Analysis I
Homework 3

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October 14, 2022

Pack Pledge: I have neither given nor received unauthorized aid on this test or assignment.

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1. **Kreyszig p.303 / Problem 4.** It is important that in Banach's theorem 5.1-2 the condition (1) cannot be replaced by $d(Tx, Ty) < d(x, y)$ when $x \neq y$. To see this, consider $X = \{x \mid 1 \leq x < \infty\}$, taken with the usual metric of the real line, and $T : X \to X$ defined by $x \to x + x^{-1}$. Show that $|Tx - Ty| < |x - y|$ when $x \neq y$ but the mapping has no fixed points.

For the given map, we see that

$$|Tx - Ty| = |x + x^{-1} - y - y^{-1}| = \left| x - y + \frac{y - x}{xy} \right|.$$

Since $(x - y)$ and $(y - x)/xy$ have different signs, $|x - y + (y - x)/xy| < |x - y|$ and we see that $|Tx - Ty| < |x - y|$.

Now, taking $T(x) = x$, we have $1/x = 0$. But no point $x \in X$ satisfies this. Thus, T has no fixed point in X.

2. **Kreyszig p.303 / Problem 6.** If T is a contraction, show that T^n, $(n \in \mathbb{N})$ is a contraction. If T^n is a contraction for an $n > 1$, show that T need not be a contraction.

If T is a contraction on a metric space X, then there is a positive real number $\alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all x, $y \in X$. Then, for $n \in \mathbb{N}$,

$$d(T^n x, T^n y) = d(T \cdot T^{n-1} x, T \cdot T^{n-1} y) \leq \alpha d(T^{n-1} x, T^{n-1} y)$$

Continuing this process, we get

$$d(T^n x, T^n y) < \alpha^n d(x, y).$$

Since, $0 < \alpha^n < 1$, T^n is a contraction. To show that T^n being a contraction does not imply that

T is a contraction, we define $T : \mathbb{R} \to \mathbb{R}$ and by

$$T(x) = \begin{cases} -2x, & x < 0 \\ x/4, & x \geq 0 \end{cases}$$

We see that T increases the distance when $x < 0$, so it is not a contraction. But $T(X) = [0, \infty)$, so T^2 decreases the distance by $1/4$. Hence T^2 is a contraction.

3. **Kreyszig p.32 / Problem 2.** If (x_n) is Cauchy and has a convergent subsequence, say, $x_{n_k} \to x$, show that (x_n) is convergent with the limit x.

If $\{x_{n_k}\}_{n_k=1}^{\infty}$ is the convergent subsequence of the sequence $\{x_n\}_1^{\infty}$ in the metric space X, then as $x_{n_k} \to x$, for every $\varepsilon > 0$ there exists $N_1 \in \mathbb{Z}$ such that for all $n_k > N_1$, we have

$$d(x_{n_k}, x) < \varepsilon/2.$$

Also, as $\{x_n\}_1^{\infty}$ is Cauchy in X, for $\varepsilon > 0$ (same as above), there exists N such that for all m, $n > N$ we have,

$$d(x_n, x_m) < \varepsilon/2.$$

Since there are infinitely many terms in the subsequence, we have M such that $n_k > N$ for all $k > M$. Then, for all $k > M$ and $n > N$ we have,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) = \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since, this is true for any arbitrary $\varepsilon > 0$, we have $x_n \to x$.

4. **Kreyszig p.56 / Problem 5.** Show that $\{x_1, \cdots, x_n\}$, where $x_j(t) = t^j$, is a linearly independent set in the space $C[a, b]$.

We note that if the linear combinations of $x_j(t) = t^j$ is identically 0, that is

$$\alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n = 0$$

then equating the corresponding coefficients, we must have each $\alpha_i = 0$. Hence the set $\{x_1, \cdots, x_n\}$ must be linearly independent.

5. **Kreyszig p.56 / Problem 10.** If Y and Z are subspaces of a vector space X, show that $Y \cap Z$ is a subspace of X.

