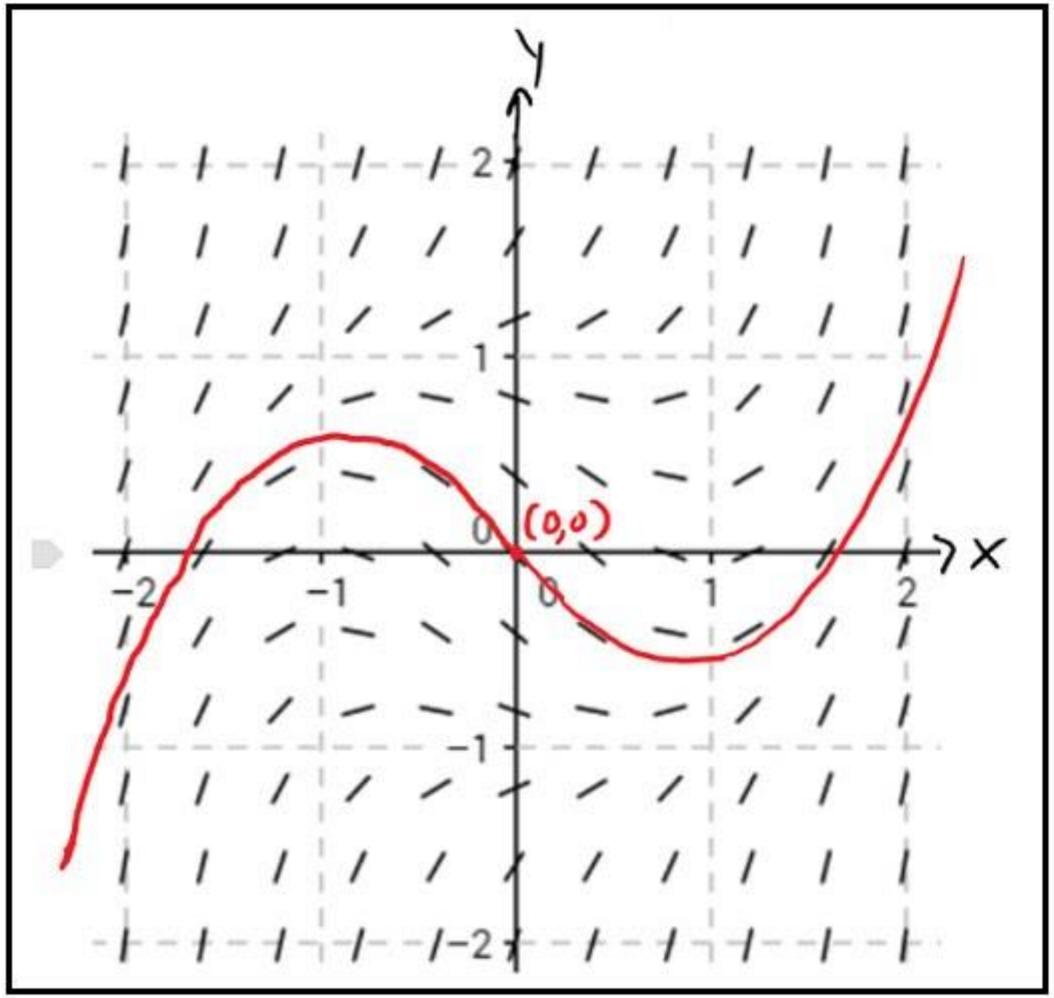
Лекция 7. Решение обыкновенных дифференциальных уравнений и задачи Коши

