Chapter Two *Basic fiber theory*

2.1 The basic equations of fiber optics

2.2 Waveguide field equation and eigen solution of guided modes

2.3 Fiber loss and the measurement methods

2.4 Fiber dispersion and the measurement methods

Waveguide field equation

Maxwell equation $\xrightarrow{\text{Electromagnetic separation}}$ Wave equation $\longrightarrow$ Transverse-longitudinal separation

$\xrightarrow{\text{Time-space separation}}$ Helmholtz equation $\longrightarrow$ Waveguide field equation

$$\nabla_t^2\psi(r,\phi)+\chi^2\psi(r,\phi)=0$$

ψ can represent each component $(E_r,E_\phi,E_z;H_r,H_\phi,H_z)$

$$\psi(r,\phi,z)=\psi(r,\phi)e^{-i\beta z}$$

Waveguide field equation

In cylindrical coordinate system, the relationship between transverse and longitudinal coordinates

$$\chi^2 H_\phi=-i\left(\omega\varepsilon\frac{\partial E_z}{\partial x}+\beta\frac{1}{r}\cdot\frac{\partial H_z}{\partial\phi}\right)$$

$$\chi^2 E_r=-i\left(\omega\mu\frac{1}{r}\cdot\frac{\partial H_z}{\partial\phi}+\beta\frac{\partial E_z}{\partial r}\right)$$

$$\chi^2 E_\phi=-i\left(-\omega\mu\frac{\partial H_z}{\partial r}+\beta\frac{1}{r}\cdot\frac{\partial E_z}{\partial\phi}\right)$$

$$\chi^2 H_r=-i\left(-\omega\varepsilon\frac{1}{r}\cdot\frac{\partial E_z}{\partial\phi}+\beta\frac{\partial H_z}{\partial r}\right)$$

$\Longrightarrow$

$$E_r=f_1(E_z,H_z)$$
$$E_\phi=f_2(E_z,H_z)$$
$$H_r=f_3(E_z,H_z)$$
$$H_\phi=f_4(E_z,H_z)$$

Waveguide field equation

Transverse propagation constant（U、W）

–Transverse propagation constant： $\chi_1=\sqrt{n_1^2k_0^2-\beta^2}$ –(core)

$\chi_2=\sqrt{n_2^2k_0^2-\beta^2}$ –(cladding)

–Transverse constant：

$$U=a\chi_1=\sqrt{n_1^2k_0^2-\beta^2}\cdot a$$

$$W=-ia\chi_2=\sqrt{\beta^2-n_2^2k_0^2}\cdot a$$

–Satisfy:

$$V^2=U^2+W^2$$

Waveguide field equation

According to above formulas, it concludes that the transverse components can be obtained if the longitudinal components E_z, H_z are given.

Scalar waveguide field equation: $\nabla_t^2\psi+\chi^2\psi=0$ $\quad \psi_{r,\phi}=f\begin{bmatrix}E_z(r,\phi)\\H_z(r,\phi)\end{bmatrix}$

$$\frac{\partial^2\psi(r,\phi)}{\partial r^2}+\frac{1}{r}\frac{\partial\psi(r,\phi)}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi(r,\phi)}{\partial\phi^2}+\left(\omega^2\varepsilon\mu_0-\beta^2\right)\psi=0$$

Assuming $g(\phi)=e^{il\phi}$ **Amplitude at angular direction is periodic**

Variables separation: $\psi(r,\phi)=F(r)g(\phi)$

We obtain $r^2\left[\frac{1}{F}\frac{d^2F}{dr^2}+\frac{1}{r}\frac{1}{F}\frac{dF}{dr}+\left(\omega^2\varepsilon\mu_0-\beta^2\right)\right]+\frac{1}{g}\frac{d^2g}{d\phi^2}=0$

$F(r)$ **satisfy**

$$\frac{d^2F}{dr^2}+\frac{1}{r}\frac{dF}{dr}+\left[\left(\omega^2\varepsilon\mu_0-\beta^2\right)-\frac{l^2}{r^2}\right]F=0$$ **Field equation**

Waveguide field equation

Eigen solution of the guided modes

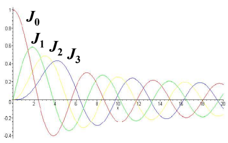
$$\frac{d^2F}{dr^2}+\frac{1}{r}\frac{dF}{dr}+\left[\left(\omega^2\varepsilon\mu_0-\beta^2\right)-\frac{l^2}{r^2}\right]F=0$$

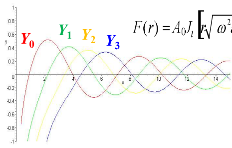
The solution of the above field equation is Bessel function

(1) when $\omega^2\varepsilon\mu_0-\beta^2>0$ core

$$F(r)=A_0\underbrace{J_l\left[r\sqrt{\omega^2\varepsilon\mu_0-\beta^2}\right]}_{\text{first kind Bessel function}}+B_0\underbrace{Y_l\left[r\sqrt{\omega^2\varepsilon\mu_0-\beta^2}\right]}_{\text{second kind Bessel function}}$$

$$\begin{cases}\lim\limits_{r\to0}J_l\left[r\sqrt{\omega^2\varepsilon\mu_0-\beta^2}\right]\to 1 \text{ or } 0\\ \lim\limits_{r\to0}Y_l\left[r\sqrt{\omega^2\varepsilon\mu_0-\beta^2}\right]\to-\infty\end{cases}$$





