# Analysis II

Homework 4

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March 13, 2023

**Pack Pledge:** I have neither given nor received unauthorized aid on this test or assignment.

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1. Let $A \subseteq \mathbb{R}$ s.t. $m^*(A) = 0$. Show that for any $B \subseteq \mathbb{R}$, we have that $m^*(B) = m^*(A \cup B) = m^*(B \setminus A)$.

   Since $A$ has outer measure 0, it is measurable. Then we have $m^*(B) = m^*(B \cap A) + m^*(B \cap A^c)$. Since $m^*(A \cap B) \leq m^*(A) = 0$, we have $m^*(B) = m^*(B \cap A^c) = m^*(B \setminus A)$. Furthermore, $m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B)$ and $m^*(B) = m^*(B \setminus A) \leq m^*(A \cup B)$. Thus, we have

   $$m^*(B) = m^*(A \cup B) = m^*(B \setminus A).$$

2. Let $A \subseteq \mathbb{R}$ non-measurable. Let $E \in \mathcal{L}(\mathbb{R})$ s.t. $A \subseteq E$. Prove that $m^*(E \setminus A) > 0$.

   If we assume that $m^*(E \setminus A) = 0$, then we see that $E \setminus A$ is a measurable set and since $E$ is measurable and the difference is measurable, $A$ must also be measurable. We have a contradiction and thus we must have $m^*(E \setminus A) > 0$.

3. A set function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is called an outer measure if $\mu^*(\emptyset) = 0$ and $\mu^*$ is countably monotone, i.e., if $E \subseteq \bigcup_1^\infty E_n$, then $\mu^*(E) \leq \sum_1^\infty \mu^*(E_n)$.

   We say that a subset $E \subset X$ is measurable if $\forall\, T \subset X$, we have

   $$\mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c).$$

   Let $\mathcal{M}$ be the collection of measurable sets in $X$. Show that $\mathcal{M}$ is a $\sigma$-algebra.

   Define $\mu = \mu^* \mid_{\mathcal{M}}$. Show that $(X, \mathcal{M}, \mu)$ is a complete measure space.

   $\mathcal{M}$ is a $\sigma$-algebra:

   (a) ...

   $\mu$ is a complete measure: We only need to show that if a set $E$ has outer measure 0, then it is measurable. We have, for any set $T$, since $T \supset T \cap E^c$ and $\mu^*(T \cap E) \leq \mu^*(E) = 0$,

$$\mu^*(T) \geq \mu^*(T \cap E^c) = \mu^*(T \cap E^c) + \mu^*(T \cap E).$$

The converse inequality holds by the property of outer measure and thus we see that $E$ is measurable.

4. Let $S$ be a collection of subsets of $X$. Let $\mu$ be a set function $\mu : S \to [0, \infty]$, with $\mu(\emptyset) = 0$. Define $\bar{\mu}(\emptyset) = 0$, and for all $E \subset X$, $E \neq \emptyset$, define

$$\bar{\mu}(E) = \inf \left\{ \sum_j \mu(E_j) \;\middle|\; E \subseteq \bigcup_j E_j,\ E_j \in S \right\}$$

Show that $\bar{\mu}$ is an outer measure on $X$ (it is called the outer measure induced by $\mu$).

We only need to show that $\bar{\mu}$ is countably subadditive. For $F \subseteq \bigcup_i F_i$, we have $\bar{\mu}(F_i) = \inf \left\{ \sum_j \mu(E^i_j) |\ \ F_i \subseteq \bigcup_i E^i_j,\ E^i_j \in S \right\}$. Then the set $\bigcup_{i,j} E^i_j$ also covers $F$. Then we have,

$$\bar{\mu}(F) = \inf \sum_{i,j} \mu(E^i_j) = \sum_j \bar{\mu}(F_j).$$

Hence $\bar{\mu}$ is an outer measure.

5. Let $E \in \mathcal{L}(\mathbb{R})$. Show that $E + \lambda = \{e + \lambda \mid e \in E\} \in \mathcal{L}(\mathbb{R})$ and $m(E) = m(E + \lambda)$.

We first note that the outer measure is translation invariant. Now, for any set $A \subset \mathbb{R}$, we have $m^*(A) = m^*(A - \lambda) = m^*([A-\lambda] \cap E) + m^*([A-\lambda] \cap E^c)$ since $E$ is a measurable set. We have $[A-\lambda] \cap E = \{x : \ x \in A-\lambda, x \in E\} = \{y+\lambda : \ y \in A, y \in E+\lambda\} = A \cap [E+\lambda]$ and similarly for $[A-\lambda] \cap E^c$, $[A-\lambda] \cap E^c = \{x : \ x \in A-\lambda, x \in E^c\} = \{y+\lambda : \ y \in A, y \in []E+\lambda]^c\} = A \cap [E+\lambda]^c$. Thus, we have $m^*(A) = m^*(A - \lambda) = m^*(A \cap [E + \lambda]) + m^*(A \cap [E + \lambda]^c)$. Hence $E + \lambda$ is measurable. Since this set is measurable, the measure is equal to the outer measure which is translation invariant and thus the Lebesgue measure is also translation invariant here.

6. Let $E \in \mathcal{L}(\mathbb{R})$ and let $\varepsilon > 0$. Show that there exist an open set $G$ and a closed set $F$ with $F \subset E \subset G$ so that $m(G \setminus F) < \varepsilon$. Then show that every Lebesgue measurable set is a union of a Borel set and a set of Lebesgue measure 0.

