Analysis I

Homework 6

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December 5, 2022

Pack Pledge: I have neither given nor received unauthorized aid on this test or assignment.

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1. Let $M \subset X$ be a closed subspace in an inner product space such that $M \neq M^{\perp\perp}$. (Note that this would not be possible in a Hilbert space.) Let $x \in M^{\perp\perp} - M$. Show that there is no best approximation to x in M.

   M and $M^{\perp\perp}$ are both closed subspaces of X. If $x \in M^{\perp\perp} - M$, suppose there is a best approximation to x in M given by $s = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$ where $\{e_i\}_1^\infty$ is an orthonormal sequence of M. Then for the sequence $\{x_i\}_1^\infty$ in $M^{\perp\perp}$ converging to x, we take $y \in M^{\perp} = M^{\perp\perp\perp}$. Then

   $$\langle s, y \rangle = \left\langle \lim_{n\to\infty} \sum_{i=1}^{\infty} \langle x_n, e_i \rangle e_i, y \right\rangle = 0.$$

   But this means that $s \in M^{\perp\perp}$ which cannot be true. Thus, x doesn't have a best approximation.

2. Show that $\{\sin nx\}_{n\geq 1}$ and $\{\cos nx\}_{n\geq 1}$ are orthogonal in $L^2[-\pi, \pi]$. Build two orthonormal sequences in $L^2[-\pi, \pi]$.

   For $n \neq 0$ and $m \neq 0$, we have

   $$\int_{-\pi}^{\pi} \cos nx \cdot \cos mx \; dx = \int_{-\pi}^{\pi} \frac{1}{2}(\cos(n-m)x + \cos(n+m)x) \; dx$$

   So when $n \neq m$,

   $$\begin{aligned}\int_{-\pi}^{\pi} \cos nx \cdot \cos mx \; dx &= \frac{1}{2}(\frac{1}{n-m}\sin(n-m)x + \frac{1}{n+m}\sin(n+m)x)\Big|_{x=-\pi}^{x=\pi} \\ &= \frac{1}{n-m}\sin(n-m)\pi + \frac{1}{n+m}\sin(n+m)\pi = 0\end{aligned}$$

   And when $n = m$,

   $$\int_{-\pi}^{\pi} \cos nx \cdot \cos mx \; dx = \frac{x}{2} + \frac{1}{4n}\sin 2nx\Big|_{x=-\pi}^{x=\pi} = \pi$$

Thus we obtain,

$$\langle \cos nx, \cos mx \rangle = \int_{-\pi}^{\pi} \cos nx \cdot \cos mx \; dx = \begin{cases} 0 & m \neq n, \\ \pi & m = n. \end{cases}$$

Thus $\{\cos(nx)/\sqrt{\pi}\}_{n \geq 1}$ is an orthonormal sequence in $L^2[-\pi, \pi]$. Similarly,

$$\langle \sin nx, \sin mx \rangle = \int_{-\pi}^{\pi} \sin nx \cdot \sin mx \; dx = \begin{cases} 0 & m \neq n, \\ \pi & m = n. \end{cases}$$

Hence $\{\sin(nx)/\sqrt{\pi}\}_{n \geq 1}$ is another orthonormal sequence in $L^2[-\pi, \pi]$.

3. Determine whether or not the following is true in a Hilbert space H:

$$[x \perp y] \quad \Longleftrightarrow \quad [\|x + y\|^2 = \|x\|^2 + \|y\|^2].$$

We know that $\|x+y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + 2\mathfrak{Re} \langle x, y \rangle + \|y\|^2$ and $\|x-y\|^2 = \langle x - y, x - y \rangle = \|x\|^2 + \langle x, -y \rangle + \langle -y, x \rangle + \|y\|^2 = \|x\|^2 + \mathfrak{Im} \langle x, -y \rangle + \|y\|^2$.

If $x \perp y$, then $\langle x, y \rangle = 0$ and hence $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Similarly, if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$, we have $\mathfrak{Re} \langle x, y \rangle = 0$. Also by parallelogram law, $\|x - y\|^2 = \|x\|^2 + \|y\|^2$, which gives $\mathfrak{Im} \langle x, -y \rangle = -\mathfrak{Im} \langle x, y \rangle = 0$. Hence, $x \perp y$.

