Prove that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{\theta_n}{n!n} \quad \text{with} \quad 0 < \theta_n < 1.$$

Proof. Write

$$z_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

Now compare z_n with $(1+\frac{1}{n})^n$. We have

$$(1+\frac{1}{n})^n = \sum_{k=0}^{n} C_n^k \frac{1}{n^k} = 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n}).$$

So we have $z_n > (1+\frac{1}{n})^n$.

Now estimate $\frac{1}{k!} - \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n})$, for $k \geq 2$. Here we use the Bernoulli inequality:

$$(1+x_1)(1+x_2)\cdots(1+x_n) \geq 1 + x_1 + x_2 \cdots x_n,$$

if $x_i > -1$ for all $1 \leq i \leq n$ and all x_i have the same sign.

By Bernoulli inequality, we have

$$(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n}) \geq 1 + \sum_{i=1}^{k-1} \frac{i}{n} = 1 + \frac{k(k-1)}{2n}.$$

This implies

$$\frac{1}{k!} - \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n}) \leq \frac{1}{k!} \cdot \frac{k(k-1)}{2n} = \frac{1}{2n(k-2)!},$$

for all $k \geq 2$. Besides,

$$\frac{1}{(k-2)!} = \frac{1}{1 \cdot 2 \cdot 3 \cdots (k-2)} \leq \frac{1}{2^{k-3}}, \quad \text{for all} \quad k \geq 3.$$

This implies

$$0 < z_n - (1+\frac{1}{n})^n \leq \sum_{k=2}^{n} \frac{1}{2n(k-2)!} \leq \frac{1}{2n}(1 + \sum_{k=3}^{n} \frac{1}{2^{k-3}}) \leq \frac{3}{2n}.$$

So we have

$$(1+\frac{1}{n})^n < z_n \leq (1+\frac{1}{n})^n + \frac{3}{2n}.$$

By Sandwich Theorem, we have $z_n \to e$ when $n \to \infty$.

Now suppose $m > n$, we have

$$\begin{aligned} z_m - z_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!} \\ &= \frac{1}{(n+1)!}(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{(n+2)\cdots m}) \\ &< \frac{1}{(n+1)!}(1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+2)^{m-n-1}}) \\ &< \frac{1}{(n+1)!}(1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+2)^{m-n-1}} + \cdots) \\ &= \frac{n+2}{(n+1)!(n+1)} = \frac{1}{n!} \cdot \frac{n+2}{(n+1)^2}. \end{aligned}$$

Now let $m \to \infty$, we get

$$0 < e - z_n \le \frac{1}{n!} \cdot \frac{n+2}{(n+1)^2} \quad \text{for all} \quad n.$$

But

$$\frac{n+2}{(n+1)^2} < \frac{1}{n},$$

so we have $0 < e - z_n < \frac{1}{n!n}$, that is

$$e = z_n + \frac{\theta_n}{n!n} \quad \text{with} \quad 0 < \theta_n < 1.$$

□