

Analysis I

Homework 2

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Pack Pledge: I have neither given nor received unauthorized aid on this test or assignment.

1. Let (X, ρ) and (Y, σ) be metric spaces, and let $f : (X, \rho) \rightarrow (Y, \sigma)$ be a map such that $f^{-1}(V)$ is open in X , for all V open in Y . Show that f is continuous on X .

Let V be the open set around the image $f(x)$ of a point $x \in X$. Then there exists an open set $f^{-1}(V) \subset X$ such that $x \in f^{-1}(V)$. Let V be the open ball $B_{x,\varepsilon}(f(x))$ around the point $f(x)$ for some given $\varepsilon > 0$. Then $f^{-1}(B_{x,\varepsilon}(f(x)))$ is also open in X . Since x is in the open set $f^{-1}(B_{x,\varepsilon}(f(x)))$, we can find a $\delta > 0$ such that $x \in B_\delta(x) \subset f^{-1}(B_{x,\varepsilon}(f(x)))$. This means that for all $z \in X$ such that $d(z, x) < \delta$ we have $d(f(z), f(x)) < \varepsilon$. Hence f is continuous at x . Since we can do this at every point $x \in X$, we see that f is continuous on X .

2. (**Continuous mapping**) Show that a mapping $T : X \rightarrow Y$ is continuous if and only if the inverse image of any closed set $M \subset Y$ is a closed set in X .

We first note that for any set $A \subset Y$, $f^{-1}(A) = \{x \in X : f(x) \in A\}$ and $f^{-1}(Y \setminus A) = \{x \in X : f(x) \notin A\}$. Then clearly, $f^{-1}(A)$ and $f^{-1}(Y \setminus A)$ are disjoint sets of X . Particularly,

$$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A).$$

(\Leftarrow) Let V be a closed set of Y . Then there exists a closed set $f^{-1}(V)$ in X . Since all open sets can be written as a complement of closed set, we see that for any open set $Y \setminus V$ in Y there exists an open set $X \setminus f^{-1}(V)$ in X . Since $X \setminus f^{-1}(V) = f^{-1}(Y \setminus V)$, the preimage of any open set of Y is open in X . Hence f is continuous.

(\Rightarrow)

3. Assume that $f : (\mathbb{R}^2, d_1 = \text{Euclidean metric}) \rightarrow \mathbb{R}$ is continuous at $x \in \mathbb{R}^2$. Show that $f : (\mathbb{R}^2, d_2 = \text{taxicab metric}) \rightarrow \mathbb{R}$ is also continuous at x .

Since $f : (\mathbb{R}^2, d_1 = \text{Euclidean metric}) \rightarrow \mathbb{R}$ is continuous, we know that $f^{-1}(V)$ is open in \mathbb{R}^2 , for all V open in \mathbb{R} . Our proof will be complete if we can show that every open set U in (\mathbb{R}^2, d_1) is also open with respect to the taxicab metric.

Let V be open in (\mathbb{R}^2, d_1) . Then for every point $(p, q) \in X$ there exists an open ball $B_\delta((p, q)) \subset V$ containing (p, q) .

$$B_\delta((p, q)) = \{(x, y) \in \mathbb{R}^2 : (x - p)^2 + (y - q)^2 < \delta^2\}$$

4. Show that the discrete metric space (X, d) is separable iff X is countable.

If X is countable, then X is the countable dense subset in X . So, it is separable.

Now, we show that if (X, d) is separable, then it is countable. Let $Y \subset X$ be the countable dense subset of X . Then every open set of X must contain a point from Y . We note that all the singleton sets of X are open in X and so Y must contain all the points from the singleton sets. Hence we get $X \subset Y \implies X = Y$. So, X is countable.

5. Show that l^p , with $1 \leq p < \infty$ is separable.

We will show that the set M of sequences in l^p with finite non-zero rational terms is the countable dense subset of l^p with $1 \leq p < \infty$. That is M has all the sequences x of the form

$$(x_1, x_2, \dots, x_n, 0, 0, \dots)$$

with each $x_i \in \mathbb{Q}$. First, we show that M is countable. Note that for a fixed n , the set of sequences in l^p with rational terms and all but the first n terms zero is countable. Then M is the countable union of countable sets and, hence, is countable.

