Chapter Two *Basic fiber theory*

2.1 The basic equations of fiber optics

2.2 Waveguide field equation and eigen solution of guided modes

2.3 Fiber loss and the measurement methods

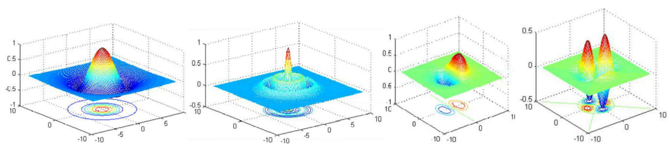
2.4 Fiber dispersion and the measurement methods

Fiber modes

Optical fiber: a kind of waveguide

- There are some certain field distributions which remain unchanged during propagation.
- Such field distributions are called modes of the fiber.
- Each mode corresponds to one solution of Maxwell equations.

TE01;TM01 mode **TE04;TM04 mode** **EH11 mode** **EH21 mode**



Approaches

Two main approaches are commonly used to analyze the mode characteristics in the optical fiber.

(1) Ray theory —— Geometrical Optics

(2) Wave theory —— Wave optics

Wave theory

- Performed by a rigorous analysis
- To solve the Maxwell equations
- Originating from electromagnetic wave thus deriving the field distribution of electromagnetic wave.

Advantage: It is available for analyzing both single- and multi-mode optical fibers with different refractive index profiles because of not making any approximations

Disadvantage: the analysis is comparatively complex

Ray theory

- Fiber core is much larger than optical wavelength λ_0
- The light wave can be regarded as some beams
- Therefore, the ray theory can be employed to investigate the light incidence, propagation path, time delay, etc.

Advantages：

- Simple
- Available for large core (multimode) fiber

Disadvantages：

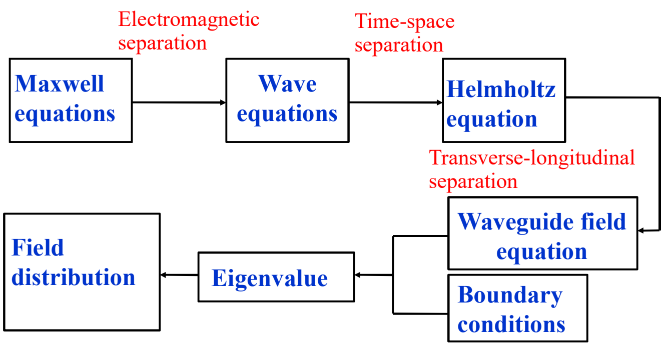
- Unable to explain some cases: mode distribution, cladding modes, mode coupling, etc.
- Significantly large deviations in the case of analyzing single mode fiber (fiber core comparable to λ_0).

Comparison of two above theories

	Ray theory	Wave theory
Condition	$\lambda_0 << d$	all
Subject	Beams	Modes
Basic equation	Ray equation	Waveguide equation
Method	Refraction/ reflection law	Boundary condition
Content	Ray trace	Modal distribution

The path of wave theory

Here we focus on investigating wave theory ?



The path of wave theory

Hamilton operator $$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

divergence $$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

curl $$\nabla\times\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - (\nabla\cdot\nabla)\vec{A}$$

The path of wave theory

a) Maxwell equations

In differential form:

$\nabla\cdot\vec{D} = \rho$ Gauss' Law of electric fields

$\nabla\cdot\vec{B} = 0$ Gauss' Law of magnetic fields

(static electric field and magnetic field)

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$ Faraday' Law of electromagnetic induction

$$\nabla\times\vec{H} = J + \frac{\partial\vec{D}}{\partial t}$$ The Law of total currents

(The relationship between dynamic electric field and magnetic field)

where E、D、B、H are electric field intensity, electric induction intensity, magnetic induction intensity and magnetic field intensity, respectively. ρ is electric charge volume density. J is surface current density.