Вычислительна математика

Весенний семестр 2022

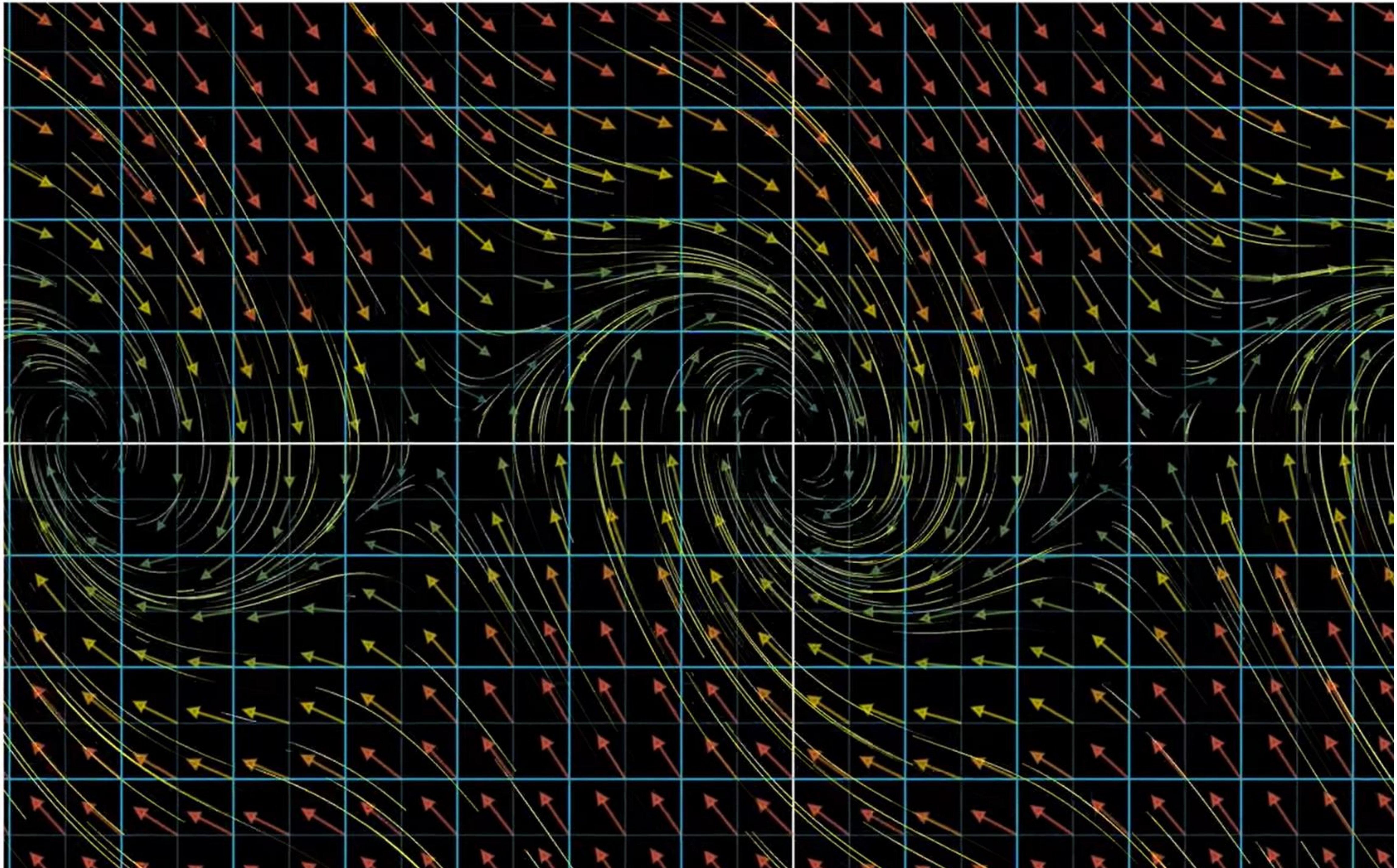
Ольга Вячеславовна Перл

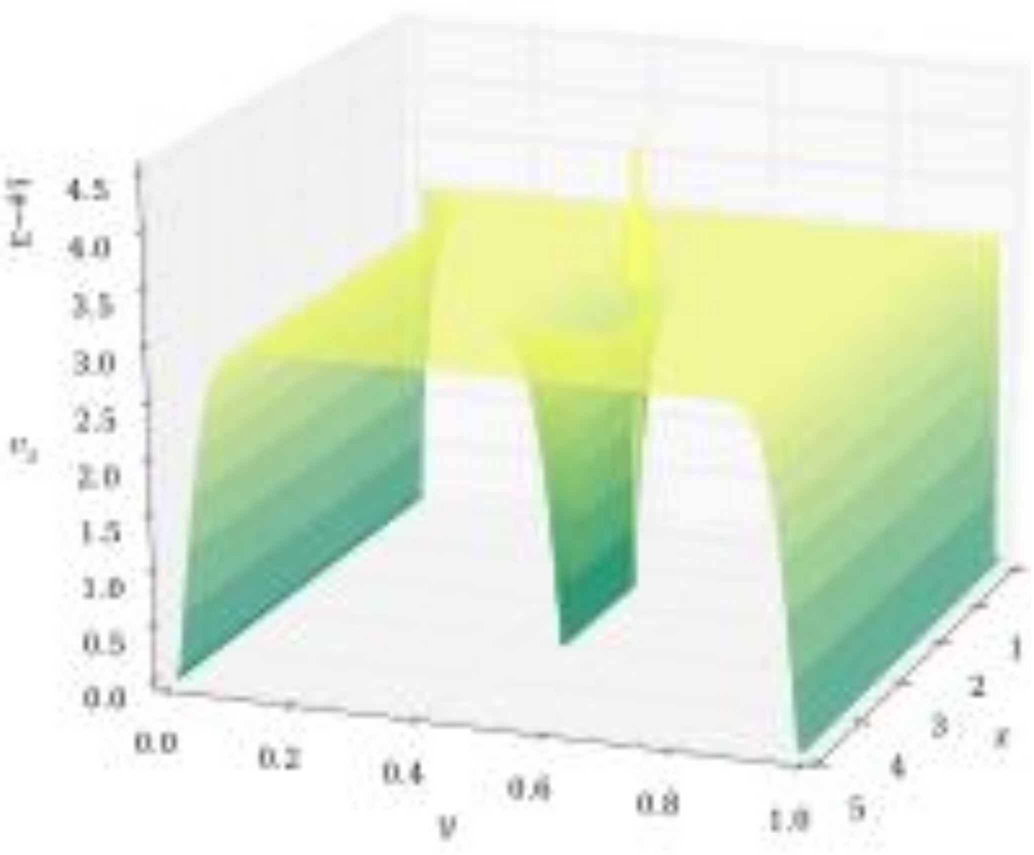
Differential Equations: Direction Fields: Example 1

