180066F - Nuwan Sriyantha Bandara

EN4553 - Assignment 01

$\boxed{Q1}$ X, Y - random variables ;

a, b - constants ; $a, b \in \mathbb{R}$

g, h - functions

(1) Prove $V[X] = E[X^2] - (E[X])^2$

By def'n ; $V[X] = E[(X-\mu_X)^2]$ (Assuming $E[X^2] < \infty$)

$= E[X^2 - 2X\mu_X + \mu_X^2]$

$= E[X^2] - 2E[X\mu_X] + E[\mu_X^2]$ ($\because E(\cdot)$ is linear)

$= E[X^2] - 2\mu_X E[X] + \mu_X^2$ ($\because \mu_X$ is a constant & $E[c] = c$ when $c \in \mathbb{R}$)

$= E[X^2] - 2E[X]E[X] + (E[X])^2$ ($\because \mu_X = E[X]$ by def'n)

$= E[X^2] - (E[X])^2$ //

(2) $E[aX+b] = aE[X] + b$; $a, b \in \mathbb{R}$ (constants)

By def'n ; $E[aX+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\, dx$ (X - RV)

$= \int_{-\infty}^{\infty} ax f_X(x)\, dx + \int_{-\infty}^{\infty} b f_X(x)\, dx$

$= a \underbrace{\int_{-\infty}^{\infty} x f_X(x)\, dx}_{E[X] \text{ (by def'n)}} + b \underbrace{\int_{-\infty}^{\infty} f_X(x)\, dx}_{=1\ (\because f_X(x) \text{ is a PDF})}$

$= aE[X] + b$ //

(3) Prove $V[aX+b] = a^2 V[X]$

By def'n; $V[X] = E[(X-\mu_x)^2]$ (assuming $V[X] < \infty$ & $a, b \in \mathbb{R}$)

$E[X] = \mu_X$ (given)

$$V[aX+b] = E[[(aX+b) - E[aX+b)]^2]$$

but, $E[aX+b] = aE[X]+b = a\mu_X + b$

$$V[aX+b] = E[[(aX+b) - (a\mu_X+b)]^2]$$

$$= E[[a(X-\mu_X)]^2]$$

$$= E[a^2(X-\mu_X)^2]$$

$$= a^2 E[(X-\mu_X)^2]$$

$$= a^2 V[X] \quad //$$

(def'n; $V[X] = E[(X-\mu_X)^2]$)

(4) Prove $E[X+Y] = E[X] + E[Y]$

Suppose X & Y are jointly continuous with joint probability density f;

By def'n; $E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f(x,y)\, dx\, dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dy\, dx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f(x,y)\, dy}_{f_X(x)}\, dx + \int_{-\infty}^{\infty} y \underbrace{\int_{-\infty}^{\infty} f(x,y)\, dx}_{f_Y(y)}\, dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx + \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$= E[X] + E[Y] \quad //$$

(5) Prove $V[X+Y] = V[X] + V[Y] + 2\,cov(X,Y)$

Assume ; $V(X) < \infty$; $V(Y) < \infty$

By def'n ; $V[X+Y] = E[[(X+Y) - (\mu_X + \mu_Y)]^2]$ where $\mu_X = E[X]$, $\mu_Y = E[Y]$

$$= E[[(X-\mu_X) + (Y-\mu_Y)]^2]$$

$$= E[(X-\mu_X)^2 + (Y-\mu_Y)^2 + 2(X-\mu_X)(Y-\mu_Y)]$$

$$= E[(X-\mu_X)^2] + E[(Y-\mu_Y)^2] + 2E[(X-\mu_X)(Y-\mu_Y)]$$

($\because$ $E[\cdot]$ is linear)

$$= V[X] + V[Y] + 2\,cov(X,Y) \;//$$

($\because$ by def'n ; $V[X] = E[(X-\mu_X)^2]$, $V(Y) = E[(Y-\mu_Y)^2]$

and $cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$)

(6) Prove $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ when X and Y are independant

By def'n ; $E[g(X)h(Y)]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y) \underbrace{f_{XY}(x,y)}_{\text{joint prob. density func.}} \, dx\, dy$$

