Analysis II

Homework 1

Nutan Nepal

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Pack Pledge: I have neither given nor received unauthorized aid on this test or assignment.

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1. (a) Show that if P is any partitions of $[a,b]$, and Q is a refinement of P (i.e. $P \subseteq Q$), then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

(b) Let P and Q be any partitions of $[a,b]$. Show that $L(f,P) \leq U(f,Q)$.

(a) Let $P = \{x_0, x_1, \ldots, x_n\}$ and $Q = \{y_0, y_1, \ldots, y_m\}$ be the given partitions. Then, each interval $I_i = [x_i, x_{i+1}]$ of P contains subintervals $J_1 = [x_i = y_\alpha, y_{\alpha+1}], \ldots, J_k = [y_{\alpha+k-1}, y_{\alpha+k} = x_{i+1}]$ of Q and the length of the intervals J_k add up to the length of I. Furthermore, $m_I = \inf_{x \in I} f(x) \leq m_{J_\beta}$ and $M_I = \sup_{x \in I} f(x) \geq M_{J_\beta}$ for each $\beta = 1, \ldots, k$. Then for each $i = 0, \ldots, n-1$,

$$m_I \cdot (x_{i+1} - x_i) \leq \sum_{p=1}^{k} m_{J_p} \cdot (x_{i+1} - x_i) = \sum_{p=1}^{k} m_{J_p} \cdot (y_{\alpha+p-1} - y_{\alpha+p}).$$

$$L(f,P) = \sum_{i=0}^{n} m_I \cdot (x_{i+1} - x_i) \leq L(f,Q).$$

For upper sums, we obtain similar result with $U(f,Q) \leq U(f,P)$ and we have $L(f,P) \leq U(f,P)$ because $m_I \leq M_I$ for all intervals I. Combining the inequalities, we obtain the required inequality

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

(b) Either P is a refinement of Q or Q is a refinement of P. For both cases we see that $L(f,P) \leq U(f,Q)$ from the result in (a).

2. Let $f : [a,b] \to \mathbb{R}$ be continuous. Show that f is Riemann integrable on $[a,b]$.

Since f is continuous on the closed interval, it is uniformly continuous on $[a,b]$. For any given $\varepsilon > 0$ we have $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon/(b-a)$ at all points

$x, y \in [a, b]$. Then, let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition on $[a, b]$ such that $x_{i+1} - x_i < \delta$. Then, if m_i and M_i denote the infinimum and supremum of $f(x)$ on the partition $[x_i, x_i + 1]$, we have $M_i - m_i < \varepsilon/(b - a)$. Multiplying by $(x_{i+1} - x_i)$ and summing over all i we get

$$U(f, P) - L(f, P) < (\varepsilon/(b - a)) \sum_{i=0}^{n-1} (x_{i+1} - x_i) = \varepsilon.$$

Since, ε was arbitrary, we see that f is Riemann integrable.

3. Show that if $f : [a, b] \to \mathbb{R}$ is monotone (without loss of generality, assume that f is increasing), then f is Riemann integrable on $[a, b]$.

Let $P_n = \{x_0, x_1, \ldots, x_n\}$ be the partition of $[a, b]$ into n equal intervals of length $(b - a)/n$. Let m_i and M_i denote the infinimum and supremum of the function f on the interval $[x_i, x_{i+1}]$. Then since f is monotonic, $m_i = f(x_i)$ and $M_i = f(x_{i+1})$. So,

$$L(f, P_n) = \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i), \quad \text{and} \quad U(f, P_n) = \sum_{i=0}^{n-1} f(x_{i+1})(x_{i+1} - x_i).$$

Since $x_{i+1} - x_i = (b - a)/n$, we have

$$U(f, P_n) - L(f, P_n) = \frac{b - a}{n} \sum_{i=0}^{n-1} f(x_{i+1}) - f(x_i) = \frac{b - a}{n}(f(b) - f(a)).$$

We see that the difference goes to 0 as $n \longrightarrow 0$. Hence f is Riemann integrable.