4. Let X be a normed space, Y a subspace in X, and $x \in X$. Show that the set M of best approximations to x out of Y is convex, i.e., $\alpha x + (1 - \alpha)y \in M$, for all $x, y \in M$ and $\alpha \in [0, 1]$.

If M has more than one point and δ is the distance from x to Y, then for y, $z \in M$ we have

$$\|x - y\| = \|x - z\| = \delta.$$

Let $w = \alpha y + (1 - \alpha)z$ for some $\alpha \in [0, 1]$. Then

$$\|x - w\| = \|x - \alpha y + (1 - \alpha)z\| = \|\alpha(x - y) + (1 - \alpha)(x - z)\| \leq \alpha\delta + (1 - \alpha)\delta = \delta.$$

But we know that $\|x - w\| \geq \delta$ since $w \in Y$. Hence $\|x - w\| = \delta$ and $w \in M$. So, M is convex.

5. Show that a Hilbert space H is strictly convex, i.e. for any $x, y \in H$ with $\|x\| = \|y\| = 1$, we have $\|x + y\| < 2$.

In a Hilbert space H, we have by parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

For x and $y \neq x$ of norm 1 in H, if $\|x - y\| = a$ then,

$$\|x + y\|^2 = 4 - a^2 < 4.$$

Hence $\|x + y\| < 2$.

6. Find the best approximation to $\sin(x)$ in $L^2[0,1]$ by a polynomial of degree ≤ 3. (You can use software to help with the calculations if necessary; or you can leave coefficients as inner products when applicable).

Let $M = \text{Span}\ \{1, x, x^2, x^3\}$ be the subspace generated by the polynomials of degree ≤ 3. We need to find the projection of $\sin x$ onto the subspace M. First we find the orthonormal basis of M by Gram-Schmidt process: $e_1 = 1$, $e_2 = v_2/\|v_2\|$ where

$$v_2 = x - \int_0^1 x\ dx = x - 1/2 \implies \|v_2\| = 1/12.$$

Similarly $e_3 = v_3/\|v_3\|$ where

$$v_3 = x^2 - e_2 \int_0^1 x^2 e_2\ dx - \int_0^1 x^2\ dx$$

and $e_4 = v_4/\|v_4\|$ where

$$v_4 = x^3 - e_3 \int_0^1 x^3 e_3\ dx - e_2 \int_0^1 x^2 e_2\ dx - \int_0^1 x^3 e_1\ dx.$$

Then the best approximation to $\sin x$ in M is given by $y = \langle e_1, sinx \rangle\, e_1 + \langle e_2, sinx \rangle\, e_2 + \langle e_3, sinx \rangle\, e_3 + \langle e_4, sinx \rangle\, e_4$.

7. Let H be a Hilbert space and $M \subset H$. Prove that

$$M \text{ is total iff } [x \perp M \Rightarrow\ x = 0]$$

If M is total, Span M is dense in H. So, $(\text{Span } M)^{\perp} = 0 \implies [x \perp M \implies x = 0]$.

Similarly, if M satisfies the given condition, then span of M is dense in H (since H is a Hilbert space). Thus M is total.

8. **Kreyszig p.159 / Problem 7.** Let (e_k) be any orthonormal sequence in an inner product space X. Show that for any x, $y \in X$,

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle \langle y, e_k \rangle| \leq \|x\| \|y\|.$$

By Cauchy-Schwarz, we have

$$\left(\sum_{k=1}^{\infty} |\langle x, e_k \rangle \langle y, e_k \rangle| \right)^2 = \left(\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \right) \cdot \left(\sum_{k=1}^{\infty} |\langle y, e_k \rangle|^2 \right).$$

By Bessel's inequality, we have $\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2$ for all $x \in X$. Thus,

$$\left(\sum_{k=1}^{\infty} |\langle x, e_k \rangle \langle y, e_k \rangle| \right)^2 \leq \|x\|^2 \|y\|^2.$$

Taking square root on both sides yields the required inequality.

