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第六次作业：证明题

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证明题

证明 1

证明以下对易关系：$[\hat{a},\hat{a}^\dagger]=\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}=1$。

证明：

根据产生算符和湮灭算符的定义，可知：

$$\begin{cases}\hat{a}=\dfrac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x}+\mathrm{i}\hat{p})\\ \hat{a}^\dagger=\dfrac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x}-\mathrm{i}\hat{p})\end{cases}\tag{.1.1}$$

将式(.1.1)代入对易关系式$[\hat{a},\hat{a}^\dagger]=\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}$中，得：

$$\begin{aligned}[\hat{a},\hat{a}^\dagger]&=\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\\&=\frac{1}{2m\hbar\omega}(m\omega\hat{x}+\mathrm{i}\hat{p})(m\omega\hat{x}-\mathrm{i}\hat{p})-\frac{1}{2m\hbar\omega}(m\omega\hat{x}-\mathrm{i}\hat{p})(m\omega\hat{x}+\mathrm{i}\hat{p})\\&=\frac{1}{2m\hbar\omega}(m^2\omega^2\hat{x}\hat{x}-\mathrm{i}m\omega\hat{x}\hat{p}+\mathrm{i}m\omega\hat{p}\hat{x}+\hat{p}\hat{p}-m^2\omega^2\hat{x}\hat{x}-\mathrm{i}m\omega\hat{x}\hat{p}+\mathrm{i}m\omega\hat{p}\hat{x}-\hat{p}\hat{p})\\&=\frac{\mathrm{i}m\omega}{2m\hbar\omega}(2\hat{p}\hat{x}-2\hat{x}\hat{p})\\&=\frac{\mathrm{i}}{\hbar}(\hat{p}\hat{x}-\hat{x}\hat{p})\end{aligned}\tag{.1.2}$$

将对易关系$[\hat{x},\hat{p}]=\mathrm{i}\hbar$代入式(.1.2)，得：

$$\begin{aligned}[\hat{a},\hat{a}^\dagger]&=\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\\&=\frac{1}{2m\hbar\omega}(m\omega\hat{x}+\mathrm{i}\hat{p})(m\omega\hat{x}-\mathrm{i}\hat{p})-\frac{1}{2m\hbar\omega}(m\omega\hat{x}-\mathrm{i}\hat{p})(m\omega\hat{x}+\mathrm{i}\hat{p})\\&=\frac{1}{2m\hbar\omega}(m^2\omega^2\hat{x}\hat{x}-\mathrm{i}m\omega\hat{x}\hat{p}+\mathrm{i}m\omega\hat{p}\hat{x}+\hat{p}\hat{p}-m^2\omega^2\hat{x}\hat{x}-\mathrm{i}m\omega\hat{x}\hat{p}+\mathrm{i}m\omega\hat{p}\hat{x}-\hat{p}\hat{p})\\&=\frac{\mathrm{i}m\omega}{2m\hbar\omega}(2\hat{p}\hat{x}-2\hat{x}\hat{p})\\&=\frac{\mathrm{i}}{\hbar}(\hat{p}\hat{x}-\hat{x}\hat{p})\\&=\frac{\mathrm{i}}{\hbar}(-\mathrm{i}\hbar)=1\end{aligned}\tag{.1.3}$$

故有对易关系$[\hat{a},\hat{a}^\dagger]=\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}=1$成立。

证明 2

利用对易关系$[\hat{a},\hat{a}^\dagger]=\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}=1$证明下式成立(定义$\hat{N}=\hat{a}^\dagger\hat{a}$):

$$\hat{H}=(\hat{a}^\dagger\hat{a}+1/2)\hbar\omega=(\hat{N}+1/2)\hbar\omega$$

证明:

利用湮灭算符和产生算符可以表达位置与动量算符,如下:

$$\begin{cases}\hat{x}=\sqrt{\dfrac{\hbar}{2m\omega}}(\hat{a}^\dagger+\hat{a})\\ \hat{p}=\mathrm{i}\sqrt{\dfrac{m\hbar\omega}{2}}(\hat{a}^\dagger-\hat{a})\end{cases}\tag{.2.1}$$

故$\hat{p}^2$可以表示为:

$$\begin{aligned}\hat{p}^2&=-\frac{m\hbar\omega}{2}(\hat{a}^\dagger-\hat{a})(\hat{a}^\dagger-\hat{a})\\&=-\frac{m\hbar\omega}{2}(\hat{a}^\dagger\hat{a}^\dagger-\hat{a}^\dagger\hat{a}-\hat{a}\hat{a}^\dagger+\hat{a}\hat{a})\end{aligned}\tag{.2.2}$$