For x, $y \in Y \cap Z$, we see that $x + y \in Y$ and $x + y \in Z$ since Y and Z are both subspaces of X. So, $x + y \in Y \cap Z$ and similarly, for $\alpha \in \mathbb{R}$, $\alpha \cdot x \in Y$ and $\alpha \cdot x \in Z$. So, $\alpha \cdot x \in Y \cap Z$. Since, $Y \cap Z$ is closed under addition and scalar multiplication, it is a subspace of X. (Since Y and Z are subset of X, they satisfy the linearity property.)

6. **Kreyszig p.66 / Problem 11.** Show that the closed unit ball

$$B(0,1) = \{x \in X : \|x\| = 1\}$$

in a normed space X is convex.

We need to show that if $x,\ y \in B(0,1)$ then any point z given by $z = \alpha x + (1-\alpha)y$ for some $\alpha \in [0,1]$ is also in $B(0,1)$. Since $x,\ y \in B(0,1)$, we see that

$$\|z\| = \|\alpha x + (1-\alpha)y\| \le |\alpha|\|x\| + |1-\alpha|\|y\| \le \alpha + 1 - \alpha = 1.$$

So, $z \in B(0,1)$ and hence $B(0,1)$ is convex.

7. **Kreyszig p.70 / Problem 2.** Show that c_0 in Prob. 1 (the space of all sequences of scalars converging to zero) is a closed subspace of l^∞, so that c_0 is complete by 1.5-2 and 1.4-7.

To show that c_0 is closed, we show that every sequence $\{x_i^k\}$ in c_0 has a limit point in c_0. Here, each x_i is a sequence of scalars that converge to 0. Since l^∞ is complete, let $y \in l^\infty$ be the limit point of the sequence $\{x_i^k\}$ in c_0. Then for all $\varepsilon > 0$, we have N such that

$$\sup_{1\le i<\infty} |x_i^k - y_i| < \varepsilon/2$$

for all $k > N$. Since for each k, $x_i^k \to 0$, we have for each k, there exists $N_1 \in \mathbb{N}$ such that

$$|x_i^k| = |x_i^k - 0| < \varepsilon/2$$

for all $i > N_1$. Then

$$|y_i - 0| = |y_i| \le |x_i^k - y_i| + |x_i^k| < \varepsilon$$

for all $i > N_1$ and $k > N$. This means that $y_i \to 0$ and thus $\{y_i\}_1^\infty \in c_0$. Thus, c_0 is closed.

8. **Kreyszig p.70 / Problem 4. (Continuity of vector space operations)** Show that in a normed space X, vector addition and multiplication by scalars are continuous operations with respect to the norm; that is, the mappings defined by $(x,y) \to x+y$ and $(\alpha, x) \to \alpha x$ are continuous.

We define the norm in $X \times X$ to be

$$\|(x,y)\|_{X\times X} = \max\{\|x\|_X, \|y\|_X\}$$

and we define the norm in $\mathbb{R} \times X$ to be

$$\|(\alpha,y)\|_{\mathbb{R}\times X} = \max\{|\alpha|, \|y\|_X\}.$$

Then, for an arbitrary point $(x_0,y_0) \in X \times X$, for every $\varepsilon > 0$, we have $\delta = \varepsilon/2$ such that

$$\|(x,y) - (x_0,y_0)\|_{X\times X} = \|(x-x_0, y-y_0)\|_{X\times X} = \max\{\|x-x_0\|_X, \|y-y_0\|_X\} < \delta$$

implies

$$\|T(x,y) - T(x_0,y_0)\|_X = \|x - x_0 + y - y_0\|_X \le \|x-x_0\| + \|y-y_0\|_X < \delta + \delta = \varepsilon.$$

Hence, the map $T(x, y) = x + y$ is continuous.