4. If f and g are Riemann-integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, show that

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition for $[a, b]$. We define a function $h(x) = g(x) - f(x)$ and see that $h(x) \geq 0$ for all x in the interval $[a, b]$. Let m_i and M_i denote the infinimum and supremum of $h(x)$ on the interval (x_i, x_{i+1}). Then $0 \leq m_i \leq M_i$ for all i and since g and f are Riemann integrable, so is h. Hence

$$\int_a^b h(x)dx = \int_a^b g(x)dx - \int_a^b f(x)dx \geq 0$$

which gives the required result.

5. Let $f : [0, 1] \to \mathbb{R}$, $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ (the Dirichlet function).

   (a) Show that f is discontinuous everywhere on $[0, 1]$

(b) Show that f is not Riemann integrable.

(a) Let $x_0 \in \mathbb{Q}$. Then, for $\varepsilon = 1$, we see that there exists no $\delta > 0$ such that $f(x) \in (0, 2)$ for all $x \in (x_0 - \delta, x_0 + \delta)$ since every interval $(x_0 - \delta, x_0 + \delta)$ contains another irrational point. Hence, f is not continuous on rational points. Similarly, f is not continuous on irrational points in $[0, 1]$ either.

(b) For any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0, 1]$, since every intervals contain rational as well as irrational points, $\inf f(x) = 0$ and $\sup f(x) = 1$ for each interval. Then $L(f, P) = 0$ and $U(f, P) = 1$ for every partition. For $\varepsilon = 1$ we see that no such partition satifying $U(f, P) - L(f, P)$ exists. Hence, f is not Riemann-integrable.

6. Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable. Suppose that $F : [a, b] \to \mathbb{R}$ is continuous and

$$F'(x) = f(x), \quad \text{for all } x \in (a, b)$$

Show that

$$F(b) - F(a) = \int_a^b f(x)dx$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition for $[a, b]$. Then for any interval $I = [x_i, x_{i+1}]$, by mean value theorem, we have

$$F(x_{i+1}) - F(x_i) = F'(x_{i_k})(x_{i+1} - x_i) = f(x_{i_k})(x_{i+1} - x_i)$$

for some $x_{i_k} \in (x_i, x_{i+1})$. Let m_i and M_i denote the infinimum and supremum of $f(x)$ on the interval (x_i, x_{i+1}). Then $m_i \leq f(x_{i_k}) \leq M_i$ and taking sum over all i from 0 to $n-1$, we obtain

$$L(f, P) \leq \sum_{i=0}^{n-1} F(x_{i+1}) - F(x_i) \leq U(f, P).$$

However, since P was an arbitrary partition and f is integrable, we have

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} F(x_{i+1}) - F(x_i) = F(b) - F(a).$$

7. Recall the Fundamental Theorem of Calculus: If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, then $F \in C^1[a, b]$ and $F'(x) = f(x)$.

Now assume that f is Riemann integrable on $[a, b]$ (and implicitly on each subinterval of $[a, b]$) and $F(x) = \int_a^x f(t)dt$. True or false: F is differentiable. Justify your answer.

We define f piecewise as follows: $f : [a, b] \to \mathbb{R}, \quad f(x) = \begin{cases} 1, & x \in [a, a + (b - a)/2] \\ 0, & x \in (a + (b - a)/2, b]\mathbb{Q} \end{cases}$ f is Riemann integrable and the function F is increasing for the first half interval. However, it stays constant for the second half and thus it is not differentiable at $x = a + (b - a)/2$.

8. Show that a countable set of real numbers has measure 0.

Given a countable set $\{x_1, x_2, \ldots\}$ and an arbitrary $\varepsilon > 0$, we choose open covers $(x_i - \varepsilon/2^{i+2}, x_i + \varepsilon/2^{i+2})$ for x_i. The open covers are each of length $\varepsilon/2^{i+1}$. We have, $\sum_{i=1}^{\infty} \varepsilon/2^{i+1} = \varepsilon/2 < \varepsilon$. Since, the epsilon was arbitrary, we see that the countable set has measure 0.

9. Suppose that $f : X \to Y$ is a function and suppose that A_α, A, and B are subsets of Y. Show that

(a) $f^{-1}\left(\bigcup_\alpha A_\alpha\right) = \bigcup_\alpha f^{-1}(A_\alpha)$

(b) $f^{-1}\left(\bigcap_\alpha A_\alpha\right) = \bigcap_\alpha f^{-1}(A_\alpha)$

(c) $f^{-1}(A^c) = (f^{-1}(A))^c$

(d) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$

Which of these remain true when f^{-1} is replaced by f (and the sets are now subsets of X)?

