Analysis II

Homework 6

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April 23, 2023

Pack Pledge: I have neither given nor received unauthorized aid on this test or assignment.

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1. In class, we showed that [If $\mu(X) < \infty$, and $f_n \to f$ pointwise a.e. $[\mu]$, then $f_n \to f$ in measure]. Give an example to show that the hypothesis $\mu(X) < \infty$ cannot be omitted.

We take $X = [0, \infty]$ which has Lebesgue measure infinity. Let $f_n : [0, \infty] \to \mathbb{R}$ be defined as $f_n(x) = \chi_{[n-1,n)}$. If f is the 0 function, then $f_n \to f$ pointwise almost everywhere. However, for all $n \in \mathbb{N}$ and $\varepsilon = 1/2$ we have,

$$\mu\{x : |f(x) - f_n(x)| > 1/2\} = \mu([n-1, n)) = 1, \qquad \text{and thus,}$$

$$\lim_{n\to\infty} (\mu\{x : |f(x) - f_n(x)| > 1/2\}) = 1.$$

This shows that f_n does not converge to f in measure.

2. Show that almost uniform convergence implies μ-convergence and piecewise convergence a.e.

The sequence $\{f_n\}$ almost uniformly converges to f on X if for every $\varepsilon > 0$, there exists a measurable set N_ε with $\mu(N_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on $M = X \setminus N_\varepsilon$, that is, given $\varepsilon' > 0$, there exists a $p \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \varepsilon'$ for all $n > p$ and all $x \in M$.

Then for every $\varepsilon > 0$ and $\varepsilon' > 0$ we have $p \in \mathbb{N}$ such that

$$\mu\left(\{x : \ |f - f_n| \geq \varepsilon' \text{ for } n > p\}\right) = \mu(N_\varepsilon) < \varepsilon.$$

This implies that f_n converges to f in measure.

Now, for each $k \in \mathbb{N}$, we take the set $N_{1/k}$ as defined above and let $N = \bigcap_{k=1}^{\infty} N_{1/k}$, the intersection of decreasing sets. Each of these sets are measurable and $N_1 < 1$. We have $\mu(N) < 1/k$ for every k and so we have $\mu(N) = 0$. For each x in the complement of N we have $x \in X \setminus N_{1/k}$ for some K and $f_n \to f$ uniformly and thus $f_n \to f$ pointwise.

3. **(Egoroff's Theorem)** Let $X \in \mathcal{L}(\mathbb{R})$ with $m(X) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions on X which converges pointwise on X to the real-valued function f. Then for each $\varepsilon > 0$,

there is a closed set $F \subset X$ for which

$$f_n \to f \text{ uniformly on } F \text{ and } \mu(X \setminus F) < \varepsilon.$$

Let A be the set where the sequence $\{f_n\}$ does not converge to f. We define the sets

$$A_k^m = \{x \in X : |f(x) - f_n(x)| \geq 1/k \text{ for all } n \geq m\}.$$

If $B_k = \bigcap_{m=1}^{\infty} A_k^m$ then we see that $A = \bigcup_{k=1}^{\infty} B_k$.

$$B_k = \{x \in X : |f(x) - f_n(x)| \geq 1/k \text{ for infinitely many } n\}.$$

We then have $\lim_{k\to\infty}$

4. Let $E \subset \mathbb{R}$ measurable, with $m(E) < \infty$. Then for all $\varepsilon > 0$, there exists a finite disjoint collection of open intervals $\{I_k\}_1^n$ for which if $\mathcal{O} = \cup_1^n I_k$, then

$$m(E \setminus \mathcal{O}) + m(\mathcal{O} \setminus E) < \varepsilon.$$

Since E is measurable, we see that for every $\varepsilon > 0$, there exists an open set U containing E such that $m(U \setminus E) < \varepsilon/2$. Let U be the countable union of disjoint open sets $\{I_k\}$. Then for each natural number n, we have,

$$\sum_{k=1}^{n} m(I_k) = m\left(\bigcup_{k=1}^{n} I_k\right) \leq m(U) < \infty \implies \sum_{k=1}^{\infty} m(I_k) < \infty.$$

Hence we can choose $n \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} m(I_k) < \varepsilon/2$ and define $\mathcal{O} = \bigcup_{k=1}^{n} I_k$. Then $m(\mathcal{O} \setminus E) \leq m(U \setminus E) < \varepsilon/2$ and we have (all these sets are measurable)

$$m(E \setminus \mathcal{O}) \leq m(U \setminus \mathcal{O}) = m\left(\bigcup_{k=n+1}^{\infty} I_k\right) < \varepsilon/2.$$

Thus, we have the required set $\mathcal{O}$ satisfying the given condition.

5. **(Lusin's Theorem)** Let f be a measurable function on $X \subset \mathbb{R}$. Show that for all $\varepsilon > 0$, there exists a continuous function g on $\mathbb{R}$ and a closed set $F \subset X$ s.t. $f = g$ on F and $m(X \setminus F) < \varepsilon$.

