Introduction to Manifold Theory

Homework 5

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1. Do Exercise 3.6: for part 1, for each of the six types of quadric surfaces listed in the problem, say which you expect to be regular level sets and thus have a smooth manifold structure. For part 2, give an equation defining an example of each of these types of quadric surfaces, view the defining equation as the level set of a function
$$f\colon \mathbb{R}^3 \to \mathbb{R},$$
and determine algebraically whether the level set is regular.

2. Do Exercise 3.5: let ξ be the function from $n \times n$-matrix-space $\mathbb{R}^{n^2}$ to itself sending a matrix A to AA^T. Show that the Jacobian of ξ at a point $A \in \mathbb{R}^{n^2}$, viewed as a linear transformation from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{n^2}$, sends a matrix $a \in \mathbb{R}^{n^2}$ to
$$(J\xi)_A(a) = aA^T + Aa^T.$$
Exercise 3.4 may be useful.

$$\begin{aligned}
\lim_{t\to 0} \frac{\|\mu(a+tA) - \mu(a) - T(tA)\|}{\|(tA)\|} &= \lim_{t\to 0} \frac{\|(a+tA)(a^T + tA^T) - aa^T - tT(A)\|}{\|(tA, tB)\|} \\
&= \lim_{t\to 0} \frac{\|aa^T + tAa^T + taA^T + t^2AA^T - aa^T - tT(A)\|}{\|(tA, tB)\|} \\
&= \lim_{t\to 0} \frac{|t|\|Aa^T + aA^T + tAA^T - T(A)\|}{|t|\|A\|} \\
&= \lim_{t\to 0} \frac{\|Aa^T + aA^T + tAA^T - T(A)\|}{\|A\|} \\
&= \frac{\|Aa^T + aA^T - T(A)\|}{\|A\|}
\end{aligned}$$

The above limit is 0 and hence the numerator

$$\|Aa^T + aA^T - T(A)\| = 0 \implies Aa^T + aA^T = T(A)$$

So $T(A)$ evaluated at the point a is given by

$$(J\xi)_A(a) = aA^T + Aa^T.$$