

Random Numbers

Nwjwr Khungur Brahma

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1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/Nwjwrbrh/
Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Nwjwrbrh/
Random_numbers/blob/main/1.1/coeffs.h
```

Use the below command in the terminal to run the code

```
cc exrand.c -lm
./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The graph 1.2 is obtained by running the below code

```
wget https://github.com/Nwjwrbrh/
Random_numbers/blob/main/1.2/uni_cdf.
py
```

Run the following command in the terminal to run the code.

```
python3 uni_cdf.py
```

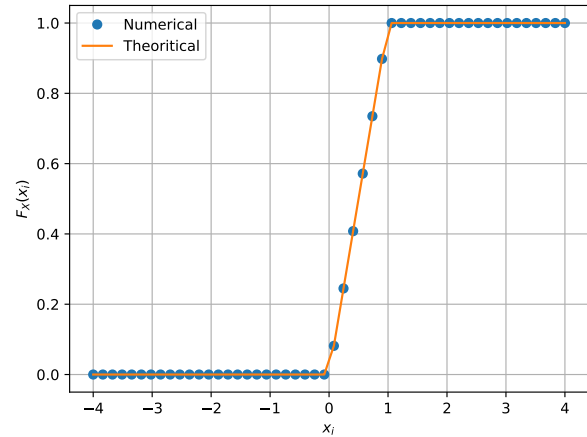


Fig. 1.2: The CDF of U

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: Since U is an uniform random variable distribution, $P_U(x_i) = P_U(x_j) = k, \forall i, j$
So CDF of $P_U(x) = F_U(x)$

$$= \int P_U(x) dx \quad (1.2)$$

$$= \int k dx \quad (1.3)$$

$$\text{wkt } \int_0^1 k dx = 1 \quad (1.4)$$

$$\therefore k = 1 \quad (1.5)$$

$$\therefore F_U(x) = x \quad (1.6)$$

- 1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.7)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.8)$$

Write a C program to find the mean and variance of U .

Solution:

```
wget https://github.com/Nwjwrbrh/
Random_numbers/blob/main/1.4/
mean_var.c
```

Use below command to run file,

```
cc mean_var.c -lm
./a.out
```

running the code gives us Mean=0.500007,
Variance=0.083301

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.9)$$

Solution:

$$dF_U(x) = dx \quad (1.10)$$

$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dx \quad (1.11)$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} \quad (1.12)$$

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \quad (1.13)$$

$$\therefore P_X(x) = 0, \forall x \in (1, \infty) \cap (-\infty, 0) \quad (1.14)$$

$$Var(X) = E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (1.15)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution:

```
wget https://github.com/Nwjwrbrh/
Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Nwjwrbrh/
Random_numbers/blob/main/1.1/coeffs.h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
cc exrand.c -lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in 2.2, Properties of the CDF:

- $\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left\{-\frac{u^2}{2}\right\} du$
- $\lim_{x \rightarrow \infty} \Phi(x) = 1, \lim_{x \rightarrow -\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

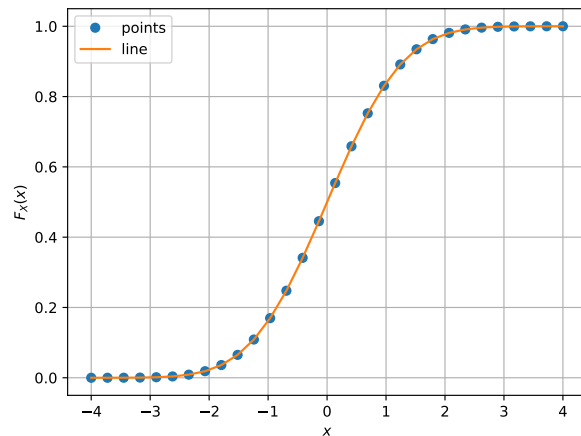


Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in 2.3 using the code below

```
wget https://github.com/Nwjwrbrh/
Random_numbers/blob/main/2.3/gau_pdf
.py
```

Use the below command to run the code:

```
python3 pdf.py
```

Properties of PDF:

- PDF is symmetric about $x = 0$
- graph is bell shaped

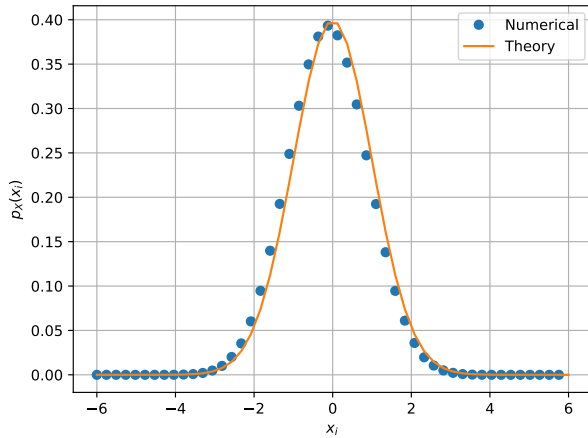


Fig. 2.3: The PDF of X

- mean of graph is situated at the apex point of the bell

2.4 Find the mean and variance of X by writing a C program.

Solution: Running the below code gives Mean = 0.000326, Variance = 1.000906

```
wget https://github.com/Nwjwrbrh/
Random_numbers/blob/main/2.4/
mean_var.c
```

Command used:

