1 Introduction

The Kazhdan-Lusztig polynomial $P_M(t)$ is a fundamental invariant associated with any matroid M, as defined by Elias, Proudfoot, and Wakefield in [2]. This polynomial, denoted $P_M(t)$, exhibits formal similarities to the Kazhdan-Lusztig polynomials defined for Coxeter groups. The coefficients of $P_M(t)$ depend only on the lattice of flats $\mathcal{L}(M)$ of the matroid, and in fact, they are integral linear combinations of the flag Whitney numbers counting chains of flats with specified ranks.

In [1], Braden and Vysogorets presented a formula that relates the Kazhdan-Lusztig polynomial of a matroid M to that of the matroid obtained by deleting an element e, denoted $M \setminus e$, as well as various contractions and localizations of M. Specifically, for a simple matroid M where e is not a coloop, their main result, Theorem 2.8, states:

$$P_M(t) = P_{M\setminus e}(t) - tP_{M_e}(t) + \sum_{F\in S} \tau(M_{F\cup e}) \cdot t^{\operatorname{crk}(F)/2} \cdot P_{M^F}(t)$$

where the sum is taken over the set S of all subsets F of $E \setminus e$ such that both F and $F \cup e$ are flats of M, and $\tau(M)$ is the coefficient of $t^{(\operatorname{rk}(M)-1)/2}$ in $P_M(t)$ if $\operatorname{rk}(M)$ is odd, and zero otherwise.

The inverse Kazhdan-Lusztig polynomial $Q_M(x)$ is another important invariant. There is a related polynomial $\hat{Q}_M(x) = (-1)^{\operatorname{rk}(M)}Q_M(x)$ which acts as the inverse of the Kazhdan-Lusztig polynomial $P_M(t)$ within the incidence algebra of the lattice of flats $\mathcal{L}(M)$.

In this paper, we aim to prove the following deletion formula for $\hat{Q}_M(x)$:

Theorem 1 (Deletion Formula for $\hat{Q}_M(x)$)**.** *Let M be a simple matroid and e an element that is not a coloop. Then*

$$\hat{Q}_M(x) = \hat{Q}_{M\setminus e}(x) - (1+x) \cdot \hat{Q}_{M_e}(x) - \sum_{G\in S'} \tau(M_e^G) \cdot x^{\operatorname{rk}(G)/2} \cdot \hat{Q}_{M_G}(x)$$

where $S' = \{F \in \mathcal{L}(M) \mid e \in F \text{ and } F \setminus e \notin \mathcal{L}(M)\}$.

The proof of this theorem is the main goal of these notes.

2 Perverse elements and the KL basis

Let M be a matroid and $\mathcal{L}(M)$ be its lattice of flats.

- **The Module $\mathcal{H}(M)$**: Let $\mathcal{H} = \mathcal{H}(M)$ be the **free $\mathbb{Z}[t,t^{-1}]$-module with basis indexed by $\mathcal{L}(M)$**. Elements of $\mathcal{H}$ are formal sums of the form

$$\alpha = \sum_{F\in\mathcal{L}(M)} \alpha_F \cdot F, \quad \alpha_F \in \mathbb{Z}[t,t^{-1}].$$

- **The Abelian Subgroup $\mathcal{H}_p$**: $\mathcal{H}_p$ is an **abelian subgroup of $\mathcal{H}$** consisting of all $\alpha \in \mathcal{H}$ such that for every flat $F \in \mathcal{L}(M)$, the following two conditions hold:
  i. $\alpha_F \in \mathbb{Z}[t]$.
  ii. $\sum_{G\geq F} t^{\operatorname{rk}(F)-\operatorname{rk}(G)}\alpha_G \in Pal(0)$, where $Pal(0)$ is the set of Laurent polynomials $f(t)$ such that $f(t) = f(t^{-1})$.
- **The Elements ζ^F**: For any flat $F \in \mathcal{L}(M)$, an element $\zeta^F \in \mathcal{H}$ is defined as

$$\zeta^F = \sum_{G\leq F} t^{\operatorname{rk}(F)-\operatorname{rk}(G)} P_{M_G^F}(t^{-2}) \cdot G.$$

- **Basis of $\mathcal{H}_p$**: Proposition 2.13 of [1] states that the set of elements $\{\zeta^F\}_{F\in\mathcal{L}(M)}$ forms a **$\mathbb{Z}$-basis for** $\mathcal{H}_p$. Any element $\beta \in \mathcal{H}_p$ can be uniquely expressed as a linear combination of the ζ^F with integer coefficients:
$$\beta = \sum_{F\in\mathcal{L}(M)} \beta_F(0)\zeta^F.$$

This algebraic framework, involving the module $\mathcal{H}(M)$ and its subgroup $\mathcal{H}_p$ with the basis $\{\zeta^F\}$, provides a foundation for studying the Kazhdan–Lusztig polynomials of matroids, as demonstrated by its role in the derivation of deletion formulas.