9. **Kreyszig p.159 / Problem 10.** Let $x_1(t) = t^2$, $x_2(t) = t$ and $x_3(t) = 1$. Orthonormalize x_1, x_2, x_3, in this order, on the interval $[-1, 1]$ with respect to the inner product given in Prob. 9.

The given inner product is

$$\langle x, y\rangle = \int_{-1}^{1} x(t)y(t)\ dt.$$

We use Gram-Schmidt process to orthonormalize the given elements. Here, $\|x_1(t)\|^2 = \int_{-1}^{1} t^4\ dt = 2/5$, so $e_1 = t^2/\sqrt{2/5}$. Now, $e_2 = v_2/\|v_2\|$ where $v_2 = x_2(t) - \langle x_2, e_1\rangle\, e_1$.

$$v_2 = t - t^2/\sqrt{2/5}\int_{-1}^{1} t^3/\sqrt{2/5}\ dt = t \qquad \text{and,}$$

$$\|v\|^2 = \int_{-1}^{1} t^2\ dt = 2/3.$$

Thus $e_2 = t/\sqrt{2/3}$. Finally, $e_3 = v_3/\|v_3\|$ where $v_3 = x_3 - \langle x_3, e_1\rangle\, e_1 - \langle x_3, e_2\rangle\, e_2$.

$$v_3 = 1 - e_1\int_{-1}^{1} t^2/\sqrt{2/5}\ dt - e_2\int_{-1}^{1} t/\sqrt{2/3}\ dt = 1 - (5t^2/2)(2/3) - 0 = 1 - 5t^2/3.$$

$$\|v_3\|^2 = \int_{-1}^{1} (1 - 5t^2/3)^2\ dt = t - \frac{10t^3}{9} + \frac{5t^5}{9}\Bigg|_{t=-1}^{t=1} = 8/9.$$

Thus $e_3 = (1 - 5t^2/3)/\sqrt{8/9}$.

10. **Kreyszig p.175 / Problem 9.** Let M be a total set in an inner product space X. If $\langle v, x\rangle = \langle w, x\rangle$ for all $x \in M$, show that $v = w$.

Here $\langle v, x\rangle = \langle w, x\rangle \implies \langle v - w, x\rangle = 0$ for all $x \in M$. Thus $v - w \in M^{\perp}$. Since M is total, there does not exist any nonzero $y \in X$ such that $y \perp M$. So, $v - w = 0 \implies v = w$.

11. **Kreyszig p.175 / Problem 10.**

Let M be a subset of a Hilbert space H, and let v, $w \in H$. Suppose that $\langle v, x\rangle = \langle w, x\rangle$ for all $x \in M$ implies $v = w$. If this holds for all v, $w \in H$, show that M is total in H.

If $\langle v, x\rangle = \langle w, x\rangle$ for all $x \in M$ implies $v = w$ then $\langle v - w, x\rangle = 0$ for all $x \in M$ implies that $v = w$. Hence, for $y \in H$, $\langle y, x\rangle = 0$ for all $x \in M$ implies $y = 0$. Thus $M^{\perp} = 0$ in the Hilbert space H. By exercise 7, we have that M is total in H.

12. Prove that every vector space $X \neq \{0\}$ has a Hamel basis.

Let M be the set of all linearly independent subsets of X. For $x \in X$, we have $\{x\} \in M$ and hence $M \neq \emptyset$. We define a partial ordering in M by set inclusion. Then every chain $C \subset M$ has an upper bound which is the union of all sets of C. By Zorn's lemma, M has a maximal element which we call B. Let $Y = \mathrm{Span}B$. Then $Y = X$ since otherwise $B \cup \{z\}$ for $z \notin Y$ would be a linearly independent set of X which contradicts the maximality of B. Hence Y is the Hamel basis of X.

