Lecture notes
Einführung in die Logik 2024W

This is a summary of the material discussed in the lecture "Mathematische Logik". It is still a work in progress and there **may be mistakes** in this work. If you find any, feel free to let me know and I will correct them
The content of this script relies on [EE01], [Van98] and [Kri98] Dieses Skript ist noch nicht vollständig und wird regelmäßig aktualisiert.

Contents

Chapter 1

Propositional logic

↰ 01.10.2024

The definitions, lemmata, propositions and theorems as well as the notes in this chapter are sourced from [EE01, chapter 1].

Language

Definition 1.1. Language of PL: The **Language** of Propositional logic is a set containing

- logical symbols: consisting of the **sentential connective** symbols $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ and parenthesis $($, $)$
- non-logical symbols: $A_1, A_2, A_3, \ldots$ (also called sentential **atoms**, variables)

atoms

from which we assume (for unique readability) that no symbol is a finite sequence of any other symbols.

Note :

1. The role of the logical symbols doesn't change, the sentential atoms we see as variables, they function as placeholders or variables.
2. we assumed the set of non-logical symbols is countable, for most of our conclusions you could use any set of propositional atoms of any size

expression

Definition 1.2. Expression / prop. sentence: An **expression**(exp.) is any finite sequence of symbols from the Language of propositional logic. We define **grammatically correct exp.** recursively by:

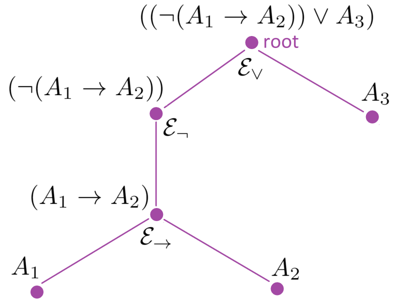
1. every propositional atom is a propositional sentence
2. if α, β are propositional sentences, then also $(\neg\alpha), (\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \rightarrow \beta), (\alpha \leftrightarrow \beta)$
3. nothing else (in particular $\varnothing$ is not a propositional fla.)

propositional sentences

and call them **propositional sentences** or **propositional fla.** . Equivalently stated every propositional sentence is built up by applying finitely many formula building operations on atoms and the propositional sent. returned from building operations.

$$\mathcal{E}_\neg, \mathcal{E}_\neg(\alpha) := (\neg\alpha) \text{ for any propositional fla. } \alpha \text{ and similarly for } \mathcal{E}_\wedge, \mathcal{E}_\vee \mathcal{E}_\rightarrow, \mathcal{E}_\leftrightarrow$$

This allows us to symbolize the **expression tree** (Here for example for $((\neg(A_1 \rightarrow A_2)) \vee A_3)$)



We will return to these construction trees in 1.2, where we answer the question of what truth value a given propositional sentence might have.

Definition 1.3. Construction sequence: Given a propositional sentence α a **construction sequence** of α is a finite sequence $\langle \alpha_1, \dots \alpha_{n-1}, \alpha \rangle$ such that for all $i \leq n$ the following holds

construction sequence

- α_i is a sentential atom
- or $\alpha_i = \mathcal{E}_\neg(\alpha_j)$ for some $j < i$
- or $\alpha_i = \mathcal{E}_\square(\alpha_j, \alpha_k)$ for some $j, k < i$ and $\square \in \{\wedge, \vee, \to, \leftrightarrow\}$

Definition 1.4. Closedness of a set: Let S be a set. We say S is **closed** under an n-ary operational symbol f iff for all $s_1, s_2, \dots s_n \in S$ it holds $f(s_1, s_2, \dots s_n) \in S$

closure

Induction principle: Suppose S is a set of propositional sentences containing all propositional atoms and closed under the 5 formula building operations, then S is the set of all propositional sentences.

Proof. let $PS =$ set of all propositional sent.

$S \subseteq PS$: is clear

$S \supseteq PS$: let $\alpha \in PS$ then α has a construction seq. $\langle \alpha_1, \dots \alpha_{n-1}, \alpha \rangle$ and $\alpha_1 \in S$. Let's assume that for $i \leq k < n$ each α_i is in S. Then α_{k+1} is either an atom and therefore in S or its obtained by one of the formula building operations and therefore $\alpha_{k+1} \in S$

□

1.1 TRUTH ASSIGNMENTS

↰ 03.10.2024

The interpretation of a propositional atom is either true or false, denoted by 0/1 or T/F or $\top/\bot$. A truth assigment is simply any map $\nu : S \mapsto \{0,1\}$, where S is a map of propositional atoms. Our goal is going to be to extend any truth assigment v to a function $\overline{v} : \overline{S} \mapsto \{0,1\}$, where $\overline{S}$ is the closure of S under the 5 fla. building operations.

Definition 1.5. Truth assignment: Let $\{0,1\}$ be the set of truth values. A truth assignment (TA) for a set S of propositional atoms is a map $\nu : S \to \{0,1\}$

Truth assigment
TA

We now want to extend ν to $\overline{\nu} : \overline{S} \to \{0,1\}$, where $\overline{S}$ is the closure of S under the 5 fla. building operations such that for all propositional atoms $A \in S$ and propositional formulas α, β in $\overline{S}$

1. $\overline{\nu}(A) = \nu(A)$
2. $\overline{\nu}(\neg\alpha) = 1 - \overline{\nu}(\alpha)$
3. $\overline{\nu}(\alpha \wedge \beta) = \begin{cases} 1 & \text{iff } \overline{\nu}(\alpha) = 1 = \overline{\nu}(\beta) \\ 0 & \text{otherwise} \end{cases}$
4. $\overline{\nu}(\alpha \vee \beta) = \begin{cases} 1 & \text{iff } \overline{\nu}(\alpha) = 1 \text{ or } \overline{\nu}(\beta) = 1 \\ 0 & \text{otherwise} \end{cases}$
5. $\overline{\nu}(\alpha \to \beta) = \begin{cases} 1 & \text{iff } \overline{\nu}(\alpha) = 0 \text{ or } \overline{\nu}(\beta) = 1 \\ 0 & \text{otherwise} \end{cases}$
6. $\overline{\nu}(\alpha \leftrightarrow \beta) = \begin{cases} 1 & \text{iff } \overline{\nu}(\alpha) = \overline{\nu}(\beta) \\ 0 & \text{otherwise} \end{cases}$

We also want the extention to be unique, that is

Theorem 1.1.1. Unique readability: *For all TA ν for a set S $\exists!\overline{\nu} : \overline{S} \to \{0,1\}$ satisfying the above properties*

We will prove this later

Definition 1.6. Satisfaction: A TA ν satisfies a propositional sent. α if $\overline{\nu}(\alpha) = 1$ (that is, provided that everery atom of α is in the domain of ν). We call α satisfiable if there exists a TA that satisfies it. satisfy satisfiable

Definition 1.7. Tautological implication: Let Σ be a set of propositional sent. and α a propositional sent. then we say: Σ tautologically imlies α if for all TA that satisfy Σ, α is also satisfied and we write $\Sigma \models \alpha$. If $\Sigma = \{\beta\}$, we simply write $\beta \models \alpha$ If $\Sigma = \varnothing$ then α is called a **tautology** and we write $\models \alpha$ instead of $\varnothing \models \alpha$ taut. implication $\models$
α, β are called **tautologically equivalent** iff $\alpha \models \beta$ and $\beta \models \alpha$, we then write $\alpha \dashv\models \beta$

Note : In other words, tautological implication $\Sigma \models \alpha$ means that you can not find a TA, that satisfy all members of Σ but not α. A tautology is satisfied by every TA. Suppose there is no TA that satisfies Σ, then we have $\Sigma \models \alpha$ for every propositional sent. α

Example 1.1. $\{\neg A \vee B\} \dashv\models A \to B$

Note : In order to check if a propositional sent. is satisfiable we need to check 2^N TAs, where $N = \#$ of atoms. It is unknown if this can be done by an algorithm in polynomial time. Answering this would settle the debate whether $P = NP$
However we can find a way to reduce satisfiability of an infinite set Σ of propositional sent. to all finite subsets of Σ. There later will be a more elementary proof of the compactness theorem, this proof is not part of the exam.

Theorem 1.1.2. Compactness theorem: *Let Σ be an infinite set of propositional sent. such that*

$$\forall \Sigma_0 \subseteq \Sigma, \Sigma_0 \textit{ finite } \exists \textit{ TA satisfying every member of } \Sigma_0 \qquad \text{(finite satisfiability)}$$

then there is a TA satsfying every member of Σ.

Proof. using topology: We have our infinite set of propositional sent. which satisfies above condition. One way to look at TA is as a sequence of 0 and 1. Let $\mathcal{A} = \{A_0, A_1, \dots\}$ be the set of all propositional atoms. We are going to identify the truth assignments on $\mathcal{A}$ with elements in $\{0,1\}^{\mathcal{A}} := \{f : \mathcal{A} \to \{0,1\}\}$ (the set of all TAs) This is a topological space with product topology, on which the basic open sets (called cylinders) are: $U \subseteq \{0,1\}^{\mathcal{A}}$ is a cylinder, such that $p_n(U) = \{0,1\}$for all but finite many n, where p_n is the n-th projection. This means U is a cylinder if the truth values of its elements are at finitely many places fixed, and are arbitrary on everything else.
Note: These basic open sets are also closed. The open sets are unions of basic open sets. The idea is to use Tychonoffs Theorem which tells us that $\{0,1\}^{\mathcal{A}}$ is compact. i.e. the intersection of a family of closed subsets w/ the finite intersection property (FIP) is non-empty. Finite intersection property means the intersection of finitely many sets is non-empty.
For $\alpha \in \Sigma$ let $T_\alpha \subseteq \{0,1\}^{\mathcal{A}}$ be the set of TA that satisfy α. This T_α is a finite union of cylinders, hence T_α is closed. For the family $\{T_\alpha : \alpha \in \Sigma\}$ of closed sets we have (FIP). Tychonoff tells us, that $\bigcap_{\alpha \in \Sigma} T_\alpha \neq \varnothing$ so there is a TA satisfying Σ. □

For a list of tautologies: useful might be book p. 26-27

1.2 A PARSING ALGORITHM

To prove Theorem 1.1.1 We essentially need to show that we have enough parenthesis to make the reading of a propositional sent. unique. That is given a TA v there is at most one truth value we can assign to a propositional sent.

Lemma 1.2.1. *Every propositional sent. has the same number of left and right parenthesis.*

Proof. Let $M =$ set of propositional sent. w/ # left parenthesis = # right parenthesis and $PS =$ set of all propositional sent. We have $M \subseteq PS$. Since atoms have no parenthesis, they are in M. we just need to show that M is closed under the 5 construction operations.
$\mathcal{E}_\neg = (\neg\alpha) \dots$ □

Lemma 1.2.2. *No proper initial segment of a propositional sent. is itself a propositional sent.*

Proof. Let $\alpha = \alpha_1\alpha_2\ldots\alpha_n$ be a propositional sent. By proper initial segment we understand $\beta = \alpha_1\ldots\alpha_i$ for $1 \leq i < n$. We will prove that every proper initial segment has an excess of left parenthesis, then we use the previous lemma. Let PS be the set of all prop. sentences and $\mathrm{lp}, \mathrm{rp} : PS \to \mathbb{N}_0$ functions that compute the number of left and right parenthesis of a given formula.

$$PF = \{\alpha = \alpha_1\alpha_2\ldots\alpha_n \in PS \,:\, \forall i < n : \mathrm{lp}(\alpha_1\alpha_2\ldots\alpha_i) \neq \mathrm{rp}(\alpha_1\alpha_2\ldots\alpha_n)\}$$

we will prove that $PF = PS$.
Let $\alpha \in PF$. By induction on the fla. building operations

- Atoms: since the empty sequence is not a propositional sent. they have no proper initial segment.
- If the above is true for α, β then the proper initial segments of $(\neg\alpha)$ are of the form

  $(\neg\alpha$
  $(\neg\alpha'$ where α' is a propper initial segment of α
  $($ or
  $(\neg$

  Therefore $\mathcal{E}_\neg$ preserves this property and under $\mathcal{E}_\wedge, \mathcal{E}_\vee, \mathcal{E}_\to, \mathcal{E}_\leftrightarrow$ this is also the case.

□

Parsing algorithm

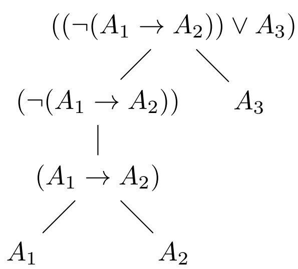
We now give a parsing algorithm procedure. For input we take some expression τ and the algorithm will determine if τ is a propositional sent. If so, it will generate a unique construction tree (in form of a rooted tree) for τ. (i.e. the construction tree gives us unique readability) That there is a unique way to perform the algorithm is implied by Lemma 1.2.2

0. create the root and label it τ
1. HALT if all leaves are labled w/ propositional atom and return: "τ is a propositional sent."
2. select a leaf of the graph which is not labled w/ propositional atom
3. if the first symbol of label under consideration is not a left parenthesis, then halt and return: "τ is not a propositional sent."
4. if the second symbol of the label is "$\neg$" then GOTO 6.
5. scan the expression from left to right
   if we reach a proper initial segment of the form "$(\beta$" where $\#lp(\beta) = \#rp(\beta)$ and β is followed by one of the five sentential connectives $\wedge, \vee, \to, \leftrightarrow$ and the remainder of the expression is of the form $\beta')$, where $\#lp(\beta') = \#rp(\beta')$

   Then: create two child nodes (left,right) to the selected element and label them (left $:= \beta$, right $:= \beta'$) GOTO 1.
   Else: HALT and return "τ is not a propositional sent."
6. if the expression is of the form $(\neg\beta)$ where $\#lp(\beta) = \#rp(\beta)$

   Then: construct one childnode and label it β and GOTO 1.
   Else: HALT and return: "τ is not a propositional sent."

Example 1.2. The parsing algorithm applied to $((\neg(A_1 \to A_2)) \vee A_3)$ returns the following construction tree.



Correctness of the parsing algorithm

- The algorithm always halts, because a child's label is shorter than the label of a parent.
- If the algorithm halts with the conclusion that τ is a propositional sent. then we can prove inductively (starting from the leaves) that each label is a propositional sent
- Unique way to make choices in the algorithm: in particular β, β' in step 5. If there was a shorter choice for β it would be a proper initial segment of β but such propositional sent. cannot exist. (This also works under the assumption that a longer choice exists).
- rejections are made correctly

Back to proving the existence and uniqueness of $\overline{\nu}$ in Theorem 1.1.1. Let α be a propositional sent. of $\overline{S}$. We apply the parsing algorithm to α to get a unique construction tree For the leaves, use ν go get the truth values then work our way up using the conditions (1-6) in Definition 1.5.

1.3 INDUCTION AND RECURSION

↰ 10.10.2024

Generalization of induction principle:

Let U be a set and $B \subseteq U$ our initial set. $\mathcal{F} = \{f, g\}$ a class of functions containing just f and g, where

$$f : U \times U \to U, \qquad g : U \to U$$

We want to construct the smallest subset $C \subseteq U$ such that $B \subseteq C$ and C is closed under all elements of $\mathcal{F}$.

Definition 1.8. Closedness, Inductiveness: We say $\mathcal{S} \subseteq U$ is

- **closed** under f and g iff for all $x, y \in \mathcal{S}$ it holds $f(x,y) \in \mathcal{S}$ and $g(x) \in \mathcal{S}$ closed
- **inductive** if $B \subseteq \mathcal{S}$ and $\mathcal{S}$ is closed under $\mathcal{F}$ inductive

One way is from the top down

$$C^* := \bigcap_{\substack{B \subseteq S \\ \text{inductive}}} S$$

Another is from bottom up: We call $C_1 := B$,

$$C_i := C_{i-1} \cup \{f(x,y) : x, y \in C_{i-1}\} \cup \{g(x) : x \in C_{i-1}\}$$

and $C_* := \bigcup_{n \geq 1} C_n$ Exercise: show that $C^* = C_* =: C$.

Example 1.3. 1. Let U be the set of all expressions, B the set of atoms and $\mathcal{F} = \{\mathcal{E}_\square : \square \in \{\neg, \wedge, \vee, \to, \leftrightarrow\}\}$ Then C would be the set of all propositional formulas.

2. Let U be $\mathbb{R}$, B the set containing 0 and $\mathcal{F} = \{S\}$, $S(x) = x + 1$ Then C would be the set of the natural numbers.

Induction principle

C generated from B by use of elements of $\mathcal{F}$ if $S \subseteq C$ such that $B \subseteq S$ and S is closed under all elements of $\mathcal{F}$, then $S = C$

Proof. $S \subseteq C$ is clear. S is inductive, so $C \subseteq S$. □

Question: under what conditions do we get "generalized unique readability?" The goal would be to define a function on C recursively i.e. to have rules for computing $\overline{h}(x)$ for $x \in B$ with some rules of computing $\overline{h}(f(x,y))$ and $\overline{h}(g(x))$ from $\overline{h}(x)$ and $\overline{h}(y)$.

Example 1.4. Suppose that G is some additive group, generated from B (the set of generators), $h = B \to H$ where $(H, \cdot, ^{-1}, 1)$ a group. When is there an extention $\overline{h}$ of h s.th. $\overline{h} : G \to H$ is a grouphomomorphism.

- $\overline{h}(0) = 1$
- $\overline{h}(a + b) = \overline{h}(a) \cdot \overline{h}(b)$
- $\overline{h}(-a) = \overline{h}(a)^{-1}$

This is not always possible. **Note:** that it is possible if G is generated freely by the elements of B and the set of atoms is independent (one element of B cannot be generated in finitely many steps by other elements of B).

Definition 1.9. **Freely generated set:** C is **freely generated** from B by f, g if

freely generated

- C is generated from B by f, g
- $f|_{C^2}$ and $g|_C$ are such that
  1. $f|_{C^2}$ and $g|_C$ are one-to-one (injective)
  2. $\operatorname{ran}(f|_{C^2})$ and $\operatorname{ran}(g|_C)$ and B are p.w. disjoint

Theorem 1.3.1. **Recursion Theorem:** *$C \subseteq U$ freely generated from B by f, g and V a set and $h : B \to V$, $F : V^2 \to V$, $G : V \to V$ Then $\exists! \overline{h} : C \to V$ s.that*

- *for all a in B it holds $\overline{h}(a) = h(a)$*
- *for all x, y in C it holds*
  1. *$\overline{h}(f(x, y)) = F(\overline{h}(x), \overline{h}(y))$*
  2. *$\overline{h}(g(x)) = G(\overline{h}(x))$*

Note : if given conditions are satisfied then h extends uniquely to a homomorphism

$$(C, f, g) \to (V, F, G)$$

Before we proof the recursion theorem, we will show how unique readability easily follows from it.
Note : Recusion Theorem implies unique readability for propositional formulas. What we need to check is that the Assumptions of recursion theorem are satisfied.
Claim: The formula building operations are one-to-one.
proof of claim. $\mathcal{F}_\vee$ is one to one, suppose $(\alpha \vee \beta) = (\delta \vee \gamma)$ then $\alpha \vee \beta) = \delta \vee \gamma)$ And α, δ are propositional formulas, so they equal to each other (else one is an initial segment of the other, hence not a propositional fla.) By the same argument we get β is equal to γ. ⊠

Claim: Disjointment of ranges
proof of claim.
- if $(\alpha \vee \beta) = A$ then A starts with (which can not be the case
- if $(\alpha \vee \beta) = (\gamma \to \delta)$ then by the same argument α is γ but $\vee$ and $\to$ are diffrent
- if $(\alpha \vee \beta) = (\neg\gamma)$, then $\alpha \vee \beta) = \neg\gamma)$, so α would start with a $\neq$, -no

For all other connectives the proof is similar. ⊠

Proof of the Rec Thm.

$v : C \rightharpoonup V$ is called acceptable if $\forall x, y \in C$

acceptable

1. if $x \in B \cap dom(v)$ then $v(x) = h(x)$
2. if $f(x, y) \in dom(v)$ then $x, y \in dom(v)$ and similarly for g
   - $v(f(x, y)) = F(v(x), v(y))$
   - $v(g(x)) = G(v(x))$

And when $U = \{\mathit{\Gamma}_v : v\text{acceptable}\}$, we define $\overline{h} :=$ function w/ graph $\bigcup \mathit{\Gamma}_v$

Claim 1: $\overline{h}$ is a function.
proof of claim.

$$S := \{x \in C : \exists \text{at most one } y \text{ with } (x, y) \in \bigcup \Gamma_v\}$$

We want $S = C$, we have $S \subseteq C$, it is enough to show that S is inductive.

- $x \in B \cap \mathrm{dom}(v)$ for some v acceptable.
  then $v(x) = h(x)$ by 1. also $x \notin \mathrm{ran}(f|_{C^2})$ and $x \notin \mathrm{ran}(g|_C)$
- $x, y \in \mathcal{S}$ We want $f(x, y), g(x) \in S$
  there are v_1, v_2 acceptable s.t. $f(x, y) \in \mathrm{dom}(v_1) \cap \mathrm{dom}(v_2)$

⊠

Claim 2: $\overline{h}$ is acceptable.
proof of claim. $\overline{h} : C \rightharpoonup V$ by definition. if $x \in B \cap \mathrm{dom}\, \overline{h}$ then there is a v acceptable, s.t. $x \in dom(v)$ then $\overline{h}(x) = v(x) = h(x)$ if $f(x, y) \in \mathrm{dom}\, \overline{h}$ then $f(x, y) \in \mathrm{dom}(v)$ form some v acceptable. Hence $x, y \in \mathrm{dom}(v)$ and therefore $x, y \in \mathrm{dom}(\overline{h})$ and we have

$$\overline{h}(f(x, y)) = v(f(x, y)) = F(v(x), v(y)) = F(\overline{h}(x), \overline{h}(y))$$

⊠

Claim 3: The domain of $\overline{h}$ equals C.
proof of claim. it is enough to show that the domain of $\overline{h}$ is inductive. $B \subseteq \mathrm{dom}(\overline{h})$ bc. $B \subseteq \mathrm{dom}(h)$ where h is acceptable. Now we need to show closure under f, g. suppose $x', y' \in \mathrm{dom}(\overline{h})$ then $x' \in \mathrm{dom}(v_1)$ for some acceptable v_i lets assume $f(x', y') \notin \mathrm{dom}(\overline{h})$ then we extend $\overline{h}$ to a function with the same graph as $\overline{h}$. Then $\Gamma \cup \{(f(x', y'), F(\overline{x'}, \overline{y'}))\}$ is the graph of an acceptable function. ⊠

Claim 4:$\overline{h}$ is uniquely constructed
proof of claim. Suppose both $\overline{h}, \overline{\overline{h}}$ work, we schow that $S = \{x \in C : \overline{h}(x) = \overline{\overline{h}}(x)\}$ is the whole set C. it is enough to show that S is inductive. Let $x \in B$ then $\overline{h}(x) = h(x) = \overline{\overline{h}}(x)$. Then for $x, y \in S$

$$\overline{h}(f(x, y)) = F(\overline{h}(x), \overline{h}(y)) = F(\overline{\overline{h}}(x), \overline{\overline{h}}(y)) = \overline{\overline{h}}(f(x, y))$$

$$\overline{h}(g(x)) = G(\overline{h}(x)) = G(\overline{\overline{h}}(x)) = \overline{\overline{h}}(g(x))$$

and $f(x, y), g(x) \in S$, therefore S is inductive. ⊠

1.4 SENTENTIAL CONNECTIVES

↰ 15.10.2024
tautological equivalence
$\sim$
$\models\!\!\!\dashv$
$\dashv\!\models$

Definition 1.10. Tautological equivalence relation: For α, β propositional sent. we define $\alpha \sim \beta$ iff $\alpha \dashv\models \beta$ (alternative notation: $\models\!\dashv$). This defines an equivalent relation.

Example 1.5. $A \to B \dashv\models \neg A \vee B$

Note : A k-place boolean function is a functon of the form $f : \{0, 1\}^k \to \{0, 1\}$ and we define $0, 1$ as the 0-place boolean functions.
If α is a propositional sent. then it determines a k-place boolean function, where k is the number of atoms, α is built up from. If α is $(A_1 \vee \neg A_2)$ then $B_\alpha : \{0, 1\}^2 \to \{0, 1\}$ and asign its values corresponding a truth value of α. That is for any TA $v : \{A_1, A_2\} \to \{0, 1\}$ we define $B_\alpha(v(A_1), v(A_2)) = \overline{v}(\alpha)$

Theorem 1.4.1. *If α, β are propositional sent. with at most n propositional Atoms (combined), then*

1. $\alpha \models \beta$ *iff* $\forall x \in \{0, 1\}^n$ *it holds* $B_\alpha(x) \leq B_\beta(x)$
2. $\alpha \dashv\models \beta$ *iff* $\forall x \in \{0, 1\}^n$ *it holds* $B_\alpha(x) = B_\beta(x)$
3. $\models \alpha$ *iff* $\forall x \in \{0, 1\}^n$ *it holds* $B_\alpha(x) = 1$

Theorem 1.4.2. (Post): [1] *Let G be an n-ary boolean function for $n \geq 1$. Then there is a propositional sent. α such that. $B_\alpha = G$. We say α realizes G.*

n-ary boolean function

Proof.

1. if G is constantly equal to 0 then set α to $A_1 \wedge \neg A_1$.
2. Otherwise the set of inputs $\{\vec{x}_1, \vec{x}_2, \dots \vec{x}_k\}$ for which $G(\vec{x}_i) = 1$ holds is not empty. We denote $\vec{x}_i = (x_{i1}, x_{i2}, \dots x_{in})$ and define a matrix $(x_{ij})_{k \times n}$ We further set

$$\beta_{ij} = \begin{cases} A_j & \text{iff } x_{ij} = 1 \\ \neg A_j & \text{iff } x_{ij} = 0 \end{cases}$$

Example:

$$(x_{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \neg A_1 & A_2 & \neg A_3 \\ A_1 & A_2 & \neg A_3 \end{pmatrix} = (\beta_{ij})$$

We define γ_i as $\beta_{i1} \wedge \beta_{i2} \wedge \dots \beta_{in}$ for $1 \leq i \leq k$ and α as $\gamma_1 \vee \gamma_2 \vee \dots \gamma_k = \vee_{i=1}^{k} \gamma_i$ Then $B_\alpha = G$ is fulfilled.

□

Note : α as constructed in the proof is in the so-called Disjunctive normal form (DNF).

DNF Disjunctive normal form

Corollary 1.4.2-A. *Every propositional sent. is tautologically equivalent to a sentence in DNF*

Corollary 1.4.2-B. *$\{\neg, \wedge, \vee\}$ is a complete set of logical connectives, i.e. every propositional sent. is tautologically equivalent to a sentence built up from atoms and $\neg, \wedge, \vee$.*

completeness (connectives)

Theorem 1.4.3. *Both $\{\neg, \wedge\}$ and $\{\neg, \vee\}$ are complete.*

Proof. Its sufficient to show that every k-place boolean function is realisable by a propositional sent. built up using only $\neg$ and $\wedge$. This is, because $\alpha \wedge \beta \mathrel{=\!\!|\!\!\models} \neg(\neg\alpha \vee \neg\beta)$ We prove this by induction on the number of disjuctions of a propositional sent. α in DNF. Suppose the statement is true for $k \leq n$. For $n+1$ and $\alpha = \bigvee_{j=1}^{n+1} \gamma_j$ there exists an $\alpha' \mathrel{=\!\!|\!\!\models} \bigvee_{j=1}^{n} \gamma_j$ and

$$\alpha = \bigvee_{j=1}^{n+1} \gamma_j \mathrel{=\!\!|\!\!\models} \alpha' \vee \gamma_{n+1} \mathrel{=\!\!|\!\!\models} \neg(\neg\alpha' \wedge \neg\gamma_{n+1})$$

□

Note : We used the observation that, if $\alpha \mathrel{=\!\!|\!\!\models} \beta$ and we replace a subsequence of α by a so called tautological equivalence then the result is also tautologically equivalent to β

Example 1.6. $\{\rightarrow, \wedge\}$ is not complete.: Let $\alpha \in PS$ built up from only $\rightarrow, \wedge$ from the atoms $A_1, \dots A_n$ then we claim

$$A_1 \wedge A_2 \wedge \cdots \wedge A_n \models \alpha$$

We can also say $\{\rightarrow, \wedge\}$ is not complete bc. $\neg A$ is not tautological equivalent to a sent. built up from $\rightarrow, \wedge$

Proof. Let $C := \{\alpha \in PS$ built up from $\rightarrow, \wedge$ and $A_1, \dots A_n$ for which $\bigwedge_{i=1}^{n} A_i \models \alpha\}$ we want to show that $C = \{\alpha \in PS$ built up from $\rightarrow, \wedge$ and $A_1, \dots A_n\}$

- We have $\{A_1, A_2 \dots, A_n\} \subseteq C$
- for $\alpha, \beta \in C$ it holds
  (1) $A_1 \wedge \cdots \wedge A_n \models \alpha \rightarrow \beta$
  (2) $A_1 \wedge \cdots \wedge A_n \models \alpha \wedge \beta$

Therefore C is closed under the fla. building operations and we have proven our claim. □

Note : $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$ is still not complete.
Note : The number of n-ary boolean functions existing is 2^{2^n} We define a notation for $n = 0$: $\perp$ (for TV $= 0$) and $\top$ (for TV $= 1$) We can conclude that $\{\neg, \rightarrow\}$ and $\{\rightarrow, \perp\}$ are both complete, it holds $\neg A \mathrel{=\!\!|\!\!\models} A \rightarrow \perp$

Definition 1.11. Satisfiability: A set of propositional sent. Σ is called **satisfiable** if there exists a TA that satisfies every member of Σ.

satisfiable

[1] Emil Post

1.5 Compactness Theorem

Theorem 1.5.1. Compactness Theorem:
Σ is satisfiable iff every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable. (i.e. Σ is finitely satisfiable)

finitely satisfiable

Proof. Let Σ be a finitely satisfiable set of propositional sent. Outline of the proof:

1. extend Σ to a maximal finitely satisfiable set Δ of propositional sent.
2. construct a thruth assigment using Δ

1. Let $\alpha_1, \alpha_2, \ldots$ be an enumeration of all propositional sent. and define Δ_n inductively by $\Delta_0 := \Sigma$

$$\Delta_{n+1} := \begin{cases} \Delta_n \cup \{\alpha_{n+1}\} & \text{if finitely satisfiable} \\ \Delta_n \cup \{\neg\alpha_{n+1}\} & \text{otherwise} \end{cases}$$

Claim: Δ_n is finitely satisfiable for each n
proof of claim. By regular induction over n. Δ_0 is finitely satisfiable. Let us assume Δ_n is finitely satisfiable. If $\Delta_{n+1} = \Delta_n \cup \{\alpha_{n+1}\}$ then we are finished. Otherwise let $\Delta' \subseteq \Delta_n$ be a finite subset that $\Delta' \cup \{\alpha_{n+1}\}$ is not satisfiable. It holds $\Delta' \models \neg\alpha_{n+1}$. Let us assume that $\Delta_n \cup \{\neg\alpha_{n+1}\}$ is not finitely satisfiable. Then there exists a finite subset $\Delta'' \subseteq \Delta_n$ such that $\Delta'' \cup \{\neg\alpha_{n+1}\}$ is not satisfiable. It therefore holds $\Delta'' \models \alpha_{n+1}$ But $\Delta' \cup \Delta''$ is a finite subset of Δ_n and by above observations $\Delta' \cup \Delta'' \models \alpha_{n+1}$ and $\Delta' \cup \Delta'' \models \neg\alpha_{n+1}$ A contradiction to the assumption that Δ_n is finitely satisfiable. ⊠

We set $\Delta := \bigcup_{i\in\mathbb{N}} \Delta_i$ and get

(a) $\Sigma \subseteq \Delta$
(b) (Maximality): for every propositional sent. α it holds $\alpha \in \Delta$ or $\neg\alpha \in \Delta$
(c) (Satisfiability): Δ is finitely satisfiable. (For every finite subset there exists a Δ_n which is a superset.)

2. Let ν be a TA for the propositional atoms $A_1, A_2, \ldots$ such that $\nu(A) = 1$ iff $A \in \Delta$
Claim: For every propositional sent. φ it holds $\overline{\nu}(\varphi) = 1$ iff $\varphi \in \Delta$.
proof of claim. Let $S = \{\varphi \in PS \text{ s.t. } \overline{\nu}(\varphi) = 1 \text{ iff } \varphi \in \Delta\}$.

