# Simulation studies

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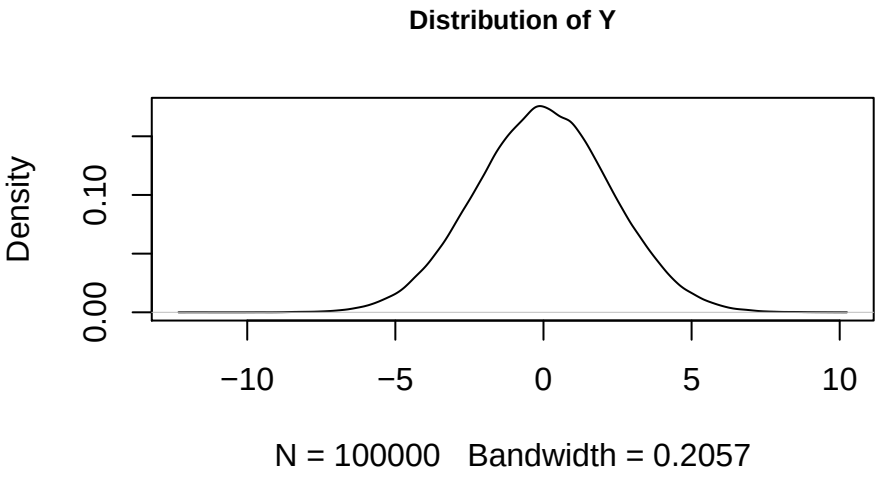
## Simulation studies

### Scenario 1: Unbiased and calibrated components

The data generating process for the observation $Y$ in the regression model is

$$Y = X_0 + a_1 X_1 + a_2 X_2 + a_3 X_3 + \epsilon, \epsilon \sim N(0, 1)$$

where $a_1 = 1, a_2 = 1$, and $a_3 = 1.1$, and $X_0, X_1, X_2, X_3$, and $\epsilon$ are independent, standard normal random variables. The TLP's PITs are approximately beta distributed in this scenario.




The individual predictive densities have partial access of the above set of covariates. $f_1$ has access to only $X_0$ and $X_1$, $f_2$ has access to only $X_0$ and $X_2$, and $f_3$ has access to only $X_0$ and $X_3$. We want to combine $f_1, f_2$, and $f_3$ to predict $Y$. In this setup, $X_0$ represent shared information, while other covariates represent information unique to each individual model.

We estimate the pooling/combination formulas on a random sample $(f_{1i}, f_{2i}, f_{3i}, Y_i) : i = 1, ..., n$ of size $n = 80,000$ and evaluate on an independent test sample of $n = 20,000$. In this scenario, $a_1 = a_2 = 1$ and $a_3 = 1.1$, so that $f_3$ is a more concentrated, sharper density forecast than $f_1$ and $f_2$ (Gneiting and Ranjan (2013)) and they are defined as follows:

$$f_1 = \mathrm{N}(X_0 + a_1 X_1, 1 + a_2^2 + a_3^2)$$
$$f_2 = \mathrm{N}(X_0 + a_2 X_2, 1 + a_1^2 + a_3^2)$$
$$f_3 = \mathrm{N}(X_0 + a_3 X_3, 1 + a_1^2 + a_2^2)$$

Table 1: Model and Beta Mixing Weight Parameters

| | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\alpha$ | $\beta$ | | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|---|---|---|---|---|---|---|
| TLP | 0.252 | 0.276 | 0.473 | NA | NA | BMC2 | 0.308 | 0.692 | NA | NA |
| BLP | 0.287 | 0.304 | 0.410 | 1.454 | 1.451 | EW-BMC2 | 0.306 | 0.694 | NA | NA |
| EW | 0.333 | 0.333 | 0.333 | NA | NA | BMC3 | 0.100 | 0.300 | 0.600 | NA |
| EW-BLP | 0.333 | 0.333 | 0.333 | 1.456 | 1.454 | EW-BMC3 | 0.099 | 0.301 | 0.600 | NA |
| BMC1-BLP | 0.287 | 0.304 | 0.410 | 1.453 | 1.451 | BMC4 | 0.053 | 0.936 | 0.011 | 0.000 |
| BMC1-EW-BLP | 0.333 | 0.333 | 0.333 | 1.455 | 1.454 | EW-BMC4 | 0.101 | 0.300 | 0.397 | 0.202 |

