# Nondeterministic Finite Automata

## Nondeterminism

## Subset Construction

The slides are created by Jeffrey D. Ullman
http://infolab.stanford.edu/~ullman/ialc/spr10/spr10.html#LECTURE NOTES

$\to A \xrightarrow{1} B$

DFA

$\delta(A, 1) = B$

$$\begin{tikzcd} \arrow[r] & A \arrow[r, "1"] \arrow[dr, "1"'] & B \\ & & C \end{tikzcd}$$

NFA

$\delta(A, 1) = \{B, C\}$

# Nondeterminism

- A *nondeterministic finite automaton* has the ability to be in several states at once.
- Transitions from a state on an input symbol can be to any set of states.

$\{\ \}$

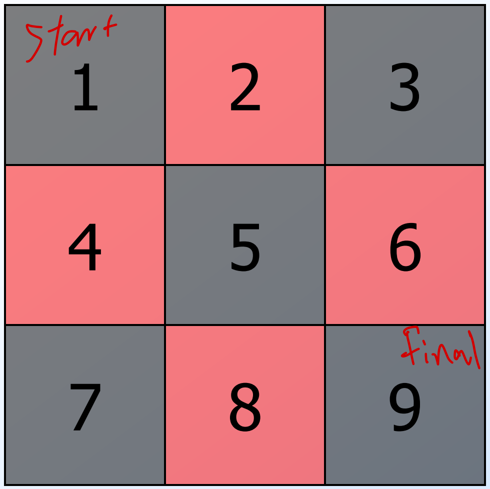
# Nondeterminism – (2)

- Start in one start state.
- Accept if any sequence of choices leads to a final state.
- Intuitively: the NFA always "guesses right."

# Example: Moves on a Chessboard

- States = squares. ช่อง
- Inputs = r (move to an ติดกันสีแดง adjacent red square) and b (move to an adjacent สีดำที่ติดกัน black square).
- Start state, final state are in ตรงข้ามกัน opposite corners.

# Example: Chessboard – (2)




| | | r | b |
|---|---|---|---|
| → | 1 | 2,4 | 5 |
| | 2 | 4,6 | 1,3,5 |
| | 3 | 2,6 | 5 |
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| | 7 | 4,8 | 5 |
| | 8 | 4,6 | 5,7,9 |
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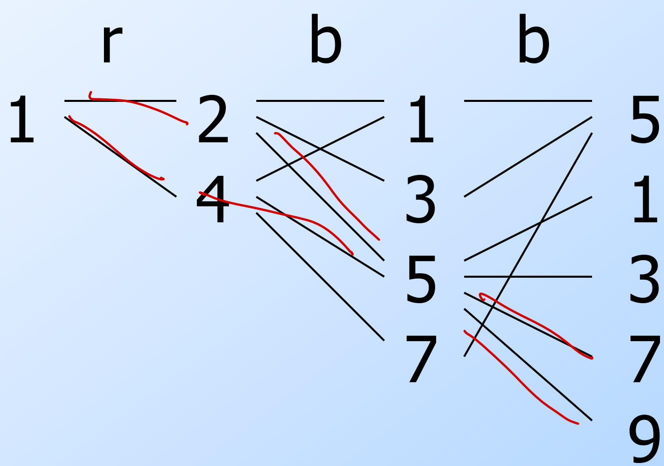



← Accept, since final state reached

# Formal NFA

- A finite set of states, typically Q.
- An input alphabet, typically $\Sigma$.
- A transition function, typically $\delta$.
- A start state in Q, typically $q_0$.
- A set of final states $F \subseteq Q$.

# Transition Function of an NFA

- $\delta(q, a)$ is a set of states.
- Extend to strings as follows:
- Basis: $\delta(q, \epsilon) = \{q\}$ NFA
- Induction: $\delta(q, wa)$ = the union over all states p in $\delta(q, w)$ of $\delta(p, a)$

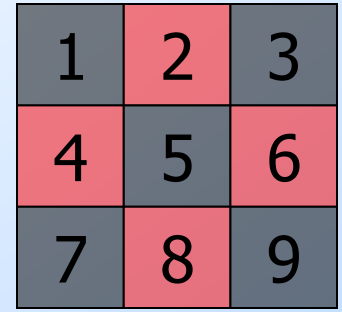
# Language of an NFA

- A string w is accepted by an NFA if $\delta(q_0, w)$ contains at least one final state.