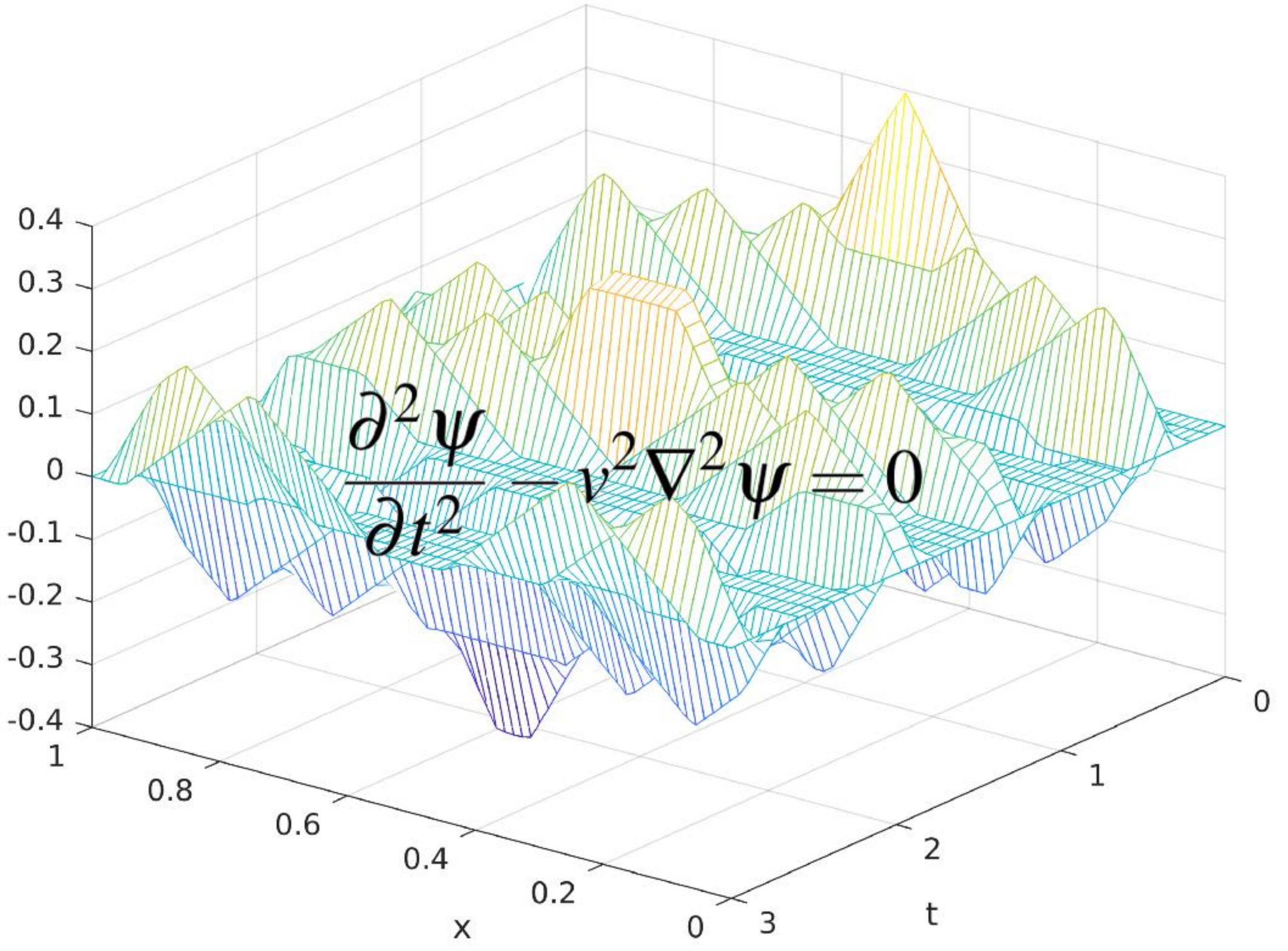


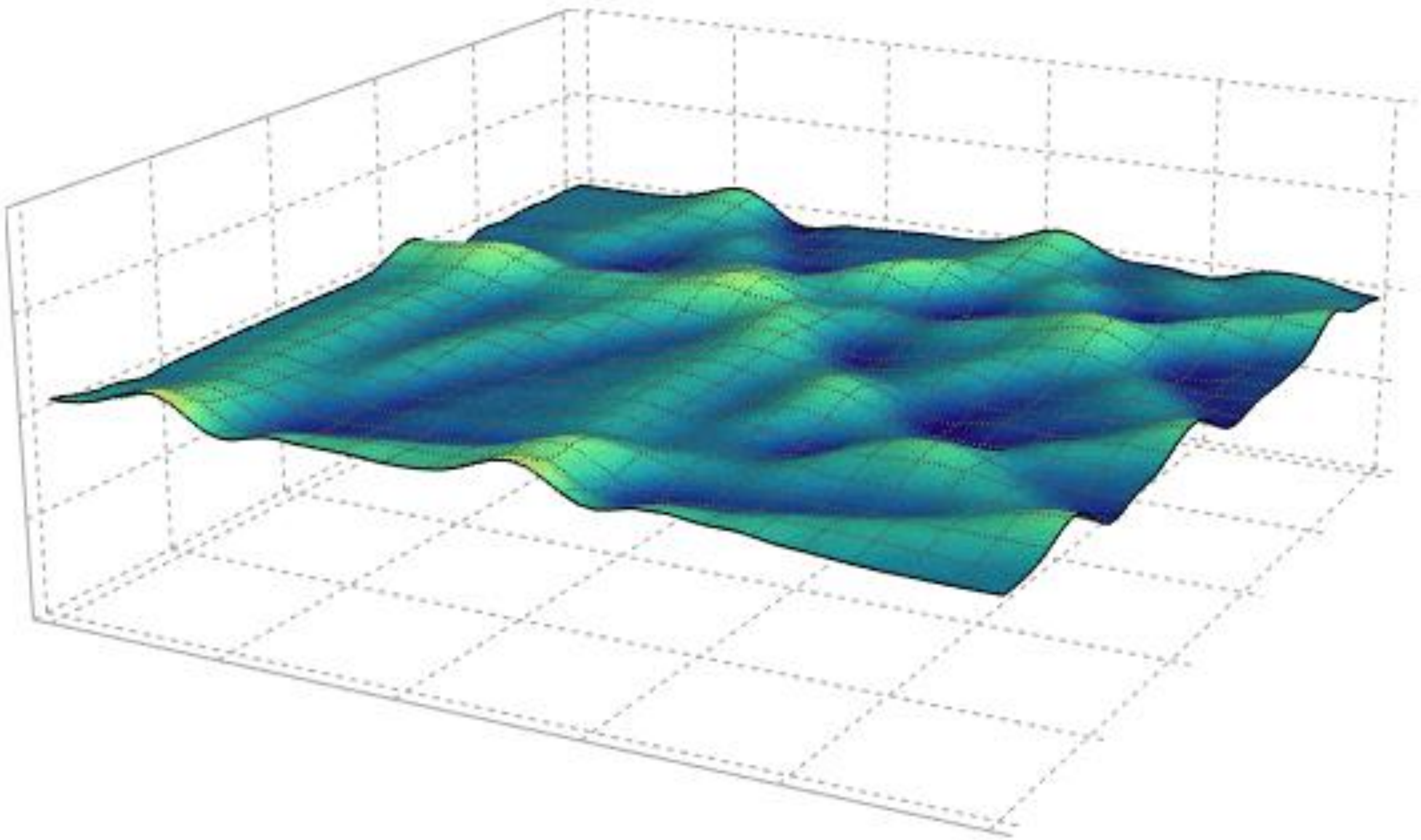
$$y' = x^2 + y^2 - 1$$

Points	y' = slope
$(0,0)$	$y' = 0 + 0 - 1 = -1$
$(0,1)$	$y' = 0 + 1^2 - 1 = 0$
$(1,0)$	$y' = 1^2 + 0 - 1 = 0$
$(0,2)$	$y' = 0^2 + 2^2 - 1 = 3$
⋮	⋮