$g(X) \xrightarrow{PDF} f_X(x)$

$h(Y) \xrightarrow{PDF} f_Y(y)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y) \underbrace{f_X(x) f_Y(y)}_{= f_{XY}(x,y) \text{ when } X \,\&\, Y \text{ are independant}} \, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)f_X(x) \cdot h(y)f_Y(y) \, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)f_X(x) \cdot h(y)f_Y(y) \, dy\, dx$$

$$= \int_{-\infty}^{\infty} g(x)f_X(x) \underbrace{\int_{-\infty}^{\infty} h(y)f_Y(y)\, dy}_{= E[h(Y)] \text{ (by def'n)}} \, dx$$

$$= E[h(Y)]\,E[g(X)] \;// \quad (\because \textstyle\int_{-\infty}^{\infty} g(x)f_X(x) = E[g(x)])$$

(7) $\text{cov}(X,Y) = E[XY] - E[X]E[Y]$

By def'n, $\text{cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$= E[XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y]$

$= E[XY] - \mu_Y E[X] - \mu_X E[Y] + \mu_X\mu_Y$

($\because$ $E[\cdot]$ is linear against addition & $E[c] = c$ when c is a constant)

$= E[XY] - \mu_Y\mu_X - \cancel{\mu_X\mu_Y} + \cancel{\mu_X\mu_Y}$

($\because$ $E[X] = \mu_X$, $E[Y] = \mu_Y$)

$= E[XY] - E[X]E[Y]$ //

(8) Prove $\text{cov}(X,Y) = 0$ when X & Y are independent

$E[XY] = E[X]E[Y]$ when X & Y are independent — (1)

$\text{cov}(X,Y) = E[XY] - E[X]E[Y]$ — (2)

$\therefore$ When X & Y are independant, from (1) & (2);

$\text{cov}(X,Y) = E[XY] - \underbrace{E[X]E[Y]}_{= E[XY]} = 0$ //

$\text{cov}(X,Y)$ vanishes when X & Y are independant //

$\boxed{Q2}$ joint PDF: $f(x_1, x_2) = \begin{cases} k(x_1 + x_2) & , \ 0 \le x_1 \le 1 \text{ and } 0 \le x_2 \le 1 \\ 0 & , \ \text{elsewhere} \end{cases}$

(1) Since $f(x_1, x_2)$ is a PDF;

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1\, dx_2 = 1$$

$$1 = \int_0^1\int_0^1 k(x_1 + x_2)\, dx_1\, dx_2 \qquad (k \text{ is a constant})$$

$$= \int_0^1 \left[\frac{kx_1^2}{2} + kx_1x_2\right]_0^1 dx_2$$

$$= \int_0^1 \left[\frac{k}{2} + kx_2\right] dx_2$$

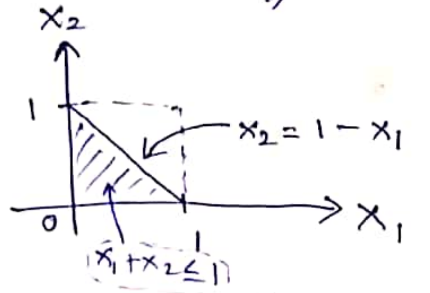
$$= \left[\frac{k}{2}x_2 + \frac{kx_2^2}{2}\right]_0^1$$

$$= \frac{k}{2} + \frac{k}{2} = k$$

$$1 = k \quad //$$

$$\therefore f(x_1, x_2) = \begin{cases} (x_1 + x_2); & 0 \le x_1, x_2 \le 1 \\ 0 & ; \text{ elsewhere} \end{cases}$$

(2)



$$\left\{ \begin{array}{l} \text{Domain of } X_1 = [0,1] \\ \text{Domain of } X_2 = [0,1] \end{array} \right\}$$

$$P(X_1 + X_2 \le 1) = P(X_1 \le 1 - X_2)$$

$$= \int_0^1\int_0^{1-x_2} (x_1 + x_2)\, dx_1\, dx_2$$

$$= \int_0^1 \left[\frac{x_1^2}{2} + x_1x_2\right]_0^{1-x_2} dx_2$$

$$= \int_0^1 \left[\frac{(1-x_2)^2}{2} + (1-x_2)x_2\right] dx_2$$

$$= \int_0^1 \left(\frac{1}{2} - \cancel{x_2} + \frac{x_2^2}{2} + \cancel{x_2} - x_2^2\right) dx_2$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{1}{2}x_2^2\right) dx_2 = \frac{1}{2}\int_0^1 (1 - x_2^2)\, dx_2$$

$$= \frac{1}{2}\left(x_2 - \frac{x_2^3}{3}\right)_0^1 = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3} \quad //$$