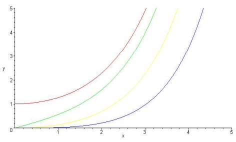
$$F(r)=A_0J_l\left[r\sqrt{\omega^2\varepsilon\mu_0-\beta^2}\right]+\cancel{B_0Y_l\left[r\sqrt{\omega^2\varepsilon\mu_0-\beta^2}\right]}$$

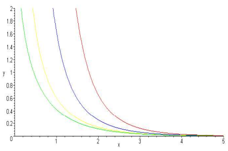
–Pseudo solution
伪解决方案

Waveguide field equation

(2) when $\omega^2\varepsilon\mu_0-\beta^2<0$ cladding

$$F(r)=C_0\underbrace{I_l\left[r\sqrt{\beta^2-\omega^2\varepsilon\mu_0}\right]}_{\text{first kind Hankel function}}+D_0\underbrace{K_l\left[r\sqrt{\beta^2-\omega^2\varepsilon\mu_0}\right]}_{\text{second kind Hankel function}}$$





Properties:

$$\begin{cases}\lim\limits_{r\to+\infty}I_l\left[r\sqrt{\beta^2-\omega^2\varepsilon\mu_0}\right]\to+\infty\\ \lim\limits_{r\to+\infty}K_l\to0\end{cases}\Longrightarrow F(r)=\cancel{C_0I_l\left[r\sqrt{\beta^2-\omega^2\varepsilon\mu_0}\right]}+D_0K_l\left[r\sqrt{\beta^2-\omega^2\varepsilon\mu_0}\right]$$

Waveguide field equation

Guided modes condition: $n_2k_0<\beta<n_1k_0$ **For step-index fibers:**

$$F(r)\begin{cases}A_0J_l\left[\frac{(k_0^2n_1^2-\beta^2)^{1/2}\cdot a}{a}r\right]=A_0J_l\left(\frac{U}{a}r\right) & (0\le r\le a)\\ D_0K_l\left[\frac{(\beta^2-k_0^2n_2^2)^{1/2}\cdot a}{a}r\right]=D_0K_l\left(\frac{W}{a}r\right) & (r>a)\end{cases}$$

$$\psi(r,\phi)=F(r)g(\phi)$$

The longitudinal components of the electric and magnetic fields in the core and cladding

$$\begin{bmatrix}E_z\\H_z\end{bmatrix}=\begin{cases}\begin{bmatrix}A\\B\end{bmatrix}J_l\left(\frac{U}{a}r\right)e^{il\phi} & (0\le r\le a) \quad \textbf{Core}\\ \begin{bmatrix}C\\D\end{bmatrix}K_l\left(\frac{W}{a}r\right)e^{il\phi} & (r>a) \quad \textbf{Cladding}\end{cases}$$

Waveguide field equation [比较重要]

$$\begin{bmatrix} E_z \\ H_z \end{bmatrix} = \begin{cases} \begin{bmatrix} A \\ B \end{bmatrix} J_l\left(\frac{U}{a}r\right)e^{il\phi} & (0 \le r \le a) \\ \begin{bmatrix} C \\ D \end{bmatrix} K_l\left(\frac{W}{a}r\right)e^{il\phi} & (r > a) \end{cases}$$

TE mode: $E_z=0$, A=0
TM mode: $H_z=0$, B=0
EH mode: $H_z \propto iE_z$
HE mode: $H_z \propto -iE_z$

–The larger the W is, the faster the mode in cladding attenuates

The determination of the eigen solution:

- Eigenvalue：β and l are determined by the continuity requirement at the fiber surface 由光纤表面的连续性要求决定
- Longitudinal components: A,B,C,D are the uncertain constants determined by boundary conditions 由边界条件确定
- Transverse components: they can be derived from longitudinal components

Waveguide field equation

Eigenvalue equation (to determine A, B, C, D, β and l)

- 线性方程组系数矩阵行列式等于0

Considering E_z、E_ϕ；H_z、H_ϕ are continuous at $r=a$

① $\frac{A}{C}=\frac{B}{D}=\frac{K_l(W)}{J_l(U)}$

② $\begin{cases} i\beta l(\frac{1}{U^2}+\frac{1}{W^2})A-\omega\mu\left[\frac{1}{U}\cdot\frac{J_l'(U)}{J_l(U)}+\frac{1}{W}\cdot\frac{K_l'(W)}{K_l(W)}\right]B=0 \\ \omega\left[\frac{\varepsilon_1}{U}\cdot\frac{J_l'(U)}{J_l(U)}+\frac{\varepsilon_2}{W}\cdot\frac{K_l'(W)}{K_l(W)}\right]A+i\beta l(\frac{1}{U^2}+\frac{1}{W^2})B=0 \end{cases}$

$E_r = f_1(E_z,H_z)$
$\boxed{E_\phi = f_2(E_z,H_z)}$
$H_r = f_3(E_z,H_z)$
$\boxed{H_\phi = f_4(E_z,H_z)}$

If they have the solutions, A and B are not all equal to zero, thus the characteristic determinant of the above equation 2 should equal to zero.

$$\boxed{l^2\beta^2(\frac{1}{U^2}+\frac{1}{W^2})^2=\left[\frac{J_l'(U)}{UJ_l(U)}+\frac{K_l'(W)}{WK_l(W)}\right]\cdot\left[\frac{k_1^2J_l'(U)}{UJ_l(U)}+\frac{k_2^2K_l'(W)}{WK_l(W)}\right]}$$

where $k_1=n_1k_0 \quad k_2=n_2k_0$

If \n_1, \n_2, a, \lambda_0
are given, there are many \beta_m for a fixed l

Waveguide field equation

$$(\frac{J_l'}{UJ_l}+\frac{K_l'}{WK_l})(\frac{k_1^2J_l'}{UJ_l}+\frac{k_2^2K_l'}{WK_l})=l^2\beta^2(\frac{1}{U^2}+\frac{1}{W^2})^2$$

It is called as characteristic equation or dispersion equation. It is actually a transcendental equation about β.