- $PS \supseteq S$ is clear.
- $PS \subseteq S$
  (a) $\{A_1, A_2 \ldots\} \subseteq S$ by definition of ν
  (b) closure under $\epsilon_\neg$: Let $\varphi \in S$ then we get by maximality and satisfiability of Δ:

$$\begin{aligned} & \overline{\nu}(\neg\varphi) = 1 \\ \text{iff} \quad & \overline{\nu}(\varphi) = 0 \\ \text{iff} \quad & \varphi \notin \Delta \\ \text{iff} \quad & (\neg\varphi) \in \Delta \end{aligned}$$

  closure under $\epsilon_\to$: Let $\varphi_1, \varphi_2 \in S$ similiarly

$$\begin{aligned} & \overline{\nu}(\varphi_1 \to \varphi_2) = 0 \\ \text{iff} \quad & \overline{\nu}(\varphi_1) = 1 \text{ and } \overline{\nu}(\varphi_2) = 0 \\ \text{iff} \quad & \varphi_1 \in \Delta \text{ and } \varphi_2 \notin \Delta \\ \text{iff} \quad & (\varphi_1 \to \varphi_2) \notin \Delta \end{aligned}$$

  The closure under the other fla. building operations are similar. ⊠

By this claim $\overline{\nu}$ satisfies Σ. □

Corollary 1.5.1-A. *If $\Sigma \models \tau$ then there exists a finite subset $\Sigma' \subseteq \Sigma$ s.t. $\Sigma' \models \tau$*

Proof. Recall: $\Sigma \models \tau$ iff $\Sigma \cup \{\neg\tau\}$ is not satisfiable. Suppose $\Sigma \models \tau$ but no finite subset does. Then $\forall \Sigma' \subseteq \Sigma$ finite $\Sigma' \cup \{\neg\tau\}$ is satisfiable. By the compactness theorem $\Sigma \cup \{\neg\tau\}$ is satisfiable which is a contradiction to $\Sigma \models \tau$. □

Note : Theorem 1.5.1 and Corollary 1.5.1-A are equivalent.

Chapter 2

Predicate - / first order logic

↰ 17.10.2024

The definitions, lemmata, propositions and theorems as well as the notes in this chapter are sourced from [EE01, chapter 2].

Definition 2.1. A First order Language: consists of infinetely many distinct symbols such that no symbol is a proper initial segment of another symbol. The symbols are divided into 2 groups:

1. logical symbols (logical symbols)
   (These elements have a fixed meaning and the equivalence symbol $=$ is optional)

   $$(,\ ),\ \neg,\ \rightarrow,\ v_1,\ v_2,\ \dots,\ =$$

2. parameters (parameters)
   (a) quantifier symbol: $\forall$ (the range is subject of interpretation)
   (b) predicate symbols: for every $n > 0$ we have a set of n-ary predicates P
   (c) constant symbols: Some set of constants
   (d) function symbols: for every $n > 0$ we have a set of n-ary function symbols

Note :

- The sets in Group 2, (b)-(d) can also be the empty set
- We could drop constants and instead introduce 0-ary function symbols
- to specify language we need to specify the parameters and say if $=$ is included
- In the book [EE01] they assume that some n-place predicate symbol is present for some n.

Example 2.1.

- $\mathcal{L}_{\text{set}} = \{\in\}$, $=$ included and the binary predicate symbol $\in$ "element in"
- $\mathcal{L}_{\text{arith}} = \{<, 0, S, E, +, \cdot\}$
  - $=$ included
  - $<$ is a binary rel. symbol
  - 0 is a constant
  - S is a unary function symbol
  - E is a binary function symbol (exponentiation)
  - $+, \cdot$ are binary function symbols
- $\mathcal{L}_{\text{ring}} = \{+, \cdot, -, 0, 1\}$
  - $=$ included
  - $0, 1$ are constants
  - $-$ is a unary function symbol (additive inverse)
  - $+, \cdot$ binary function symbols

2.1 Formulas

Definition 2.2. Expression: An **expression** is any finite sequence of symbols. There exist two kinds of expressions that make sense "grammatically" The intuition should be: expression

Terms:
- points to an object
- they are built up from variables and constants using function symbols

Formulas:
- They express assertions about objects,
- they are built up from atomic formulas
- atomic formulas these are built up from terms using predicate symbols and $=$, if included

The set of all expressions is denoted by EXPR.

Definition 2.3. Term Building Operations: For every $n > 0$ and for every n-place function symbol f, let $\mathcal{F}_f$ be an n-place term building operation, that is $\mathcal{F}_f(t_1, \ldots t_n) := ft_1 \ldots t_n$ (polish notation for $f(t_1, \ldots t_n)$). TERMS , the set of all terms is defined as the set of expressions that are built up from variables and constants by applying the term building operations finitely many times. TERMS

Example 2.2. Let $\mathcal{L} = \mathcal{L}_{arith}$, the set of terms will contain 0, v_{42}, $S0$, $SSS0$, Sv_1, $+SOv_1$

Definition 2.4. Atomic formula: Any expression of the form

$$= t_1 t_2 \text{ or } Pt_1, \ldots t_n, \text{ where } t_1, \ldots t_n \text{ are terms and } P \text{ is an } n\text{-ary predicate symbol}$$

is called **atomic formula**. atomic formula

Note : Atomic formulas are not defined inductively.

Example 2.3. *cont.* $= v_1 v_{42}$ and $< S0SS0$ are atomic formulas, but $\neg = v_1 v_{42}$ is not.

Definition 2.5. Formulas: We define $\varepsilon_\neg$, $\varepsilon_\to, Q_i$ to be the fla. building operations, defined as follows $\varepsilon_\neg(\alpha) := (\neg\alpha)$, $\varepsilon_\to := (\alpha \to \beta)$ and $Q_i(\gamma) := \forall v_i \gamma$. The set of formulas is the set of expressions built up from atomic formulas by applying the fla. building operations finitely many times. formula

Example 2.4. *cont.* $\forall v_1 (= Sv_1 0)$ is a formula we get by applying Q_1 on the atomic formula $= Sv_1 0$.

Free variables

Example 2.5. We introduce the $\exists$ **quantifier** as an abbreviation: $\exists y \alpha$ means $\neg\forall y \neg\alpha$. $\exists$ quantifier
"Every non-zero natual number is a succsesor" $\forall x(x \neq 0 \to \exists y S(y) = x)$ is different then "if a number is not 0, then it is a succsesor" $x \neq 0 \to \exists y S(y) = x$. We say, x occurs **bounded** in the first formula, for the latter x occures free in the formula. If you have an expression without free variables, it is either true or false, on the other hand if a variable occurs free in a formula, the truth value of it depends on the variable itself. bounded variable

Definition 2.6. Free variables: Let x be a variable. We say "x occurs **free in** φ ", if

1. If φ is an atomic fla. then x occurs **free** in φ iff x occurs in φ
2. If $\varphi \equiv (\neg\alpha)$ then x occurs free in φ iff x occurs free in α
3. If $\varphi \equiv (\alpha \to \beta)$ then x occurs free in φ iff x occurs free in α or β
4. If $\varphi \equiv \forall v_i \alpha$ then x occurs free in φ iff x occurs free in α and $x \neq v_i$

A formula α is called a **sentence**, if no variable occurs free in α sentence

Note : The above definition is well defined thanks to the recursion theorem.
To prove this, define the function h on the set of atoms: $h(\alpha) = \{v \in V : v \text{ occurs free in } \alpha\}$, which is the set of all variables v_i that occur free in α. we now want to extend h to $\overline{h}$ on the set of all formulas. We observe

- $\overline{h}(\neg\alpha) = \overline{h}(\alpha)$
- $\overline{h}(\alpha \to \beta) = \overline{h}(\alpha) \cup \overline{h}(\beta)$
- $\overline{h}(Q_i(\alpha)) = \overline{h}(\alpha)\backslash\{v_i\}$

Then x occurs free in α iff $x \in \overline{h}(\alpha)$.
Note : From now on, we will now use $\neq, \wedge, \vee, \leftrightarrow, \exists v_i$ as abbreviations in our formulas (all can be expressed in terms of $\neg, \to, \forall_i$), so there is no ambiguity on the in the sense of the meaning of the formulas. Note: We will sometimes drop the $(,)$ and not always be using polish notation.

2.2 SEMANTICS OF FIRST ORDER LOGIC

The equivalent scheme to our TA in predicate logic. The meaning of formulas is given by structures, which also determine the scope of the quantifier $\forall$, the meaning of all parameters.

Definition 2.7. structure: A **structure** $\mathcal{A}$ for a first order language $\mathcal{L}$ is a non-empty set A called **universe** or **underlying set of** $\mathcal{A}$ together with an interpretation of each parameters of $\mathcal{L}$ i.e. structure

- $\forall$ ranges over the universe A
- for an n-ary pred. symbol $P \in \mathcal{L}$ its interpretation $P^{\mathcal{A}}$ is a subset of A^n interpretation
- for a constant $c \in \mathcal{L}$ its interpretation $c^{\mathcal{A}}$ is an element of A
- for an n-ary function symbol $f \in \mathcal{L}$ its interpretation $f^{\mathcal{A}}$ is a total function

$$f^{\mathcal{A}} : A^n \to A$$

Note : $A \neq \varnothing$, and all functions $f^{\mathcal{A}}$ are total.

Example 2.6. Let $\mathcal{L} = \{\in\}$ where $\in$ is a binary relation " An example of an $\mathcal{L}$ structure is $(\mathbb{N}, \in^{\mathbb{N}})$ where $\in^{\mathbb{N}} = \{(x, y) \in \mathbb{N}^2 : x < y\}$

Definition 2.8. Assigment: Let φ be a $\mathcal{L}$-fla. and $\mathcal{A}$ a $\mathcal{L}$-structure. Let V be the set of all variables in $\mathcal{L}$. Any function $s : V \to A$ is called assignment. We define the extention $\overline{s}$ of s to the set of all $\mathcal{L}$-terms by assigment

- if $x \in V$ then $\overline{s}(x) := s(x)$
- if $c \in \mathcal{L}$ is a constant symbol, then $\overline{s}(c) := c^{\mathcal{A}}$
- for $t_1, \dots t_n$ $\mathcal{L}$-terms and $f \in \mathcal{L}$ an n-ary function symbol,

$$\overline{s}(ft_1 \dots t_n) := f^{\mathcal{A}}(\overline{s}(t_1), \dots \overline{s}(t_n))$$

Note : in the previous definition point 3. for $n = 1$ yields a commutative diagram.

Theorem 2.2.1. *For any given assignment s there exists a unique extention $\overline{s}$ as in the previous definition.*

Proof. will follow from recursion theorem and unique decomposition of terms. □

Satisfaction and Models

Definition 2.9. Satisfy: We define '$\mathcal{A}$ satisfies φ with s' and write $\mathcal{A} \models \varphi\,[s]$ or $\models_{\mathcal{A}} \varphi\,[s]$, if (inductively over the complexity of the formula φ) $\models_{\mathcal{A}}$

(i) if φ is atomic:

- $\mathcal{A} \models = t_1, t_2\,[s]$ iff $\overline{s}(t_1) = \overline{s}(t_2)$
- $\mathcal{A} \models Pt_1, \dots t_n\,[s]$ iff $(\overline{s}(t_1), \dots \overline{s}(t_2)) \in P^{\mathcal{A}}$

(ii) suppose $\mathcal{A} \models \varphi\,[s]$ and $\mathcal{A} \models \psi\,[s]$ are defined, then

- $\mathcal{A} \models \neg\varphi\,[s]$ iff $\mathcal{A} \not\models \varphi\,[s]$
- $\mathcal{A} \models \varphi \to \psi\,[s]$ iff $\mathcal{A} \models \psi\,[s]$ or $\mathcal{A} \not\models \varphi\,[s]$
- $\mathcal{A} \models \forall x\varphi\,[s]$ iff for all $a \in A$ we have $\mathcal{A} \models \varphi[s(x|a)]$, where

$$s(x|a)(v) := \begin{cases} s(v) \text{ if } v \neq x \\ a \text{ if } v = x \end{cases}$$

Example 2.7. Let $\mathcal{L} = \{\forall, \leq, S, 0\}$ be our language. An example of an $\mathcal{L}$-structure is $\mathcal{N} = (\mathbb{N}, \leq^{\mathcal{N}}, S^{\mathcal{N}}, 0^{\mathcal{N}})$. Together with a specified assignment $s : v_n \mapsto n-1$ we can analyse

- $s(v_1) = 0$
- $\overline{s}(0) = 0^{\mathcal{N}}$ (a constant is always mapped to its realisation, the interpretation of constant 0 in the structure $\mathcal{N}$)
- $\overline{s}(Sv_1) = S^{\mathcal{N}}(\overline{s}(v_1)) = S^{\mathcal{N}}(0) = 1$
- $\mathcal{N} \models \forall v_1(S(v_1) \neq v_1)\,[s]$
  iff for all $a \in \mathbb{N}$ we have that $\mathcal{N} \models (S(v_1) \neq v_1)[s(v_1|a)]$
  iff ...
  iff for all $a \in \mathbb{N}$ we have $S^{\mathcal{A}}(a) \neq a$, which is true in our structure of the natural numbers.
- Is it true in $\mathcal{N}$ that $\mathcal{N} \models S(0) \leq S(v_1)\,[s]$? Yes because

$$\begin{aligned} &\mathcal{N} \models S(0) \leq S(v_1)\,[s] \\ &\text{iff } 1 \leq 1 \end{aligned}$$

To know wheter $\mathcal{A} \models \varphi\,[s]$ it suffices to know where s maps the variables that occur free in φ. This results from the Coincidence Theorem 2.2.2

Theorem 2.2.2. Coincidence Lemma: *Suppose $s_1, s_2 : V \to A$ agree on all variables that occur free in φ then*

$$\mathcal{A} \models \varphi\,[s_1] \textit{ iff } \mathcal{A} \models \varphi\,[s_2]$$

Proof. By complexity of φ

1. if φ is $Pt_1, \dots t_n$ note: any var that occur in φ occur free in φ, so s_1, s_2 agree on all variables that occur in the terms $t_1, \dots t_n$.
   So we Claim: for t a term, s_1, s_2 assignments that agree on all variables of t then $\overline{s}_1(t) = \overline{s}_2(t)$

   proof of claim. By complexity of t

   - $t = v_m$ then $\overline{s}_1(t) = s_1(v_m) = s_2(v_m) = \overline{s}_2(t)$
   - $t = c$ then $\overline{s}_1(t) = c^{\mathcal{A}} = \overline{s}_2(t)$
   - $t = ft_1 \dots t_n$ inductively, assume $\overline{s}_1(t_i) = \overline{s}_2(t_i)$ for all $1 \leq i \leq n$ then TODO

   ⊠

2. if φ is $= t_1, t_2$ is similar
3. if φ is $\neg\alpha$ then $\mathcal{A} \models \neg\alpha\,[s_1]$ iff $\mathcal{A} \not\models \alpha\,[s_1]$ iff $\mathcal{A} \not\models \alpha\,[s_2]$ iff $\mathcal{A} \models \neg\alpha\,[s_1]$ ↰ 22.10.2024

4. if φ is $\alpha \to \beta$ then $\mathcal{A} \models \alpha \to \beta\,[s_1]$ iff .. or .. iff for s2 iff ... or ..
5. if φ is $\forall x \alpha$ then the assumption is that s_1, s_2 .. the free variables of α are the free variables of φ exept for x. but because $s_1(x|a) = s_2(x|a)$ they both agree on all free variables of α.

$$\begin{aligned}\mathcal{A} \models \forall x\varphi\,[s_1] &\text{ iff forall } a \in A \mathcal{A} \models \varphi\,[s_1(x|a)] \\ &\text{ iff forall } a \in A \mathcal{A} \models \varphi\,[s_2(x|a)] \\ &\text{ iff } \mathcal{A} \models \forall x\varphi\,[s_2]\end{aligned}$$

□

Notation: if all free variables of φ are among $v_1, \ldots v_n$ and the assigment $s(v_i) = a_i$ for all $1 \leq i \leq n$ we write

$$\mathcal{A} \models \varphi[\![a_1, a_2, \ldots, a_n]\!] := \mathcal{A} \models \varphi\,[s]$$

Note : If σ is a sentence then we can not have $\mathcal{A} \models \sigma$ and $\mathcal{A} \not\models \sigma$ because $A \neq \varnothing$.

Corollary 2.2.2-A. *If σ is a sentence then either $\mathcal{A} \models \sigma\,[s]$ for all $s : V \to A$ or $\mathcal{A} \not\models \sigma\,[s]$ for all $s : V \to A$.*

Notation: $\mathcal{A} \models \sigma$ and we say “σ is true in $\mathcal{A}$” or “$\mathcal{A}$ is a model of σ” or “σ holds in $\mathcal{A}$”.

Definition 2.10. Model: $\mathcal{A}$ is sayed to be “a model of a set of sentences Σ” $\mathcal{A} \models \Sigma$, if for every sentence $\sigma \in \Sigma$ it holds $\mathcal{A} \models \sigma$

Example 2.8. $\mathcal{L} = \{0, 1, +, -, \cdot\}$ A Structure could be $\mathcal{R} = (\mathbb{R}, 0, 1, +, -, \cdot)$ or $\mathcal{C} = (\mathbb{C}, 0, 1, +, -, \cdot)$ then the sentence $\sigma :\quad \exists x(x \cdot x = -1)$ then $\mathcal{R} \not\models \sigma$ but $\mathcal{C} \models \sigma$

Note : $\wedge, \vee, \leftrightarrow, \exists$ work as expected. That is $\mathcal{A} \models (\alpha \wedge \beta)\,[s]$ iff $\mathcal{A} \models \alpha\,[s]$ and $\mathcal{A} \models \beta\,[s]$ $\mathcal{A} \models (\alpha \vee \beta)\,[s]$ iff $\mathcal{A} \models \alpha\,[s]$ or $\mathcal{A} \models \beta\,[s]$ $\mathcal{A} \models \exists x\alpha\,[s]$ iff $\mathcal{A} \models \neg\forall x \neg\alpha\,[s]$
iff $\mathcal{A} \not\models \forall x \neg\alpha\,[s]$
iff it is not true that forall $a \in A$ $\mathcal{A} \models \neg\alpha[s(x|a)]$
iff there is $a \in A$ such that $\mathcal{A} \models \alpha[s(x|a)]$

Logical implication

Let Γ be a set of $\mathcal{L}$-formulas, φ a $\mathcal{L}$-formula.

Definition 2.11. Logical implication: We say “Γ logically implies φ” $\Gamma \models \varphi$, if for every $\mathcal{L}$-structure $\mathcal{A}$ and for every assigment $s : V \to A$

$$\text{if } \mathcal{A} \models \gamma[s] \text{ for every } \gamma \in \Gamma, \text{ then } \mathcal{A} \models \varphi\,[s]$$

Definition 2.12. Logical equivalence: Two formulas φ, ψ are called logically equivalent, if $\varphi \models \psi$ and $\psi \models \varphi$.

Definition 2.13. Valid: A formula φ is called valid, if $\models \varphi$ i.e. $\varnothing \models \varphi$ i.e. for every $\mathcal{L}$-structure $\mathcal{A}$ and every $s : V \to A$ it is $\mathcal{A} \models \varphi\,[s]$

Example 2.9.

1. $\forall x_1 P x_1 \models P x_2$
   Suppose $\mathcal{A} \models \forall x_1 P x_1\,[s]$. then for all $a \in A$ it is $\mathcal{A} \models P x_1[s(x_1|a)]$ in particular, $a \in P^{\mathcal{A}}$ for $a = s(x_2)$
2. $P x_2 \not\models \forall x_1 P x_1$
   We need a counterexample to $P x_2 \models \forall x_1 P x_1$. Let $A = \{a_1, a_2\}$ $s(x_2) = a_1$ and $P^{\mathcal{A}} = \{a_1\}$ then $\mathcal{A} \models P x_2\,[s]$.
3. Is the following valid? $\models \exists x(Px \to \forall y P y)$ yes
4. $\Gamma, \alpha \models \varphi$ iff $\Gamma \models \alpha \to \varphi$. (on next problem set, quite important)

2.3 DEFINABILITY IN A STRUCTURE

↰ 24.10.2024

By choosing a language $\mathcal{L}$, we specify which constant, function and relation symbols we can use when constructing formulas. If we then fix a structure $\mathcal{A}$, some n-ary relations that may are not in our structure, can still be expressed by a formula φ. For example given the structure $\mathcal{N} = (\mathbb{N}; 0, +)$ with equality and the usual addition on $\mathbb{N}$, we can view the set

$$P_< := \{(n,m) \in \mathbb{N} : \varphi(n,m)\}, \quad \varphi(n,m) \equiv \exists x\,(\neg(x=0) \wedge n + x = m)$$

That gives us the usual interpretation of $<$ in the natural numbers, namely $n < m$ iff $\varphi(n,m)$ iff $(n,m) \in P_<$.

Definition 2.14. definability in a structure: We say that a general n-ary relation P on A (we will just call it P, it does not have to be in the language) is definable in $\mathcal{A}$, if there is a $\mathcal{L}$-formula φ with free variables among $\{v_1, \ldots, v_n\}$ such that

$$P = \left\{(a_1, \ldots a_n) \in A^n : \mathcal{A} \models \varphi\,[\![a_1, \ldots a_n]\!]\right\}$$

We also say that φ defines P in the structure $\mathcal{A}$.
TODO definability by points: We say that a general n-ary relation P on A is definable **by points** in $\mathcal{A}$, if there is a $\mathcal{L}$-formula φ with free variables among $\{v_1, \ldots, v_{n+m}\}$ and $b_1, \ldots b_m \in A$ such that definable by points

$$P = \left\{(a_1, \ldots a_n) \in A^n : \mathcal{A} \models \varphi\,[\![a_1, \ldots a_n, b_1, \ldots b_m]\!]\right\}$$

Example 2.10.
1. $x = x$ would define the entire universe.
2. $\neg x = x$ would define the empty set.

Example 2.11.
1. TODO
2. $\mathcal{R} = (\mathbb{R}, 0, 1, +, -, \cdot)$ Q: is $[0, \infty)$ definable in $\mathcal{R}$ Yes because $\exists y(y \cdot y = x)$ Indeed we can even define the $\leq$ relation on $\mathbb{R}^2$ by $x \leq z :\Leftrightarrow \exists y(x + y \cdot y = z)$

Definition 2.15. definability of classes of structures: Let Σ be a set of sentences. τ a sentence. The class of models of Σ is the class $\operatorname{Mod}\Sigma = \{\mathcal{A} : \mathcal{A} \models \Sigma\}$. Let K be a class of structures. We are going to call K an elementary class (EC) if there is a single sentence τ such that $K = \operatorname{Mod}\tau$. K is called an elementary class in the wider sence (EC_Δ) if there is a set of sentences Σ such that $K = \operatorname{Mod}\Sigma$

Example 2.12. Let $\mathcal{L} = \{0, 1, +, \cdot\}$ and τ be a sentence that expresses the field axioms (the unary inverse functions are not in our language but are definable.) $\operatorname{Mod}\tau$ is the class of all the fields, which is EC. The class of all fields of characteristic 0: Let $\sigma_p : \neg(1 + \cdots + 1 = 0)$ and $\Sigma = \{\tau\} \cup \{\sigma_p : p \in \mathbb{P}\}$ yields that $\operatorname{Mod}\Sigma$ is the class of fields with characteristic 0, therefore it is EC_Δ, we will later see that it is not EC.

Example 2.13. Let E be a binary relation, $\mathcal{L} = \{E\}$ then a graph is a realisation $\mathcal{G} = (V, E^{\mathcal{G}})$ such that $V \neq \varnothing$, $E^{\mathcal{G}}$ is irreflexive and symmetric. By definition the universe is not empty, we still have to check irreflexive and symmetric.

- irreflexive: $\forall x(\neg xEx)$
- symmetric: $\forall x \forall y(xEy \to yEx)$

We take τ to be $\forall x \forall y((\neg xEx) \wedge (xEy \to yEx))$ Then $\operatorname{Mod}\tau$ is the class of all graphs and is therefore EC Note: the class of all finite graphs is neither EC nor EC_Δ. proof later.

Until now, we have no notion that tells us when two graphs are the same or at least similar.

2.4 Homomorphisms of structures

Definition 2.16. Homomorphism: Suppose that $\mathcal{A}, \mathcal{B}$ are two $\mathcal{L}$-structures. then a Homomorphism of $\mathcal{A}$ into $\mathcal{B}$ is a map $h : A \to B$ satisfying the below conditions

- for every n-ary predicate $P \in \mathcal{L}$ it is
$$(a_1, \dots a_n) \in P^{\mathcal{A}} \text{ iff } (h(a_1), \dots h(a_n)) \in P^{\mathcal{B}}$$
(this def. a strong Homomorphism, other textbooks maybe only reqire "$\to$" direction)
- for every n-ary function $f \in \mathcal{L}$ and for all $\underline{a} = (a_1, \dots a_n) \in A^n$ it holds
$$h(f^{\mathcal{A}}(\underline{a})) = f^{\mathcal{B}}(h(a_1), \dots h(a_n))$$
- for every constant symbol $c \in \mathcal{L}$ it is $h(c^{\mathcal{A}}) = c^{\mathcal{B}}$
(could also skip this if we consider constants as 0-ary functions)

Note : Intuatively a Homomorphism of $\mathcal{A}$ into $\mathcal{B}$ is a map $A \to B$ that preserve all funciton and relation symbols in some sense, (in general however not the definable relations)

Definition 2.17. Isomorphism:

- $h : A \to B$ is called isomorphism of $\mathcal{A}$ **into** $\mathcal{B}$ if h is an injective homomorphism $A \to B$ (in other textbooks it may be called an isomorphic embedding of $\mathcal{A}$ into $\mathcal{B}$) into
- $h : A \to B$ is called isomorphism of $\mathcal{A}$ **onto** $\mathcal{B}$, if h is a bijective homomorphism $A \to B$ onto
- $\mathcal{A}$ and $\mathcal{B}$ are called isomorphic if there exists an isomorphism $h : A \to B$ of $\mathcal{A}$ onto $\mathcal{B}$ isomorphic

Note :

Example 2.14. $\mathcal{L} = \{+, \cdot\}$ $\mathcal{N} = (\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}})$ and $\mathcal{B} = (B, +^{\mathcal{B}}, \cdot^{\mathcal{B}})$ where $B = \{0, 1\}$ and

$+^{\mathcal{B}}$	e	0
e	e	0
0	0	e

$\cdot^{\mathcal{B}}$	e	0
e	e	e
0	e	0

let $h : \mathbb{N} \to B$ a Homomorphism? $h(n) = \begin{cases} e & \text{if } n \text{ is even} \\ 0 & \text{else} \end{cases}$ need at first that $h(m+n) = h(m) +^{\mathcal{B}} h(n)$ and $h(m \cdot n) = h(m) \cdot^{\mathcal{B}} h(n)$. it is indeed a Homomorphism.

Definition 2.18. Substructure: Suppose we have two $\mathcal{L}$ structures with $A \subseteq B$. We call $\mathcal{A}$ a substructure of $\mathcal{B}$ (notation: $\mathcal{A} \subseteq \mathcal{B}$ or we might say $\mathcal{B}$ is an extention of $\mathcal{A}$), if

- for every n-ary relation $P^{\mathcal{A}} = P^{\mathcal{B}}|_A$
- for every n-ary function $f^{\mathcal{A}} = f^{\mathcal{B}}|_A$
- for every constant symbol c in $\mathcal{L}$ it is $c^{\mathcal{A}} = c^{\mathcal{B}}$

Example 2.15. $\mathcal{L} = \{\leq\}$ then $\mathcal{N} = (\mathbb{N}, \leq)$ and $\mathcal{P} = (\mathbb{N}^+, \leq^{\mathcal{P}})$ where $\leq^{\mathcal{P}}$ is the restriction of $\leq$ to the positive natual numbers. $\mathcal{P} \subseteq \mathcal{N}$ and there exists a isomorphic embedding $id : \mathbb{N}^+ \to \mathbb{N}$ from $\mathcal{P}$ into $\mathcal{N}$ They are even isomorphic $(h : \mathbb{N} \to \mathbb{N}^+, h(n) = n + 1)$ so in fact $\mathcal{P} \cong \mathcal{N}$.

Example 2.16. $(\mathbb{Q}, +) \subseteq (\mathbb{C}, +)$

Note : If $\mathcal{A} \subseteq \mathcal{B}$ then in particular $\mathcal{A}$ is closed under all constant and functions in $\mathcal{B}$ So suppose that $\mathcal{B}$ is a substructure and $A \subseteq B$ and $A \neq \varnothing$ and A is closed under $f^{\mathcal{B}}, c^{\mathcal{B}}$ Can then A be made into a substructure $\mathcal{A}$ of $\mathcal{B}$. $f^{\mathcal{A}}$ would be the restriction of $f^{\mathcal{B}}$ to A^n, constants $c^{\mathcal{A}} = c^{\mathcal{B}}$ and if $P \in \mathcal{L}$ is an n-ary predicate then $P^{\mathcal{A}}$ should be $P^{\mathcal{B}} \cap A^n$. If $\mathcal{L}$ has no const. or fuction symbols then any subset can be made into a substructure of a structure on $\mathcal{L}$.
Our next question will be: what is the relation of the above notions with truth and satisfiability The answer will be given by the so called Homomorphism theorem.

Theorem 2.4.1. Homomorphism Theorem: *Let $h : A \to B$ be a homomorphism of $\mathcal{A}$ into $\mathcal{B}$ and $s : V \to A$ an assignment then $h \circ s : V \to B$ is an assigment and it holds* ↰ 29.10.2024

1. *for all terms t it is $h(\overline{s}(t)) = \overline{(h \circ s)}(t)$*
2. *for all quantifier free formulas φ that do not contain $=$ we have*

$$\mathcal{A} \models \varphi\,[s] \text{ iff } \mathcal{B} \models \varphi\,[h \circ s]$$

3. *if h is additionally injective then we can drop the requirement "no $=$" in (2)*
4. *if h is homomorphism of $\mathcal{A}$ onto $\mathcal{B}$ then we can drop the requirement "q.f." in (2)*

Proof. 1. problem set

2. - case $\varphi \equiv Pt$ then $\mathcal{A} \models Pt\,[s]$ iff $\overline{s}(t) \in P^{\mathcal{A}}$ iff $h(\overline{s}(t)) \in P^{\mathcal{B}}$ iff $\overline{(h \circ s)}(t) \in P^{\mathcal{B}}$ iff $\mathcal{B} \models Pt\,[h \circ s]$
   - case $\varphi \equiv \neg\psi$ $\mathcal{A} \models \neg\psi\,[s]$ iff $\mathcal{A} \not\models \psi\,[s]$ iff $\mathcal{A} \not\models \psi\,[s]$ iff
   - case $\varphi \equiv \psi \to \alpha$
3. $\mathcal{A} \models = t_1 t_2\,[s]$ iff $\overline{s}(t_1) = \overline{s}(t_2)$ iff $h(\overline{s}(t_1)) = h(\overline{s}(t_2))$ iff (by (a))$\overline{(h \circ s)}(t_1) = \overline{(h \circ s)}(t_2)$ iff $\mathcal{B} \models = t_1 t_2\,[h \circ s]$
4. $\varphi\, \forall s : V \to A$ $\mathcal{A} \models \varphi[s]$ iff $\mathcal{B} \models \varphi[h \circ s]$, want $\mathcal{A} \models \forall x\varphi[s]$ iff $\mathcal{B} \models \forall x\varphi[h \circ s]$ 1. $\mathcal{B} \models \forall x\varphi[(h \circ s)]$ iff for all $s : V \to A$, $a \in A$ (req. surjectivity) it is $\mathcal{B} \models \varphi[(h \circ s)(x|h(a))]$ iff $\mathcal{B} \models \varphi[h \circ (s(x|a))]$ iff (inductive assumption)$\mathcal{A} \models \varphi[s(x|a)]$ because a was arbitrary it is $\mathcal{A} \models \forall x\varphi[s]$ 2. Suppose $\mathcal{B} \not\models \forall x\varphi\,[(h \circ s)]$ then there exists a $b \in B$ such that $\mathcal{B} \models \neg\varphi\,[(h \circ s)(x|b)]$ by surjectivity we can find $a \in A$ such that $h(a) = b$ and it is $\mathcal{B} \models \neg\varphi\,[(h \circ s)(x|h(a))]$ By the inductive assumption $\mathcal{A} \models \neg\varphi\,[s(x|a)]$ and $\mathcal{A} \not\models \forall x\varphi\,[s]$

□

Definition 2.19. elementarily equivalent: $\mathcal{A}$ and $\mathcal{B}$ are called elementarily equivalent (notation: $\mathcal{A} \equiv \mathcal{B}$) if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences.

Note : $\mathcal{A} \cong \mathcal{B}$ implies $\mathcal{A} \equiv \mathcal{B}$ The converse is not true. For instance DLO (dense linear order) w/o endpoints is complete, so two structures on DLO are equivalent $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ but they are not isomorphic because the universes have diffrent cardinality.

Example 2.17. $\mathcal{N} = (\mathbb{N}, \leq)$ and $\mathcal{P} = (\mathbb{N}^{>0}, \leq)$ $h : n \mapsto n - 1 : \mathcal{P} \to \mathcal{N}$ isomorphism. so i.p. $\mathcal{N} \equiv \mathcal{P}$. but $id : \mathcal{P} \to \mathcal{N}$ is only isomorphic embedding, so for example
$\mathcal{P} \models \forall y(x \neq y \to x \leq y)\,[\![1]\!]$ and $\mathcal{N} \not\models \forall y(x \neq y \to x \leq y)\,[\![1]\!]$, but $\mathcal{N} \models \forall y(x \neq y \to x \leq y)\,[\![h(1)]\!]$

Definition 2.20. Automorphism: An automorphism is an isomorphism of the form $h : A \to A$ from $\mathcal{A}$ onto $\mathcal{A}$

Note : Every structure has a trivial automorphism $id : A \to A$

Definition 2.21. Rigid: $\mathcal{A}$ is called rigid, if the only automorphism on $\mathcal{A}$ is the trivial automorphism.

Example 2.18. If every element is definable then the structure is rigid. For example $(\mathbb{N}, 0, S)$ and $(\mathbb{N}, <)$ every element is definable, therefore the structures are rigid.

