Algebra II

Homework 1

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1. (10.1 - 5) For any left ideal I of R define

$$IM = \left\{ \sum_{\text{finite}} a_i m_i \mid a_i \in I,\ m_i \in M \right\}$$

to be the collection of all finite sums of elements of the form am where $a \in I$ and $m \in M$. Prove that IM is a submodule of M.

> We show that IM is a submodule of M and is closed under the action of the ring elements:
>
> Clearly, $0 \in IM$. If $p = \alpha_1 f_1 + \cdots + \alpha_n f_n$ and $q = \beta_1 g_1 + \cdots + \alpha_m g_m$ in IM with α, $\beta \in I$ and f_i, $g_i \in M$, then $p + q = \alpha_1 f_1 + \cdots + \alpha_n f_n + \cdots + \beta_1 g_1 + \cdots + \alpha_m g_m$ is a finite sum with coefficients in the ideal and hence belongs to the set IM. For any finite sum $p = \alpha_1 f_1 + \cdots + \alpha_n f_n \in IM$, its inverse $-p$ is also a finite sum with coefficients in the ideal and hence belongs to IM. So, IM is a subgroup. Now, for any ring element $r \in R$ and $p = \alpha_1 f_1 + \cdots + \alpha_n f_n \in IM$, we have $rp = r(\alpha_1 f_1 + \cdots + \alpha_n f_n) = r\alpha_1 f_1 + \cdots + r\alpha_n f_n$. Since $rI \subset I$, we see that IM is closed under the ring action and hence is a submodule of M.

2. (10.1 - 6) Show that the intersection of any nonempty collection of submodules of an R-module is a submodule.

> Let $N = \bigcap_{\lambda \in \Lambda} N_\lambda$ be the intersection of the nonempty collection of submodules N_λ of M indexed by the set Λ. Clearly, $0 \in N$, so N is nonempty. Now, for any p, $q \in N$, $p + q \in N_\lambda$ for every $\lambda \in \Lambda$ since each N_λ is a submodule and thus $p + q \in N$. Also, $p \in N \implies p \in N_\lambda \implies -p \in N_\lambda$ for all λ which implies that $-p \in N$. Thus, N is a subgroup of M. Now, for $p \in N$ and $r \in R$, we note that $rp \in N_\lambda$ for all λ. So $rp \in N$ and N is closed under the action of ring elements. Thus N is a submodule.

3. (10.2 - 6) Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/(n,m)\mathbb{Z}$.

> We first note that if $\alpha \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ with $\alpha(1) = p$, then $0 = \alpha(n) = pn$. Since $m \mid pn$ we have, $m/(m,n) \mid p$. Thus all elements $\alpha \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ are given by:

$$\alpha(1) = \frac{mx}{(m,n)}$$

for $x \in \mathbb{Z}/n\mathbb{Z}$.
Clearly, any map defined as above is a homomorphism in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$.

We define a map $\varphi : \mathbb{Z}/(n,m)\mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ defined by $\varphi(x) = \alpha_x$ where $\alpha_x(1) = mx/(m,n)$. Since these are all the unique homomorphisms in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, φ is surjective. If $\varphi(x) = \varphi(y)$, then $mx/(m,n) = my/(m,n) \implies x = y$. Thus φ is injective. φ is a homomorphism since $\varphi(x+y)(1) = m(x+y)/(m,n) = \varphi(x)(1) + \varphi(y)(1)$ and $\varphi(ry)(1) = mry/(m,n) = r\varphi(y)(1)$. Thus, this is an isomorphism of $\mathbb{Z}$-modules.

4. (10.2 - 10) Let R be a commutative ring. Prove that $\mathrm{Hom}_R(R,R)$ and R are isomorphic as rings.

We first note that if $\varphi \in \mathrm{Hom}_R(R,R)$ then for any $x \in R$, $\varphi(x) = \varphi(x \cdot 1) = r \cdot \varphi(1)$. Since, $\varphi(1)$ can be any element of R we have $\varphi(x) = x \cdot r$ for some $r \in R$. Hence φ is completely determined by its value on the identity of R.

