Signals and Systems (0512-2835)

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MATLAB Project

Submission Date (No Extensions): 9.1.2022

"There is no substitute for hard work – Thomas Edison".

1. Analytical and exact computation of the Fourier coefficients of signal is typically possible only in simple cases. In this exercise, you will learn how to use a very simple technique to get quick and dirty estimates of the Fourier coefficients.

   Consider a signal $x(t)$ with period T that can be expanded in a trigonometric Fourier series

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\omega_0 k t) + \sum_{k=1}^{\infty} b_k \sin(\omega_0 k t), \tag{1}$$

   where $\omega_0 = 2\pi/\mathsf{T}$. Recall that the Fourier coefficients are given by

$$a_0 = \frac{1}{\mathsf{T}} \int_{-\mathsf{T}/2}^{\mathsf{T}/2} \mathrm{d}t\, x(t), \tag{2}$$

$$a_k = \frac{2}{\mathsf{T}} \int_{-\mathsf{T}/2}^{\mathsf{T}/2} \mathrm{d}t\, x(t) \cos(\omega_0 k t), \tag{3}$$

$$b_k = \frac{2}{\mathsf{T}} \int_{-\mathsf{T}/2}^{\mathsf{T}/2} \mathrm{d}t\, x(t) \sin(\omega_0 k t), \tag{4}$$

   for $k \in \mathbb{N} \setminus 0$. To approximate these coefficients, we can use the fact that, for any sufficiently "nice" function $f(t)$, the integral $\int_{-\mathsf{T}/2}^{\mathsf{T}/2} \mathrm{d}t\, f(t)$ can be approximated by the sum

$$\frac{\mathsf{T}}{\mathsf{N}} \sum_{n=1}^{\mathsf{N}} f\left(-\frac{\mathsf{T}}{2} + \frac{2n-1}{2\mathsf{N}}\mathsf{T}\right), \tag{5}$$

where $\mathsf{N} \in \mathbb{N}$ is under our control. Approximation methods of this kind are known as *Monte Carlo methods*.

Consider a signal $x(t)$ which is a periodic extension of the function

$$s(t) = \begin{cases} -1 + \sqrt{-t}e^{t} & -1 \leq t \leq 0 \\ 1 - \sqrt{t}e^{-t} & 0 < t \leq 1. \end{cases} \tag{6}$$

Note that $\mathsf{T} = 2$. Fix an integer $M \in \mathbb{N}$. Denote by

$$x_{\mathsf{M}}(t) \triangleq a_0 + \sum_{k=1}^{\mathsf{M}} a_k \cos(\omega_0 k t) + \sum_{k=1}^{\mathsf{M}} b_k \sin(\omega_0 k t), \tag{7}$$

the approximation of $x(t)$ using the first M coefficients. Use MATLAB to solve the following items.

(a) Plot $x(t)$ for $-4 \leq t \leq 4$.

(b) Using the above Monte Carlo method approximate $x_5(t)$, $x_{15}(t)$, and $x_{30}(t)$, with $\mathsf{N} = 200$. Plot the obtained signals for $-4 \leq t \leq 4$.

(c) Define the squared error as

$$e_{\mathsf{M}} \triangleq \frac{1}{\mathsf{T}} \int_{-\mathsf{T}/2}^{\mathsf{T}/2} \mathrm{d}t \, (x(t) - x_{\mathsf{M}}(t))^2. \tag{8}$$

Use the Monte Carlo method with $\mathsf{N} = 200$ to approximate and plot $e_1, e_2, \ldots, e_{40}$. Explain your results.

2. Let $x_c(t)$ be a continuous-time signal, and let T_s denote the sampling rate. The sampled signal is denoted by $x[n]$. In the following items, consider the each one of the signals below:

$$x_{1,c}(t) = \mathrm{sinc}(t/6), \tag{9}$$

$$x_{2,c}(t) = \mathrm{sinc}^2(t/12), \tag{10}$$

$$x_{3,c}(t) = \cos(\pi t/12), \tag{11}$$

where $\mathrm{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$.

(a) Find an analytic expression for the Fourier transform $X_c(j\Omega)$ of each of the above signals.

For the following item recall that: if a signal $x[n]$ has a low pass spectrum of finite support, that is, $X(\Omega) = 0$, for $|\Omega| > \Omega_{max}$, where Ω_{max} is the maximum frequency present in the signal, then $x[n]$ is called band limited.

(b) Determine if they above signals are band-limited or not. If not, determine the frequency for which the energy of the non-band-limited signal corresponds to 99% of its total energy, and use this result to approximate its maximum frequency Ω_M. In case that one of the integrals is difficult to find analytically, consider using MATLAB. If they are band limited, find the maximum frequency Ω_M and maximum T_s for which the signal could be reconstruct from the sampled signal $x[n]$.

(c) Find $x[n]$ as a function of T_s. Plot the signal $x[n]$ and $x_c(t)$ in MATLAB using 'hold on'. *Two sampling rates are needed: one being the frequency rate under study, f_s, and the other being the one used to simulate the continuous signal, $f_{\mathsf{sim}} \gg \mathsf{f}_s$.*

(d) Plot $|X(e^{j\omega})|$ using MATLAB for $\mathsf{T}_s = 4$ and $\mathsf{T}_s = 8$. Explain your results. *The computation of the Fourier transform of x(t) can be done approximately using the fast Fourier transform (FFT). Think of the FFT as an algorithm to compute the Fourier transform of a discretized signal.*

(e) Plot the reconstructed signal $x_r(t)$ using the *sinc interpolation*:

$$x_r(t) = \sum_n x[n]\,\mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right).$$

3. Consider the following finite discrete time signal:

$$x[n] = 1 \text{ for } 0 \le n \le N-1.$$

(a) Compute the DTFT $X(e^{j\omega})$ of $x[n]$ analytically.

(b) We now define:

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN].$$

Compute the DFS coefficients $\tilde{X}[k]$ of $\tilde{x}[n]$ analytically. What is the length of a period of $\tilde{X}[k]$?

(c) In Matlab the function 'fft' (fast Fourier transform) calculates the DFT of a given vector. Using the functions 'subplot' and 'stem', plot the finite signal $x[n]$ and its DFT absolute value $|X[k]|$, for $N = \{20, 50\}$. Explain your results.

(d) Now, zero-pad the signal $x[n]$, as follows:

$$x_{\mathsf{pad}}[n] = 1 \ \text{ for } \ 0 \leq n \leq N - 1,$$

and

$$x_{\mathsf{pad}}[n] = 0 \ \text{ for } \ N \leq n \leq N + K - 1,$$

for some $K \in \mathbb{N}$. What is the length of $x_{\mathsf{pad}}[n]$? Repeat part (c) for $x_{\mathsf{pad}}[n]$ and $X_{\mathsf{pad}}[k]$, using $N = 50$ and $K = \{5, 10\}$. Explain your results.