Corollary 2.4.1-A. *Let h be an automorphism on $\mathcal{A}$, $R \subseteq A^n$ definable in $\mathcal{A}$ then for all $a \in A^n$ $a \in R$ iff $(h(a_1), \dots h(a_n)) \in R$. Suppose φ defines R in $\mathcal{A}$, then*

$$\mathcal{A} \models \varphi\,[\![a_1, \dots a_n]\!] \quad \text{iff} \quad \mathcal{A} \models \varphi\,[\![h(a_1), \dots h(a_n)]\!]$$

Note : Corollary 2.4.1-A can be used to show that some $R \subseteq A^n$ is not definable in $\mathcal{A}$

Example 2.19. $\mathcal{R} = (\mathbb{R}, <)$ then $\mathbb{N}$ is not definable in $\mathcal{R}$. What do automorphisms of $\mathcal{R}$ look right? $h : \mathbb{R} \to \mathbb{R}$ is a bijection and $x < y$ iff $h(x) < h(y)$ so h is strictly increasing. for example $x \mapsto x + \frac{1}{2}$ or $x \mapsto x^3$.

2.5 Unique readability for terms and formulas

Unique readability for terms

Definition 2.22. : We define a function K : TERMSYMB $\to \mathbb{Z}$ on the set TERMSYMB of symbols from which terms are built up (variables, constants, function symbols).

$$\text{TERMSYMB} = \text{VARIABLES} \cup \text{CONSTANTS} \cup \text{FUNCTSYMB}$$

$K(s) := 1 - n$ where $s \in$ TERMSYMB and n is the number of terms that need to follow s in order to obtain a term. More precicely

(i) $K(x) := 1$ for $x \in$ VARIABLES

(ii) $K(c) := 1$ for $c \in$ CONSTANTS

(iii) and $K(f) = 1 - n$ for $f \in$ FUNCTSYMB, where n is the arity of the function symbol f.

K extends to a function on the set $\overline{\text{TERMSYMB}}$ of all expressions build up from the set TERMSYMB by setting

$$K(s_1, \dots s_n) = K(s_1) + \cdots + K(s_n)$$

The idea is that K tells us the number of symbols that at least need to follow to obtain a formula. (This number is unique because no symbol is a finite sequence of other symbols) Note that the following set inclusions hold: TERMS $\subseteq \overline{\text{TERMSYMB}} \subseteq$ EXPR.

Lemma 2.5.1. *Let $t \in$ TERMS be a term. Then $K(t) = 1$*

Proof. $K(x) = 1 = K(c)$ and $K(ft_1, \dots t_n) = 1 - n + n = 1$ □

Definition 2.23. terminal segment: A terminal segment of string of symbols $(s_1, \dots s_n)$ is $(s_k, s_{k+1}, \dots s_n)$ for some $1 \leq k \leq n$.

Lemma 2.5.2. *Any terminal segment of terms is a concatenation of one or more terms.*

Proof. True for variables and constants. $ft_1 \dots t_n$ the only non trivial case is $t'_k t_{k+1} \dots t_m$ where t_k is $t''_k t'_k$ □

Corollary 2.5.2-A. *If t_1 is a proper initial segment of a term t then $K(t_1) < 1$.*

Proof. let t be $t_1 t_2$ where t_1 is a proper initial segment then $K(t) = 1$ and $K(t_2) \geq 1$ therefore $K(t_1) \leq 0$ □

Proposition 2.5.3. Unique readability for terms: *The set of terms is freely generated from the set of variables(Var) and the set of constant symbols (Const) by the term building operations $\mathcal{F}_f$ (for each function symbol f).*

Proof. By Definition Definition 1.9:

- disjointment of ranges: Let f and g be two distinct funciton symbols then

$$\begin{aligned} &\operatorname{ran} \mathcal{F}_f \cap \operatorname{ran} \mathcal{F}_g = \varnothing \\ &\operatorname{ran} \mathcal{F}_f \cap Var = \varnothing \\ &\operatorname{ran} \mathcal{F}_f \cap Const = \varnothing \end{aligned}$$

- $\mathcal{F}_f|_{\text{terms}}$ are one-to-one (injective): assume $ft_1 \dots t_n = ft'_1 \dots t'_n$ and assume $t_1 \neq t'_1$ then one is a proper initial segment of the other. By Corollary 2.5.2-A its K-value has to be less than 1 so it is not a term. By induction we get $t_1 = t'_1, \dots, t_n = t'_n$.

□

Unique readability for formulas

Lemma 2.5.4. *$K : \overline{\text{TERMSYMB}} \to \mathbb{Z}$ extends to a function $K : \text{EXPR} \to \mathbb{Z}$ on the set of all expressions by adding to the above definition:*

(iv) $K(() := -1$ *and* $K()) := 1$

(v) $K(\forall) := 1$, $K(\neg) := 0$ *and* $K(\to) := -1$

(vi) $K(P) := 1 - n$ *for an n-ary rel. symb. P.*

(vii) $K(=) := -1$

(viii) $K(s_1 \dots s_n) := K(s_1) + K(s_n)$ *for* $s_i \in \text{TERMS} \cup \{(,), \forall, \neg, \to, P, \dots, =\}$

(ix) $K(s_1, \dots s_n) := K(s_1) + \dots + K(s_n)$ *for* $s_i \in \text{EXPR}$ *if none of the above definitions apply.*

Lemma 2.5.5. *for every formula φ it is $K(\varphi) = 1$*

Proof. induction on φ □

Lemma 2.5.6. *for every proper initial segment α' of a formula α we have $K(\alpha') < 1$*

Corollary 2.5.6-A. *No proper initial segment of a formula is a formula. The set of flas. is freely generated from the set of atomic flas. by operations $\mathcal{E}_\neg, \mathcal{E}_\to, Q_i$*

Proof.
- $\mathcal{E}_\neg, Q_i$ are one to one
- $\mathcal{E}_\to|_{\text{Flas.}}$ then itemwise and use of prev. lemmas
- p.w. disjointness of ranges

□

2.6 Deductions (formal proofs)

↰ 31.10.2024 MP

Definition 2.24. **Modus Ponens:** We will use one rule of interference, Modus Ponens(MP). Our notation will be:

$$\frac{\alpha, \alpha \to \beta}{\beta}$$

And it reads as follwos: "If α and $\alpha \to \beta$ then β." This rule is the formalisation of the rather informal statement: "If we know a statement α is true, and this statement implies another statement β, then β must also be true."

Definition 2.25. **Deduction:** A formal proof (decuction) of a fla φ from a set of formulas Σ is a finite sequence of formulas $(\alpha_0, \alpha_1, \dots, \alpha_n)$ such that $\alpha_n = \varphi$ and for every $i \le n$ α_i is either a logical axiom or $\alpha_i \in \Sigma$ or α_i is obtained from α_k and α_l where $0 \le k, l < i$ by the use of MP, in particular $\alpha_k = \beta \to \alpha_i$ and $\alpha_l = \beta$. If a deduction of φ from Σ exists, we say "φ is deducible from Σ" or "φ is a theorem of Σ".

Note : Deductions are not unique. However we do have an induction principle: If a set of formulas contains all logical axioms and all of Σ and is closed under MP, then it contains all theorems of Σ.

Logical axioms

Definition 2.26. **Generalization:** ψ is a generalization of φ if $\psi = \forall x_{i_1} \dots \forall x_{i_k} \varphi$

Definition 2.27. **Logical axioms:** Let x, y be variables and α, β formulas. then the logical axioms are generalizations of the following formulas:

1. tautologies
2. $\forall x \alpha \to \alpha^x_t$ where t is substitutable for x in α
3. $\forall x(\alpha \to \beta) \to (\forall x \alpha \to \forall x \beta)$
4. $\alpha \to \forall x \alpha$ where x does not occur free in α

if our language contains $=$ then

1. $x = x$
2. $x = y \to (\alpha \to \alpha')$ where α' is obtained from α by replacing some of the occurences of x with y.

Ad axiom group (2), Substitution:

Definition 2.28. **Substitution:** Let α, β be formulas, x a variable and t a term then α^x_t is expression obtained from α by substituting t for x We define substitution inductive as follows:

1. if α is atomic then $\alpha =$ expression obtained from α by replacing all x's by t's
2. $(\neg\alpha)^x_t = \neg(\alpha^x_t)$
3. $(\alpha \to \beta)^x_t = (\alpha^x_t) \to (\beta^x_t)$
4. $(\forall y \alpha)^x_t = \begin{cases} \forall y(\alpha^x_t) & \text{iff } x \neq y \\ \forall x \alpha & \text{iff } x = y \end{cases}$

Example 2.20.

- $\alpha^x_x = \alpha$
- Let $\alpha = \neg\forall y x = y$ what is $\forall x \alpha \to \alpha^x_z$?

$$\forall x \neg \forall y x = y \rightsquigarrow \neg \forall y z = y$$

  What is $\forall x \alpha \to \alpha^x_y$ $\forall x \neg\forall y x = y$ is true in all structures with a universe A with $|A| \geq 2$.

$$\forall x \neg \forall y x = y \rightsquigarrow \neg \forall y y = y$$

  and $\neg\forall y y = y$ is an antitautology (it is always false).

-

So we have to define substitutable

Definition 2.29. **substitutable:** Let x be a variable, t a term. Then t is substitutable for x in α if

1. α atomic then t is SUB for x in α
2. t is SUB for x in $\neg\alpha$ iff t is SUB for x in α
3. t is SUB for x in $\alpha \to \beta$ iff t is SUB for x in α and β
4. t is SUB for x in $\forall y \alpha$ iff either
   - x does not occur free in $\forall y \alpha$ or
   - y does not occur in t and t is SUB for x in α

Example 2.21. For instance the following is a logical axiom.

$$\forall x_3(\forall x_1(Ax_1 \to \forall x_2 Ax_2) \to (Ax_2 \to \forall x_2 Ax_2))$$

It is a generalization of $\forall x_1(Ax_1 \to \forall x_2 Ax_2) \to (Ax_2 \to \forall x_2 Ax_2)$ which is by point two a substitution with $\alpha = Ax_1 \to \forall x_2 Ax_2$. Then $\alpha^{x_1}_{x_2} = Ax_2 \to \forall x_2 Ax_2$ And x_2 is indeed substitutable for x_1 in α because it does not get bounded.

$$\forall x_1(\forall x_2 Bx_1x_2 \to \forall x_2 Bx_2x_2)$$

is a generalization of point (2), but x_2 is not substitutable for x_1 in $\alpha =$, therefore it is not a logical axiom.

Ad (1): tautologies

Definition 2.30. **Tautologies of first order language:** Tautologies are the formulas obtained from tautologies of propositional logic by replacing all propositional atoms by formulas of first order logic.
An alternative definition is: Divide all formulas of first order logic into two groups:

1. atomic formulas and generalizations of first order formulas (these are called prime formulas)
2. all other formulas i.e. of the form $\neg\alpha$ and $\alpha \to \beta$ (non-prime formulas)

So any first order formula is built up from the prime formulas using finitly many times the formula building operations. $\mathcal{E}_\neg$ $\mathcal{E}_\to$ We have unique readability because the set of formulas is freely generated.

Example 2.22. Consider the formula

$$\neg(\forall y(Px \to Py)) \to (Px \to \forall y \neg Py)$$

which is built up from $\varphi \equiv \neg(\forall y(Px \to Py))$ and $\psi \equiv Px \to \forall y \neg Py$. Furthermore, φ is built up from $\forall y(Px \to Py)$ and ψ is built up from Px and $\forall y \neg Py$. The latter beeing prime formulas.

Example 2.23. Is the following a tautology?

$$(\forall y(\neg Py) \to \neg Px) \to (Px \to \neg\forall y \neg Py)$$

We construct the construction tree into prime formulas and then assign truth values to them and evalue the truth value of the whole formula. It is indeed a tautology.

Note :

- $\forall x(Px \to Px)$ is a prime formula which corresponds to a propositional atom, and therefore not a tautology. But it is a generalization of a tautology and therefore by (1) a logical axiom.
- $\forall x Px \to Px$ is not a tautology but is a logical axiom by group (2).

Note : The definition of tautological implication $\Gamma \models_{\text{taut}} \varphi$ from propositional logic can be translated to **tautological implication** in first order logic.
Let Γ be a set of first order formulas, φ a first order formula and $\mathbb{A} = \{A_1, A_2, \dots\}$ our set of atoms. We aim to construct a propositional sentence β that behaves in some sense similar to φ and a propositional sentence α_γ for each $\gamma \in \Gamma$ in the same way. If we

Definition 2.31. **tautological implication:** Let Γ be a set of first order formulas and φ a first order formula. We say Γ tautologically implies φ, iff TODO

Lemma 2.6.1. *If* $\Gamma \models_{taut} \varphi$ *then* $\Gamma \models \varphi$

Proof. Problem set. □

Note : The converse fails. For instance $\forall x Px \models Pc$. However Pc is a diffrent propositional atom then $\forall x Px$ they have no connection between them when viewed in propositional logic.
We will prove $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$ (the first direction is completeness and the converse soundness.)

Theorem 2.6.2. $\Gamma \vdash \varphi$ *iff* $\Gamma \cup \Lambda \models_{taut} \varphi$

Proof.

- Let $\Gamma \vdash \varphi$ and v be a truth assigment that satiesfies every element in $\Gamma \cup \Lambda$. Induction on deduction of φ from Γ.
  - if $\varphi \in \Gamma \cup \Lambda$ then we are done
  - if φ is obtained from α, $\alpha \to \varphi$ by MP then v satiesfies α and $\alpha \to \varphi$ $\{\alpha, \alpha \to \varphi\} \models_{\text{taut}} \varphi$
- Assume $\Gamma \cup \Lambda \models_{\text{taut}} \varphi$. Then by the compactness theorem for propositional logic there are $\gamma_1 \dots, \gamma_n \in \Gamma$ and $\lambda_1, \dots \lambda_m \in \Lambda$ such that

$$\gamma_1 \to \gamma_2 \to \dots \to \gamma_n \to \lambda_1 \to \dots \to \lambda_m \to \varphi$$

is a tautology (always grouped to the right) because $\Gamma \cup \{\alpha\} \models_{\text{taut}} \beta$ iff $\Gamma \models_{\text{taut}} (\alpha \to \beta)$

□

2.7 Generalization and Deduction Theorem

↰ 05.11.2024

Note : Intuativly if Γ does not assume anything about x and Γ proves φ then Γ proves $\forall x\varphi$

Theorem 2.7.1. Generalization theorem: *If $\Gamma \vdash \varphi$ and x does not occur free in Γ, then $\Gamma \vdash \forall x\varphi$*

Proof. We use axiom group 4, $\alpha \to \forall x\alpha$ if x is not occuring free in α. Since x does not occur free in $\sigma \in \Gamma$, if $\varphi \in \operatorname{Thm}\Gamma$ then $\forall x\varphi \in \operatorname{Thm}\Gamma$. Induction principle: S the set of flas. If $\Lambda \cup \Gamma \subseteq S$ and S is closed under MP then S contains $\operatorname{Thm}(\Gamma)$. It is enough to show that $\{\varphi : \Gamma \vdash \forall x\varphi\}$ contains $\Gamma \cup \Lambda$. and is closed under MP.

1. if φ is a logical axiom then $\forall x\varphi$ is a generalization and therefore also a logical axiom, so $\Gamma \vdash \forall x\varphi$
2. Lets assume $\varphi in \Gamma$. then x does not occur free in any element of Γ, then $\varphi \to \forall x\varphi$ is a logical axiom and $\Gamma \vdash \forall x\varphi$ by MP.
3. Closedness under MP. suppose φ is obtained from ψ, $\psi \to \varphi$ by MP. Then by induction hyphothesis $\Gamma \vdash \forall x\psi$ and $\Gamma \vdash \forall x(\psi \to \varphi)$ Then $\forall x(\psi \to \varphi) \to (\forall x\psi \to \forall x\varphi)$ is a logical axiom in group 3. Then by MP $\Gamma \vdash \forall x\psi \to \forall x\varphi$
   By MP again $\Gamma \vdash \forall x\varphi$

□

Note : Suppose x has free occurence in Γ for example $Px \not\models \forall xPx$ so we can not have $Px \vdash \forall xPx$ (want $\models$ iff $\vdash$)
Note : Proof of Generalization theorem can be used to obtain a deduction of $\forall x\varphi$ from Γ from a deduction of φ from Γ.

Lemma 2.7.2. Rule T: *If $\Gamma \vdash \alpha_1$, $\Gamma \vdash \alpha_2, \dots \Gamma \vdash \alpha_n$ and $\{alpha_1, \alpha_2, \dots \alpha_n\} \models_{taut} \beta$ then $\Gamma \vdash \beta$.*

Proof. $\alpha_1 \to \alpha_2 \to \cdots \to \alpha_n \to \beta$ is a logical axiom because it is a tautology. Apply MP n-times. □

Theorem 2.7.3. Deduction theorem: *If $\Gamma \cup \{\gamma\} \vdash \varphi$ then $\Gamma \vdash (\gamma \to \varphi)$*

Proof. Assume $\Gamma \cup \{\gamma\} \vdash \varphi$. $\Gamma \cup \{\gamma\} \vdash \varphi$ iff $\Gamma \cup \{\gamma\} \cup \Lambda \models_{\text{taut}} \varphi$
iff $\Gamma \cup \Lambda \models_{\text{taut}} \gamma \to \varphi$ (exercise sheet 1, ex 7)
iff $\Gamma \vdash (\gamma \to \varphi)$ □

Note : Deduction theorem is an equivalence. $\Gamma \vdash \gamma \to \varphi$ then $\Gamma \cup \{\gamma\} \vdash \gamma$. the statement follows by MP.

Corollary 2.7.3-A. *(Contraposition): If $\Gamma \cup \{\varphi\} \vdash \neg\psi$ then $\Gamma \cup \{\psi\} \vdash \neg\varphi$*

Proof. Suppose $\Gamma \cup \{\varphi\} \vdash \neg\psi$ then by deduction theorem $\Gamma \vdash \varphi \to \neg\psi$ We observe that $\{\varphi \to \neg\psi\} \models_{\text{taut}} \psi \to \neg\varphi$.
By rule T: $\Gamma \vdash \psi \to \neg\varphi$ and by the converse of the deduction theorem, by MP we have $\Gamma \cup \{\psi\} \vdash \neg\varphi$ □

Definition 2.32. Inconsistence: A set of flas. Γ is called inconsistent, if for some (euqivalent to all) fla. β it is $\beta, \neg\beta \in \operatorname{Thm}\Gamma$.

Note : If Γ is inconsistent, then for $\alpha \in \operatorname{Thm}\Gamma$. Then $(\beta \to (\neg\beta \to \alpha))$ is a tautology. Use β from definition of inconsistence and use MP twice.

Corollary 2.7.3-B. *(Reductio ad absurdum): If $\Gamma; \varphi$ inconsistent, then $\Gamma \vdash \neg\varphi$.*

Proof. Suppose that $\Gamma; \varphi$ is inconsistent. then for any β $\Gamma; \varphi \vdash \beta$ and $\Gamma; \varphi \vdash \neg\beta$ By the deduction theorem $\Gamma \vdash \varphi \to \beta$ and $\Gamma \vdash \varphi \to \neg\beta$, therefore $\{\varphi \to \beta, \varphi \to \neg\beta\} \models_{\text{taut}} \neg\varphi$ By Rule T: $\Gamma \vdash \neg\varphi$. □

Note : strategies for finding deductions can be found in the textbook [EE01].

Theorem 2.7.4. Generalization on constants: *Suppose $\Gamma \vdash \varphi$ and c is a constant symbol that does not occur in Γ. Then there is a variable y (y does not occur in φ) s.th. $\Gamma \vdash \forall y(\varphi)^c_y$. and moreover also there is a deduction of $\forall y(\varphi)^c_y$ in which c does not occur.*

Proof. We will take a deduction $\langle \alpha_1, \ldots \alpha_n \rangle$ of φ from Γ. Pick the variable y as the first variable in any α_i for each i. **Claim:** $\langle (\alpha_1)^c_y, \ldots (\alpha_n)^c_y \rangle$ is a deduction of $(\varphi)^c_y$ from Γ.
proof of claim. We need to verify that every member $(\alpha)^c_y$ is actually provable from Γ.

- if $\alpha_k \in \Gamma$ then c does not occur in α_k then $(\alpha)^c_y = \alpha_k$
- if $\alpha_k \in \Lambda$ then $(\alpha_k)^c_y$ is also a logical axiom.
- lets say α_k was obtained by α_i, $\alpha_i \to \alpha_k$ $i < k$ by MP. Now take $(\alpha_i \to \alpha_k)^c_y = (\alpha_i)^c_y \to (\alpha_k)^c_y$. (induction hyphothesis) $(\alpha_k)^c_y$ is obtained from $(\alpha_i)^c_y$ nad $(\alpha_i \to \alpha_k)^c_y$. by MP.

⊠

Because formal proofs are finite, there is a $\Gamma_0 \subseteq \Gamma$ finite such that Γ_0 consists of the elements of Γ used in our deduction $\langle (\alpha_1)^c_y, \ldots (\alpha_n)^c_y \rangle$ (is therefore deduction of $(\varphi)^c_y$ from Γ_0). And because we assumed that y does not occur in Γ_0, so we can use the generalization theorem on $\Gamma_0 \vdash (\varphi)^c_y$ and yield $\Gamma_0 \vdash \forall y(\varphi)^c_y$ □

Alphpabetic Variants

We will formalize and proof the statement "You can always rename your bound variables". Why is that impoirtant? Suppose we want to proof that it is provable that $\forall x \forall y P(x, y) \to \forall y P(y, y)$ If we want to use a logical axiom of group 2, we would need to check if y is actually SUB for x. We obviously do not have that because y would get bounded. $\vdash \forall x \forall y P(x, y) \to \forall x \forall z P(x, z)$ $\vdash \forall x \forall z P(x, z) \to \forall y P(y, y)$

Theorem 2.7.5. Existence of alphabetic variants: *Let φ be a fla., x a variable, t a term. Then there exists a fla. φ' such that φ differs from φ only in the choice of names of the bound variables. And*

1. *$\varphi' \vdash \varphi$ as well as $\varphi \vdash \varphi'$*
2. *t is SUB for x in φ'*

Proof. Define φ' inductively on complexity of φ.

- if φ is atomic, then $\varphi' = \varphi$
- $(\neg\varphi)' = \neg\varphi'$
  1. $\varphi' \vdash \varphi$ and $\varphi \vdash \varphi'$, we want: $\neg\varphi' \vdash \neg\varphi$ as well as $\neg\varphi \vdash \neg\varphi'$ Ok by Contraposition.
  2. ok by definition of SUB
- $(\varphi \to \psi)' = \varphi' \to \psi'$
  1. By assumption: We want $(\varphi \to \psi) \vdash (\varphi \to \psi)'$, it is enough to show $\varphi \to \psi; \varphi' \vdash \psi'$ We have
  $$\varphi \to \psi; \varphi' \vdash \varphi$$
  $$\varphi \to \psi; \varphi' \vdash \psi$$
  2. ok by definition of SUB
- $(\forall y \varphi)'$

Case 1: No occurence of y in t. or $x = y$ (that is, t is substitutable for x in φ). We define $(\forall y \varphi)' = \forall y \varphi'$. All we need to check is part (a). We have that $\forall y \varphi \vdash \varphi$ because $\forall y \varphi \to \varphi$ is an axiom group 2. So $\forall y \varphi \vdash \varphi'$ and therefore by the generalization theorem $\forall x \varphi \vdash \forall y \varphi'$

Case 2: If y does occur in t and $x \neq y$. let z be the variable that is the first variable that does not occur in φ', x, t then set $(\forall y \varphi)' = \forall z (\varphi')^y_z$

2. want t SUB for x in $(\forall y \varphi)'$
z does not occur in t (choice of z) t is SUB for x in φ'. (ind assumption) Then t is SUB für x in $\forall z(\varphi')^y_z$ iff t is SUB for x in $(\varphi')^y_z$ because $x \neq z$.

1. $\varphi \vdash \varphi'$ (by ind. assumption) Then $\forall y\varphi \vdash \forall y\varphi'$, because
$$\vdash \forall y(\varphi \to \varphi') \to (\forall y\varphi \to \forall y\varphi')\text{(axiom of group 3)}$$
then
$$\forall y(\varphi \to \varphi')\text{gen thm}$$
and by MP:
$$\forall y\varphi \to \forall y\varphi'$$
We have $\forall y\varphi' \vdash (\varphi')^y_z$ (axiom of group 2, z does not occur in φ') By Gen Thm. $\forall y\varphi' \vdash \forall z(\varphi')^y_z$ Then
Want $\forall z(\varphi')^y_z \vdash \forall y\varphi$
$\forall z(\varphi')^y_z \vdash ((\varphi')^y_z)^z_y$ (ax of group 2.), y is SUB for z in $(\varphi')^y_z$ bc. φ' does not contain z so all occurences of z in $(\varphi')^y_z$ are free. (we substituted z for free occ of y.) (Re-replacement lemma $((\varphi')^y_z)^z_y = \varphi'$, see problem set.) So we have $\forall z(\varphi')^y_z \vdash \varphi$ We also know that $\varphi' \vdash \varphi$ by the inductive hyphothesis. So $\forall z(\varphi')^y_z \vdash \varphi$ So $\forall z(\varphi')^y_z \vdash \forall y\varphi$ (Gen Thm.)

□

Note : φ' constructed in proof is also called an alphabetic variant of φ
if our language contains equality:

1. $\vdash \forall x x = x$ (ax 5.)
2. $\vdash \forall x\forall y(x = y \to y = x)$ p.122
3. $\vdash \forall x\forall y\forall z(x = y \to (y = z \to x = z))$ ([Exercise 11 of 2.4 EE01, p. 130])
4. $\vdash \forall x_1\forall x_2\forall y_1\forall y_2(x_1 = y_1 \to (x_2 = y_2 \to (Px_1x_2 \to Py_1y_2)))$, similarly for any n-ary predicate. (see [EE01, p.128])
5. $\vdash \forall x_1\forall x_2\forall y_1\forall y_2(x_1 = y_1 \to (x_2 = y_2 \to (fx_1x_2 = fy_1y_2)))$, similarly for n-ary formula symbol, (see [EE01, p.122])

In first order logic it holds: ↰ 07.11.2024

- soundness: If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$
- completeness: If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$

2.8 Soundness of first order logic

For the proof of soundness we will have to show that all our axioms are valid. For this we will need the following two lemmas.

Lemma 2.8.1. pre-substitution lemma: *Let be a map TODO*

Lemma 2.8.2. Substitution lemma: *If t SUB x in φ then $\mathcal{A} \models \varphi^x_t\,[s]$ iff $\mathcal{A} \models \varphi\,[s(x|\overline{s}(t))]$*

Proof. 1. φ atomic: use pre-substitution lemma.

2. φ is of the form $\neg\psi$ or $\psi \to \theta$ - use induction
3. φ is of the form $\forall y\psi$ and x does not occur free in φ
$\varphi^x_t = \varphi$ wts. $\mathcal{A} \models \varphi^x_t\,[s]$ iff $\mathcal{A} \models \varphi\,[s(x|\overline{s}(t))]$
By Theorem 2.2.2, this is indeed the case, so the lemma holds.
4. φ is $\forall y\psi$ where x occurs free in φ and t is SUB for x in φ. Then it must be: y does not occur in t and t is SUB for x in ψ.
then $\overline{s}(t) = \overline{s(y|a)}(t)$ for every $a \in A$. Moreover we also have, that $\varphi^x_t = \forall y\psi^x_t$ bc. $x \neq y$
Then $\mathcal{A} \models \varphi^x_t\,[s]$ iff $\mathcal{A} \models \forall y\psi^x_t\,[s]$
iff $\mathcal{A} \models \psi^x_t\,[s(y|a)]$ and for all $a \in A$.
iff $\mathcal{A} \models \psi\,[s(y|a)(x|\overline{s(y|a)}(t))]$ (inductive assumption) and for all $a \in A$
By above: iff $\mathcal{A} \models \psi\,[s(y|a)(x|\overline{s}(t))]$ for all $a \in A$
iff $\mathcal{A} \models \forall y\psi\,[s(x|\overline{s}(t))]$

□

Theorem 2.8.3. Soundness Theorem: *If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$*

Proof. Proof by induction on φ. We have to show:

(i) that every logical axiom is valid

(ii) logical implication is preserved by MP

On (ii) Assume 1. we have to show that if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

- $\varphi \in \Lambda$ by 1.
- $\varphi \in \Gamma$ then $\Gamma \models \varphi$
- φ follows by MP from $\psi, \psi \to \varphi$ then by assumption $\Gamma \models \psi$ and $\Gamma \models \psi \to \varphi$ Therefore $\Gamma \models \varphi$

On (i) [EE01, exercise 6, section 2.2] consists in showing that if a logical axiom is valid, then also its generalization. So generalizations of valid formulas are valid, we therefore may only consider logical axioms that are not generalizations of another logical axiom.

Ax of 1. can be found in [EE01, exercise 3, section 2.3]

Ax of 2. wts. $\forall x\varphi \to \varphi^x_t$ is valid, where t is SUB for x in φ.
simple case: $\forall xPx \to Pt$ is valid. Let $\mathcal{A} \models \forall xPx[s]$ then $\mathcal{A} \models \forall xPx[s(x|a)]$ for every $a \in A$. so i.p. for $a = \overline{s}(t)$ this means $\overline{s}(t) \in P^{\mathcal{A}}$ that is $\mathcal{A} \models Pt$. In more generality we will need the substitution lemma: We have $\mathcal{A} \models \forall x\varphi\,[s]$
this is equivalent to $\forall a \in A$ we have $\mathcal{A} \models \varphi[s(x|a)]$ then in particular $\mathcal{A} \models \varphi[s(x|\overline{s}(t))]$ and by the substitution lemma we have the equivalence to $\mathcal{A} \models \varphi^x_t\,[s]$

Ax of 3. can be found in [EE01, exercise 3, section 2.2]

Ax of 4. can be found in [EE01, exercise 4, section 2.2]

Ax of 5. $x = x$: $\mathcal{A} \models x = x[s]$ because $s(x) = s(x)$

Ax of 6. $x = y \to (\alpha \to \alpha')$ where α is atomic fla, and α' is obtained from α by remplacing some occurances of x's with y's. By the deduction theorem, is enough to show that the set of formulas $\{x = y, \alpha\} \models \alpha'$. Let $\mathcal{A}$ be a structure, s an assigment such that $\mathcal{A} \models x = y[s]$
Claim: for every term t if t' is obtained from t by replacing some x's by y's, then $\overline{s}(t) = \overline{s}(t')$.
proof of claim. Induction on terms. ⊠

- α of the form $t_1 = t_2$ then α' is $t_1' = t_2'$, use prev. claim.
- α of the form $Pt_1 \dots t_n$ similar

□

↰ 12.11.2024

Corollary 2.8.3-A. *$\vdash \varphi \leftrightarrow \psi$ then φ, ψ are logically equivalent.*

Corollary 2.8.3-B. *φ' an alphabetic variant of φ then φ, φ' are logically equivalent.*

Definition 2.33. : A set of formulas Γ is called satisfiable, whenever there is a structure $\mathcal{A}$ with an assigment into A that for all $\sigma \in \Gamma$ $\mathcal{A} \models \sigma\,[s]$

Corollary 2.8.3-C. *If Γ is satisfiable then Γ is consistent*

Note : Corollary 2.8.3-C is equivalent to Soundness (see Exercises)

2.9 Completeness of First Order Logic

Theorem 2.9.1. Completness Theorem: $\Gamma \models \varphi \implies \Gamma \vdash \varphi$

Theorem 2.9.2. Completness Theorem': *Every consistent set of formulas is satisfiable.*

Note :

- The completeness Theorem is equivalent to completeness theorem'
- The completeness Theorem holds for language of any cardinality.

- We will assume for simplicity that the Language is countable.