(3) Domain of $X_1 = [0,1] \rightarrow x_1 \in [0,1]$

Domain of $X_2 = [0,1] \rightarrow x_2 \in [0,1]$

let marginal density function of X_1 is $f_{X_1}(x_1)$;

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2)\, dx_2$$

$$= \int_0^1 (x_1 + x_2)\, dx_2 = \left(x_1 x_2 + \frac{x_2^2}{2}\right)_0^1$$

$$= \left(x_1 + \tfrac{1}{2}\right)$$

$$f_{X_1}(x_1) = \begin{cases} (x_1 + 0.5) & , \quad 0 \le x_1 \le 1 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Similarly, if marginal density function of X_2 is $f_{X_2}(x_2)$;

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1 X_2}(x_1, x_2)\, dx_1$$

$$= \int_0^1 (x_1 + x_2)\, dx_1 = \left(x_1 x_2 + \frac{x_1^2}{2}\right)_0^1$$

$$= \left(x_2 + \tfrac{1}{2}\right)$$

$$f_{X_2}(x_2) = \begin{cases} (x_2 + 0.5) & , \quad 0 \le x_2 \le 1 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

(4) Let the conditional density function is $f_{X_1|X_2}(x_1 | x_2 = x_2')$;

$$f_{X_1|X_2}(x_1 | x_2 = x_2') = \frac{f_{X_1 X_2}(x_1, x_2')}{f_{X_2}(x_2')}$$

$$= \frac{x_1 + x_2'}{x_2' + 0.5} \quad , \quad 0 \le x_1 \le 1$$

$$\therefore f_{X_1|X_2}(x_1 | x_2 = x_2') = \begin{cases} \dfrac{x_1 + x_2'}{x_2' + 0.5} & ; \quad 0 \le x_1 \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(5) $P(X_1 > 0.75) = 1 - P(X_1 \leq 0.75)$ — ①

Since $f_{X_1}(x_1) = \begin{cases} (x_1 + 0.5) & , 0 \leq x_1 \leq 1 \\ 0 & , \text{elsewhere} \end{cases}$

$\therefore$ from ① ;

$$P(X_1 > 0.75) = 1 - F_{X_1}(0.75) = 1 - \int_{0.0}^{0.75} f_{X_1}(x_1)\, dx_1$$

$$= 1 - \int_0^{0.75} (x_1 + 0.5)\, dx_1$$

$$= 1 - \left(\frac{x_1^2}{2} + 0.5x_1 \right)_0^{0.75}$$

$$= 1 - \left(\frac{0.75^2}{2} + 0.5 \times 0.75 - 0 \right)$$

$$= 0.34375$$

(6) $f_{X_1|X_2}(x_1 | x_2 = x_2') = \begin{cases} \dfrac{x_1 + x_2'}{x_2' + 0.5} & ; 0 \leq x_1 \leq 1 \\ 0 & , \text{otherwise} \end{cases}$

$$P\left(X_1 > 0.75 / X_2 = 0.5\right) = 1 - P\left(X_1 < 0.75 / X_2 = 0.5\right)$$

$$= 1 - \int_0^{0.75} \frac{x_1 + x_2'}{x_2' + 0.5} \Bigg|_{x_2' = 0.5} dx_1$$

$$= 1 - \int_0^{0.75} \frac{x_1 + 0.5}{0.5 + 0.5}\, dx_1$$

$$= 1 - \int_0^{0.75} (x_1 + 0.5)\, dx_1$$

$$= 1 - \left[\frac{x_1^2}{2} + 0.5x_1 \right]_0^{0.75}$$

$$= 1 - \left(\frac{0.75^2}{2} + 0.5 \times 0.75 - 0 \right)$$

$$= 0.34375$$

$\boxed{Q3}$ $X_1 \sim N(\mu_1, \sigma_1)$, $X_2 \sim N(\mu_2, \sigma_2)$

Define a Bernoulli - distributed RV - I such that,

$$Y = IX_1 + (1-I)X_2 \quad \sim (1)$$ (I is independent of both X_1 & X_2)

let $E[I] = p \sim (2)$

(1) $E[Y] = E[IX_1 + (1-I)X_2]$

$= E[IX_1] + E[(1-I)X_2]$ ($\because E[\cdot]$ is linear)