3 The module homomorphism $\mathcal{H}(M) \to \mathcal{H}(M \setminus e)$

Let M be a simple matroid and e be an element of its ground set that is not a coloop. There is a surjective map $\mathcal{L}(M) \to \mathcal{L}(M\setminus e)$ sending a flat $F \in \mathcal{L}(M)$ to $F \setminus e \in \mathcal{L}(M \setminus e)$. There also exists a module homomorphism $\Delta : \mathcal{H}(M) \to \mathcal{H}(M \setminus e)$ which is $\mathbb{Z}[t, t^{-1}]$-linear and is defined on the basis elements $F \in \mathcal{L}(M)$ by

$$\Delta(F) = \begin{cases} F & \text{if } F \not\ni e \\ F \setminus e & \text{if } F \ni e \text{ and } F \setminus e \notin \mathcal{L}(M) \\ t^{-1} \cdot (F \setminus e) & \text{if } F \ni e \text{ and } F \setminus e \in \mathcal{L}(M). \end{cases}$$

This definition is taken from Section 2.6 of Braden and Vysogorets [1]. In [1], the authors also prove the following lemma:

Lemma 2. *Let $F \in \mathcal{L}(M)$ be a flat such that $e \in F$ and $F \setminus e \notin \mathcal{L}(M)$. Then*
$$\Delta(\zeta^F) = \zeta^{F\setminus e} + \sum_{G\in S(M^F)} \tau(M^F_{G\cup e}) \cdot \zeta^G$$
where $S(M^F) = \{G \in \mathcal{L}(M^F) \mid e \notin G \text{ and } G \cup e \in \mathcal{L}(M^F)\}$.

Theorem 3. *Let $F \in \mathcal{L}(M)$ be a flat such that $e \in F$ and $F \setminus e \in \mathcal{L}(M)$, then*
$$\Delta(\zeta^F) = (t + t^{-1})\zeta^{F\setminus e}$$
where $\zeta^{F\setminus e}$ on the right-hand side is the ζ-element in $\mathcal{H}(M \setminus e)$ associated with the flat $F \setminus e \in \mathcal{L}(M \setminus e)$.

Proof. Let $F_0 = F \setminus e$. We are given $F_0 \in \mathcal{L}(M)$. By the definition of $\Delta(G)$:

- If $G \le F$ and $e \notin G$: Then $G \le F_0$. Since $G \in \mathcal{L}(M)$ and $e \notin G$, G is also a flat in $M \setminus e$, so $G \in \mathcal{L}(M \setminus e)$. In this case, $\Delta(G) = G$.
- If $G \le F$ and $e \in G$: Let $G_0 = G \setminus e$. Then $G_0 \le F_0$. Since $F_0 \in \mathcal{L}(M)$, it follows that $G_0 = G \cap F_0 \in \mathcal{L}(M)$. Thus $G_0 \in \mathcal{L}(M \setminus e)$. In this case, $\Delta(G) = t^{-1}G_0$.

Applying this to $\Delta(\zeta^F)$:

$$\begin{aligned}
\Delta(\zeta^F) &= \Delta\left(\sum_{G\le F} t^{\mathrm{rk}_M(F)-\mathrm{rk}_M(G)} P_{M^F_G}(t^{-2}) \cdot G\right) \\
&= \sum_{G\le F, G\not\ni e} t^{\mathrm{rk}_M(F)-\mathrm{rk}_M(G)} P_{M^F_G}(t^{-2}) \cdot \Delta(G) \\
&\quad + \sum_{G\le F, G\ni e} t^{\mathrm{rk}_M(F)-\mathrm{rk}_M(G)} P_{M^F_G}(t^{-2}) \cdot \Delta(G) \\
&= \sum_{G\le F_0} t^{\mathrm{rk}_M(F)-\mathrm{rk}_M(G)} P_{M^F_G}(t^{-2}) \cdot G \\
&\quad + \sum_{\substack{G_0\le F_0 \\ (G=G_0\cup\{e\})}} t^{\mathrm{rk}_M(F)-\mathrm{rk}_M(G_0\cup\{e\})} P_{M^F_{G_0\cup\{e\}}}(t^{-2}) \cdot (t^{-1}G_0).
\end{aligned}$$

We use the rank relations:

- $\mathrm{rk}_M(F) = \mathrm{rk}_{M\setminus e}(F_0) + 1$.
- For $G \le F_0$: $\mathrm{rk}_M(G) = \mathrm{rk}_{M\setminus e}(G)$.
- For $G_0 \le F_0$: $\mathrm{rk}_M(G_0 \cup \{e\}) = \mathrm{rk}_{M\setminus e}(G_0) + 1$.