Table 2: Mixture Parameters

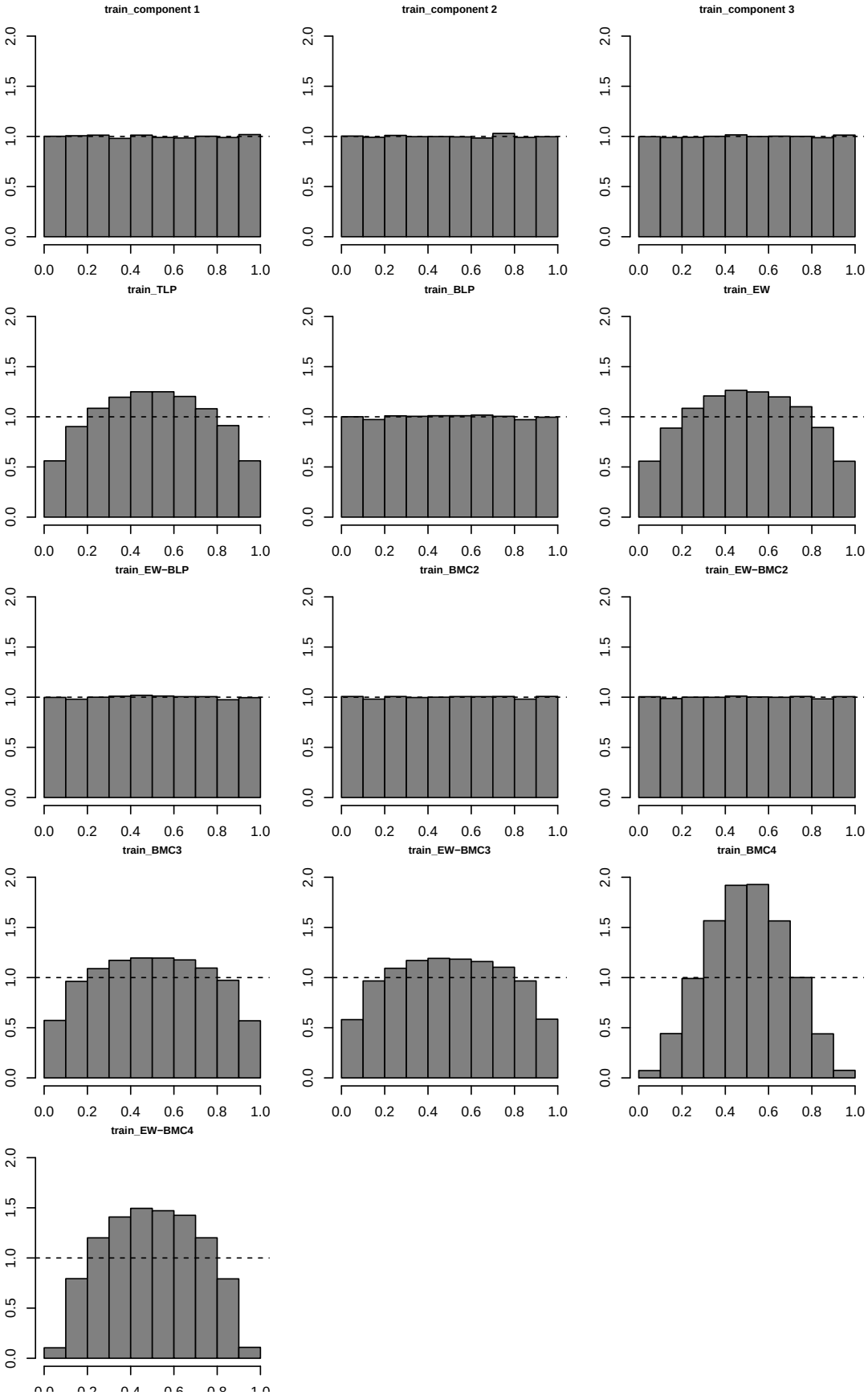
| | $\alpha_1$ | $\beta_1$ | $\alpha_2$ | $\beta_2$ | $\alpha_3$ | $\beta_3$ | $\alpha_4$ | $\beta_4$ |
|---|---|---|---|---|---|---|---|---|
| BMC2 | 1.191 | 1.173 | 1.617 | 1.625 | NA | NA | NA | NA |
| EW-BMC2 | 1.208 | 1.186 | 1.607 | 1.619 | NA | NA | NA | NA |
| BMC3 | 1.179 | 1.121 | 1.253 | 1.260 | 1.656 | 1.662 | NA | NA |
| EW-BMC3 | 1.212 | 1.153 | 1.258 | 1.259 | 1.647 | 1.658 | NA | NA |
| BMC4 | 0.805 | 0.801 | 1.529 | 1.553 | 8.288 | 1.367 | 0.000 | 0.000 |
| EW-BMC4 | 1.269 | 1.173 | 1.336 | 1.398 | 1.767 | 1.765 | 1.297 | 1.259 |

Table 3: Mixture Parameters

| | $\omega_{11}$ | $\omega_{12}$ | $\omega_{13}$ | $\omega_{21}$ | $\omega_{22}$ | $\omega_{23}$ | $\omega_{31}$ | $\omega_{32}$ | $\omega_{33}$ | $\omega_{41}$ | $\omega_{42}$ | $\omega_{43}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| BMC2 | 0.261 | 0.262 | 0.477 | 0.297 | 0.320 | 0.383 | NA | NA | NA | NA | NA | NA |
| EW-BMC2 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | NA | NA | NA | NA | NA | NA |
| BMC3 | 0.254 | 0.264 | 0.482 | 0.277 | 0.274 | 0.450 | 0.296 | 0.323 | 0.381 | NA | NA | NA |
| EW-BMC3 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | NA | NA | NA |
| BMC4 | 0.000 | 0.000 | 1.000 | 0.301 | 0.313 | 0.387 | 0.000 | 1.000 | 0.000 | 1.000 | 0.000 | 0.000 |
| EW-BMC4 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 |