```
cc mean_var(gau).c -lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution: Given $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$E[x] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.4)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}} dx \quad (2.5)$$

$$\because x e^{-\frac{x^2}{2}} \text{ is a odd function,} \quad (2.6)$$

$$E[x] = 0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.7)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x (x e^{-\frac{x^2}{2}}) dx \quad (2.8)$$

Using integration by parts:

$$= x \int x e^{-\frac{x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{x^2}{2}} dx \quad (2.9)$$

$$I = \int x e^{-\frac{x^2}{2}} dx \quad (2.10)$$

$$\text{Let } \frac{x^2}{2} = t \quad (2.11)$$

$$\Rightarrow x dx = dt \quad (2.12)$$

$$\Rightarrow \int e^{-t} dt = -e^{-t} + c \quad (2.13)$$

$$\therefore \int x e^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}} + c \quad (2.14)$$

Using (2.14) in (2.9)

$$= -x e^{-\frac{x^2}{2}} + \int e^{-\frac{x^2}{2}} dx \quad (2.15)$$

$$\text{Also, } \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} \quad (2.16)$$

$$\therefore \text{ substituting limits we get, } E[x^2] = 1 \quad (2.17)$$

$$\text{Var}(X) = E[x^2] - (E[x])^2 = 1 - 0 \quad (2.18)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Running the below code generates samples of V from file uni.dat(U).

```
wget https://github.com/Nwjwrbrh/
Random_numbers/blob/main/3.1/V.py
```

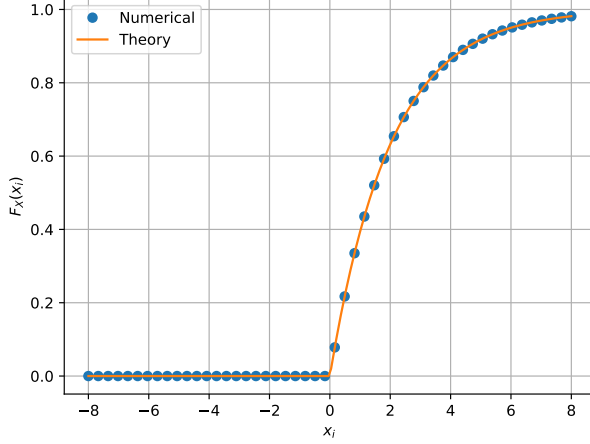


Fig. 3.1: CDF for (3)

Use the below command in the terminal to run the code:

```
python3 V.py
```

Now these samples are used to plot (3.1) by running the below code,

```
wget https://github.com/Nwjwrbrh/
Random_numbers/blob/main/3.1/V_cdf.
py
```

Use the below command to run the code:

```
python3 V_cdf.py
```

3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V \leq x) \quad (3.2)$$

$$= P(-2\ln(1 - U) \leq x) \quad (3.3)$$

$$= P(1 - e^{-\frac{x}{2}} \geq U) \quad (3.4)$$

$$P(U < x) = \int_0^x dx = x \quad (3.5)$$

$$\therefore P(1 - e^{-\frac{x}{2}} \geq U) = 1 - e^{-\frac{x}{2}}, \forall x \geq 0 \quad (3.6)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Run the below code to generate T.dat

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/4.1/
T_gen_dat.c
```

Run the command below in the terminal

```
cc T_gen_dat.c -lm
./a.out
```

4.2 Find the CDF of T .

$$F_T(t) = P(T < t) \quad (4.2)$$

$$= P(U_1 + U_2 < t) \quad (4.3)$$

we know that $0 \leq U_1 \leq 1$ and $0 \leq U_2 \leq 1$

$\therefore 0 \leq U_1 + U_2 \leq 2$, so

$\forall t > 2, P(U_1 + U_2 < t) = 1$

$\forall t < 0, P(U_1 + U_2 < t) = 0$

for $0 \leq t \leq 2$ let us split it into 2 cases, for $0 \leq t \leq 1$ and $1 < t \leq 2$

The above figure is produced by the following code

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/4.2/T_plot.
py
```

Run the following command in the terminal to run the code

```
python3 T_plot.py
```

From Fig (??)

$$P(U_1 + U_2 < t, 0 \leq t \leq 1) = \frac{\Delta(Eof)}{\Delta(AEOD)} \quad (4.4)$$

$$= \frac{t^2}{2} \quad (4.5)$$

$$P(U_1 + U_2 < t, 1 \leq t \leq 2) = \frac{\Delta(ABC)}{\Delta(AEOD)} \quad (4.6)$$

$$= 1 - \frac{(2-t)^2}{2} \quad (4.7)$$

$$\therefore F_T(t) = P(U_1 + U_2 < t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \quad (4.8)$$

4.3 Find the PDF of T .

Solution:

$$P_T(t) = \frac{d(F_T(t))}{dt} \quad (4.9)$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t \leq 2 \\ 0 & t > 2 \end{cases} \quad (4.10)$$

4.4 Find the theoretical expressions for the PDF and CDF of T .

Solution:

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t \leq 2 \\ 0 & t > 2 \end{cases} \quad (4.11)$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \quad (4.12)$$

4.5 Verify your results through a plot.

Solution: Run the below code to get the cdf

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/4.5/T_cdf.
py
```

Use the following command in the terminal to run the code

```
python3 T_cdf.py
```

Run the below code to get the pdf

```
https://github.com/Pranavb060504/
Random_numbers/blob/main/4.5/T_pdf.
py
```

Use the following command in the terminal to run the code

```
python3 T_pdf.py
```