Introduction to Manifold Theory

Homework 2

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1. Do Exercise 2.6 (show that for topological spaces X, Y, Z, the "rearrange-the-parentheses" map from $(X \times Y) \times Z$ to $X \times (Y \times Z)$ is a homeomorphism).

Let the function $f : (X \times Y) \times Z \to X \times (Y \times Z)$ be defined as

$$f((x, y), z) = f(x, (y, z))$$

where x, y and z are respective points of the topological spaces. We see that the map is clearly bijective and hence invertible.

Now, for each open set $U_x \times (U_y \times U_z)$, the preimage of f is given by $(U_x \times U_y) \times U_z$ which is open in $(X \times Y) \times Z$. Similarly, for each open set $(U_x \times U_y) \times U_z$, the preimage of f^{-1} is given by $U_x \times (U_y \times U_z)$ which is open in $X \times (Y \times Z)$. Hence f and f^{-1} are both continuous and the "rearrange the parentheses" map is homeomorphism.

2. Do Exercise 2.7 (show that the product topology and the usual topology on $\mathbb{R}^n$ agree).

Suppose $\mathscr{P}$ be the product topology and $\mathscr{T}$ be the usual topology in $\mathbb{R}^n$. Let U be open in the product topology, then for all $x = (x_1, \dots, x_n) \in U$ there exists open neighborhoods $U_i \in \mathbb{R}$ such that $x_i \in U_i$ and $U_1 \times \cdots \times U_n \subset U$. Then for all x_i, there exists an open interval $(x_i - \delta_i, x_i + \delta_i)$ for some $\delta_i > 0$. Let $\delta = \min\{\delta_i\}$ taken over all i from 1 to n. Clearly, $\delta > 0$ and $x \in B_\delta(x) \subset U_1 \times \cdots \times U_n \subset U$. This shows that $\mathscr{P} \subset \mathscr{T}$.

Now let U be open with respect to the usual topology. Then for all $x \in U$, there exists an open ball $B_\delta(x)$ containing x such that $B_\delta(x) \subset U$ for some $\delta > 0$. Let $\delta_i = \delta/\sqrt{2}$. Then each x_i is contained in the interval $U_i = (x_i - \delta_i, x_i + \delta_i)$ and we see that $B_\delta(x) \supset U_1 \times \cdots \times U_n$. Then $x \in U_1 \times \cdots \times U_n \subset B_\delta(x) \subset U$. Hence U is open in the product topology and $\mathscr{P} \supset \mathscr{T}$. So we see that the two topologies agree.

3. The following exercises are about the "line with two origins" of Example 2.44, which we will call X.

   (a) Show that the construction in Example 2.44 defines a topology on X.

The construction in Example 2.44 is reproduced below:

Let $\mathscr{B}$ be the set of subsets of X that have one of the following two forms:

i. open intervals $(a, b) \subset \mathbb{R}$ (with a and b finite and $a < b$);

ii. sets of the form $((a, b)\backslash 0) \cup \bar{0}$ whenever $a < 0 < b$.

Then we declare a subset U of X to be open if, for all $x \in U$, there exists a subset B of $\mathscr{B}$ with $x \in B$ and $B \subset U$.

Let $\mathscr{T}$ be the collection of open sets as defined above. We now show that it is a topology.

a. Clearly, $\phi \in \mathscr{T}$ and also $X \in \mathscr{T}$.

b. Let $A = \bigcup_i U_i$ be the union of arbitrary collection of indexed open sets. For all $x \in A$ then there exists a U_i such that $x \in U_i$. So, there exists a subset B of $\mathscr{B}$ with $x \in B$ and $B \subset U_i \subset A$. Hence, A is open.

c. Let $A = U_1 \cap U_2$ be the finite intersection of open sets of X. For any $x \in A$ we see that $x \in U_1$ and $x \in U_2$. Then there exists a subset B_1 of $\mathscr{B}$ with $x \in B_1$ and $B_1 \subset U_1$ and there exists a subset B_2 of $\mathscr{B}$ with $x \in B_2$ and $B_2 \subset U_2$. If $x \neq \bar{0}$ then the problem reduces to $\mathbb{R}$ which implies that A is open. If $x = \bar{0}$ then we see that $B_1 \cap B_2$ is the intersection of open intervals and $\bar{0}$ which is again open in X.

Thus X is a topological space with the topology $\mathscr{T}$.

(b) Show that with this topology, X is locally homeomorphic to R.

For any point $x \neq \bar{0}$ in X, we observe that there is an open ball $(x - \delta, x + \delta)$ around x for some $\delta > 0$. Since any open intervals of $\mathbb{R}$ are homeomorphic to $\mathbb{R}$ itself, we see that X is locally homeomorphic $\mathbb{R}$ for every point $x \neq \bar{0}$.

Now, when $x = \bar{0}$ we take $Y = (-\delta, 0) \cup (0, \delta) \cup \{\bar{0}\}$ and define a function $f : Y \to \mathbb{R}$ by $f(\bar{0}) = 0$ and $f(y) = \tan(\pi y/2\delta)$. We see that f is invertible, continuous and has a continuous inverse and hence is a homeomorphism. Thus, X is locally homeomorphic to $\mathbb{R}$.

(c) Show that X is not Hausdorff.

For every $\epsilon > 0$, the neighborhood $N_\epsilon(0)$ of the point 0 intersects with the neighborhood around the point $\bar{0}$ non-trivially. So, X is not Hausdorff.