Since f is measurable, let $\{f_n\}$ be a sequence of simple functions on X that converges pointwise to f. By Proposition 11 (page 66), we choose a continuous function g_n on $\mathbb{R}$ and a closed set F_n with $f_n = g_n$ on F_n and $m(X \setminus F_n) < \varepsilon/2^{n+1}$. By Egoroff's theorem, there is a closed set F_0 in X such that $f_n \to f$ uniformly on F_0 and $m(X \setminus F) < \varepsilon/2$. Defining $F = \bigcap_{n=0}^{\infty} F_n$ we have

$$m(X \setminus F) = m\left((X \setminus F_0) \cup \bigcup_{n=1}^{\infty} (E \setminus F_n)\right) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

F is closed and $f_n \to f$ uniformly on $F \subset F_0$. The corresponding functions g_n restricted to F

equals f and is continuous on $\mathbb{R}$.

6. Let $X \in \mathcal{L}(\mathbb{R})$. Show that $\overline{L_s^\infty(X)} = L^\infty(X)$.

Let $f \in L^\infty(X)$. Then f is bounded on the complement E of a set of measure 0 in X. By simple approximation lemma, for every $\varepsilon > 0$, there exists simple functions φ_ε and ψ_ε on E such that $\varphi_\varepsilon \leq f \leq \psi_\varepsilon$ and $0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon$ on E. Thus, for any $\varepsilon > 0$, we have a simple function φ such that

$$\|f - \varphi\|_\infty = \sup_{x \in E} |f - \varphi| < \varepsilon.$$

Thus, $L_s^\infty(X)$ is dense in $L^\infty(X)$.

7. Let $X \in \mathcal{L}(\mathbb{R})$. Let $1 \leq p < \infty$. Show that $L^p(X)$ is separable.

For a closed interval $[a, b]$ in $\mathbb{R}$ we define $S[a, b]$ to be the collection of step functions on $[a, b]$. We also define $S'[a, b]$ to be the step functions f on $[a, b]$ that take rational values and for which there is a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ with x_i rational and f constant on each partition (x_{i-1}, x_i). Clearly, $S'[a, b]$ is dense in $S[a, b]$ since rationals are dense in real numbers. Furthermore, the graph of each f in $S'[a, b]$ is a partition of a line in $\mathbb{Q}^2$ and hence $S'[a, b]$ is countable. Since the step functions $S[a, b]$ are dense in $L^p[a, b]$, we see that $S'[a, b]$ is also dense in $L^p[a, b]$.

Now for each natural number n, we define $\mathcal{F}_n$ to be the collection of functions that are 0 on the complement of $[-n, n]$ and restrict to some function in $S'[-n, n]$ in the interval $[-n, n]$. We define $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ which, we note, is countable. For each $f \in L^p(\mathbb{R})$, we see that, by monotone convergence theorem,

$$\lim_{n \to \infty} \int_{[-n,n]} |f|^p = \int_{\mathbb{R}} |f|^p$$

where each function on the left is an element of $\mathcal{F}$. Thus $\mathcal{F}$ is dense in $L^p(\mathbb{R})$. For any measurable set X, the restriction of the functions in $\mathcal{F}$ is also countable and dense in X and hence $L^p(X)$ is separable.

8. Let $X \in \mathcal{L}(\mathbb{R})$. Show that $L^\infty(X)$ is not separable.

We show that $L^\infty[a, b]$ is not separable which would imply that $L^\infty(X)$ is not separable for any measurable set X.

Suppose to the contradiction that there exists a countable set $\{f_n\}$ that is dense in $L^\infty[a, b]$. For each $x \in [a, b]$, we take natural number $\eta(x)$ for which $\|\chi_{[a,x]} - f_{\eta(x)}\|_\infty < 1/2$. We see that

$$\|\chi_{[a,x_1]} - \chi_{[a,x_2]}\|_\infty = 1 \qquad \text{whenever } x_1 \neq x_2.$$

Thus η is an injective mapping of $[a, b]$ onto the natural numbers which cannot be true. So, $L^\infty[a, b]$ is not separable.

9. Show that $C_c(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R})$. Hint: Take $f = \chi_{(0,1)}$ and suppose there is a function $g \in C_c(\mathbb{R})$ close to it.

> Let g be a function in $C_c(\mathbb{R})$ such that g is non-zero on a compact set X containing $I = (0,1)$. Then g must necessarily restrict to 1 on the set I, otherwise the norm $\|f - g\|_\infty$ would be non-zero on the set I. Since g is continuous, it must attain every value in [0,1]on the set X. Then we have, $\|f - g\|_\infty > \delta$ for any $0 < \delta < 1$ on the set $X \setminus E$. Thus $C_c(\mathbb{R})$ is no dense in $L^\infty(\mathbb{R})$.

10. Fix $1 \le p < \infty$ and let $f_n \in L^p([0,1])$ be a sequence of step functions defined as follows:

$$f_n(x) = (-1)^k, \text{ for } \frac{k}{2^n} \le x < \frac{k+1}{2^n}, \text{ and } 0 \le k \le 2^n - 1.$$

Show that $\{f_n\}$ is bounded in $L^p([0,1])$, but there is no subsequence of f_n that is Cauchy in $L^p([0,1])$. Can f_n have a pointwise a.e. convergent subsequence?