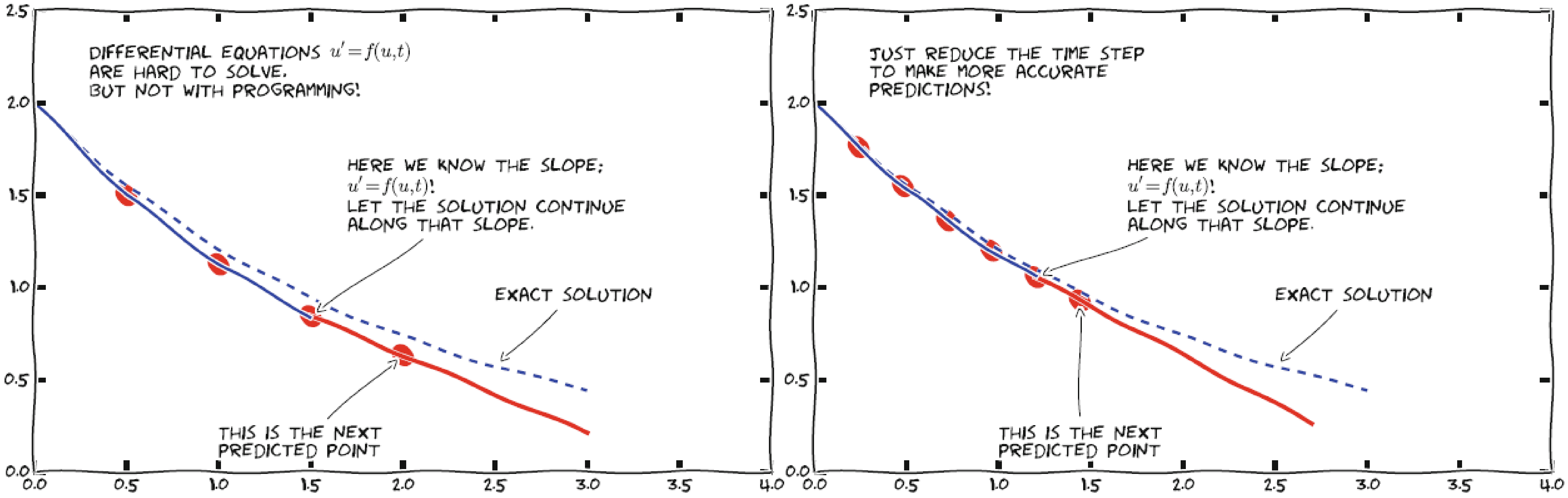

4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
y
1
2
3
4
5


$$\frac{\partial^2\psi}{\partial t^2} - v^2\nabla^2\psi = 0$$
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
1
0.8
0.6
0.4
0.2
0
x
0
1
2
3
t



Численная устойчивость Numerical stability

- In the mathematical subfield of numerical analysis, numerical stability is a generally desirable property of numerical algorithms.
- The precise definition of stability depends on the context.
- One is numerical linear algebra and the other is algorithms for solving ordinary and partial differential equations by discrete approximation.
- In numerical linear algebra the principal concern is instabilities caused by proximity to singularities of various kinds, such as very small or nearly colliding eigenvalues.
- On the other hand, in numerical algorithms for differential equations the concern is <u>the growth of round-off errors and/or small fluctuations in initial data which might cause a large deviation of final answer from the exact solution.</u>


DIFFERENTIAL EQUATIONS $u' = f(u,t)$
ARE HARD TO SOLVE.
BUT NOT WITH PROGRAMMING!
HERE WE KNOW THE SLOPE;
$u' = f(u,t)$!
LET THE SOLUTION CONTINUE
ALONG THAT SLOPE.
EXACT SOLUTION
THIS IS THE NEXT
PREDICTED POINT
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
JUST REDUCE THE TIME STEP
TO MAKE MORE ACCURATE
PREDICTIONS!
HERE WE KNOW THE SLOPE;
$u' = f(u,t)$!
LET THE SOLUTION CONTINUE
ALONG THAT SLOPE.
EXACT SOLUTION
THIS IS THE NEXT
PREDICTED POINT

Ordinary differential equations VS Cauchy problem

TO SOLVE ORDINARY DIFFERENTIAL EQUATIONS

- To solve ordinary differential equations, for example:

  $y' = \frac{xy}{2}$

- Means to find set (family) of functions, which have different constant.

TO SOLVE CAUCHY PROBLEM

- To solve ordinary differential equations by <u>initial condition</u>, for example: $y' = \frac{xy}{2}$ if $y(0) = 1$
- Means to find only one function (with constant value), which passing through a given point.

Численные методы

- Одношаговые методы
  - Метод Эйлера
  - Усовершенс-твованный метод Эйлера
  - Методы Рунге-Кутты
  - Улучшенный метод Эйлера
- Многошаговые методы
  - Метод Милна
  - Метод Адамса
  - Метод Адамса-Башфорта