The path of wave theory

Optical fiber is one type of optical waveguides having the following characteristics:

① no conduction current: J=0

② no free charge: ρ=0

③ linear isotropic 线性各向同性

$$\nabla\cdot\vec{D} = 0 \quad \nabla\cdot\vec{B} = 0 \quad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$$

b) material equations

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_0\varepsilon_r\vec{E}$$

$$\vec{B} = \mu\vec{H} = \mu_0\mu_r\vec{H}$$

ε_0 is the dielectric constant in vacuum （8.8542×10^{-12}Fm^{-1}）

ε_r is the relative dielectric constant

μ_0 is the magnetic conductivity in vacuum（$4\pi\times10^{-7}$Hm^{-1}）

μ_r is the relative magnetic conductivity

The path of wave theory

c) Boundary conditions

$$\vec{n}\cdot(\vec{D}_1-\vec{D}_2)=\rho \qquad \vec{n}\cdot(\vec{D}_1-\vec{D}_2)=0$$
$$\vec{n}\cdot(\vec{B}_1-\vec{B}_2)=0 \qquad \vec{n}\cdot(\vec{B}_1-\vec{B}_2)=0$$
$$\vec{n}\times(\vec{E}_1-\vec{E}_2)=0 \qquad \vec{n}\times(\vec{E}_1-\vec{E}_2)=0$$
$$\vec{n}\times(\vec{H}_1-\vec{H}_2)=\vec{J} \qquad \vec{n}\times(\vec{H}_1-\vec{H}_2)=0$$

At the dielectric surface $\xrightarrow[J=0,\ \rho=0]{}$

- The tangential components of H and E are continuous 切向分量
- The normal components of B and D are continuous

The path of wave theory

Electromagnetic separation

Maxwell equations $\longrightarrow$ wave equations

$$\nabla\cdot\vec{D}=0$$
$$\nabla\cdot\vec{B}=0$$
$$\nabla\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t} \;\rightarrow\; \nabla\times(\nabla\times\vec{E})=-\mu\frac{\partial(\nabla\times\vec{H})}{\partial t}$$
$$\nabla\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t} \;\rightarrow\; \nabla\times(\nabla\times\vec{E})=-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}\ .....(a)$$

The path of wave theory

$$\nabla\times(\nabla\times\vec{E})=\nabla(\nabla\cdot\vec{E})-(\nabla\cdot\nabla)\vec{E}$$
$$=\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}\ ......(b)$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$ **Laplace operator**

$$\nabla\cdot\vec{E}=\nabla\cdot\left(\frac{\vec{D}}{\varepsilon}\right)=\frac{1}{\varepsilon}\nabla\cdot\vec{D}+\vec{D}\cdot\nabla\left(\frac{1}{\varepsilon}\right)=-\vec{E}\cdot\frac{\nabla\varepsilon}{\varepsilon}$$

Therefore

$$\nabla^2\vec{E}+\nabla\left(\vec{E}\cdot\frac{\nabla\varepsilon}{\varepsilon}\right)=\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

The path of wave theory

Similarly, we can derive the wave equation relating to H and obtain

Vector wave equations:

$$\nabla^2\vec{E}+\nabla\left(\vec{E}\cdot\frac{\nabla\varepsilon}{\varepsilon}\right)=\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}$$
$$\nabla^2\vec{H}+\frac{\nabla\varepsilon}{\varepsilon}\times(\nabla\times\vec{H})=\varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2}$$

- The precise equations of electromagnetic wave.
- In fiber, the refractive index variation is significantly moderate, thus $\nabla\varepsilon\approx 0$ .The above vector wave equations can be simplified to scalar wave equations

$$\nabla^2\vec{E}=\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2},\quad \nabla^2\vec{H}=\varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2}$$

$\boxed{\text{The formulas are valid for each component of } E\,(x,y,z,t) \text{ and } H\,(x,y,z,t)}$

Notice: These equations are the approximated results only valid for solving the general problems in fiber.

If more precise analysis is required, the vector equations are needed.

2.1.2 Helmholtz equation

Wave equations $\longrightarrow$ Helmholtz equation

Time-space separation

set $\Phi(x,y,z,t)=\Psi(x,y,z)e^{i\omega t}$

$$\nabla^2\vec{E}=\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}$$
$$\nabla^2\vec{H}=\varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2}$$
scalar wave equations

$$\rightarrow\ \nabla^2\Psi(x,y,z)+k^2\Psi(x,y,z)=0$$

This is the Helmholtz equation

k is the wave number $\quad k=2\pi/\lambda=\omega\sqrt{\varepsilon\mu_0}=n\omega\sqrt{\varepsilon_0\mu_0}=nk_0$,

$k_0=\omega\sqrt{\varepsilon_0\mu_0}=2\pi/\lambda_0$ is the wave number in vacuum

2.1.3 Waveguide field equation

Helmholtz equation $\longrightarrow$ Waveguide field equation

Transverse-longitudinal separation

Light transmission characteristics in optical fibers:

In the transverse direction: standing wave

In the longitudinal direction: travelling wave

Only phase without amplitude variations exists for the field distribution along longitudinal direction

Transverse-longitudinal separation

set $\Psi(x,y,z)=\psi(x,y)e^{-i\beta z}$

Substituting the formula into the Helmholtz equation

$$\nabla^2\Psi(x,y,z)+k^2\Psi(x,y,z)=0$$
$$\rightarrow\ \nabla_t^2\psi(x,y)+\chi^2\psi(x,y)=0$$

$$\nabla_t^2=\nabla^2-\frac{\partial^2}{\partial z^2}$$ Laplacian operator at transverse direction

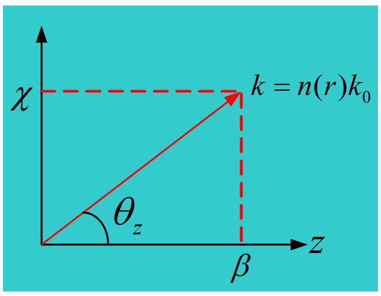
2.1.3 Waveguide field equation

$$\nabla_t^2\psi(x,y)+\chi^2\psi(x,y)=0$$

χ propagation constant at transverse direction

β propagation constant at longitudinal direction

$$\chi^2=k^2-\beta^2=n^2k_0^2-\beta^2$$
$$\beta=nk_0\cos\theta_z$$



2.1.3 Waveguide field equation

The transverse distribution of electric field E and magnetic field H should meet waveguide field equation

$$\nabla_t^2\begin{bmatrix}\vec{E}(x,y)\\ \vec{H}(x,y)\end{bmatrix}+\chi^2\begin{bmatrix}\vec{E}(x,y)\\ \vec{H}(x,y)\end{bmatrix}=0$$

It is the basic equation of wave theory and also a typical eigen equation, the eigenvalue is χ or β.

If the boundary condition is given, the eigen solution and its corresponding eigenvalue can be obtained. In general, the eigen solution is defined as the mode.

2.1.4 Mode and its basic properties

Field distribution （i.e., eigen solution）

- The mode field distribution is uniquely determined by six field components:
  rectangular coordinate system: $E_x, E_y, E_z, H_x, H_y, H_z$
  cylindrical coordinate system: $E_r, E_\varphi, E_z, H_r, H_\varphi, H_z$

Time-space separation:

Maxwell equations:
$$\begin{cases}\nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}\\ \nabla\times\vec{H}=\dfrac{\partial\vec{D}}{\partial t}\end{cases}\Rightarrow\begin{cases}\nabla\times\vec{E}=i\omega\mu_0\vec{H}\\ \nabla\times\vec{H}=i\omega\varepsilon\vec{E}\end{cases}$$

According to：

$$\nabla\times\vec{A}=\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}\qquad \nabla\times\vec{A}=\frac{1}{r}\begin{vmatrix}\vec{e}_r & r\vec{e}_\phi & \vec{e}_z\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z}\\ A_r & rA_\phi & A_z\end{vmatrix}$$

2.1.4 Mode and its basic properties

In rectangular coordinate system, each component can be expressed as the following

x component $\quad \boxed{\dfrac{\partial H_z}{\partial y}-\dfrac{\partial H_y}{\partial z}=i\omega\varepsilon E_x},\ \dfrac{\partial E_z}{\partial y}-\dfrac{\partial E_y}{\partial z}=-i\omega\mu H_x$

y component $\quad \dfrac{\partial H_x}{\partial z}-\dfrac{\partial H_z}{\partial x}=i\omega\varepsilon E_y,\ \boxed{\dfrac{\partial E_x}{\partial z}-\dfrac{\partial E_z}{\partial x}=-i\omega\mu H_y}$

z component $\quad \dfrac{\partial H_y}{\partial x}-\dfrac{\partial H_x}{\partial y}=i\omega\varepsilon E_z,\ \dfrac{\partial E_y}{\partial x}-\dfrac{\partial E_x}{\partial y}=-i\omega\mu H_z$

After space separation

$$\Psi(x,y,z)=\psi(x,y)e^{-i\beta z}\Rightarrow\boxed{\frac{\partial\Psi}{\partial z}=-i\beta\Psi(x,y,z)}$$

