# Algebraic Topology

Midterm Review Sheets

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## 1. Homotopy and Homotopy Type

- Definition and basic intuition for homotopy of maps between topological spaces

- Homotopy of maps is an equivalence relation: statement and proof (can assume "restriction to closed subsets" lemma without proof)

- Homotopy of maps relative to a subset of their domain: definition and basic intuition

- Special case of homotopy rel endpoints for paths: homotopy relative to $\{0,1\} \subset [0,1]$

- Definition of homotopy equivalence between topological spaces

- Intuition for homotopy equivalence vs. homeomorphism, e.g. classifying letters of the alphabet

- Homotopy of maps is preserved when pre- or post-composing with some other map: statement and proof

- Homotopy equivalence of topological spaces is an equivalence relation: statement and proof

- Definitions of contractible topological space and nullhomotopic map

- Proof that $X$ is contractible iff every map into $X$ is nullhomotopic iff every map out of $X$ is nullhomotopic (HW 1)

- House with two rooms: basic idea (is it contractible? how would you describe it to a friend?)

- Definition of retraction of a topological space $X$ onto a subset $A$ (idempotent map $r$ from $X$ to itself with image $A$)

- Definition of deformation retraction of a topological space X onto a subset A (homotopy rel A from the identity on $X$ to a retraction onto $A$)

- Pictoral and formulaic descriptions of deformation retractions in examples (HW 1)

- Example of a retraction from a space $X$ onto a subset $A$ that is not homotopic rel $A$ to the identity map on $X$, i.e. doesn't come as the ending map of a deformation retraction (take $X =$ two-point set and send both points to the same point of $X$)

- Definition of quotient topology on the set of equivalence classes $X/\sim$ where $X$ is a topological space and $\sim$ is any equivalence relation on $X$

- Proof that the quotient topology is a valid topology, via the interaction of preimages with unions / intersections

- Definition of mapping cylinder $M_f$ for a map $f : X \to Y$ of topological spaces

- Proof that $M_f$ deformation retracts onto $Y$ (HW 1): you won't be asked to reproduce this on the exam, but you should remember that the statement is true.

- Statement (no proof) that if $f$ is a homotopy equivalence then $M_f$ deformation retracts onto $X$

- Corollary that two spaces $X$ and $Y$ are homotopy equivalent if and only if there exists a third space $Z$ deformation retracting onto both $X$ and $Y$ ; you should be able to prove this assuming the above statements.

- Example of a deformation retraction that's not "$M_f$ retracting onto $Y$ :" anything where distinct points of the larger space "collide" at an earlier time than $t = 1$, e.g. thickened letter $X$ deformation retracting onto a point by first deformation retracting onto an ordinary $X$ and then shrinking the legs

## 2. Cell Complexes

- Definition of CW complex / cell complex, including definitions of $n$-skeleton and attaching map $\varphi_\alpha$ for a cell index $\alpha$

- Definitions of finite-dimensionality and finiteness for CW complexes

- Definition of Euler characteristic of a finite CW complex; computing in examples by counting vertices, edges, faces, higher cells if they exist

- Topological space associated to a finite-dimensional CW complex

- Topological space associated to an infinite-dimensional CW complex

- Constructing CW decompositions of $S^2$ with any allowable number of vertices, edges, and faces (HW 2)

- Constructing standard CW structure on $\mathbb{R}P^n$, number of cells in each dimension, Euler characteristic, definition of $\mathbb{R}P^\infty$

- Constructing standard CW structure on $\mathbb{C}P^n$, number of cells in each dimension, Euler characteristic, definition of $\mathbb{C}P$

- CW decompositions of closed oriented surfaces of genus $g$; description of these surfaces by identifying sides of $4g$-gon

- Proof that for a CW complex $X$, the natural maps $X^n \to X^{n+1}$ are injective (HW 2); you won't be asked to reproduce problem 3 on HW 2.

- Understand the statement (no proof) that on the $n$-skeleton $X^n$, the quotient topology it began its life with agrees with the topology it acquires as a subspace of $X$ due to the injectivity of the map $X^n \to X$

- Definition of characteristic map $\Phi_\alpha$ and cell $e^n_\alpha$ for an $n$-cell index $\alpha$

- Statement (no proof) that $\Phi_\alpha$ gives a homeomorphism from $\mathrm{int}(D^n)$ to $e^n_\alpha \subset X$

- Identifying the cells $e^n_\alpha$ in the usual CW structure on the torus: which cells are closed subsets of the torus? Only the vertex!

- If $X$ has a CW structure then every point of $X$ is in a unique cell: statement only since that's all we covered, although the proof isn't so bad.

- Definition of subcomplex of CW complex $X$ as subset $A$ of underlying topological space that's closed and consists of a union of cells

- Proposition enabling us to build a CW structure on a subcomplex $A$: technical, won't be asked statement or proof, just know that if a subset $A$ is closed and consists of a union of cells then "everything works" when viewing $A$ itself as a CW complex.

- Examples of subcomplexes: $\mathbb{R}P^k \subset \mathbb{R}P^n$ in the usual CW structure, same for complex case; equators of spheres not subcomplexes in the simplest CW structure but you can choose CW structure so that they're subcomplexes

- Example of a CW complex where the closure of some cell is not a subcomplex: attaching a closed interval in a sphere to a proper subinterval of a larger closed interval

## 3. Paths and Homotopy

- Definition of straight line homotopy between any two paths with same endpoints in a convex subset $X$ of $\mathbb{R}^n$

- Definition of reparametrization used in this section, and homotopy between a path and any reparametrization

- Definition of composite path $f * g$ (and condition required for $f * g$ to make sense)

- Definition of reverse path $\bar{f}$ (namely: $\bar{f}(s) := f(1-s)$)

- Composition respects homotopy rel endpoints: statement and proof.

- Definition of fundamental group $\pi_1(X, x_0)$ with group operation given by $*$ on homotopy classes

- The group axioms hold for $\pi_1(X, x_0)$: statement and proof (but you'll be asked at most a proper subset of the proof)

- Using straight-line homotopies to show $\pi_1(X, x_0)$ is trivial when $X$ is a convex subset of $\mathbb{R}^n$ (usual topology)

- Definition of change-of-basepoint isomorphism $\beta_h$ from $\pi_1(X, x_1)$ to $\pi_1(X, x_0)$ given a homotopy class $[h]$ of paths from $x_0$ to $x_1$; proof that $\beta_h$ is an isomorphism

- Definition of simply connected

- $X$ is simply connected iff for any $x_0$, $x_1$ in $X$ there is a unique homotopy class of paths joining them: statement and proof

- $\pi_1(X)$ is abelian iff all change-of-basepoint isomorphisms $\beta_h$ depend only on the end-points of $h$ (HW 3)

- You won't be asked about problems 2 or 3 on HW 3.

## 4. The Fundamental Group of the Circle

- Definition of covering space

- Definition of homotopy lifting property with respect to a space $Y$

- Definition of fibration (homotopy lifting property holds with respect to every space $Y$ )

- Statement that covering spaces are fibrations with unique lifts of homotopies (no proof)

- Special case: path lifting principle with unique lifts of paths holds for covering spaces

(take $Y$ to be a single point)

- Corollary about lifting homotopies rel endpoints between paths

- Statement (no proof) that $p(s) := (\cos(2\pi s), \sin(2\pi s))$ gives a covering space $R \xrightarrow{p} S^1$

- $\pi_1(S^1) \simeq \mathbb{Z}$: statement and proof assuming all previous lemmas (just the final proof itself)