Analysis I
Homework 4

Nutan Nepal

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Pack Pledge: I have neither given nor received unauthorized aid on this test or assignment.

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1. Let X and Y be vector space. Let $T : \mathcal{D}(T) \subseteq X \to Y$ be a linear operator. Prove that
   (a) The inverse $T^{-1} : \mathcal{R}(T) \to \mathcal{D}(T)$ exists iff $\mathcal{N}(T) = \{0\}$.
   (b) If T^{-1} exists, then T^{-1} is linear.
   (c) If $\dim(\mathcal{D}(T)) = n < \infty$ and T^{-1} exists, then $\dim(\mathcal{R}(T)) = \dim(\mathcal{D}(T))$.

(a) If $\mathcal{N} = \{0\}$ then for $x, y \in X$ we have $T(x) = T(y) \iff T(x - y) = 0 \iff x - y = 0 \iff x = y$. Hence, T is injective. Since $T : \mathcal{D}(Y) \to \mathcal{R}(T)$ is a surjective map (by definition), $T^{-1} : \mathcal{R}(T) \to \mathcal{D}(T)$ exists.

Now, if T^{-1} exists, then T must be injective. So, $T(x) = T(y) \iff x = y$. But, by linearity of T, we have $T(x - y) = 0 \iff x = y$. Hence, the null space $\mathcal{N}(T) = \{0\}$.

(b) If T^{-1} exists, then every element of $\mathcal{R}(T)$ can be written as $T(x)$ for some $x \in X$. Then, for two elements $T(x)$ and $T(y)$ in the range and $\alpha \in \mathbb{R}$, we have

$$T^{-1}(\alpha T(x) + T(y)) = T^{-1}T(\alpha x + y) = \alpha x + y = \alpha T^{-1}(T(x)) + T^{-1}(T(y)).$$

Hence, T^{-1} is linear.

(c) It suffices to show that if $\{e_1, \ldots, e_n\}$ is the basis of the domain, then $\{Te_1, \ldots, Te_n\}$ is a linearly independent set. We show that if

$$\lambda_1 Te_1 + \cdots + \lambda_n Te_n = 0$$

then each $\lambda_i = 0$. Suppose, otherwise that some λ_i are non-zero in the above equation. Without loss of generality, we can assume that $\lambda_1, \ldots, \lambda_m$ are non-zero when $\lambda_1 Te_1 + \cdots + \lambda_n Te_n = 0$. Then we have, $\lambda_1 Te_1 + \cdots + \lambda_m Te_m = 0$ and $T(\lambda_1 e_1 + \cdots + \lambda_m e_m) = 0$. This means that $\lambda_1 e_1 + \cdots + \lambda_m e_m \in \mathcal{N}(T)$, but $\lambda_1 e_1 + \cdots + \lambda_m e_m \neq 0$ in X. This contradicts our assumption that T^{-1} exists. Hence dimension of the range is equal to the dimension of the domain.

2. Let X, Y and Z be vector space. Let $T : X \to Y$ and $S : Y \to Z$ be bijective. Then $(ST)^{-1} : Z \to X$ exists and $(ST)^{-1} = T^{-1}S^{-1}$.

Since S and T are bijective, they are both injective. So $\mathcal{N}(ST) = \{0\}$ and $ST : X \to Z$ is a surjective map, hence the inverse $(ST)^{-1} : Z \to X$ exists. We see that for every $z \in Z$ there exists a $y \in Y$ such that $S(y) = z$ and for the same $y \in Y$ there exists an $x \in X$ such that $T(x) = y$. Then for each such $z \in Z$, we define $(ST)^{-1}(z) = x$. By the procedure of defining this map we see that $x = T^{-1}(S^{-1}(z))$. Hence $(ST)^{-1} = T^{-1}S^{-1}$.

3. Let X be a finite dimensional normed space and Y a normed space. Let $T : X \to Y$ be linear. Show that T is bounded.

Let the dimension of X be n and the basis is given by $\{e_1, \ldots, e_n\}$. Then for any $x = \sum_{i=1}^{n} \lambda_i e_i \in X$, we have $T(x) = T\left(\sum_{i=1}^{n} \lambda_i e_i\right) = \sum_{i=1}^{n} \lambda_i T(e_i)$. We see that

$$\|T(x)\| = \left\| \sum_{i=1}^{n} \lambda_i T(e_i) \right\| \leq \left| \sum_{i=1}^{n} \lambda_i \right| \cdot \max_{i=\overline{1,n}} T(e_i).$$

We also know that for any $x \in X$, $\|x\| \geq c \cdot |\sum_{i=1}^{n} \lambda_i|$ for some positive c. Thus $\|T(x)\| \leq k \cdot \|x\|$ for $k = (\max_{i=\overline{1,n}} T(e_i))/c$ and T is bounded.

