June 1, 2024

ML-4360

ex_01_image_formation

Solutions.

1 Pen and Paper

1.1 Homogeneous Coordinates

a) Proof that in homogeneous coordinates, the intersection point $\tilde{x}$ of the two lines$\tilde{I}_1$ and $\tilde{I}_2$ is given by $\tilde{x} = \tilde{I}_1 \times \tilde{I}_2$

$$\begin{cases} \tilde{x}^T \tilde{I}_1 = (\tilde{I}_1 \times \tilde{I}_2)^T \tilde{I}_1 = 0 \\ \tilde{x}^T \tilde{I}_2 = (\tilde{I}_1 \times \tilde{I}_2)^T \tilde{I}_2 = 0 \end{cases} \quad => \begin{cases} \tilde{x} \in \tilde{I}_1 \\ \tilde{x} \in \tilde{I}_2 \end{cases}$$

so $\tilde{x}$ is the intersection of $\tilde{I}_1$ and $\tilde{I}_2$.

b) Similarly, proof that the line that joins two points $\tilde{x_1}$ and $\tilde{x_2}$ is given by $\tilde{I} = \tilde{x_1} \times \tilde{x_2}$.

$$\begin{cases} \tilde{I}^T \tilde{x_1} = (\tilde{x_1} \times \tilde{x_2})^T \tilde{x_1} = 0 \\ \tilde{I}^T \tilde{x_2} = (\tilde{x_1} \times \tilde{x_2})^T \tilde{x_2} = 0 \end{cases} \quad => \begin{cases} \tilde{x_1} \in I \\ \tilde{x_2} \in I \end{cases}$$

so the line $\tilde{I} = \tilde{x_1} \times \tilde{x_2}$ joins the two points $\tilde{x_1}$ and $\tilde{x_2}$.

b) You are given the following two lines:

$$I_1 = \{(x, y)^T \in \mathbb{R}^3 | x + y + 3 = 0\}$$
$$I_2 = \{(x, y)^T \in \mathbb{R}^3 | -x - 2y + 7 = 0\}$$

First, find the intersection point of the two lines by solving the system of linear equations.
Next write the lines using homogeneous coordinates and calculate the intersection point using the cross product.
Do you obtain the same intersection point?

Linear equation:

$$\begin{cases} x + y + 3 = 0 \\ -x - 2y + 7 = 0 \end{cases} \quad => \begin{cases} x = -13 \\ y = 10 \end{cases}$$

the inhomogeneous intersection point is $(-13, 10)^T$.

Homogeneous Coordinates:

First we write the 2D points in Homogeneous Coordinate:

$$\tilde{I}_1 = (1, 1, 3)^T, \ \tilde{I}_2 = (-1, -2, 7)^T$$

The intersection point $\tilde{x}$ is given by $\tilde{x} = \tilde{I}_1 \times \tilde{I}_2$

$$\tilde{x} = \tilde{I}_1 \times \tilde{I}_2 = \begin{bmatrix} 0 & -3 & 1 \\ 3 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 13 \\ -10 \\ -1 \end{bmatrix} \backsimeq \begin{bmatrix} -13 \\ 10 \\ 1 \end{bmatrix}$$

Hence, the two approach are equivalent

d) Write down the line whose normal vector is pointing into the direction $(3,4)^T$ and which has a distance of 3 from the origin.

First we should normalize the normal vector $\mathbf{n}$

$$n' = \begin{bmatrix} \frac{3}{5} \\ \\ \frac{4}{5} \end{bmatrix}$$

$$\tilde{I} = \begin{bmatrix} \mathbf{n'} \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \\ \frac{4}{5} \\ \\ 3 \end{bmatrix}$$

e) What distance from the origin and what (normalized) normal vector does the homogeneous line $\tilde{I} = (2, 5, \frac{\sqrt{29}}{5})$ have?

$$\mathbf{n} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

so that we can divide $\tilde{I}$ by $||n||$

$$\tilde{I} \backsimeq \begin{bmatrix} \frac{2}{\sqrt{29}} \\ \frac{5}{29} \\ \frac{1}{5} \end{bmatrix}$$

As a result, we obtain

$$\mathbf{n} = (\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}})^T$$
$$d = \frac{1}{5}$$

1.2 Transformations

a) Write down the 2×3 translation matrix which maps $(1,2)^T$ onto $(0,3)^T$

$$T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

b) Let's assume that you are given N 2D correspondence pairs

$$(x_i, y_i) = \left(\begin{pmatrix} x_1^i \\ x_2^i \end{pmatrix}, \begin{pmatrix} y_1^i \\ y_2^i \end{pmatrix} \right)$$

Find the 2×3 translation matrix mapping x_i onto y_i which is optimal in the least square sense.