Proof. Let Γ be a consisten t set of flas in some language $\mathcal{L}$ The idea of the proof:

- 1.-3. build a new set of formulas Δ
  - $\Gamma \subseteq \Delta$
  - Δ consistent and maximal
  - For every fla φ and every variable x there is constant c $\neg\forall x\varphi \to \neg\varphi^x_c \in \Delta$
- 4. Build $\mathcal{A}$ by A is the set of terms (in expanded language) such that Every fla in Δ w/o. equality ($=$) is satisfiable in $\mathcal{A}$
- accomodate $=$

1. Add a countable infinite set of new constant symbols to the language $\mathcal{L}$ and call it $\mathcal{L}'$
   Claim: Γ is a consistent set of flas. in $\mathcal{L}'$.
   proof of claim. Why? If not, then $\Gamma \vdash \beta \wedge \neg\beta$ where deduction is in $\mathcal{L}'$ and there occurs finitly many new constant symbols in this deduction. By generalization on constants the new constants in the proof can be replaced by new variables. We get a deduction in the old language $\mathcal{L}$ and that contradicts the assumption that Γ is consistent. ⊠

2. Want to add for every formula φ and every variable x $\neg\forall x\varphi \to \varphi^x_c$
   and need to stay consistent. Fix enumeration of pairs (φ, x) where φ' is a $\mathcal{L}'$-fla., x variable.

$$\theta_1 := \forall x_1\varphi_1 \to \neg{\varphi_1}^{x_1}_{c_1}$$

   where c_1 is the first new constant that does not occur in φ_1 $\vdots$

$$\theta_n := \forall x_n\varphi_n \to \neg{\varphi_n}^{x_n}_{c_n}$$

   where c_n is the first new constant that does not occur in φ_n and does not occur in θ_k for $k < n$.
   $\Theta = \{\theta_1, \dots\}$
   Claim: $\Gamma \cup \Theta$ is consistent.
   proof of claim. Suppose it is not. Then let m be minimal such that $\Gamma \cup \{\theta_1 \dots \theta_{m+1}\} \vdash \beta \wedge \neg\beta$. Then by (Raa) $\Gamma \cup \{\theta_1 \dots \theta_m\} \vdash \neg\theta_{m+1}$ θ_{m+1} is of the form

$$\forall x_m\varphi_m \to \neg{\varphi_n}^{x_m}_{c_m}$$

   then by (Rule T)

$$\Gamma \cup \{\theta_1 \dots \theta_m\} \vdash \neg\forall x\varphi$$

   and

$$\Gamma \cup \{\theta_1 \dots \theta_m\} \vdash \varphi^x_c$$

   (star..TODO)
   star: $\Gamma \cup \{\theta_1, \dots \theta_m\} \vdash \forall x\varphi$ By generalization on constants: $\Gamma \cup \{\theta_1, \dots \theta_m\} \vdash \forall x(\varphi^x_c)^c_x$ since c does not occur on the left. also $(\varphi^x_c)^c_x = \varphi$ bc c does not occur in φ. Now we have

$$\Gamma \cup \{\theta_1 \dots \theta_m\} \vdash \neg\forall x\varphi$$

   and

$$\Gamma \cup \{\theta_1, \dots \theta_m\} \vdash \forall x(\varphi^x_c)^c_x$$

   which is a contradiction to minimality of m or the consistentness of Γ. ⊠

3. Extend $\Gamma \cup \Theta$ to maximal consistent set. Λ is the set of logical axioms in $\mathcal{L}'$ we know that $\Gamma \cup \Theta$ is consistent. so we know that there is no β

$$\Gamma \cup \Theta \cup \Lambda \models_{\text{taut}} \beta \wedge \neg\beta$$

   So we find v a truth assigment on prime flas. that satisfies $\Gamma \cup \Theta \cup \Lambda$ and we are going to use this truth assigment to find the maximal set

$$\Delta := \{\varphi : \overline{v}(\varphi) = 1\}$$

   Then for every φ either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$ so we have maximality and we also have consistency bc. $\Delta \vdash \varphi$ then $\Delta \models_{\text{taut}} \varphi$ because $\Lambda \subseteq \Delta$ and that means $\overline{v}(\varphi) = 1$ so $\varphi \in \Delta$. So we have that Δ is consistent. and we say that Δ is deductively closed i.e. $\Delta \vdash \varphi$ then $\varphi \in \Delta$.

4. Construction of an $\mathcal{L}'$ structure $\mathcal{A}$ from Δ. We will firstly replace $=$ with E bin. predicate symbol. $A =$ set of all $\mathcal{L}'$-terms
$E^{\mathcal{A}}$ def. by $uE^{\mathcal{A}}t$ iff $u = t \in \Delta$
$f^{\mathbf{A}}$ def by $f^{\mathbf{A}}(t_1, \dots t_n) = ft_1 \dots t_n$
$c^{\mathcal{A}} := c$
$P^{\mathcal{A}}$ then $P^{\mathcal{A}}t_1, \dots t_n$ iff $Pt_1 \dots t_n \in \Delta$ We take the assignment $s : Var \to A$ by $s(x) = x$
Claim 1: $\overline{s}(t) = t$ for every term t **Claim 2:** for every φ let φ^* be obtained from φ by replacing each $=$ with E then $\mathcal{A} \models \varphi^*\,[s]$ iff $\varphi \in \Delta$

proof of claim. • φ atomic then φ is Pt

$$\mathcal{A} \models \varphi^*\,[s] \text{ iff } \mathcal{A} \models Pt\,[s] \text{ iff } \overline{s}(t) \in P^{\mathcal{A}} \text{ iff } t \in P^{\mathcal{A}}$$

φ is uEt then

$$\mathcal{A} \models \varphi^*\,[s] \text{ iff } \mathcal{A} \models uEt\,[s] \text{ iff } \overline{s}(u)E\overline{s}(t) \text{ iff } u = t \in \Delta$$

- $\neg\varphi$

$$\mathcal{A} \models \neg\varphi^*\,[s] \text{ iff } \mathcal{A} \not\models \varphi\,[s] \text{ iff } \varphi \notin \Delta \text{ iff } \neg\varphi \in \Delta$$

- $\varphi \to \psi$

$$\begin{aligned}\mathcal{A} \models \varphi^* \to \psi^*\,[s] &\text{ iff } \mathcal{A} \not\models \varphi^*\,[s] \text{ or } \mathcal{A} \models \psi^*\,[s] \\ &\text{ iff } \mathcal{A} \models \neg\varphi^*\,[s] \text{ or } \mathcal{A} \models \psi^*\,[s] \\ &\text{ iff } \neg\varphi \in \Delta \text{ or } \psi \in \Delta \\ &\text{ iff} (\varphi \to \psi) \in \Delta\end{aligned}$$

- $\forall x\varphi$ wts. $\mathcal{A} \models \forall x\varphi^*\,[s]$ iff $\forall x\varphi \in \Delta$ Suppose $\mathcal{A} \models \forall x\varphi^*\,[s]$ then $\mathcal{A} \models \varphi^*\,[s(x|c)]$ where c is such that $\neg\forall x\varphi \to \neg\varphi^x_c \in \Delta$ Provided that we have substitutability we have by substitution lemma we know $\mathcal{A} \models (\varphi^x_c)^*\,[s]$ By the inductive hyphothesis $\varphi^x_c \in \Delta$ and $\neg\varphi^x_c \notin \Delta$ so we do not have $\neg\forall x\varphi \notin \Delta$ and by maximality of Δ we have $\forall x\varphi \in \Delta$.
  Suppose $\mathcal{A} \not\models \forall x\varphi^*\,[s]$ then $\mathcal{A} \not\models \varphi^*\,[s(x|t)]$ for some t. By the substitution lemma (providet that t is SUB for x in φ) we can replace x by t in the formula.
  $\mathbf{A} \not\models (\varphi^x_t)^*\,[s]$ by the inductive hyphothesis $\varphi^x_t \notin \Delta$ then $\forall x\varphi \notin \Delta$ becasue Δ is deductively closed. If t is not SUB for x in φ, we know that there exists a logically equivalent alphabetic variant φ' of φ such that t is SUB for x in φ'.

⊠

So at this point we have: If $\mathcal{L}$ does not contain $=$ then take $\mathcal{A}$ reduction to $\mathcal{L}$ and $\mathcal{A}$ w/s satisfies Δ.

5. Define $\mathcal{A}/E$ and assigment
Claim: $E^{\mathcal{A}}$ is a congruence on the structure $\mathcal{A}$ compatible with the predicates and formulas.
   - $E^{\mathcal{A}}$ is equivalence relation
   - $P^{\mathcal{A}}$ compatible w/ $E^{\mathcal{A}}$ i.e. $P^{\mathcal{A}}t_1, \dots t_n$ iff $P^{\mathcal{A}}s_1, \dots s_n$ whenever $t_iE^{\mathcal{A}}s_i$ for all $1 \leq i \leq n$.
   - $f^{\mathcal{A}}$ compatible w/ $E^{\mathcal{A}}$ i.e. $f^{\mathcal{A}}(\underline{t})E^{\mathcal{A}}f^{\mathcal{A}}(\underline{s})$ iff $\underline{t}E^{\mathcal{A}}\underline{s}$

Definition 2.34. : $\mathcal{A}/_E$ is the structure w/ universe $A/_E$ and $([t_1], \dots [t_n]) \in P^{\mathcal{A}/_E}$ iff $(t_1, \dots t_n) \in P^{\mathcal{A}}$ $f^{\mathcal{A}/_E}([t_1], \dots [t_n]) = [f^{\mathcal{A}}(t_1, \dots t_n)]$ Let $h : A \to A/_E : t \mapsto [t]$ quotient map. note h is surjective. $E^{\mathcal{A}/_E}$ realized by equality on $A/_E$: $[t]E^{\mathcal{A}/_E}[s]$ iff $tE^{\mathcal{A}}s$ iff $[t] = [s]$

Claim: $\mathcal{A}/_E$ satisfies Δ w/ $h \circ s$.

proof of claim. Let $\varphi \in \Delta$, $\mathcal{A} \models \varphi^*[s]$ by (4) Want to show $\mathcal{A} \models \varphi^*[s]$ iff $\mathcal{A}/_E \models \varphi^*[h \circ s]$ by Homomorphism Thm.(φ^* has no occurence of $=$, surjectivity) realisation of E in $A/_E$ is the equality in $A/_E$. Take the reduct of $\mathcal{A}/_E$ to $\mathcal{L}$. ⊠

□

↰ 14.11.2024

Corollary 2.9.2-A. *compactness statements*

1. *$\Gamma \models \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ s.th. $\Gamma_0 \models \varphi$*
2. *every finitly satisfiable set of formulas is satisfiable*

Proof. 1. $\Gamma \models \varphi$ then by completeness $\Gamma \vdash \varphi$ where the deduction uses only formulas from some $\Gamma_0 \subseteq \Gamma$ finite. By soundness, $\Gamma_0 \models \varphi$

2. Γ finitly satisfiable. Suppose Γ is not satisfiable then by completeness Γ is not consistent. So there has to be some $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \beta \wedge (\neg\beta)$ so Γ is not finitly satisfiable (by soundness).

□

2.9.1 Sizes of models / completness for uncountable languages

Let Γ be a consistent set of formulas. Is it possible to

Example 2.24. 1. For each $n \in \mathbb{N}$ there is Γ such that all models of Γ that have size n.

2. DLO (Dense liniear order) w/o endpoints: no finite models

Lemma 2.9.3. *Let Γ be a set of formulas such that all models are finite. Then there has to be an $m \in \mathbb{N}$ such that $|\mathcal{A}| \leq m$ for every model $\mathcal{A} \in \operatorname{Mod}\Gamma$*

Proof. Suppose Γ has models of arbitrarily large finite size.
Idea: Expand language by new constant symbols $c_0, c_1, \ldots$.

$$\theta_1 := \; c_0 \neq c_1$$

$$\theta_2 := \; c_0 \neq c_1 \wedge c_1 \neq c_2 \wedge c_0 \neq c_2$$

$\vdots$

$$\theta_n := \bigwedge_{i,j=0 i\neq j}^{n} c_i \neq c_j$$

$$\Theta := \{\theta_1, \theta_2 \ldots\}$$

$\Gamma \cup \theta$ finitely satisfiable. By the compactness theorem there exists a $\Theta_0 \subseteq \Theta$ finite TODO. then there is a maximal element θ_k By the compactness theorem reduct to language of Γ is an infinite models of Γ which is a contradiction to all models of Γ are finite. □

Note : There is no sentence in the language of groups / rings /... that would be satisfied in all finite groups/rings/... and not satisfied in all infinite groups/rings/...

Lemma 2.9.4. *1. FIN, the class of all finite $\mathcal{L}$-structures is not EC_Δ*

2. INF, the class of infinite $\mathcal{L}$-structures is EC_Δ but not EC.

Proof. 1. Suppose FIN is EC_Δ. By definition there is a set of formulas Γ such that

$$\operatorname{Mod}\Gamma = \{\text{ the collection of all finite } \mathcal{L}\text{-structures}\} = \text{ FIN}$$

But then Γ has only finite models, but of arbitrarily large size. That contradicts our previous lemma.

2.

$$\begin{array}{ll} \varphi_1 & \exists x x = x \\ \varphi_2 & \exists x_1 \exists x_2 x_1 \neq x_2 \\ \vdots & \vdots \\ \varphi_n \exists x_1 \ldots \exists x_n & \bigwedge_{i,j=0 i\neq j}^{n} x_i \neq x_j \end{array}$$

and $\Gamma := \{\varphi_1, \ldots\}$ and INF is indeed EC_Δ. Suppose INF $= \operatorname{Mod}(\tau)$ then $\operatorname{Mod}(\neg\tau)$ would be FIN, a contradiction to (a).

□

Recall the proof of completeness theorem. $\mathcal{L}$, $|\mathcal{L} = \aleph_0$ Γ consistent set of $\mathcal{L}$-formulas. $\mathcal{A}/_E$ countable.

Completness for uncountable languages

Use (AC) in the form of Zorn's Lemma and Zermelo's Theorem

Theorem 2.9.5. **Zorn's Lemma:** *P partially ordered set such that every chain has an upperbound in P then P contains a maximal element.*

Theorem 2.9.6. **Zermelos's Theorem:** *Every set can be well-ordered. That is linearly ordered such that every non-empty set has a smallest element.*

ω is the first infinite ordinal. then it is also a cardinal and is called $\aleph_0$

$$A_0 = \{(\varphi_\alpha, x_\alpha) : \alpha < \lambda\}$$

$\vdots$

$$|\mathcal{A}/_E| \leq \lambda$$

Chapter 3

Model Theory

The sections 3.1 to 3.4 are sourced from [EE01, chapter 2] and and the theory of o-minimality (from 3.5 onwards) can be found in [Van98].

3.1 LST-Theorem

LST stands for Löwenheim-Skolem-Tarski and is the combination of the "upward Löwenheim-Skolem theorem" with the "downward Löwenheim-Skolem theorem".

Theorem 3.1.1. **LST-Theorem:** *Let Γ be a set of $\mathcal{L}$-formulas. $|\mathcal{L}| = \lambda$ and lets assume Γ is satisfiable in some infinite structure.*
Then for every cardinal $\kappa \geq \lambda$, Γ is satisfiable in a structure of cardinality κ.

Proof. add κ many new constants to the language $\mathcal{L}$.
$\mathcal{L}' = \mathcal{L} \cup \{c_\alpha : \alpha < \kappa\}$
$\Sigma = \{c_\alpha \neq c_\beta : \alpha \leq \beta,\ \alpha, \beta \leq \kappa\}$
Then $\Gamma \cup \Sigma$ is finitly satisfiable in $\mathcal{L}'$. This is because Γ is satisfiable in some infinite structure. By compactness $\Gamma \cup \Sigma$ is satisfiable. We have $\mathcal{A} \models \Gamma \cup \Sigma$ then $|\mathcal{A}| \geq \kappa$.
By the proof of completeness theorem, $\Gamma \cup \Sigma$ has a model of size $\leq \kappa$. Hence it is exactly of size κ. Take the reduct of $\mathcal{A}$ to the language $\mathcal{L}$. □

Example 3.1. The language of ZFC $\mathcal{L} = \{\in\}$ is countable, so Löwenheim-Skolem guaranties that ZFC has a countable model. But ZFC knows that there are uncountable sets (see Cantor's Theorem 5.5.4). This is called skolems paradox. explanation: some bijections are missing

Example 3.2.

1. $\overline{\mathbb{R}}$ real field. Thm($\overline{\mathbb{R}}$) has a countable model. $\mathbb{R}_{\text{alg}}$
2. $\mathcal{N} = (\mathbb{N}, 0, S, +, \cdot)$
   Claim: there exists a countable structure $\mathcal{M}$ such that $\mathcal{N} \equiv \mathcal{M}$ but $\mathcal{N} \not\cong \mathcal{M}$ One way is to add new constant c to language $\Sigma = \{0 < c, S0 < c, \dots\}$ is fin satisfiable. So $\Sigma \cup Th(\mathcal{N})$ is fin satisfiable by compactness it is satisfiable

   Take the reduct to original language. $\mathcal{M}$. and $\mathcal{M}$ not isomorphic to $\mathcal{N}$, bc A bijection of $M \to \mathbb{N}$ would have to map c somewhere but for every $S^k 0 < c$ for every k wont be preserved by any map.

3.2 Theories and completeness

↰ 14.11.2024

Definition 3.1. **Theory:** A theory T is a set of sentences that is closed under logical implication.

$$T \models \sigma \implies \sigma \in T$$

Note : If $\mathcal{L}$ is a language. Then

- there is a smallest $\mathcal{L}$-theory. The set of all valid $\mathcal{L}$-sentences.
- and also a largest $\mathcal{L}$-theory. The set of all $\mathcal{L}$-sentences.

Definition 3.2. **Theory of structures:** Let $\mathcal{K}$ some class of $\mathcal{L}$- structures. Then

$$\text{Th}(\mathcal{K}) = \{\sigma : \sigma\ \mathcal{L}\text{-sentence and for every } K \in \mathcal{K}\ \sigma \in \text{Th}(K)\}$$

Note : Th($\mathcal{K}$) is a theory.

$$\text{if } \text{Th}(\mathcal{K}) \models \sigma \text{ then } \sigma \in \text{Th}(\mathcal{K})$$

Example 3.3. • $\mathcal{L} = \{0, 1, +, \cdot, -\}$ $\mathcal{F}$ the class of fields then $\mathrm{Th}(\mathcal{F})$ is the set of sentences true in every field.

Recall that $\mathrm{Mod}(\Sigma)$ is the class of all models of Σ. $\mathrm{Th}(\mathrm{Mod}\,\Sigma)$ might not be the set Σ but it is the set of all sentences true in all models of Σ. Which is the set of all sentences that are logically implied by Σ
Or in other words: The set of all consequences of Σ

Definition 3.3. C_n**:** $C_n(\Sigma) := \mathrm{Th}(\mathrm{Mod}\,\Sigma)$

Note : Σ is a theory iff $C_n(\Sigma) = \Sigma$

Definition 3.4. : We say that a theory T is complete, if for every sentence σ either $\sigma \in T$ or $\neg\sigma \in T$.

Example 3.4. $\mathcal{A}$ a $\mathcal{L}$-structure, then $\mathrm{Th}(\mathcal{A})$ is complete.

Note : $\mathrm{Th}(\mathcal{K})$ is complete, iff any $K_1, K_2 \in \mathcal{K}$ are elementarily equivalent.
A theory T is complete iff any to models are elementarily equivalent. ↰ 19.11.2024

Example 3.5. • The theory of fields is not complete.

- The theory of algebraically closed fields of characteristic 0 is complete (That is non-trivial)

Definition 3.5. **axiomatizability:**

- A theory T is finitely axiomatizable if there is a sentence σ such that $C_n(\sigma) = T$.
- A theory T is axiomatizable, if there is a decidable set Σ such that $C_n(\Sigma) = T$.

Example 3.6. • The theory of fields (common theory of all fields) is finitely axiomatizable.

- The theory of fields of characteristic 0 is axiomatizable. Let Ψ be the finitely many axioms of fields. $\Psi \cup \{1 + 1 \neq 0, 1 + 1 + 1 \neq 0, \dots\}$ axiomatizes the fields of characteristic 0. It is however not finitely axiomatizable. If $\Psi_0 \subseteq \Psi \cup \{1 + 1 \neq 0, 1 + 1 + 1 \neq 0, \dots\}$ finite, then Ψ_0 has a model of characteristic p for some sufficiently large p.

Theorem 3.2.1. *If $C_n(\Sigma)$ is finitlely axiomatizable then there exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $C_n(\Sigma_0) = C_n(\Sigma)$*

Proof. Suppose $C_n(\Sigma)$ is finitely axiomatizable. So $C_n(\sigma) = C_n(\Sigma)$. Then there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \sigma$. And we get $C_n(\Sigma_0) = C_n(\Sigma)$ □

Definition 3.6. : A theory T is $\aleph_0$-categorical, if any two infinite countable models of T are isomorphic. Futhermore for some infinite cardinal κ a theory T is called κ-categorical, if every two models of cardinality κ are isomorphic.

Theorem 3.2.2. **Los-Vaught test***:* *For a theory T in a countable language with only infinite models it holds*
If T is κ-categorical for some infinite cardinality κ then T is complete.

Proof. Let T be κ-categorical. Want: If $\mathcal{A}, \mathcal{B} \models T$ then $\mathcal{A} \equiv \mathcal{B}$. Note: both $\mathcal{A}$ and $\mathcal{B}$ are infinite. By LST there exists structures $\mathcal{A}'$ and $\mathcal{B}'$ with $\mathcal{A} \equiv \mathcal{A}'$ and $\mathcal{B} \equiv \mathcal{B}'$ and $|\mathcal{A}'|, |\mathcal{B}'| = \kappa$. By κ-categorical we have $\mathcal{A}' \cong \mathcal{B}'$ so $\mathcal{A} \equiv \mathcal{B}$ □

Note : completness does not imply categorical.

- The theory of natural numbers is not $\aleph_0$-categorical. See Example 3.2
- RCF not κ-categorical for all infinite cardinalities κ Not $\aleph_0$ categorical real clo of $\mathbb{Q}(\pi)$, real closure of $\mathbb{Q}$ not uncountable categorical $\overline{\mathbb{R}}$, $\overline{\mathbb{R}(\varepsilon)}, 0 < \varepsilon < \frac{1}{n}$ for every $n \in \mathbb{N}$.

3.3 THEORY OF ALGEBRAIC CLOSED FIELDS

Theorem 3.3.1. *The theory of algebraic closed fields of characteristic p ACF_p, where p is either 0 or prime is complete.[EE01, Theorem 26J, p.158]*

Proof. Note that we have a

- countable language
- with no finite models

Let $\mathcal{K}_1, \mathcal{K}_2 \models \mathrm{ACF}_p$ such that $|K_1| = |K_2| = \kappa$ uncountable. F_1 prime field of $\mathcal{K}_1$, F_2 prime field of $\mathcal{K}_2$.
Note F_1, F_2 are determined by p if $p = 0$ then $F_1 = F_2 = \mathbb{Q}$ and if p prime then $F_1 = F_2 = \mathbb{F}_p$
Define $F := F_1 = F_2$. B_1 trancendence base of $\mathcal{K}_1$ over F B_2 trancendence base of $\mathcal{K}_2$ over F

- B is trancendence base of K over F if B is a $\subseteq$-maximal subset of K which is algebraically closed then
- $B \subseteq K$ is algebraically TODO

$F(B_1)$, $F(B_2)$ subfields of $\mathcal{K}_1, \mathcal{K}_2$

- alg cl $F(B_1) = \mathcal{K}_1$
- alg cl $F(B_2) = \mathcal{K}_2$

Fact: Let F subfield of K. if F is countable and K uncountable, then any transe basis B of K oer F is of cardinality $|K|$, hence uncountable.
Steinitz: Two ACF are isomorphic iff they have the same characteristic and there trancendence spaces have the same cardinality. □

Lefschetz Principle

Proposition 3.3.2. Lefschetz Principle: *Let $\mathcal{C} = (\mathbb{C}, 0, 1, +, \cdot, -)$ For a sentence in the language of $\mathcal{C}$ Then the following are equivalent:*

(a) $\mathcal{C} \models \sigma$

(b) $\mathcal{A} \models \sigma$ *for every* $\mathcal{A} \models ACF_0$

(c) $ACF_0 \models \sigma$

(d) for all sufficiently large primes p it is $ACF_p \models \sigma$

(e) For infinitely many primes p it is ACF_p modelsσ

Proof. Sketch:

(a), (b), (c) are equivalent by completeness of ACF_0

(c) $\implies$ (d) $\mathrm{ACF}_0 \models \sigma$ so there is $T_0 \subseteq \mathrm{ACF}_0$ such that $T_0 \models \sigma$ therefore there exists a sufficiently large prime p such that $\mathrm{ACF}_p \models \sigma$.

(d) $\implies$ (e) If it is true for all sufficiently large primes that it holds for infinitely many.

(e) $\implies$ (c) If $\mathrm{ACF}_0 \vDash \sigma$ than $\mathrm{ACF}_0 \models \sigma$

□

Example of the Lefschetz Principle: ↰ 21.11.2024

Proposition 3.3.3. Ax-Grothendieck: [1] *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If f is injective, then f is surjective.*

[1] Named by Alexander Grothendieck and James Burton Ax

Proof. Our language is $\mathcal{L} = \{0, 1, +, -, \cdot\}$. Note that there is an $\mathcal{L}$-sentence Φ_d such that a Field F
$F \models \Phi_d$ iff for every polynomial map $f : F^n \to F^n$ whose TODO coord. function is of degree at most d, if f is injective then f is surjective.
By Lefschetz principle it is enough to show for sufficiently large primes p, $\mathrm{ACF}_p \models \Phi_d$ for all $d \in \mathbb{N}$. Since ACF_p is complete, it is enough to show that every injective polynomial map $f : K^n \to K^n$ is surjective, where $K = TODO$ Let $f : K^n \to K^n$ be a polynomial map.
Then there is a finite subfield K_0 of K such that all coefficients of f come from K_0. Let $y \in K^n$. Then there is a finite subfield K_1 of K such that $y \in K_1$ and $K_0 \subseteq K_1 \subseteq K$. Since $f : K_1^n \to K_1^n$ is injective and K_1 finite, $f|_{K_1}$ is surjective onto K_1. So there is $x \in K_1^n$ such that $f(x) = y$. □

Note : Later, a purely geometric proof was found by Borel.
Another use of Łoś-Vaught

Proposition 3.3.4.

$$(\mathbb{Q}, <_{\mathbb{Q}}) \equiv (\mathbb{R}, <_{\mathbb{R}})$$

Proof. $\mathcal{L} = \{<\}$ and note that both $(\mathbb{Q}, <_{\mathbb{Q}}), (\mathbb{R}, <_{\mathbb{R}})$ are DLO without endpoints, i.e. they satisfy the following axioms

1. $\forall x \forall y (x < y \lor x = y \lor y < x)$
2. $\forall x \forall y (x < y \to \neg(y < x))$
3. $\forall x \forall y \forall z ((x < y \land y < z) \to x < z)$
4. $\forall x \forall y (x < y \to \exists z (x < z \land z < y))$
5. $\forall x \exists y \exists z (y < x \land x < z)$

TODO □

3.4 Nonstandard Analysis

1. Language $\mathcal{L}$: $=, \forall$ ranging over $\mathbb{R}$,
   - P_R TODO
2. standard structure for $\mathcal{L}$: $\mathcal{R}$ with universe $\mathbb{R}$, $c_r^{\mathcal{R}} = r$, $P_R^{\mathcal{R}} = R$, $f_F^{\mathcal{R}} = F$.
3. Nonstandard structure for $\mathcal{L}$: $\mathcal{R}^*$, which is constructed using the compactness theorem

$$\Gamma := \mathrm{Th}(\mathcal{R}) \cup \{c_r P_< v_1 : r \in \mathbb{R}\}$$

   Compactness theorem $\implies$ there exists a $\mathcal{L}$-structure $\mathcal{R}^*$ with $\mathcal{R}^* \models \Gamma[(v_1|a)]$ for some $a \in R^*$. We have $\mathcal{R} \equiv \mathcal{R}^*$. Moreover, $h : \mathbb{R} \to R^*$ defined by $r \mapsto c_r^*$ is an isomorphism of $\mathcal{R}$ into $\mathcal{R}^*$
   - h is injective:
   - TODO

Note : WMA $\mathcal{R}$ substructure of $\mathcal{R}^*$ (se PS)
Notation: We will write ${}^*B$ instead of $P_B^{\mathcal{R}^*}$.

Example 3.7. what is ${}^*\mathbb{R}$? We have that $\mathcal{R} \models \forall x P_{\mathbb{R}}$, hence $\mathcal{R}^* \models \forall x^*\mathbb{R}$, so ${}^*\mathbb{R} = R^* =$ universe of $\mathcal{R}^*$. Note : Let F be an n-ary operator on $\mathbb{R}$. Then F is the restriction of ${}^*F$ to $\mathbb{R}$. ${}^*c_r = r$.
Idea: If we want to show that ${}^*R$ or ${}^*F$ has certain property, then we show

- R or F have that property.
- property can be expressed in $\mathcal{L}$

TODO ↰ 26.11.2024
$\mathcal{R}^* \supseteq \mathcal{R}$ such that $\mathcal{R}^* \equiv \mathcal{R}$. $\mathcal{F} = \{x \in \mathcal{R}^* : \exists r \in \mathbb{R}\ {}^*|x|^* \leq r\}$ $\mathcal{I} = \{x \in \mathcal{R}^* : \forall r \in \mathbb{R}\ {}^*|x|^* < r\}$

Proposition 3.4.1. *1. $\mathcal{F}$ is a subring of $\mathcal{R}^*$*

2. *$\mathcal{I}$ is an ideal in $\mathcal{R}*$*

Proof. 1. Let $x, y \in \mathcal{F}$ then there exists $a, b \in \mathbb{R}^{>0}$ such that ${}^*|x|^* \leq a$ and ${}^*|y|^* \leq b$. then

$$ {}^*|x \pm^* y|^* \leq^* |x|^* +^* |y|^* \leq a + b \in \mathbb{R}^{>0} $$
$$ {}^*|x \cdot^* y|^* =^* |x|^* \,{}^* \cdot^* |y|^* \leq a \cdot b \in \mathbb{R}^{>0} $$

2. $x, y \in \mathcal{I}$ then $\forall a \in \mathbb{R}^{>0}$ we have $|x| < \frac{a}{2}$ Then

$$ |x \pm y| \leq \frac{a}{2} + \frac{a}{2} = a $$

Let z be finite then $|z| < b \in \mathbb{R}^{>0}$ Let $a \in \mathbb{R}^{>0}$ then $|x| < \frac{a}{b}$ so

$$ |xz| < \frac{a}{b} b = a $$

□

Definition 3.7. infinitely close: x, y are called to be infinitely close ($x \simeq y$), if $y - x \in \mathcal{I}$ $\simeq$

Proposition 3.4.2. *1. $\simeq$ is an equivalence relation*

2. $\simeq$ is congruent with ${}^*+, {}^*\cdot, {}^*-$

Lemma 3.4.3. *Suppose $\neg x \simeq y$ and at least one of x, y is finite then there exists $q \in \mathbb{R}$ such that q is betweeen x and y*

Proof. $y - x \notin \mathcal{I}$, wlog. $x < y$ then there exists $b \in \mathbb{R}$ such that $0 < b < y - x$ and by the archimedian property there is $m \in \mathbb{N}^{>0}$ such that $x < mb$. Let m be the smallest such. i.e. $(m-1)b \leq x < mb$. And $mb < y$. □

Proposition 3.4.4. *For every $x \in \mathcal{F}$ there is exactly one $r \in \mathbb{R}$ such that $x \sim r$*

Proof. Let $S := \{r \in \mathbb{R} : r < x\}$. S is bounded in $\mathbb{R}$ because $|x| < r_0$ for some $r_0 \in \mathbb{R}^{>0}$. Then $r := \sup S$. Claim: $r \simeq x$. Lets assume by contradiction that this is not the case. By the previous lemma, there is $q \in \mathbb{R}$ such that $r < q < x$ or $x < q < r$. but neither of this things can happen.

- $r < q < x$ is contradiction to r not being an upper bound.
- $x < q < r$ is contradiction to r is not the least upper bound.

□

A concequence of that:

Corollary 3.4.4-A. *for each $x \in \mathcal{F}$ there is a unique way of writing of x in the form $r + i$ where $r \in \mathbb{R}$ and $i \in \mathcal{I}$*

Note : If $x = r + i$ then we also write $\mathrm{st}(x) = r$.

Proposition 3.4.5.
- $\mathrm{st} : \mathcal{F} \twoheadrightarrow \mathbb{R}$
- $\mathrm{st}(x) = 0$ *iff* $x \in \mathcal{I}$
- $\mathrm{st}(x \;{}^* + y) = \mathrm{st}(x) + \mathrm{st}(y)$
- $\mathrm{st}(x \;{}^* \cdot y) = \mathrm{st}(x) \cdot \mathrm{st}(y)$

Note : this says that st is a homomorphism of $\mathcal{F}$ onto field $\overline{\mathbb{R}}$ with $\ker(\mathrm{st}) = \mathcal{I}$ and $\mathcal{F}/_{\mathcal{I}} \cong \overline{\mathbb{R}}$

Definition 3.8. Convergence (non-standard definition): F converges at a to b if whenever $x \simeq a$ and $x \neq a$ then ${}^*F(x) \simeq b$.

Note : This definition is equivalent to $\varepsilon - \delta$ definition of convergence in Analysis.

- Suppose F converges to b at a in $\varepsilon - \delta$-sense

$$\mathcal{R} \models \forall \varepsilon > 0 \exists \delta > 0 \forall z(|z-a| < \delta \implies |F(z) - b| < \varepsilon)$$

$$\mathcal{R}^* \models \forall \varepsilon > 0 \exists \delta > 0 \forall z(|z-a| < \delta \implies |F(z) - b| < \varepsilon)$$

  Let $\varepsilon > 0$ and $\delta > 0$ corresponding to ε. Let $x \simeq a$ then $|x - a| < r$ for all positive $r \in \mathbb{R}^{>0}$ so in particular $|x-a| < \delta$, therefore $|F(x) - b| < \varepsilon$ but ε was arbitrarily, so $st(F(x)) = b$.

- Suppose F convergences to b at a in the non-standard-sense. Then $\forall \varepsilon \in \mathbb{R}^{>0}$

$$\mathcal{R}^* \models \exists \delta > 0 \forall x(|x-a| < \delta \to |F(x) - b| < \varepsilon)$$

  Because $\delta \in \mathcal{I}$ works. But then

$$\mathcal{R} \models \exists \delta > 0 \forall x(|x-a| < \delta \to |F(x) - b| < \varepsilon)$$

Note : If F converges to b at a then b is unique such that for every $i \in \mathcal{I}$ the standard part $\mathrm{st}(F(a+i)) = b$. And we use the general notation $\lim_{x \to a} F(x) = b$

Corollary 3.4.5-A. *F continuous at a then $x \simeq a \implies {}^*F(x) \simeq^* F(a)$*

Derivatives From Analysis: If $F : \mathbb{R} \to \mathbb{R}$ then $F'(a) = \lim_{h \to 0} \frac{F(a+h) - F(a)}{h}$

Definition 3.9. : We will say that $F'(a) = b$, iff for all $dx \in \mathcal{I}$, $dx \neq 0$ we have

$$\mathrm{st}\left(\frac{F(a+dx) - F(a)}{dx}\right) = b$$

$dF :=^* F(a+dx) - F(a)$ then $F'(a) = b$ iff $\forall dx \in \mathcal{I}, dx \neq 0$ we have $\mathrm{st}(\frac{dF}{dx}) = b$ $\frac{dF}{dx}$ is an actual division.

