (x, f'(x)) = (x, g'(x)) \implies f' = g' \implies f = g.$$

   The map is also surjective since every $f \in \mathcal{B}$ fixes all the points of $x \in X$ and for $f \in \mathcal{B}$ such that $f(x) = (x, f'(x))$ we have $f \in \mathcal{A}$ such that $f(x) = y$. Thus this identity map is a bijection.