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第七次作业：证明题

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证明题

证明 1

利用：

$$[\hat{a}, \hat{N}] = \hat{a}, [\hat{a}^\dagger, \hat{N}] = -\hat{a}^\dagger, [(\hat{N}+1)^{-1/2}, \hat{N}] = 0$$

证明：$[\hat{N}, \sin\hat{\phi}] = \mathrm{i}\cos\hat{\phi}$.
证明：

由$\cos\hat{\phi}$和$\sin\hat{\phi}$的定义可知：

$$\begin{cases} \cos\hat{\phi} = [(\hat{N}+1)^{-1/2}\hat{a} + \text{h.c.}]/2 \\ \sin\hat{\phi} = [(\hat{N}+1)^{-1/2}\hat{a} - \text{h.c.}]/2i \end{cases} \tag{.1.1}$$

将式(.1.1)代入$[\hat{N}, \sin\hat{\phi}] = \hat{N}\sin\hat{\phi} - \sin\hat{\phi}\hat{N}$对易关系式中，得：

$$\begin{aligned}
&[\hat{N}, \sin\hat{\phi}] = \hat{N}\sin\hat{\phi} - \sin\hat{\phi}\hat{N} \\
&= \frac{1}{2\mathrm{i}}[\hat{N}(\hat{N}+1)^{-1/2}\hat{a} - \hat{N}\hat{a}^\dagger(\hat{N}+1)^{-1/2}] - \frac{1}{2\mathrm{i}}[(\hat{N}+1)^{-1/2}\hat{a}\hat{N} - \hat{a}^\dagger(\hat{N}+1)^{-1/2}\hat{N}] \\
&= \frac{1}{2\mathrm{i}}[\hat{N}(\hat{N}+1)^{-1/2}\hat{a} - \hat{N}\hat{a}^\dagger(\hat{N}+1)^{-1/2} - (\hat{N}+1)^{-1/2}\hat{a}\hat{N} + \hat{a}^\dagger(\hat{N}+1)^{-1/2}\hat{N}] \\
&= \frac{1}{2\mathrm{i}}[(\hat{N}+1)^{-1/2}\hat{N}\hat{a} - \hat{N}\hat{a}^\dagger(\hat{N}+1)^{-1/2} - (\hat{N}+1)^{-1/2}\hat{a}\hat{N} + \hat{a}^\dagger\hat{N}(\hat{N}+1)^{-1/2}] \\
&= \frac{1}{2\mathrm{i}}[(\hat{N}+1)^{-1/2}\hat{N}\hat{a} - (\hat{a}^\dagger\hat{N} + \hat{a}^\dagger)(\hat{N}+1)^{-1/2} - (\hat{N}+1)^{-1/2}(\hat{N}\hat{a} + \hat{a}) + \hat{a}^\dagger\hat{N}(\hat{N}+1)^{-1/2}] \\
&= \frac{1}{2\mathrm{i}}[(\hat{N}+1)^{-1/2}\hat{N}\hat{a} - \hat{a}^\dagger\hat{N}(\hat{N}+1)^{-1/2} - \hat{a}^\dagger(\hat{N}+1)^{-1/2} \\
&\quad - (\hat{N}+1)^{-1/2}\hat{N}\hat{a} - (\hat{N}+1)^{-1/2}\hat{a} + \hat{a}^\dagger\hat{N}(\hat{N}+1)^{-1/2}] \\
&= \frac{1}{2\mathrm{i}}[-\hat{a}^\dagger(\hat{N}+1)^{-1/2} - (\hat{N}+1)^{-1/2}\hat{a}] \\
&= \mathrm{i}\frac{1}{2}[\hat{a}^\dagger(\hat{N}+1)^{-1/2} + (\hat{N}+1)^{-1/2}\hat{a}] \\
&= \mathrm{i}[(\hat{N}+1)^{-1/2}\hat{a} + \text{h.c.}]/2 = \mathrm{i}\cos\hat{\phi}.
\end{aligned} \tag{.1.2}$$

综上，故有$[\hat{N}, \sin\hat{\phi}] = \mathrm{i}\cos\hat{\phi}$成立。

证明 2

利用：

$$\cos\hat{\phi} = (\hat{A}+\hat{A}^\dagger)/2, \hat{A} = (\hat{N}+1)^{-1/2}\hat{a} \tag{.2.1}$$
$$\hat{N}|n\rangle = n|n\rangle, (\hat{N}+1)^{-1/2}|n\rangle = (n+1)^{-1/2}|n\rangle \tag{.2.2}$$
$$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a}+1 = \hat{N}+1 \tag{.2.3}$$
$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \langle n|\hat{a}^\dagger = \langle n-1|\sqrt{n} \tag{.2.4}$$
$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \langle n|\hat{a} = \langle n+1|\sqrt{n+1} \tag{.2.5}$$