Метод Эйлера

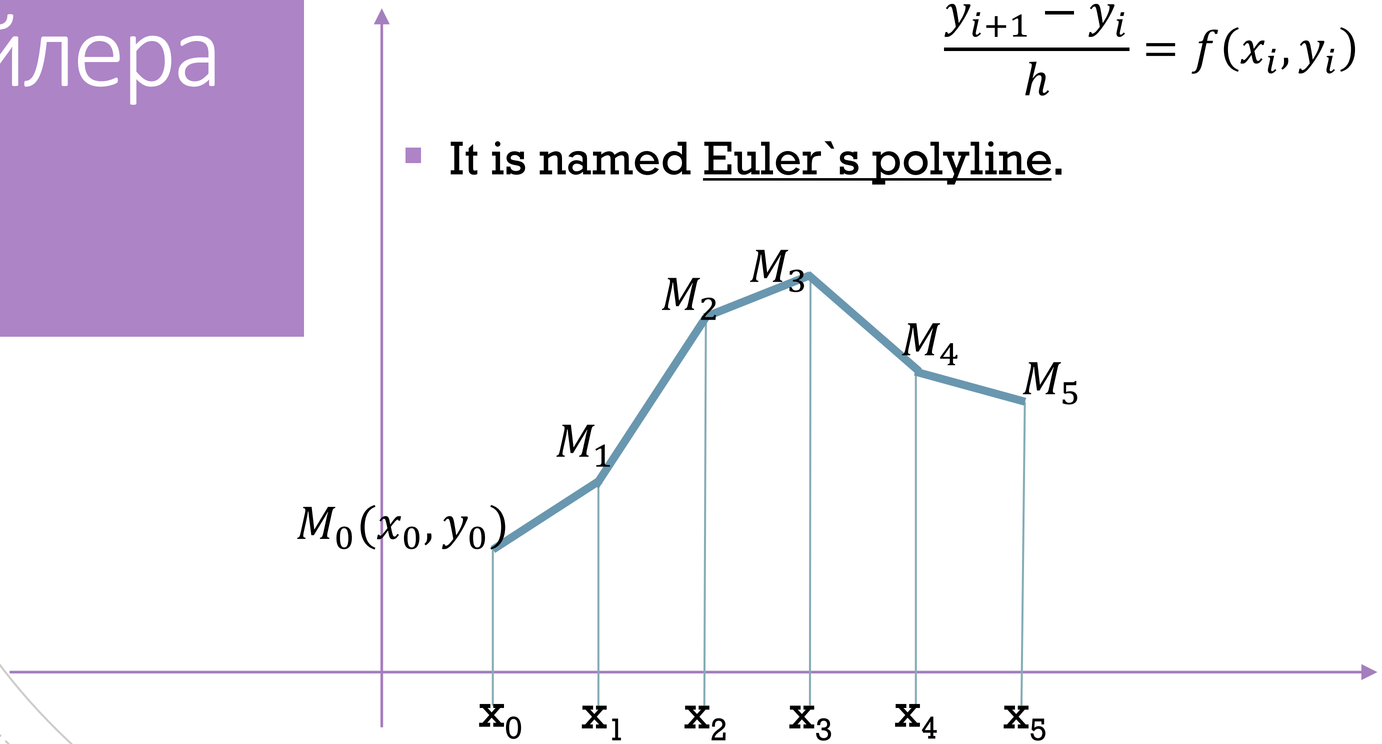
- Let we have ordinary differential equation: $y' = f(x, y)$ with initial condition $y(x_0) = y_0$. And let choose any small step h, then we may build system of equidistant points:

$$x_i = x_0 + ih \ (i = 0, 1, 2, \dots)$$

- Then we may replace correct integral line $y = y(x)$, which passing through a point $M_0(x_0, y_0)$, by polyline $M_0 M_1 M_2 \dots$ with vertices $M_i(x_i, y_i)(i = 0, 1, 2, \dots)$,where segments $M_i M_{i+1}$ are straight line between $x = x_i, \mathrm{x} = x_{i+1}$ and have ascent:

$$\frac{y_{i+1} - y_i}{h} = f(x_i, y_i)$$

- It is named Euler`s polyline.