$= E[I]E[X_1] + E[X_2] - E[I]E[X_2]$

($\because$ I is independent of both X_1 & X_2)

$= pE[X_1] + E[X_2] - pE[X_2]$ ($\because$ from (2))

$= pE[X_1] + (1-p)E[X_2]$

$E[Y] = p\mu_1 + (1-p)\mu_2$ // ($\because E[X_1] = \mu_1$, $E[X_2] = \mu_2$)

(2) $E[I^2] = \sum_{i=0}^{1} i^2 P(I=i) = 0 \cdot P(I=0) + 1 \cdot P(I=1) = p$

$\therefore E[I] = E[I^2] = p \sim (3)$ (I - Bernoulli - distributed RV)

$V(Y) = E[Y^2] - [E[Y]]^2$ (as proved in $\boxed{Q01}$)

$= E[(IX_1 + (1-I)X_2)^2] - \{p\mu_1 + (1-p)\mu_2\}^2$

$= E[I^2X_1^2 + (1-I^2)X_2^2 + 2IX_1X_2(1-I)] - \{p\mu_1 + (1-p)\mu_2\}^2$

$= E[I^2X_1^2] + E[(1-I)X_2^2] + 2E[IX_1X_2(1-I)] - \{p\mu_1 + (1-p)\mu_2\}^2$

$= E[I^2]E[X_1^2] + E[X_2^2] - E[I^2]E[X_2^2] - 2E[I^2X_1X_2] + 2E[IX_1X_2] - \{p\mu_1 + (1-p)\mu_2\}^2$

$= E[I^2]E[X_1^2] + E[X_2^2] - E[I^2]E[X_2^2] - 2E[I^2]E[X_1]E[X_2] + 2E[I]E[X_1]E[X_2] - \{p\mu + (1-p)\mu_2\}^2$ ($\because I^2$ is indep. of both X_1 & X_2)

(Assuming X_1 is independent of X_2)

$$(*) = pE[X_1^2] + E[X_2^2] - pE[X_2^2] + \cancel{2p\mu_1\mu_2} - \cancel{2p\mu_1\mu_2} - \{p\mu_1 + (1-p)\mu_2\}^2$$

$(\because E[X_1] = \mu_1,\ E[X_2] = \mu_2,\ E[Z^2] = p)$

$$= p(\sigma_1^2 + \mu_1^2) + \sigma_2^2 + \mu_2^2 - p(\sigma_2^2 + \mu_2^2) - p^2\mu_1^2 - (1-p)^2\mu_2^2 - 2p\mu_1(1-p)\mu_2$$

$\{\because V[X_1] = \sigma_1^2,\ V[X_2] = \sigma_2^2,\ E[X_1^2] = V[X_1] + (E[X])^2\}$

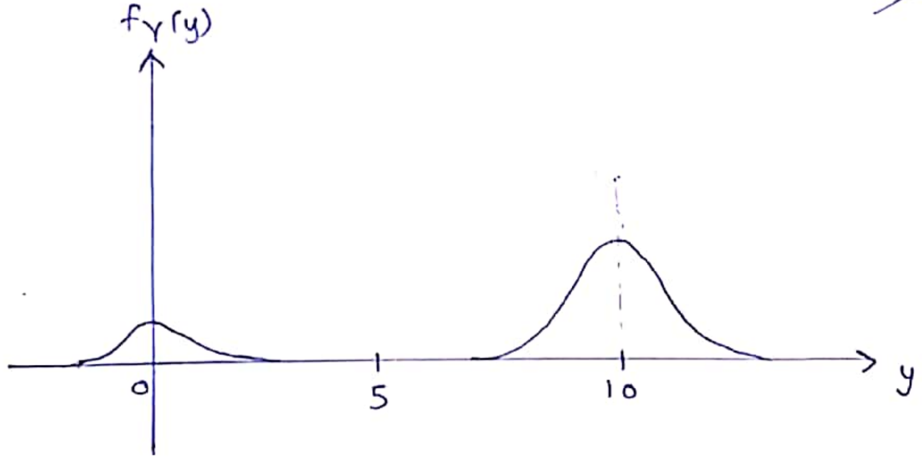
$$= p\sigma^2 + (1-p)\sigma_2^2 + p\mu_1^2 + \mu_2^2 - \mu_2^2 p - p^2\mu_1^2 - (1 + p^2 - 2p)\mu_2^2 - 2p\mu_1(1-p)\mu_2$$