(a) Let $x \in f^{-1}\left(\bigcup_\alpha A_\alpha\right)$. Then, there is $y \in \bigcup_\alpha A_\alpha$ such that $f(x) = y$. So y is in some A_α. Thus $x \in \bigcup_\alpha f^{-1}(A_\alpha)$. Now, let $x \in \bigcup_\alpha f^{-1}(A_\alpha)$, then x is in some $f^{-1}(A_\alpha)$. So there exists some $y \in A_\alpha$ such that $f(x) = y$. Since $y \in \bigcup_\alpha A_\alpha$, we have $x \in f^{-1}\left(\bigcup_\alpha A_\alpha\right)$.

(b) Let $x \in f^{-1}\left(\bigcap_\alpha A_\alpha\right)$. Then, there is $y \in \bigcap_\alpha A_\alpha$ such that $f(x) = y$. So y is in all A_α and x belongs to all $f^{-1}(A_\alpha)$. Thus $x \in \bigcap_\alpha f^{-1}(A_\alpha)$. Now, let $x \in \bigcap_\alpha f^{-1}(A_\alpha)$, then x is in all $f^{-1}(A_\alpha)$ and there exists y in every A_α such that $f(x) = y$. Since $y \in \bigcap_\alpha A_\alpha$, we have $x \in f^{-1}\left(\bigcap_\alpha A_\alpha\right)$.

(c) Let $x \in f^{-1}(A^c)$. Then there is some $y \notin A$ such that $f(x) = y$. So $x \notin f^{-1}(A)$ and hence $x \in (f^{-1}(A))^c$. Now, let $x \in (f^{-1}(A))^c$. Then $f(x) \notin A$ and hence $f(x) \in A^c \implies x \in f^{-1}(A^c)$.

(d) Let $x \in f^{-1}(A \setminus B)$. Then there is some $y \in A$ and $y \notin B$ such that $f(x) = y$. So $x \notin f^{-1}(B)$ and hence $x \in f^{-1}(A) \setminus f^{-1}(B)$. Now, let $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then $f(x) \notin B$ and hence $f(x) \in A \setminus B \implies x \in f^{-1}(A \setminus B)$.

10. (a) If I and J are open intervals in $\mathbb{R}$ and if $f : X \to \mathbb{R}^2$ is any function with $f(x) = (u(x), v(x))$, show that

$$f^{-1}(I \times J) = u^{-1}(I) \cap v^{-1}(J).$$

If $x \in f^{-1}(I \times J)$, then there exists $(y, z) \in I \times J$ such that $y = u(x)$ and $z = v(x)$. Hence $x \in u^{-1}(I)$ and $v^{-1}(J)$. Now, let $x \in u^{-1}(I) \cap v^{-1}(J)$. Then $x \in u^{-1}(I)$ and there exists $y \in I$ such that $y = f(x)$. Also $x \in v^{-1}(J)$ and so there exists $z \in J$ such that $z = f(x)$. Thus x is in the preimage of (y, z) and so in $f^{-1}(I \times J)$. Hence we have the required result:

$$f^{-1}(I \times J) = u^{-1}(I) \cap v^{-1}(J).$$

(b) Show that if $u : X \to \mathbb{R}$ and $v : X \to \mathbb{R}$ are measurable functions, then so is $f = (u, v) : X \to \mathbb{R}^2$. [You can assume that any open set in $\mathbb{R}^2$ can be written as countable union of open rectangles $R_k = I_k \times J_k$, where I_k and J_k are open intervals in $\mathbb{R}$.]

Since any open set in $\mathbb{R}^2$ can be witten as countable union of open rectangles $R_k = I_k \times J_k$, we have, for open set $U \subset \mathbb{R}^2$,

$$U = \bigcup_{k=1}^{\infty} I_k \times J_k.$$

From (a), we see that $f^{-1}(I \times J) = u^{-1}(I) \cap v^{-1}(J)$. Since u and v are measurable, $u^{-1}(I)$ and $v^{-1}(J)$ are measurable sets and so is their intersection. Hence f is a measurable function.