Table 4: Log Score

| | Training | Test | | Training | Test |
|---|---|---|---|---|---|
| TLP | -1.910 | -1.908 | BMC2 | -1.869 | -1.866 |
| BLP | -1.869 | -1.866 | EW-BMC2 | -1.871 | -1.868 |
| EW | -1.913 | -1.911 | BMC3 | -1.918 | -1.916 |
| EW-BLP | -1.871 | -1.868 | EW-BMC3 | -1.919 | -1.917 |
| | | | BMC4 | -2.164 | -2.163 |
| | | | EW-BMC4 | -2.066 | -2.065 |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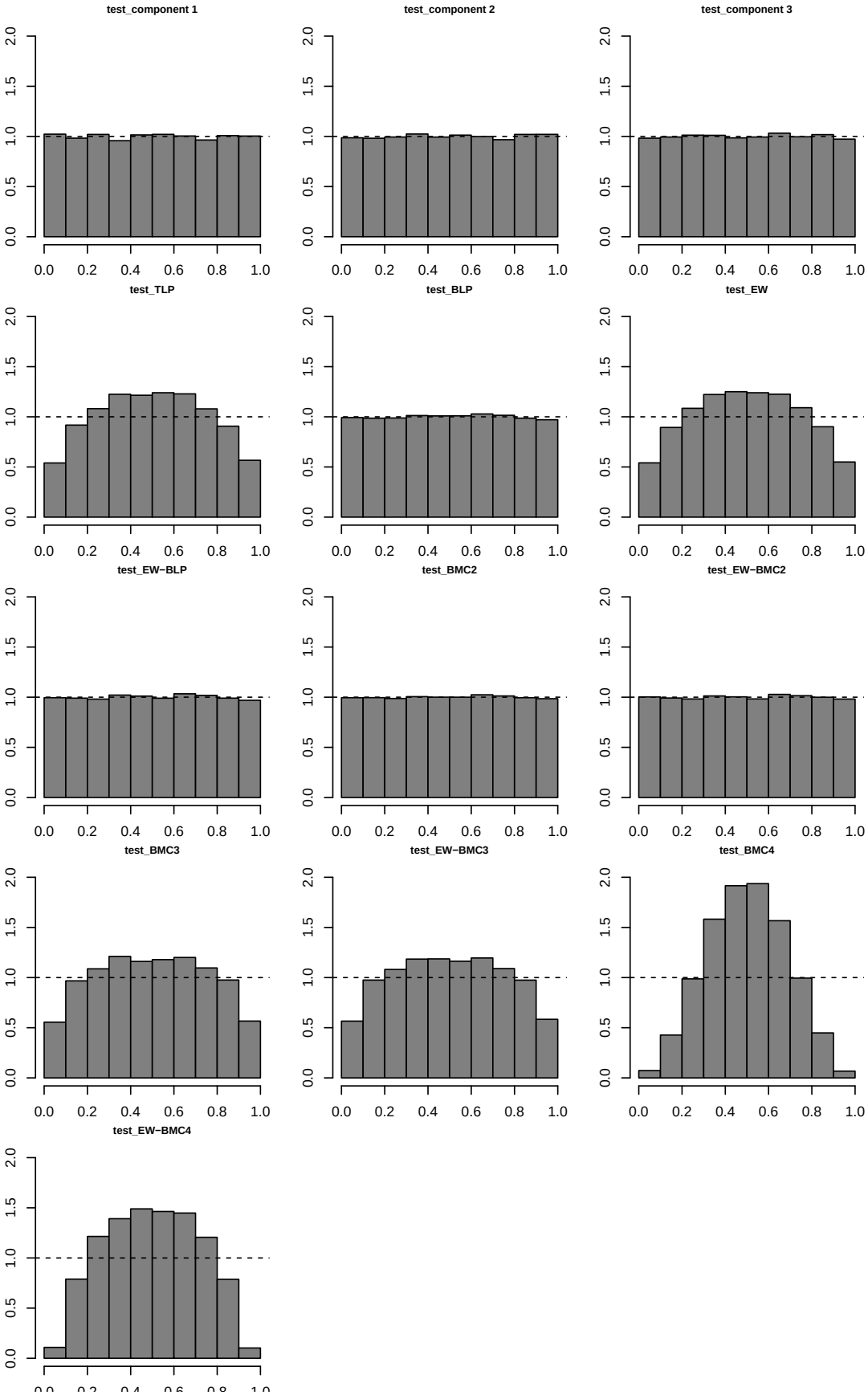
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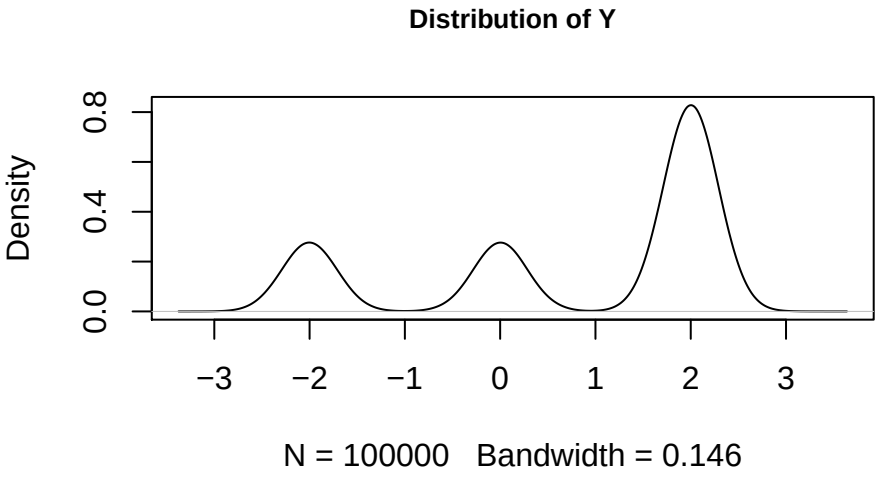
## Scenario 2: Multimodal DGP (Normal mixture) and close-$\mathcal{M}$