$\hat{x}^2$可以表示为:

$$\begin{aligned}\hat{x}^2&=\frac{\hbar}{2m\omega}(\hat{a}^\dagger+\hat{a})(\hat{a}^\dagger+\hat{a})\\&=\frac{\hbar}{2m\omega}(\hat{a}^\dagger\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger+\hat{a}\hat{a})\end{aligned}\tag{.2.3}$$

将式(.2.2)和式(.2.3)代入谐振子的Hamiltonian算符,可得:

$$\begin{aligned}\hat{H}&=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2\\&=-\frac{m\hbar\omega}{4m}(\hat{a}^\dagger\hat{a}^\dagger-\hat{a}^\dagger\hat{a}-\hat{a}\hat{a}^\dagger+\hat{a}\hat{a})+\frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}(\hat{a}^\dagger\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger+\hat{a}\hat{a})\\&=-\frac{\hbar\omega}{4}(\hat{a}^\dagger\hat{a}^\dagger-\hat{a}^\dagger\hat{a}-\hat{a}\hat{a}^\dagger+\hat{a}\hat{a})+\frac{\hbar\omega}{4}(\hat{a}^\dagger\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger+\hat{a}\hat{a})\\&=\frac{\hbar\omega}{4}(2\hat{a}^\dagger\hat{a}+2\hat{a}\hat{a}^\dagger)\\&=\frac{\hbar\omega}{4}(2\hat{a}^\dagger\hat{a}+2\hat{a}^\dagger\hat{a}+2)\\&=\frac{\hbar\omega}{4}(4\hat{a}^\dagger\hat{a}+2)\\&=(\hat{a}^\dagger\hat{a}+1/2)\hbar\omega=(\hat{N}+1/2)\hbar\omega\end{aligned}\tag{.2.4}$$

证明 3

利用对易关系$[\hat{N},\hat{a}^\dagger]=\hat{a}^\dagger$及数学归纳法证明,对于任意正整数$l$,有:

$$\hat{N}(\hat{a}^\dagger)^l\left|n\right\rangle=(n+l)(\hat{a}^\dagger)^l\left|n\right\rangle$$

证明:

根据题目已知的对易关系$[\hat{N},\hat{a}^\dagger]=\hat{a}^\dagger$可以得到:

$$\hat{N}\hat{a}^\dagger=\hat{a}^\dagger\hat{N}+\hat{a}^\dagger\tag{.3.1}$$

(1)当$l=1$时，结合$\hat{N}\left|n\right\rangle = n\left|n\right\rangle$有：

$$\begin{aligned}\hat{N}\hat{a}^\dagger\left|n\right\rangle &= (\hat{a}^\dagger\hat{N}+\hat{a}^\dagger)\left|n\right\rangle \\ &= \hat{a}^\dagger\hat{N}\left|n\right\rangle + \hat{a}^\dagger\left|n\right\rangle \\ &= (\hat{a}^\dagger n+\hat{a}^\dagger)\left|n\right\rangle \\ &= (n+1)\hat{a}^\dagger\left|n\right\rangle \end{aligned} \tag{.3.2}$$

即原式成立。

(2)假设当$l=m\geqslant 1$时，原式也成立，即有下列等式成立：

$$\hat{N}(\hat{a}^\dagger)^m\left|n\right\rangle = (n+m)(\hat{a}^\dagger)^m\left|n\right\rangle \tag{.3.3}$$

则当$l=m+1$时，有：

$$\begin{aligned}\hat{N}(\hat{a}^\dagger)^{m+1}\left|n\right\rangle &= \hat{N}\hat{a}^\dagger(\hat{a}^\dagger)^m\left|n\right\rangle \\ &= (\hat{a}^\dagger\hat{N}+\hat{a}^\dagger)(\hat{a}^\dagger)^m\left|n\right\rangle \\ &= \hat{a}^\dagger\hat{N}(\hat{a}^\dagger)^m\left|n\right\rangle + (\hat{a}^\dagger)^{m+1}\left|n\right\rangle \\ &= \hat{a}^\dagger(n+m)(\hat{a}^\dagger)^m\left|n\right\rangle + (\hat{a}^\dagger)^{m+1}\left|n\right\rangle \\ &= (n+m)(\hat{a}^\dagger)^{m+1}\left|n\right\rangle + (\hat{a}^\dagger)^{m+1}\left|n\right\rangle \\ &= (n+m+1)(\hat{a}^\dagger)^{m+1}\left|n\right\rangle \end{aligned} \tag{.3.4}$$

即原式也成立。

故综上所述，对于任意正整数l，有$\hat{N}(\hat{a}^\dagger)^l\left|n\right\rangle = (n+l)(\hat{a}^\dagger)^l\left|n\right\rangle$成立。