Take E_x component as an example

$$i\omega\varepsilon E_x=\frac{\partial H_z}{\partial y}+i\beta H_y=\frac{\partial H_z}{\partial y}+i\beta\frac{1}{-i\omega\mu}\left(-i\beta E_x-\frac{\partial E_z}{\partial x}\right)$$

$$\Rightarrow\chi^2E_x=-i\left(\omega\mu\frac{\partial H_z}{\partial y}+\beta\frac{\partial E_z}{\partial x}\right)$$

2.1.4 Mode and its basic properties

In rectangular coordinate system, the relationship between transverse and longitudinal coordinates

$$\chi^2E_x=-i\left(\omega\mu\frac{\partial H_z}{\partial y}+\beta\frac{\partial E_z}{\partial x}\right)$$
$$\chi^2E_y=-i\left(-\omega\mu\frac{\partial H_z}{\partial x}+\beta\frac{\partial E_z}{\partial y}\right)$$
$$\chi^2H_x=-i\left(-\omega\varepsilon\frac{\partial E_z}{\partial y}+\beta\frac{\partial H_z}{\partial x}\right)$$
$$\chi^2H_y=-i\left(\omega\varepsilon\frac{\partial E_z}{\partial x}+\beta\frac{\partial H_z}{\partial y}\right)$$

$$\Rightarrow\quad E_x=f_1(E_z,H_z),\quad E_y=f_2(E_z,H_z),\quad H_x=f_3(E_z,H_z),\quad H_x=f_4(E_z,H_z)$$

$$\Downarrow\qquad \psi_{x,y}=f\begin{bmatrix}E_z\\H_z\end{bmatrix}$$

2.1.4 Mode and its basic properties

In cylindrical coordinate system

$$\begin{cases}\dfrac{1}{r}\dfrac{\partial E_z}{\partial\phi}-\dfrac{\partial E_\phi}{\partial z}=-i\omega\mu_0H_r\\ \dfrac{\partial E_r}{\partial z}-\dfrac{\partial E_z}{\partial r}=-i\omega\mu_0H_\phi\\ \dfrac{\partial E_\phi}{\partial r}+\dfrac{1}{r}E_\phi-\dfrac{1}{r}\dfrac{\partial E_r}{\partial\phi}=-i\omega\mu_0H_z\end{cases}\qquad\begin{cases}\dfrac{1}{r}\dfrac{\partial H_z}{\partial\phi}-\dfrac{\partial H_\phi}{\partial z}=i\omega\varepsilon E_r\\ \dfrac{\partial H_r}{\partial z}-\dfrac{\partial H_z}{\partial r}=i\omega\varepsilon E_\phi\\ \dfrac{\partial H_\phi}{\partial r}+\dfrac{1}{r}H_\phi-\dfrac{1}{r}\dfrac{\partial H_r}{\partial\phi}=i\omega\varepsilon E_z\end{cases}$$

2.1.4 Mode and its basic properties

In cylindrical coordinate system, the relationship between transverse and longitudinal coordinates

$$\chi^2H_\phi=-i\left(\omega\varepsilon\frac{\partial E_z}{\partial x}+\beta\frac{1}{r}\cdot\frac{\partial H_z}{\partial\phi}\right)$$
$$\chi^2E_r=-i\left(\omega\mu\frac{1}{r}\cdot\frac{\partial H_z}{\partial\phi}+\beta\frac{\partial E_z}{\partial r}\right)$$
$$\chi^2E_\phi=-i\left(-\omega\mu\frac{\partial H_z}{\partial r}+\beta\frac{1}{r}\cdot\frac{\partial E_z}{\partial\phi}\right)$$
$$\chi^2H_r=-i\left(-\omega\varepsilon\frac{1}{r}\cdot\frac{\partial E_z}{\partial\phi}+\beta\frac{\partial H_z}{\partial r}\right)$$

$$\Rightarrow\quad E_r=f_1(E_z,H_z),\quad E_\phi=f_2(E_z,H_z),\quad H_r=f_3(E_z,H_z),\quad H_\phi=f_4(E_z,H_z)$$

$$\Downarrow\qquad \psi_{r,\phi}=f\begin{bmatrix}E_z\\H_z\end{bmatrix}$$

2.1.4 Mode and its basic properties

- The transverse components can be obtained from the longitudinal components, thus the 6 field components can be simplified to 2 longitudinal components
- E_z and H_z satisfy waveguide field equation independently

$$\nabla_t^2\begin{bmatrix}E_z\\H_z\end{bmatrix}+\chi^2\begin{bmatrix}E_z\\H_z\end{bmatrix}=0$$