4. Let X and Y be normed spaces. Let $T : \mathcal{D}(T) \subseteq X \to Y$ be a bounded linear operator. Show that T is continuous.

If T is bounded and linear then for all $x \in X$ we have, $\|T(x)\| \leq c\|x\|$ for some $c > 0$ in $\mathbb{R}$. Then, for an arbitrary $x \in X$, for every $\varepsilon > 0$ we have $0 < \delta < \varepsilon/c$ such that $\|x - y\| < \delta$ implies $\|T(x) - T(y)\| = \|T(x - y)\| \leq c\|x - y\| < \varepsilon$. Hence T is continuous.

5. Let X be a normed space, and Y be a Banach space. Let $T : \mathcal{D}(T) \subseteq X \to Y$ be a bounded linear operator. Show that T has an extension

$$\widetilde{T} : \overline{\mathcal{D}(T)} \to Y$$

which is a bounded linear operator such that $\|\widetilde{T}\| = \|T\|$.

For any $x \in \overline{\mathcal{D}(T)}$, we take a sequence $\{x_i\}_1^\infty$ in $\overline{\mathcal{D}(T)}$ such that $x_i \longrightarrow x$. Then we have

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq \|T\|\|x_n - x_m\|.$$

Since $\{x_i\}_1^\infty$ is a convergent sequence in $\overline{\mathcal{D}(T)}$, we see that $\{Tx_i\}_1^\infty$ is Cauchy and hence (because Y is Banach), convergent in Y. We can call this limit y and define a function $\widetilde{T} : \overline{\mathcal{D}(T)} \to Y$ by $Tx = y$. We now show that this is a well-defined function, that is, the value of Tx does not depend on our choice of the sequence $\{x_i\}_1^\infty$ converging to x. Suppose $\{x_i\}_1^\infty$ and $\{z_i\}_1^\infty$ both converge to x. Then the sequence $\{v_i\}_1^\infty$ given by

$$x_1, z_1, x_2, z_2, \ldots\ldots$$

also converges to x. With similar argument as above we say that $\{Tv_i\}_1^\infty$ is a convergent sequence that converges to y since the subsequence $\{Tx_i\}_1^\infty$ converges to y. This shows that $\{Tz_i\}_1^\infty$ also converges to the same point and the function is well-defined. $\widetilde{T}$ is linear and $\widetilde{T}(x) = T(x)$ for all $x \in \mathcal{D}(T)$, so $\widetilde{T}$ is an extension of T.

Now, since T is bounded, we have

$$\|Tx_n\| \leq \|T\|\|x_n\|.$$

As $n \to \infty$, we have $Tx_n \to \widetilde{T}x$ and $x_n \to x$. Since $\|\cdot\|$ is a continuous function, we obtain

$$\|\widetilde{T}x\| \leq \|T\|\|x\|.$$

Hence, $\widetilde{T}$ is bounded with $\|\widetilde{T}\| \leq \|T\|$. But $\|\widetilde{T}\| \leq \|T\|$ because the supremum cannot decrease in an extension. Hence $\|\widetilde{T}\| = \|T\|$.

6. **Kreyszig p.102 / Problem 10.** On $C[0,1]$ define S and T by

$$y(s) = s\int_0^1 x(t)\ dt, \qquad y(s) = sx(s),$$

respectively. Do S and T commute? Find $\|S\|$, $\|T\|$, $\|ST\|$ and $\|TS\|\cdot$

We have,

$$ST(x) = S(sx(s)) = s\int_0^1 (sx(s))(t)\ dt = s\int_0^1 tx(t)\ dt$$

and

$$TS(x) = T\left(s\int_0^1 x(t)\ dt\right) = s^2\int_0^1 x(t)\ dt.$$

For the constant function $x = 1$ we see that, $ST(x) = s\int_0^1 t\cdot 1\ dt = s/2$ and $TS(x) = s^2(\int_0^1 1\ dt) = s^2$ and so S and T do not commute.

Now,

$$\|S(x)\| = \left\|s\int_0^1 x(t)\ dt\right\| \leq \|s\|\int_0^1 \|x(t)\|\ dt = \|s\|\|x\| = \|x\|,$$

So, $\|S\| \leq 1$. But, for $x = 1$ we get $S(1) = \|s\cdot 1\| = 1$, so $\|S\| = 1$.