13. Prove Hahn-Banach Theorem (Real Version). Use the ideas discussed in class.

Theorem 1 (Hahn-Banach Theorem (Real)). *Let X be a real vector space and Z be a subspace of X. Let $p : X \longrightarrow \mathbb{R}$ be a sublinear functional on X and $f : Z \longrightarrow \mathbb{R}$ a linear functional satisfying $f(x) \leq p(x)$ for all $x \in Z$. Then f has a linear extension $\tilde{f} : X \longrightarrow \mathbb{R}$ such that $\tilde{f}(x) \leq p(x)$ for all $x \in X$.*

Proof. We consider the set P of all linear extensions $g : \mathcal{D}(g) \longrightarrow \mathbb{R}$ of f which satisfy $g(x) \leq p(x)$. Since $f \in P$, $P \neq \emptyset$ and thus we can define a partial order on P. For g_1 and g_2 in P, we say that $g_1 \leq g_2$ if and only if $\mathcal{D}(g_1) \subset \mathcal{D}(g_2)$ and $g_2|_{\mathcal{D}(g_1)} = g_1$ (i.e. g_2 is an extension of g_1).

For any chain $C \subset P$ and $g \in C$ we define $\tilde{g}$ by $\tilde{g}(x) = g(x)$ if $x \in \mathcal{D}(g)$. Then $\tilde{g}$ is a linear functional whose domain is $\bigcup_{g \in C} \mathcal{D}(g)$ which is a vector space since C is a chain. Here $g \leq \tilde{g}$ for all $g \in C$ and so $\tilde{g}$ is an upper bound of C. Then, since C was arbitrary, by Zorn's lemma P has a maximal element $\tilde{f}$ such that $\tilde{f} \leq p(x)$ for $x \in \mathcal{D}(\tilde{f})$.

Now we show that $\mathcal{D}(\tilde{f}) = X$. Suppose that this is false and there exists $y \in X - \mathcal{D}(\tilde{f})$. Then we consider the subspace Y spanned by $\mathcal{D}(\tilde{f})$ and y. Any $z \in Y$ can be uniquely represented as $z = x + \alpha y$ where $x \in \mathcal{D}(\tilde{f})$ and we can define a linear functional g by

$$g(x + \alpha y) = \tilde{f}(x) + \alpha c \tag{1}$$

where c is any real constant. Since g is a linear extension of $\tilde{f}$ if we can show that $g(x) \leq p(x)$ then this would contradicts the maximality of $\tilde{f}$ and show that the domain of $\tilde{f} = X$.

Now we show that $g(x) \leq p(x)$ with a suitable c. For y, $z \in \mathcal{D}(\tilde{f})$ we have,

$$\tilde{f}(y) - \tilde{f}(z) \leq p(y - z) = p(y + y_1 - y_1 - z) \leq p(y + y_1) + p(-y_1 - z)$$

which gives $-p(-y_1 - z) - \tilde{f}(z) \leq p(y + y_1) - \tilde{f}(y)$ where y_1 is fixed. Since the left side doesn't depend on y and the right doesn't depend on z, the inequality continues to hold if we take supremum (call it m_0) over z and infinum (call it m_1) over y in $\mathcal{D}(\tilde{f})$. Then $m_0 \leq m_1$ and for a c with $m_0 \leq c \leq m_1$, we have $-p(-y_1 - z) - \tilde{f}(z) \leq c$ and $c \leq p(y + y_1) - \tilde{f}(y)$ for all y, $z \in \mathcal{D}(\tilde{f})$.

For $\alpha < 0$ in (1), we write $z = \alpha^{-1} y$ and obtain $-p(-y_1 - y/\alpha) - \tilde{f}(y/\alpha) \leq c$. Multiplying by $-\alpha > 0$ gives, $\alpha p(-y_1 - y/\alpha) + \tilde{f}(y) \leq -\alpha c$. Using this in (1), we obtain for $x = y + \alpha y_1$,

$$g(x) = \tilde{f}(y) + \alpha c \leq -\alpha p(-y_1 - y/\alpha) = p(x).$$

Similarly, we obtain required inequality $g(x) \leq p(x)$ for $\alpha = 0$ and $\alpha > 0$.

□