Introduction to Manifold Theory

Homework 8

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1. Do Exercise 6.9: show that the matrix representative of a total derivative Df_p of a smooth map $f : X \to Y$, in the coordinate bases for TX_p and $TY_{f(p)}$ arising from charts φ near p and ψ near $f(p)$, is the Jacobian matrix of $\psi \circ f \circ \varphi^{-1}$ at $\varphi(p)$.

2. Do Exercise 6.11: show that the usual identification of a finite-dimensional vector space V with its double dual V^{**} is an isomorphism.

The usual identification is given by the map $T : V \to V^{**}$ that takes $v \in V$ to the linear functional $T(v) \in V^{**}$ defined by

$$(T(v))(\alpha) = \alpha(v)$$

for all $\alpha \in V^*$. We now check that the map T is linear and bijective to show that it is an isomorphism of vector spaces.

(a) For $p \in \mathbb{R}$ and x, $y \in V$, we have $(T(px + y))(\alpha) = \alpha(px + y) = p(T(x))(\alpha) + (T(y))(\alpha)$ for all $\alpha \in V^*$. Thus, T is linear.

(b) $(T(v))(\alpha) = \alpha(v) = 0$ for all $\alpha \in V^*$ implies that $v = 0$. Thus the kernel of T is trivial and hence T is injective.

(c) The dimension of V, V^* and V^{**} are all same and since T is injective, it follows that it is also surjective.

Hence, T is an isomorphism of vector spaces.

3. Do Exercise 6.12: show that, given a basis for a finite-dimensional vector space V , the corresponding "dual basis" as defined in this exercise is a basis for the dual space V^*.

If $\beta = \{e_1, e_2, \dots, e_n\}$ is a basis for the finite dimensional vector space V, then we show that the set $\beta^* = \{e_1^*, \dots, e_n^*\}$ with e_i^* defined by

$$e_i^*(a^1 e_1 + \cdots + a^n e_n) = a^i$$

is a dual basis for the dual space V^*.

First we show that the given set is linearly independent. Indeed if $\sum_{i=1}^n a^i e_i^* = 0$, then for each $k \in \{1, \ldots, n\}$, we see that $(\sum_{i=1}^n a^i e_i^*)(e_k) = a^k = 0$. Thus the set β^* is linearly independent. Furthermore, since the dimension of the dual space V^* is equal to n we see that the set β^* containing n linearly independent element is a basis for V^*.

4. Do Exercise 6.14: roughly, show that the "transforms like a covector" condition makes the definition of an abstract element $\alpha \in T^* X_p$ given in the exercise well-defined.

I could not wrap my head around ques. 1 and 4. Sorry for the incomplete submission.