MA515 - Analysis I

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August 28, 2022

Problems

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1. Prove the following inequality:

$$(a+b)^p \leq 2^{p-1}(a^p+b^p) \qquad \forall a,b \geq 0, \;\; p \geq 1$$

Proof. Let $f(x) = x^p$, $f : [0,\infty) \to \mathbf{R}$ and $p \geq 1$. Since f is a *convex* function, we have

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \qquad \text{for } \alpha \in [0,1].$$

For $\alpha = 1/2$, we have

$$\begin{aligned} & f\left(\frac{a}{2}+\frac{b}{2}\right) \leq \frac{f(a)}{2} + \frac{f(b)}{2} \\ \text{or,} \quad & \frac{1}{2^p} f(a+b) \leq \frac{1}{2}\left(f(a)+f(b)\right) \\ \text{or,} \quad & a^p + b^p \leq 2^{p-1}\left(a^p+b^p\right). \end{aligned}$$

□

2. **Young's inequality:** Let $p > 1$ and q its conjugate (i.e. $\frac{1}{p}+\frac{1}{q}=1$). Then

$$u \cdot v \leq \frac{u^p}{p} + \frac{v^q}{q} \qquad \forall u, v \geq 0.$$

Proof. If either u or v equals 0, then the inequality follows immediately. Suppose $u > 0$, $v > 0$ and let $f(x) = e^x$. Since f is a *convex* function,

$$\begin{aligned} u \cdot v &= \exp\left(\log u + \log v\right) \\ &= f\left(\frac{1}{p}\log u^p + \frac{1}{q}\log v^q\right) \\ &\leq \frac{1}{p} f(\log u^p) + \frac{1}{q} f(\log v^q) \\ &= \frac{u^p}{p} + \frac{v^q}{q}. \end{aligned}$$

□

3. **Holder's inequality for sums:** Let $p, q \geq 1$ be conjugate exponents. Let $x = \{x_i\}_1^\infty \in l^p$ and $y = \{y_i\}_1^\infty \in l^q$. Then

   a. $xy = \{x_i y_i\}_1^\infty \in l^1$ and

   b. $\sum_1^\infty |x_i y_i| \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} \cdot (\sum_1^\infty |y_i|^q)^{\frac{1}{q}}$.

Proof. Let $u_i = \frac{x_i}{\left(\sum_1^\infty |x_i|^p\right)^{1/p}}$ and $v_i = \frac{y_i}{\left(\sum_1^\infty |y_i|^q\right)^{1/q}}$. Then by Young's inequality,

$$\begin{aligned} u_i \cdot v_i &= \frac{x_i}{\left(\sum_1^\infty |x_i|^p\right)^{1/p}} \cdot \frac{y_i}{\left(\sum_1^\infty |y_i|^q\right)^{1/q}} \\ &\le \frac{x_i^p}{p\sum_1^\infty |x_i|^p} + \frac{y_i^q}{q\sum_1^\infty |y_i|^q} \end{aligned}$$

Let $m = \left(\sum_1^\infty |x_i|^p\right)^{1/p}$ and $n = \left(\sum_1^\infty |y_i|^q\right)^{1/q}$. Then from above we have

$$\begin{aligned} \sum_1^\infty |x_i y_i| = mn\sum_1^\infty |u_i v_i| \le mn\sum_1^\infty \left|\frac{1}{pm^p}x_i^p + \frac{1}{qn^q}\cdot y_i^q\right| &\le mn\left(\frac{1}{pm^p}\cdot\sum_1^\infty |x_i^p| + \frac{1}{qn^q}\sum_1^\infty |y_i^q|\right) \\ &= mn\left(\frac{1}{pm^p}\cdot m^p + \frac{1}{qn^q}\cdot n^q\right) = mn \end{aligned}$$

Hence $\sum_1^\infty |x_i y_i| \le mn = \left(\sum_1^\infty |x_i|^p\right)^{1/p} \cdot \left(\sum_1^\infty |y_i|^q\right)^{1/q}$ which proves (b). Since $0 \le \sum_1^\infty |x_i y_i| < \infty$, we also have (a) by definition. $\square$

4. **Minkowski's inequality :** Let $p \ge 1$ and $x = \{x_i\}_1^\infty \in l^p$ and $y = \{y_i\}_1^\infty \in l^p$. Then
   a. $x + y = \{x_i + y_i\}_1^\infty \in l^p$ and
   b. $\left(\sum_1^\infty |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_1^\infty |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_1^\infty |y_i|^p\right)^{\frac{1}{p}}$.

   Proof. If $p = 1$ then the Minkowski inequality follows from the triangle inequality. Let $p > 1$ then

$$\begin{aligned} \sum_1^\infty \left(|x_i + y_i|^p\right) = \sum_1^\infty \left(|x_i + y_1||x_i + y_i|^{p-1}\right) &\le \sum_1^\infty \left(|x_i||x_i + y_i|^{p-1} + |y_i||x_i + y_i|^{p-1}\right) && (1) \\ &= \sum_1^\infty \left(|x_i||x_i + y_i|^{p-1}\right) + \sum_1^\infty \left(|y_i||x_i + y_i|^{p-1}\right) && (2) \end{aligned}$$

   Now let q be the conjugate exponent of p, then we have $\frac{1}{p} + \frac{1}{q} = 1 \iff p + q = pq \iff p = p(q-1)$. Then, from 2

$$\left(\sum_1^\infty |x_i + y_i|^{(p-1)q}\right)^{1/q} = \left(\sum_1^\infty |x_i + y_i|^p\right)^{1/q} < \infty$$

$$\sum_1^\infty |x_i||x_i + y_i|^{p-1} \le \left(\sum_1^\infty |x_i|^p\right)^{1/p} \cdot \left(\sum_1^\infty |x_i + y_i|^{(p-1)q}\right)^{1/q} = \left(\sum_1^\infty |x_i|^p\right)^{1/p} \cdot \left(\sum_1^\infty |x_i + y_i|^p\right)^{1/q}$$

$\square$

5. **Jensen's inequality for sums :**

$$\left(\sum_1^\infty |x_i|^{p_2}\right)^{\frac{1}{p_2}} \le \left(\sum_1^\infty |x_i|^{p_1}\right)^{\frac{1}{p_1}} \qquad \text{for all } 1 \le p_1 < p_2 < \infty.$$

6. Define l^p as the set of all sequences $x = \{x_i\}_1^\infty$ such that $\left(\sum_1^\infty |x_i|^p\right)^{\frac{1}{p}} < \infty$ and define the metric d in l^p by

$$d_p(x, y) = \left(\sum_1^\infty |x_i - y_i|^p\right)^{\frac{1}{p}} \qquad x, y \in l^p.$$

   Prove that l^p is a metric space.