  ต้องมี อย่างน้อย 1 final

- The language of the NFA is the set of strings it accepts.

# Example: Language of an NFA




- For our chessboard NFA we saw that rbb is accepted.
- If the input consists of only b's, the set of accessible states alternates between {5} and {1,3,7,9}, so only even-length, nonempty strings of b's are accepted.
- What about strings with at least one r?

# Equivalence of DFA's, NFA's

- A DFA can be turned into an NFA that accepts the same language.
- If $\delta_D(q, a) = p$, let the NFA have $\delta_N(q, a) = \{p\}$.
- Then the NFA is always in a set containing exactly one state – the state the DFA is in after reading the same input.

# Equivalence – (2)

- Surprisingly, for any NFA there is a DFA that accepts the same language.
- Proof is the *subset construction*.
- The number of states of the DFA can be exponential in the number of states of the NFA.
- Thus, NFA's accept exactly the regular languages.

# Subset Construction

- Given an NFA with states $\overset{3}{Q}$, inputs $\Sigma$, transition function $\delta_N$, state state $q_0$, and final states F, construct equivalent DFA with:

  2×2×2 = 8

  - States $2^{\overset{3}{Q}}$ (Set of subsets of Q).
  - Inputs $\Sigma$.
  - Start state $\{q_0\}$.
  - Final states = all those with a member of F.

# Critical Point

- The DFA states have *names* that are sets of NFA states.
- But as a DFA state, an expression like {p,q} must be read as a single symbol, not as a set.
- Analogy: a class of objects whose values are sets of objects of another class.

# Subset Construction – (2)

- The transition function $\delta_D$ is defined by:

$\delta_D(\{q_1,\ldots,q_k\}, a)$ is the union over all $i = 1,\ldots,k$ of $\delta_N(q_i, a)$.

- Example: We'll construct the DFA equivalent of our "chessboard" NFA.

# Example: Subset Construction

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| * | 9 | 6,8 | 5 |

| | r | b |
|---|---|---|
| → {1} | {2,4} | {5} |
| {2,4} | | |
| {5} | | |

Alert: What we're doing here is the *lazy* form of DFA construction, where we only construct a state if we are forced to.

# Example: Subset Construction

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|---|---|---|
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| {5} | | |
| {2,4,6,8} | | |
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| | {2,4,6,8} | {2,4,6,8} | {1,3,5,7,9} |
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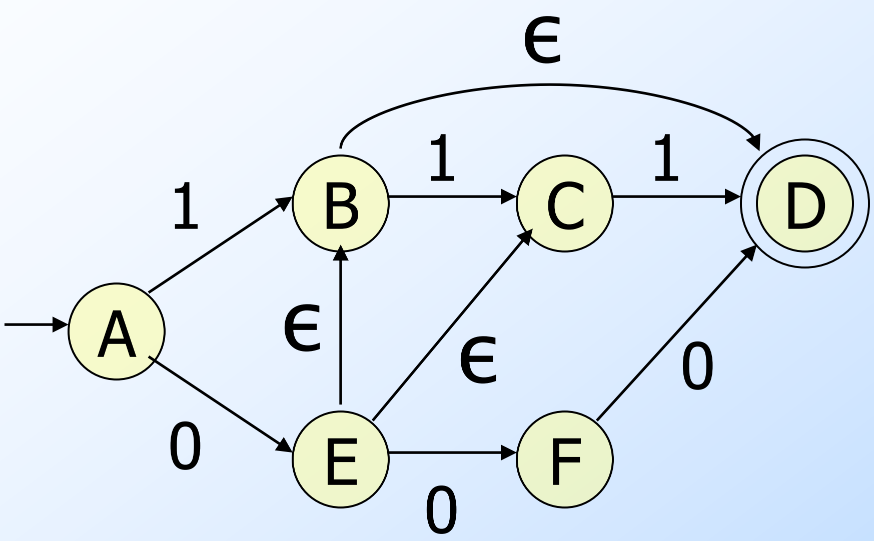
| | {1,3,5,7} | {2,4,6,8} | {1,3,5,7,9} |
| * | {1,3,7,9} | {2,4,6,8} | {5} |
| * | {1,3,5,7,9} | {2,4,6,8} | {1,3,5,7,9} |

# NFA's With ϵ-Transitions

- We can allow state-to-state transitions on ϵ input.
- These transitions are done spontaneously, without looking at the input string.
- A convenience at times, but still only regular languages are accepted.