$$= p\sigma_1^2 + (1-p)\sigma_2^2 + p(1-p)\mu_1^2 + p(1-p)\mu_2^2 - 2p(1-p)\mu_1\mu_2$$

$$= p\sigma_1^2 + (1-p)\sigma_2^2 + p(1-p)(\mu_1^2 + \mu_2^2 - 2\mu_1\mu_2)$$

$$V[Y] = p\sigma_1^2 + (1-p)\sigma_2^2 + p(1-p)(\mu_1 - \mu_2)^2$$

(3)



180066F_EN4553_Assignment1_Q3

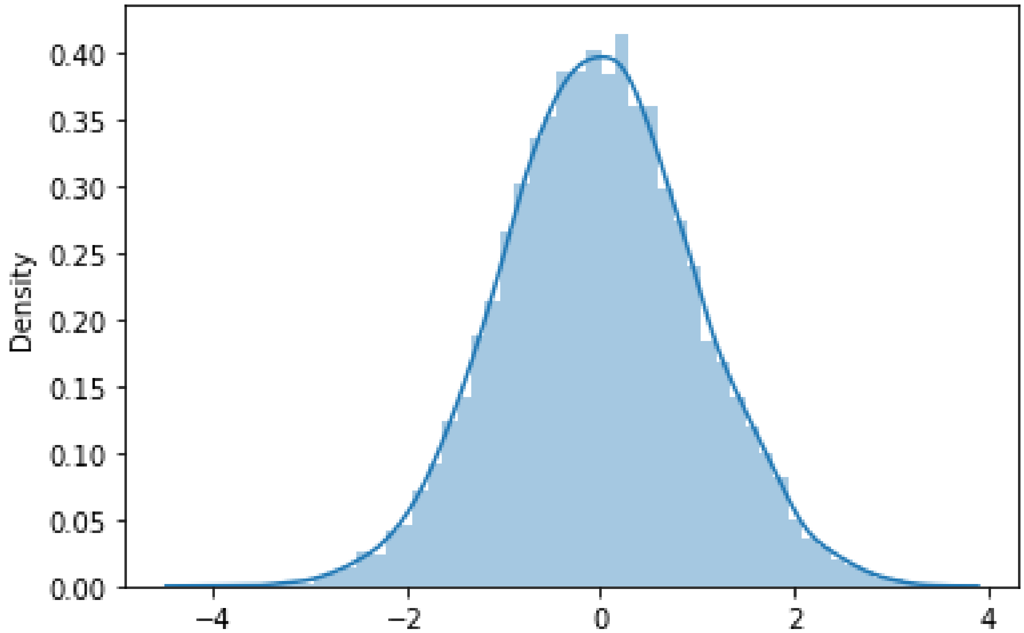
November 27, 2022

[45]:
```
import numpy as np
import seaborn as sns

import warnings
warnings.filterwarnings('ignore')
```

[46]:
```
mu_1 = 0
mu_2 = 10
sigma_1 = 1
sigma_2 = 1
p = 0.2

x1 = sigma_1 * np.random.randn(1,10000) + mu_1  #PDF of X1
x2 = sigma_2 * np.random.randn(1,10000) + mu_2  #PDF of X2

I=np.random.binomial(n=1, p=p, size=(100,1))  #Binomial RV

y = I*x1+(1-I)*x2                             #defined Y
```

