Closure Properties of Regular Languages

Union, Intersection, Difference, Concatenation, Kleene Closure, Reversal, Homomorphism, Inverse Homomorphism

Closure Properties

- Recall a closure property is a statement that a certain operation on languages, when applied to languages in a class (e.g., the regular languages), produces a result that is also in that class.
- For regular languages, we can use any of its representations to prove a closure property.

Closure Under Union

- If L and M are regular languages, so is $L \cup M$.
- Proof: Let L and M be the languages of regular expressions R and S, respectively.
- Then R+S is a regular expression whose language is $L \cup M$.

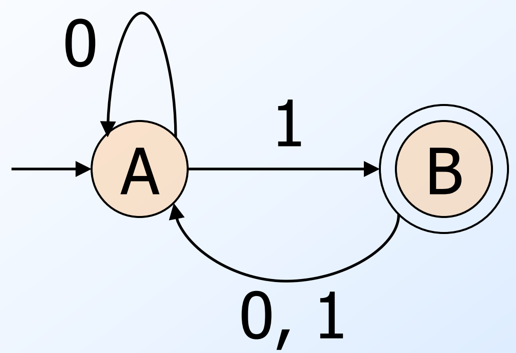
Closure Under Concatenation and Kleene Closure

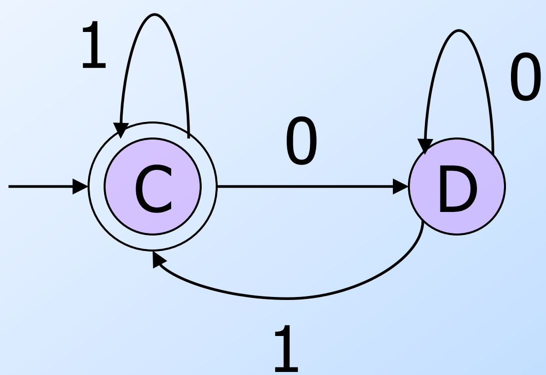
- Same idea:
  - RS is a regular expression whose language is LM.
  - R* is a regular expression whose language is L*.

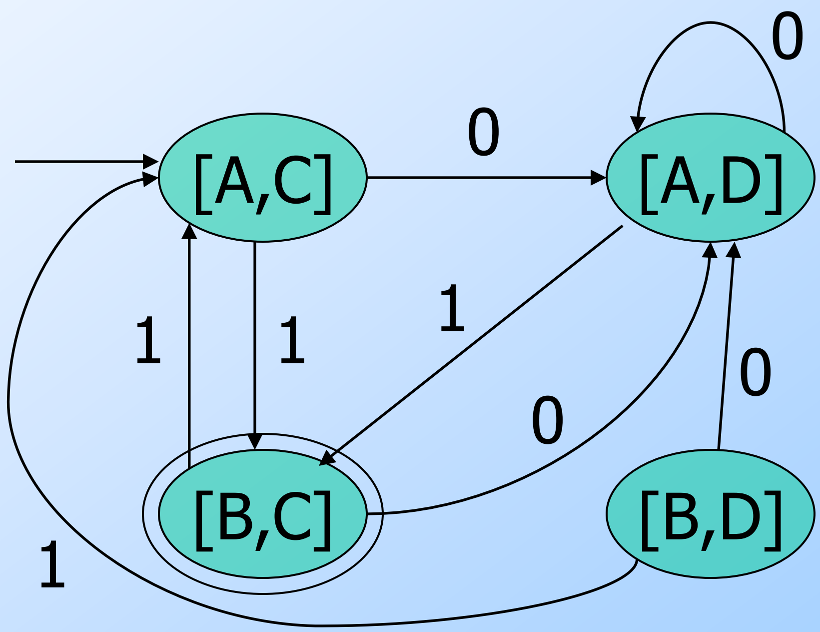
Closure Under Intersection

- If L and M are regular languages, then so is $L \cap M$.
- Proof: Let A and B be DFA's whose languages are L and M, respectively.
- Construct C, the product automaton of A and B.
- Make the final states of C be the pairs consisting of final states of both A and B.

Example: Product DFA for Intersection







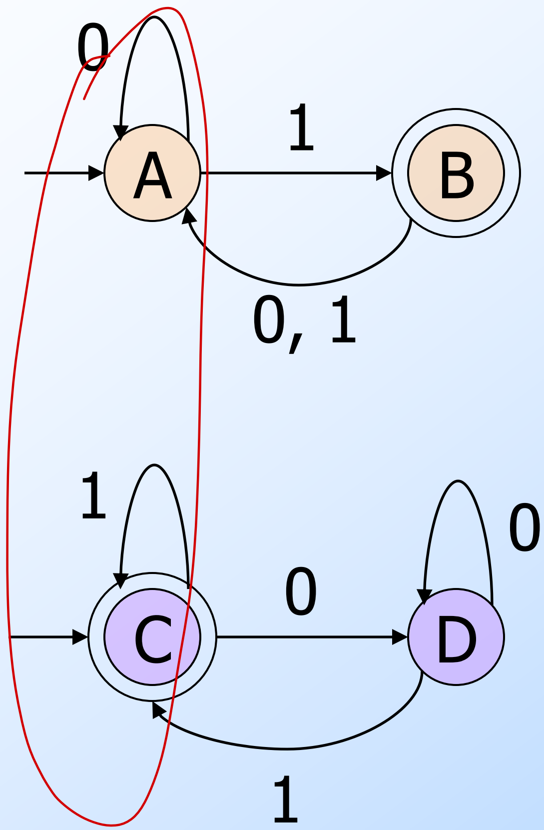
String [illegible] / $A \cap B$?