And the Kazhdan-Lusztig polynomial identities under these conditions:

- For $G \le F_0$: $M_G^F \cong (M \setminus e)_G^{F_0} \oplus U_{1,1}(\{e\})$, so $P_{M_G^F}(t^{-2}) = P_{(M\setminus e)_G^{F_0}}(t^{-2})$.
- For $G_0 \le F_0$: $M_{G_0\cup\{e\}}^F \cong (M \setminus e)_{G_0}^{F_0}$, so $P_{M_{G_0\cup\{e\}}^F}(t^{-2}) = P_{(M\setminus e)_{G_0}^{F_0}}(t^{-2})$.

Substituting these into the sums:

$$\begin{aligned}
\Delta(\zeta^F) &= \sum_{G\le F_0} t^{(\mathrm{rk}_{M\setminus e}(F_0)+1)-\mathrm{rk}_{M\setminus e}(G)} P_{(M\setminus e)_G^{F_0}}(t^{-2}) \cdot G \\
&\quad + t^{-1} \sum_{G_0\le F_0} t^{(\mathrm{rk}_{M\setminus e}(F_0)+1)-(\mathrm{rk}_{M\setminus e}(G_0)+1)} P_{(M\setminus e)_{G_0}^{F_0}}(t^{-2}) \cdot G_0 \\
&= t \sum_{G\le F_0, G\in\mathcal{L}(M\setminus e)} t^{\mathrm{rk}_{M\setminus e}(F_0)-\mathrm{rk}_{M\setminus e}(G)} P_{(M\setminus e)_G^{F_0}}(t^{-2}) \cdot G \\
&\quad + t^{-1} \sum_{G_0\le F_0, G_0\in\mathcal{L}(M\setminus e)} t^{\mathrm{rk}_{M\setminus e}(F_0)-\mathrm{rk}_{M\setminus e}(G_0)} P_{(M\setminus e)_{G_0}^{F_0}}(t^{-2}) \cdot G_0.
\end{aligned}$$

Using the definition of $\zeta_{M\setminus e}^{F_0}$:

$$\begin{aligned}
\Delta(\zeta^F) &= t \cdot \zeta_{M\setminus e}^{F_0} + t^{-1} \cdot \zeta_{M\setminus e}^{F_0} \\
&= (t + t^{-1}) \zeta_{M\setminus e}^{F_0}.
\end{aligned}$$

□

4 The Deletion Formula for $\hat{Q}_M(x)$

Lemma 4. *The standard basis $\{F\}_{F\in\mathcal{L}(M)}$ of $\mathcal{H}(M)$ satisfies*

$$F = \sum_{G\le F} t^{\mathrm{rk}(F)-\mathrm{rk}(G)} \hat{Q}_{M_G^F}(t^{-2}) \cdot \zeta^G. \tag{1}$$

Proof of 1. Applying the homomorphism Δ to equation 1 when $F = E$, we get

$$E \setminus e = \sum_{G\le E} t^{\mathrm{rk}(E)-\mathrm{rk}(G)} \hat{Q}_{M_G^E}(t^{-2}) \cdot \Delta(\zeta^G).$$

Using lemma 4, the coefficient of $\zeta^\emptyset$ in the left-hand side is $t^{\mathrm{rk}(E\setminus e)-\mathrm{rk}(\emptyset)} \cdot \hat{Q}_{M_\emptyset^{E\setminus e}}(t^{-2}) = t^{\mathrm{rk}(M)} \cdot \hat{Q}_{M\setminus e}(t^{-2})$.

By Theorem 3, the coefficient $\zeta^\emptyset$ on the right-hand side is non-zero only when $G \in \{\emptyset, e\} \cup S'$. In each of these cases, we have the following:

- For $G = \emptyset$, we have $\Delta(\zeta^\emptyset) = \zeta^\emptyset$.
- For $G = e$, we have $\Delta(\zeta^e) = (t + t^{-1})\zeta^\emptyset$.
- For $G \in S'$, we have $\Delta(\zeta^G) = \tau(M_e^G) \cdot \zeta^\emptyset + \text{terms not including } \zeta^\emptyset$.