Now, we show that the set M is dense in l^p with $1 \leq p < \infty$. Let $x = \{x_i\}_1^\infty \in l^p$. Then $\sum_1^\infty |x_i|^p$ is convergent and so for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that,

$$\sum_{N+1}^\infty |x_i|^p < \varepsilon^p/2.$$

Then, for the first N terms, since rational numbers are dense in \mathbb{R} , we can choose rational numbers y_i such that $|x_i - y_i|^p < \varepsilon^p/2N$. So $(y_1, \dots, y_N, 0, 0, \dots)$ is a point in M and we see that

$$d(x, y) = \left(\sum_{i=1}^\infty |x_i - y_i|^p \right)^{1/p} = \left(\sum_{i=1}^N |x_i - y_i|^p + \sum_{i=N+1}^\infty |x_i - y_i|^p \right)^{1/p} < (\varepsilon^p/2 + \varepsilon^p/2)^{1/p} = \varepsilon$$

So, for every $\varepsilon > 0$, we can find a point of M in the ε -neighborhood of every point $x \in l^p$. Hence l^p with $1 \leq p < \infty$ is separable.

6. Show that l^∞ is not separable.

Let $x = \{x_i\}_{i=0}^{\infty}$ be the sequence of zeros and ones. Then since the sequence is bounded, it is in l^{∞} . Now we will show that the set of such sequences x are uncountably many and have disjoint open neighborhoods for some radius.

For each sequence x we associate a real number y whose binary representation is

$$\sum_{i=1}^{\infty} \frac{x_i}{2^i}.$$

Then for each $y \in [0, 1]$, there exists a unique sequence of zeros and ones as each y has a unique binary representation. Since there are uncountably many y , the sequences associated with them are also uncountable. The metric on l^{∞} given by

$$d(x, y) = \sup_{i \in \mathbb{N}} |x_i - y_i|$$

implies that any two distinct binary sequences $\{x_i\}_{i=0}^{\infty}$ must be 1 distance apart. Then we can take $r = 1/2$ to get the disjoint neighborhoods in l^{∞} associated with each sequence $\{x_i\}_{i=0}^{\infty}$.

Now, if M is any dense set in l^{∞} , then every open set in l^{∞} must contain a point of M . So, for each disjoint open neighborhood constructed above, M contains a point in the neighborhood. Hence, any dense set M is uncountable in l^{∞} and l^{∞} is not separable.

7. Let $\{x_n\}_{n \geq 1}$ be a sequence in a m.s. (X, d) which converges to x . Show that $\{x_n\}_{n \geq 1}$ is a bounded sequence. Then let $\{y_n\}_{n \geq 1}$ be a sequence in (X, d) which converges to y . Show that $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$.

Since the sequence $\{x_n\}_{n \geq 1}$ converges to x , for every $\varepsilon > 0$, we can find a N such that $d(x_n, x) < \varepsilon$ for all $n \geq N$. If we take $\varepsilon = 1$, we see $x_n \leq |x| + 1$ for all $n \geq N$. Let $M = \max\{x_1, \dots, x_{N-1}\}$. Then

$$x_n \leq \max\{M, |x| + 1\} \text{ for all } n \in \mathbb{Z}.$$

Hence, $\{x_n\}_{n \geq 1}$ is bounded.

If $\{x_n\}_{n \geq 1}$ converges to x and $\{y_n\}_{n \geq 1}$ converges to y , then for every $\varepsilon > 0$, there exists N_1 and N_2 such that $d(x, x_n) < \varepsilon/2$ for $n > N_1$ and $d(y, y_n) < \varepsilon/2$ for $n > N_2$. Then for $n > \max\{N_1, N_2\}$,

$$d(x_n, y_n) \leq d(x, x_n) + d(x, y_n) \leq d(x, x_n) + d(y, y_n) + d(x, y) \leq \varepsilon + d(x, y).$$

As $n \rightarrow \infty$, $d(x_n, y_n) \rightarrow d(x, y)$.

8. Show that any nonempty set $A \subset (X, d)$ is open if and only if it is a union of open balls.

If A is open in X then for all $x \in A$ we exists an open ball $B_{\delta_x}(x)$ such that $x \in B_{\delta_x}(x) \subset A$.