The data generating process for the observation $y_t$ is

$$y_t \overset{i.i.d.}{\sim} p_1\mathrm{N}(-2, 0.25) + p_2\mathrm{N}(0, 0.25) + p_3\mathrm{N}(2, 0.25), t = 1, ..., 100,0000$$

where $p_1 = 0.2, p_2 = 0.2$, and $p_3 = 0.6$. In this scenario, the three component models are in the data generating process and the TLP's PITs are approximately beta distributed (uniformly distributed, specifically).




The individual predictive densities are defined as follows:

$$f_1 \overset{i.i.d.}{\sim} \mathrm{N}(-2, 0.25)$$
$$f_2 \overset{i.i.d.}{\sim} \mathrm{N}(0, 0.25)$$
$$f_3 \overset{i.i.d.}{\sim} \mathrm{N}(2, 0.25)$$

Table 5: Model and Beta Mixing Weight Parameters

| | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\alpha$ | $\beta$ | | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|---|---|---|---|---|---|---|
| TLP | 0.201 | 0.198 | 0.601 | NA | NA | BMC2 | 0.300 | 0.700 | NA | NA |
| BLP | 0.201 | 0.199 | 0.600 | 0.997 | 0.992 | EW-BMC2 | 0.272 | 0.728 | NA | NA |
| EW | 0.333 | 0.333 | 0.333 | NA | NA | BMC3 | 0.100 | 0.300 | 0.600 | NA |
| EW-BLP | 0.333 | 0.333 | 0.333 | 1.237 | 0.781 | EW-BMC3 | 0.228 | 0.665 | 0.106 | NA |
| BMC1-BLP | 0.201 | 0.199 | 0.600 | 0.996 | 0.992 | BMC4 | 0.100 | 0.300 | 0.400 | 0.200 |
| BMC1-EW-BLP | 0.333 | 0.333 | 0.333 | 1.237 | 0.781 | EW-BMC4 | 0.106 | 0.350 | 0.319 | 0.225 |