证明：

$$\langle\cos\hat{\phi}\rangle = \langle n|\cos\hat{\phi}|n\rangle = 0 \tag{.2.6}$$

$$\langle\cos^2\hat{\phi}\rangle = \langle n|\cos^2\hat{\phi}|n\rangle = \begin{cases} 1/2, n\neq 0 \\ 1/4, n=0 \end{cases} \tag{.2.7}$$

1、式(.2.6)证明：

$$\begin{aligned}\langle\cos\hat{\phi}\rangle &= \langle n|\cos\hat{\phi}|n\rangle \\ &= \frac{1}{2}\langle n|[(\hat{N}+1)^{-1/2}\hat{a}+\hat{a}^\dagger(\hat{N}+1)^{-1/2}]|n\rangle \\ &= \frac{1}{2}\langle n|(\hat{N}+1)^{-1/2}\hat{a}|n\rangle + \frac{1}{2}\langle n|\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle\end{aligned} \tag{.2.8}$$

若$n=0$，则有$\hat{a}|0\rangle = \langle 0|\hat{a}^\dagger = 0$，则式(.2.8)为0；若$n\neq 0$，即$n\geqslant 1$，则有：

$$\begin{aligned}\langle\cos\hat{\phi}\rangle &= \langle n|\cos\hat{\phi}|n\rangle \\ &= \frac{1}{2}\langle n|[(\hat{N}+1)^{-1/2}\hat{a}+\hat{a}^\dagger(\hat{N}+1)^{-1/2}]|n\rangle \\ &= \frac{1}{2}\langle n|(\hat{N}+1)^{-1/2}\hat{a}|n\rangle + \frac{1}{2}\langle n|\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle \\ &= \frac{1}{2}\langle n|(n+1)^{-1/2}\sqrt{n}|n-1\rangle + \frac{1}{2}\langle n-1|\sqrt{n}(n+1)^{-1/2}|n\rangle \\ &= \frac{1}{2}(n+1)^{-1/2}\sqrt{n}\langle n|n-1\rangle + \frac{1}{2}\sqrt{n}(n+1)^{-1/2}\langle n-1|n\rangle = 0\end{aligned} \tag{.2.9}$$

故有$\langle\cos\hat{\phi}\rangle = \langle n|\cos\hat{\phi}|n\rangle = 0$始终成立。

2、式(.2.7)证明：

根据题目已知的关系式，可以得到$\cos^2\hat{\phi}$的表达式为：

$$\begin{aligned}\cos^2\hat{\phi} &= \frac{1}{4}[(\hat{N}+1)^{-1/2}\hat{a}+\hat{a}^\dagger(\hat{N}+1)^{-1/2}][(\hat{N}+1)^{-1/2}\hat{a}+\hat{a}^\dagger(\hat{N}+1)^{-1/2}] \\ &= \frac{1}{4}[(\hat{N}+1)^{-1/2}\hat{a}(\hat{N}+1)^{-1/2}\hat{a} + (\hat{N}+1)^{-1/2}\hat{a}\hat{a}^\dagger(\hat{N}+1)^{-1/2} \\ &+ \hat{a}^\dagger(\hat{N}+1)^{-1}\hat{a} + \hat{a}^\dagger(\hat{N}+1)^{-1/2}\hat{a}^\dagger(\hat{N}+1)^{-1/2}]\end{aligned} \tag{.2.10}$$

根据式(.2.10)，可知：

$$\begin{aligned}\langle n|\cos^2\hat{\phi}|n\rangle = \frac{1}{4}[&\langle n|(\hat{N}+1)^{-1/2}\hat{a}(\hat{N}+1)^{-1/2}\hat{a}|n\rangle + \langle n|(\hat{N}+1)^{-1/2}\hat{a}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle \\ &+ \langle n|\hat{a}^\dagger(\hat{N}+1)^{-1}\hat{a}|n\rangle + \langle n|\hat{a}^\dagger(\hat{N}+1)^{-1/2}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle]\end{aligned} \tag{.2.11}$$

(1) 求解$\langle n|(\hat{N}+1)^{-1/2}\hat{a}(\hat{N}+1)^{-1/2}\hat{a}|n\rangle$：

若$n=0$，则由于$\hat{a}\,|0\rangle=0$，故$\langle n|(\hat{N}+1)^{-1/2}\hat{a}(\hat{N}+1)^{-1/2}\hat{a}|n\rangle=0$；若$n\neq 0$，则有：

$$\begin{aligned}\langle n|(\hat{N}+1)^{-1/2}\hat{a}(\hat{N}+1)^{-1/2}\hat{a}|n\rangle &= \langle n|(n+1)^{-1/2}\hat{a}(\hat{N}+1)^{-1/2}\sqrt{n}|n-1\rangle \\ &= (n+1)^{-1/2}\sqrt{n}\,\langle n|\hat{a}(\hat{N}+1)^{-1/2}|n\rangle \\ &= (n+1)^{-1/2}\sqrt{n}\,\langle n+1|\sqrt{n+1}(n+1)^{-1/2}|n\rangle = 0 \end{aligned} \tag{.2.12}$$

