Analysis II

Homework 1

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Pack Pledge: I have neither given nor received unauthorized aid on this test or assignment.

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1. Let $f : X \to \mathbb{R}$, where X is a measurable space. Show that if $V_r := \{x \mid f(x) \geq r\}$ is measurable for every rational number r, then f is measurable.

We note that for the open set $U_r := (-\infty, r)$, $V_r = f^{-1}(U_r^c) = (f^{-1}(U_r))^c$. Thus, $(f^{-1}(U_r))$ is measurable in X. Now, for any open set $W \subset \mathbb{R}$, we know that W is an arbitrary union or finite intersections of the sets of the form (a, b), so it suffices to show that $f^{-1}(a, b)$ is measurable.

If $(a_n) \to a$ is a sequence in $\mathbb{Q}$ that decreases to a then

$$(a,b) = (-\infty, b) \cap \bigcup_{n=1}^{\infty} (-\infty, a_n)^c \implies f^{-1}(a,b) = f^{-1}(-\infty, b) \cap \bigcup_{n=1}^{\infty} (f^{-1}(-\infty, a_n))^c.$$

Since each of the sets on the right are measurable, f is measurable.

2. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}(c)$ is measurable for each number c. Is f necessarily measurable?

Let E be a non-measurable set if $\mathbb{R}$ and define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2^x$ if $x \in E$ and $f(x) = -2^x$ if $x \notin E$. Then $f^{-1}(c)$ is empty if $c = 0$ and a singleton set if $c \neq 0$. Thus $f^{-1}(c)$ is measurable for each c. However, f is not measurable since $f^{-1}(0, \infty) = E$ is not a measurable set.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Is the composition $f \circ g$ necessarily measurable?

For each measurable set $W \subset \mathbb{R}$, $f^{-1}(W)$ is measurable. However the preimage of the measurable set $f^{-1}(W)$ under the continuous function g may not be measurable. Thus $f \circ g$ is not necessarily measurable.

4. Give an alternate proof that if $f, g : X \to \mathbb{R}$ are measurable, then so is $f + g$, by showing directly

that

$$(f+g)^{-1}(a,\infty) = \{x \mid (f+g)(x) > a\}$$

is measurable for every $a \in \mathbb{R}$.
Hint: Show that $\{x \mid (f+g)(x) > a\} = \cup_{b\in\mathbb{Q}}\big(\{x \mid f(x) > b\} \cap \{x \mid g(x) > a-b\}\big)$.

Similarly show directly that cf (for c constant) is measurable, as is f^2. Using these results, show that fg is measurable.

Let $A_a = (f+g)^{-1}(a,\infty) = \{x \mid (f+g)(x) > a\}$ for some $a \in \mathbb{R}$. Since $(f+g)(x) = f(x)+g(x)$, if $f(x) > b$ for some $b \in \mathbb{Q}$, then $(f+g)(x) > a \implies g(x) > a-b$. Then $A_a = \{x \mid f(x) > b \text{ and } g(x) > a-b\}$. So, $A_a = \{x \mid (f+g)(x) > a\} = \bigcup_{b\in\mathbb{Q}} (\{x \mid f(x) > b\} \cap \{x \mid g(x) > a-b\})$ is a countable union of intersections of measurable sets. Thus, A_a is measurable for every $a \in \mathbb{R}$ and $f+g$ is measurable.

Now, let $A_a = (cf)^{-1}(a,\infty) = \{x \mid (cf)(x) > a\} = \{x \mid f(x) > a/c\}$. But this is a measurable set since $f^{-1}(a/c,\infty)$ is measurable in X for all $a \in \mathbb{R}$. Thus cf is measurable.

Now, let $A_a = (f^2)^{-1}(a,\infty) = \{x \mid (f^2)(x) > a\} = \{x \mid f(x) > \sqrt{a}\} \cup \{x \mid f(x) > \sqrt{a}\}$. But this is a union measurable sets since $f^{-1}(\sqrt{a},\infty)$ and $f^{-1}(\sqrt{a},\infty)$ are measurable in X for all $a > 0 \in \mathbb{R}$. For $a < 0$, we have $(f^2)^{-1}(a,\infty) = X$. Thus f^2 is measurable.

5. Let X be an uncountable set. Let

$$M = \{E \subset X \text{ such that either } E \text{ or } E^c \text{ is countable}\}.$$

Set $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if E^c is countable. Show that M is a σ-algebra and that μ is a measure on M.