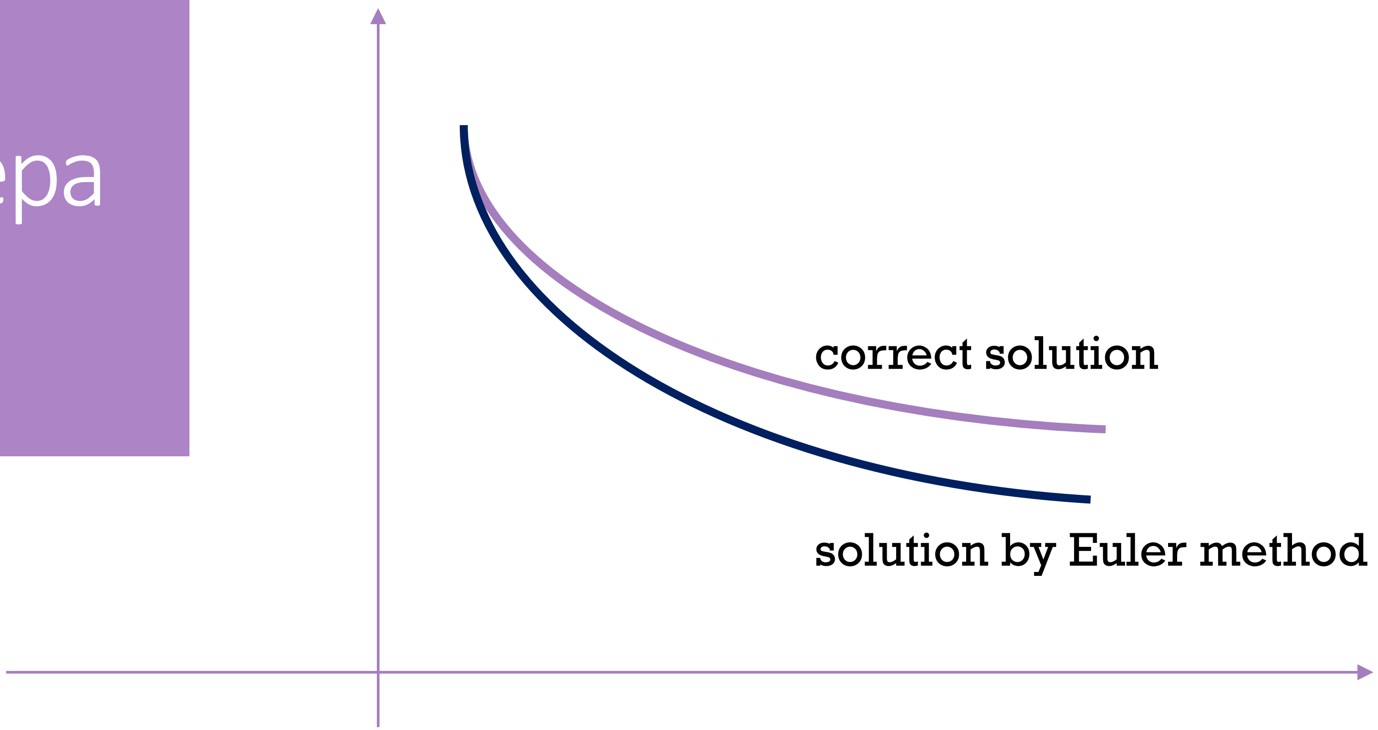


Метод Эйлера

- Formula for Euler method:

$$y_{i+1} = y_i + \Delta y_i$$
$$\Delta y_i = hf(x_i) \quad (i = 0, 1, 2, \dots)$$

- Method has low accuracy
- Method is not stable because systematically increase errors.
- Order of accuracy: $O(h^2)$



Улучшенный Метод Эйлера

- At Euler method we should to calculate intermediate point:

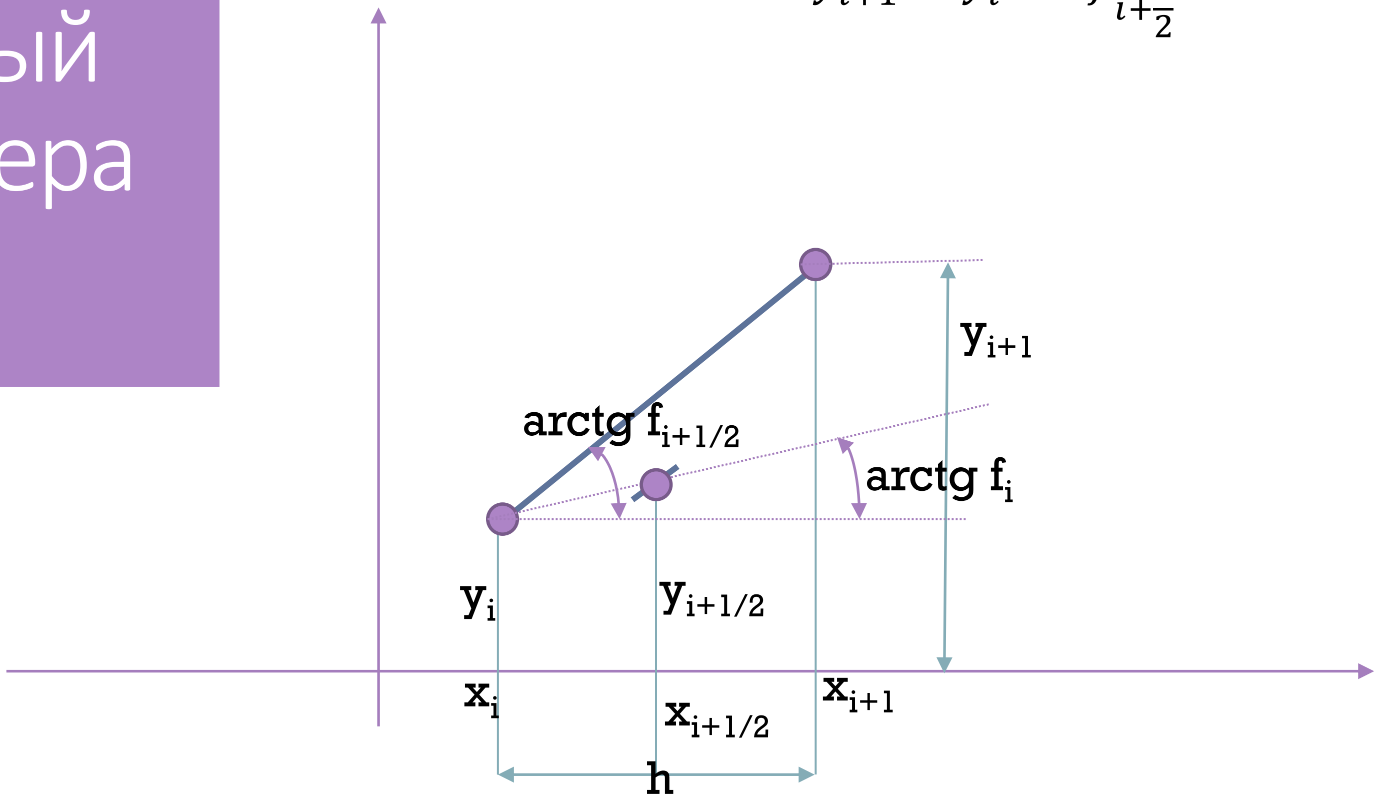
$$x_{i+\frac{1}{2}} = x_i + \frac{h}{2};\ \ y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f_i;$$