Table 6: Mixture Parameters

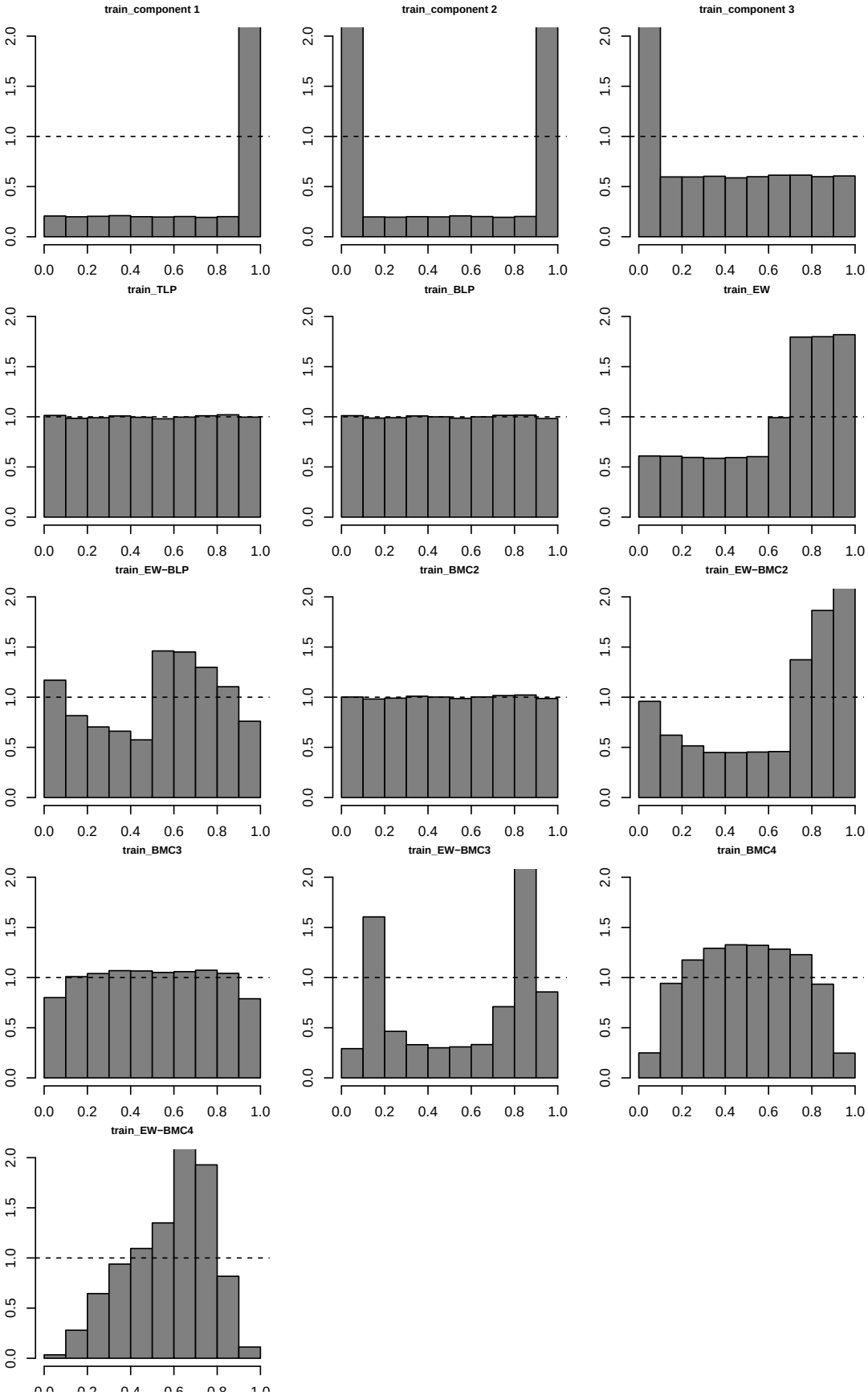
| | $\alpha_1$ | $\beta_1$ | $\alpha_2$ | $\beta_2$ | $\alpha_3$ | $\beta_3$ | $\alpha_4$ | $\beta_4$ |
|---|---|---|---|---|---|---|---|---|
| BMC2 | 0.998 | 0.992 | 0.996 | 0.992 | NA | NA | NA | NA |
| EW-BMC2 | 1.161 | 3.329 | 4.171 | 1.217 | NA | NA | NA | NA |
| BMC3 | 0.998 | 0.997 | 0.996 | 0.998 | 0.996 | 0.989 | NA | NA |
| EW-BMC3 | 12.832 | 2.216 | 0.936 | 0.781 | 72.349 | 26.868 | NA | NA |
| BMC4 | 1.000 | 0.992 | 0.994 | 0.994 | 0.997 | 0.989 | 0.996 | 0.995 |
| EW-BMC4 | 0.929 | 0.743 | 0.931 | 0.743 | 10.696 | 2.667 | 0.930 | 0.743 |

Table 7: Mixture Parameters

| | $\omega_{11}$ | $\omega_{12}$ | $\omega_{13}$ | $\omega_{21}$ | $\omega_{22}$ | $\omega_{23}$ | $\omega_{31}$ | $\omega_{32}$ | $\omega_{33}$ | $\omega_{41}$ | $\omega_{42}$ | $\omega_{43}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| BMC2 | 0.201 | 0.199 | 0.600 | 0.202 | 0.199 | 0.599 | NA | NA | NA | NA | NA | NA |
| EW-BMC2 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | NA | NA | NA | NA | NA | NA |
| BMC3 | 0.201 | 0.198 | 0.600 | 0.201 | 0.199 | 0.600 | 0.202 | 0.199 | 0.599 | NA | NA | NA |
| EW-BMC3 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | NA | NA | NA |
| BMC4 | 0.201 | 0.198 | 0.600 | 0.201 | 0.199 | 0.600 | 0.202 | 0.199 | 0.599 | 0.201 | 0.199 | 0.600 |
| EW-BMC4 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 |