There are many different β_{lm} (l=0,1,2,3... m=1,2,3...) values, each β_{lm} corresponds to a guided mode.

Modes analysis [比较重要]

(1) Types of guided modes

TE mode (only H_z exists, E_z=0)
TM mode (only E_z exists, H_z=0) $\longrightarrow l=0 \longrightarrow$ orthogonal linear polarized mode 正交线性偏振光

EH mode (electric field dominates占主导, H_z is phase-leading相位超前)
HE mode (magnetic field dominates, E_z is phase-leading) $\longrightarrow l \neq 0$
elliptical polarized light 椭圆偏振光

Define a parameter q which can reflect the mode type

$$q=\frac{\omega\mu_0}{i\beta}\cdot\frac{H_z}{E_z}=\frac{\omega\mu_0}{i\beta}\cdot\frac{B}{A}$$

q represents the relationship of phase and amplitude between E_z and H_z

TE mode: A=0 $\Rightarrow$ $q=\infty$
TM mode: B=0 $\Rightarrow$ $q=0$
EH mode: $H_z \propto iE_z$ $\Rightarrow$ $q=1$
HE mode: $H_z \propto -iE_z$ $\Rightarrow$ q = -1

The property of Bessel function

Differential: 微分

$J_\iota'(U)=(1/2)\left[J_{\iota-1}(U)-J_{\iota+1}(U)\right] \qquad K_\iota'(W)=-\frac{1}{2}\left[K_{\iota-1}(W)+K_{\iota+1}(W)\right]$

Recursion: 递归 $\quad (\iota/U)J_\iota(U)=(1/2)\left[J_{\iota-1}(U)+J_{\iota+1}(U)\right]$

$\frac{\iota}{W}K_\iota(W)=-\frac{1}{2}\left[K_{\iota-1}(W)-K_{\iota+1}(W)\right]$

Approximation: 近似

$\lim_{U\to\infty}J_\iota(U)=\sqrt{\frac{2}{\pi U}}\cos\left(U-\frac{\pi}{4}-\frac{\iota\pi}{2}\right) \qquad \lim_{W\to\infty}K_\iota(W)=\sqrt{\frac{1}{W}}e^{-W}$

$\lim_{U\to 0}J_\iota(U)=\frac{1}{\iota!}\left(\frac{U}{2}\right)^\iota \qquad \lim_{W\to 0}K_\iota(W)=\begin{cases}(\iota-1)!2^{\iota-1}W^{-\iota} & (\iota\ge 1)\\ \ln\left(\frac{2}{W\gamma}\right)=\ln\left(\frac{1.123}{W}\right) & (\iota=0)\end{cases}$

The property of Bessel function

(1) For TE mode： $E_z=0,\ \ell=0,\ H_z\neq 0$

To make the equation have the solutions not all equaling to zero

$$\frac{J_\ell'(U)}{UJ_\ell(U)}+\frac{K_\ell'(W)}{WK_\ell(W)}=0$$

The eigenvalue equation of the TE mode： $\frac{J_0'(U)}{UJ_0(U)}+\frac{K_0'(W)}{WK_0(W)}=0$

(2) For TM mode： $H_z=0,\ \ell=0,\ E_z\neq 0$

To make the equation have the solutions not all equaling to zero

$$\frac{k_1^2J_\ell'(U)}{UJ_\ell(U)}+\frac{k_2^2K_\ell'(W)}{WK_\ell(W)}=0$$

The eigenvalue equation of the TM mode： $\frac{k_1^2J_0'(U)}{UJ_0(U)}+\frac{k_2^2K_0'(W)}{WK_0(W)}=0$

The property of Bessel function

(3) The eigenvalue equation of the EH or HE modes

$$\ell^2\beta^2\left(\frac{1}{U^2}+\frac{1}{W^2}\right)^2=\left(\frac{J_\ell'(U)}{UJ_\ell(U)}+\frac{K_\ell'(W)}{WK_\ell(W)}\right)\left(\frac{k_1^2J_\ell'(U)}{UJ_\ell(U)}+\frac{k_2^2K_\ell'(W)}{WK_\ell(W)}\right)$$

set $\frac{J_\ell'(U)}{UJ_\ell(U)}=\bar{J}_\ell,\quad \frac{K_\ell'(W)}{WK_\ell(W)}=\bar{K}_\ell$

we obtain $\bar{J}_\ell=-\frac{1}{2}\left(1+\frac{k_2^2}{k_1^2}\right)\bar{K}_\ell\pm\sqrt{\left(1+\frac{k_2^2}{k_1^2}\right)^2\bar{K}_\ell-4\left[\frac{k_2^2}{k_1^2}\bar{K}_\ell^2-\ell^2\left(\frac{1}{U^2}+\frac{k_2^2}{k_1^2}\frac{1}{W^2}\right)\left(\frac{1}{U^2}+\frac{1}{W^2}\right)\right]}$

set “＋” as EH mode and “－” as HE mode
The eigenvalue equation of EH mode : 上
The eigenvalue equation of HE mode : 下

$$\frac{J_\ell'(U)}{UJ_\ell(U)}+\frac{K_\ell'(W)}{WK_\ell(W)}=\ell\left(\frac{1}{U^2}+\frac{1}{W^2}\right)$$

$$\frac{k_1^2}{k_2^2}\frac{J_\ell'(U)}{UJ_\ell(U)}+\frac{K_\ell'(W)}{WK_\ell(W)}=-\ell\left(\frac{k_1^2}{k_2^2}\frac{1}{U^2}+\frac{1}{W^2}\right)$$

