

Pack Pledge:

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1. Choose either d_1 or d_2 below and show that it is a metric on \mathbb{R}^n .

$$d_1(x, y) = \max\{|x_i - y_i|\} \quad \text{and} \quad d_2(x, y) = \sum_{i=1}^n |x_i - y_i| \quad (\text{taxicab metric}).$$

2. Let $C[0, 1]$ be the space of continuous functions on $[0, 1]$.

Show that $d(f, g) = \int_0^1 |f(x) - g(x)| dx$ is a metric on $C[0, 1]$

3. Let $x = \{x_n\}_{n=1}^\infty$ be a sequence.

(a) True or False: If $x_n \in l^p$ for some $1 \leq p < \infty$, then $x_n \rightarrow 0$ as $n \rightarrow \infty$. Justify your answer.

(b) True or False: If $x_n \rightarrow 0$ as $n \rightarrow \infty$, then $x_n \in l^p$, for some $1 \leq p < \infty$. Justify your answer.

4. Let $a, b \geq 0$, and $p \geq 1$. Prove that

$$(a + b)^p \leq 2^{p-1}(a^p + b^p)$$

Use the hints from class.

Let $f(x) = x^p$, $f : [0, \infty) \rightarrow \mathbf{R}$ and $p \geq 1$. Since f is a *convex* function, we have

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \quad \text{for } \alpha \in [0, 1].$$

For $\alpha = 1/2$, we have

$$\begin{aligned} f\left(\frac{a}{2} + \frac{b}{2}\right) &\leq \frac{f(a)}{2} + \frac{f(b)}{2} \\ \text{or, } \frac{1}{2^p} f(a + b) &\leq \frac{1}{2} (f(a) + f(b)) \\ \text{or, } (a + b)^p &\leq 2^{p-1} (a^p + b^p). \end{aligned}$$

5. For $p > 1$, let q be its conjugate, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Prove the following inequality:

$$u \cdot v \leq \frac{1}{p} u^p + \frac{1}{q} v^q, \quad \forall u, v \geq 0$$

Use the hints from class.

If either u or v equals 0, then the inequality follows immediately. Suppose $u > 0$, $v > 0$ and let $f(x) = e^x$. Since f is a *convex* function,

$$\begin{aligned} u \cdot v &= \exp(\log u + \log v) \\ &= f\left(\frac{1}{p} \log u^p + \frac{1}{q} \log v^q\right) \\ &\leq \frac{1}{p} f(\log u^p) + \frac{1}{q} f(\log v^q) \\ &= \frac{u^p}{p} + \frac{v^q}{q}. \end{aligned}$$

6. Prove Holder's Inequality for Sums. Use the hints from class.

Holder's inequality: Let $p, q \geq 1$ be conjugate exponents. Let $x = \{x_i\}_1^\infty \in l^p$ and $y = \{y_i\}_1^\infty \in l^q$. Then

a. $xy = \{x_i y_i\}_1^\infty \in l^1$ and

b. $\sum_1^\infty |x_i y_i| \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} \cdot (\sum_1^\infty |y_i|^q)^{\frac{1}{q}}.$

Let $u_i = \frac{x_i}{(\sum_1^\infty |x_i|^p)^{1/p}}$ and $v_i = \frac{y_i}{(\sum_1^\infty |y_i|^q)^{1/q}}$. Then by Young's inequality,

$$\begin{aligned} u_i \cdot v_i &= \frac{x_i}{(\sum_1^\infty |x_i|^p)^{1/p}} \cdot \frac{y_i}{(\sum_1^\infty |y_i|^q)^{1/q}} \\ &\leq \frac{x_i^p}{p \sum_1^\infty |x_i|^p} + \frac{y_i^q}{q \sum_1^\infty |y_i|^q} \end{aligned}$$

Let $m = (\sum_1^\infty |x_i|^p)^{1/p}$ and $n = (\sum_1^\infty |y_i|^q)^{1/q}$. Then from above we have

$$\begin{aligned} \sum_1^\infty |x_i y_i| &= mn \sum_1^\infty |u_i v_i| \leq mn \sum_1^\infty \left| \frac{1}{pm^p} x_i^p + \frac{1}{qn^q} y_i^q \right| \leq mn \left(\frac{1}{pm^p} \cdot \sum_1^\infty |x_i|^p + \frac{1}{qn^q} \sum_1^\infty |y_i|^q \right) \\ &= mn \left(\frac{1}{pm^p} \cdot m^p + \frac{1}{qn^q} \cdot n^q \right) = mn \end{aligned}$$

Hence $\sum_1^\infty |x_i y_i| \leq mn = (\sum_1^\infty |x_i|^p)^{1/p} \cdot (\sum_1^\infty |y_i|^q)^{1/q}$ which proves (b). Since $0 \leq \sum_1^\infty |x_i y_i| < \infty$, we also have (a) by definition.