Now, if $\alpha : R \to R$ is any map such that $\alpha(x) = x \cdot r$ for some $r \in R$ then we have $\alpha(px+qy) = (px+qy) \cdot r = p\alpha(x) + q\alpha(y)$. Thus, $\alpha \in \mathrm{Hom}_R(R,R)$ and we see that every $\varphi \in \mathrm{Hom}_R(R,R)$ is of this form. Hence we can define a map $\psi : \mathrm{Hom}_R(R,R) \to R$ by $\psi(\varphi) = \varphi(1)$ which is surjective as we saw above. Furthermore, if $\psi(\varphi) = \varphi(1) = 0$ then φ is a 0 map and hence ψ is injective. We now show that ψ is a ring homomorphism:

(a) $\psi(\alpha + \beta) = (\alpha + \beta)(1) = \psi(\alpha) + \psi(\beta)$
(b) $\psi(\alpha \circ \beta) = (\alpha \circ \beta)(1) = \alpha(\beta(1)) = \alpha(1 \cdot \beta(1)) = \alpha(1) \cdot \beta(1) = \psi(\alpha) \cdot \psi(\beta)$
(c) $\psi(e) = e(1) = 1$

where $e \in \mathrm{Hom}_R(R,R)$ is the identity map. Thus, ψ is a ring isomorphism.

5. (10.3 - 4) An R-module M is called a torsion module if for each $m \in M$ there is a nonzero element $r \in R$ such that $rm = 0$, where r may depend on m (i.e., $M = \mathrm{Tor}(M)$ in the notation of Exercise 8 of Section 1). Prove that every finite abelian group is a torsion $\mathbb{Z}$-module. Give an example of an infinite abelian group that is a torsion $\mathbb{Z}$-module.

If $p \neq 0$ is the order of the any given finite abelian group then $p \cdot m = 0$ for every element $m \in M$. Thus every finite abelian group is a torsion $\mathbb{Z}$-module.

We take $M = \mathbb{F}_2[x]$, the group of polynomials over the finite field $\mathbb{F}_2$ to be the our abelian group. Clearly, it's infinite as it has elements of every degree. Considered as a $\mathbb{Z}$-module, we see that $2 \cdot f(x) = 0$ for all $f(x) \in M$. Thus it is an infinite abelian torsion $\mathbb{Z}$-module.

6. (10.3 - 9) An R-module M is called *irreducible* if $M \neq 0$ and if 0 and M are the only submodules of M. Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as generator. Determine all the irreducible $\mathbb{Z}$-modules.

We first prove that if M is irreducible then $M \neq 0$ and M is a cyclic module with any nonzero element as generator. Assume that $M \neq 0$ is irreducible and $m \in M$ is any element. Then since $Rm \subset M$ is a submodule of M and M contains no non-trivial submodule, $Rm = M$ and m is the nonzero generator. Now, assume that $M \neq 0$ is a cyclic module with any nonzero generator. Then if $m \neq 0$, $Rm \subset M$. But M is generated by any nonzero element of M so $M \subset Rm$. Thus $M = Rm$, meaning that M and 0 are the only submodules of M. Hence M is irreducible.

From above, we see that the only irreducible $\mathbb{Z}$-modules are cyclic groups of prime order.

7. (10.4 - 2) Show that the element "$2 \otimes 1$" is 0 in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ but is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.

We note that in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, $2 \otimes 1 = 2 \cdot 1 \otimes 1 = 1 \otimes 2 \cdot 1$. But since $2 \equiv 0$ in $\mathbb{Z}/2\mathbb{Z}$, we have, $2 \otimes 1 = 1 \otimes 2 = 1 \otimes 0 = 0$.
To show that it is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, we consider the map $\alpha : 2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ defined by $\alpha(2m, n) = mn \mod 2$ for $2m \in 2\mathbb{Z}$ and $n \in \mathbb{Z}/2\mathbb{Z}$. Clearly, the following holds:

(a) $\alpha(r_1 \cdot 2m + r_2 \cdot 2p, n) = r_1 \cdot \alpha(2m, n) + r_2 \cdot \alpha(2p, n) = r_1 mn + r_2 pn \mod 2$.
(b) $\alpha(2m, r_1 \cdot n + r_2 \cdot q) = r_1 \cdot \alpha(2m, n) + r_2 \cdot \alpha(2m, q) = r_1 mn + r_2 mq \mod 2$.

for m, $p \in 2\mathbb{Z}$ and n, $q \in \mathbb{Z}/2\mathbb{Z}$. Then, α is a $\mathbb{Z}$-bilinear map and induces a $\mathbb{Z}$-linear map $\beta : 2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ such that $\beta(2m \otimes n) = mn \mod 2$.

$$\begin{tikzcd}
2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[dr, "\alpha"] & 2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \arrow[d, "\beta"] \\
& \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

Since $\beta(2 \otimes 1) = 1 \cdot 1 \mod 2 = 1 \neq 0$ in $\mathbb{Z}/2\mathbb{Z}$, we see that $2 \otimes 1 \neq 0$ in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ since β is an R-linear map.