Table 8: Log Score

| | Training | Test | | Training | Test |
|---|---|---|---|---|---|
| TLP | -0.982 | -0.980 | BMC2 | -0.982 | -0.980 |
| BLP | -0.982 | -0.980 | EW-BMC2 | -1.249 | -1.247 |
| EW | -1.131 | -1.132 | BMC3 | -1.003 | -1.002 |
| EW-BLP | -1.043 | -1.043 | EW-BMC3 | -1.768 | -1.763 |
| | | | BMC4 | -1.110 | -1.111 |
| | | | EW-BMC4 | -1.355 | -1.360 |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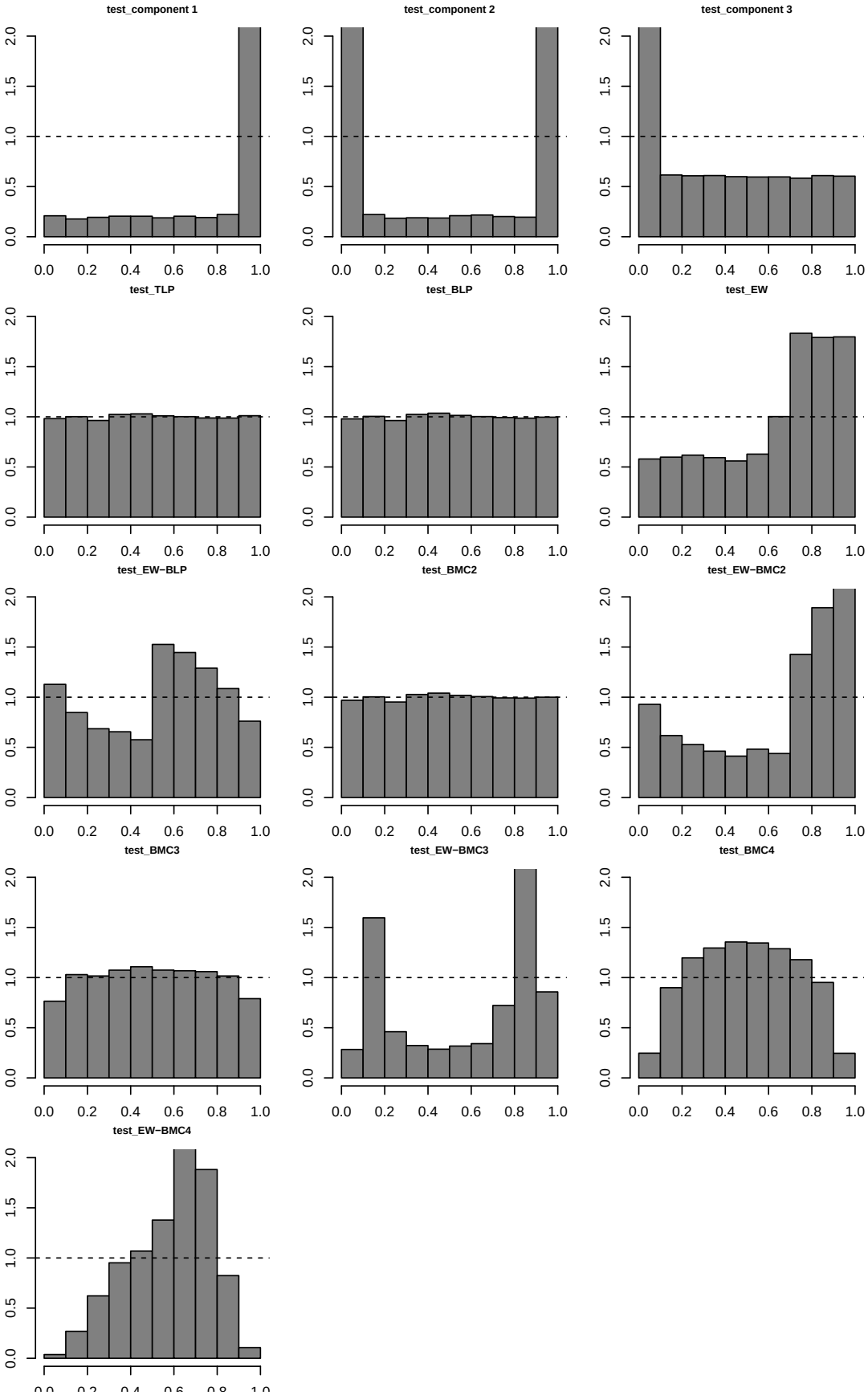
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0.0 0.5 1.0 1.5 2.0
0.0 0.2 0.4 0.6 0.8 1.0

## Scenario 3: Multimodal DGP (Normal mixture) and open-$\mathcal{M}$

The data generating process for the observations in this scenario is the same as in Scenario 2. There are two component models defined as follows

$$f_1 \overset{i.i.d.}{\sim} \mathrm{N}(2,1)$$
$$f_2 \overset{i.i.d.}{\sim} \mathrm{N}(-1,1).$$

The component models are not part of the data generating process. In this scenario, the TLP's PITs are not approximately beta distributed.

Table 9: Model and Beta Mixing Weight Parameters

| | $\omega_1$ | $\omega_2$ | $\alpha$ | $\beta$ | | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|---|---|---|---|---|---|
| TLP | 0.660 | 0.340 | NA | NA | BMC2 | 0.399 | 0.601 | NA | NA |
| BLP | 0.784 | 0.216 | 1.032 | 1.634 | EW-BMC2 | 0.201 | 0.799 | NA | NA |
| EW | 0.500 | 0.500 | NA | NA | BMC3 | 0.201 | 0.198 | 0.601 | NA |
| EW-BLP | 0.500 | 0.500 | 1.386 | 1.281 | EW-BMC3 | 0.186 | 0.253 | 0.561 | NA |
| BMC1-BLP | 0.784 | 0.216 | 1.032 | 1.634 | BMC4 | 0.201 | 0.601 | 0.198 | 0.000 |
| BMC1-EW-BLP | 0.500 | 0.500 | 1.386 | 1.280 | EW-BMC4 | 0.040 | 0.158 | 0.601 | 0.201 |