Now, for the map $T : \mathbb{R} \times X \to X$ given by $T(\alpha, x) = \alpha x$, for every $\varepsilon > 0$, we take $\delta = \min\{\varepsilon/2(|\alpha_0| + 1), \varepsilon/2(\|x_0\|_X + 1), 1\}$, we get

$$\|(\alpha, x) - (\alpha_0, x_0)\| = \|(\alpha - \alpha_0, x - x_0)\| = \max\{|\alpha - \alpha_0|, \|x - x_0\|_X\} < \delta$$

implies

$$\begin{aligned}\|T(\alpha, x) - T(\alpha_0, x_0)\|_X = \|\alpha x - \alpha_0 x_0\|_X \leq & |\alpha_0|\|x - x_0\| + |\alpha - \alpha_0|\|x\|_X \\ \leq & |\alpha_0|\|x - x_0\| + |\alpha - \alpha_0|(\|x - x_0\|_X + \|x_0\|_X) \\ < & \delta|\alpha_0| + \delta(\delta + \|x_0\|_X) \\ \leq & \frac{\varepsilon|\alpha_0|}{2(|\alpha_0| + 1)} + \frac{\varepsilon}{2(\|x_0\|_X + 1)}(\|x_0\|_X + 1) < \varepsilon.\end{aligned}$$

Hence, T is continuous.

9. **Kreyszig p.70 / Problem 5.** Show that $x_n \to x$ and $y_n \to y$ implies $x_n + y_n \to x + y$. Show that $\alpha_n \to \alpha$ and $x_n \to x$ implies $\alpha_n x_n \to \alpha x$.

If $x_n \to x$, $y_n \to y$ and $\alpha_n \to \alpha$, then for every $\varepsilon > 0$ we have $N \in \mathbb{Z}$ such that for all $n > N$,

$$\|x_n - x\| < \varepsilon/2, \quad \|y_n - y\| < \varepsilon/2 \quad \text{and} \quad \|\alpha_n - \alpha\| < \varepsilon/2.$$

Then for the same N,

$$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) - (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\| < \varepsilon.$$

Hence, $x_n + y_n \to x + y$ and similarly,

$$\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| = \|\alpha_n(x_n - x) + x(\alpha_n - \alpha)\| \leq |\alpha_n|\|x_n - x\| + \|y\||\alpha_n - \alpha| < \varepsilon.$$

Hence $\alpha_n x_n \to \alpha x$.

10. **Kreyszig p.71 / Problem 9.** Show that in a Banach space, an absolutely convergent series is convergent.

If a series $\sum_1^\infty x_i$ in X absolutely converges, then the sequence of partial sums $\{y_i\}_1^\infty$ given by

$$y_k = \sum_{i=1}^{k} \|x_i\|$$

converges to some point $y \in \mathbb{R}$. Hence, $\{y_i\}_1^\infty$ is Cauchy and for every $\varepsilon > 0$, we have $N \in \mathbb{Z}$ such that for all $m > n > N$,

$$|y_m - y_n| = \sum_{i=n+1}^{m} \|x_i\| < \varepsilon.$$

Now, if $\{s_i\}_1^\infty$ is the sequence of partial sums of the sequence $\{x_i\}_1^\infty$ given by $s_k = \sum_{i=1}^k x_i$, then for the same $\varepsilon > 0$ and $m > n > N$ as above, we have

$$\|s_m - s_n\| = \left\| \sum_{i=n+1}^{m} x_i \right\| \leq \sum_{i=n+1}^{m} \|x_i\| < \varepsilon.$$

Hence, the sequence $\{s_i\}_1^\infty$ is Cauchy in X and since X is complete, the sequence converges in X. Hence, the absolutely convergent series is convergent.

11. **Kreyszig p.71 / Problem 12.** A seminorm on a vector space X is a mapping $p : X \to \mathbb{R}$ satisfying (N1), (N3), (N4) in Sec. 2.2 (Some authors call this a pseudonorm.) Show that

$$p(0) = 0,$$
$$|p(y) - p(x)| \leq p(y - x).$$

(Hence if $p(x) = 0$ implies $x = 0$ then p is a norm.)

