# Introduction to Manifold Theory

Homework 2

Nutan Nepal

September 29, 2022

---

1. Do Exercise 2.6 (show that for topological spaces $X$, $Y$, $Z$, the "rearrange-the-parentheses" map from $(X \times Y) \times Z$ to $X \times (Y \times Z)$ is a homeomorphism).

   Let the function $f : (X \times Y) \times Z \to X \times (Y \times Z)$ be defined as

   $$f((x, y), z) = f(x, (y, z))$$

   where $x$, $y$ and $z$ are respective points of the topological spaces. We see that the map is clearly bijective and hence invertible.

   Now, for each open set $U_x \times (U_y \times U_z)$, the preimage of $f$ is given by $(U_x \times U_y) \times U_z$ which is open in $(X \times Y) \times Z$. Similarly, for each open set $(U_x \times U_y) \times U_z$, the preimage of $f^{-1}$ is given by $U_x \times (U_y \times U_z)$ which is open in $X \times (Y \times Z)$. Hence $f$ and $f^{-1}$ are both continuous and the "rearrange the parentheses" map is homeomorphism.

2. Do Exercise 2.7 (show that the product topology and the usual topology on $\mathbb{R}^n$ agree).

   Suppose $\mathscr{P}$ be the product topology and $\mathscr{T}$ be the usual topology in $\mathbb{R}^n$. Let $U$ be open in the product topology, then for all $x = (x_1, \ldots, x_n) \in U$ there exists open neighborhoods $U_i \in \mathbb{R}$ such that $x_i \in U_i$ and $U_1 \times \cdots \times U_n \subset U$. Then for all $x_i$, there exists an open interval $(x_i - \delta_i, x_i + \delta_i)$ for some $\delta_i > 0$. Let $\delta = \min\{\delta_i\}$ taken over all $i$ from 1 to $n$. Clearly, $\delta > 0$ and $x \in B_\delta(x) \subset U_1 \times \cdots \times U_n \subset U$. This shows that $\mathscr{P} \subset \mathscr{T}$.

   Now let $U$ be open with respect to the usual topology. Then for all $x \in U$, there exists an open ball $B_\delta(x)$ containing $x$ such that $B_\delta(x) \subset U$ for some $\delta > 0$. Let $\delta_i = \delta/\sqrt{2}$. Then each $x_i$ is contained in the interval $U_i = (x_i - \delta_i, x_i + \delta_i)$ and we see that $B_\delta(x) \supset U_1 \times \cdots \times U_n$. Then $x \in U_1 \times \cdots \times U_n \subset B_\delta(x) \subset U$. Hence $U$ is open in the product topology and $\mathscr{P} \supset \mathscr{T}$. So we see that the two topologies agree.

3. The following exercises are about the "line with two origins" of Example 2.44, which we will call $X$.

   (a) Show that the construction in Example 2.44 defines a topology on $X$.

The construction in Example 2.44 is reproduced below:

Let $\mathscr{B}$ be the set of subsets of $X$ that have one of the following two forms:

i. open intervals $(a,b) \subset \mathbb{R}$ (with $a$ and $b$ finite and $a < b$);

ii. sets of the form $((a,b)\backslash 0) \cup \overline{0}$ whenever $a < 0 < b$.

Then we declare a subset $U$ of $X$ to be open if, for all $x \in U$, there exists a subset $B$ of $\mathscr{B}$ with $x \in B$ and $B \subset U$.

Let $\mathscr{T}$ be the collection of open sets as defined above. We now show that it is a topology.

a. Clearly, $\phi \in \mathscr{T}$ and also $X \in \mathscr{T}$.

b. Let $A = \bigcup_i U_i$ be the union of arbitrary collection of indexed open sets. For all $x \in A$ then there exists a $U_i$ such that $x \in U_i$. So, there exists a subset $B$ of $\mathscr{B}$ with $x \in B$ and $B \subset U_i \subset A$. Hence, A is open.

c. Let $A = U_1 \cap U_2$ be the finite intersection of open sets of $X$. For any $x \in A$ we see that $x \in U_1$ and $x \in U_2$. Then there exists a subset $B_1$ of $\mathscr{B}$ with $x \in B_1$ and $B_1 \subset U_1$ and there exists a subset $B_2$ of $\mathscr{B}$ with $x \in B_2$ and $B_2 \subset U_2$. If $x \neq \bar{0}$ then the problem reduces to $\mathbb{R}$ which implies that $A$ is open. If $x = \bar{0}$ then we see that $B_1 \cap B_2$ is the intersection of open intervals and $\bar{0}$ which is again open in $X$.

Thus $X$ is a topological space with the topology $\mathscr{T}$.

(b) Show that with this topology, $X$ is locally homeomorphic to $R$.

For any point $x \neq \overline{0}$ in $X$, we observe that there is an open ball $(x-\delta, x+\delta)$ around $x$ for some $\delta > 0$. Since any open intervals of $\mathbb{R}$ are homeomorphic to $\mathbb{R}$ itself, we see that $X$ is locally homeomorphic $\mathbb{R}$ for every point $x \neq \overline{0}$.

Now, when $x = \overline{0}$ we take $Y = (-\delta, 0) \cup (0, \delta) \cup \{\overline{0}\}$ and define a function $f : Y \to \mathbb{R}$ by $f(\overline{0}) = 0$ and $f(y) = \tan(\pi y/2\delta)$. We see that $f$ is invertible, continuous and has a continuous inverse and hence is a homeomorphism. Thus, $X$ is locally homeomorphic to $\mathbb{R}$.

(c) Show that $X$ is not Hausdorff.

For every $\epsilon > 0$, the neighborhood $N_\epsilon(0)$ of the point 0 intersects with the neighborhood around the point $\overline{0}$ non-trivially. So, $X$ is not Hausdorff.