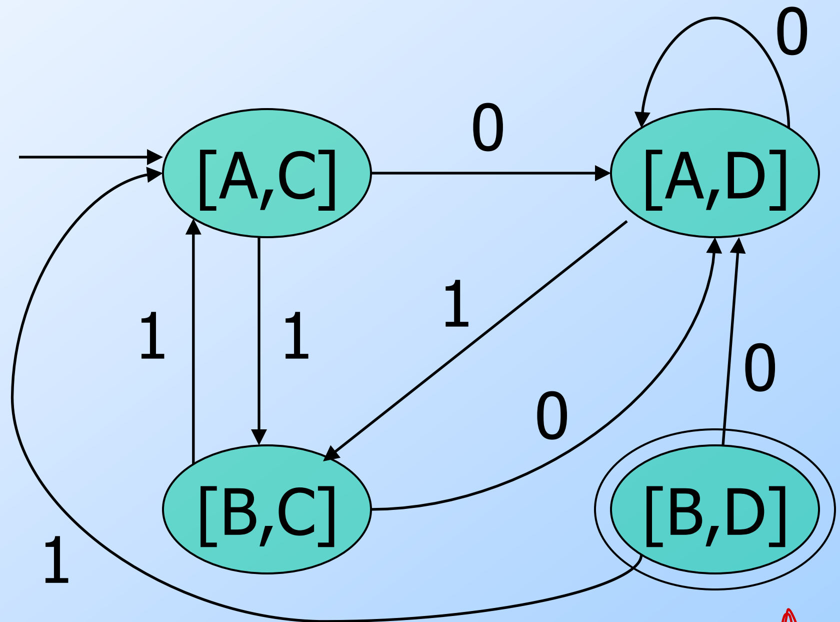
$=$ Reject

Closure Under Difference

- If L and M are regular languages, then so is $L - M$ = strings in L but not M.
- Proof: Let A and B be DFA's whose languages are L and M, respectively.
- Construct C, the product automaton of A and B.
- Make the final states of C be the pairs where A-state is final but B-state is not.

Example: Product DFA for Difference





BD and Diff = AC

Notice: difference is the empty language

Closure Under Complementation

- The *complement* of a language L (with respect to an alphabet Σ such that Σ^* contains L) is $\Sigma^* - L$.
- Since Σ^* is surely regular, the complement of a regular language is always regular.

Closure Under Reversal

- Recall example of a DFA that accepted the binary strings that, as integers were divisible by 23.
- We said that the language of binary strings whose reversal was divisible by 23 was also regular, but the DFA construction was very tricky.
- Good application of reversal-closure.

Closure Under Reversal – (2)

- Given language L, L^R is the set of strings whose reversal is in L.
- Example: L = {0, 01, 100}; L^R = {0, 10, 001}.
- Proof: Let E be a regular expression for L.
- We show how to reverse E, to provide a regular expression E^R for L^R.

R

Reversal of a Regular Expression

- **Basis**: If E is a symbol a, ϵ, or ∅, then $E^R = E$.
- **Induction**: If E is
  - E = F+G, then $E^R = F^R + G^R$.
  - E = FG, then $E^R = G^R F^R$
  - E = F*, then $E^R = (F^R)^*$.

Remember pattern.

Example: Reversal of a RE

concat สลับหน้าหลัง

- Let E = **01*** + **10***.
- $E^R = (\mathbf{01}^*+\mathbf{10}^*)^R = (\mathbf{01}^*)^R + (\mathbf{10}^*)^R$
- $= (\mathbf{1}^*)^R\mathbf{0}^R + (\mathbf{0}^*)^R\mathbf{1}^R$
- $= (\mathbf{1}^R)^*\mathbf{0} + (\mathbf{0}^R)^*\mathbf{1}$
- $= \mathbf{1}^*\mathbf{0} + \mathbf{0}^*\mathbf{1}$.

R สลับเข้าใน ( )

E^R

Homomorphisms

- A *homomorphism* on an alphabet is a function that gives a string for each symbol in that alphabet.
- Example: $h(0) = ab$; $h(1) = \epsilon$.
- Extend to strings by $h(a_1 \ldots a_n) = h(a_1) \ldots h(a_n)$.
- Example: $h(01010) = ababab$.