Example 3.8. $F(x) = x^2$

$$\frac{dF}{dx} = \frac{(a+dx)^2 - a^2}{dx} = \frac{2dxa + (dx)^2}{dx} = 2a + dx$$

and $\mathrm{st}\,\frac{dF}{dx} = 2a$

Proposition 3.4.6. *(standard) If $F'(a)$ exists, then F is continuous at a.*

Proof. Assume $F'(a)$ (in the standard sense) exist, then $F'(a)$ is a finite number and $F'(a) \simeq \frac{F(a+dx)-F(a)}{dx}$. Therefore $F(a+dx) - F(a)$ has to be infinitesimal ($\in \mathcal{I}$). Which means $F(a+dx) \simeq F(a)$. □

Proposition 3.4.7. Chain rule: *Suppose $G'(a)$ and $F'(G(a))$ exist then $(F \circ G)'(a) = F'(G(a)) \cdot G'(a)$*

Proof. Note: ${}^*(F \circ G) =^* F \circ^* G$ because $\mathcal{R} \models \forall x F_{f \circ g}(x) = (F_f \circ F_g)(x)$

$$dG :=^* G(a+dx) -^* G(a)$$

$$\begin{aligned} dF :=& ^*(F \circ G)(a+dx) -^* (F \circ G)(a) \\ =& ^*F(^*G(a+dx)) -^* F(^*G(a)) \\ =& ^*F(^*G(a) + dG)) -^* F(^*G(a)) \end{aligned}$$

We know that $G(a)$ exists so G is continuous at a and therefore $dG \simeq 0$

- case $dG \neq 0$ then $\frac{dF}{dG} \simeq F'(G(a))$. We can re-write

$$\frac{dF}{dx} = \frac{dF}{dG}\frac{dG}{dx} = F'(G(a)) \cdot G'(a)$$

- case $dG = 0$ then $dF = 0$ and $G'(a) = \frac{dG}{dx} = 0$ and therefore $(dx \neq 0)$

$$\frac{dF}{dx} = 0 = F'(G(a))\frac{dG}{dx}$$

□

3.5 O-MINIMALITY

↰ 28.11.2024

Example 3.9. $\overline{\mathbb{R}} = (\mathbb{R}, +, -, \cdot, 0, 1, \leq)$ $\overline{\mathbb{R}} = (\mathbb{R}, \leq)$ TODO or at least in a very similar language, then by quantifier elimination (QE, Tarski) all the definable sets of $\mathbb{R}$ are finite unions of points and intervals.

Definition 3.10. o-minimality: Let $\mathcal{L} = \{\leq, \dots\}$, $\mathcal{M}$ is an $\mathcal{L}$-structure such that $\mathcal{M} \models \mathrm{DLO}$ and the only definable subsets of M are finite union of points and intervalls. Then $\mathcal{M}$ or equivalent $\mathrm{Th}(\mathcal{M})$ is called o-minimal.

o-minimal is not a first order property so to say that a theory is o-minimal is non trivial. Note : Cell decomposition means Suppose X is definable in an o-minimal structure $\mathcal{M}$, $X \subseteq M^n$ then X is a finite union of cells (in dimension 1 these are points or intervalls) in M^2 it is either the graph of a continuous function or everything inbetween two graphs of continuous functions. (its an inductive definition)
Note : Have Dedekind complete for definable subets of M: For $X \subseteq M$ definable then $\inf X, \sup X$ exist in $M_{\pm\infty}$.
Note : If M contains infinitely small elements, for example $M = {}^*\mathbb{R}$ then $(0, 1) \subseteq M$ is not connected. $O_1 = \{x : \forall n \in \mathbb{N}^* 0 < x < \frac{1}{n}\}$ is open and so is its complement in $(0, 1)$. We have Note that O_1 is however not definable in M. If O_1 would be definable it would be a finite union of points and intervals. It is convex, and not a point. But it is also not an intervall, because then it would have by Dedekind completness that $\sup O_1$ exists in $M_{\pm\infty}$, a contradiction. TODO:

Definition 3.11. definably connectedness: $X \subseteq M^m$ is said to be definably connected, if X is definable and X is not the disjoint union of two definable, non-empty open sets.

Lemma 3.5.1.

1. *The definably connected subsets of M are the intervalls (including singletons) and $\varnothing$.*
2. *The image of a definable connected subset $X \subseteq M^n$ under a definable continuous map $f : X \to M^n$ is definably connected. (f is called to be definable, if its graph $\Gamma f \subseteq M^{mn}$ is).*
3. *(IVP) If $f : [a, b] \to M$ definable and continuous, then f assumes all values between $f(a)$ and $f(b)$.*

Proof. Exercise □

Definition 3.12. ordered group: A ordered group is a group with a linear order such that

$$\forall x \forall y \forall z x < y \to (zx < zy \wedge xz < yz)$$

Example 3.10. (i) $(\mathbb{R}, <, +)$ (ii) $(\mathbb{R}^{>0}, <, \cdot)$ (iii) non-example: $(\mathbb{R}^*, <, \cdot)$

Recall:

- $(G, \cdot)$ is divisible, if $\forall n \forall g \exists x g = x^n$, equivalent to $\forall n G^n = G$.
- $(G, \cdot)$ is torsion-free, if no element has finite order except for 1.

Proposition 3.5.2. *$(M, <, \cdot, \dots)$ o-minimal such that $(M, <, \cdot)$ ordered group, then $(M, <, \cdot)$ abelian, divisible and torsion-free.*

Lemma 3.5.3. *If G is a definable subgroup of M then G is convex.*

Proof. Suppose G is not convex, then there exists $1 < a < g$ for some $g \in G$ and $a \in M \backslash G$. Then

$$1 < a < g < ag < g^2 < ag^2 < g^3 < \dots$$

but elements alternate being in G and outside of G so G is not definable (finite union). □

Lemma 3.5.4. *The only definable subsets of M that are subgroups are $\{1\}$ and M.*

Proof. Suppose $G \neq \{1\}$ wts. $G = M$. From the previous lemma we know that G is convex. The idea is $s := \sup G$ then $1 < s$ and $(1, s) \subseteq G$. If $s = +\infty$ then $G = M$. Suppose $s \neq +\infty$ then Take $1 < g < s$ then $g^{-1}s \in (1, s)$ So $s = gg^{-1}s \in G$ and $s < sg$ thats a contradiction with $s = \sup G$. □

Proof. of Proposition.

- $(M, \cdot)$ abelian: For any $a \in M$ we can look at $C_a = \{x \in M : xa = ax\}$ it is a definable subgroup and contains a it therefore is non-trivial and we have $C_a = M$ for every $a \in M$, so abelian.
- For any $n \in \mathbb{N}^{>0}$ look at $\{x^n : x \in M\}$ non trivial definable subgroup of M, hence $= M$.
- Every ordered group is torsion-free.

□

Definition 3.13. Ordered ring: A ring (assumed to always be associative, with 1) equipped with a linear order $<$ such that

1. $0 < 1$
2. $<$ translation invariant
3. $<$ invariant under multiplication by positive elements

Note :

- The additive group $(R, <, +)$ of an ordered ring is a ordered group.
- Ordered rings have no zero-divisors $\forall x \forall y xy = 0 \to (x = 0 \vee y = 0)$
- $x^2 \geq 0$
- $k \mapsto k \cdot 1 : \mathbb{Z} \to$ ring is a strictly increasing embedding with resprect to the usual ordering on $\mathbb{Z}$ that means our characteristic is 0.

Note :

- A division ring is a field without commutativity of multiplication, so
$$\forall x x \neq 0 \to \exists y xy = 1$$
- Suppose ordered ring is also a division ring. Then such y are unique and $yx = 1$. Further, $x > 0 \to y > 0$.
Also the additive group is divisible, the underlying set is DLO w/o endpoints and $(x, y) \to x \cdot y$ $x \to x^{-1}$ are continuous with respect to intervall topology.

Definition 3.14. ordered field: An ordered field is an ordered division ring with commutative multiplication.

Definition 3.15. real closed field: ordered field R such that if $f(X) \in R[X]$ and $a < b$ are such that $f(a) < 0 < f(b)$ then there exists a $c \in (a, b)$ such that $f(c) = 0$ RCF

Example 3.11. (i) $(\mathbb{R}, +, \cdot, <)$ is RCF (ii) $(\mathbb{Q}, +, \cdot, <)$ is not a RCF

Proposition 3.5.5. *Let $(M, <, +, \cdot, \dots)$ be an o-minimal structure, such that $(M, <, +, \cdot)$ is an ordered ring. Then $(M, <, +, \cdot)$ is already an RCF.*

Proof.
- wts. $(M; <, +, \cdot)$ is ordered division ring. For all $a \in M$ aM is additive subgroup of $(M, +)$ hence $aM = M$ if $a \neq 0$.
- wts. commutativity of $\cdot$. $\mathrm{Pos}(M) := \{a \in M : a > 0\}$ is a subgroup of the multiplicative group of M. Let $a \in M$ then bc $M = aM$ we have $b \in M$ such that $1 = a \cdot b$ and by note $0 < a$ then $0 < a^{-1}$. so multiplication is commutative on M
- IVP property for polynomials: The ring operations are continuous, see note and use lemma about IVP (c).

□

3.6 Monotonicity Theorem

Proposition 3.6.1. Monotonicity Theorem: *Suppose $f : (a, b) \to M$ definable, then there are $a < c_1 < \cdots < c_k < b$ such that for (a, c_1), (c_i, c_{i+1}), (c_k, b) subsets of (a, b) we have: f is either constant or strictly monotinic and continuous.*

Lemma 3.6.2. *$\exists$ subinterval on which f is const or injective.*

Lemma 3.6.3. *If f injective, then strictly monotone on a subinterval.*

Lemma 3.6.4. *If f strictly monotone, then f continuous on a subinterval.*

Proof. Proof of Monotonicity theorem: Consider

$$X := \left\{ x \in (a, b) : \begin{array}{l} \text{on some subinterval containing } x, \\ f \text{ is either constant or strictly monotone and continuous} \end{array} \right\}$$

remark: X is a definable set. Look at $(a, b) - X$ is finite. If not, it would contain subinterval use lemma to get contradiction. WMA: $X = (a, b)$ in particular we may assume f continuous. By subdividin (a, b) further WMA that we are in one of the following cases

Case 1: $\forall x \in (a, b)$ f constant on some neighborhood of x

Case 2: $\forall x \in (a, b)$ f is strictly monotone increasing on some neighborhood of x

Case 3: $\forall x \in (a, b)$ f is strictly monotone decreasing on some neighborhood of x

Case 1: $x_0 \in (a, b)$ then $s := \{x : x_0 < x < b \wedge f \text{ cont. on } [x_0, x)\}$ wts $s = b$ suppose $s < b$ then f constant on neighborhood of s contradiction with definition of s so f continuous on $[x_0, b)$. f constant on $(a, x_0]$ similar.

Case 2: $x_0 \in (a, b)$ then $s := \{x : x_0 < x < b \wedge f \text{strictly incr. on } [x_0, x)\}$. wts: $s = b$ assume $s < b$ then f is strictly increasing on some neighborhood of s so f strictly increasing on $[x_0, s + \delta)$ for some $\delta > 0$, a contradiction to definition of s.

Case 3: similar to Case 2.

□

↰ 03.12.2024

Proof of Lemma 1:

Proof. Statement: "There exists a subinterval on which f is constant or injective."

- If $y \in R$ so that f^{-1} is infinite (it has to be a finite union of points and intervals), then $f^{-1}(y)$ contains an interval and $f(x) = y$ on that interval.
- Suppose $f^{-1}(y)$ is finite for every $y \in R$.
  $f(I)$ is infinite and is definable because f is definable, so it contains an interval J. We can define an inverse to f on J $g : J \to I$, $g(y)$ is the first $x \in I$ such that $f(x) = y <$ (this is definable). g is necessarily injective. $g(J)$ infinite, so contains a subinterval on which f is injective.

□

Proof of Lemma 2:

Proof. Statement: "If f injective, then strictly monotone on a subinterval."
Suppose f is injective. $f : I = (a, b) \to R$ pick $x \in (a, b)$ then $(a, x) = \{y \in (a, x) : f(y) < f(x)\} \uplus \{y \in (a, x) : f(x) < f(y)\}$ is definable disjoint union of definable sets, so one of the subsets has to contain an interval (c, x) with $a \leq c$, similarly for (x, d). So for all $x \in I$ we have x satisfies one of the following:

- $\Phi_{++}(x)$ iff $\exists c_1, c_2 (c_1 < x < c_2 \wedge \forall c \in (c_1, x) f(c) > f(x) \wedge \forall c \in (x, c_2) f(c) > f(x)$
- $\Phi_{--}(x)$
- $\Phi_{+-}(x)$

- $\Phi_{-+}(x)$

The set of all x that satisfy each Φ is definable, it therefore is a finite union of points and intervalls. After passing to subinterval (a,b) of I WMA that each $x \in I$ satisfies the same $\Phi_{\pm\pm}$.

- $\Phi_{-+}(x)$, on the left everybody is smaller, on the right everybody is bigger.

  $$\forall x \in I s(x) := \sup\{s \in (x,b) : f(x) < f(s)\}$$

  If $s(x) < b$ then $\Phi_{-+}(s(x))$, and therefore there is an element $s' > s(x)$ such that $f(x) \leq f(s(x)) < f(s')$ so $s(x) \geq s'$ which is a contradiction to definition to $s(x)$, therefore $s(x) = b$ for every $x \in (a,b)$. Then f has to be strictly increasing on (a,b).

- $\Phi_{+-}(x)$ similar (monotinic decreasing)

- $\Phi_{++}(x)$ $\forall x \in I$.

  $$B := \{x \in I : \forall y \in I(x < y \to f(y) > f(x))\}$$

  B is definable, if B is infinite, it has to contain a subinterval on which f is strictly increasing.

  WMA B is finite. We restrict ourselves to subinterval and may assume $B = \varnothing$. So by injectivity:

  $$\circledast \quad \forall x \in I \exists y \in I x < y \wedge f(x) > f(y)$$

  Let $c \in I$. Claim: for every large enough $y \in I$ we have $f(y) < f(c)$.

  proof of claim. By contradiction. suppose we can not find a neighborhood of b such that for all elements in this neighborhood $f(y) < f(c)$ otherwise $f(y) > f(c)$ for all large enough y. Let $d < b$ be minimal such that

  $$\forall y \in (d,b) f(y) > f(c)$$

  - case $f(d) > f(c)$: $\Phi_{++}(d)$, contradiction with minimality of d.
  - case $f(d) < f(c)$: By $\circledast$ there has to be an e with $d < e < b$ and $f(e) < f(d)$. So $f(e) < f(c)$ which is a contradiction to $\Phi_{++}(d)$

  ☒

  Define $y(c)$ to be the least element of $[c,b)$ for which

  $$\forall y y(c) < y < b \, f(c) > f(y)$$

  c satisfies Φ_{++}, therefore $c < y(c)$ and $f(y(c)) < f(c)$ if $y(c) < y < b$. The minimality of $y(c)$ implies that $y(c)$ satisfies Ψ_{+-}, where

  $$\Psi_{+-}(v) \text{ iff } \exists v_1, v_2 \in I\big(v_1 < v < v_2 \wedge \forall z_1, z_2(v_1 < z_1 < v \wedge v < z_2 < v_2) \to f(z_1) > f(z_2)\big)$$

  But c was arbitrarily so $\forall x \in I \exists v \in I(x < v \wedge \Psi_{+-}(v))$ On subinterval Ψ_{+-} we have a contradiction with Φ_{++}, similarly on subinterval for Ψ_{-+}.

- Φ_{--} similar to above

□

Proof of Lemma 3:

Proof. Statement: "If f strictly monotone, then f continuous on a subinterval."
WMA: $f : (a,b) \to R$ strictly monotone increasing. $f(I)$ infinite and definable, so $f(I)$ contains an interval J. Let $r, s \in J$ $r < s$ and $d, e \in I$ with $d < e$ and $f(d) = r$ and $f(e) = s$. restrict f to (d,e) and we get an increasing bijection $(d,e) \to (r,s)$. Our topology is the order topology, so f is continuous on (d,e) □

So we have proved the monotonicity Theorem. Note : If $f : (a,b) \to R$ is definable, then $\lim_{x \to c^-} f(x)$ exists in R_∞ for $c \in (a,b]$. And further $\lim_{x \to c^+} f(x)$ exists in R_∞ for $c \in [a,b)$ If furthermore $f : [a,b] \to R$ is continuous and definable, then f assumes a minimum and maximum on $[a,b]$
On of the important tools in o-minimality theory is the cell cecomposition theorem:

Definition 3.16. Cell: Let $(i_1, \dots i_n)$ a sequence in $\{0,1\}$. An $(i_1, \dots i_n)$-cell is defined inductively:

- (0)-cell: $\{r\} \subseteq R$,
- (1)-cell: $(a, b) \subseteq R$, $a < b$, $a, b \in R$.
- $(i_1, \dots i_k, 0)$-cell: $\Gamma f \subseteq R^{k+1}$, where f is definable and continuous function $f : X \to R$, where X is a $(i_1, \dots, i_k)$-cell
- $(i_1, \dots i_k, 1)$-cell: is a the set

$$(f, g) = \{(\underline{x}, x_{k+1}) \in R^{k+1} : f(\underline{x}) < x_{k+1} < g(\underline{x})\}$$

  $f : X \to R, g : X \to R$, $f < g$ f, g are definable and continuous on X, which is a $(i_1, \dots i_k)$-cell. f may be constantly $-\infty$ and g may be constantly ∞.

Note `Cells have nice topological properties:`

- every $(1, \dots 1)$-cell are precisely the cells that are open in their ambient space. continuity of the functions is important.
- The union of finitely many non-open cells has empty interior.
- Each cells is locally closed i.e. open in its closure.
- Each cell is homeomorphic to an open cell under a coordinate projection Example $(1, 0, 0, 1)$-cell or $(1, 0)$-cell with $(x_1, x_2) \mapsto x_2$
- If $A \subseteq R^{n+1}$, then $\pi A \subseteq R^n$ cell $\pi(x_1, \dots x_{n+1}) \mapsto (x_1, \dots x_n)$
- Every cell is definalby connected. You can proof this by induction on the cell. $\{r\}$ and open intervals are definably connected. If the projection of a cell is definably connected, then the fibre above it is either an open interval or a single point. It is even definable path connected. If there would exist an open disjoint cover there exists an open disjoint cover of the fibre, which is not possible.

3.7 Cell decomposition Theorem

Definition 3.17. `decomposition:` A decomposition of R^m is a finite partition of R^m into cells defined inductively:

- decomposition of $R^1 = R$:

$$\{(-\infty, a_1), \{a_1\}, (a_1, a_2) \dots (a_k, \infty)\}$$

- A decomposition of R^{n+1} is a finite partition of R^{n+1} into cells C such that the collection of πC is a decomposition of R^n.

Theorem 3.7.1. `Cell decomposition`*:*

(I_m) *Let $A_1, \dots A_k \subseteq R^m$ definable sets. Then there is a decomposition of R^m partitioning each A_i.*

(II_m) *Given a definable function $f : A \to R$, $A \subseteq R^m$ there is a decomposition $\mathcal{D}$ of R^m partitioning A such that for every $B \in \mathcal{D}$ $f|_B : B \to R$ is continuous.*

Proof. By induction on m. Base step:

- (I_1) o-minimality
- (II_1) monotonicity theorem.

Proof idea: Suppose we have

$$\begin{cases} (\mathrm{I}_1) \dots (\mathrm{I}_m) \\ (\mathrm{II}_1) \dots (\mathrm{II}_m) \end{cases} \Big\} \implies (\mathrm{I}_{m+1}), (\mathrm{II}_{m+1})$$

$\square$

↰ 05.12.2024

Definition 3.18. : A definably connected component of a non-empty definable Subset $X \subseteq R^m$ is a definably-connected subset of X which is maximal wrt being definably connected

Example 3.12. $X \subseteq R^m$ definable. Then it is definably connected iff X definably path connected i.e.

$$\forall x, y \in X \exists f : [0,1] \to X \text{ definable and continuous with } f(0) = x \wedge f(1) = y$$

Proposition 3.7.2. *Suppose $X \subseteq R^m$ is definable and non-empty, then X has only finitlely many definably connected components. The components are both open and closed in X and they form a finite partition of X.*

Proof. Let $\{C_1, \dots C_k\}$ be a partition of X into cells. $I \subseteq \{1, \dots k\}$ then $C_I := \bigcup_{i \in I} C_i$. Let C' be the maximal among the C_I that is definable connected. Claim: For $Y \subseteq X$ definable connected such that $Y \cap C' \neq \varnothing$ then $Y \subseteq C'$.
proof of claim. $C_Y := \bigcup\{C_i : C_i \cap Y \neq \varnothing\}$ Then $Y \subseteq C_Y$. and C_Y is definably (finite union) connected union of definably connected set Y and finitly meny cells that have non-empty intersection with Y. Then $C_Y \cap C'$ contains $Y \cap C' \neq \varnothing$. So if we take $C_Y \cup C'$ has to be again definably connected. By maximality $C_Y \cup C' = C'$ and $Y \subseteq C_Y \subseteq C'$. ⊠

Hence

- C' definably connected component of X
- The sets C' form a finite partition of X
- C' are the only definable connected components of X

The closure in X of a definably connected subset of X is definable connected. (see topology) So the C' are closed in X. They are also open because the complement in X is a finite union of closed subsets. □

Note : The above Proposition is not true if we drop the requirement "definable"

Definition 3.19. `Definable families:` Let $S \subseteq R^{m+n}$ definable. For $a \in R^m$ we put

$$S_a = \{\underline{x} \in R^n : (a, \underline{x}) \in S\} \subseteq R^n$$

S describes the family of sets $(S_a)_{a \in R^m}$. And the sets S_a are called the fibers of S.

Example 3.13. $\mathcal{R} = (\mathbb{R}, <, +, \cdot)$

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

defines $S \subseteq R^6 \times R^2$. The fibers are:

- points, circles, ellipse, hyperbola, parabola
- and the limiting cases: $\varnothing$, 2 lines intersecting each other, 2 parallel lines, one line, RR^2

Note : In o-minimal structures there are only finitly many homomorphism types in a definably family. (If there are infinitely many fibres, then only finitlely many are not homeomorphic to each other).

Proposition 3.7.3. *1. C cell in R^{m+n}, $a \in \pi_m^{m+n} C$ (where $\pi_m^{m+n}(x_1, \dots x_{m+n}) = (x_1, \dots x_m)$) Then C_a is a cell in R^n*

2. $\mathcal{D}$ decomposition of R^{m+n}, and $a \in R^m$ then

$$\mathcal{D}_a = \{C_a : C \in \mathcal{D} \wedge a \in \pi_m^{m+n}(C)\}$$

is a decomposition of R^m.

Proof. 1. induction on n. If $n = 1$, $a \in \pi_m^{m+1} C$ Then C_a is one of the below

- If C is a $(i_1, \dots i_m, 0)$-cell then $C = \Gamma f$, $f : \pi_m^{m+1} C \to R$ definalby continuous. $a \in \pi_m^{m+1} C$ then $C_a = \{f(a)\} \subseteq R$
- If C is a $(i_1, \dots i_m, 1)$-cell then $C = (f, g)$, $C_a = (f(a), g(a))$

Suppose the statement holds for some n then let $C \subseteq R^{m+n+1}$ be a cell. Consider the two projections $\pi^{m+n+1}_{m+n}, \pi^{m+n}_{m}$ and

$$\pi^{m+n}_{m} \circ \pi^{m+n+1}_{m+n} : R^{m+n+1} \to R^m$$

Two options: Either $C = \Gamma f$, then

$$C_a = \Gamma f_a \text{ where } f_a : (\pi^{m+n+1}_{m+n} C)_a \to R$$

and $f_a(x) = f(a, x)$
Or $C = (f, g)_D$ i.e. $f, g : D \to R$, $D \subseteq R^{m+n}$ cell, $D = \pi^{m+n+1}_{m+n}$ Then $C_a = (f_a, g_a)_E$, $E = D_a$. in both cases, C_a is a cell.

2. Exercise.

□

Corollary 3.7.3-A. *Let $S \subseteq R^m \times R^n$ a definable family then there exists $M_S \in \mathbb{N}$ such that for all $a \in R^m$ $S_a \subseteq R^m$ has a partition into M_S many cells.*

Proof. $S \subseteq R^m \times R^n$ $\mathcal{D}$ decomposition of $R^m \times R^n$ that partions S. Then S is a finite union of cells from $\mathcal{D}$, each fiber S_a is a finite union of $C_a, C \in \mathcal{D}$ but C_a is a cell by Proposition. A bound: $|\mathcal{D}|$. □

Note : There is a uniform bound on # of definable connected somponents of sets in definable family.

Theorem 3.7.4. *$\mathcal{R} = (R; <, \dots)$ o-minimal $\mathcal{L}$-structure, $R' = (R'; <, \dots)$ $\mathcal{L}$-structure. If $R \equiv R'$ then R' is o-minimal.*

Proof. $S \subseteq R$ definable, $S = \{r \in R : \mathcal{R} \to \varphi(x)[r]\}$ might use parameters from R. If φ is a $\mathcal{L}$-fla. over $\varnothing$. Then

$$\begin{aligned} \mathcal{R} \models \exists x_1, x_2, x_3 (&x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3 \\ &\wedge \forall c((x_1 < c < x_2 \to \varphi(c)) \wedge (c = x_3 \to \varphi(c)) \\ &\wedge \neg(x_1 < c < x_2 \vee c = x_3) \to \neg\varphi(c))) \end{aligned}$$

But if φ uses parameters, we TODO For all $S \subseteq R^{m+1}$ need formula $\forall \underline{a} \in R^m$ "S_a is finite union of points and intervals" Idea: Subset of R definableby formula w/ param is just a fiber in a definable family that is parameter-free definable. $S_a \subseteq R$ definable. By the note, there is some number M_S that only depends on the TODO Such that for each $\underline{a} \in R^m$ S_a is a finite union of at most M_S cells.
Then

$$\begin{aligned} \mathcal{R} \models \forall \underline{z} \exists x_1 \dots \exists x_{M_S+1} &\big(\forall y(y < x_1 \to \varphi_{\underline{z}}(y)) \vee \forall y(y < x_1 \to \neg\varphi_{\underline{z}}(y))\big) \\ &\wedge \big(\forall y(x_1 < y < x_2 \to \varphi_{\underline{z}}(y)) \vee \forall y(x_1 < y < x_2 \to \neg\varphi_{\underline{z}}(y))\big) \\ &\dots \end{aligned}$$

By elementarily equivalence: $\mathcal{R}' \models \dots$. □

Chapter 4

Boolean Algebra

↰ 10.12.2024

From [Kri98]? Our language in this chapter will be $\mathcal{L} = \{0, 1, +, \cdot, \overline{}\}$, where $+, \cdot$ are binary operations and $\overline{}$ is a unary operation. We will sometimes write $x + y \cdot z$ instead of $x + (y \cdot z)$ for more clarity. The axioms for boolean algebras are

$$\forall x, y, z \left(x + (y + z) = (x + y) + z \wedge x \cdot (y \cdot z) = (x \cdot y) \cdot z\right) \quad \text{(Associativity } +, \cdot)$$
$$\forall x, y \left(x + y = y + x \wedge x \cdot y = y \cdot x\right) \quad \text{(Commutativity of } +, \cdot)$$
$$\forall x \left(x + x = x \wedge x \cdot x = x\right) \quad \text{(Idempotence)}$$
$$\forall x, y, z \left(x \cdot (y + z) = x \cdot y + x \cdot z \wedge x + (y \cdot z) = (x + y) \cdot (x + z)\right) \quad \text{(Distributivity)}$$
$$\forall x, y \left(x \cdot (x + y) = x = x + (x \cdot y)\right) \quad \text{(Absorbtion)}$$
$$\forall x, y \left(\overline{x + y} = \overline{x} \cdot \overline{y} \wedge \overline{x \cdot y} = \overline{x} + \overline{y}\right) \quad \text{(De Morgan's Laws)}$$
$$\forall x \left(x + 0 = x \wedge x \cdot 0 = 0\right) \quad \text{(Laws of 0)}$$
$$\forall x \left(x + 1 = 1 \wedge x \cdot 1 = x\right) \quad \text{(Laws of 1)}$$
$$\forall x \left(x + \overline{x} = 1 \wedge x \cdot \overline{x} = 0 \wedge \overline{\overline{x}} = x\right) \quad \text{(Laws of } \overline{})$$

BA denotes the $\mathcal{L}$-theory of boolean algebras, i.e. the deductive closure of the axioms above. Every model of BA is called a boolean algebra.

Note : Every boolean algebra $\mathcal{B}$ can be partially ordered by

$$x \leq y \quad \text{iff} \quad x + y = y$$

It is easy to see that $\leq$ is reflexive, antisymmetric and transitive. In this ordering the smallest set is 0 and the largest one is 1. In this notion the supremum and infimum of two elements are equal to

$$\sup\{x, y\} = x + y, \quad \inf\{x, y\} = x \cdot y \quad \text{(see Exercises)}$$

That makes every boolean algebra to a complemented, distributive **lattice**, see appendix. lattice

Example 4.1. $X \neq \varnothing$ and $S \subseteq \mathcal{P}(X)$ such that

- $\varnothing \in S$
- $X \in S$
- S is closed under (finite) intersections and unions and complements.

Then $(S; \varnothing, X, \cup, \cap, \overline{})$ is called a boolean algebra of sets and $\leq$ corresponds to $\subseteq$.

Note : Conversly, it is true that every boolean algebra is isomorphic to a boolean algebra of sets. In fact it can be embedded in a boolean algebra of all subsets of a set. The next section yields the choice of the set.

4.1 Stone Representation Theorem

Definition 4.1. (Ultra-)Filter: Suppose $\mathcal{B} \models BA$. A non-empty subset $F \subseteq B$ is called a filter on $\mathcal{B}$, if

(F1) $0 \notin F$

(F2) $\forall a \forall b \, (a \in F \wedge b \in F) \to a \cdot b \in F$

(F3) $\forall a \forall b \, (a \in F \wedge a \leq b) \to b \in F$

An **Ultrafilter** of $\mathcal{B}$ is a filter $\mathcal{F}$ that satisfies Ultrafilter

(UF) $\forall a \in B \; a \in \mathcal{F} \vee \overline{a} \in \mathcal{F}$.

The congruence with filters defined in functional analisis seems unambiguous, however the one key difference here is that a filter on $\mathcal{B}$ is a subset of B. If $\mathcal{B}$ is a boolean algebra of the set X then a filter in our definition would coincide with a filter (in the sense of the usual definition) on the maximal element X.
Recall: Let $X \neq \varnothing$ be a set, $\tau \subseteq \mathcal{P}(X)$ then (X, τ) is called a **topological** space, if

(T1) $\varnothing, X \in \tau$

(T2) $\forall I \, \forall (\sigma_i)_i \in \tau^I \; \bigcup_{i \in I} (\sigma_i) \in \tau$

(T3) $\forall n \in \mathbb{N} \, \forall (\sigma_i)_i \in \tau^{\{1,\ldots n\}} \; \bigcap_{1 \leq i \leq n} (\sigma_i) \in \tau$

And $\tau' \subseteq \mathcal{P}(X)$ is called a base for the topology τ on X, if every (open) set in τ is the union of sets in τ'. In other words, if $\tau = \{\bigcup_{U \in \mathcal{S}} U : \mathcal{S} \subseteq \tau'\}$

Definition 4.2. **Stone space:** A **Stone space** is a non-empty topological space (X, τ) which

Stone space

1. has a basis $\sigma \subseteq \tau$ (for the topology) of clopen sets

$$\begin{aligned} &\forall V \in \sigma \; X \setminus V \in \tau && \text{(clopen)} \\ &\forall U \in \tau \forall x \in U \exists V \in \sigma \; x \in V \subseteq U && \text{(basis)} \end{aligned}$$

2. is compact (every open cover of X contains a finite subcover)
3. and hausdorff (every two distinct points can be seperated by open sets)

$$\forall x, y \in X \; x \neq y \to \exists U_x, U_y \in \tau \; x \in U_x \wedge y \in U_y \wedge U_x \cap U_y = \varnothing$$

Let $\mathcal{B}$ be a boolean algebra. We define $S(\mathcal{B}) :=$ {the set of all ultrafilters on $\mathcal{B}$} and call it the Stone space of $\mathcal{B}$, equipped with the so called <u>Stone topology</u> : The topology generated by $\sigma = \{\langle a \rangle : \; a \in B\}$, where $\langle a \rangle := \{x \in S(\mathcal{B}) : a \in x\}$

$S(\mathcal{B})$
Stone topology

The clarification that such a topology even exists (and is unique) has to be proven (see: Theorem A.4.7) and that $S(\mathcal{B})$ is indeed a Stone space in the sense of the above definition will be given by the Stone-Representation theorem below. There exists other equivalent definitions of Stone spaces and some equivalent ones will be shown in the appendix A.4.1.
Note : Let $\mathcal{B} \models BA$ then

(i) for a filter F on $\mathcal{B}$ it holds $1 \in F$ (by $F \neq \varnothing$ and (F3))

(ii) Any subset $F' \subseteq B$ satisfying (F3) satisfies (F1) iff $F' \neq B$.