Let

$$B = \bigcup_{x \in A} B_{\delta_x}(x).$$

Clearly, $B \subset A$ since each $B_{\delta_x}(x)$ is contained in A . Also, since B contains all the points of A , we have $A = B$. So, A is a union of open balls.

Now, if $A = \bigcup B_\alpha$ is a union of open balls B_α then for each $x \in A$, we have $x \in B_\alpha$ for some B_α . That means that for every $x \in A$ we have some open ball B_α such that $x \in B_\alpha \subset A$. Hence A is open in X .

9. Let (X, ρ) be a metric space, $E \subset X$, and $x \in X$. Prove that the following are equivalent:

- (a) $x \in \overline{E}$
- (b) $B(x, r) \cap E \neq \emptyset, \forall r > 0$
- (c) $\exists \{x_n\} \in E$ s.t. $x_n \rightarrow x$

(a) \implies (b). Let $x \in \overline{E} = X \setminus (X \setminus E)^\circ$. Then $x \notin (X \setminus E)^\circ$. Negating this statement, we see that for all $r > 0$, $B(x, r) \cap E \neq \emptyset$.

(b) \implies (c). We first define the radii $r_n = 1/n$ of the open balls around the point $x \in X$. Since $B(x, r) \cap E \neq \emptyset, \forall r > 0$, we can take a sequence of points $\{x_i\}_1^\infty$ in E such that $x_i \in B(x, r_i)$ for each i . Then this is the sequence in E which converges to the point x .

(c) \implies (a). Since $x_n \rightarrow x$, for every $\varepsilon > 0$ we can find a N such that $d(x, x_n) < \varepsilon$ all $n > N$. That is, every open ball around x contains a point of E . So $x \notin (X \setminus E)^\circ \implies x \in X \setminus (X \setminus E)^\circ$. Thus x is in the closure \overline{E} .

10. If d_1 and d_2 are metrics on the same set X and there are positive numbers a and b such that for all $x, y \in X$,

$$ad_1(x, y) \leq d_2(x, y) \leq bd_1(x, y),$$

show that the Cauchy sequences in (X, d_1) and (X, d_2) are the same.

Let $\{x_n\}_1^\infty$ be a Cauchy sequence in (X, d_1) , then for every $\varepsilon' > 0$, we can find an integer N such that $d_1(x_m, x_n) < \varepsilon'$ for all $m, n > N$. Then taking $\varepsilon = \varepsilon'/b$ we have $d_2(x_m, x_n) \leq \varepsilon$. Hence the sequence $\{x_n\}_1^\infty$ is Cauchy in (X, d_2) .

Now, let $\{y_n\}_1^\infty$ be a Cauchy sequence in (X, d_2) , then for every $\varepsilon' > 0$, we can find an integer N such that $d_2(y_m, y_n) < \varepsilon'$ for all $m, n > N$. Then taking $\varepsilon = a\varepsilon'$ we have $ad_1(y_m, y_n) \leq a\varepsilon \implies d_1(y_m, y_n) \leq \varepsilon$. Hence the sequence $\{y_n\}_1^\infty$ is Cauchy in (X, d_1) .

11. Show that l^p , with $1 \leq p < \infty$ is complete.

Let $\{\{x_i^n\}_{i=1}^\infty\}_{n=1}^\infty$ be a Cauchy sequence in the space l^p where for each n , $\{x_i^n\}_{i=1}^\infty$ is a convergent

sequence in \mathbb{R} . Then for every $\varepsilon > 0$, there exists an N such that

$$d(x_i^m, x_i^n) = \left(\sum_{i=1}^{\infty} |x_i^m - x_i^n|^p \right)^{1/p} < \varepsilon$$

for all $m, n > N$. Then for each fixed i , the term $|x_i^m - x_i^n|^p < \varepsilon^p \implies |x_i^m - x_i^n| < \varepsilon$. So $\{x_i^j\}_1^\infty$ is a Cauchy sequence of real numbers for each fixed i and it converges to some number that we can call x_i . We define $x = \{x_i\}_{i=1}^\infty$ and show that this is the limit of our sequence $\{\{x_i^n\}_{i=1}^\infty\}_{n=1}^\infty$ in l^p .

