Assignment-8

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Outline

Question

x and y are independent identically distributed normal random variables with zero mean and common variance σ^2, that is, $x \sim N(0, \sigma^2)$, $y \sim N(0, \sigma^2)$ and $f_{xy}(x, y) = f_x(x).f_y(y)$.
Find the pdf of

1. $x = \sqrt{x^2 + y^2}$
2. $w = x^2 + y^2$
3. $u = x - y$

Solution

1. For $z = \sqrt{x^2 + y^2}$
   The case corresponds to a circle with radius z. Thus

$$F_z(z) = \int_{y=-z}^{z} \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f_{xy}(x,y)\,dx\,dy \tag{1}$$

and by differentiation,

$$f_z(z) = \int_{-z}^{z} \frac{z}{\sqrt{z^2-y^2}} \left(f_{xy}(\sqrt{z^2-y^2}, y) + f_{xy}(-\sqrt{z^2-y^2}, y) \right) dy \tag{2}$$

As x and y are zero mean independent Gaussian random variables with common variance σ^2

$$f_z(z) = 2\int_{0}^{z} \frac{z}{\sqrt{z^2-y^2}} \frac{2}{2\pi\sigma^2} \exp^{\frac{-(z^2-y^2+y^2)}{2\sigma^2}}\, dy \tag{3}$$

$$= \frac{2z}{\pi\sigma^2} \exp^{-\frac{z^2}{2\sigma^2}} \int_0^z \frac{1}{\sqrt{z^2 - y^2}} \, dy \tag{4}$$

$$= \frac{2z}{\pi\sigma^2} \exp^{\frac{-z^2}{2\sigma^2}} \int_0^{\frac{\pi}{2}} \frac{z\cos\theta}{z\cos\theta} \, d\theta \tag{5}$$

$$= \frac{2z}{\pi\sigma^2} \exp^{\frac{-z^2}{2\sigma^2}} U(z) \tag{6}$$

2. We know that for $z = \sqrt{x^2 + y^2}$

$$f_z(z) = \int_{-z}^{z} \frac{z}{\sqrt{z^2 - y^2}} \left(f_{xy}(\sqrt{z^2 - y^2}, y) + f_{xy}(-\sqrt{z^2 - y^2}, y) \right) dy \tag{7}$$

As $w = x^2 + y^2$, x and y are independent random variables with zero mean and common variance σ^2.

$$f_w(w) = \int_{-\sqrt{w}}^{\sqrt{w}} \frac{1}{2\sqrt{w - y^2}} \left(2.\frac{1}{2\pi\sigma^2} \exp^{\left(\frac{w - y^2 + y^2}{2\sigma^2}\right)} \right) dy \tag{8}$$

$$= \frac{\exp^{\frac{-w}{2\sigma^2}}}{\pi\sigma^2} \int_0^{\sqrt{w}} \frac{1}{\sqrt{w - y^2}} dy \tag{9}$$

$$= \frac{\exp^{\frac{-w}{2\sigma^2}}}{\pi\sigma^2} \int_0^{\frac{\pi}{2}} \frac{\sqrt{w}\cos\theta}{\sqrt{w}\cos\theta} d\theta \tag{10}$$

$$= \frac{1}{2\sigma^2} \exp^{\frac{-w}{2\sigma^2}} U(w) \tag{11}$$

3. For $u = x - y$
Since linear combinations of jointly Gaussian random variables are Gaussian random Variables. Here $var(u) = var(x) + var(y) = 2\sigma^2$
So $u = x - y \sim N(0, 2\sigma^2)$

$$f_u(u) = \int_{-\infty}^{\infty} f_x(z + y) f_y(y)\, dy \tag{12}$$