Back on $\langle a \rangle = \{x \in S(\mathcal{B}) : a \in x\}$. It holds

- $\varnothing = \langle 0 \rangle$
- $S(\mathcal{B}) = \langle 1 \rangle$
- $\langle a \rangle \cap \langle b \rangle = \langle a \cdot b \rangle$

Every filter on B can be extended to an ultrafilter on $\mathcal{B}$ (by use of Zorn's Lemma). Thinking like in example 4.1 "boolean algebra of sets", the binary function multiplication corresponds to the set intersection. This motivates viewing finite intersection property (FIP) in the setting of Boolean algebras as such: A subset $F \subseteq B$ is said to have (FIP), if for any $n \in \mathbb{N}$ and for all $f_1, \ldots f_n \in F$ it is $f_1 \cdot f_2 \cdot \cdots \cdot f_n \neq 0$

$$\forall n \in \mathbb{N} \forall f_1, \ldots f_n \in F \; f_1 \cdot f_2 \cdot \ldots \cdot f_n \neq 0$$

It can be proven that any subset $\mathcal{F} \subseteq B$ having (FIP) can be extended to an ultrafilter on $\mathcal{B}$. (see Exercises)

Theorem 4.1.1. Stone Representation Theorem:

(i) If $\mathcal{B} \models BA$, then $S(\mathcal{B})$ is a Stone-space

(ii) If $\mathcal{S}$ is a Stone space then the clopen subsets of $\mathcal{S}$ form a boolean algebra denoted by $B(\mathcal{S})$.

(iii) Every boolean algebra $\mathcal{B}$ is isomorphic to the boolean algebra $B(S(\mathcal{B}))$ with $a \mapsto \langle a \rangle$. Hence $\mathcal{B}$ is isomorphic to a subalgebra of the boolean algebra $\mathcal{P}(S(\mathcal{B}))$ of sets

(iv) Every Stone space $\mathcal{S}$ is homeomorphic to the Stone space $S(B(\mathcal{S}))$

$$x \mapsto \{a \in S(\mathcal{B}) : x \in a\}$$

Proof. (ii)-(iv) are proven in the appendix in Theorem A.4.8. Let $\langle a \rangle = \{x \in S(\mathcal{B}) : a \in x\}$

(i) **Claim:** $\sigma = \{\langle a \rangle : a \in B\}$ is a base for a topology consisting of clopen sets.
proof of claim. The first part is left as an exercise. For a hint, see Theorem A.4.7. From now on let τ be the Stone topology on $S(\mathcal{B})$. Let $a \in B$.

$$\langle a \rangle^c = \{x \in S(\mathcal{B}) : a \notin x\} \overset{(UF)}{=} \{x \in S(\mathcal{B}) : \overline{a} \in x\} = \langle \overline{a} \rangle \in \sigma \subseteq \tau$$

So $\langle a \rangle$ is closed. It is open by definition, hence $\langle a \rangle$ is clopen. ⊠

Claim: $(S(\mathcal{B}), \tau)$ is a T2-space.
proof of claim. Let $x, y \in S(\mathcal{B})$ such that $x \neq y$. Because x and y are ultrafilters, there exists some $a \in x \subseteq B$ such that $a \notin y$. Otherwise we would have $x \subsetneq y$ but then there would be a $b \in y$ such that $b \notin x$, by (UF) $\overline{b} \in x \subseteq y$, by (F2) $0 = b \cdot \overline{\in} y$, a contradiction to (F1).

By (UF) once again: $\exists a \in Ba \in x \wedge \overline{a} \in y$ and $x \in \langle a \rangle =: U, y \in \langle \overline{a} \rangle =: V$. From our previos observations we notice that $U \cap V = \varnothing$. ⊠

Claim: $(S(\mathcal{B}), \tau)$ is a compact space.
proof of claim. We use the following fact: "A topological space is compact, iff any family of closed sets which has (FIP), has non-empty intersection."

Let $(F_i)_{i \in I}$ be a family of closed subsets of $S(\mathcal{B})$ such that $I \neq \varnothing$ and for all $n \in \mathbb{N}$ and for all $i_1, \dots i_k \in I \bigcap_{1 \leq m \leq k} F_{i_m} \neq \varnothing$ (FIP). We show that $\bigcap_{i \in I} F_i \neq \varnothing$.
For $i \in I$ wlog. $F_i = \langle a_i \rangle$ for some $a_i \in B$. Otherwise $F_i^c = \bigcup_{b \in S_i} \langle b \rangle$ for some $S_i \subseteq B$, because F_i is closed and σ is base for τ. Hence

$$F_i = \bigcap_{b \in S_i} \langle b \rangle^c = \bigcap_{b \in S_i} \langle \overline{b} \rangle$$

Choosing $J = \{(i, b_i) : i \in I \wedge b_i \in S_i\}$ and associating b above with b_i gives us

$$\bigcap_{i \in I} F_i = \bigcap_{(i,b_i) \in J} \langle b_i \rangle$$

We can replace I with J since we assumed nothing about I, except beeing non-empty. And if J was to be empty, then every set F_i would be empty and the collection of F_i would not have (FIP).

Let $n \in \mathbb{N}$ and let $i_1, \dots i_k \in I \bigcap_{k \leq n} \langle a_{i_k} \rangle \neq \varnothing$, so there exists a ultrafilter containing all finitely many a_{i_k}. By an argument with (F2) we get If $a_{i_1} \cdot \dots \cdot a_{i_k} \neq 0$ and $\{a_i : i \in I\} \subseteq B$ has (FIP), this time in the sense of Boolean algebras and by an extra argument (using Zorns lemma) it extends to an ultrafilter on $\mathcal{B}$. Hence $\bigcap_{i \in I} \langle a_i \rangle \neq \varnothing$. ⊠

□

Definition 4.3. Atomic, Atomless: An atom is an element of a boolean algebra such that $a \neq 0$ and there is no element in the boolean algebra that is strictly inbetween 0 and a.

$$\forall y (0 \leq y \leq a \to (y = 0 \vee y = a))$$

$$\forall a (a \neq 0 \to \exists y (y \leq x \wedge y \neq 0 \wedge \forall z (0 \leq z \leq y \to (z = 0 \vee z = y))))$$

A boolean algebra $\mathcal{B}$ is called atomic, if

$$\forall a (a \neq 0 \to \exists y (y \leq a \wedge y \text{ is atomic}))$$

A boolean algebra is atomless if it contains no atoms.

$$\forall y y \neq 0 \to \exists z (0 < z < y)$$

Note : There exists boolean algebras that are neither atomic nor atomless.

4.2 Lindenbaum-Tarski Algebras

↰ 12.12.2024

Let $\mathcal{L}$ be a first order Language, $\mathcal{L}_0$ the set of all $\mathcal{L}$-sentences and $\sim$ the logical equivalence relation. On the quotient set $\mathcal{L}_0/_\sim$ we can define $\wedge, \vee, \neg$ py passing to representatives. This is well defined and does not depend on the choice of representatives.

Definition 4.4. Lindenbaum-Tarski algebra: With the above notation

$$B_L = (\mathcal{L}_0/_\sim; \perp/_\sim, \top/_\sim, \vee, \wedge, \neg)$$

forms then a boolean algebra. (it is called Lindenbaum-Tarski algebra for $\mathcal{L}$)
Note that $\perp$ is logically equivalent to $\exists x x \neq x$ and $\top$ is logically equivalent to $\forall x x = x$
The construction can be extended to equivalence modulo some $\mathcal{L}$-theory T (or $T \subseteq \mathcal{L}_0$) Let $n \in \mathbb{N}$ and $\underline{x} = (x_1, \dots x_n)$ For $\varphi, \psi \in \mathcal{L}_{\underline{x}}$, where $\mathcal{L}_{\underline{x}}$ are the $\mathcal{L}$-formulas with free variables among $\{x_1, \dots x_n\}$
Define $\varphi \leq_T \psi$ iff $T \models \forall \underline{x}(\varphi \to \psi)$
We can define T-equivalence: $\varphi \sim_T \psi$ iff $\varphi \leq_T \psi$ and $\psi \leq_T \varphi$

$$\mathcal{B}_n = (\mathcal{L}_{\underline{x}}/_{\sim_T}; \perp/_{\sim_T}, \top/_{\sim_T}, \wedge, \vee, \neg)$$

Is then again a boolean algebra, whose isomorphism type depends only on T and it is called the n-th Lindenbaum-Tarski-algebra of T. In the case we take the 0-th L-T algebra of $\varnothing$ $B_L = B_0(\varnothing^{\models 0})$

As a recap from the previous chapters:

- The deductive closure of a set of sentences Σ is $\{\varphi : \sigma \models \varphi\}$
- A contradiction is any sentence of the form $\varphi \wedge \neg\varphi$
- A set of sentences is consistent, if its deductive closure does not contain a contradiction.
- A $\mathcal{L}$-theory is a set of sentences that is consistent and deductively closed.

The question we now ask ourselves is: what is the stone space of a Lindenbaum-Tarski algebra?
Note :

- $\mathcal{L}$-theories are indeed exactly the filters of $\mathcal{B}_L$
- complete $\mathcal{L}$-theories are exactly the ultrafilters of $\mathcal{B}_L$

Let S_L equal the set of all complete $\mathcal{L}$-theories then our compactness theorem

$$\Gamma \models \varphi \implies \exists \Gamma' \subseteq \Gamma \text{ finite } \Gamma \models \varphi$$

is equivalent to

Theorem 4.2.1. Compactness Theorem *: *S_L with stone topology is compact.*

Two things we would like to show:

- Compactness theorem $\implies$ Compactness theorem *
  By showing that $S_L = S(B_\mathcal{L})$

  proof of claim. $\subseteq$ Let T be complete $\mathcal{L}$-Theory. by consistency and abuse of notation $0 \notin T$. and T is closed under conjunction. So for all $\varphi, \psi \in T$ we have $\varphi \wedge \psi \in T$. $\varphi \in T$ and $\varphi \leq \psi$ then $\models \varphi \to \psi$ so $\varphi \models \psi$ and $\psi \in T$ bc. T is deductively closed.

  $\supseteq$ Let $x \in S(\mathcal{B}_\mathcal{L})$ completeness: By maximality of x, $\forall \varphi$ either $\varphi/_\sim \in x$ or $\neg\varphi/_\sim \in x$.
  deductively cloesdness: $x \models \gamma$ then by compactness Theorem $\exists x' \subseteq x x' \models \gamma$ and $x' \in x$, so by if $x' \in x$ and $x' \leq \gamma$, then $\gamma \in x$ hence deductive closure
  consistency: $0 \notin x$ and x is deductively closed.

  ⊠

- Compactness theorem * $\implies$ compactness theorem $\Gamma = \{\gamma_i : i \in I\}$ set of $\mathcal{L}$-sentences. We want: $\Gamma \models \varphi$ then $\exists \Gamma' \subseteq \Gamma$ finite $\Gamma' \models \varphi$

proof of claim. Suppose, by contradiction that it is not the case.

$\forall I' \overset{\text{fin}}{\subseteq} I\{\gamma_i : i \in I'\} \cup \{\neg\varphi\}$ is consistent.

$$\implies \forall I' \overset{\text{fin}}{\subseteq} I \bigcap_{i \in I'} \langle\varphi_i\rangle \cap \langle\neg\varphi\rangle \neq \varnothing$$

$$\{\langle\varphi_i\rangle : i \in I\} \cup \{\langle\neg\varphi\rangle\}$$

is a collection of closed sets with (FIP). By using compactness of S_L with stone topology,

$$\bigcap_{i \in I} \langle\varphi_i\rangle \cap \langle\neg\varphi\rangle \neq \varnothing$$

hence $\Gamma \not\models \varphi$. ⊠

CHAPTER 5

Set Theory of ZFC

The contents on this chapter are at least partially sourced on [Kri98].

Example 5.1. **Russel's paradox:** Let $A = \{a : a \notin a\}$. If any collection of elements is a set, then A would be a set. Question: is $A \in A$? if yes, then $A \notin A$, if not then $A \in A$

Trying to resolve this, we will introduce the ZFC (Zermelo-Frankel axioms w/ choice) System. Let $\mathcal{L} = \{\in\}$ be a Language of first order, where $\in \ldots$ binary relation "beeing element of" For $(\mathcal{U}, \in)$ If $\mathcal{A} = (\mathcal{U}, \in^{\mathcal{A}}) \models$ ZFC, then the elements of the universe $\mathcal{U}$ are called sets. We will show roughly that some definably sets are not sets (in the sense of ZFC), others are not. The latter will be called classes.

5.1 First axioms of ZFC

Axiom 1. (Def. 5.1) **Axiom of extensionality:**

$$\forall x \forall y (x = y \leftrightarrow \forall u (u \in x \leftrightarrow u \in y))$$

In other words, two sets are the same if they have the same elements. This will give us later uniqueness in construction of other sets.

Axiom 2. (Def. 5.2) **Pairing Axiom:** for any two sets a, b one can form a set whose elements are precicely a, b

$$\forall x \forall y \exists z \forall u\, u \in z \leftrightarrow (u = x \vee u = y)$$

Our notation will be $z = \{x, y\}$

In words: For any two sets there exists a set whose members are those two sets.
Note :

- $\{x, y\}$ is unique by Axiom 1
- $\{x\}$ is a set. from Axiom 2, take $x = y$

Let $x_1, x_2, \ldots x_n$ be sets We define the **n-tuple** $(x_1, \ldots x_n)$ inductively: n-tuple

- $(x_1, x_2) := \{\{x\}, \{x, y\}\}$.
- $(x_1, \ldots x_n) := (x_1(x_2, \ldots x_n))$

Note : The set (x, y) exists, because its obtained by repeatedly using Axiom 2

Lemma 5.1.1. *Let x, y, a, b be sets. Then $(x, y) = (a, b)$ iff $x = a$ and $y = b$*

Proof. If $x = a$ and $y = b$ then by Axiom 1 $(x, y) = (a, b)$. The other direction by cases:

- case $x = y$, then $(x, y) = \{\{x\}\}$ is a singleton then (a, b) is a singleton, wlog $\{a\} = \{a, b\}$ then $a = b = x$.
- case $x \neq y$ and $\{\{x\}, \{x, y\}\} = \{\{a\}, \{a, b\}\}$ then $\{x\} = \{a\}$ and $\{x, y\} = \{a, b\}$ because by Axiom 1 a singleton can not be equal to a set of size 2.

□

Lemma 5.1.2. *For all $n, m > 1$ and for all sets $x_1, \ldots, x_n, y_1, \ldots, y_m$:*
$(x_1, \ldots, x_n) = (y_1, \ldots, y_m)$ iff $n = m$ and $\forall i \leq n x_i = y_i$

Proof. By induction, left as an exercise □

Axiom 3. (Def. 5.3) **Union Axiom:** For every set x there is a set z consisting of all elements of the elements of x.

$$\forall x \exists z \forall y (y \in z \leftrightarrow \exists u (u \in x \wedge y \in u))$$

We call z the union of x, notation: $\cup_x := z$
The union of two sets is often abbreviated with $x \cup y := \cup_{\{x,y\}}$

Example 5.2. 1. $\bigcup_{(x,y)} = \{x, y\}$.

2. $(x_1, x_2, \ldots x_n) = \bigcup_{\{x_1\}, x_2, \ldots x_n}$

Note :

- For all sets $x_1, \ldots x_n$ there is exactly one set with elements $x_1, \ldots x_n$
- The union is asociative $x \cup (y \cup z) = (x \cup y) \cup z$

Axiom 4. (Def. 5.4) **Power set Axiom:** Let $x \subseteq y$ be the abbreviation for $\forall z (z \in x \to z \in y)$. For every set x there exists a set z consisting of all subsetes $y \subseteq x$ that are themselve sets.

$$\forall x \exists z \forall y (y \in z \leftrightarrow y \subseteq x)$$

Notation: $\mathcal{P}(x) := z$.

Or in words: "For every set x there is a set z consisting of all subcollections of x that are themselve sets."

5.1.1 Classes and functions

Definition 5.5. **Classes:** All the unary $\mathcal{L}$-definable relations (w/ parameters) are called classes.

Example 5.3.

- $\varphi(x) \equiv x = x$ defines the universe $\mathcal{U}$, a class that is not a set
- $\varphi(x) \equiv \exists u (u \in x \wedge \forall v \, (v \in u \to v \in x))$

Definition 5.6. **Class functions:** Suppose we have a formula $\phi(x_1, \ldots x_n, y)$. Then we say ϕ defines a class function R_ϕ iff

$$\forall x_1 \ldots \forall x_n \forall y \forall y' ((\phi(\underline{x}, y) \wedge \phi(\underline{x}, y')) \to y = y')$$

We can then define the domain and image of the class function.

$$\operatorname{dom} R_\phi := \{(x_1, \ldots x_n) : \exists y \phi(\underline{x}, y)\}$$

$$\operatorname{im} R_\phi := \{y : \exists \underline{x} \phi(\underline{x}, y)\}$$

Note that $R_\phi(\underline{x}) = y$ iff $\phi(\underline{x}, y)$

Axiom 5. (Def. 5.7) **Axiom of replacement / substitution:** Let $\varphi(x, y, \underline{a})$ a $\mathcal{L}$-fla., w/ free variables among x, y and set-parameters $\underline{a}$. Suppose φ defines a class function on $\mathcal{U}$, than the followoing is an axiom:

$$\forall u \exists z \forall y \, (y \in z \leftrightarrow \exists x (x \in u \wedge \varphi(x, y, \underline{a})))$$

i.e. the image of a set under a class function is a set.

Axiom 6. (Def. 5.8) **Axiom scheme of comprehension:** Let $\psi(x, \underline{a})$ be an $\mathcal{L}$-formula. Then the followoing is an axiom:

$$\forall u \exists z \forall v \, (v \in z \leftrightarrow (v \in u \wedge \forall \psi(v, \underline{a})))$$

i.e. all elements of a set that satisfy a given $\mathcal{L}$-formula form a set.

Note : Axiom 6 follows from Axiom 5

Axiom 7. (Def. 5.9) **Set existence:**

$$\exists x x = x$$

i.e. $U \neq \varnothing$. - this is clear when we view it as a universe of a structure.

Note on the existence of the empty set: Let u be any set (there exists one by Axiom 7), $\psi(x) \equiv x \neq x$ then by Axiom 6 $\varnothing := \{x \in u : \psi(x)\}$ is a set. ↰ 17.12.2024
Note : We can derive pairing from replacement Axiom 5, extensionality, powerset and set existence. From set existence: $\varnothing$ is a set By powerset, replacement(comprehension): $\{\varnothing\}$ is a set.

$$\mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$$

is a set. Then by defining a class function $R_\phi(x) = y$,

$$\phi(x, y) \equiv (x = \varnothing \wedge y = a) \vee (x = \{\varnothing\} \wedge y = b)$$

Note : If the domain of a class function happens to be a set then the graph of the class function is a set. R_ϕ defined by $\phi(x, y, \underline{a})$ then the domain $u := \operatorname{dom} R_\phi \in \mathcal{U}$ Image $v := \operatorname{im} R_\phi \in \mathcal{U}$ by replacement

$$\{(x, y) : x \in u \wedge y \in v \wedge \phi(x, y, \underline{a})\}$$

is the graph of R_ϕ The above would be a set if $u \times v$ which can be shown by using comprehenson (exercise).

Definition 5.10. Function: A function $f : a \to b$ where a, b are sets is a subset of $a \times b$ that satisfies the followoing

- $\forall x\, (x \in a \to \exists y \in b\, (x, y) \in f)$
- $\forall x \forall y \forall y'\, (((x, y) \in f \wedge (x, y') \in f) \to y = y')$

Families of sets and cartesian products

Suppose we have a function $a : I \to X$. Let a_i be the unique $x \in X$ s.th. $(i, x) \in a$. We define the following sets:

$$\bigcup_{i \in I} a_i := \{z \in \bigcup X : \exists i \in I z \in a_i\}$$

$$\bigcap_{i \in I} a_i := \{z \in \bigcup X : \forall i \in I z \in a_i\}$$

$$\prod_{i \in I} a_i := \{f : I \to \bigcup X : \forall i \in I z \in a_i\}$$

Note : If $I = \varnothing$ then $\bigcap_{i \in I} a_i = \bigcup X$

Class relations and well ordering

Types of well ordered sets

Definition 5.11. Strict (linear) order: Let R be a class relation, C be a class. Then R defines a strict ordering on C, if

(i) $\forall x \forall y \forall z\, R(x, y) \to (C(x) \wedge C(y))$

(ii) $\forall x \forall y\, \neg(R(x, y) \wedge R(y, x))$

(iii) $\forall x \forall y \forall z\, (R(x, y) \wedge R(y, z)) \to R(x, z)$

The ordering is linear, if additionally

(iv) $\forall x \forall y\, (C(x) \wedge C(y)) \to (x = y \vee R(x, y) \vee R(y, x))$

Definition 5.12. Well ordering: Let R be a strict ordering on C and x be a set such that $\forall y \in x\, C(y)$ Then x is called well-ordered by R, if

$$\forall \varnothing \neq y \subseteq x \ y \text{ has a smallest element}$$

i.e.

$$\forall y \big((\varnothing \neq y \wedge y \subseteq x) \to \exists y' \big(y' \in y \wedge \forall z (z \in y \to (R(z, y') \vee y' = z))\big)\big)$$

Definition 5.13. Initial segment: Let x be a set, well ordered by R. Then $y \subseteq x$ is called an initial segment of x, if

$$\forall s \forall t\, (s \in x \wedge t \in x) \to \big((t \in y \wedge R(s,t)) \to s \in y\big)$$

Let x be well-ordered by $<$ and $y \in x$. Then $\delta_y^{<}(x) := \{z \in x : z < y\}$. If there is no ambiguity among the well ordering, we abbreviate $\delta_z(x) := \delta_y^{<}(x)$. With $<$ above strict it holds $y \notin \delta_y(x)$

Note : If x is well-ordered by $<$ and $y \subseteq x$ then

$$y \text{ is an initial segment of } x \text{ iff } y = x \text{ or } y = \delta_z(x) \text{ for some } z \in x$$

Proof. $\delta_z(x)$ is well ordered for $z \in x$.
Let $y \subseteq x$ an initial segment. Suppose $x \neq y$ that means $x \setminus y \neq \varnothing$ Let z be the smallest element of $x \setminus y$ (exists by well-ordering of x). Suppose $y \neq \delta_z(x)$. Then there is $a \in y$ $z < a$ and y is not an initial segment. □

Definition 5.14. Propper class: A class C is called a proper class if it is not a set.
i.e. if C is given by $\phi(x, \underline{a})$ then there is no $z \in \mathcal{U}$ such that $\forall x x \in z$ iff $\phi(x, \underline{a})$

Example 5.4. $\mathcal{U}$ is a proper class: If $\mathcal{U}$ was a set then $\{x : x \notin x\}$ would be a set.
Ord, the class of all ordinals is a proper class.

Definition 5.15. well-ordering (class): A class relation R defining a strict ordering on a class C is called a well-ordering, if
for every $x \in C$ the class initial segment $\delta_x^R(C) = \{y : R(y,x)\}$ is a set that is well ordered by R.

$$\forall x C(x) \to \exists z\, z = \delta_x^R(C) \wedge \ z \text{ has a smallest element}$$

5.2 Ordinals

Definition 5.16. Tranistivity of sets: A set x is called transitive, if $\forall y\, (y \in x \to y \subseteq x)$

Note : It corresponds to Tranistivity of the belonging relation "$\in$". $z \in y \in x \to z \in x$

Definition 5.17. Ordinal: An ordinal is a transitive set which is well ordered by $\in$.

Note : The collection of all ordinals form a class relation, notation: Ord, On Ord
Proof: Write down formula On

Example 5.5. • $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$ are ordinals

Lemma 5.2.1. Characterization of ordinals: *Let α be a set. α is an ordinal, iff*

- *the initil segments of α are α itself and the elements of α*
- *if $\beta \in \alpha$ then β is an ordinal.*
- $\alpha \notin \alpha$

Proof. Problem set □

Lemma 5.2.2. *Let $\alpha, \beta \in \mathrm{Ord}$ then either $\alpha = \beta$, or $\alpha \in \beta$ or $\beta \in \alpha$.*

Proof. Let $\gamma := \alpha \cap \beta$
Claim: γ is initial segment of both α and β
proof of claim. $x \in y \in \gamma$ then $x \in y \in \alpha$ and $x \in y \in \beta$. but α, β are ordinals, so $x \in \alpha$ and $x \in \beta$ and $x \in \gamma$ ⊠

Then by previous lemma, either

- $\gamma = \alpha$ and $\gamma = \beta$ and we are done
- $\gamma = \alpha$ and $\gamma \in \beta$, so $\alpha \in \beta$
- $\gamma \in \alpha$ and $\gamma = \beta$, so we have $\beta \in \alpha$.

- $\gamma \in \alpha$ and $\gamma \in \beta$ we have $\gamma \in \alpha \cap \beta = \gamma$ which is impossible

Hence the statement follows. □

Proposition 5.2.3. Ord *is well-ordered by* $\in$

Proof. We need to show that if $\alpha \in \mathrm{Ord}$ then $\delta_\alpha(\mathrm{Ord})$ is a set which is well ordered by $\in$.
$\delta_\alpha(\mathrm{Ord}) = \{\beta \in \mathrm{Ord} : \beta \in \alpha\} = \alpha$ And α is a well-ordered set. By Lemma 5.2.2 Ord is even linearly ordered by $\in$. □

Lemma 5.2.4. Ord, *the class of all ordinals is a proper class.*

Proof. Suppose Ord would be a set z.
Ord is well ordered by $\in$ Ord is transitive: $y \in x \in \mathrm{Ord}$ then $y \in \mathrm{Ord}$ by Lemma 5.2.1 so Ord would be an ordinal itself and we would have $\mathrm{Ord} \in \mathrm{Ord}$ which is not possible by Lemma 5.2.1. □

Note :

- If $\alpha \in \mathrm{Ord}$ then the initial segments of α are α and the elements of α.
- If $\alpha \in \mathrm{Ord}$ and $\beta \in \alpha$ then $\beta \in \mathrm{Ord}$
- $\alpha, \beta \in \mathrm{Ord}$ then $\alpha \subseteq \beta$ iff $\alpha \in \beta$ or $\alpha = \beta$
- $\alpha \subseteq \beta$ iff $\alpha = \beta$ or $\alpha \in \beta$.

↰ 07.01.2025

Lemma 5.2.5. : *If* $\alpha \in \mathrm{Ord}$ *then* $\alpha \cup \{\alpha\} \in \mathrm{Ord}$ *and* $\alpha \cup \{\alpha\}$ *is the successor of* α *in the ordering* $\in$

Proof.
- $\alpha \cup \{\alpha\}$ transitive:
$x \in y \in \alpha \cup \{\alpha\}$ if $y \in \alpha$ then $x \in \alpha$ hence $x \in \alpha \cup \{\alpha\}$. else $y = \alpha$ then $x \in \alpha \cup \{\alpha\}$
- $\alpha \cup \{\alpha\}$ well-ordered:
$\varnothing \neq x \subseteq \alpha \cup \{\alpha\}$ if $x \cap \alpha \neq \varnothing$ then there is $x_0 \in x \cap \alpha$ smallest. $x_0 \in \alpha$ and is smallest element in $\alpha \cup \{\alpha\}$ otherwise $\varnothing \neq x \subseteq \{\alpha\}$ then α is the smallest element.
- $\alpha \cup \{\alpha\}$ is successor of α
$\alpha \in \alpha \cup \{\alpha\}$ assume $\alpha \in \beta \in \alpha \cup \{\alpha\}$ if $\beta \in \alpha$ then $\alpha \in \beta \in \alpha$, so by transitivity, $\alpha \in \alpha$ which is not possible.
else $\beta = \alpha$

□

Note : If α, β are ordinals then we will use $\alpha \in \beta$, $\alpha < \beta$, $\alpha \subsetneq \beta$ interchangable.
$\gamma \in \mathrm{Ord}$ then $\gamma = \{\alpha \in \mathrm{Ord} : \alpha \in \gamma\}$

Lemma 5.2.6. *X a set of ordinals, then* $\sup X = \bigcup X$ *is an ordinal and* $\forall \alpha \in X \alpha \subseteq \bigcup X$ *and* $\bigcup X$ *is smallest with this property.*

Proof.
- $\bigcup X$ transitive: $x \in y \in \bigcup X$. then $\exists \alpha \in X$ such that $y \in \alpha$. then $x \in \alpha$ hence $x \in \bigcup X$
- $\bigcup X$ well-ordered by $\in$: $\bigcup X$ contained in Ord and is a set, but Ord is well-ordered, so $\bigcup X$ is well-ordered.
- $\alpha \in X$ then $\alpha \subseteq \bigcup X$.
- $\alpha \in X$ then $\alpha \subseteq \bigcup X$. Let $\beta \in \bigcup X$ then there exists $\alpha \in X$ such that $\beta \in \alpha$ so β is not an upper bound for X.

□

Lemma 5.2.7. *Suppose that* $\alpha, \beta \in \mathrm{Ord}$ *and* $f : \alpha \to \beta$ *that is strictly increasing i.e.* $\forall \gamma, \delta \in \alpha \gamma < \delta \to f(\gamma) < f(\delta)$ *Then* $\alpha \subseteq \beta$ *and* $\forall \gamma\, \gamma \leq f(\gamma)$

Proof. By contradiction, Let $\gamma \in \alpha$ be the smallest element with $f(\gamma) < \gamma$ then by minimality of γ, $f(\gamma) \leq f(f(\gamma))$. Because f is strictly increasing $f(f(\gamma)) < f(\gamma)$. So we get $f(\gamma) \leq f(f(\gamma)) < f(\gamma)$ but $f(\gamma) \notin f(\gamma)$ because $f(\gamma) \in \beta$.
Suppose $\beta \in \alpha$ then $f(\beta) < f(\alpha) < \beta$ so $f(\beta) < \beta$, a contradiction. □

Theorem 5.2.8. *$f : \alpha \to \beta$ isomorphism between $(\alpha, \in)$, $(\beta, \in)$ and $\alpha, \beta \in \mathrm{Ord}$ then $\alpha = \beta$ and f is unique such isomorphism, hence $f = id_\alpha$.*

Proof. $\alpha = \beta$:
Apply previous lemma to f, f^{-1} hence $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$ uniqueness:
$\gamma \in \alpha$ then $\gamma \leq f(\gamma)$ and $\gamma \leq f^{-1}(\gamma)$ by prev lemma we get $\gamma \leq f(\gamma) \leq \gamma$ so $f(\gamma) = \gamma$ □

Note : $y \mapsto \beta_y$ for $y \in Y$ is function defined on Y and maps to $Z = \{\beta(x) : x \in Y\}$ (is a set by replacement)

Theorem 5.2.9. *Let $(X, <_X)$ be well-ordered, then there is a unique isomorphism onto an ordinal $(\alpha, \in)$*

Proof. uniqueness:
Suppose we have $f : (X, <_X) \to (\alpha, \in)$, $g : (X, <_X) \to (\beta, \in)$ isomorphisms then $f \circ g^{-1}$ and by prev thm: $\alpha = \beta$ and $f \circ g^{-1} = id_\alpha$ so $f = g$.
Existence:
define $y = \{x \in X : (\delta_x, <_X)$ is isomorphic to an ordinal$\}$ where $\delta_x := \delta_x(X)$.
For each $y \in Y$ there is a unique ordinal $\beta_y \in \mathrm{Ord}$ such that $(\delta_y, <_X)$ and $(\beta(y), \in)$ are isomorphic.
Claim: Y is initial segment of X
proof of claim. If $x <_X y \in Y$, $f : \delta_y \to \beta(y)$ isomorphism, then f maps $\delta_x \subseteq \delta_y$ to initial segment of β, hence to an ordinal. ⊠

$y \mapsto \beta_y$ for $y \in Y$ is function defined on Y and maps to $Z = \{\beta(x) : x \in Y\}$ (is a set by replacement)
Claim: $Z = \{\beta(x) : x \in Y\}$ is an initial segment in Ord
proof of claim. if $\gamma \in \beta(x)$, $x \in Y$ have isomorphism between $(\delta_x, <_X)$ and $(\beta_x, \in)$ so its preimage y and $(\delta_y, <_X)$ is mapped to the initial segment determined by gamma, so to $(\gamma, \in)$ hence $(\delta_y, <_X)$ isom to γ ⊠

So Z is initial segment of Ord and Z is a set. So $\alpha := Z$ is itself an ordinal and $y \mapsto \beta_y$ isomorphism between Y and α.
Assuming $Y \subsetneq X$, then there is a minimal $x_0 \in X \backslash Y$ $\delta_{x_0} = Y$ (Y is initial segment) $Y \cong \alpha$ hence $x_0 \in Y$, a contradiction. □

5.3 Transfinite induction/recursion

Suppose $\phi(x)$ (possibly with parameters) to prove

$$\forall \alpha \in \mathrm{Ord}\, \phi(\alpha) \text{ iff } \forall \alpha \forall \beta ((\beta < \alpha \to \phi(\beta)) \to \phi(\alpha)) \tag{5.1}$$

5.1 $\implies \forall \alpha \in \mathrm{Ord}\, \phi(\alpha)$
Suppose ther is $\alpha \in \mathrm{Ord}$ such that $\neg\phi(\alpha)$ then let α be smallest with the property.
proof by induction on ordinals (proof by transfinite induction) is a proof of $\forall \alpha \in \mathrm{Ord}\, \phi(\alpha)$ by proving 5.1
Let F be a class function in one variable and a a set contained in $\mathrm{dom}(F)$ Then $F|_a = \{(x, y) \in (a, b) : F(x) = y\}$ where $b = \{F(x) : x \in a\}$ (which is a set by replacement).
Let H be any class function in one variable.