From (N3), we have $p(\mathbf{0}) = p(0 \cdot \mathbf{0}) = 0 \cdot p(\mathbf{0}) = 0$.

Again, from (N4), we have $p(x + z) \leq p(x) + p(z)$ for all x, $z \in X$. Then, taking $z = y - x$, we get $p(y) \leq p(x) + p(y - x) \implies p(y) - p(x) \leq p(y - x)$. Similarly, we have $p(x) = p(x - y + y) \leq p(x - y) + p(y) \implies p(x) - p(y) \leq p(x - y) = p(y - x)$. Combining these two inequalities, we get

$$|p(y) - p(x)| \leq p(y - x).$$

12. **Kreyszig p.76 / Problem 9.** If two norms $\|\cdot\|$ and $\|\cdot\|_0$ on a vector space X are equivalent, show that (i) $\|x_n - x\| \to 0$ implies (ii) $\|x_n - x\|_0 \to 0$ (and vice versa, of course).

If the two norms $\|\cdot\|$ and $\|\cdot\|_0$ on a vector space X are equivalent, then for some $c > 0$, we have

$$0 \leq \|x_n - x\|_0 \leq c\|x_n - x\|.$$

Taking limit as $n \to \infty$, we see that, by squeeze theorem $\|x_n - x\| \to 0 \implies |x_n - x\|_0 \to 0$. Similarly for some $c_1 > 0$, we have

$$0 \leq \|x_n - x\| \leq c_1\|x_n - x\|_0$$

which after taking limit as $n \to \infty$ proves the converse statement.

13. Let X be a finite dimensional v.s. over $\mathbb{R}$, with basis $\{e_1, ..., e_n\}$.

    (a) Show that for any $1 \leq p \leq \infty$, the map $\|\cdot\|_p$ defined by

$$x = \sum_1^n x_i e_i \to \|x\|_p = \Big(\sum_1^n |x_i|^p \Big)^{1/p}, \quad \text{for} \quad 1 \leq p < \infty$$

$$x = \sum_1^n x_i e_i \to \|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \quad \text{for} \quad p = \infty$$

is a norm on X.

i. For $p < \infty$, $\|x\|_p = \left(\sum_1^n |x_i|^p\right)^{1/p}$ and each $|x| \geq 0$, so we have $\|x\| \geq 0$ for all $x \in X$. Similarly, for $p = \infty$, since $|x_i| \geq 0$, $\|x\|_\infty \geq 0$.

ii. If $\|x\|_p = \left(\sum_1^n |x_i|^p\right)^{1/p} = 0$, then each $|x_i| = 0 \implies x_i = 0 \implies x = 0$. And, if $x = 0$ then each $|x_i| = 0$ and so $\|x\| = 0$.

Similarly, for if $\|x\|_p = \max_{1 \leq i \leq n} |x_i| = 0$, then each $|x_i|$ must be 0 and so $x = 0$. And, if $x = 0$ then each $|x_i| = 0$ and so $\|x\|_\infty = 0$.

iii. For $\alpha \in \mathbb{R}$,

$$\|\alpha x\|_p = \left(\sum_1^n |\alpha x_i|^p\right)^{1/p} = \left(\alpha^p \sum_1^n |x_i|^p\right)^{1/p} = |\alpha|\|x\|.$$

$$\|\alpha x\|_\infty = \max_{1 \leq i \leq n} |\alpha x_i| = |\alpha| \max_{1 \leq i \leq n} |x_i| = |\alpha|\|x\|_\infty.$$

iv. If another element $y \in X$ is given by $y = \sum_{i=1}^n y_i e_i$, then we take sequences $x' = \{x_i'\}_1^\infty$ and $y' = \{y_i'\}_1^\infty$ in l^p such that $x_i' = x_i$ and $y_i' = y_i$ for $i = 1, \ldots, n$ and $x_i' = y_i' = 0$ for $i > n$. Then, we have

$$\left(\sum_{i=1}^n |x_i|^p\right)^{1/p} = \left(\sum_{i=1}^n |x_i + y_i|^p\right)^{1/p}$$

and by Minkowski's inequality, we have

$$\|x+y\|_p = \left(\sum_{i=1}^n |x_i + y_i|^p\right)^{1/p} = \left(\sum_{i=1}^\infty |x_i' + y_i'|^p\right)^{1/p} \leq \left(\sum_{i=1}^\infty |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^\infty |y_i|^p\right)^{1/p}$$

Hence $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ for $1 \leq p < \infty$.