Table 10: Mixture Parameters

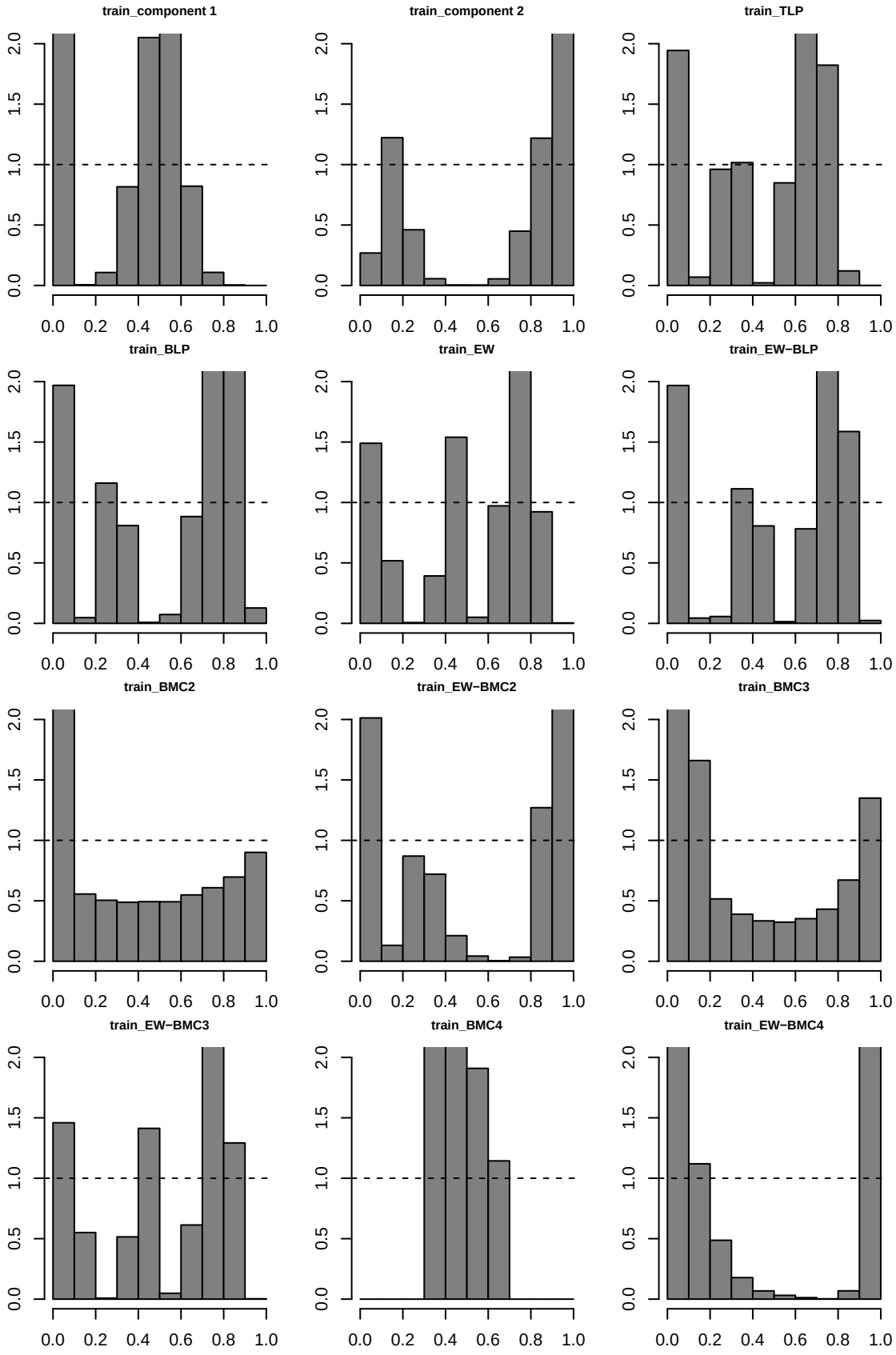
| | $\alpha_1$ | $\beta_1$ | $\alpha_2$ | $\beta_2$ | $\alpha_3$ | $\beta_3$ | $\alpha_4$ | $\beta_4$ |
|---|---|---|---|---|---|---|---|---|
| BMC2 | 0.948 | 31.616 | 12.981 | 12.863 | NA | NA | NA | NA |
| EW-BMC2 | 6.805 | 75.723 | 7.264 | 3.634 | NA | NA | NA | NA |
| BMC3 | 6.718 | 77.380 | 3.090 | 114.803 | 12.793 | 12.761 | NA | NA |
| EW-BMC3 | 1.112 | 2.630 | 1.112 | 2.630 | 63.686 | 21.089 | NA | NA |
| BMC4 | 6.753 | 83.373 | 12.793 | 12.761 | 3.091 | 114.810 | 0.000 | 0.000 |
| EW-BMC4 | 54.266 | 81.145 | 104.401 | 133.992 | 56.135 | 18.767 | 6.678 | 73.997 |

