Introduction to Manifold Theory

Homework 6

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1. Do Exercise 3.7: show that if X is a smooth manifold and $f\colon X \to \mathbb{R}$ is a smooth function, then **any** coordinate representation of f is smooth.

Since X is a smooth n-manifold, it is equipped with a maximal atlas, say, $\mathcal{A}$ and since f is smooth, for all $p \in X$, there exists some chart (U, φ) with $p \in U$ in some smooth atlas $\mathcal{B}$ representing the smooth structure on X, such that the composite function

$$f \circ \varphi^{-1} : D_\varphi^\circ \to \mathbb{R}$$

from an open neighborhood $D_\varphi^\circ \subset \mathbb{R}^n$ of $\varphi(p)$ to $\mathbb{R}$ is smooth. We need to show that, if $\mathcal{C}$ is any smooth atlas on X, then for all $(U, \alpha) \in \mathcal{C}$ and $p \in X$ with $p \in U$, the map $f \circ \alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth.

We have $\mathcal{A} \supset \mathcal{B}$ and $\mathcal{A} \supset \mathcal{C}$ since $\mathcal{A}$ is the maximal atlas of the equivalence class of the atlases that X is equipped with. Then, for every point $p \in X$, there exists $(U, \varphi) \in \mathcal{B}$ and $(V, \alpha) \in \mathcal{C}$ with $p \in U, V$ such that the transition maps $\alpha \circ \varphi^{-1}$ and $\varphi \circ \alpha^{-1}$ are smooth. Then, the function $f \circ \alpha^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \alpha^{-1})$ is the composite of smooth functions and hence, is smooth itself. So, any coordinate representation of f is smooth.

2. Do Exercise 3.8: define the standard vector space structure on the stalk of smooth functions C_x^∞ at a point x of a smooth manifold X, define the usual product of elements of C_x^∞ (i.e. of germs of smooth functions at x), and prove that this product is compatible with the vector space structure.

We consider the functions $f_{[p]}, g_{[p]} : C_x^\infty \to C_x^\infty$ by $f_{[p]}([q]) = [p \cdot q]$ and $g_{[p]}([q]) = [q \cdot p]$ and show that they are linear maps between the $\mathbb{R}$-vector spaces. Since the product of functions are commutative, we see that $f_{[p]} = g_{[p]}$ and it is enough to show that one of them, say $f_{[p]}$, is linear.

We first note that for $q \in [q]$, $r \in [r]$, $\alpha \in \mathbb{R}$ and $x \in X$,

$$(q + r)(x) = q(x) + r(x) \qquad \text{and} \qquad (\alpha q)(x) = \alpha q(x)$$

and hence, $q + r \in [q + r] = [q] + [r]$ and $\alpha q \in [\alpha q] = \alpha[q]$. Now,

$$\begin{aligned} f_{[p]}(\alpha[q] + [r]) &= f_{[p]}([\alpha q + r]) \\ &= [p(\alpha q + r)] \\ &= [\alpha p \cdot q + p \cdot r] \\ &= \alpha[p \cdot q] + [p \cdot r] = \alpha f_{[p]}([q]) + f_{[p]}([r]) \end{aligned}$$

Hence, the given functions are linear maps.

3. Do Exercise 3.9: for a point x in a smooth manifold X, show that the set $\mathfrak{m}$ of "germs vanishing at x" gives a maximal ideal of C_x^∞ and that any other ideal of C_x^∞ is either equal to all of C_x^∞ or contained in $\mathfrak{m}$.

For any smooth function $f : V \to \mathbb{R}$ defined on some open neighborhood of x, if $f(x) \neq 0$ then we know that, since f is continuous, there exists some open neighborhood $U \ni x$ such that $f(y) \neq 0$ for all $y \in U$. Then, the function $1/f : U \to \mathbb{R}$ defined by $(1/f)(x) = 1/f(x)$ exists. We note that $1/f$ is a smooth function and belongs to some germ $[1/f]$ at x. We call such germs $[f]$ as **unit** elements of C_x^∞. Then if I is an ideal of the ring C_x^∞ containing $[f]$, then $[1/f][f] = [1]$ is in the ideal I. So, I must be the whole ring C_x^∞.

Now, we take the set I of all germs $[g]$ such that $g(x) = 0$ and show that it is an ideal. For any element $[f] \in C_x^\infty$ and $[g] \in I$, we have $[f][g] = [f \cdot g]$. Since, $(f \cdot g)(x) = f(x) \cdot g(x) = f(x) \cdot 0 = 0$, we see that $[f][g] \in I$. Hence, I is an ideal of C_x^∞. Furthermore, since I contains **all** the non-unit elements of C_x^∞, I is the unique maximal ideal of C_x^∞. Hence, C_x^∞ is a local ring with the unique maximal ideal I as defined above.