For $p = \infty$, we have

$$\|x+y\|_p = \max\{|x_1|, \ldots, |x_n|, |y_1|, \ldots, |y_n|\} \leq \max\{|x_1|, \ldots, |x_n|\} + \max\{|y_1|, \ldots, |y_n|\}$$

So, $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ for $p = \infty$.

(b) Show that for $1 \leq p \leq \infty$, $(X, \|\cdot\|)$ is separable.

We take the set M consisting of elements of X with rational coordinates given by

$$M = \left\{x \in X : \ x = \sum_{i=1}^n \lambda_i e_i, \ \lambda_i \in \mathbb{Q}\right\}.$$

M is countable since M is a countable union of countable sets. To show that M is dense in X, we show that every open neighborhood of an arbitrary point contains a point of M. For a point $y = \sum_{i=1}^{n} y_i e_i$ we note that its ε-neighborhood is given by

$$N_\varepsilon(y) = \{x \in X : \ \|y - x\| < \varepsilon\}.$$

For $1 \leq p < \infty$,

$$N_\varepsilon(y) = \left\{ x \in X : \ \sum_{i=1}^{n} |y_i - x_i|^p < \varepsilon^p \right\}.$$

Then since rational numbers are dense in $\mathbb{R}$, we can choose rational numbers x_i such that $|y_i - x_i| < \varepsilon / n^{1/p}$. Then clearly, $x = \sum_{i=1}^{n} x_i e_i \in N_\varepsilon(y)$. Hence, X is separable.

Now, or $p = \infty$,

$$N_\varepsilon(y) = \left\{ x \in X : \ \max_{i=\overline{1,\ldots,n}} \{|y_i - x_i|\} < \varepsilon \right\}.$$

Then, we can choose each rational number x_i such that $x_i \in (y_i, y_i + \varepsilon)$. Then $x \in N_\varepsilon(y)$ and hence M is dense in X. So, X is separable.

14. Prove that any vector space can be normed.

We note that every vector space V has a Hamel basis B such that every element x of the vector space V can be written as a linear combination of finitely many elements of B with non-zero scalars as coefficients. Then, if for an element $x \in V$, $b_1, \ldots, b_k$ are the finitely many elements of B with non-zero coefficients, then

$$x = \lambda_1 b_1 + \cdots + \lambda_k b_k$$

and we define the norm of x to be

$$\|x\| = \max_{i=\overline{1,k}} |\lambda_i|.$$

Clearly, $\|x\| \geq 0$ and if $\|x\| = 0$ then $x = 0$. Also, $x = 0 \implies \|x\| = 0$ and $\|\alpha x\| = \max_{i=\overline{1,k}} |\alpha \lambda_i| = |\alpha| \|x\|$. Furthermore, if $y = \beta_1 c_1 + \cdots + \beta_n c_n$ is another element of V, then $x + y = \lambda_1 b_1 + \cdots + \lambda_k b_k + \beta_1 c_1 + \cdots + \beta_n c_n$. And

$$\|x + y\| = \max\{\lambda_1, \ldots, \lambda_k, \beta_1, \ldots, \beta_n\} \leq \max\{\lambda_1, \ldots, \lambda_k\} + \max\{\beta_1, \ldots, \beta_n\} = \|x\| + \|y\|.$$

We see that $\|\cdot\|$ satisfies all the conditions of a norm on the vector space V.