故$\langle n|(\hat{N}+1)^{-1/2}\hat{a}(\hat{N}+1)^{-1/2}\hat{a}|n\rangle$始终为0。

(2) 求解$\langle n|(\hat{N}+1)^{-1/2}\hat{a}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle$：

$$\begin{aligned}\langle n|(\hat{N}+1)^{-1/2}\hat{a}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle &= \langle n|(n+1)^{-1/2}\hat{a}\hat{a}^\dagger(n+1)^{-1/2}|n\rangle \\ &= (n+1)^{-1}\,\langle n|\hat{a}\hat{a}^\dagger|n\rangle \\ &= \frac{\langle n+1|\sqrt{n+1}\sqrt{n+1}|n+1\rangle}{n+1} \\ &= \frac{(n+1)\,\langle n+1|n+1\rangle}{n+1} = 1 \end{aligned} \tag{.2.13}$$

故$\langle n|(\hat{N}+1)^{-1/2}\hat{a}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle$始终为1。

(3) 求解$\langle n|\hat{a}^\dagger(\hat{N}+1)^{-1}\hat{a}|n\rangle$：

若$n=0$，则由于$\hat{a}\,|0\rangle=0$，故$\langle n|\hat{a}^\dagger(\hat{N}+1)^{-1}\hat{a}|n\rangle=0$；若$n\neq 0$，则有：

$$\begin{aligned}\langle n|\hat{a}^\dagger(\hat{N}+1)^{-1}\hat{a}|n\rangle &= \langle n-1|\sqrt{n}(\hat{N}+1)^{-1}\sqrt{n}|n-1\rangle \\ &= n\,\langle n-1|(\hat{N}+1)^{-1}|n-1\rangle \\ &= n(n-1+1)^{-1}\,\langle n-1|n-1\rangle = 1 \end{aligned} \tag{.2.14}$$

故$\langle n|\hat{a}^\dagger(\hat{N}+1)^{-1}\hat{a}|n\rangle$在$n=0$时为0，在$n\neq 0$时为1。

(4) 求解$\langle n|\hat{a}^\dagger(\hat{N}+1)^{-1/2}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle$：

若$n=0$，则由于$\hat{a}\,|0\rangle=\langle 0|\,\hat{a}^\dagger=0$，故$\langle n|\hat{a}^\dagger(\hat{N}+1)^{-1/2}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle=0$；若$n\neq 0$，则有：

$$\begin{aligned}\langle n|\hat{a}^\dagger(\hat{N}+1)^{-1/2}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle &= \langle n-1|\sqrt{n}(\hat{N}+1)^{-1/2}\hat{a}^\dagger(n+1)^{-1/2}|n\rangle \\ &= \sqrt{n}(n+1)^{-1/2}\,\langle n-1|(\hat{N}+1)^{-1/2}\hat{a}^\dagger|n\rangle \\ &= \sqrt{n}(n+1)^{-1/2}\,\langle n-1|n^{-1/2}\sqrt{n+1}|n+1\rangle \\ &= \langle n-1|n+1\rangle = 0 \end{aligned} \tag{.2.15}$$

故$\langle n|\hat{a}^\dagger(\hat{N}+1)^{-1/2}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle$始终为0。

将以上结果代入式(.2.11)：

根据以上讨论，当$n=0$时，有：

$$\begin{aligned}\langle n|\cos^2\hat{\phi}|n\rangle &= \frac{1}{4}[\langle n|(\hat{N}+1)^{-1/2}\hat{a}(\hat{N}+1)^{-1/2}\hat{a}|n\rangle + \langle n|(\hat{N}+1)^{-1/2}\hat{a}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle \\ &\quad + \langle n|\hat{a}^\dagger(\hat{N}+1)^{-1}\hat{a}|n\rangle + \langle n|\hat{a}^\dagger(\hat{N}+1)^{-1/2}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle] \\ &= \frac{1}{4}(0+1+0+0) = \frac{1}{4}. \end{aligned} \tag{.2.16}$$

根据式(.2.12)、(.2.13)、(.2.14)和(.2.15)，当$n\neq 0$时，有：

$$\begin{aligned}\langle n|\cos^2\hat{\phi}|n\rangle &= \frac{1}{4}[\langle n|(\hat{N}+1)^{-1/2}\hat{a}(\hat{N}+1)^{-1/2}\hat{a}|n\rangle + \langle n|(\hat{N}+1)^{-1/2}\hat{a}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle \\ &\quad + \langle n|\hat{a}^\dagger(\hat{N}+1)^{-1}\hat{a}|n\rangle + \langle n|\hat{a}^\dagger(\hat{N}+1)^{-1/2}\hat{a}^\dagger(\hat{N}+1)^{-1/2}|n\rangle] \\ &= \frac{1}{4}(0+1+1+0) = \frac{1}{2}. \end{aligned} \tag{.2.17}$$

综上所述，有式(.2.7)成立。