Table 11: Mixture Parameters

| | $\omega_{11}$ | $\omega_{12}$ | $\omega_{21}$ | $\omega_{22}$ | $\omega_{31}$ | $\omega_{32}$ | $\omega_{41}$ | $\omega_{42}$ |
|---|---|---|---|---|---|---|---|---|
| BMC2 | 0.967 | 0.033 | 0.997 | 0.003 | NA | NA | NA | NA |
| EW-BMC2 | 0.500 | 0.500 | 0.500 | 0.500 | NA | NA | NA | NA |
| BMC3 | 0.517 | 0.483 | 1.000 | 0.000 | 1.0 | 0.0 | NA | NA |
| EW-BMC3 | 0.500 | 0.500 | 0.500 | 0.500 | 0.5 | 0.5 | NA | NA |
| BMC4 | 0.547 | 0.453 | 1.000 | 0.000 | 1.0 | 0.0 | 1.0 | 0.0 |
| EW-BMC4 | 0.500 | 0.500 | 0.500 | 0.500 | 0.5 | 0.5 | 0.5 | 0.5 |

Table 12: Log Score

| | Training | Test | | Training | Test |
|---|---|---|---|---|---|
| TLP | -1.743 | -1.740 | BMC2 | -21.970 | -21.755 |
| BLP | -1.681 | -1.675 | EW-BMC2 | -2.387 | -2.375 |
| EW | -1.787 | -1.785 | BMC3 | -1.912 | -1.915 |
| EW-BLP | -1.759 | -1.755 | EW-BMC3 | -1.819 | -1.816 |
| | | | BMC4 | -3.794 | -3.801 |
| | | | EW-BMC4 | -3.861 | -3.860 |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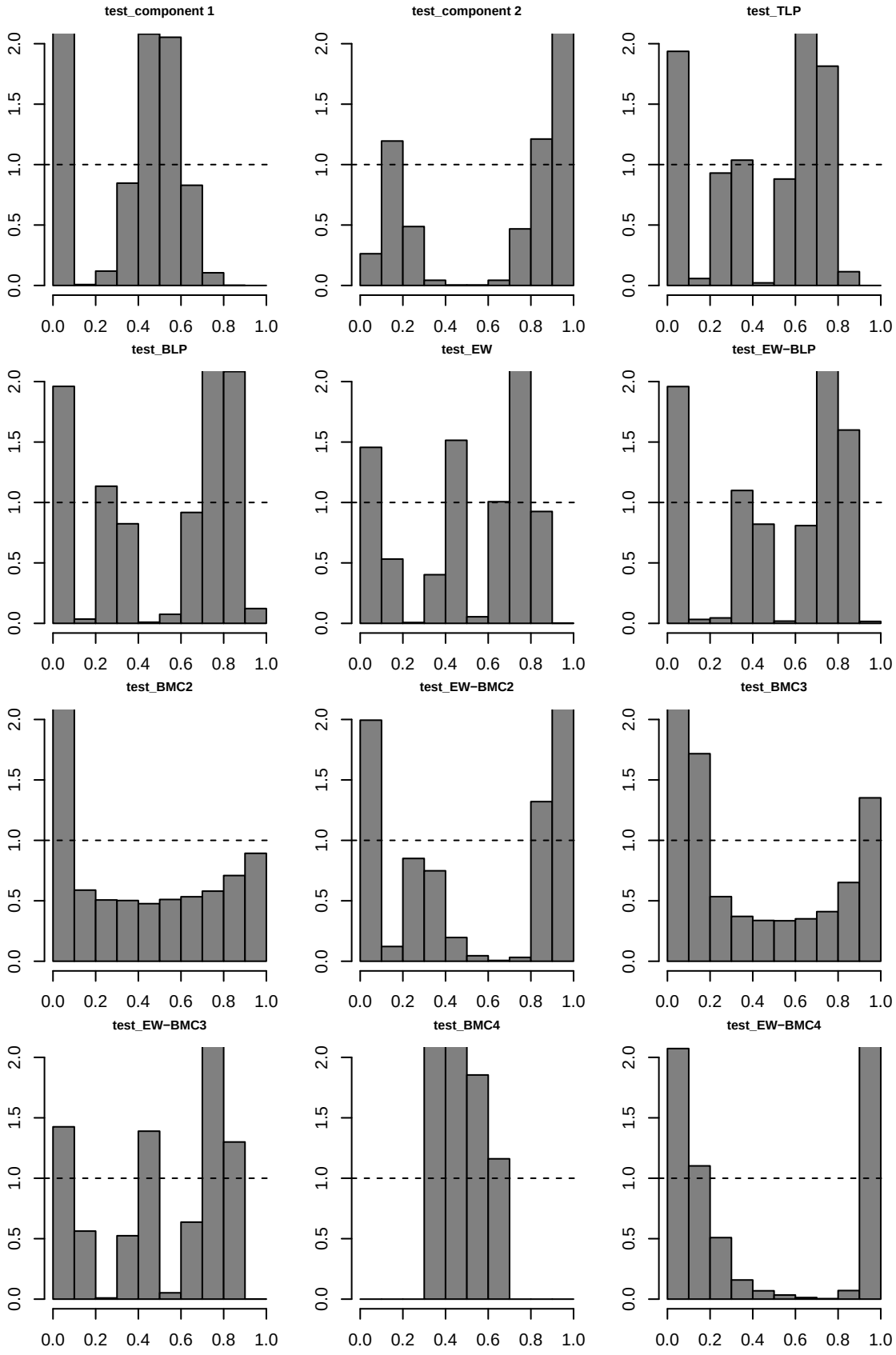
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