The property of Bessel function

- The mode eigen value β can be determined by U or W

The expression of the electromagnetic field component in the cladding:

$$E_z=CK_l(\frac{W}{a}r)e^{jl\varphi}$$

K_l is an attenuated exponent function

When $W\to\infty$, the electromagnetic wave attenuates very quickly in the cladding thus is well restricted in the core —— Far from cut-off

When $W\to 0$, the electromagnetic wave attenuates very slowly in the cladding thus escapes from the cladding —— Near from cut-off

The property of Bessel function

2. The conditions of guided modes cut-off and far from cut-off

$W \to 0$ cladding mode dose not attenuate, 导模截止cut-off

$W \to \infty$ 包层模衰减很快cladding mode attenuates severely 远离截止far from cut-off

①TE mode (TE_{0m}) $(l = 0, q = \infty)$ $\dfrac{J_0'(U)}{UJ_0(U)} + \dfrac{K_0'(W)}{WK_0(W)} = 0$

$J_0' = (1/2)(J_{-1} - J_1) = -J_1$

$K_0' = (-1/2)(K_{-1} + K_1) = -K_1$

$\Rightarrow \dfrac{J_1(U)}{UJ_0(U)} + \dfrac{K_1(W)}{WK_0(W)} = 0$

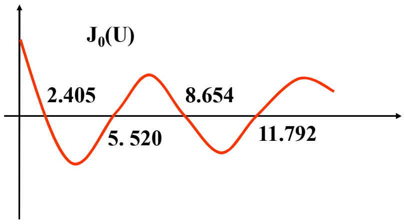
Far from cut-off condition:

$W \to \infty, \dfrac{K_1(W)}{WK_0(W)} \to 0 \Rightarrow \dfrac{J_1(U)}{UJ_0(U)} \to 0 \Rightarrow J_1(U) = 0, U_{0m}^{\infty}$ $(U_{0m}^{\infty} \neq 0)$

cut-off condition:

$W \to 0, \dfrac{K_1(W)}{WK_0(W)} \to \infty \Rightarrow \dfrac{J_1(U)}{UJ_0(U)} \to \infty \Rightarrow J_0(U) = 0, U_{0m}^{c}$

The property of Bessel function



$U_{01}^c = 2.405, U_{02}^c = 5.520$

$U_{03}^c = 8.654, U_{04}^c = 12.492$

$\Downarrow$

TE_{01}, TE_{02}, TE_{03}, TE_{04}

Guided modes:

$U \in (U_{0m}^c, U_{0m}^{\infty}), W \in (0, \infty),$

$V \in (U, \infty)$

$TE_{01}(V > V_c = 2.405);\ \ TE_{02}(V > V_c = 5.52)\cdots\cdots$

The property of Bessel function

②TM mode TM_{0m} $(l = 0)$

Eigen equation $\dfrac{n_1^2 J_1(U)}{n_2^2 UJ_0(U)} + \dfrac{K_1(W)}{WK_0(W)} = 0$

Far from cut-off and cut-off conditions are same as that of TE mode

TE、TM modes appear and disappear in pairs

$TM_{01}(V > V_c = 2.405);\ \ TM_{02}(V > V_c = 5.52)\cdots\cdots$

Degenerate state

- TE_{0m} and TM_{0m} modes have the same eigenvalue near and far from cut-off，i.e., two modes are at the degenerate state;
- They have different eigenvalue between cut-off and far from cut-off, this phenomenon is called as degenerate states split 简并态分离

The property of Bessel function

③ HE_{lm} mode l=1, q= -1

Eigenvalue equation $\dfrac{n_1^2}{n_2^2} \cdot \dfrac{J_1'(U)}{UJ_1(U)} + \dfrac{K_1'(W)}{WK_1(W)} = -\left(\dfrac{n_1^2}{n_2^2} \cdot \dfrac{1}{U^2} + \dfrac{1}{W^2}\right)$

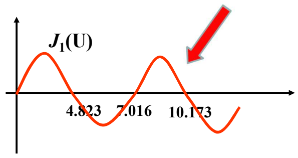
According to Bessel function property $\dfrac{n_1^2}{n_2^2} \dfrac{J_0(U)}{UJ_1(U)} - \dfrac{K_0(W)}{WK_1(W)} = 0$

Near cut-off condition:

$W \to 0, \dfrac{K_1(W)}{WK_0(W)} \to \infty \Rightarrow \dfrac{n_1^2}{n_2^2} \cdot \dfrac{J_0(U)}{UJ_1(U)} = \infty \Rightarrow J_1(U_{lm}^c) = 0$

Far from cut-off condition:

$J_0(U_{lm}^{\infty}) = 0$



The property of Bessel function

③ HE_{lm} mode l>1, q=-1

Eigenvalue equation $\dfrac{n_1^2}{n_2^2} \cdot \dfrac{J_l'(U)}{UJ_l(U)} + \dfrac{K_l'(W)}{WK_l(W)} = -l\left(\dfrac{n_1^2}{n_2^2} \cdot \dfrac{1}{U^2} + \dfrac{1}{W^2}\right)$