From above, we have

$$\sum_{i=1}^k |x_i^m - x_i^n|^p < \varepsilon^p$$

for all $m, n > N$. Then as $n \rightarrow \infty$, we have (by definition) for $m > N$

$$\sum_{i=1}^k |x_i^m - x_i|^p \leq \varepsilon^p.$$

Then as $k \rightarrow \infty$, we have

$$\sum_{i=1}^{\infty} |x_i^m - x_i|^p \leq \varepsilon^p$$

12. Prove that $(\mathbb{R}, d(x, y) = |x - y|)$ is complete.

Hint: Follow the steps provided in class. You can use Bolzano-Weierstrass Theorem without proving it.

First, we show that every Cauchy sequence in (\mathbb{R}, d) is bounded. Let $\{x_n\}_1^\infty$ be a Cauchy sequence in (\mathbb{R}, d) , then for every $\varepsilon > 0$, we can find an integer N such that $d(x_m, x_n) = |x_m - x_n| < \varepsilon$ for all $m, n > N$. Then when $m = N + 1$ and $\varepsilon = 1$, we see that $|x_m| - |x_{N+1}| \leq |x_m - x_{N+1}| < 1$. So $|x_m| \leq |x_{N+1}| + 1$ for all $m > N$. Hence the sequence $\{x_n\}_1^\infty$ is bounded by M where $M = \max\{|x_1|, \dots, |x_N|, |x_{N+1}| + 1\}$. Then, since $\{x_n\}_1^\infty$ is bounded, by Bolzano-Weierstrass Theorem, we know that $\{x_n\}_1^\infty$ has a convergent subsequence $\{x_{n_i}\}_{i=1}^\infty$. Let $\{x_{n_i}\}_{i=1}^\infty \rightarrow x$ in \mathbb{R} .

Now, we show that our sequence $\{x_n\}_1^\infty$ itself converges to the point $x \in \mathbb{R}$. In the Cauchy sequence $\{x_n\}_1^\infty$, we have, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon/2$ for all $m, n > N$. Similarly, in the convergent subsequence $\{x_{n_i}\}_{i=1}^\infty$, there exists $N' \in \mathbb{N}$ such that $|x_{n_i} - x| < \varepsilon/2$ for all $i > N'$. Then for all $\varepsilon > 0$, there exists $N, N' \in \mathbb{N}$ such that, for all $n > N$ and $i > N'$ with $n_i > N$, we have

$$|x_n - x| \leq |x_n - x_{n_i}| + |x_{n_i} - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $\{x_n\}_{n=1}^\infty$ converges in (\mathbb{R}, d) and \mathbb{R} is complete.

13. Prove that $(\mathbb{Q}, d(x, y) = |x - y|)$ is incomplete.

We take the sequence $\{x_n\}_1^\infty$ of rational numbers given by $x_n = \left(1 + \frac{1}{n}\right)^n$. Note that this is a convergent sequence in \mathbb{R} with the same metric and hence it is Cauchy in \mathbb{Q} too. However, it does not converge to any number in \mathbb{Q} . So, \mathbb{Q} is not complete.

14. Prove that $\left(C[-1, 1], d(f, g) = \int_{-1}^1 |f(t) - g(t)| dt\right)$ is incomplete.

Hint: Follow the steps provided in class.

We take the sequence $\{f_n\}_1^\infty$ of piecewise defined functions in $C[-1, 1]$ such that

$$f_n(x) = \begin{cases} 1, & x > \frac{1}{n} \\ nx, & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ -1, & x < -\frac{1}{n} \end{cases}$$

We first observe that $d(f_m, f_n) = 2 \times \text{area of the triangle with base } \left|\frac{1}{m} - \frac{1}{n}\right| \text{ and height } 1 = |1/m - 1/n|$. Then, the sequence is Cauchy since for every $\varepsilon > 0$ we have an integer $N > 1/\varepsilon$ such that for all $m, n > N$, $d(f_m, f_n) < \varepsilon$.