7. Prove Minkowski's Inequality for Sums. Use the hints from class.

Minkowski's inequality : Let $p \geq 1$ and $x = \{x_i\}_1^\infty \in l^p$ and $y = \{y_i\}_1^\infty \in l^p$. Then

a. $x + y = \{x_i + y_i\}_1^\infty \in l^p$ and

b. $(\sum_1^\infty |x_i + y_i|^p)^{\frac{1}{p}} \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} + (\sum_1^\infty |y_i|^p)^{\frac{1}{p}}.$

First we show that $x + y \in l^p$ by showing that

$$\left(\sum_1^\infty |x_i + y_i|^p \right)^{1/p} < \infty$$

We have,

$$\sum_1^\infty |x_i + y_i|^p \leq \sum_i^\infty (|x_i| + |y_i|)^p \leq 2^{p-1} \left(\sum_i^\infty |x_i|^p + \sum_i^\infty |y_i|^p \right) < \infty.$$

Now, since $x, y \in l^p$, $d_p(x, y) < \infty$. If $p = 1$ then the Minkowski inequality follows from the triangle inequality. Let $p > 1$ then

$$\sum_1^\infty |x_i + y_i|^p = \sum_1^\infty (|x_i + y_i| |x_i + y_i|^{p-1}) \leq \sum_1^\infty (|x_i| |x_i + y_i|^{p-1} + |y_i| |x_i + y_i|^{p-1}) \quad (1)$$

$$= \sum_1^\infty (|x_i| |x_i + y_i|^{p-1}) + \sum_1^\infty (|y_i| |x_i + y_i|^{p-1}) \quad (2)$$

Now let q be the conjugate exponent of p , then we have $\frac{1}{p} + \frac{1}{q} = 1 \iff p + q = pq \iff p = p(q - 1)$. Then, at line 2

$$\left(\sum_1^\infty |x_i + y_i|^{(p-1)q} \right)^{1/q} = \left(\sum_1^\infty |x_i + y_i|^p \right)^{1/q} < \infty$$

which shows that $\{|x_i + y_i|^{p-1}\}_i^\infty \in l^q$. Then by Holder's inequality,

$$\sum_1^\infty |x_i| |x_i + y_i|^{p-1} \leq \left(\sum_1^\infty |x_i|^p \right)^{1/p} \cdot \left(\sum_1^\infty |x_i + y_i|^{(p-1)q} \right)^{1/q} \quad (3)$$

$$= \left(\sum_1^\infty |x_i|^p \right)^{1/p} \cdot \left(\sum_1^\infty |x_i + y_i|^p \right)^{1/q} \quad (4)$$

Using the results from line 2 and line 4 on line 1,

$$\sum_1^\infty (|x_i + y_i|^p) \leq \sum_1^\infty (|x_i| |x_i + y_i|^{p-1}) + \sum_1^\infty (|y_i| |x_i + y_i|^{p-1}) \quad (5)$$

$$\text{or, } \sum_1^\infty (|x_i + y_i|^p) \leq \left(\sum_1^\infty |x_i + y_i|^p \right)^{1/q} \cdot \left(\left(\sum_1^\infty |x_i|^p \right)^{1/p} + \left(\sum_1^\infty |y_i|^p \right)^{1/p} \right) \quad (6)$$

Dividing both sides by $(\sum_1^\infty |x_i + y_i|^p)^{1/q}$, we get the Minkowski's inequality (since $1 - \frac{1}{q} = \frac{1}{p}$).

8. For $1 \leq p < \infty$, let $l^p = \{x = \{x_i\}_1^\infty \mid \sum_1^\infty |x_i|^p < \infty\}$. For any $x, y \in l^p$, define

$$d_p(x, y) = \left(\sum_1^\infty |x_i - y_i|^p \right)^{1/p}$$

Prove that (l^p, d_p) is a metric space.

- i. Since $d_p(x, y)$ is the p th root of a sum of positive numbers, $d_p \geq 0$. Also from Minkowski inequality (a.), we have $d_p < \infty$.

ii.

9. Prove Jensen's Inequality for Sums. Use the hints from class.
10. Show that $l^1 \subset l^2$ without using Jensen's inequality. Then show that inclusion is strict, i.e., find an element in l^2 that is not in l^1 .