Closure Under Homomorphism

- If L is a regular language, and h is a homomorphism on its alphabet, then h(L) = {h(w) | w is in L} is also a regular language.

  string.
- Proof: Let E be a regular expression for L.
- Apply h to each symbol in E.
- Language of resulting RE is h(L).

Example: Closure under Homomorphism

- Let $h(0) = ab$; $h(1) = \epsilon$.
- Let L be the language of regular expression $\mathbf{01}^* + \mathbf{10}^*$.
- Then h(L) is the language of regular expression $\mathbf{ab}\epsilon^* + \epsilon(\mathbf{ab})^*$.

Note: use parentheses to enforce the proper grouping.

Example – Continued

- **ab**ϵ* + ϵ(**ab**)* can be simplified.
- ϵ* = ϵ, so **ab**ϵ* = **ab**ϵ.
- ϵ is the identity under concatenation.
  - That is, ϵE = Eϵ = E for any RE *E*.
- Thus, **ab**ϵ* + ϵ(**ab**)* = **ab**ϵ + ϵ(**ab**)* = ~~**ab**~~ + (**ab**)*.
- Finally, L(**ab**) is contained in L((**ab**)*), so a RE for h(L) is (**ab**)*.

→ เพราะ ab เป็น ตัว ใน ab หลังอยู่แล้วเลยตัด.

Inverse Homomorphisms

- Let h be a homomorphism and L a language whose alphabet is the output language of h.
- $h^{-1}(L)$ = {w | h(w) is in L}.

Example: Inverse Homomorphism

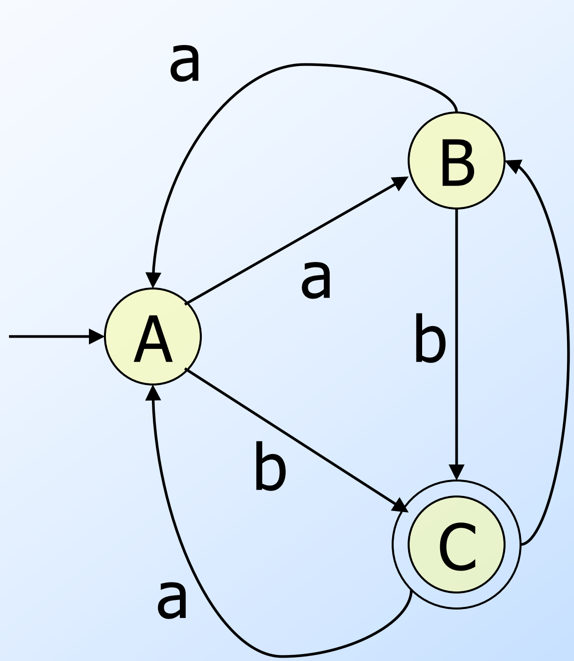
- Let $h(0) = ab$; $h(1) = \epsilon$.
- Let $L = \{abab, baba\}$.
- $h^{-1}(L)$ = the language with two 0's and any number of 1's = $L(\mathbf{1^*01^*01^*})$.

Notice: no string maps to baba; any string with exactly two 0's maps to abab.

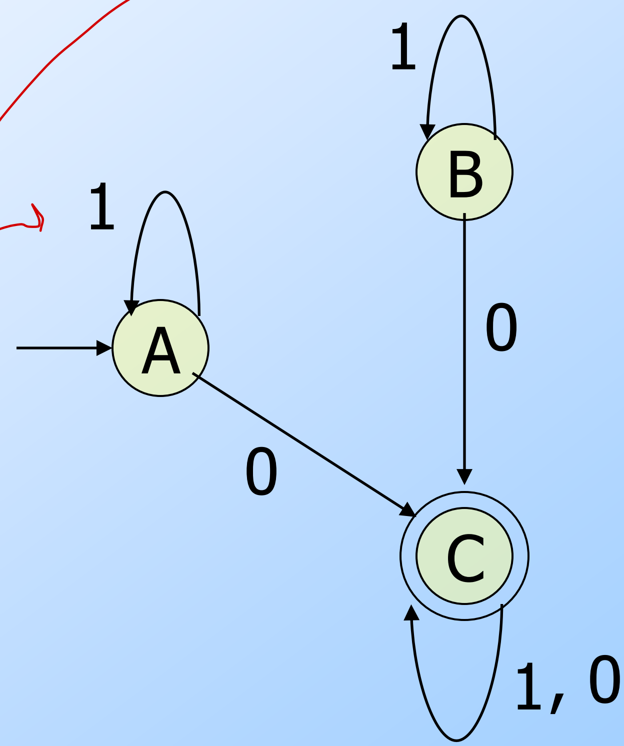
Example: Inverse Homomorphism Construction

clausure ε

วนเข้าหาตัวเอง



$h(0) = ab$
$h(1) = \epsilon$



Since
$h(1) = \epsilon$

Since
$h(0) = ab$