- Then:

$$f_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

- And then we let:

$$y_{i+1} = y_i + h f_{i+\frac{1}{2}}$$



Метод Рунге-Кутты (4-го порядка)

- Runge-Kutta change way to calculate Δy_i in formula $y_{i+1} = y_i + \Delta y_i$:

$$\Delta y_i = \frac{1}{6}\left(k_1^{(i)} + 2k_2^{(i)} + 2k_3^{(i)} + k_4^{(i)}\right) (i = 0, 1, 2, \dots)$$

- where:

$$\left.\begin{aligned} k_1^{(i)} &= hf(x_i, y_i), \\ k_2^{(i)} &= hf(x_i + \frac{h}{2}, y_i + \frac{k_1^{(i)}}{2}) \\ k_3^{(i)} &= hf(x_i + \frac{h}{2}, y_i + \frac{k_2^{(i)}}{2}) \\ k_4^{(i)} &= hf(x_i + h, y_i + k_3^{(i)}) \end{aligned}\right\}$$

- Method has good accuracy and well used. Method is applicable for changed step.
- Order of accuracy: $O(h^5)$

Метод Адамса

- For all multistep methods we need set on initial points. For Cauchy problem (when we have only 1 initial point) we may calculate initial segment (acceleration points) by any one-step methods (usually it is Runge-Kutta method).
- Adams method used Newton interpolation method to approximate solution of Cauchy problem with accuracy of 4th order:

$$y' = y'_i + q\Delta y'_{i-1} + \frac{q(q+1)}{2!}\Delta^2 y'_{i-2} + \frac{q(q+1)(q+2)}{3!}\Delta^2 y'_{i-3}$$

where $q = \frac{x-x_i}{h}$

- Adams extrapolation formula:

$$\Delta y_i = hy'_i + \frac{1}{2}\Delta\left(hy'_{i-1}\right) + \frac{5}{12}\Delta^2\left(hy'_{i-2}\right) + \frac{3}{8}\Delta^3\left(hy'_{i-3}\right)$$

- Adams method:

$$y_{i+1} = y_i + \frac{h}{24}\left(55y'_i - 59y'_{i-1} + 37y'_{i-2} - 9y'_{i-3}\right)(i = 4, 5, \dots)$$

- Order of accuracy: $O(h^4)$

Метод Милна (предиктора и корректора)

- This method also based on Newton interpolation method.
- Prediction formula:

$$y_i = y_{i-4} + \frac{4h}{3}(2y'_{i-1} - y'_{i-2} + 2y'_{i-3})$$

- Correction formula:

$$y_i = y_{i-2} + \frac{h}{3}(y'_i + 4y'_{i-1} + y'_{i-2})$$

- Controlling formula for Milne method:

$$\varepsilon_i = \left|\varepsilon_i^{(2)}\right| = \frac{\left|y_i^{(1)} - y_i^{(2)}\right|}{29}$$

- Order of accuracy: $O(h^4)$

Спасибо за внимание!

В случае вопросов по лекции задавайте их через форму:
https://forms.yandex.ru/u/61ffab0425b437e0e3410e9b/

Мы обязательно обсудим их на следующем занятии.