Introduction to Manifold Theory

Homework 7

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1. Do Exercise 3.11: show that if a second-countable Hausdorff topological space X admits an n-dimensional smooth atlas, then X is an n-dimensional topological manifold (and thus a smooth manifold equipped with the equivalence class of this atlas).

It suffices to show that the topological space X is locally homeomorphic to an open set of $\mathbb{R}^n$. Let $x \in X$. Since X admits an n-dimensional smooth atlas, we know that there exists a chart (U, φ) of open set U and a smooth map φ with $x \in U$ such that $\phi(U)$ is an open disk in $\mathbb{R}^n$. Then we see that x has an open neighborhood $N_\varepsilon(x)$ such that $\varphi(N_\varepsilon(x))$ is an open disk in $\mathbb{R}^n$. Hence, X is locally homeomorphic to $\mathbb{R}^n$ and is a smooth manifold.

2. Do Exercise 3.12: in imprecise terms, show that if a set X has a "sets-and-bijections smooth atlas" then X can be turned into a smooth manifold in a natural way.

We first check that the given topology $\mathscr{T}$ is indeed a valid topology on X:

(a) For all coordinate patch (U, φ) in the atlas $\mathcal{A}$, $\varphi(\emptyset \cap U) = \emptyset$ and $\varphi(X \cap U) = \varphi(U)$ is open in $\mathbb{R}^n$. Hence $\emptyset$ and X are in the topology.

(b) If $V = \bigcup_{\alpha \in A} U_\alpha$ is an arbitrary union of indexed (by A) open sets, then

$$\varphi(V \cap U) = \bigcup_{\alpha \in A} \varphi(U_\alpha) \cap \varphi(U).$$

Since eaach $\varphi(U_\alpha)$ and $\varphi(U)$ are open in $\mathbb{R}^n$, the arbitrary union $\bigcup_{\alpha \in A} \varphi(U_\alpha)$ is open and thus V is open in X.

(c) If $V = U_1 \cap U_2$ is intersection of open sets of X, then $\varphi(V) = \varphi(U_1) \cap \varphi(U_2)$ is open in $\mathbb{R}^n$. Thus $V \in \mathscr{T}$.

Thus the given topology is indeed a topology. If we start with finitely many U_i, then we see that all the open sets of X are generated by these U_i. Thus X has finitely many (hence, countable) basis and is second countable.

We note that every $U \subset X$ from the chart (U, φ) is open since $\varphi(U \cap U) = \varphi(U)$ is open in $\mathbb{R}^n$. Here, φ is a bijection and hence has an inverse φ^{-1}. Now we show that this is a homeomorphism by showing that both φ and φ^{-1} are continuous. Clealy for every open set V in $\varphi(X)$ and a chart (U, ψ) of X, $\psi \circ \varphi^{-1}$ is a transition map and is smooth. Hence $\varphi^{-1}(V)$ is open in X, Thus, φ is continuous. Similarly, For open set $V \subset \varphi^{-1}(X)$ and chart (U, ψ), $\psi(V \cap U)$ is open in $\mathbb{R}^n$. Thus $\varphi \circ \psi^{-1}$ being a transition function is smooth and thus, $\varphi(V)$ is open in $\mathbb{R}^n$. So φ^{-1} is also continuous.

3. Do Exercise 3.13: prove the analogue of Exercise 3.11 where the charts in your smooth atlas are allowed to have general smooth manifolds (rather than just open disks) as codomains.

$\mathcal{A}$ is given by the collection of pairs

$$(\varphi^{-1}(V), \psi \circ \varphi)$$

where (V, ψ) is a coordinate patch from the atlas of the smooth n-manifold $X_{U,\varphi}$. φ is a homeomorphism and hence is continuous. Thus $\varphi^{-1}(V)$ is open in X. Since every $p \in X$ is in some (U, φ), p is contained in some $\varphi^{-1}(V)$. For each $p \in X$, we have an open set U containing p and since $\varphi(U)$ is open in the smooth manifold $X_{U,\varphi}$, there is an open disk around $\varphi(p)$ contained in $\varphi(U)$. Thus we have the map $\psi \circ \varphi : X \to \mathbb{R}^n$ such that the image of some open set U of X in $\mathbb{R}^n$ an open disk. Thus, X is locally homeomorphic to $\mathbb{R}^n$. (I could not prove for smoothness of transition functions.)

4. Do Exercise 3.14: prove the analogue of Exercise 3.12 where the charts in your "sets-and-bijections smooth atlas" are allowed to have general smooth manifolds (rather than just open disks) as codomains.

We note that this topology is the same as the topology in problem 3.12 with the modification that rl^n is replaced by a general n-manifold. The proof follows similarly.

We now check that for every given pair (U, φ), U is open in X and φ is a homeomorphism from U to $X_{U,\varphi}$. Clearly, U is open since $\varphi(U \cap U) = \varphi(U)$ is open in $X_{U,\varphi}$. Now, for every open set V in $\varphi(X)$ and a chart (U, ψ) of X, $\psi \circ \varphi^{-1}$ is a transition map and is smooth. Hence $\varphi^{-1}(V)$ is open in X, Thus, φ is continuous. Similarly, For open set $V \subset \varphi^{-1}(X)$ and chart (U, ψ), $\psi(V \cap U)$ is open in $X_{U,\varphi}$. Thus $\varphi \circ \psi^{-1}$ being a transition function is smooth and thus, $\varphi(V)$ is open in $X_{U,\varphi}$. So φ^{-1} is also continuous.