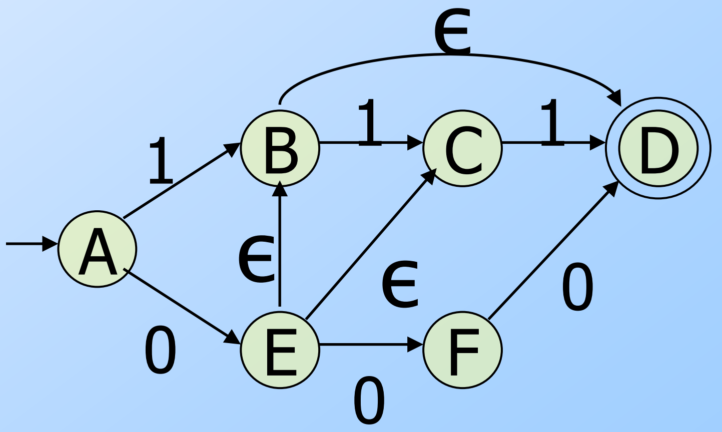
# Example: ϵ-NFA




| | | 0 | 1 | ϵ |
|---|---|---|---|---|
| → | A | {E} | {B} | ∅ |
| | B | ∅ | {C} | {D} |
| | C | ∅ | {D} | ∅ |
| * | D | ∅ | ∅ | ∅ |
| | E | {F} | ∅ | {B, C} |
| | F | {D} | ∅ | ∅ |

# Closure of States

- CL(q) = set of states you can reach from state q following only arcs labeled ϵ.
- Example: CL(A) = {A};
  CL(E) = {B, C, D, E}.



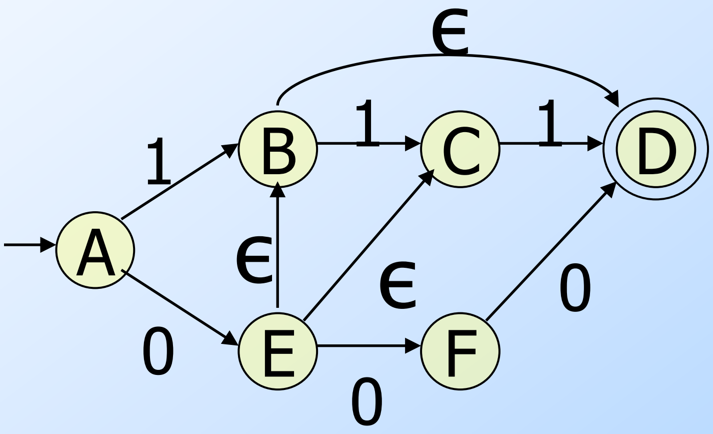

- Closure of a set of states = union of the closure of each state.

# Extended Delta

- **Basis**: $\hat{\delta}(q, \epsilon) = CL(q)$.
- **Induction**: $\hat{\delta}(q, xa)$ is computed as follows:
  1. Start with $\hat{\delta}(q, x) = S$.
  2. Take the union of $CL(\delta(p, a))$ for all p in S.
- **Intuition**: $\hat{\delta}(q, w)$ is the set of states you can reach from q following a path labeled w.

And notice that $\delta(q, a)$ is *not* that set of states, for symbol a.

# Example: Extended Delta




- $\hat{\delta}(A, \epsilon) = CL(A) = \{A\}$.
- $\hat{\delta}(A, 0) = CL(\{E\}) = \{B, C, D, E\}$.
- $\hat{\delta}(A, 01) = CL(\{C, D\}) = \{C, D\}$.
- *Language* of an ϵ-NFA is the set of strings w such that $\hat{\delta}(q_0, w)$ contains a final state.

# Equivalence of NFA, ϵ-NFA

- Every NFA is an ϵ-NFA.
  - It just has no transitions on ϵ.
- Converse requires us to take an ϵ-NFA and construct an NFA that accepts the same language.
- We do so by combining ϵ–transitions with the next transition on a real input.

Warning: This treatment is a bit different from that in the text.