Definition 5.18. `H-inductive:` A function f is called H-inductive, if 1. $\alpha := \mathrm{dom}(f) \in \mathrm{Ord}$ and 2. $\forall \beta \in \alpha f|_\beta \in \mathrm{dom}(H)$ and 3. $f(\beta) = H(f|_\beta)$

$f : \alpha \to X$ then H gives you a way to extend the f. H extends f to a function on $\alpha \cup \{\alpha\}$

$$f(\alpha) = H(f)$$

Lemma 5.3.1. *For every class function H and ordinal α there is at most one H-inductive function on α with domain α.*

Proof. Suppose not. $f, g : \alpha \to X$ different H-inductive functions. Let x_0 the smallest element of α such that $f(x_0) \neq g(x_0)$. By x_0 smallest, $f|_{x_0} = g|_{x_0}$ By H-inductiveness

$$f(x_0) = H(f|_{x_0} = H(g|_{x_0}) = g(x_0)$$

A contradiction. □

Lemma 5.3.2. *Let H be a class function, $\alpha \in \mathrm{Ord}$ such that any function $f : \beta \to X$ where $\beta \in \mathrm{Ord}$ belongs to $\mathrm{dom}(H)$ then there is an H-inductive function $f : \alpha \to X$.*

Proof. $\tau = \{\beta < \alpha : \text{ there is H-inductive } f_\beta : \alpha \to X\}$ τ is a set and initial segment of α hence $\tau \in \mathrm{Ord}$ and $\tau \subseteq \alpha$
$\beta \mapsto f_\beta$ for $\beta \in \tau$ is well-defined function by uniqueness of f_β.
Moreover for $\gamma < \beta < \tau$ we have $f|_\beta|_\gamma = f|_\gamma$ (H-ind, uniqueness)
$f := \bigcup_{\beta<\tau} f_\beta$ is H-inductive function (graphs agree on intersection, each of the f_β are H-ind). The domain

$$\mathrm{dom}(f) = \sup_{\beta<\tau}(\beta) = \bigcup_{\beta<\tau}(\beta) = \sigma \in \mathrm{Ord}$$

If $\sigma = \alpha$ we are finished, otherwise we can define
$\tilde{f}$ such that $\tilde{f}|_\sigma = f$ and $\tilde{f}(\sigma) = H(f)$ $\tilde{f}$ is now H-inductive, and $\mathrm{dom}(\tilde{f}) = \sigma \cup \{\sigma\}$ a contradiction σ, the domain of f. □

↰ 09.01.2025

Theorem 5.3.3. Transfinite Recursion: *Let A be a class, M be a class of all functions $f : \alpha \to X$ for $\alpha \in \mathrm{Ord}$ arb. X a subset of A and H a class function in one variable defined on all of M with values in A. Then there exists a unique class function F defined on Ord such that $\forall \alpha F(\alpha) = H(F|_\alpha)$*

Proof. define F by

$$F(\alpha) = y \text{ iff there is an H-inductive function } f : \alpha \to X,\ X \text{ subset of } A \text{ and } y = H(f)$$

It is well defined by the prev two lemmas TODO □

5.4 Axiom of Choice and Zermelo's Theorem

Axiom 8. (Def. 5.19) Axiom of Choice (AC): For every set X and $A \subseteq \mathcal{P}(X)$ that consists of pairwise disjoint, non-empty subsets of X there is a set $T \subseteq X$ such that $\forall a \in A\ \# a \cap T = 1$.
T as above is called a transversal.

Definition 5.20. (AC'): (Existence of choice function.) For every set X there exists a function $\pi : \mathcal{P} \setminus \{\varnothing\} \to X$ such that for every non-empty subset $a \subseteq X$ $\pi(a) \in a$

Definition 5.21. (AC''): Let $(X_i)_{i\in I}$ be an indexed family of non-empty sets then $\prod_{i\in I} X_i \neq \varnothing$. ($(X_i)_{i\in I}$ could also be thought of as a function $i \mapsto X_i : I \to X$)

In the next Problemset: $AC \leftrightarrow AC' \leftrightarrow AC''$

Theorem 5.4.1. Zermelo: *"Well-ordering theorem": Every set can be well-ordered.*

Proof. By contradiction. Suppose X is a set which can not be well-ordered. Choose a choice function $\pi : \mathcal{P}(X) \setminus \varnothing \to X$ define a class function H by $H(f) = y$ iff f is a function with (i) $\mathrm{dom}\, f = \alpha \in \mathrm{Ord}$ (ii) $\mathrm{im}\, f \subsetneq X$ (iii) $y = \pi(X \setminus \mathrm{im}\, f)$
note:

- H is defined on class of all H-inductive functions, whose image is a proper subset of X. $\mathrm{im}\, f \subsetneq X$.
- each H-inductive function is injective

If f is a H-inductive function that is also surjective, then we are done because f induces well-ordering on X.
By our assumption, every H-inductive function has to be not surjective. So H is defined on all H-inductive functions, and we can use Transfinite recursion theorem and get an H-inductive class function $F : \mathrm{Ord} \to X$ which is injective
Suppose $\alpha < \beta < \gamma$ and $F(\alpha) = F(\beta)$. $F(\alpha) = \pi(X \setminus \mathrm{im}\, F|_\alpha)$ $F(\beta) = \pi(X \setminus \mathrm{im}\, F|_\beta)$
Then F is an injection from a proper class into $\mathrm{im}\, F$, a set, which is impossible. □

Theorem 5.4.2. Zorn's Lemma: *Let $(X, \leq)$ be a partially ordered set (poset) such that all linearly ordered subsets (called chains) have an upperbound. Then $(X, \leq)$ has a maximal element. i.e. $\exists y \in X \forall x \in X y \not< x$*

Proof. Let $A := \{Y \subseteq X : \exists x \in X \forall y \in Y y < x\}$ Take $\pi : \mathcal{P}(X) \backslash \{\varnothing\} \to X$ a choice function.
Define
$$p : A \to X, \quad p(Y) := \pi(\{x \in X : \forall y \in Y y < x\})$$
Define a class function H by $H(f) = y$ iff (i) f is a function with $\operatorname{dom} f \in \mathrm{Ord}$ (ii) $\operatorname{im} f \in A$ (iii) $y = \pi(\operatorname{im} f)$
We get

- any H-inductive $f : \alpha \to X$ is strictly increasing. $(\star)$ $f : \alpha \to X, f(\alpha \cup \{\alpha\}) = H(f) > \operatorname{im} f$
- The image of any H-inductive function $f : \alpha \to X$ is linearly ordered, so by assumption has an upperbound.

  i.e. $\exists x_f \in X \, \forall \beta < \alpha f(\beta) < x_f$

The idea now is similar to above therorem. Suppose $f : \alpha \to X$ is H-inductive but H is not defined on f, then the image of f has no strict majorant, so there is $\beta < \alpha$ such that $x_f = f(\beta)$. Then x_f has to be maximal for X.
If H is actually defined on all H-inductive functions, then there is an H-inductive class function $F : \mathrm{Ord} \to X$. F is strictly increasing by $\star$. So have injective of proper class into set. a contradiction. □

5.5 Ordinal arithmetic and the size of a set

Or how to think of the natural numbers to be contained in Ord

Definition 5.22. successor / limit ordinals:

- $0 := \varnothing$ is the smallest ordinal
- $\beta \in \mathrm{Ord}$ then its successor ordinal is defined by $\beta + 1 := \beta \cup \{\beta\}$
- β is called a **successor ordinal** if there is $\alpha \in \mathrm{Ord}$ such that $\beta = \alpha + 1$ successor ordinal
- $\beta \in \mathrm{Ord}$ is called a **limit ordinal** if $\beta \neq 0$ and β is not a successor ordinal limit ordinal
- $\beta \in \mathrm{Ord}$ is called a natual number / finite ordinal, if $\beta = 0$ or for every $\alpha \leq \beta$ we have "α is a successor ordinal or $\alpha = 0$"

Example 5.6. $0 = \varnothing$, $1 = \varnothing \cup \{\varnothing\} = \{\varnothing\}, \ldots$

Note : If $(X, <_X), (Y, <_Y)$ are well-ordered sets then we can well-order both their cartesian product $X \times Y$ and **disjoint union** $X \sqcup Y := (X \times \{0\}) \cup (Y \times \{1\})$ by the reverse lexiographical order: disjoint union
$$(x_0, y_0) \lhd (x_1, y_1) \text{ iff } y_0 <_Y y_1 \vee (y_0 = y_1 \wedge x_0 <_X x_1)$$
For $(x_0, y_0), (x_1, y_1)$ in $X \times Y$ or $X \sqcup Y$, in the latter case $y_0 <_Y y_1$, if $y_0 = 0$ and $y_1 = 1$. Note that in this case the the ordering of $X \sqcup Y$ corresponds to
$$(a, i) \lhd (b, j) \text{ iff } \begin{cases} i = j = 0 \text{ and } a <_X b, \text{ or} \\ i = j = 1 \text{ and } a <_Y b, \text{ or} \\ i < j \end{cases}$$

Definition 5.23. Sum and product of ordinals: Let $\alpha, \beta \in \text{Ord}$, then

(i) $\alpha + \beta$ is the unique $\gamma \in \text{Ord}$ such that γ is order-isomorphic to the sum / disjoint union $\alpha \sqcup \beta$

(ii) $\alpha \cdot \beta$ is the unique $\gamma \in \text{Ord}$ such that γ is order-isomorphic to the product of α and β

Properties of sum and product.

Lemma 5.5.1. *1. $+$ is associative, 0 is a 2-sided add. identity*

2. $\cdot$ is associative

3. $\alpha \cdot 0 = 0$, $\alpha \cdot 1 = \alpha = 1 \cdot \alpha$

4. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

5. λ limit ordinal then $\alpha \cdot \lambda = \sup_{\beta < \lambda} \alpha \cdot \beta$

Proof. Problem set ☐

Note : Right now it would be consistent to assume $\cdot$ is commutative, but no longer after the next axiom

Axiom 9. (Def. 5.24) Axiom of infinity: There exists an infinite ordinal.
i.e. there exists an ordinal that is not a natural number.

Note :

- The natural numbers form an initial segment in Ord.
- Let ω be the smallest infinite ordinal, i.p. ω is a limit ordinal

Example 5.7. 1. $\omega \cdot 2 \stackrel{?}{=} 2 \cdot \omega$, observations:

  - $\omega \cdot 2 = \omega + \omega$
  - $2 \cdot \omega = \omega$
  - $\omega + \omega \neq \omega$

2. $\omega + 2 \neq 2 + \omega$, observations:

  - $2 + \omega = \omega$
  - $\omega + 2$ has maximal element

↰ 14.01.2025

Definition 5.25. Exponentiation: α^β is defined recursively on β:

1. $\alpha^0 := 1$
2. $\alpha^{\beta+1} := \alpha^\beta \cdot \alpha$
3. $\alpha^\lambda := \sup_{\delta < \lambda} \alpha^\delta$ for a limit ordinal λ

Note : Alternatively, we can define exponentiation as given by the class function EXP : $\text{Ord} \times \text{Ord} \to \text{Ord}$ defined by

$$Exp(\alpha, \beta) = \gamma \quad \text{iff} \quad \text{There exists a function } f : \beta + 1 \to \text{Ord} \text{ such that for all } \xi < \beta \text{ it holds}$$

- $f(\beta) = \gamma$
- if $\xi = 0$ then $f(\xi) = 1$
- if $\xi = \delta + 1$ then $f(\xi) = f(\delta) \cdot \alpha$
- if ξ is a limit ordinal then $f(\xi) = \sup_{\delta < \xi} f(\delta)$

TODO uniqueness follows from recursion theorem.

Definition 5.26. Cardinality: Given a set X, the cardinality of X (denoted by $|X|$, $card(X)$) is the smallest ordinal for which there is a bijection with X.

Note : Every set has a cardinality. This is bc every set can be well-ordered (by Zermelo's Theorem, which is equivalent to AC) and then we get an order-preserving bijection. In fact the statement "$card(X)$ is defined for each set X" is equivalent to AC.

Definition 5.27. Equinumerous sets: We call two sets X, Y equinumerous, if there is a bijection between them.

By the previous Definition 5.26 we have: X, Y equinumerous iff $|X| = |Y|$

Theorem 5.5.2. *Let X, Y be non-empty sets Then the following are equivalent.*

(i) there is an injection of X into Y

(ii) there is a surjection of Y into X

(iii) $|X| \leq |Y|$

Proof. "$i \implies ii$" don't need AC "$ii \implies i$" need AC

□

Theorem 5.5.3. Cantor-Schröder-Bernstein: *$|X| = |Y|$ iff there is an injection $X \to Y$ and an injection $Y \to X$*

Proof. □

Proof is clear with AC, but can proof it without AC need to do some kind of back and fourth argument

Theorem 5.5.4. Cantor: *For every set X we have $|\mathcal{P}(X)| > |X|$*

Proof. Suppose its not, then there exists a set X with $|X| \geq |\mathcal{P}(X)|$.
We can find a surjection $\pi : X \to \mathcal{P}(X)$ $Y = \{x \in X : x \notin \pi(x)\}$ Let $y \in X$ be such that $\pi(y) = Y$
Either $y \in Y$ or $y \notin Y$ If $y \in Y$ then by definition of Y, $y \notin Y$ If $y \notin Y$ then by definition $y \in Y$ □

Note this says: there is no largest set.

Definition 5.28. Cardinal: $\kappa \in \text{Ord}$ is called a cardinal, if $\kappa = |\kappa|$

Note : **Claim:** The class Card of all cardinals is a proper class.
proof of claim. Suppose Card is a set. then $\sup \text{Card} = \gamma \in \text{Ord}$ Then $|\mathcal{P}(\gamma)| > |\gamma| \geq |\lambda|$ for every cardinals $\lambda \in \text{Card}$ that is a contradiction ⊠

Definition 5.29. finite sets: A set X is called finite, if $|X|$ is a finite ordinal, otherwise X is called infinite.

Note : In particular a set is infinite iff ω injects into it.

Proposition 5.5.5. Galileo: *A set a is infinite iff it properly injects into itself.*

Proof. $\implies$ Suppose a is infinite, then by note use ω injects into it show ω injects properly in it self
$\impliedby$ want: set finite then it does not inject properly into itself show: every finite ordinal is a cardinal inductively, then $\kappa = |\kappa|$

□

The $\aleph$-function

Note : Card is cofinial in Ord. That is for every $\alpha \in$ Ord there is $\gamma \in$ Card with $\alpha < \gamma$
And Card is a proper subclass of Ord, well ordered by the same ordering $\in$ as in Ord
There is a unique function (class function) from the class of all ordinals to the infinite cardinals that preserves $\in \ldots$ this function is called $\aleph$

Definition 5.30. : Instead of $\aleph(0)$ we will write subscript $\aleph_0$

- $\aleph_0 = |\omega|$
- $\aleph_{\alpha+1}$ is the smallest cardinal larger than $\aleph_\alpha$

Note : For every cardinal κ there is a smallest cardinal κ^+ such that $\kappa < \kappa^+$ this is not the successor in the sense of the ordinals. It is

- $n^+ = n + 1$
- $\aleph_\alpha^+ = \aleph_{\alpha+1}$

Note : For every ordinal γ we have $|\gamma| \leq \gamma < |\gamma^+|$
Similarly as with ordinals we will call cardinals of the form κ^+ successor cardinals. and non-zero, non-successor cardinals will be called limit cardinals. successor cardinals limit cardinals

Proposition 5.5.6. *The function $\aleph$ is continuous with respect to the interval topology induced by $\in$. i.e. If λ is a limit ordinal then $\aleph_\lambda = \sup_{\delta<\lambda} \aleph_\delta$*

Proof. let $\gamma := \sup_{\delta<\lambda} \aleph_\delta$ have $|\gamma| \leq \gamma < |\gamma|^+$ Assume $\gamma < \aleph_\lambda$ then there exist a $\xi_0 < \lambda$ such that $|\gamma| = \alpha_{\xi_0}$
Then we would have $\gamma < |\gamma|^+ = \aleph_{\xi_0+1} \leq \sup_{\delta<\lambda} \aleph_\delta$ which is a contradiction.
we have shown that $\aleph_\lambda \leq \sup_{\delta<\lambda} \aleph_\delta$ (TODO: other direction?) □

Definition 5.31. `countable sets`: A set X is called to be countable iff $|X| \leq \aleph_0$ and we say X is uncountable otherwise.

Continuum hypothesis (CH)

Hypothesis 5.5.7. `(CH)`:

$$|\mathcal{P}(\omega)| = \aleph_1$$

i.e. there is no cardinality between $|\mathbb{R}|$ and $|\mathbb{N}|$ or $|\mathbb{R}|$ is the first uncountable cardinality.

It can be shown that CH is independent of ZFC. (method to show this is called forcing, very popular method)

5.6 Cardinal Arithmetic

Definition 5.32. : Let κ, λ be cardinals then $\kappa \otimes \lambda := |\kappa \times \lambda|$ $\kappa \oplus \lambda := |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$

Note : $\oplus, \otimes$ are commutative and associative.

Theorem 5.6.1. *If κ is infinite, then $\kappa \otimes \kappa = \kappa$*

Proof. By induction on κ. What is ment by that is induction on α where $\aleph_\alpha = \kappa$.
Base case: We should check it for ω $\aleph_0 \times \aleph_0 = \aleph_0$ is like finding bijection of ω^2 onto ω.(diagonal)
Now suppose: $\forall \beta < \kappa |\beta| \otimes |\beta||\beta \times \beta| = |\beta|$
On The cartesian product define order such that we get order isomorphism, then this is also a bijection.
Define ordering on $\kappa \times \kappa$

$$(\alpha, \beta) \prec (\alpha', \beta') \iff \begin{cases} \max\{\alpha, \beta\} < \max\{\alpha', \beta'\}, \text{ or} \\ \max\{\alpha, \beta\} = \max\{\alpha', \beta'\} \text{ and } \alpha < \alpha', \text{ or} \\ \max\{\alpha, \beta\} = \max\{\alpha', \beta'\} \text{ and } \alpha = \alpha' \text{ and } \beta < \beta' \end{cases}$$

Claim:$\prec$ is a well-ordering (Exercise)
Claim: $(\kappa \times \kappa, \prec)$ is order-isomorphic to $(\kappa, \in)$ From the claim we imediately get $|\kappa \times \kappa| = |\kappa|$

proof of claim. $\kappa\times\kappa = \bigcup_{\alpha<\kappa}\alpha\times\alpha$ increasing union. then $\{\xi\times\xi\}$ is an initial segment of $\kappa\times\kappa$ (with respect to $\prec$) Consider $(\xi\times\xi,\prec)$ then by our inductive assumption this will have to be isomorphic to some ordinal $\gamma\in\mathrm{Ord}$. We know that the cardinality of that ordinal γ $|\gamma| = |\xi\times\xi| \overset{\text{ind. Hyp.}}{=} |\xi| < |\kappa|$ So also $|\gamma|<|\kappa|$. so then $(\kappa\times\kappa,\prec)$ is order-isomorphic to $(\kappa,\in)$ ⊠

□

Corollary 5.6.1-A. *For every infinite cardinal κ we have $\kappa\oplus\kappa=\kappa$*

Proof. Proof of corollary: $\kappa\oplus\kappa = |\kappa\times 2| \leq |\kappa\times\kappa| = |\kappa|$ □

Definition 5.33. : Define for κ,λ cardinals $\kappa^\lambda := |\{f : f \text{ is a function from } \lambda \text{ to } kappa\}|$

Note $2^\kappa = |\mathcal{P}(X)|$

Lemma 5.6.2. *If $\lambda\geq\omega$ and $2\leq\kappa\leq\lambda$ then $\kappa^\lambda = 2^\lambda$.*

Proof.
$$2^\lambda = 2^{\lambda\otimes\lambda} = 2^{\lambda^\lambda} \geq \lambda^\lambda \geq \kappa^\lambda$$
$2^{\lambda^\lambda}\geq\lambda^\lambda$ think of function that constantly maps to 0 is part TODO image 5 The other implication is obvious. □

↰ 16.01.2025

Definition 5.34. : A function $f:\alpha\to\beta$, where α,β are ordinals is said to be cofinal, if $\operatorname{im} f$ is unbounded in β i.e.
$$\forall\gamma\in\beta\exists\xi\in\alpha\gamma\leq f(\xi)$$
The cofiniality of $\beta\in\mathrm{Ord}$ (denoted by $cof(\beta)$) is the smallest ordinal α such that there exists a function $f:\alpha\to\beta$ that is cofinal. Note: $cof(\beta)\leq\beta$,

Example 5.8. $cof 1 = cof(\{\varnothing\}) = 1$ and in more generality $cof(\alpha+1) = cof(\alpha\cup\{\alpha\}) = 1$

Note : $cof(\beta)$ is always a cardinal. Let $\beta\in\mathrm{Ord}$ and let $f : cof(\beta)\to\beta$ be cofinal. Then $|cof(\beta)|\leq cof(\beta)|$ and there exists a bijection $h:|cof(\beta)|\to cof\beta$, so $f\circ h$ yields cofinal map and by minimality of $cof(\beta)$ have $|cof(\beta)| = cof(\beta)|$
If β limit ordinal then there is a strict increasing cofinal map $h: cof(\beta)\to\beta$ Let $f: cof(\beta)\to\beta$ be cofinal define $h: cof(\beta)\to\beta$ by
$$h(\xi) := \max\{f(\xi), \sup_{\gamma<\xi}(h(\gamma)+1)\}$$

Proposition 5.6.3. *Suppose α,β are limit ordinals, $f:\alpha\to\beta$ strictly increasing and cofinal. Then $cof(\alpha) = cof(\beta)$*

Proof. One side is obvious

- $cof(\alpha)\geq cof(\beta)$ is clear
- $cof(\alpha)\leq cof(\beta)$ Let $g: cof(\beta)\to\beta$ be cofinal define $h: cof(\beta)\to\alpha$ by
$$h(\xi) = \min\{\gamma<\alpha : f(\gamma) > g(\xi)\}$$
cofinal in α

□

Corollary 5.6.3-A. *For a limit ordinal α, $cof(\aleph_\alpha) = cof(\alpha)$*

Proof. Use Proposition 5.5.6 and Proposition 5.6.3 on $f=\aleph$ □

Corollary 5.6.3-B. *For every ordinal β, $cof(\beta) = cof(cof(\beta))$*

Proof. By cases

- β, $cof(\beta)$ are limit ordinals: there exists a strictly increasing map $cof(\beta)\to\beta$ result follows from Proposition 5.6.3

- $\beta, cof(\beta)$ are not limits then $\beta = \alpha \cup \{\alpha\}$, $cof(\beta) = 1$ and $cof(cof(\beta)) = cof(1) = 1$

$\square$

Definition 5.35. regular ordinals: An ordinal β is called regular, if $cof(\beta) = \beta$ (it is fixed point of cofininality map)

Note : regular ordinals are cardinals and the first regular, infinite cardinal is ω

Lemma 5.6.4. *κ^+ is reglular for $\kappa \geq \omega$*

Proof. If $f : \alpha \to \kappa^+$ is cofinal, then $\kappa^+ = \bigcup_{\gamma<\alpha} f(\gamma) = \sup_{\gamma<\alpha} f(\gamma)$. Each $f(\gamma)$ is an ordinal of cardinality less then κ^+ hence is less or equal than κ

$$\kappa^+ = |\bigcup_{\gamma<\alpha} f(\gamma)| \leq |\alpha \times \kappa| \leq \max\{|\alpha|, |\kappa|\}$$

because $\kappa < \kappa^+$ we have $\kappa^+ \leq |\alpha| = \alpha$ TODO check $\square$

Note : α limit ordinal then $cof(\aleph_\alpha) = \alpha$
If $\aleph_\alpha$ is regular then $cof \aleph_\alpha = cof \alpha \leq \alpha \leq \aleph_\alpha$ So $\alpha = \aleph_\alpha$

Definition 5.36. : Let κ be a cardinal.

- κ is called weakly inaccessible, if α is a regular limit cardinal strictly greater than ω
- κ is called (strongly) inaccessible, if $\kappa > \omega$, κ regular and for every $\lambda < \omega$ we have $2^\lambda < \kappa$

We will see that κ inaccessible implies that κ is not a union of fewer that κ sets each of cardinality less that κ. Note : Existence of inaccessible cardinals does not follow from ZFC.
The question is, why care then?
Axiom of existence of inaccessible cardinals:

$$(IC) \quad \text{There exists some inaccessible cardinal.} \tag{5.2}$$

(AC) has less desirable consequences e.g. Banach-Tarski-Paradox
If we drop AC we could not prove that the lebesgue meassure is countably additive, so we may want to replace AC by something weaker (e.g. DC, dependent choice). Various nice results that depent on AC still hold.
LM the axiom that states: every set of reals is lebesgue meassurable

Theorem 5.6.5. *Con... consistency. Con(ZF+DC+LM) is equal to Con(ZF + IC)*

Note: We do now that ZF can not prove IC, but we dont know yet if ZF can prove the negation of IC.

Lemma 5.6.6. *If κ is an infinite cardinal and $\lambda \geq cof(\kappa)$ then $\kappa^\lambda > \kappa$*

Proof. Fix a cofinal map $f : \lambda \to \kappa$. Consider any function $G : \kappa \to \kappa^\lambda$ It suffices to show that G can not be surjective.
κ^λ is technically the set of functions $\lambda \to \kappa$ define $h : \lambda \to \kappa$ by

$$h(\xi) = \min\{\kappa \setminus (G(\alpha)(\xi)) : \alpha \leq f(\xi)\}$$

If $h \in \operatorname{im} G$ then there is $\alpha \in \kappa$ s.t. $G(\alpha) = \kappa$ pick $\xi < \lambda$ s.t. $f(\xi) \geq \alpha$ Then $h(\xi) = G(\alpha)(\xi)$ but by construction of h, $G(\alpha)(\xi) \neq h(\xi)$ so contradiction. $\square$

Corollary 5.6.6-A. *If $\lambda \geq \omega$ then $cof(2^\lambda) > \lambda$*

Proof. We know that $(2^\lambda)^\lambda = 2^{(\lambda \otimes \lambda)} = 2^\lambda$ So if $cof(2^\lambda) \leq \lambda$ then by lemma $(2^\lambda)^\lambda > 2^\lambda$, a contradiction. $\square$

Theorem 5.6.7. (König) : *Let I be a set, $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ indexed families of sets. If $\forall i \in I |A_i| < |B_i|$ then $|\bigsqcup_{i\in I} A_i| < |\prod_{i\in I} B_i|$*

Proof. Exercise with hints. $\square$

Note: above thm is equivalent to AC
One can use König theorem to show that: "κ inaccessible implies that κ is not a union of fewer that κ sets each of cardinality less that κ"TODO: check if actually true
Next time : ZF can not proof ZFC

5.7 Consistency of a theory

↰ 21.01.2025

Today we are going to state our last axiom, the axiom of foundation. We are also going to show a relative consistency theorem. We are going to finish prob next course thursday

Axiom 10. (Def. 5.37) Axiom of foundation (AF): Beeing an element of, does not admit an infinite decreasing chain.

$$\forall x x \neq \varnothing \to \exists y(y \in x \wedge \forall z \in y z \notin x)$$

Note : Suppose we have a sequence of sets $(u_n)_{n\in\omega}$ s.t. $\forall n u_{n+1} \in u_n$ then $\{u_n : n \in \mathbb{N}\}$ would contradict AF. and have $\forall x(x \notin x)$ too.

Definition 5.38. : Class function $V : \text{Ord} \to \mathcal{U}$ by transfinite induction on ordinals and set $V_\beta := \bigcup_{\alpha<\beta} \mathcal{P}(V_\alpha)$ $V_0 = \varnothing$ If $\alpha \leq \beta$ then $V_\alpha \subseteq V_\beta$ (an example of increasing sequence, in contrast to note on axiom AF) $V_{\beta+1} = \mathcal{P}(V_\beta)$
If λ is a limit ordinal then $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$
V is also a class given by $V(x) \equiv \exists\alpha \in \text{Ord } x \in V_\alpha$

V also gives us a way to associate a rank to each set.

Definition 5.39. : For every x such that $V(x)$ we define $\text{rk}(x) := \min\{\alpha : x \in V_\alpha\}$

Note : The rank $\text{rk}(x)$ is always a successor ordinal.

Lemma 5.7.1. *$V(x)$ iff $\forall y\, y \in x \to V(y)$ And also if $V(x)$ then $\forall y(y \in x \to \text{rk}(y) < \text{rk}(x))$*

Proof. " $\Longrightarrow$ " direction: $V(x)$ and $\text{rk}(x) = \beta + 1$ then $x \in V_{\beta+1}$ so $x \subseteq V_\beta$ hence $\forall y \in x\, y \in V_\beta$
Also $\text{rk}(y) \leq \beta < \beta + 1 = \text{rk}(x)$
" $\Longleftarrow$ "-direction: Suppose $\forall y \in x V(y)$ Note that $\text{rk} : V \to \text{Ord}$ is bounded on x. Else, $\{\text{rk}\, y : y \in x\}$ is unbounded in Ord and it is a set (image of function rk of a set) Take $\bigcup\{\text{rk}\, y : y \in x\} \in \text{Ord}$ and it would be an ordinal bigger than any other set, TODO
Suppose $\{\text{rk}\, y : y \in x\}$ is bounded by β then $\forall y \in x y \in V_\beta$ so $x \in V_{\beta+1}$. □

Lemma 5.7.2. *For every ordinal $\alpha \in \text{Ord}$ we have $V(\alpha)$ and $\text{rk}(\alpha) = \alpha + 1$.*

Proof. Exercise □

Definition 5.40. inductive closure: For any set x we define the function with domain ω $f(0) := x$ $f(n+1) := \bigcup_{y\in f(n)} y = \bigcup f(n)$ and we define the closure of x to be the union $\text{cl}\, x := \bigcup_{n<\omega} f(n)$

Note :

- $x \subseteq \text{cl}\, x$
- cl is transitive
- If z is transitive set that contains x then then $\text{cl}\, x \subseteq z$ ($\text{cl}\, x$ is the unique transitive closure of x)

Theorem 5.7.3. *(AF) holds iff $\forall x V(x)$*

Proof. " $\Longleftarrow$ "-direction: Suppose $\forall x V(x)$. We need to show that any set contains an element Let $a \neq \varnothing$. Let $y \in a$ be of minimal rank. Then for every $c \in y$ we know $\text{rk}\, c < \text{rk}\, y$ so $c \notin a$ by minimality of $\text{rk}\, y$. So $y \in a$ s.t. $y \cap a = \varnothing$
" $\Longrightarrow$ " direction: Lets assume the axiom of foundation and by contradiction that x is a set for which $\neg V(x)$ Have $x \subseteq \text{cl}\, x$ **Claim:** $Y = \{y \in \text{cl}\, x : \neg V(y)\} \neq \varnothing$
proof of claim. If $Y = \varnothing$ then $\forall y \in Y V(y)$ and rk bounded on Y. ⊠

Let $y \in Y$. Then $\neg V(y)$ so i.p. $y \not\subseteq V$, so for some $z \in y$ have $\neg V(z)$ but because $\text{cl}\, x$ is transitive, $z \in \text{cl}\, x$ hence $z \in Y$ hence $\forall y \in Y y \cap Y \neq \varnothing$, a contradiction with (AF). □

By Gödels 2-nd incompletness theorem ZFC can not prove its own consistency. There is a way to express consistency in the formal level, by coding and using peano arithmetic, and the above statement says that ZFC can not prove this sentence. All we can hope for are relative consistency results, and therefore relate two theories with each other. For example some theories are less debated about and it shows a way to prove independence of certain axioms from others.