(a) Since the empty set $\emptyset$ is countable, and $X^c = \emptyset$, $\emptyset \in M$ and $X \in M$.
(b) If $\{E_k\}_{k=1}^{\infty}$ is a countable collection of sets in M where either E_k is countable or E_k^c is countable, then if all E_k are countable, then $E = \bigcup_{k=1}^{\infty} E_k$ is a countable union of countable sets and hence is countable itself. Thus $E \in M$. If there exists one E_k such that E_k^c is countable, then E^c is countable and thus $E \in M$.
(c) If $\{E_k\}_{k=1}^{\infty}$ is a countable collection of sets in M where each E_k^c is countable, then $E^c = \left(\bigcap_{k=1}^{\infty} E_k\right)^c$ is a countable intersection of countable sets and hence is countable itself and $E \in M$. If one of the E_k is countable, then E is countable and so $E \in M$.

All sets $E \in X$ are either countable or uncountable, so image of $\mu = \{0,1\}$. If $\{E_k\}_{k=0}^{\infty}$ is a countable collection of measurable pairwise disjoint sets then, either

(a) all E_k are countable and hence $\bigcup_{k=1}^{\infty} E_k$ is countable, or
(b) one E_k^c is countable and hence $\left(\bigcup_{k=1}^{\infty} E_k\right)^c$ is countable.

We note that two distinct E_i and E_j cannot both have countable complements since $E_i \subset E_j^c$. Then,

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \begin{cases} 0 = \displaystyle\sum_{i=1}^{\infty} E_k & \text{all } E_k \text{ are countable} \\ 1 = 0 + 1 = \displaystyle\sum_{i=1}^{\infty} E_k & \text{one } E_k^c \text{ is countable.} \end{cases}$$

Thus μ is a measure on M.

6. Let A and B be any sets. Show that

$$\chi_{A\cap B} = \chi_A \cdot \chi_B, \quad \chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B, \quad \chi_{A^c} = 1 - \chi_A.$$

(a) $\chi_{A\cap B}(x) = 1 \iff x \in A \wedge x \in B \iff (\chi_A(x) = 1) \wedge (\chi_B(x) = 1)$.
And, $\chi_A \cdot \chi_B(x) = 1 \iff (\chi_A(x) = 1) \wedge (\chi_B(x) = 1)$. Thus $\chi_{A\cap B} = \chi_A \cdot \chi_B$.

(b) $\chi_{A\cup B}(x) = 0 \iff (x \notin A) \wedge (x \notin B)$.
$\chi_A + \chi_B - \chi_A \cdot \chi_B = 0 \iff (x \notin A) \wedge (x \notin B)$. Thus $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$.

(c) Taking $B = A^c$ in the above formula, we have, $\chi_{A\cup A^c} = \chi_A + \chi_{A^c} - \chi_A \cdot \chi_{A^c}$. But $\chi_{A\cup A^c} = 1$ and $\chi_A \cdot \chi_{A^c} = 0$ in all cases, and hence, $\chi_{A^c} = 1 - \chi_A$.

7. "Continuity Property of Decreasing Intersections": Let $A_n \in \mathcal{M}$ s.t. $A_1 \supseteq A_2 \supseteq A_3 ...$, and $\mu(A_1) < \infty$. Show that $\mu(\cap_1^\infty A_n) = \lim_{n\to\infty} \mu(A_n)$.

We define $\{B_k\}_{k=1}^\infty$ collection of measurable sets by $B_k = A_1 - A_k$. Then $\{B_k\}_{k=1}^\infty$ is an ascending collection of measurable sets with

$$\mu\left(\bigcup_{k=1}^\infty B_k\right) = \lim_{k\to\infty} \mu(B_k).$$

We have $\bigcup_{k=1}^\infty B_k = \bigcup_{k=1}^\infty (A_1 - A_k) = A_1 - \bigcap_{k=1}^\infty A_k$. Since $\mu(A_k) \le \mu(A_1) < \infty$, we write $\mu(B_k) = \mu(A_1) - \mu(A_k)$. So,

$$\mu\left(\bigcup_{k=1}^\infty B_k\right) = \mu\left(A_1 - \bigcap_{k=1}^\infty A_k\right) = \mu(A_1) - \mu\left(\bigcap_{k=1}^\infty A_k\right) = \mu(A_1) - \lim_{k\to\infty} \mu(A_k).$$

Hence we have, $\mu\left(\bigcap_{k=1}^\infty A_k\right) = \lim_{k\to\infty} \mu(A_k)$ as required.

8. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Show that if $A_1, A_2, ... \in \mathcal{M}$, but not necessarily pairwise disjoint, with $\mu(A_i) = 0$ for each i, then $\mu(\cup_j A_j) = 0$.

Let $B_i = \bigcup_{j=1}^i A_j$, then $\{B_i\}_{i=1}^\infty$ is an ascending collection of measurable sets and for each $i \in \mathbb{N}$,

$$\mu(B_i) = \mu\left(\bigcup_{j=1}^i A_j\right) \le \sum_{j=1}^i \mu(A_j) = 0.$$

Then, by the continuity of measure,

$$\mu\left(\bigcup_{j=1}^\infty A_j\right) = \mu\left(\bigcup_{j=1}^\infty B_j\right) = \lim_{j\to\infty} \mu(B_j) = 0.$$

9. Let $(X, \mathcal{M}, \mu)$ be a measure space. If A and B are disjoint measurable sets, and $\mu(A \cup B) = \mu(A)$, must $\mu(B) = 0$?

Since A and B are disjoint, $\mu(A) = \mu(A \cup B) = \mu(A) + \mu(B) \implies \mu(B) = 0$.

10. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $B \in \mathcal{M}$. Define $\nu(A) = \mu(A \cap B)$ for $A \in \mathcal{M}$. Show that ν is a measure.

Since $\nu(A) = \mu(A \cap B) \leq \mu(A) \leq \infty$ and $\nu(A) \geq 0$.

Now if $\{A_k\}_{k=0}^{\infty}$ is a countable pairwise disjoint collection of sets then $\{B \cap A_k\}_{k=0}^{\infty}$ is a countable collection of pairwise disjoint sets. Then

$$\nu\left(\bigcup_{n=0}^{\infty} A_k\right) = \mu\left(B \cap \bigcup_{n=0}^{\infty} A_k\right) = \sum_{n=0}^{\infty} B \cap A_k = \sum_{n=0}^{\infty} \nu(A_k).$$

Thus, ν is countably additive and is a measure.

11. Let $f : X \to [0, \infty]$ be a measurable function. Let

$$s_n(x) = \begin{cases} n, & f(x) \geq n \\ \frac{i-1}{2^n}, & \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}, i = \overline{1, n \cdot 2^n} \end{cases}$$

Show that $\{s_n\}$ is a monotone increasing sequence and $s_n \to f$ pointwise as $n \to \infty$.

We see that $s_n(x) \leq f(x)$ for all x. Clearly, $s_{n+1}(x) \geq s_n(x)$ for $f(x) \geq n$. Now, if $f(x) < n$, then $s_n = p/2^n$ where p is the greatest number that is less than $2^n f(x)$. Then if $q = s_{n+1}(x)$ is the greatest integer less than $2^{n+1} f(x) = 2 \cdot 2^n f(x)$ then $q > p$. Thus $\{s_n\}$ is a monotone increasing sequence.

From above, we have $s_n(x) = p/2^n > (2^n f(x) - 1)/2^n = f(x) - 1/2^n$ since $p > 2^n f(x)$ for all x. Then $f(x) - s_n(x) < 1/2^n$ and as $n \to \infty$, $s_n \to f$ pointwise.

12. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $A \in \mathcal{M}$. Let s and v be non-negative, simple, measurable functions. Let $\alpha, \beta \geq 0$. Show that

$$\int_A (\alpha s + \beta v) \, d\mu = \alpha \int_A s \, d\mu + \beta \int_A v \, d\mu$$

Moreover, if $s \leq v$ on A, then show that

$$\int_A s \, d\mu \leq \int_A v \, d\mu.$$

Now, we note that if $f = \alpha s + \beta v$, we choose a finite disjoint collection of measurable subsets $\{E_i\}_{i=1}^{n}$ of A such that their union is A and s and v are constant on each E_i. Let p_i and q_i be the values taken by s and v for each i. Then

$$\int_A s \, d\mu = \sum_{i=1}^{n} p_i \cdot \mu(E_i) \quad \text{and} \quad \int_A v \, d\mu = \sum_{i=1}^{n} q_i \cdot \mu(E_i).$$

Then clearly,

$$\int_A \alpha s + \beta v \, d\mu = \sum_{i=1}^{n} (\alpha p_i + \beta q_i) \cdot \mu(E_i) = \alpha \int_A s \, d\mu + \beta \int_A v \, d\mu.$$

Now, if $s \leq v$ on A, then we take $r = v - s$ to be the simple non-negative function and by linearity, we have,

$$\int_A v \, d\mu - \int_A s \, d\mu = \int_A r \, d\mu \leq 0$$

as required.