# Equivalence – (2)

- Start with an ϵ-NFA with states $Q$, inputs $\Sigma$, start state $q_0$, final states $F$, and transition function $\delta_E$.
- Construct an "ordinary" NFA with states $Q$, inputs $\Sigma$, start state $q_0$, final states $F'$, and transition function $\delta_N$.

# Equivalence – (3)

- Compute $\delta_N(q, a)$ as follows:
  1. Let S = CL(q).
  2. $\delta_N(q, a)$ is the union over all p in S of $\delta_E(p, a)$.
- F' = the set of states q such that CL(q) contains a state of F.
- Intuition: $\delta_N$ incorporates $\epsilon$–transitions before using *a* but not after.

# Equivalence – (4)

- Prove by induction on |w| that

  $$CL(\delta_N(q_0, w)) = \hat{\delta}_E(q_0, w).$$

- Thus, the ϵ-NFA accepts w if and only if the "ordinary" NFA does.

# Example: ϵ-NFA-to-NFA

Interesting closures: CL(B) = {B,D}; CL(E) = {B,C,D,E}

| | | 0 | 1 | ϵ |
|---|---|---|---|---|
| → | A | {E} | {B} | ∅ |
| | B | ∅ | {C} | {D} |
| | C | ∅ | {D} | ∅ |
| * | D | ∅ | ∅ | ∅ |
| | E | {F} | ∅ | {B, C} |
| | F | {D} | ∅ | ∅ |

ϵ-NFA

| | | 0 | 1 |
|---|---|---|---|
| → | A | {E} | {B} |
| * | B | ∅ | {C} |
| | C | ∅ | {D} |
| * | D | ∅ | ∅ |
| * | E | {F} | {C, D} |
| | F | {D} | ∅ |

Since closures of B and E include final state D.

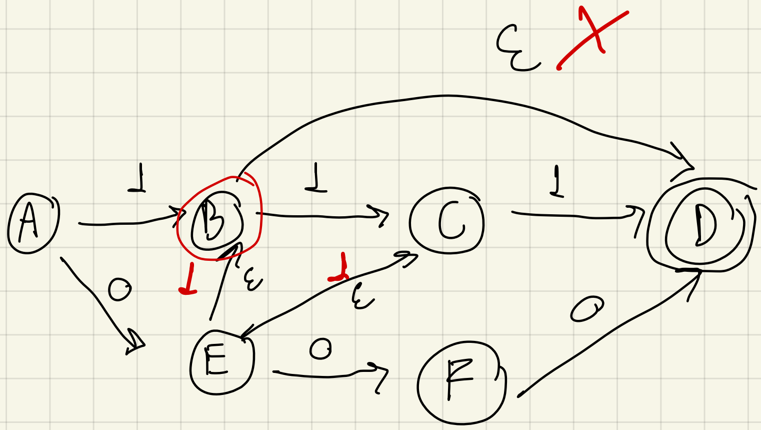
Since closure of E includes B and C; which have transitions on 1 to C and D.

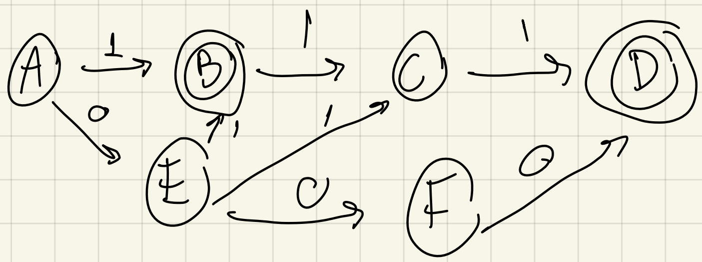
| | | 0 | 1 | ϵ |
|---|---|---|---|---|
| → | A | {E} | {B} | ∅ |
| | B | ∅ | {C} | {D} |
| | C | ∅ | {D} | ∅ |
| * | D | ∅ | ∅ | ∅ |
| | E | {F} | ∅ | {B, C} |
| | F | {D} | ∅ | ∅ |

ϵ-NFA

$$Cl(E) = \{B, C, D, E\}$$

| | | 0 | 1 |
|---|---|---|---|
| → | A | {E} | {B} |
| * | B | ∅ | {C} |
| | C | ∅ | {D} |
| * | D | ∅ | ∅ |
| * | E | {F} | {C, D} |
| | F | {D} | ∅ |