According to Bessel function property: $\dfrac{n_1^2 J_{l-1}(U)}{n_2^2 UJ_l(U)} - \dfrac{K_{l-1}(W)}{WK_l(W)} = 0$

Near cut-off: $W \to 0, \dfrac{K_{l-1}(W)}{WK_l(W)} \to \dfrac{1}{2(l-1)} \Rightarrow \dfrac{J_{l-1}(U)}{UJ_l(U)} = \dfrac{n_2^2}{n_1^2} \cdot \dfrac{1}{2(l-1)} \Rightarrow$

$\dfrac{2(l-1)J_{l-1}}{UJ_l} = \dfrac{n_2^2}{n_1^2} \Rightarrow \dfrac{J_{l-2} + J_l}{J_l} = \dfrac{n_2^2}{n_1^2} \Rightarrow \dfrac{J_{l-2}}{J_l} = \dfrac{n_2^2}{n_1^2} - 1 \approx 0 \Rightarrow$

$J_{l-2}(U_{lm}^c) = 0$

$(U_{lm}^c \neq 0)$

The property of Bessel function

Far from cut-off condition:

$W \to \infty, \dfrac{K_{l-1}(W)}{WK_l(W)} \to 0 \Rightarrow \dfrac{n_1^2}{n_2^2} \cdot \dfrac{J_{l-1}(U)}{UJ_l(U)} = 0 \Rightarrow J_{l-1}(U_{lm}^{\infty}) = 0$ $(U_{lm}^{\infty} \neq 0)$

Note that: $U_{lm}^{\infty} = 0$ **is also not the eigenvalue, so when** $U \to 0, \dfrac{J_{l-1}}{UJ_l} \to \dfrac{2l}{U^2} \to \infty$ **instead of zero**

Considering the above two cases ,for HE_{lm} mode（l >0, q= -1 ）

Near cut-off: $J_{l-2}(U_{lm}^c) = 0$ (when $l > 1$, $U_{lm}^c \neq 0$)

Far from cut-off: $J_{l-1}(U_{lm}^{\infty}) = 0$ $(U_{lm}^{\infty} \neq 0)$

The property of Bessel function

④ EH_{lm} mode

Eigenvalue equation $\dfrac{J_l'(U)}{UJ_l(U)} + \dfrac{K_l'(W)}{WK_l(W)} = l\left(\dfrac{1}{U^2} + \dfrac{1}{W^2}\right)$

According to Bessel function property : $\dfrac{J_{l+1}(U)}{UJ_l(U)} = \dfrac{K_{l+1}(W)}{WK_l(W)}$

Similarly we can obtain

Near cut-off: $J_l(U_{lm}^c) = 0$ $(U_{lm}^c \neq 0)$

Far from cut-off: $J_{l+1}(U_{lm}^{\infty}) = 0$ $(U_{lm}^c \neq 0)$

For some EH_{lm} modes, the cut-off conditions are

mode	EH_{11}	EH_{12}	EH_{13}	……	EH_{21}	EH_{22}	EH_{23}
V_c	3.832	7.016	10.173	……	5 .136	8 . 417	11.620

The property of Bessel function

Mode	Eigen equation	Cut-off condition	Far from cut-off condition
TE_{0m}	$\dfrac{J_1(U)}{UJ_0(U)} + \dfrac{K_1(W)}{WJ_0(W)} = 0$	$J_0(U_{0m}^c) = 0$	$J_1(U_{0m}^{\infty}) = 0(U_{0m}^{\infty} \neq 0)$
TM_{0m}	$\dfrac{\varepsilon_1}{\varepsilon_2} \dfrac{J_1(U)}{UJ_0(U)} + \dfrac{K_1(W)}{WJ_0(W)} = 0$	$J_0(U_{0m}^c) = 0$	$J_1(U_{0m}^{\infty}) = 0(U_{0m}^{\infty} \neq 0)$
HE_{1m}	$\dfrac{\varepsilon_1}{\varepsilon_2} \dfrac{J_0(U)}{UJ_1(U)} - \dfrac{K_0(W)}{WJ_1(W)} = 0$	$J_1(U_{1m}^{\infty}) = 0(U_{1m}^{\infty} = 0)$	$J_0(U_{1m}^{\infty}) = 0$
$HE_{lm}(l \geq 2)$	$\dfrac{\varepsilon_1}{\varepsilon_2} \dfrac{J_{l-1}(U)}{UJ_l(U)} - \dfrac{K_{l-1}(W)}{WJ_l(W)} = 0$	$J_{l-2}(U_{lm}^{\infty}) = 0(U_{lm}^{\infty} \neq 0)$	$J_{l-1}(U_{lm}^{\infty}) = 0(U_{lm}^{\infty} \neq 0)$
$EH_{lm}(l > 0)$	$\dfrac{J_{l+1}(U)}{UJ_l(U)} + \dfrac{K_{l+1}(W)}{WJ_l(W)} = 0$	$J_l(U_{lm}^{\infty}) = 0(U_{lm}^{\infty} \neq 0)$	$J_{l+1}(U_{lm}^{\infty}) = 0(U_{lm}^{\infty} \neq 0)$