8. (10.4 - 4) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic left $\mathbb{Q}$-modules. [Show they are both 1-dimensional vector spaces over $\mathbb{Q}$.]

By Theorem 8, letting $\iota : \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ be the ring homomorphism defined by $\iota(q) = 1 \otimes q$, we have the commutative diagram:

$$\begin{tikzcd}
\mathbb{Q} \arrow[r, "\iota"] \arrow[dr, "\varphi"] & \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \arrow[d, "\Phi"] \\
& \mathbb{Q}
\end{tikzcd}$$

If we let φ to be the identity map then we get $\Phi \circ \iota$ to be the identity map. Then ι is an isomorphism of $\mathbb{Q}$-modules. To show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$ we define a map $\beta : \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$\beta(q) = 1 \otimes q$. We can prove that this is a $\mathbb{Q}$-linear map by

$$\beta\left(\frac{a}{b}q\right) = 1 \otimes \frac{a}{b}q = a \otimes \frac{q}{b} = \frac{ab}{b} \otimes \frac{q}{b} = \frac{a}{b} \otimes q = \frac{a}{b}(1 \otimes q) = \frac{a}{b}\beta(q).$$

Now, we can define a map $\alpha : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ by $\alpha(p \otimes q) = pq$. It is a well-defined map and we have $\alpha \circ \beta(q) = q$ and

$$\beta \circ \alpha(p \otimes q) = \beta(pq) = 1 \otimes pq = p \otimes q.$$

Thus, since the maps are inverse of each other, we see that β is an isomorphism of $\mathbb{Q}$-modules. Since, both $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic to $\mathbb{Q}$ as a bimodule, we see that they are isomorphic to each other as a left $\mathbb{Q}$-module.

9. (10.5 - 1) Suppose that

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
A' \arrow[r, "\psi'"] & B' \arrow[r, "\varphi'"] & C'
\end{tikzcd}$$

is a commutative diagram of groups and that the rows are exact. Prove that

(a) if φ and α are surjective, and β is injective then γ is injective. [If $c \in \ker(\gamma)$, show there is a $b \in B$ with $\varphi(b) = c$. Show that $\varphi'(\beta(b)) = 0$ and deduce that $\beta(b) = \psi'(a')$ for some $a' \in A'$. Show there is an $a \in A$ with $\alpha(a) = a'$ and that $\beta(\psi(a)) = \beta(b)$. Conclude that $b = \psi(a)$ and hence $c = \varphi(b) = 0$.]

Let $c \in \ker(\gamma)$. Since φ is surjective, there exists $b \in B$ with $\varphi(b) = c$. The given diagram is commutative and hence we have $\varphi' \circ \beta(b) = \gamma \circ \varphi(b) = 0$. Hence $\beta(b) \in \ker(\varphi') = \mathrm{im}(\psi')$ and we have $\beta(b) = \psi'(a')$ for some $a' \in A'$. Since α is surjective, there exists $a \in A$ such that $\alpha(a) = a'$. Thus $\psi'(\alpha(a)) = \beta(\psi(a)) = \beta(b) \implies b = \psi(a)$. Hence $c = \varphi(b) = \varphi \circ \psi(a) = 0$. So, γ is injective.

(b) if ψ', α, and γ are injective, then β is injective,

Since $\psi' \circ \alpha$ is injective, we see that $\beta \circ \psi$ is injective and so ψ is injective. Let $b \in \ker(\beta)$. Then $\varphi' \circ \beta(b) = 0 = \gamma \circ \varphi(b)$. Since, γ is injective and $\varphi(b) \in \ker(\gamma)$, $\varphi(b) = 0 \implies b \in \mathrm{im}(\psi)$. Thus there exists $a \in A$ with $b = \psi(a)$. But ψ is injective and $\beta(b) = \beta \circ \psi(a) = 0$. So $a = 0$ and $b = \psi(a) = 0$. β is injective.

(c) if φ, α, and γ are surjective, then β is surjective, Let $b' \in B'$ be any element. We want to show that there exists a $p \in B$ with $\beta(p) = b'$.