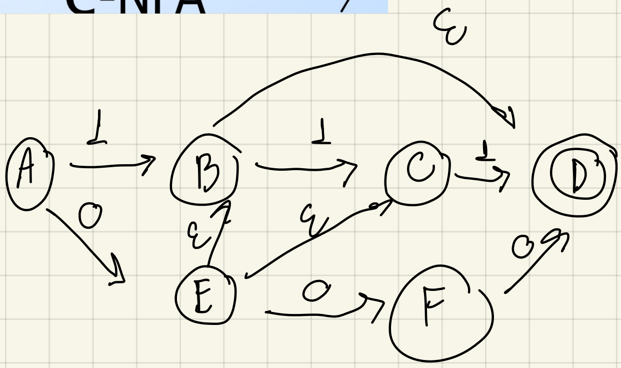


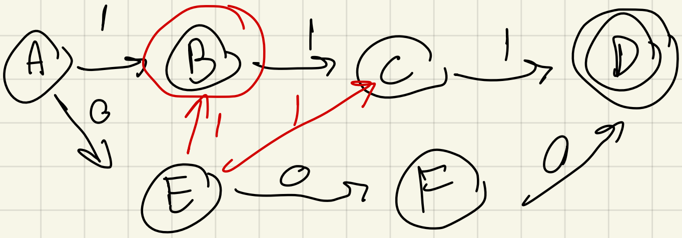
$$
\begin{array}{ccccc}
 & X & X & X & F \\
 & \uparrow & \uparrow & \uparrow & \uparrow \\
CL(E) = \{ & B, & C, & D, & E \}^{0} \\
 & \downarrow & \downarrow & \downarrow & \downarrow \\
 & C & D & X & X
\end{array}
\quad 1
$$

| | 0 | 1 | ∈ |
|---|---|---|---|
| → A | {E} | {B} | ∅ |
| B | ∅ | {C} | {D} |
| C | ∅ | {D} | ∅ |
| * D | ∅ | ∅ | ∅ |
| E | {F} | ∅ | {B, C} |
| F | {D} | ∅ | ∅ |

∈-NFA

| | 0 | 1 |
|---|---|---|
| → A | {E} | {B} |
| * B | ∅ | {C} |
| C | ∅ | {D} |
| * D | ∅ | ∅ |
| * E | {F} | {C, D} |
| F | {D} | ∅ |



# Example: ϵ-NFA-to-NFA

ε-NFA

NFA

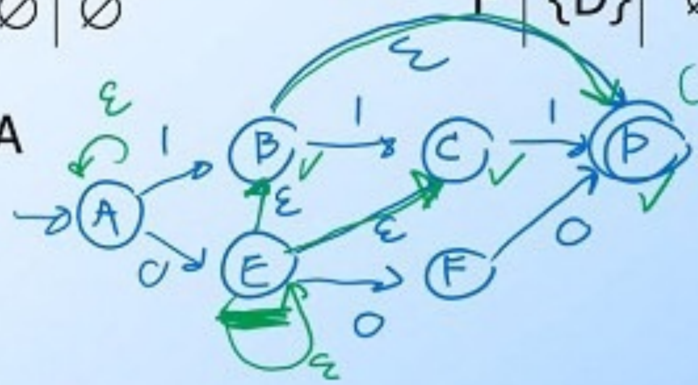
| | | 0 | 1 | ϵ |
|---|---|---|---|---|
| → | A | {E} | {B} | ∅ |
| | B | ∅ | {C} | {D} |
| | C | ∅ | {D} | ∅ |
| * | D | ∅ | ∅ | ∅ |
| | E | {F} | ∅ | {B, C} |
| | F | {D} | ∅ | ∅ |

ϵ-NFA

| | | 0 | 1 |
|---|---|---|---|
| → | A | {E} | {B} |
| * | B | ∅ | {C} |
| | C | ∅ | {D} |
| * | D | ∅ | ∅ |
| * | E | {F} | {C, D} |
| | F | {D} | ∅ |

CL(A) = {A}

CL(E) = {B, C, D, E}



# Summary

- DFA's, NFA's, and ϵ–NFA's all accept exactly the same set of languages: the regular languages.
- The NFA types are easier to design and may have exponentially fewer states than a DFA.
- But only a DFA can be implemented!