Similarly,

$$\|T(x)\| = \|sx(s)\| \leq \|s\|\|x\| = \|x\| \implies \|T\| = 1,$$

$$\|ST(x)\| = \left\|s\int_0^1 tx(t)\ dt\right\| \leq \|s\|\left\|\int_0^1 tx(t)\ dt\right\| \leq \|s\|\int_0^1 \|tx(t)\|\ dt = \int_0^1 t\|x(t)\|\ dt \leq \|x\|/2.$$

With similar argument as above, we obtain $\|ST\| = 1/2$. Now,

$$\|TS(x)\| = \left\|s^2\int_0^1 x(t)\ dt\right\| \leq \|s^2\|\cdot\|x(t)\| = \|x\|.$$

So, $\|TS\| \leq 1$ but for $x = 1$ we get $\|TS(x)\| = 1$. Hence $\|TS\| = 1$.

7. **Kreyszig p.109 / Problem 2.** Show that the functionals defined on $C[a, b]$ by

$$f_1(x) = \int_a^b x(t)y_0(t)\ dt \qquad\qquad (y_0 \in C[a, b])$$

$$f_2(x) = \alpha x(a) + \beta x(b) \qquad\qquad (\alpha, \beta \text{ fixed})$$

are linear and bounded.

For $x,\ y \in C[a, b]$ and $\alpha \in \mathbb{R}$, we see that $f_1(\alpha x + y) = \int_a^b (\alpha x + y)(t)y_0(t)\ dt = \int_a^b (\alpha x(t) + y(t))y_0(t)\ dt = \alpha \int_a^b x(t)y_0(t)\ dt + \int_a^b y(t)y_0(t)\ dt = \alpha f_1(x) + f_1(y)$. Furthermore,

$$\|f_1(x)\| = \left\| \int_a^b x(t)y_0(t)\ dt \right\| \le \int_a^b \|x(t)y_0(t)\|\ dt \le (b - a)\,\|x(t)y_0(t)\| = (b - a)\|y_0\|\|x\|.$$

Hence, f_1 is linear and bounded.

Similarly, for $x,\ y \in C[a, b]$ and $\gamma \in \mathbb{R}$, $f_2(\gamma x + y) = \alpha(\gamma x + y)(a) + \beta(\gamma x + y)(b) = \alpha\gamma x(a) + \alpha y(a) + \beta\gamma x(b) + \beta y(b) = \gamma(\alpha x(a) + \beta x(b)) + \alpha y(a) + \beta y(b) = \gamma f_2(x) + f_2(y)$. And,

$$\|f_2(x)\| = \|\alpha x(a) + \beta x(b)\| \le \|\alpha x(a)\| + \|\beta x(b)\| \le \alpha\|x\| + \beta\|x\| = (\alpha + \beta)\|x\|.$$

Hence, f_2 is also linear and bounded.

8. **Kreyszig p.109 / Problem 6. (Space $C'[a, b]$)** The space $C^1[a, b]$ or $C'[a, b]$ is the normed space of all continuously differentiable functions on $J = [a, b]$ with norm defined by

$$\|x\| = \max_{t\in J} |x(t)| + \max_{t\in J} |x'(t)|.$$

Show that the axioms of a norm are satisfied. Show that $f(x) = x'(c)$, $c = (a + b)/2$, defines a bounded linear functional on $C'[a, b]$. Show that f is not bounded, considered as a functional on the subspace of $C[a, b]$ which consists of all continuously differentiable functions.

We first check that the axioms for the norms are satisfied.

(a) $\|x\| = \max_{t\in J} |x(t)| + \max_{t\in J} |x'(t)| \ge \max_{t\in J} |x(t)| \ge 0$ and since $x \in C[a, b]$ is continuous on a bounded set, we have $\|x\| \le \max_{t\in J} |x(t)| < \infty$.

(b) If $x = 0$, then clearly $\|x\| = 0$ and if $\|x\| = 0$, we have $\max_{t\in J} |x(t)| + \max_{t\in J} |x'(t)| = 0 \implies \max_{t\in J} |x(t)| = 0 \implies x = 0$.

(c) If $\alpha \in \mathbb{R}$, then $\|\alpha x\| = \max_{t\in J} |\alpha x(t)| + \max_{t\in J} |(\alpha x)'(t)| = \alpha \max_{t\in J} |x(t)| + \max_{t\in J} |\alpha x'(t)|$. So we have $\|\alpha x\| = \alpha(\max_{t\in J} |x(t)| + \max_{t\in J} |x'(t)|) = \alpha\|x\|$.

(d) (Triangle Inequality) $\|x + y\| = \max_{t\in J} |x(t) + y(t)| + \max_{t\in J} |(x + y)'(t)|$. Since we have $(x + y)' = x' + y'$ and $\max_{t\in J} |x(t) + y(t)| \le \max_{t\in J} |x(t)| + \max_{t\in J} |y(t)|$, we get

$$\|x + y\| \le \max_{t\in J} |x(t)| + \max_{t\in J} |x'(t)| + \max_{t\in J} |y(t)| + \max_{t\in J} |y'(t)| = \|x\| + \|y\|.$$

We now check that f is a linear bounded functional. We see that $f(x + y) = (x + y)'(c) = x'(c) + y'(c) = f(x) + f(y)$ and hence f is linear. Now, $|f(x)| = |x'(c)| \le \max_{t\in J} |x'(t)| \le$

$\max_{t\in J}|x(t)| + \max_{t\in J}|x'(t)| = \|x\|$ and so f is also bounded.