The property of Bessel function

The eigenvalues of low-order modes at cut off and far from cut-off

l \ m (U_{lm})	1: U^C_{lm}	1: U^∞_{lm}	2: U^C_{lm}	2: U^∞_{lm}	3: U^C_{lm}	3: U^∞_{lm}	4: U^C_{lm}	4: U^∞_{lm}	Modes cut-off and far from cut-off	
0	2.405	3.823	5.520	7.016	8.654	10.173	11.792	13.324	TE_{0m} TM_{0m}	$J_0=0$ $J_1=0$
1	0	2.405	3.823	5.520	7.016	8.654	10.173	11.792	HE_{1m}	$J_1=0$ $J_0=0$
1	3.823	5.136	7.016	8.417	10.173	11.620	13.324	14.796	EH_{1m}	$J_1=0$ $J_2=0$
2	2.405	3.823	5.520	7.016	8.654	10.173	11.792	13.324	HE_{2m}	$J_0=0$ $J_1=0$
2	5.136	6.380	8.417	9.761	11.620	13.015	14.796	16.223	EH_{2m}	$J_2=0$ $J_3=0$
3	3.823	5.136	7.016	8.417	10.173	11.620	13.324	14.796	HE_{3m}	$J_1=0$ $J_2=0$
3	6.380	7.588	9.761	11.065	13.015	14.700	16.223	17.616	EH_{3m}	$J_3=0$ $J_4=0$
4	5.136	6.380	8.417	9.761	11.620	13.015	14.796	16.223	HE_{4m}	$J_2=0$ $J_3=0$

Solve the electromagnetic field distribution

- How to solve the electromagnetic field distribution of a given mode?

Taking TM_{0m} mode as an example:

$$\frac{J_0'(U)}{UJ_0(U)}+\frac{K_0'(W)}{WK_0(W)}=0\ (l=0,\ q=0)$$

$$U^2+W^2=V^2$$

$\Rightarrow$ U and W **can be obtained**

① $$\frac{A}{C}=\frac{B}{D}=\frac{K_l(W)}{J_l(U)}$$

② $$\begin{cases} i\beta l\left(\frac{1}{U^2}+\frac{1}{W^2}\right)A-\omega\mu\left[\frac{1}{U}\cdot\frac{J_l'(U)}{J_l(U)}+\frac{1}{W}\cdot\frac{K_l'(W)}{K_l(W)}\right]B=0 \\ \omega\left[\frac{\varepsilon_1}{U}\cdot\frac{J_l'(U)}{J_l(U)}+\frac{\varepsilon_2}{W}\cdot\frac{K_l'(W)}{K_l(W)}\right]A+i\beta l\left(\frac{1}{U^2}+\frac{1}{W^2}\right)B=0 \end{cases}$$

$\Rightarrow$ ***A*, B, C, D can be obtained**

According to

$$\begin{bmatrix}E_z\\H_z\end{bmatrix}=\begin{cases}\begin{bmatrix}A\\B\end{bmatrix}J_l\left(\frac{U}{a}r\right)e^{jl\phi} & (0\le r\le a)\\ \begin{bmatrix}C\\D\end{bmatrix}K_l\left(\frac{W}{a}r\right)e^{jl\phi} & (r>a)\end{cases}$$

E_z and H_z can be obtained

Then the other components such as E_r, E_φ, H_r, H_φ can be expressed.

(3) Dispersion curve and single mode condition

- Dispersion curve

In a fiber with the given structure parameters, the modes distribution is fixed.

Based on the eigenvalue equation, the propagation constant of each guided mode β (or normalized propagation constant b) as a function of the fiber normalized frequency V can be obtained by numerical calculation.

(3) Dispersion curve and single mode condition

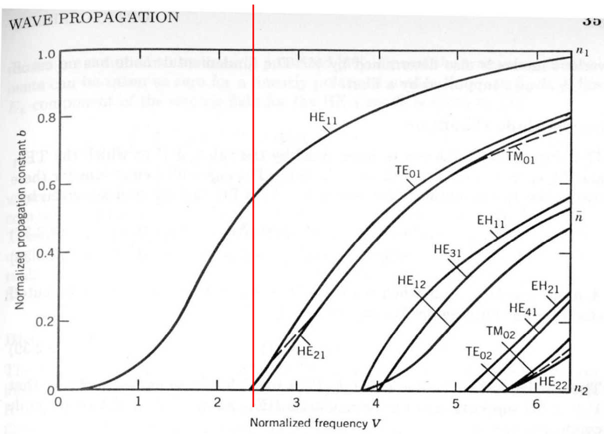


Figure 2.5 Normalized propagation constant b as a function of normalized frequency V for a few low-order fiber modes. The right scale shows the mode index $\bar{n}$. (After Ref. [31]. ©1981 Academic Press. Reprinted with permission.)

(3) Dispersion curve and single mode condition

The analysis of dispersion curve

- Each curve corresponds to a guided mode.
- The number of the cross points for the line parallel to vertical axis with the dispersion curve represents the number of the guided modes allowed in the fiber.
- Larger V leads to more modes
- When $V<2.405$, only HE_{11} mode is allowed in fiber, other modes are cut-off, thus the fiber operates at the single mode propagation state.