15. Let X be a finite dimensional normed space. Prove that any closed and bounded subset is compact.

Let M be a closed and bounded subset of the finite dimensional vector space X with dim $X = n$ and basis $\{e_1, \ldots, e_n\}$. Let $\{x_i\}_1^\infty$ be a sequence in M, then each x_i has a unique linear representation

$$x_i = \lambda_i^1 e_1 + \cdots + \lambda_i^n e_n.$$

By the lower bound theorem, we have $\|x_i\| \geq c \sum_{k=1}^{n} |\lambda_i^k|$ for some $c > 0$ and since M is a bounded set we have, for some real positive number c_2, $\|x_i\| \leq c_2$. Taking $c_1 = \sum_{k=1}^{n} |\lambda_i^k|$, we get

$$c_1 \leq \|x_i\| \leq c_2.$$

Then for each k, we have $c_1 \leq \|\lambda_i^k\| \leq c_2$ and for a fixed k, we have the sequence $\{\lambda_i^k\}_1^\infty$ of real numbers which is bounded. By Bolzano-Weierstrass theorem, we know that this sequence has a convergent subsequence which converges to a point β^k in $\mathbb{R}$. And hence, we have a corresponding convergent subsequence $\{x_{i_k}\}$ of $\{x_i\}$ which converges to $z = \sum \beta_i e_i$. Since, M is closed, this limit point is in M. So, M is compact.

16. Prove Riesz's Lemma.

Theorem 1 (Riesz's Lemma)**.** *Let Y and Z be subspaces of a normed space X (of any dimension), and suppose that Y is closed and is a proper subset of Z. Then for every real number θ in the interval $(0,1)$ there is a $z \in Z$ such that*

$$\|z\| = 1, \qquad \|z - y\| \geq \theta \text{ for all } y \in Y.$$

Proof. For $v \in Z - Y$, we denote the distance between v and the subspace Y by $a = \inf_{y \in Y} \|v - y\|$. Since $v \notin Y$, $a > 0$ and for some $\theta \in (0,1)$, there exists a $y_0 \in Y$ such that

$$a \leq \|v - y_0\| \leq a/\theta.$$

Let $z = c(v - y_0)$ where $c = 1/\|v - y_0\|$. Then $\|z\| = 1$ and

$$\|z - y\| = \|c(v - y_0) - y\| = c\|v - y_0 - c^{-1}y\| = c\|v - y_1\|$$

where $y_1 = y_0 + c^{-1}y$ which is in Y (since Y is a subspace). Hence $\|v - y_1\| \geq a$ and

$$\|z - y\| = c\|v - y_1\| \geq \frac{a}{\|v - y_0\|} \geq a/(a/\theta) = \theta.$$

$\square$

17. If a normed space X has the property that the closed unit ball $\overline{B}(0,1)$ is compact, then X is finite dimensional.

We show that if the closed unit ball $\overline{B}$ is compact, and X is of infinite dimension, then we get a contraction. Let $x_1 \in X$ such that $\|x\| = 1$ for some norm $\|\cdot\|$. Then the span of x_1 is a subspace M_1 of dimension 1 and hence, is closed and proper subspace of X. Then, by Riesz's Lemma, taking $\theta = 1/2$, there exists $x_2 \in X$ of norm 1 such that $\|x_2 - x_1\| \geq 1/2$. The elements x_1 and x_2 now generate a 2 dimensional closed and proper subspace M_2 of X and again, by Riesz's lemma, there exists an x_3 of norm 1 such that $\|x_3 - x_1\| \geq 1/2$ and $\|x_3 - x_2\| \geq 1/2$. Then inductively, we obtain a sequence $\{x_n\}_1^\infty$ in $\overline{B}$ such that for all integers $m \neq n$, we have

$$\|x_m - x_n\| \geq \frac{1}{2}.$$

But this sequence cannot have a convergent subsequence even though $\overline{B}$ is compact. Hence, our assumption that the dimension of X is infinite cannot be true. $\square$