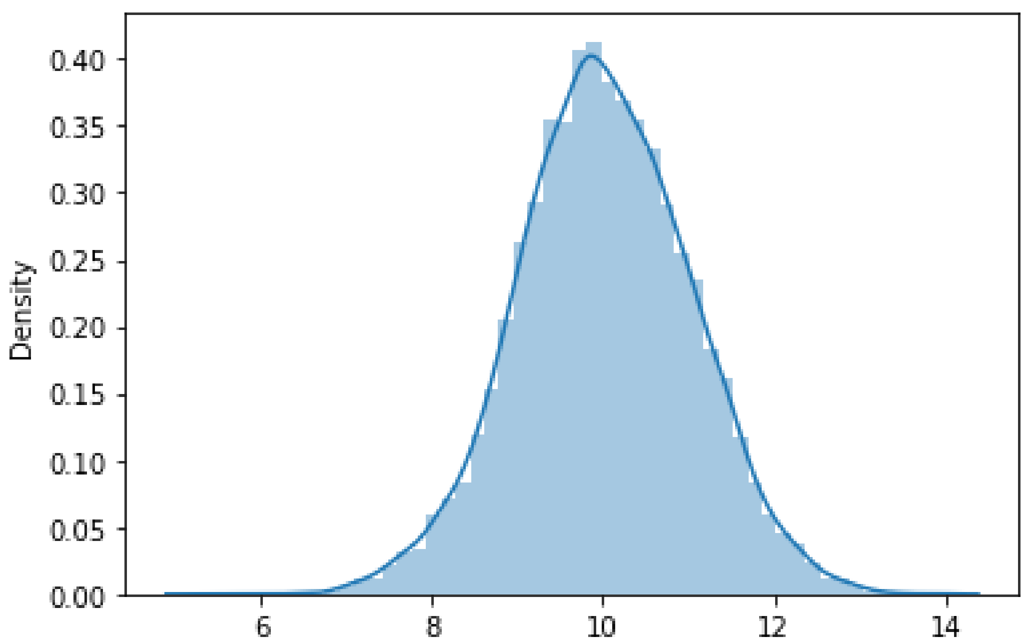
[47]:
```
sns.distplot(x1)
```

[47]:
```
<matplotlib.axes._subplots.AxesSubplot at 0x7fc4a0b68a10>
```



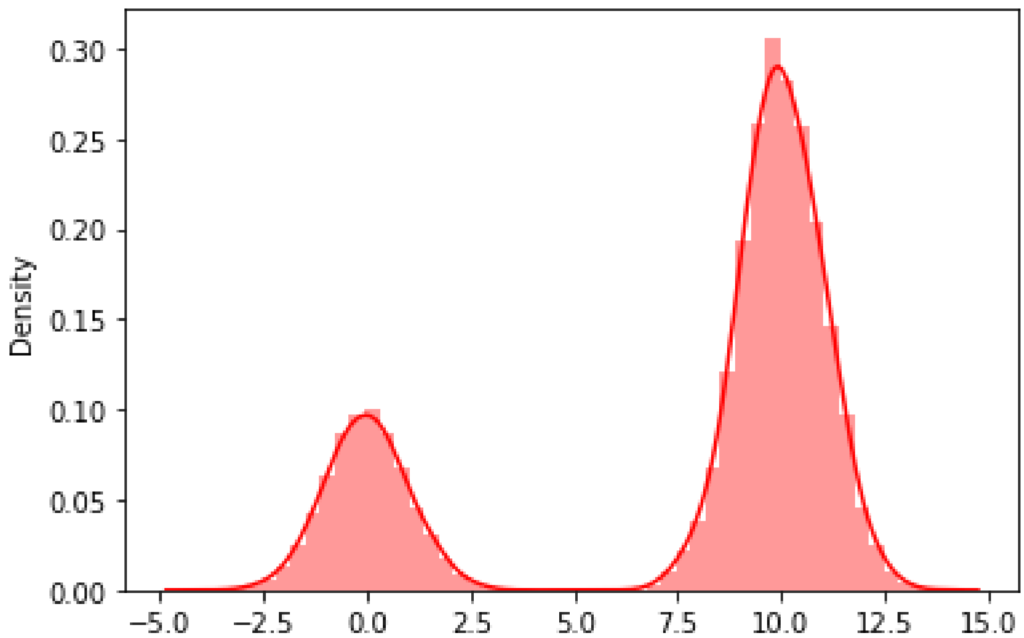
[48]:
```
sns.distplot(x2)
```

[48]: <matplotlib.axes._subplots.AxesSubplot at 0x7fc4a3cec710>



[49]:
```
sns.distplot(y, color ='red')
```

[49]: <matplotlib.axes._subplots.AxesSubplot at 0x7fc4a0d9a810>



[ ]:
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