C[ontinuations]:

← continued

and

to be continued →

At 関数型まつり 2025 June 14, 2025

Satoru Kawahara

[*1]

Q. What is the most expressive concept of functional programming?

[*1] https://arkw.net/products/web/hakase/

Twitter絵師ハカセの質問コーナー[*1]

Q. What is the most expressive concept of functional programming?

A. Definitely, it's a

CONTINUATION!

[要出典]

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Today's Topic

Learn about

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Learn about

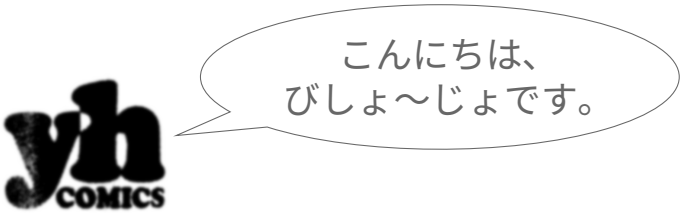
Continuations:

The Concept, History, and Practice

Continuations:
continued and to be continued

Who talks

- eiicon, co.,ltd.
- Nymphium
- OCaml
- Interested in:
  - Programming language theory and implementations
  - Control flow and its operators



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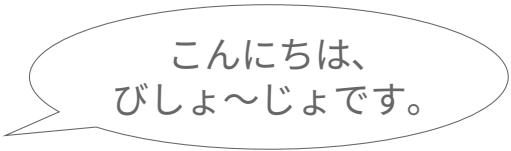

💪 Motto:

継　続　は　力　な　り

Continuation is power

Continuations?

Do you *remember* continuations?

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Hmm. . . . . I forget. Help me remember!

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- 🤷‍♂️ Hmm..... I forget. Help me remember!
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- 👱 I'm one with continuations.

Yes, they are just

callback functions!

Continuations?

E.g. Read a file, then pass the result to a callback

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readFile(file).flatMap { data ⇒ ..... }
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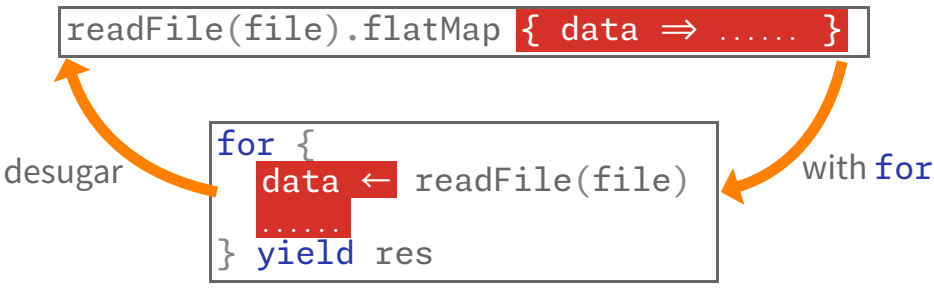

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Pseudo Continuation-Passing Style Conversion

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The Rest of The Computation!!

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Continuations:
continued and to be continued

Dump and J operator, SECD machine

The first abstract machine for evaluating functional programs[4]:

Stack	:	Stores intermediate results
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And J operator **captures the** Dump[10]

This marks the origin of **continuations!**[17][7]

●call/cc: goto Practical

Operators in Scheme[16], notably `call/cc`[15][*2]:
Captures the current continuation as a first-class value

```
(let {[(x (call/cc (λ (K)
             (+ 2 (K 4)))))]}
    (+ x 3))
; ⇒ (let {[x 4]} (+ x 3))
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ANYTHING related to *control transfer*

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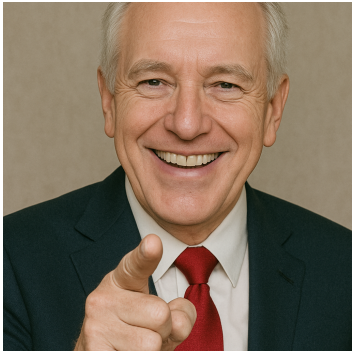
Solution:

Use
Delimited Continuations![9]

delimited
✅ capturing *scoped* continuations
✅ more structured and composable



Delimited Continuations

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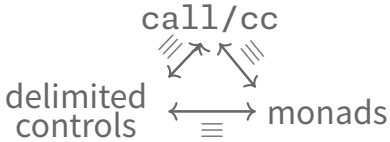
…… and **vice-versa!**[8]



Continuation-Passing Style

A program representation where control flow is made *explicit* by chaining computations as continuations[12]:

```
(define (add1 x)
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CPS Conversion! →

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①②

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①②

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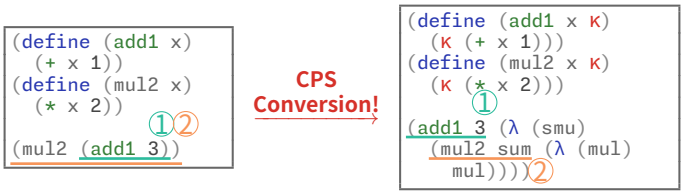

CPS *fixes* the order of evaluation and control flow, so that it's a good choice for

an intermediate representation

for language implementations!

Compiling with Continuations

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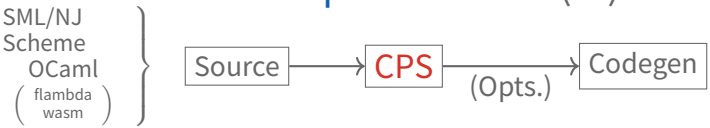

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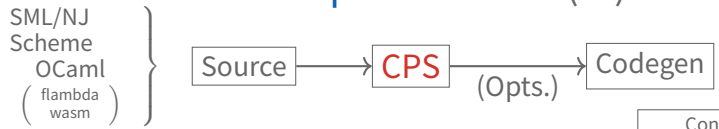

- ✅ Good for **functional languages**
  CPS operates functions as first-class values!
- ✅ Good for **optimizations**
  Several optimizations can be done by β/η, and each values are single-assignment!

Constant Folding & Inlining	⇒ β reduction
Defunctionalization	⇒ η reduction
Common Subexpression Elimination	⇒ *EZ*:

```
let z = a * b + a * b
in e
```

CPS Conversion ($[\![-]\!]$)

same!

```
( *' ) a b (fun v1 ->
( *' ) a b (fun v2 ->
( +' ) v1 v2 (fun z ->
[[e]]
```

Correspondence between procedural

How do *control transfers* correspond in functional and procedural?

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And also,
CPS ⇔ **SSA!** [2]

[*3] Static Single-Assignment form

Continuations:
continued and to be continued

Why continuations for effects?

Effects, side effects or computational effects:

- Exception
- Async I/O
- Coroutines
- etc.

Handling effects is about what happens——

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- Exception ..... *halt?*
- Async I/O ..... *when to resume?*
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So, it's time for

Continuations!

Monads

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An approach to modeling calculi with *effects*[11]:

```
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throw away continuation
to stop computation!

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Algebraic Effect Handlers

A new way to model effectful computations, with high *modularity* and *composablility*[3]:

```
type _ eff += Print : string -> unit eff

match perform @@ Print "hello" with
| effect (Print msg), k ->
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Handlers make it *easier* to **compose** and **modularise** than monad transformers**!**

[a]

Monad Transformers

Algebraic Effect Handlers

[a] https://www.youtube.com/watch?v=uxpDa-c-4Mc

Conclusion

Summary

- Explain from the history to recent trends of **continuations**
- Continuations are powerful, joyful, and useful**!**
- Research about continuations is **still hot topic**, and **continued!**

Couldn’t talk today 😭

- Type Systems
  - Answer-Type Modification
  - Linear Types for efficient runtime repr
  - Type-preserving conversion
- For Formal Languages
- Recursion
- ····· and *anything* about computation wiht continuations.

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