Since $\gamma \circ \varphi$ is surjective, there exists a $b \in B$ such that $\gamma \circ \varphi(b) = \varphi'(b') = \varphi' \circ \beta(b)$. Then $\varphi'(\beta(b) - b') = 0 \implies b' - \beta(b) \in \operatorname{im}(\psi')$. So there exists $a' \in A'$ with $\psi'(a') = \beta(b) - b'$. Since α is surjective, we have $a \in A$ such that $\psi' \circ \alpha(a) = \beta(b) - b'$. So, $\beta \circ \psi(a) = \psi' \circ \alpha(a) = \beta(b) - b' \implies \varphi'(\beta \circ \psi(a)) = \varphi'(\beta(b) - b')$. If we let $p = \psi(a) + b$,

$$\beta(\psi(a) + b) = \beta \circ \psi(a) + \beta(b) = b' = \beta(b) + \beta(b) = b'.$$

So, β is surjective.

(d) if β is injective, α and φ are surjective, then γ is injective,

Let $c \in \ker(\gamma)$. Then since φ is surjective, there exists $b \in B$ with $c = \varphi(b)$. Then $\varphi' \circ \beta(b) = \gamma \circ \varphi(b) = 0$. Then $b' = \beta(b) \in \ker(\varphi') = \operatorname{im}(\psi')$. So we have, $a' \in A$ with $\psi'(a) = b'$. Since α is surjective, we have $a \in A$ with $\alpha(a) = a'$ and $\psi' \circ \alpha(a) = \beta \circ \psi(a) = b' = \beta(b)$. Since β is injective, we have $\psi(a) = b \implies b \in \operatorname{im}(\psi) \implies c = \varphi(b) = 0$. Thus γ is injective.

(e) if β is surjective, γ and ψ' are injective, then α is surjective.

Let $a' \in A'$ be any element. We want to show that there exists a $a \in A$ with $\alpha(a) = a'$.

Since β is surjective, there exists $b \in B$ with $\beta(b) = \psi'(a') = b'$ where $b' \in B'$ is some element of B'. Then $\gamma \circ \varphi(b) = \varphi' \circ \beta(b) = \varphi' \circ \psi'(a') = 0$ follows from the exactness of the lower row. Since γ is injective, $\gamma \circ \varphi(b) = 0 \implies \varphi(b) = 0 \implies b \in \operatorname{im}(\psi)$. So there exists $a \in A$ such that $b = \psi(a)$. Then $\beta \circ \psi(a) = \psi' \circ \alpha(a) = \psi'(a')$. But since ψ' is injective, we have $\alpha(a) = a'$ as required.

10. (10.5 - 4) Let Q_1 and Q_2 be R-modules. Prove that $Q_1 \oplus Q_2$ is an injective R-module if and only if both Q_1 and Q_2 are injective.

If $Q_1 \oplus Q_2$ is injective, then for any R-modules L, M, if $0 \to L \xrightarrow{\psi} M$ is exact, then every R-module homomorphism f from L to $Q_1 \oplus Q_2$ lifts to an R-module homomorphism F of M into $Q_1 \oplus Q_2$. Now, if $g : L \to Q_1$ is any R-module homomorphism, then g factors through $Q_1 \oplus Q_2$ with $g = \pi_1 \circ f$ for some $f \in \operatorname{Hom}_R(L, Q_1 \oplus Q_2)$. If F is the lift of f, then $\pi_1 \circ F \in \operatorname{Hom}_R(M, Q_1)$ is the lift of g. Hence, Q_1 is injective. We see that Q_2 is also injective similarly.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "\psi"] \arrow[d, "f"] \arrow[ddl, bend right, "g"'] & M \arrow[dl, dashed, "F"] \arrow[ddll, dashed, bend left, "G"] \\
& Q_1 \oplus Q_2 \arrow[dl, "\pi_1"] & \\
Q_1 & &
\end{tikzcd}$$

Now, if Q_1 and Q_2 are injective modules and $f : L \to Q_1 \oplus Q_2$ is any module homomorphism, then $\pi_1 \circ f$ and $\pi_2 \circ f$ are homomorphisms from L to Q_1 and Q_2 respectively. If F_1 and F_2 are the lift of these homomorphisms to $\mathrm{Hom}_R(M, Q_1)$ and $\mathrm{Hom}_R(M, Q_2)$ respectively, then we define $F : M \to Q_1 \oplus Q_2$ by $F(m) = (F_1(m), F_2(m))$. Then this is a lift of the map f and hence the direct sum is injective.