> For any f_n we have, $\|f_n\|_p^p = \int_{[0,1]} |f_n| = 1$ and hence the sequence is bounded. For $n \neq m$, we have $|f_n - f_m| = 2$ on a set of measure 1/2. Hence $\|f_n - f_m\|_p >\geq 2^{1-1/p}$ and hence there is no subsequence of $\{f_n\}$ that is Cauchy. There is also no subsequence that converges pointwise since such a sequence need to necessarily converge in L^p itself.

11. Show that $L^p(\mu)$ is not a Hilbert space for $p \neq 2$. Hint: Show that the parallelogram law fails for every $p \neq 2$.

> We know that if L^p with the p-norm is a Hilbert Space, it must satisfy the parallelogram law:
>
> $$\|x + y\|_p^2 + \|x - y\|_p^2 = 2(\|x\|_p^2 + \|y\|_p^2)$$
>
> for all x, $y \in L^p(\mu)$.
>
> We take $x = \chi_{[0,1/2)}$ and $y = \chi_{[1/2,1]}$ and note that $xy = 0$ and x, $y \in L^p(\mu)$ for all $p > 0$. Furthermore, $\|x\|_p^2 = \|y\|_p^2 = \left(\int_{[0,1/2)} 1\right)^{2/p} = (1/2)^{2/p}$. Similarly, $\|x + y\|_p^2 = \|x - y\|_p^2 = \left(\int_{[0,1]} 1\right)^{2/p} = 1$. Substituting these values in the equality, we have
>
> $$2 = 2((1/2)^{2/p} + (1/2)^{2/p}) \implies 1/2 = (1/2)^{2/p}.$$
>
> This satisfies only when $p = 2$. Thus, $L^p(\mu)$ is not a Hilbert space for any $p \neq 2$.

12. Prove Clarkson's 1st inequality (for real-valued functions).

> **Clarkson's first inequality**:
>
> $$\|x + y\|_p^p + \|x - y\|_p^p \le 2^{p-1}(\|x\|_p^p + \|y\|_p^p) \qquad \text{for all } x, y \in L^p(\mu), \quad 2 \le p < \infty.$$
>
> We first note the inequality:

Let $a, b \geq 0$, and $p \geq 1$, then $(a+b)^p \leq 2^{p-1}(a^p + b^p)$.

We have, with respect to the p-norm,

$$\left\|\frac{x+y}{2}\right\|^p = \int \left|\frac{x+y}{2}\right|^p \leq \int \left(\left|\frac{x}{2}\right| + \left|\frac{y}{2}\right|\right)^p \leq 2^{p-1}\left(\int \left|\frac{x}{2}\right|^p + \int \left|\frac{y}{2}\right|^p\right) = \frac{1}{2}(\|x\|^p + \|y\|^p).$$

The same inequality holds for the other term and by adding the two, we have,

$$\left\|\frac{x+y}{2}\right\|^p + \left\|\frac{x-y}{2}\right\|^p \leq \|x\|_p^p + \|y\|_p^p.$$

13. Use Clarkson's 2nd inequality to prove that L^p is uniformly convex, for $1 < p \leq 2$.

Clarkson's second inequality:

$$\left\|\frac{x+y}{2}\right\|_p^q + \left\|\frac{x-y}{2}\right\|_p^q \leq \left(\frac{1}{2}\|x\|_p^p + \frac{1}{2}\|y\|_p^p\right)^{q/p} \qquad \text{for all } x,\ y \in L^p(\mu), \quad 1 < p < 2.$$

Let $\varepsilon > 0$; x, $y \in L^p(\mu)$ with $\|x\| = \|y\| = 1$ and $\|x - y\| < \varepsilon$. We see that

$$\left\|\frac{x+y}{2}\right\|_p^q \leq 1 - (\varepsilon/2)^q.$$

Taking $\delta = (1 - (\varepsilon/2)^q)^{1/q}$, we see that $\left\|\frac{x+y}{2}\right\|_p^q < 1 - \delta$. Hence, L^p is uniformly convex for $1 < p < 2$.

14. Let X and Y be normed space, and $T \in B(X, Y)$. If $x_n \xrightarrow{w} x$ in X, show that $Tx_n \xrightarrow{w} Tx$.

If f is a continuous (hence, bounded) linear functional on Y, then we note that $f \circ T$ must be a continuous (hence, bounded) linear functional on X since the composition of linear (resp. continuous) operators is linear (resp. continuous).

Now, iF $x_n \xrightarrow{w} x$, then for every bounded linear functional S on X we have $Sx_n \longrightarrow Sx$. Then, for every bounded linear functional f on Y, since $f \circ T$ is a bounded linear functional on X, we have

$$(f \circ T)x_n \longrightarrow (f \circ T)x \implies f(Tx_n) \longrightarrow f(Tx).$$

Thus $\{Tx_n\}$ converges weakly to Tx by definition.