(3) Dispersion curve and single mode condition

☆The condition of single mode operation 单模工作条件

- The condition of single mode operation $V=\frac{2\pi a\sqrt{n_1^2-n_2^2}}{\lambda_0}<2.405$
- The size of single mode fiber $a_c=1.202\lambda_0\Big/\left(\pi\sqrt{n_1^2-n_2^2}\right)$
- The cur-off wavelength of single mode fiber $\lambda_c=a\pi\sqrt{n_1^2-n_2^2}\Big/1.202$
- The cur-off frequency of single mode fiber $f_c=1.202c\Big/\left(a\pi\sqrt{n_1^2-n_2^2}\right)$
- Only when $\lambda>\lambda_c$ or $f<f_c$, the fiber operates at the single mode propagation state. The transmission mode is HE_{11} mode, named as fundamental mode

(LP modes) Lineally polarized modes and weakly guiding fiber

Longitudinal component $\longrightarrow$ **Transverse component**

$$\longrightarrow \left|\frac{E_z}{E_t}\right|\sim\left|\frac{H_z}{H_t}\right|\sim\sqrt{\Delta}$$ **where** $\Delta=\frac{n_1^2-n_2^2}{2n_1}$ **Relative refractive index difference**

$$\xrightarrow[\sqrt{\Delta}\in[1/30,1/10]]{\Delta\in[0.1\%,1.0\%]} |E_z|<<|E_t|,|H_z|<<|H_t|$$

The transmission field in fiber is almost transverse. Thus the transverse component can better reveal the field characteristics.

However, the complex expression of transverse components leads to a difficult analysis

weakly guiding fiber

What is weakly guiding approximation?

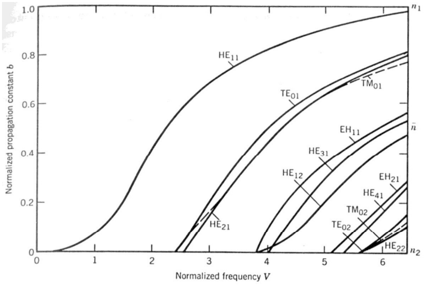
$$\nabla\varepsilon\approx 0,\ n_1=n_2$$

The refractive indexes of core (n_1) and cladding (n_2) are very similar.

Therefore, the analysis of optical fiber is significantly simplified, this kind of fiber is called as the weakly guiding fiber.

weakly guiding fiber

- Under the weakly guiding approximation, we can find $HE_{l+1,m}$ and $EH_{l-1,m}$ modes have the similar dispersion curves thus are degenerate.
- Therefore we can offset one transverse component by linear superposition of the above two modes, hence significantly simplify field expression.



A type of new mode is proposed. It is the linearly polarized (LP_{lm}) mode which has only four field components

weakly guiding fiber

Mathematical proof that $EH_{l-1,m}$ and $HE_{l+1,m}$ modes are degenerate

According to the eigen equation when $n_1=n_2$:

$$\left[\frac{1}{U}\cdot\frac{J_l'(U)}{J_l(U)}+\frac{1}{W}\cdot\frac{K_l'(W)}{K_l(W)}\right]^2=l^2\left(\frac{1}{U^2}+\frac{1}{W^2}\right)^2$$

The eigen equations of EH and HE modes can be obtained:

$$\begin{cases}\dfrac{J_l'(U)}{UJ_l(U)}+\dfrac{K_l'(W)}{WK_l(W)}=l\left(\dfrac{1}{U^2}+\dfrac{1}{W^2}\right)\rightarrow(EH)_{lm}\\[2ex]\dfrac{J_l'(U)}{UJ_l(U)}+\dfrac{K_l'(W)}{WK_l(W)}=-l\left(\dfrac{1}{U^2}+\dfrac{1}{W^2}\right)\rightarrow(HE)_{lm}\end{cases}$$

weakly guiding fiber

$$\Rightarrow\begin{cases}\dfrac{UJ_l(U)}{J_{l+1}(U)}+\dfrac{WK_l(W)}{K_{l+1}(W)}=0 & (EH)_{lm}\ \text{mode}\\[2ex]\dfrac{UJ_l(U)}{J_{l-1}(U)}-\dfrac{WK_l(W)}{K_{l-1}(W)}=0 & (HE)_{lm}\ \text{mode}\end{cases}$$

Substitute l-1,m into the eigen equation of EH_{lm}, while substitute l+1,m into the eigen equation of HE_{lm}. Consequently, the two equations become same indicating $EH_{l-1\ m}$ and $HE_{l+1\ m}$ modes are degenerate

weakly guiding fiber

Scalar mode = superposition of vector modes

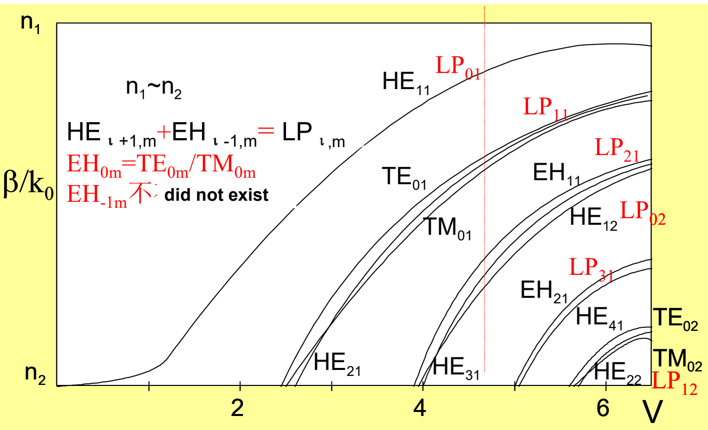
标量模式 = 矢量模式的叠加

$$\begin{pmatrix}E\\H\end{pmatrix}[LP_{lm}]=\begin{pmatrix}E\\H\end{pmatrix}[EH_{l-1,m}]+\begin{pmatrix}E\\H\end{pmatrix}[HE_{l+1,m}],\ l>1$$

$$\begin{pmatrix}E\\H\end{pmatrix}[LP_{1m}]=\begin{pmatrix}E\\H\end{pmatrix}[TE_{0m}]+\begin{pmatrix}E\\H\end{pmatrix}[TM_{0m}]+\begin{pmatrix}E\\H\end{pmatrix}[HE_{2m}],\ l=1$$

$$\begin{pmatrix}E\\H\end{pmatrix}[LP_{0m}]=\begin{pmatrix}E\\H\end{pmatrix}[HE_{1m}],\ l=0$$

weakly guiding fiber



weakly guiding fiber

The eigen solutions of LP mode are demonstrated below indicating the simply field expressions