Regard that

$$T = [I \mid t]$$

Define the cost function

$$e = \begin{bmatrix} T\tilde{x_1} - y_1 \\ T\tilde{x_2} - y_2 \\ \vdots \\ T\tilde{x_N} - y_N \end{bmatrix}$$

Then we should minize the function $E = e^T e$

$$\begin{aligned} J_E &= \frac{\partial E}{\partial t} = \frac{\partial e^T}{\partial t^T}\frac{\partial E}{\partial e^T} \\ &= \left[1, 1, \ldots, 1\right] 2e \end{aligned}$$

set $J_E = 0$, we can obtain

$$t_1 = \frac{1}{N}\sum_{i=1}^{n}(y_1^i - x_1^i)$$
$$t_2 = \frac{1}{N}\sum_{i=1}^{n}(y_2^i - x_2^i)$$

c) You are given the following three correspondence pairs:

$$\left(\begin{pmatrix} 0 & 3 \\ 1' & -5 \end{pmatrix}\right)$$
$$\left(\begin{pmatrix} 5 & 7 \\ 7' & 6 \end{pmatrix}\right)$$
$$\left(\begin{pmatrix} 4 & 5 \\ 1' & -4 \end{pmatrix}\right)$$

Using your derived equation, calculate the optimal 2×3 translation matrix T^*.

$$\boldsymbol{t_1} = (3, -6)^T$$
$$\boldsymbol{t_2} = (2, -1)^T$$
$$\boldsymbol{t_3} = (1, -5)^T$$
$$\boldsymbol{t^*} = (2, -4)^T$$
$$\boldsymbol{T^*} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \end{bmatrix}$$

1.3 Camera Projections

a) Calculate the full rank 4×4 projection matrix $\tilde{\boldsymbol{P}}$ for the following scenario:
The camera pose consists of a 90^o rotation around the x axis and translation of $(1, 0, 2)$
The focal lengths f_x, f_y are 100.
The principal point (c_x, c_y) is $(25, 25)$
The camera matrix is

$$\boldsymbol{K} = \begin{bmatrix} 100 & 0 & 25 \\ 0 & 100 & 25 \\ 0 & 0 & 1 \end{bmatrix}$$

From Ridrigues formula, we can easily obtain that:

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{t} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Then the projection matrix is:

$$\begin{aligned} \tilde{\boldsymbol{P}} &= \boldsymbol{K}[\boldsymbol{I} \mid \boldsymbol{t}] \\ &= \begin{bmatrix} 100 & 0 & 25 & 0 \\ 0 & 25 & -100 & 50 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

b) For the previously defined projection, find the world point in inhomogeneous coordinates x_w which corresponds to the projected homogeneous point in screen space $\tilde{x_s} = (25, 50, 1, 0.25)^T$.

$$\begin{aligned} \frac{1}{0.25}\tilde{x_s} &= \tilde{P}\tilde{x_w} \\ => \tilde{x_w} &= \frac{1}{0.25}\tilde{P}^{-1}\tilde{x_s} \end{aligned}$$

so we can obtain

$$\tilde{\boldsymbol{x_w}} = \begin{bmatrix} -0.25 \\ 0.5 \\ -0.25 \\ 0.25 \end{bmatrix} \cong \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

1.4 Photometric Image Formation

a) Write down the thin lens formula and calculate the focus distance z_c in meters for focal length f = 100r
The thin lense formula is

$$\begin{aligned} \frac{1}{z_s} + \frac{1}{z_c} &= \frac{1}{f} \\ z_c &= 2.6m \end{aligned}$$

b) Write the diameter of the circle of confusion c as a function of the focal length f, the image plane distance z_s as well as the distance Δz_s and the f-number N.

$$\frac{c}{\Delta z_s} = \frac{d}{z_s} \tag{1}$$

$$N = \frac{f}{d} \tag{2}$$

From (1)(2), we can conclude that

$$c = \frac{f \Delta z_s}{N z_s}$$