Collecting the coefficients of $\zeta^{\emptyset}$ on the right-hand side, we have:

$$\begin{aligned} & t^{\mathrm{rk}(E)-\mathrm{rk}(\emptyset)}\hat{Q}_{M^E_{\emptyset}}(t^{-2}) + t^{\mathrm{rk}(E)-\mathrm{rk}(e)}(t+t^{-1})\hat{Q}_{M^E_e}(t^{-2}) + \sum_{G\in S'} t^{\mathrm{rk}(E)-\mathrm{rk}(G)}\hat{Q}_{M^E_G}(t^{-2})\cdot\tau(M^G_e) \\ = {} & t^{\mathrm{rk}(M)}\hat{Q}_M(t^{-2}) + t^{\mathrm{rk}(M)-1}(t+t^{-1})\hat{Q}_{M_e}(t^{-2}) + \sum_{G\in S'} t^{\mathrm{rk}(M)-\mathrm{rk}(G)}\hat{Q}_{M_G}(t^{-2})\cdot\tau(M^G_e). \end{aligned}$$

Equating the coefficients of $\zeta^{\emptyset}$ on both sides, we obtain:

$$\hat{Q}_{M\setminus e}(t^{-2}) = \hat{Q}_M(t^{-2}) + (1+t^{-2})\hat{Q}_{M_e}(t^{-2}) + \sum_{G\in S'} t^{-\mathrm{rk}(G)}\hat{Q}_{M_G}(t^{-2})\cdot\tau(M^G_e).$$

Finally, taking $x = t^{-2}$ and rearranging yields the desired statement:

$$\hat{Q}_M(x) = \hat{Q}_{M\setminus e}(x) - (1+x)\cdot\hat{Q}_{M_e}(x) - \sum_{G\in S'}\tau(M^G_e)\cdot x^{\mathrm{rk}(G)/2}\cdot\hat{Q}_{M_G}(x).$$

□

The inverse Kazhdan-Lusztig polynomial $Q_M(x) = (-1)^{\mathrm{rk}(M)}\hat{Q}_M(x)$ then satisfies the following:

Corollary 5. *Let M be a simple matroid and e an element that is not a coloop. Then*

$$Q_M(x) = Q_{M\setminus e}(x) + (1+x)\cdot Q_{M_e}(x) - \sum_{G\in S'}\tau(M^G_e)\cdot x^{\mathrm{rk}(G)/2}\cdot Q_{M_G}(x)$$

where $S' = \{F\in\mathcal{L}(M) \mid e\in F \text{ and } F\setminus e\notin\mathcal{L}(M)\}$.

Corollary 6. *Let $M := U_{m,d}$ be the uniform matroid of rank d on $m+d$ elements with $d\geq 1$. Then*

$$Q_{U_{m,d}}(x) = Q_{U_{m-1,d}}(x) + (1+x)\cdot Q_{U_{m,d-1}}(x) - \tau(U_{m,d-1})\cdot x^{d/2}.$$

Proof. As none of the elements in the ground set are coloops, we may apply the deletion formula for a generic element e in the ground set. The set S' then consists only of the top flat E and the sum over S' reduces to

$$\tau(M_e)\cdot x^{\mathrm{rk}(M)/2} = \tau(U_{m,d-1})\cdot x^{d/2}.$$

□

♣ Jacob: [For the previous corollary, can you now use induction to get a formula for $Q_{m,d}$ for any m and d that does not have any Q's in it? (Perhaps starting with the fact the inverse KL polynomials of Boolean matroids are equal to 1?)]

Corollary 7. *Let M be the parallel connection of the cycle matroids $U_{1,m}$ and $U_{1,n}$ along the element e. Then*

$$\begin{aligned} Q_M(x) = {} & Q_{U_{1,m+n-1}}(x) + (1+x)\cdot Q_{U_{1,m-1}}(x)\cdot Q_{U_{1,n-1}}(x) \\ & - \Big[\tau(U_{1,n-1})\cdot x^{m/2}\cdot Q_{U_{1,m-1}}(x) + \tau(U_{1,m-1})\cdot x^{n/2}\cdot Q_{U_{1,n-1}}(x) + \tau(M_e)\cdot x^{(m+n-1)/2}\Big]. \end{aligned}$$

Specifically, when the ranks m and n are odd, the τ terms are 0 and hence the formula simplifies to:

$$Q_M(x) = Q_{U_{1,m+n-1}}(x) + (1+x)\cdot Q_{U_{1,m-1}}(x)\cdot Q_{U_{1,n-1}}(x).$$

♣ Nutan: [Corrected an error in the statement of above corollary.]