To show that f is not bounded as a functional on the subspace of $C[a,b]$ we define a sequence of functions in $C[a,b]$ such that the derivative of the limit at c is unbounded. Since the space $C[a,b]$ is complete, the limit should also exist in the space but will have unbounded derivative at c,

9. **Kreyszig p.116 / Problem 2.** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $(\xi_1, \xi_2, \xi_3) \to (\xi_1, \xi_2, -\xi_1 - \xi_2)$. Find $\mathcal{R}(T)$, $\mathcal{N}(T)$ and a matrix which represents T.

For any $(x, y, z) \in \mathbb{R}^3$, we see that every element of $\mathcal{R}(T)$ can be represented as $(x, y, -x - y) = x(1, 0, -1) + y(0, 1, -1)$. Hence, every element of $\mathcal{R}(T)$ can is a linear combination of $(1, 0, -1)$ and $(0, 1, -1)$. So, $\mathcal{R}(T) = \text{Span}\{(1, 0, -1), (0, 1, -1)\}$.

We see that an element in $\mathcal{R}(T)$ is $(0, 0, 0)$ whenever $x = 0$ and $y = 0$. So T maps every element $(0, 0, z)$ to $(0, 0, 0) \in \mathcal{R}(T)$. Hence $\mathcal{N}(T) = \text{Span}\{(0, 0, 1)\}$. We have

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

that takes an input $x \in \mathbb{R}^3$ as a column vector.

10. **Kreyszig p.116 / Problem 4.** Let $\{f_1, f_2, f_3\}$ be the dual basis of $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$, where $e_1 = (1, 1, 1)$, $e_2 = (1, 1, -1)$, $e_3 = (1, -1, -1)$. Find $f_1(x)$, $f_2(x)$, $f_3(x)$, where $x = (1, 0, 0)$.

First we write $x = (1, 0, 0)$ as $x = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ for some $\lambda_i \in \mathbb{R}^3$. To find the values of λ_i, we solve the system of equations,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}.$$

Solving the system of equations, we obtain $\lambda_1 = 1/2$, $\lambda_2 = 0$ and $\lambda_3 = 1/2$. Then,

$$\begin{aligned} f_1(x) &= \lambda_1 f_1(e_1) = 1/2 \\ f_2(x) &= \lambda_2 f_2(e_2) = 0 \\ f_3(x) &= \lambda_3 f_3(e_3) = 1/2. \end{aligned}$$

11. Show that if Y is a Banach space, then $B(X, Y)$ is a Banach space.

The vector space $B(X, Y)$ of all bounded linear operators from a normed space X into a normed space Y is itself a normed space with the operator norm.

We take a Cauchy sequence $\{T_i\}_1^\infty$ of bounded linear operators from X to Y and show that it is convergent in $B(X, Y)$ to show that this is a Banach space. Since $\{T_i\}_1^\infty$ is Cauchy, for every

$\varepsilon > 0$ we have $N \in \mathbf{N}$ such that for all $m, n > N$ we have $\|T_n - T_m\| < \varepsilon$. Then for all $x \in X$ and $m, n > N$ we have, $\|T_n x - T_m x\| \leq \|T_n - T_m\|\|x\| < \varepsilon\|x\|$. Then for any fixed $x \in X$, since $\|x\|$ is a fixed number, we see that $\{T_i x\}_1^\infty$ is a Cauchy sequence in Y. Since Y is complete, $\{T_i x\}_1^\infty$ converges to a point (say y) in Y. This limit point depends on the choice of x and defines an operator $T : X \to Y$ where $T(x) = y$. T is linear since

$$\lim_{n\to\infty} T_n(\alpha x + z) = \lim_{n\to\infty} (\alpha T_n x + T_n z) = \alpha \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n z.$$

Using the continuity of norm, we obtain

$$\|T_n x - Tx\| = \left\| T_n x - \lim_{m\to\infty} T_m x \right\| = \lim_{m\to\infty} \|T_n x - T_m x\| < \varepsilon\|x\|.$$

This shows that $(T_n - T)$ with $n > N$ is a bounded linear operator. Since T_n is bounded, we see that T is also bounded. Furthermore, taking supremum over all $x \in X$ of norm 1, we obtain, $\|T_n - T\| < \varepsilon$. Hence T_n converges to T in $B(X, Y)$ and it is a Banach space.