Now, for every $g \in C[0, 1]$, we have

$$d(f_n, g) = \int_{-1}^1 |f_n(t) - g(t)| dt = \int_{-1}^{-\frac{1}{n}} |-1 - g(t)| dt + \int_{-\frac{1}{n}}^{\frac{1}{n}} |f_n(t) - g(t)| dt + \int_{\frac{1}{n}}^1 |1 - g(t)| dt$$

Since $d \geq 0$ for all $g \in C[0, 1]$, $d(f_n, g) \rightarrow 0$ implies that each integral on the right should also approach 0. Then we should have

$$g(t) = -1 \quad \text{for } t \in [-1, 0) \quad \text{and} \quad g(t) = 1 \quad \text{for } t \in (0, 1].$$

But then g cannot be continuous so we have a contradiction. So, the Cauchy sequence $\{f_n\}_1^\infty$ does not converge in $C[0, 1]$ and the space is not complete.

15. Determine whether or not the discrete metric space is complete. Justify your answer.

Let $\{x_i\}_{i=0}^\infty$ be a Cauchy sequence in the discrete metric space. Then for every $\varepsilon > 0$ there exists an N such that for all $m, n > N$, we have

$$d(x_m, x_n) < \varepsilon.$$

If we take $\varepsilon < 1$ then we see that $x_m = x_n = x$ for all $m, n > N$ for some N . So the sequence $\{x_i\}_{i=0}^\infty$ converges to x and the space is complete.

16. Prove the Completion of a Metric Space Theorem.

Theorem 1 (Completion of a Metric Space). *For a metric space $X = (X, d)$ there exists a complete metric space $X' = (X', d')$ which has a subspace W that is isometric with X and is dense in X' . This space X' is unique except for isometries, that is, if \tilde{X} is any complete metric space having a dense subspace \tilde{W} isometric with X , then X' and \tilde{X} are isometric.*

Proof. We prove the theorem in the following steps.

- (a) We first define a relation \sim on Cauchy sequences of X and show that it is a well-defined equivalence relation. For the Cauchy sequences $x = \{x_i\}_1^\infty$, $y = \{y_i\}_1^\infty$ in X , we say that $x \sim y$ if

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

Clearly, this relation is symmetric and reflexive. For transitivity we see that if $x \sim y$ and $y \sim z$ then

$$0 \leq \lim_{n \rightarrow \infty} d(x_n, z_n) \leq \lim_{n \rightarrow \infty} d(x_n, y_n) + d(y_n, z_n) = 0.$$

Hence \sim is an equivalence relation on the Cauchy sequences of X . Now, let X' be the set of all equivalence classes of Cauchy sequences on X and define the function $d' : X' \rightarrow \mathbb{R}$ by

$$d'(x', y') = \lim_{n \rightarrow \infty} d(x_n, y_n)$$

where x' and y' are the equivalence classes of x and y respectively. We show that this limit exists and the function d' is well defined. We have,

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_n, y_m) \\ \text{or, } |d(x_n, y_n) - d(x_m, y_m)| &\leq d(x_n, x_m) + d(y_n, y_m). \end{aligned}$$

Taking limit as m, n go to ∞ on both sides we obtain,

$$\lim_{n \rightarrow \infty} |d(x_n, y_n) - d(x_m, y_m)| \leq \lim_{n \rightarrow \infty} [d(x_n, x_m) + d(y_n, y_m)] = 0$$

Hence the limit $d'(x', y') = \lim_{n \rightarrow \infty} d(x_n, y_n)$ exists.

Now, if $x \sim x'$ and $y \sim y'$, then

$$d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y_n, y'_n)$$

And as before,

$$0 \leq \lim_{n \rightarrow \infty} |d(x_n, y_n) - d(x'_n, y'_n)| \leq \lim_{n \rightarrow \infty} [d(x_n, x'_n) + d(y_n, y'_n)] = 0$$

which implies that $\lim_{n \rightarrow \infty} d(x_n, y_n) = \lim_{n \rightarrow \infty} d(x'_n, y'_n)$. Hence d' is a well-defined function on X' .