Corollary 8. *Let $M = PG(r-1,q)$ be the projective geometry of rank r over the finite field $GF(q)$, for $r \geq 2$. Then the deletion formula for its inverse Kazhdan-Lusztig polynomial is:*

$$Q_M(x) = Q_{M\setminus e}(x) + (1+x)\cdot Q_{PG(r-2,q)}(x) - \begin{bmatrix} r-1 \\ 1 \end{bmatrix}_q \cdot x \cdot Q_{PG(r-3,q)}(x)$$

where $\begin{bmatrix} r-1 \\ 1 \end{bmatrix}_q = \frac{q^{r-1}-1}{q-1}$ *is the number of lines through the point e in M.*

Proof. We start from the formula for $Q_M(x)$ in Corollary 5 and substitute $M = PG(r-1,q)$. The lattice of flats of a projective geometry is modular. For any matroid N with a modular lattice of flats, its Kazhdan-Lusztig polynomial $P_N(t)$ is 1. [3][Proposition 2.14] The coefficient $\tau(N)$ is the coefficient of $t^{(\mathrm{rk}(N)-1)/2}$ in $P_N(t)$, which is zero if $\mathrm{rk}(N) \geq 2$.

The sum in Corollary 5 is over the set $S' = \{G \in \mathcal{L}(M) \mid e \in G \text{ and } G \setminus e \notin \mathcal{L}(M)\}$. For the projective geometry M, a flat G containing e is a linear subspace. If $\mathrm{rk}(G) \geq 2$, removing the point e results in a set $G \setminus e$ that is not a subspace, and thus not a flat. If $\mathrm{rk}(G) = 1$, then $G = \{e\}$ and $G \setminus e = \emptyset$, which is a flat. Thus, for $M = PG(r-1,q)$, the set S' consists of all flats containing e of rank 2 or greater.

For each $G \in S'$, the term in the sum is $\tau(M_e^G) \cdot x^{\mathrm{rk}(G)/2} \cdot Q_{M_G}(x)$. The matroid M_e^G is the interval $[e,G]$ isomorphic to $PG(k-2,q)$ where $k = \mathrm{rk}(G)$. The rank of M_e^G is $k-1$ and $\tau(M_e^G)$, therefore, is non-zero only if the rank $k-1=1$, which implies $k=2$.

Thus, the sum is over the set of rank-2 flats containing the point e. The number of such flats is the number of 2-dimensional subspaces of $V(r,q)$ containing a given 1-dimensional subspace, which is equal to the number of 1-dimensional subspaces in the quotient space $V(r,q)/\langle e \rangle \cong V(r-1,q)$. This number is $\begin{bmatrix} r-1 \\ 1 \end{bmatrix}_q$. For each such rank-2 flat G:

- $\tau(M_e^G) = \tau(PG(0,q)) = 1$ since $PG(0,q)$ has rank 1.
- $Q_{M_G}(x) = Q_{PG(r-3,q)}(x)$, since the contraction M_G of $M = PG(r-1,q)$ by a rank-2 flat G is isomorphic to $PG(r-3,q)$.

The entire sum therefore reduces to:

$$\sum_{G \in S', \mathrm{rk}(G)=2} \tau(M_e^G) \cdot x^{\mathrm{rk}(G)/2} \cdot Q_{M_G}(x) = \begin{bmatrix} r-1 \\ 1 \end{bmatrix}_q \cdot x \cdot Q_{PG(r-3,q)}(x).$$

Substituting this and the fact that the contraction M_e is isomorphic to $PG(r-2,q)$ back into the general formula from Corollary 5, we get:

$$Q_{PG(r-1,q)}(x) = Q_{M\setminus e}(x) + (1+x)\cdot Q_{PG(r-2,q)}(x) - \begin{bmatrix} r-1 \\ 1 \end{bmatrix}_q \cdot x \cdot Q_{PG(r-3,q)}(x). \quad \square$$

□

Remark. The inverse Kazhdan-Lusztig polynomial of a projective geometry $PG(k-1,q)$ of rank k is known to be the constant polynomial $q^{\binom{k}{2}}$. This allows for substituting the known values for the Q polynomials in the formula to obtain an explicit expression for $Q_{M\setminus e}(x)$. Rearranging the terms and substituting gives:

$$\begin{aligned} Q_{M\setminus e}(x) &= Q_{PG(r-1,q)}(x) - (1+x)Q_{PG(r-2,q)}(x) + \begin{bmatrix} r-1 \\ 1 \end{bmatrix}_q x\, Q_{PG(r-3,q)}(x) \\ &= q^{\binom{r}{2}} - (1+x)q^{\binom{r-1}{2}} + \frac{q^{r-1}-1}{q-1} q^{\binom{r-2}{2}} \cdot x. \end{aligned}$$

References

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