The relationship between transverse and longitudinal components

If the transverse components are known, the longitudinal components can be solved

In rectangular coordinate system

$$H_z=-i\frac{1}{\beta}\left(\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}\right)=\frac{i}{\omega\mu}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)$$

$$E_z=-i\frac{1}{\omega\varepsilon}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)$$

In cylindrical coordinate system

$$E_z=-i\frac{1}{\omega\varepsilon}\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}(rH_\phi)-\frac{1}{r}\cdot\frac{\partial H_r}{\partial\phi}\right]$$

$$H_z=-i\frac{1}{\beta}\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}(rH_r)+\frac{1}{r}\cdot\frac{\partial H_\phi}{\partial\phi}\right]=\frac{1}{\omega\mu}\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}(rE_\phi)-\frac{1}{r}\cdot\frac{\partial E_r}{\partial\phi}\right]$$

(1) The name of modes

- According to that whether longitudinal component E_z and H_z exist or not, the modes can be named as:
  (1) TEM: $E_z=H_z=0$;
  (2) TE: $E_z=0$, $H_z\neq0$;
  (3) TM: $E_z\neq0$, $H_z=0$;
  (4) HE or EH: $E_z\neq0$, $H_z\neq0$。
- Most modes in fiber are HE (EH) modes, sometimes there are some TE (TM) modes

(2) Longitudinal propagation constant (β)

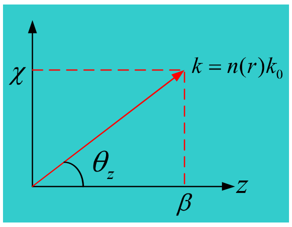
Phase variation rate per unit length along z direction

The component of wave vector k along z direction

$$\beta=\vec{k}\cdot\vec{e}_z=nk_0\cos\theta_z$$

- The values of $\boldsymbol{b}$ are separated and correspond to a group of guided modes
- If different guided modes correspond to a same $\boldsymbol{b}$, they are degenerate

$n_2k_0<\beta<n_1k_0$



(3) Normalized frequency (V)

In a given fiber, the allowed guided modes are determined by the structure parameter which is characterized by normalized frequency V. Larger V leads to more guided modes.

$$V = \frac{2\pi}{\lambda_0} a\sqrt{n_1^2 - n_2^2} = k_0 a \cdot NA = k_0 a n_1 \sqrt{2\Delta}$$

$$NA = \sqrt{n_1^2 - n_2^2} = n_1\sqrt{\frac{(n_1+n_2)(n_1-n_2)}{n_1^{\ 2}}} \approx n_1\sqrt{\frac{2(n_1-n_2)}{n_1}} = n_1\sqrt{2\Delta}$$

Numerical aperture

$$\Delta = \frac{n_1 - n_2}{n_1}$$

Relative refractive index difference

(3) Normalized frequency (V)

Guided mode cutoff: For a certain V, the modes except fundamental mode maybe not allowed and the guided mode will be converted to radiation mode. This V (labeled V_c) is called the cutoff frequency for a specific guided mode.

Different guided modes have different V_c

$V < V_c$ or $\lambda > \lambda_c$ $\Longrightarrow$ Guided mode cutoff

Guided mode far from cutoff: If the guided mode eigen value β approaches $n_1 k_0$, the guided mode field will be restricted tightly within the core. This is called as guided mode far from cutoff condition.

(4) Transverse propagation constant（U,W）

Transverse propagation constant：

$$\chi_1 = \sqrt{n_1^2 k_0^2 - \beta^2} \qquad \text{(core)}$$

$$\chi_2 = \sqrt{n_2^2 k_0^2 - \beta^2} \qquad \text{(cladding)}$$

Transverse constant:

$$U = a\chi_1 = \sqrt{n_1^2 k_0^2 - \beta^2} \cdot a$$

$$W = -ia\chi_2 = \sqrt{\beta^2 - n_2^2 k_0^2} \cdot a$$

Satisfy: $V^2 = U^2 + W^2$

Normalized propagation constant:

$$b = \frac{W^2}{V^2} = \frac{\beta^2 - n_2^2 k_0^2}{n_1^2 k_0^2 - n_2^2 k_0^2}$$