Since $E$ and $E^c$ are measurable, there are countable collection of open intervals $\{I_k\}$ and $\{J_k\}$ that cover $E$ and $E^c$ respectively. Then there exist the open set $G = \bigcup_k I_k$ and the closed set $F = \left(\bigcup_k J_k\right)^c$ such that $m^*(G - E) < \varepsilon/2$ and $m^*(E - F) < \varepsilon/2$ for some $\varepsilon > 0$. Since all these sets are measurable, we have $m(G \setminus F) < \varepsilon$.

7. **Royden p. 43/ Problem 19** Let $E$ have finite outer measure. Show that if $E$ is not measurable, then there is an open set $O$ containing $E$ that has finite outer measure and for which

$$m^*(O \setminus E) > m^*(O) - m^*(E).$$

For some $O \supset E$, we have $O = (O \setminus E) \cup E$ and thus $m^*(O) \le m^*(O \setminus E) + m^*(E)$. But $E$ is non-measurable and so we have strict inequality for at least one open set $O$ containing $E$. We subtract $m^*(E)$ from both sides (as the outer measure is finite) to get the required inequality.

8. **Royden p. 47/ Problem 26** Let $\{E_k\}_k$ be a countable disjoint collection of measurable sets. Prove that for any set $A$,
$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

We see that $A \cap \bigcup_{k=1}^{\infty} E_k = \bigcup_k A \cap E_k$ and $\{A \cap E_k\}_k$ is a countable collection of disjoint sets. Clearly, $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m^*(A \cap E_k)$ by the countably subadditivity of the outer measure. Conversely, we have
$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) \ge m^*\left(A \cap \bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*\left(A \cap E_k\right)$$
for all $n$. The last equality follows from proposition 6 in Royden by induction on $n$. Thus, since the inequality is true for all $n$ we have $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) \ge \sum_{k=1}^{\infty} m^*(A \cap E_k)$ and thus we obtain the required equality.

9. Let $0 < \varepsilon < 1$. Show how to construct an open set in $[0,1]$ which is dense in $[0,1]$ and which has Lebesgue measure $\varepsilon$.

We take the countable sequence $(a_n)$ of rational numbers in $I_1 = [0,1]$ and choose a subsequence $(b_k)$ as follows: Take $b_1$ such that $(b_1 - \varepsilon/2^2, b_1 + \varepsilon/2^2) \subset I_1$. Now we consider $I_2 = I_1 \setminus (b_1 - \varepsilon/2^2, b_1 + \varepsilon/2^2)$ and choose $b_2$ such that $(b_2 - \varepsilon/2^3, b_2 + \varepsilon/2^3) \subset I_2$ and now we take $I_3 = I_2 \setminus (b_2 - \varepsilon/2^3, b_2 + \varepsilon/2^3)$ so on. Thus we take $b_k$ such that $(b_k - \varepsilon/2^{k+1}, b_k + \varepsilon/2^{k+1}) \subset I_k$. Now we consider all the sets $J_k = I_{k+1} \setminus I_k$. This construction is possible since the total length of the intervals is just $\varepsilon$ which is less than 1. Thus we take the set $\left(\mathbb{Q} \cup \bigcup_k J_k\right) \cap I_1$. The measure of the collection is clearly $\varepsilon$ since $J_k$ are disjoint and countable and $\mathbb{Q} \cap I$ has measure 0.

10. With the same notation that we used in class for the Cantor set $C$, let $O_k = [0,1] \setminus C_k$. Define $O = \cup_1^{\infty} O_k$ (i.e. $[0,1] = C \cup O$). Note that for each $k$, $O_k$ is a union of $2^k - 1$ open intervals. Define a function $\phi$ on $[0,1]$ in the following way:

- On each $O_k$, $\phi$ takes the values $\frac{1}{2^k}$, $\frac{2}{2^k}$, $\frac{3}{2^k}$, ..., $\frac{2^k-1}{2^k}$ (i.e. on the 1st subinterval of $O_k$, it takes the value $\frac{1}{2^k}$, and so on...)
- On $C$: $\phi(0) = 0$ and $\phi(x) = \sup\{\phi(t) \mid t \in O \cap [0,x)\}$ if $x \in C \setminus \{0\}$.

$\phi$ is called the Cantor-Lebesgue function. Show that $\phi$ is an increasing continuous function that maps $[0,1]$ into $[0,1]$. Moreover, its derivative exists on the open set $O$, $\phi' = 0$ on $O$, and $M(O) = 1$.

We note that, for each $nth$ subinterval of $O_k$ it becomes the $2nth$ subinterval of $O_{k+1}$. Then for each $x$ in this interval, we have $\phi(x) = n/2^k = 2n/2^{k+1}$. Thus $\phi$ is constant on each open subintevals of $O_k$ and and it is continuous there. $phi$ is also increasing on $O$ and because it is defined in terms of supremum in the interval $[0, x)$, it is increasing on $[0, 1]$. If $x$ is not in any subintervals of any $O_k$ then, we consider the $nth$ and $(n + 1)th$ subintervals of $O_k$ and note that $|\phi(y) - \phi(z)| < 1/2^k$ for all $y$ and $z$ in a sufficiently small neighborhood of $x$. Thus $\phi$ is indeed continuous. We have $\phi(0) = 0$ and $\phi(1) = 1$ for all $O_k$ and it is continuous and so $\phi$ maps $[0, 1]$ to itself.

For each subintervals of $O_k$, the function is constant and thus the derivative exists and equals 0 on each subintervals. So the derivative exists on $O$ and since $M(C) = 0$ we have $M(O) = 1$.