5.7.1 relative consistency

- ZFC The axioms 1. extensionality 2. union 3. power-set 4. ax scheme of replacement 5. set ax 6. axiom of infinity 7. AC 8. AF (recall pairing and comprehension follow from the other)
- ZFC^- is ZFC without (AF)
- ZF is ZFC without (AC)
- ZF^- is ZF without (AF)

We will use a tool that is in set theory called Relativization
Let C be a class, $\phi(\underline{x}, \underline{a})$, $\underline{a} \in C$
$\phi^C(\underline{x}, \underline{a})$ defined by induction on compl of ϕ

- If ϕ is atomic then $\phi^C = \phi$
- $(\neg\phi)^C = \neg(\phi^C)$, $(\phi \vee \psi)^C = (\phi^C) \vee (\psi^C)$
- $(\exists y\phi)^C = \exists y(C(y) \wedge \phi^C)$
- $(\forall y\phi)^C = \forall y(C(y) \to \phi^C)$

Theorem 5.7.4. *Suppose $(\mathcal{U}, \in) \models ZF^-$ then V constructed in $\mathcal{U}$ is such that $(V, \in) \models ZF$ i.e. assume ZF^- is consistent (has a model) and we get a model of ZF*

Proof. ZF^- Take 1. extensionality 2. union 3. power-set 4. ax scheme of replacement 5. set ax 6. axiom of infinity (recall pairing and comprehension follow from the other) Need to check if $(\mathcal{U}, \in) \models ZF^-$ and V class defined by $V = \bigcup_{\alpha\in\text{Ord}} V_\alpha$, $V_\beta := \bigcup_{\alpha<\beta} \mathcal{P}(V_\alpha)$ then $(V, \in) \models ZF^-$ (AF will follow from prev. result)

1. Let $x, y \in V$ wts $(\forall z \in V z \in x \leftrightarrow z \in y) \leftrightarrow x = y$ $x, y \in V$ so $x, y \subseteq V$ and can use Axiom of extensionality in $\mathcal{U}$ $(\forall z \in \mathcal{U} z \in x \leftrightarrow z \in y) \leftrightarrow x = y$
2. Let $x \in V$ $U_x = \{z : \exists y \in x z \in y\} \subseteq V$ so $\bigcup_x \in V$.
3. Let $x \in V$ then $x \subseteq V$. so every subset of x is a subset of V hence is an element of V. Powerset $\mathcal{P}(x) \subseteq V$ hence $\mathcal{P}(x) \in V$
4. $\varphi(x, y)$ with parameters from V and assume that $\varphi(x, y)$ defines a class function in V i.e.
$$\left(\forall x \exists^{\leq 1} y \varphi(x, y)\right)^V$$
i.e.
$$\forall x(V(x) \to (\exists^{\leq 1} y \varphi(x, y)))$$
Then $\psi(x, y) :\Leftrightarrow V(x) \wedge V(y) \wedge \varphi^V(x, y)$ defines a class function in $\mathcal{U}$, use replacement in $\mathcal{U}$ That yields
$$\forall a \exists b y \in b \,\text{iff}\, (\exists x \in A \varphi(x, y)) \,\text{iff}\, \exists x \in a \varphi^V(x, y) \wedge V(y)$$
$b \subseteq V$ hence $b \in V$ and b is the image of a under the class function given by φ.
5. Since we proved that every ordinal is in V, $\varnothing \in V$.
6. enough to show that $\omega \in V$. in fact we know that any ordinal is in V.

□

An ordinal α is called regular, if it is equal to its own cardinality $\operatorname{cof}\alpha = \alpha$ A cardinal κ is called inaccessible if $\kappa > \omega$ and κ is regular.
It suffices to say $\forall\lambda < \omega 2^\lambda < \kappa$ Suppose now that $(\mathcal{U}, \in) \models ZFC$

Lemma 5.7.5. *If κ is an inaccessible cardinal, then $|V_\kappa| = \kappa$. Moreover for every $a \subseteq V_\kappa$ we have $a \in V_\kappa$ iff $|a| < \kappa$*

Proof. Recall, if $\alpha \in \text{Ord}$ then $\alpha \subseteq V_\alpha$. i.p. $\kappa \subseteq V_\kappa$ so $|\kappa| \leq |V_\kappa|$
The other direction: For this we are going to show inductively that for all $\xi < \kappa$ that $|V_\xi| < \kappa$ Then $\kappa \geq |V_\kappa|$ follows. If $|V_\xi| < \kappa$ then $|V_{\xi+1}| = |\mathcal{P}(V_\xi)| \leq |2^{|V_\xi|}| < \kappa$ by κ inaccessible. Suppose $|V_\xi| < \kappa$ for all $\xi < \lambda < \kappa$ where λ is a limit ordinal.

$$|V_\lambda| = |\bigcup_{\xi<\lambda} V_\xi| \leq \sup \xi < \lambda |V_\xi| < \kappa$$

here we use regularity of κ.
So have $|V_\kappa| = \kappa$. Left to show: for every $a \subseteq V_\kappa$ have $a \in V_\kappa$ iff $|a| < \kappa$

- "$\Longleftarrow$"-direction: Assume $a \subseteq V_\kappa$ and $|a| < \kappa$ rk : $a \to \text{Ord}$ is not cofinal in κ because $cof\kappa = \kappa > |a|$ So for some ordinal β have $a \subseteq V_\beta$ and then $a \in V_{\beta+1} \subseteq V_\kappa$
- "$\Longrightarrow$"-direction: it suffices $a \in V_\kappa$ then $|a| < \kappa$ Exercise.

□

↰ 23.01.2025

Lemma 5.7.6. *Let $(\mathcal{U}, \in) \models ZFC$. If κ is inaccessible, then $V_\kappa \models ZFC$.*

Proof. Will check (AC) and replacement, the remaining axioms are an exercise.

(AC) : Suppose $a \in V_\kappa$ and a is a family of pairwise disjoint, non-empty sets. By (AC) in $\mathcal{U}$ there is a transversal T in $\mathcal{U}$ for the set a. What is left to show is $T \in V_\kappa$. Have $a \subseteq V_\kappa$ then every subset of a is a subset of V_κ so in particular $T \subseteq V_\kappa$. We do know that $a \in V_\kappa$ so $|a| < \kappa$ and $T \subseteq a$ we have $|T| \leq |a|$ so by the previous lemma $T \in V_\kappa$.

(RE) : $\varphi(x, y)$ a formula with parameters in V_κ that defines a class function in V_κ. i.e.

$$\forall x \in V_\kappa \exists^{\leq 1} y \in V_\kappa \varphi^{V_\kappa}(x, y)$$

let $a \in V_\kappa$ then $\psi(x, y) \equiv x \in V_\kappa \wedge y \in V_\kappa \wedge \varphi(x, y)$ does define a class function on $\mathcal{U}$ with domain contained in V_κ So $f[a] \subseteq V_\kappa$ and $|f[a]| < \kappa$ so $f[a] \in V_\kappa$.

□

Now we are ready to state our meta-theorem, it is not a statement in first order language of set theory.

Theorem 5.7.7. *If ZFC is consistent then ZFC + "There are no strongly inaccessible cardinals" is also consistent*

Proof. Assume we have a model of ZFC, $(\mathcal{U}, \in) \models ZFC$.

- If $\mathcal{U}$ does not contain any inaccessible cardinals then we are done.
- Assume that $\mathcal{U}$ does contain inaccessible cardinals. Let κ be the smallest inaccessible cardinal.

What we want to show is that there are no inaccessible cardinals in V_κ. An ordinal is by definition a transitive set, well-ordered by $\in$. By (AF) α is transitive and α is linearly ordered by $\in$. i.e. α ordinal iff

$$\forall x, y \in \alpha(x \in y \vee y \in x \vee x = y) \wedge \forall x(x \in \alpha \to x \subseteq \alpha) \tag{5.3}$$

Claim 1: The ordinals in V_κ are the ordinals below κ i.e. $Ord^{V_\kappa} = \kappa$.

proof of claim. If $\alpha < \kappa$ then $\alpha \subseteq V_\kappa$ (rk $\alpha = \alpha + 1$ and $V_{\alpha+1} \subseteq V_\kappa$ so $\alpha \in V_\kappa$ $(\alpha + 1 < \kappa)$) α is an ordinal so 5.3 holds. and $\alpha \in \text{Ord}^{V_\kappa}$ If $\alpha \in \text{Ord}^{V_\kappa}$ then $\alpha \in V_\kappa$ and α is transitive and totally ordered by $\in$. Hence α is an ordinal. Left to show α is below κ. Have $|\alpha| < \kappa$, by the Lemma 5.7.5 $\alpha < \kappa$. ⊠

Claim 2: The cardinals in V_κ are the cardinals that are below κ.

proof of claim. Suppose λ is a cardinal in V_κ. in particular λ is an ordinal in V_κ and therefore an ordinal in $\mathcal{U}$ and by the Lemma 5.7.5 $\lambda = |\lambda| < \kappa$. left to show: λ is an actual cardinal in $\mathcal{U}$ Suppose there is a bijection $f : \lambda \to \alpha$ to some smaller ordinal α. Then by Claim 1, $\alpha \in \text{Ord}^{V_\kappa}$ and $f \subseteq V_\kappa$ (the graph $f \subseteq \lambda \times \alpha \subseteq \lambda \times \lambda$ so $f \subseteq \mathcal{PPP}(V_{\lambda+1}) =: V_\beta$, rk $\lambda = \lambda + 1$ and have $\beta < \kappa$ by inaccessability). $f \subseteq V_\kappa$ and $|f| < \kappa$ so by Lemma 5.7.5 $f \in V_\kappa$. f is a bijection in V_κ between λ and a smaller ordinal, a contradiction.
Suppose $\lambda < \kappa$ and λ cardinal, then by Lemma 5.7.5 $\lambda \in V_\kappa$ and $\lambda \in \text{Ord}^{V_\kappa}$. If there would be a bijection in V_κ between λ and a strictly smaller ordinal α then α would be an actual ordinal. By an argument as before would get a bijection in $\mathcal{U}$ between λ and smaller ordinal. ⊠

If $\lambda \in V_\kappa$ is a cardinal in V_κ then $\lambda < \kappa$ and by claim 2 it is a cardinal in $\mathcal{U}$, λ can not be inaccessible. by choice of κ.
Note that only $\mathcal{U}$ knows that λ is not inaccessible. We therefore need to check that also V_κ knows that λ is inaccessible. Reasons for λ to not be inaccessible:

- $\lambda \leq \omega$ (in $\mathcal{U}$) hence $(\lambda \leq \omega)^{V_\kappa}$ bc. $\omega \in V_\kappa$
- $\lambda \leq 2^\xi$ for some $\xi < \lambda < \kappa$ hence $\xi, 2^x i \in V_\kappa$ and have $(\xi < \lambda < 2^\xi)^{V_\kappa}$
- λ is not regular, if we have a function $f : \alpha \to \lambda$ cofinal and α an ordinal $\alpha < \lambda$, then $\alpha \in V_\kappa$ hence $f \in V_\kappa$ then λ is not regular in V_κ.

□

Note: $Ord^{V_\kappa} = \kappa$. but Ord^{V_κ} is no set in V_κ.

Chapter A

Appendix

A.3 On Model theory

Theorem A.3.1. Löwenheim-Skolem: *Let $\mathcal{L}$ be a language of cardinality λ. Γ a set of formulas and Σ a set of sentences.*

(i) If Γ is satisfiable, then it is satisfiable in some structure of cardinality at most λ

(ii) If Σ has any model, then it has a model of cardinality at most λ.

Proof. by using the LST theorem.

□

A.4 On Boolean Algebras

Definition A.1. lattice: A **lattice** is a set L with two binary, commutative and associative operations $\vee, \wedge$ satisfying the absorbtion axioms. lattice

$$\forall a \forall b\, a \wedge (a \vee b) = a$$
$$\forall a \forall b\, a \vee (a \wedge b) = a$$

A lattice is called distributive, if the distributive axioms hold. distributive

$$\forall a \forall b \forall c\, a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$\forall a \forall b \forall c\, a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

A lattice is called bounded, if it has a least element 0 and a greatest element 1. bounded

$$\exists 0 \forall a\, a \vee 0 = a$$
$$\forall 1 \forall a\, 1 \vee a = 1$$

A lattice is called complemented, if it is bounded and every element a has a complement b satisfying $a \vee b = 1$ and $a \wedge b = 0$. complemented

$$\forall a \exists b\, a \vee b = 1 \text{ and } a \wedge b = 0$$

Definition A.2. Alternative Def: Boolean Algebra: A **boolean algebra** is a set B with

- distinguished elements $0, 1$ (called zero and unit of B)
- a unary operation $'$ on B (called **complementation**)
- two binary operations $\vee$ called **join** and $\wedge$ called **meet** s.t. for all $x, y, z \in B$

  (i) $x \vee 0 = x \qquad x \wedge 1 = x$

  (ii) $x \vee x' = 1 \qquad x \wedge x' = 0$

  (iii) $x \vee y = y \vee x \qquad x \wedge y = y \wedge x$

  (iv) $(x \vee y) \vee z = x \vee (y \vee z) \qquad (x \wedge y) \wedge z = x \wedge (y \wedge z)$

  (v) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

With this definition a boolean algebra is exactly a complemented distributive lattice closed under the additional complementation map. The definition is compatible with the definition given in the chapter of boolean algebras. However we must be careful, a subalgebra of a Boolean algebra must again be closed under the restricted complementation map.[1]

[1] See https://math.nmsu.edu/people/personal-pages/files/ESSLLI2.pdf on slide 7 for example.

Example A.1. Let $X \neq \varnothing$ be a set, $B := \mathcal{P}(X)$ the power set of X, $0 := \varnothing$ and $1 := S$,

$$' : \mathcal{P}(S) \to \mathcal{P}(S), x' := S \backslash x \qquad x \vee y := x \cup y, \quad x \wedge y := x \cap y \text{ for } x, y \in \mathcal{P}(S)$$

Lemma A.4.1. *Let $(B,', \vee, \wedge, 0, 1)$ be a boolean algebra. Then it holds*

a) $0' = 1$, $1' = 0$

b) $x \vee x = x$, $x \wedge x = x$

c) $(x')' = x$

d) $(x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$

e) $x \vee y = y$ *iff* $x \wedge y = x$

Lemma A.4.2. *a) $x \leq y :\Leftrightarrow x \vee y = y$ defines a partial ordering on B (inclusion) and it holds*

b) $x \vee y$ is the least upper bound of $\{x, y\}$ in B
$x \wedge y$ is the greatest lower bound of $\{x, y\}$ in B

c) $0 \leq x \leq 1$ for all $x \in B$

Definition A.3. Opposite of boolean algebra: Let $(B,', \vee, \wedge, 0, 1)$ be a boolean algebra. The boolean algebra B^{op} is defined by

$$B^{\text{op}} := B, \quad 0^{\text{op}} := 1, \quad 1^{\text{op}} := 0, \quad ' \text{ stayes the same as for} B, \quad \vee^{\text{op}} := \wedge, \quad \wedge^{\text{op}} := \vee$$

Note: $(B^{\text{op}})^{\text{op}} = B$

Definition A.4. Subalgebra: A **subalgebra** of B is a subset $A \subseteq B$ s.t. $0, 1 \in A$ and A is closed under $', \wedge, \vee$. The subalgebra generated by $P \subseteq B$ is defined to be the smallest subalgebra containing P. Equivalently it is the intersection of all Subalgebras of B that contain P.

Example A.2. Power set algebra*:* Let S be a set then $\mathcal{P}(S)$ defines a boolean algebra on S. $B := \{x \in \mathcal{P}(S) : x \text{is finite or cofinite}\}$ is a subalgebra of $\mathcal{P}(S)$ w/ set of generators $\{\{s\} : s \in S\}$

Note : We will prove the Tarski-Stone Theorem: every boolean algebra is isomorphic to an algebra on a set.

Example A.3. Lindenbaum Algebra of Σ*:* Let A be a set of prop. atoms, Prop(A) the set of prop. generated by A. Further let $\Sigma \subseteq$ Prop(A) and p, q, r range over Prop(A).
We say p is Σ-equivalent to q iff $\Sigma \models_{\text{taut}} p \leftrightarrow q$ Σ-Equivalence is an equivalent relation on Prop(A) and Prop$(A)/\Sigma$ is a boolean algebra with

$$0 := \perp/\Sigma, \quad 1 := \top/\Sigma, \quad (p/\Sigma)' := (\neg p)/\Sigma, \quad (p/\Sigma \vee q/\Sigma) := (p \vee q)/\Sigma, \quad (p/\Sigma \wedge q/\Sigma) := (p \wedge q)/\Sigma$$

a set of generators is $\{a/\Sigma : a \in A\}$

Definition A.5. Homomorphisms of boolean algebras: Let B, C be boolean algebras. A map $\phi : B \to C$ is a (homo)morphism of boolean algebras iff $\forall x, y \in B$ it holds

- $\phi(0_B) = 0_C$
- $\phi(x') = \phi(x)'$
- $\phi(x \vee y) = \phi(x) \vee \phi(y)$
- $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$

If $\phi : B \to C$ is bijective too , we call ϕ an isomorphism and $\phi^{-1} : C \to B$ is also a morphism of boolean algebras.

Note : $\phi(B)$ is subalgebra of C

Example A.4. Let S, T be sets then a function $f : S \to T$ induces a morphism of boolean algebras $\mathcal{P}(T) \to \mathcal{P}(S) : y \mapsto f^{-1}(y)$ If $S \subseteq T$ and f the inclusion map $S \hookrightarrow T$ then we get a boolean algebra morphism $Y \to Y \cap S$.
• $id_B : B \to B$ • $x \mapsto x' : B \to B^{\text{op}}$ are both isomorphism

Note : A boolean algebra morphism $\phi : B \to C$ is injective iff $\ker f = 0_B$

Lemma A.4.3. *Let $X_1, \dots X_m \subseteq S$ and $\mathcal{A}$ a boolean algebra on S generated by $\{X_1, \dots X_m\}$. Then $\mathcal{A}$ is finite and isomorphic to $\mathcal{P}(\{1, 2, \dots n\})$ for some $n \leq 2^m$.*

Proof. TODO □

Definition A.6. Trivial algebras:

- B is trivial if $|B| = 1$ (equivalently $0 = 1 \in B$) according to Lemma A.4.3 B is isomorphic to $\mathcal{P}(\varnothing)$
- If $|S| = 1$ then $|\mathcal{P}(S)| = 2$ TODO

Definition A.7. Ideal: An ideal of B is a subset of $I \subseteq B$ s.t.

(I1) $0 \in I$

(I2) $\forall a, b \in B$ it holds $a \leq b$ and $b \in I \implies a \in I$ and $a, b \in I \implies a \vee b \in I$

Example A.5. $F_{\text{in}} = \{F \subseteq S : F \text{ finite}\}$ is ideal in $\mathcal{P}(S)$.

Note : If I is an ideal of B then $I \vee b := \{x \in B : x = a \vee b \text{ for some } a \in I\}$ is the smallest ideal w/ respect of $\subseteq$ of B that contains $I \cup \{b\}$.

Example A.6.
- For a boolean algebra morphism $\phi : B \to C$ the kernel $\ker(\phi)$ is an ideal in B.
- If I is an ideal in B then $a =_I b :\Leftrightarrow a \vee x = b \vee x$ for some $x \in I$ defines an equivalent relation and $B/_{=_I}$ is a boolean algebra w/

  $0 := 0/_{=_I} \quad 1 := 1/_{=_I} \quad (a/_{=_I})' := a'/_{=_I} \quad a/_{=_I} \vee b/_{=_I} := (a \vee b)/_{=_I} \quad a/_{=_I} \wedge b/_{=_I} := (a \wedge b)/_{=_I}$

  Then $\phi : B \to B/_{=_I} : b \mapsto b/_{=_I}$ is a boolean algebra morphism w/ $\ker(\phi) = I$

A.4.1 Notes on Stone spaces

Definition A.8. topological properties: Let X be a **topological space**. X is said to

topological space

(i) be compact, if every open cover of X has a finite subcover. A subset $K \subseteq X$ of a topological space is called compact if it is a compact subspace of X.

compact

(ii) be a T0-space or equivalently hausdorff, if

T0-space

$$\forall x \forall y\, x \neq y \rightarrow \exists U, V \in \tau\, (x \in U \wedge y \notin U) \vee (x \notin V \wedge y \in V)$$

(iii) be a T2-space, if

T2-space

$$\forall x \forall y\, x \neq y \rightarrow \exists U, V \in \tau\, (x \in U \wedge y \in V \wedge U \cap V = \varnothing)$$

(iv) be totally seperated, if

totally seperated

$$\forall x \forall y\, x \neq y \rightarrow \exists U, V \in \tau\, (x \in U \wedge y \in V \wedge U \cap V = \varnothing \wedge U \cup V = X)$$

(v) be zero-dimensional with respect to the small inductive dimension, if it has a base for the topology consisting of clopen sets.

zero-dimensional

$$\forall U \in \tau\, \forall x \in U\, \exists W \in \tau\, (x \in W \subseteq U \wedge W^c \in \tau)$$

(vi) be irreducible, if one of the equivalent conditions below is satisfied [2]

irreducible

- (a) No two nonempty open sets are disjoint.
- (b) X cannot be written as the union of two proper closed subsets. Proper means that none of the sets are equal to X or the empty set.
- (c) Every nonempty open set is dense in X.
- (d) The interior of every proper closed subset of X is empty.
- (e) Every subset is dense or nowhere dense in X.
- (f) No two points can be separated by disjoint neighbourhoods.

An irreducible set is a subset of a topological space for which the subspace topology is irreducible.

(vii) have a generic point, if there is a singleton $\{p\} \subseteq X$, whose closure is X. A subset $Z \subseteq$ of a topological space is said to have a gerneric point, if

generic point

$$\exists! p \in Z\, \overline{\{p\}} = Z$$

(viii) be sober, if every nonempty irreducible closed subset of X has a (necessarily unique) generic point.

sober

$$\exists p \in X\, \overline{\{p\}} = X$$

(ix) be coherent, if all of the following conditions are satisfied.
Let $K^\circ(X) = \{V \subseteq X : V \in \tau \wedge V \text{ is compact}\}$

coherent

- (a) X is compact, T0 and sober
- (b) $K^\circ(X)$ is a basis for the topology
- (c) $K^\circ(X)$ is closed under finite intersections

[2] from: https://en.wikipedia.org/wiki/Hyperconnected_space

Lemma A.4.4. *The definitions in Definition A.8 (vi) are indeed equivalent.*

Proof. $(c) \implies (d)$, $(d) \implies (e)$ and $(f) \implies (a)$ are left as an exercise.

$(a) \implies (b)$ Suppose X can be written es the union of two proper closed sets. Then $X = A \cup B$ with $A, B \neq \varnothing$ and $A, B \neq X$. Then A^c, B^c are open, nonempty and disjoint $(A^c \cap B^c = (A \cup B)^c = \varnothing)$.

$(b) \implies (c)$ Suppose there is a nonempty open set U with $\overline{U} \neq X$. Take $A := U^c$ and $B := \overline{U}$. We have $\varnothing \neq \overline{U}^c \subseteq U^c \neq X$ and $\varnothing \neq U \subseteq \overline{U} \neq X$ so A, B are proper closed sets with $X = A \cup B$.

$(e) \implies (f)$ Suppose (f) fails, then there exist $x, y \in X$ with $x \neq y$ and there exist neighbourhoods $U_x \in \mathcal{U}(x)$ and $U_y \in \mathcal{U}(y)$ with $U_x \cap U_y = \varnothing$. By definition there exist $V_x, V_y \in \tau$ with $x \in V_x \subseteq U_x$ and $y \in V_y \subseteq U_y$. Hence $x \in V_x^\circ = V_x$ and $y \notin \overline{V_x} \subseteq V_y^c$ and (e) fails too. □

Definition A.9. **Stone space:** A Stone space is a zero-dimensional, compact, hausdorff topological space X.

There are other equivalent equivalent definitions of Stone spaces in other mathematical works.

Proposition A.4.5. *The following statements are equivalent:*

(i) X is a Stone space

(ii) X is compact and totally seperated

(iii) X is compact, T0 and zero-dimensional

(iv) X is T2 and coherent

Proof. We show (i) $\iff$ (ii), (iii) $\implies$ (i), (i) $\implies$ (iv), (iv) $\implies$ (iii)

(i) $\implies$ (ii): We just have to show that X is totally seperated. Let $x, y \in X$ with $x \neq y$. Since X is hausdorff there exists a open set U such that $x \in U$ and $y \notin U$. For this U there exists a $W \subset U$ with $x \in W$ which is clopen. Therefore W^c is clopen too and $y \in U^c \subseteq W^c$, $W^c \cup W = X$.

(i) $\impliedby$ (ii): T2 follows from totally seperatedness, so let U be an open set and $x \in U$. We have to show that there exists a clopen subset $W \subseteq U$ with $x \in W$. We define V_y to be the clopen set V we get from totally seperatedness with respect to x and y. From $y \in V_y$ we get that $\{U\} \cup \{V_y : y \in U^c\}$ is an open cover of X, which is compact so there is a natural number n and $y_1, \ldots y_n \in U^c$ with $U \cup \bigcup_{i \leq n} V_{y_i} = X$, hence $\bigcup_{i \leq n} V_{y_i} \supseteq U^c$ and therefore $\bigcap_{i \leq n} V_{y_i}^c \subseteq U$. $W := \bigcap_{i \leq n} V_{y_i}^c$ is clopen, since finite intersections of clopen sets are clopen. Furthermore for every $i \leq n$ we have $x \in V_{y_i}^c$, so $x \in W$.

(iii) $\implies$ (i): Let $x, y \in X$ with $x \neq y$. Since X is T0, There exists open sets U, V with $x \in U \wedge y \notin U$ or $x \notin V \wedge y \in V$ Assuming $x \in U \wedge y \notin U$ does not hold, then $x \notin V \wedge y \in V$. Since X is zero-dimensional with respect to the small inductive dimension, there exists a clopen set $W \subseteq V$ with $y \in W$. Therefore $W^c \in \tau$, $x \in W^c$ and we have shown that X is a T2-space.

(i) $\implies$ (iv):

(a) compact and T2 are clear, we have to show that X is sober. Let $Z \subseteq X$ be a nonempty irreducible closed subset. Subspaces of T2 are T2, and since Z is irreducible, using condition (vi)f "No two points can be separated by disjoint neighbourhoods." we get that Z has to be a singleton, and for T2-spaces, singletons have a unique generic point, hence X is sober.

(b) Let σ be the basis for the topology τ consisting of clopen sets. Since X is T2, the closed sets are exactly the compact sets. Therefore $\sigma \subseteq K^\circ(X) \subseteq \tau$ and $K^\circ(X)$ is a basis for the topology.

(c) Let $n \in \mathbb{N}$, $A_1, \ldots A_n \in K^\circ(X)$. Like above, all A_i are closed, hence clopen. The finite intersection of clopen sets $\bigcap_{i \leq n} A_i$ is clopen and therefore open and compact and we have $\bigcap_{i \leq n} A_i \in K^\circ(X)$

(iv) $\implies$ (iii): Compact and T0 are given by definition. $K^\circ(X)$ is a basis for the topology and since X is T2, every set of $K^\circ(X)$ is closed, hence X is zero-dimensional. □

Note that in the previous lemma (i) $\implies$ (iii) is for free.

Lemma A.4.6. *Let $\mathcal{B}$ be a boolean Algebra and $U \subseteq B$. Then U is a ultrafilter on $\mathcal{B}$ if and only if $h : B \to \{0,1\}$, $h(x) := \chi_U(x)$ is a homeomorphism from $\mathcal{B}$ to the two-element boolean algebra.*

Proof. Let $x, y \in B$.

$\implies$: Clearly $x + y \geq x$ and $x + y \geq y$. Suppose $h(x) + h(y) = 1$ then $x \in U$ or $y \in U$, by (F3) $h(x + y) = 1$. Otherwise $x, y \notin U$. By (UF) we have $\overline{x}, \overline{y} \in U$. By (F2) we have $\overline{x} \cdot \overline{y} \in U$, but $\overline{x} \cdot \overline{y} = \overline{x + y}$. By (F1) we have $x + y \notin U$.

If $h(x) \cdot h(y) = 1$ then $x, y \in U$, by (F2) $x \cdot y \in U$ and $h(x \cdot y) = 1$. Similarly $x \cdot y \leq x, y$, so if $h(x \cdot y) = 1$ then $x \cdot y \in U$ and by (F3) $h(x) \cdot h(y) = 1$.

$h(0) = 0$ follows from (F1).

$h(1) = 1$ because $U \neq \varnothing$ and (F3).

$\overline{h(x)} = h(\overline{x})$ by (UF) and (F1). Suppose $1 = \overline{h(x)}$ then $x \notin U$ by (UF) we have $\overline{x} \in U$, so $h(\overline{x}) = 1$ Suppose $1 = h(\overline{x})$ then $\overline{x} \in U$ by (F1) $x \notin U$ and therefore $\overline{h(x)} = 1$

$\impliedby$: (F1): Suppose $0 \in U$ then $h(0) = 1 \neq 0$.

(F2): Let $x, y \in U$ then $1 = h(x) \cdot h(y) = h(x \cdot y)$, so $x \cdot y \in U$.

(F3): Let $x \in U$ and $y \in B$ with $x \leq y$. Then by definition $x+y = y$, hence $h(y) = h(x+y) = h(x + y) = h(x) + h(y) = 1 + h(y) = 1$.

(UF): Suppose there is a $x \in B$ with $x, \overline{x} \notin U$ then $0 = h(x) + h(\overline{x}) = h(x + \overline{x}) = h(1) = 1$.

□

Existence and uniqueness of a topology generated by a subset of the power set

Theorem A.4.7. *Let X be a set, $\sigma \subseteq \mathcal{P}(X)$. Then it is equivalent:*

1. *σ is a basis for a uniquely determined topology $\tau \supseteq \sigma$.*
2. *$X = \bigcup_{B \in \sigma} B$ and*

   $\forall B_1, B_2 \in \sigma \, \forall p \in B_1 \cap B_2 \, \exists B_3 \in \sigma \, (p \in B_3 \subseteq B_1 \cap B_2)$

Theorem A.4.8. *(i) If $\mathcal{B} \models BA$, then $S(\mathcal{B})$ is a Stone-space*

(ii) If $\mathcal{S}$ is a Stone space then the clopen subsets of $\mathcal{S}$ form a boolean algebra of sets denoted by $B(\mathcal{S})$.

(iii) Every boolean algebra $\mathcal{B}$ is isomorphic to the boolean algebra $B(S(\mathcal{B}))$ with $a \mapsto \langle a \rangle$. Hence $\mathcal{B}$ is isomorphic to a subalgebra of the boolean algebra $\mathcal{P}(S(\mathcal{B}))$ of sets

(iv) Every Stone space $\mathcal{S}$ is homeomorphic to the Stone space $S(B(\mathcal{S}))$

$$x \mapsto \{a \in S(\mathcal{B}) : x \in a\}$$

Proof. (i) is proven in Theorem 4.1.1

(ii) Let $B(S) = \{V \subseteq S : V \in \tau \wedge V^c \in \tau\}$ The functions on a boolean algebra of sets are defined as in 4.1. Hence we can, for simplicity write $\cap$ instead of $\cdot$ . . . .

Clearly clopen sets are closed under finite operations of $\cup$, $\cap$, $^{-}$. Checking the axioms of boolean algebras

$$\forall x, y, z \left(x + (y + z) = (x + y) + z \wedge x \cdot (y \cdot z) = (x \cdot y) \cdot z\right) \quad \text{(Associativity } +, \cdot)$$
$$\forall x, y \left(x + y = y + x \wedge x \cdot y = y \cdot x\right) \quad \text{(Commutativity of } +, \cdot)$$
$$\forall x \left(x + x = x \wedge x \cdot x = x\right) \quad \text{(Idempotence)}$$
$$\forall x, y, z \left(x \cdot (y + z) = x \cdot y + x \cdot z \wedge x + (y \cdot z) = (x + y) \cdot (x + z)\right) \quad \text{(Distributivity)}$$
$$\forall x, y \left(x \cdot (x + y) = x = x + (x \cdot y)\right) \quad \text{(Absorbtion)}$$
$$\forall x, y \left(\overline{x + y} = \overline{x} \cdot \overline{y} \wedge \overline{x \cdot y} = \overline{x} + \overline{y}\right) \quad \text{(De Morgan's Laws)}$$
$$\forall x \left(x + 0 = x \wedge x \cdot 0 = 0\right) \quad \text{(Laws of 0)}$$
$$\forall x \left(x + 1 = 1 \wedge x \cdot 1 = x\right) \quad \text{(Laws of 1)}$$
$$\forall x \left(x + \overline{x} = 1 \wedge x \cdot \overline{x} = 0 \wedge \overline{\overline{x}} = x\right) \quad \text{(Laws of } ^{-})$$

reveals that they imediately follow from the properties of $\cup$, $\cap$ and $^{-}$.

(iii) Let $\mathcal{B}$ be a boolean algebra, $C := B(S(\mathcal{B}))$ like above and $h : B \to C$, $h(a) = \langle a \rangle$.

$h(0_B) = \langle 0 \rangle = \varnothing =: 0_C$
$h(a)^c = \langle a \rangle^c = \langle \overline{a} \rangle = h(\overline{a})$
$h(a \cdot b) = \langle a \cdot b \rangle = \langle a \rangle \cap \langle b \rangle$
$h(a + b) = h(\overline{a} \cdot \overline{b})^c = (h(\overline{a}) \cap h(\overline{b}))^c = h(\overline{a}) \cup h(\overline{b})$

Hence h is a homomorphism of boolean algebras.

(iv) Let τ be the Stone topology on S.

$$S(B(S)) = \{F \subseteq B(S) : F \text{ is ultrafilter on } B(S)\}$$

Since $B(S)$ is a boolean algebra, we have $S(B(S))$ is a stone space. Let τ' be the Stone topology on $S(B(S))$. Let $h : S \to S(B(S))$, $h(x) = \{a \in S(\mathcal{B}) : x \in a\}$.

Claim: h is a bijection.

proof of claim. Let $x, y \in S$ with $h(x) = h(y)$. Therefore

$$\{a \in S(\mathcal{B}) : x \in a\} = \{a \in S(\mathcal{B}) : y \in a\}$$

⊠

Claim:

proof of claim. ⊠

□

Outernotes, Abbreviations, Overview

List of Outernotes

List of Abbreviations

List of definitions

List of Theorems

Total number of Definitions:	117
Total number of Lemmatas:	43
Total number of Propositions:	20
Total number of Theorems:	45
Total number of Corollaries:	20
Total number of Hypotheses:	1
Total number of Examples:	58
Total number of Notes:	102

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