Introduction to Manifold Theory

Homework 7

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1. Do Exercise 3.11: show that if a second-countable Hausdorff topological space X admits an n-dimensional smooth atlas, then X is an n-dimensional topological manifold (and thus a smooth manifold equipped with the equivalence class of this atlas).

It suffices to show that the topological space X is locally homeomorphic to an open set of $\mathbb{R}^n$. Let $x \in X$. Since X admits an n-dimensional smooth atlas, we know that there exists a chart (U, φ) of open set U and a smooth map φ with $x \in U$ such that $\phi(U)$ is an open disk in $\mathbb{R}^n$. Then we see that x has an open neighborhood $N_\varepsilon(x)$ such that $\varphi(N_\varepsilon(x))$ is an open disk in $\mathbb{R}^n$. Hence, X is locally homeomorphic to $\mathbb{R}^n$ and is a smooth manifold.

2. Do Exercise 3.12: in imprecise terms, show that if a set X has a "sets-and-bijections smooth atlas" then X can be turned into a smooth manifold in a natural way.

We first check that the given topology $\mathscr{T}$ is indeed a valid topology on X:

(a) For all coordinate patch (U, φ) in the atlas $\mathcal{A}$, $\varphi(\emptyset \cap U) = \emptyset$ and $\varphi(X \cap U) = \varphi(U)$ is open in $\mathbb{R}^n$. Hence $\emptyset$ and X are in the topology.

(b) If $V = \bigcup_{\alpha \in A} U_\alpha$ is an arbitrary union of indexed (by A) open sets, then

$$\varphi(V \cap U) = \bigcup_{\alpha \in A} \varphi(U_\alpha) \cap \varphi(U).$$

Since eaach $\varphi(U_\alpha)$ and $\varphi(U)$ are open in $\mathbb{R}^n$, the arbitrary union $\bigcup_{\alpha \in A} \varphi(U_\alpha)$ is open and thus V is open in X.

(c) If $V = U_1 \cap U_2$ is intersection of open sets of X, then $\varphi(V) = \varphi(U_1) \cap \varphi(U_2)$ is open in $\mathbb{R}^n$. Thus $V \in \mathscr{T}$.

Thus the given topology is indeed a topology. If we start with finitely many U_i, then we see that all the open sets of X are generated by these U_i. Thus X has finitely many (hence, countable) basis and is second countable.

We note that every $U \subset X$ from the chart (U, φ) is open since $\varphi(U \cap U) = \varphi(U)$ is open in $\mathbb{R}^n$. Here, φ is a bijection and hence has an inverse φ^{-1}. Now we show that this is a homeomorphism by showing that both φ and φ^{-1} are continuous. Clealy for every open sets V in X, i

3. Do Exercise 3.13: prove the analogue of Exercise 3.11 where the charts in your smooth atlas are allowed to have general smooth manifolds (rather than just open disks) as codomains.

$\mathcal{A}$ is given by the collection of pairs

$$(\varphi^{-1}(V), \psi \circ \varphi)$$

where (V, ψ) is a coordinate patch from the atlas of the smooth n-manifold $X_{U,\varphi}$. φ is a homeomorphism and hence is continuous. Thus $\varphi^{-1}(V)$ is open in X. Since every $p \in X$ is in some (U, φ), p is contained in some $\varphi^{-1}(V)$. For each $p \in X$, we have an open set U containing p and since $\varphi(U)$ is open in the smooth manifold $X_{U,\varphi}$, there is an open disk around $\varphi(p)$ contained in $\varphi(U)$. Thus we have the map $\psi \circ \varphi : X \to \mathbb{R}^n$ such that the image of some open set U of X in $\mathbb{R}^n$ an open disk. Thus, X is locally homeomorphic to $\mathbb{R}^n$.

4. Do Exercise 3.14: prove the analogue of Exercise 3.12 where the charts in your “sets-and-bijections smooth atlas” are allowed to have general smooth manifolds (rather than just open disks) as codomains.