We now show that d' is a metric on X' . Clearly $0 \leq d' < \infty$ since the limit exists and $d'(x', x') = 0$. Furthermore,

$$d'(x', y') = 0 \implies x \sim y \implies x' \sim y'.$$

And,

$$d'(x', z') = \lim_{n \rightarrow \infty} d(x_n, z_n) \leq \lim_{n \rightarrow \infty} d(x_n, y_n) + \lim_{n \rightarrow \infty} d(y_n, z_n) = d'(x', y') + d'(y', z').$$

So, d' satisfies the definition of a metric.

- (b) Now, we construct an isometry $T : X \rightarrow W$ where W is a dense subset of X' . Let T be a function that takes each element to the equivalence class x' in X' of the Cauchy sequence $\{x\}_1^\infty = (x, x, x, \dots)$ associated with that element. Then T is an isometry since for each $x, y \in X$,

$$d'(T(x), T(y)) = \lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y).$$

We note that isometry is injective map and if $W = T(X)$, then $T : X \rightarrow W$ is surjective. So, W and X are isometric.

We need to show that W is dense in X' . Let $x' \in X'$ be the equivalence class of the Cauchy sequence $\{y_i\}_1^\infty$. Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(y_n, y_m) < \varepsilon$ for all $m, n > N$. Let $z = y_{N+1}$. Then, if z' is the image of z under T ,

$$d'(y', z') = \lim_{n \rightarrow \infty} d(y_n - z) < \varepsilon.$$

Thus we see that every open neighborhood around the point x' in X' contains a point z' of W and so, W is dense in X' .

- (c) We now show that X' is a complete metric space. Let $\{x'_i\}_1^\infty$ be a Cauchy sequence in X' . Since W is dense in X' , every open neighborhood in X' contains a point of W . So, for each i , there exists $z'_i \in W$ such that

$$d'(x'_i, z'_i) < 1/n$$

Then

$$d'(z'_j, z'_i) \leq d'(z'_j, x'_j) + d'(x'_j, x'_i) + d'(x'_i, z'_i) < 1/j + 1/i + d'(x'_j, x'_i)$$

So, since $\{x'_i\}_1^\infty$ is Cauchy, as $i, j \rightarrow \infty$, $d'(z'_j, z'_i) \rightarrow 0$. So the sequence $\{z'_i\}_1^\infty$ is Cauchy in X' . Since T is an isometry, we see that the sequence $T^{-1}(\{z'_i\}_1^\infty) = (T^{-1}(z'_1), T^{-1}(z'_2), T^{-1}(z'_3), \dots) = (z_1, z_2, z_3, \dots)$ is Cauchy in X . Let x' be the equivalence class of the Cauchy sequence (z_1, z_2, z_3, \dots) . We now show that x' is the limit of our Cauchy sequence $\{x'_i\}_1^\infty$ in X' . We have

$$d'(x'_i, x') \leq d'(x'_i, z'_i) + d'(x', z'_i) < 1/n + d'(x', z'_i)$$

for $z'_i \in W$. Then, since x' is the equivalence class of the Cauchy sequence (z_1, z_2, z_3, \dots) and z'_i is the equivalence class of (z_i, z_i, z_i, \dots) , $d'(x', z'_i) = \lim_{n \rightarrow \infty} d(z_n, z_i)$. Then

$$d'(x'_i, x') < 1/n + \lim_{n \rightarrow \infty} d(z_n, z_i).$$

The right hand side goes to zero as $n, i \rightarrow \infty$. So the Cauchy sequence $\{x'_i\}_1^\infty$ is convergent in X' and X' is complete.

- (d) Now, we show that the space X' is unique upto isometry. If (\tilde{X}, \tilde{d}) is another space that contains a dense subset \tilde{W} isometric to X , then for any $\tilde{x}, \tilde{y} \in \tilde{X}$, we have sequences $\{\tilde{x}_n\}_1^\infty$ and $\{\tilde{y}_n\}_1^\infty$ in \tilde{W} such that $\tilde{x}_n \rightarrow \tilde{x}$ and $\tilde{y}_n \rightarrow \tilde{y}$ with

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n \rightarrow \infty} \tilde{d}(\tilde{x}_n, \tilde{y}_n).$$

Since W and \tilde{W} are isometric and the closure of W in X' is X' itself, X' and \tilde{X} must be isometric.

□