The eigen solutions along y polarization direction

$$E_y=A\begin{cases}\left[J_l(Ur/a)/J_l(U)\right]\cos l\varphi & (0\le r\le a)\\ \left[K_l(Wr/a)/K_l(W)\right]\cos l\varphi & (r>a)\end{cases}$$

$$H_x=-(An/Z_0)\begin{cases}\left[J_l(Ur/a)/J_l(U)\right]\cos l\varphi & (0\le r\le a)\\ \left[K_l(Wr/a)/K_l(W)\right]\cos l\varphi & (r>a)\end{cases}$$

$$E_x=0$$

$$H_y\approx 0$$

weakly guiding fiber

The eigen solutions along x polarization direction

$$E_x=A\begin{cases}\left[J_l(Ur/a)/J_l(U)\right]\cos l\varphi & (0\le r\le a)\\ \left[K_l(Wr/a)/K_l(W)\right]\cos l\varphi & (r>a)\end{cases}$$

$$H_y=-(An/Z_0)\begin{cases}\left[J_l(Ur/a)/J_l(U)\right]\cos l\varphi & (0\le r\le a)\\ \left[K_l(Wr/a)/K_l(W)\right]\cos l\varphi & (r>a)\end{cases}$$

$$E_y=0$$

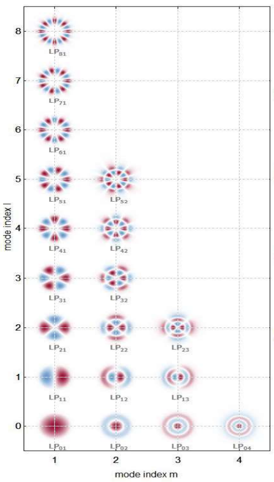
$$H_x\approx 0$$

$$and\ \ \cos l\varphi\Rightarrow\sin l\varphi$$

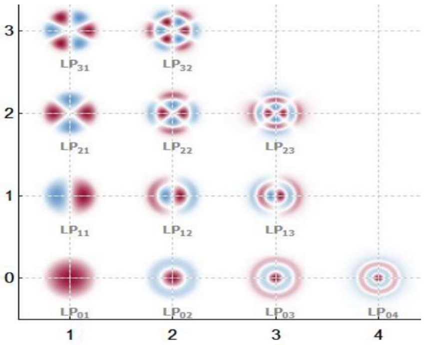
weakly guiding fiber

LP mode	Accurate modes 精确模式
LP_{01}	HE_{11}
LP_{11}	HE_{21}, TE_{01},TM_{01}
LP_{02}	HE_{12}
LP_{21}	HE_{31}, EH_{11}
LP_{31}	HE_{41},EH_{21}
LP_{12}	HE_{22},TE_{02},TM_{02}
LP_{41}	HE_{51},EH_{31}
LP_{03}	HE_{13}
LP_{22}	HE_{32},EH_{12}
LP_{51}	HE_{61},EH_{41}
LP_{32}	HE_{42},EH_{22}
LP_{13}	HE_{23},TE_{03},TM_{03}
LP_{61}	HE_{71},EH_{51}

weakly guiding fiber



y-l, x-m, 读模式时顺序为**LP_*lm***



***LP_lm*(l不等于0) 线偏振模的模斑为径向亮斑m，角向亮斑2l，中心为暗**

Mode group

Based on the approximation of Bessel function

$$U_{lm}^{\infty} \approx \left(l + 2m - 1/2 \right) \pi / 2$$

Same l+2m values lead to a same propagation constant β_{lm}
indicating these LP modes are degenerate.
Set $P=l+2m$, the same P leads to a same propagation constant β_P
Therefore, we defining the LP modes with the same P as a “mode group”

Mode group

For the Pth mode group, how many LP modes there are?

$$\begin{cases} P=6 \quad (l,m)=(0,3),(2,2),(4,1) \quad \text{three types} \\ P=7 \quad (l,m)=(1,3),(3,2),(5,1) \quad \text{three types} \\ P=8 \quad (l,m)=(0,4),(2,3),(4,2),(6,1) \quad \text{four types} \end{cases}$$

Taking $P=6$ as an example，

$(l,m)=(0,3),(2,2),(4,1)$

$$\begin{cases} \mathrm{LP}_{03} \Rightarrow \mathrm{HE}_{13} \\ \mathrm{LP}_{22} \Rightarrow \mathrm{HE}_{32},\ \mathrm{EH}_{12} \\ \mathrm{LP}_{41} \Rightarrow \mathrm{HE}_{52},\ \mathrm{EH}_{31} \end{cases}$$

Review of the knowledge points

- •The development and applications of optical fiber
- •The types of optical fibers
- •Silica Fiber Manufacture
- •Basic equation of optical fiber——waveguide field equation
- •Modes and their basic properties
- •The eigen solutions of guided mode
- •The eigenvalue equation
- •Mode analysis including types, eigenvalues and dispersion curves, etc.
- •Single mode condition
- •Linearly polarized mode and weakly guiding fiber